Finite quantum groups and quantum permutation groups

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Talk based on joint work with Teodor Banica and Sonia Natale
Quantum permutation algebras

We work over $k$, an algebraically closed field of characteristic zero.

**Definition**

A quantum permutation algebra is a Hopf algebra generated (as an algebra) by the coefficients of a matrix $x = (x_{ij}) \in M_n(H)$ such that

1. $x$ is a permutation matrix: for all $i, j, k \in \{1, \ldots, n\}$
   \[
   \sum_{l=1}^{n} x_{li} = 1 = \sum_{l=1}^{n} x_{il}, \quad x_{ij}x_{ik} = \delta_{kj}x_{ij}, \quad x_{ji}x_{ki} = \delta_{jk}x_{ji}
   \]

2. $x$ is a multiplicative matrix: for all $i, j \in \{1, \ldots, n\}$
   \[
   \Delta(x_{ij}) = \sum_{l=1}^{n} x_{il} \otimes x_{lj}, \quad \varepsilon(x_{ij}) = \delta_{ij}, \quad S(x_{ij}) = x_{ji}
   \]
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2. $x$ is a multiplicative matrix

Example

$k^{S_n}$ is a quantum permutation algebra with $x_{ij}(\sigma) = \delta_{i,\sigma(j)}$, for all $\sigma \in S_n$.

Definition

Let $A_s(n)$ be the universal algebra generated by the coefficients of a permutation matrix of size $n$. $A_s(n)$ is a quantum permutation algebra.

A Hopf algebra $H$ is a quantum permutation algebra if and only if $A_s(n) \to H$ for some $n$.

**Theorem**

$A_s(n)$ is the universal cosemisimple Hopf algebra coacting on the algebra $k^n$. This means:

1. $A_s(n)$ is cosemisimple and $k^n$ is an $A_s(n)$-comodule algebra via

   $$
   k^n \to k^n \otimes A_s(n)
   $$

   $$
   e_i \mapsto \sum_{k=1}^n e_k \otimes x_{ki}
   $$

2. If $k^n$ is a comodule algebra over a cosemisimple Hopf algebra $H$ with coaction $\beta : k^n \to k^n \otimes H$, then there is a unique Hopf algebra map $f : A_s(n) \to H$ with $(1 \otimes f) \circ \alpha = \beta$

Thus we write $A_s(n) = \mathcal{O}(S_n^+)$, where $S_n^+$ is the quantum permutation group on $n$ points, and quantum permutation algebras correspond to quantum permutation groups.
Theorem

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We observe that

1. $A_s(n) \cong k^S_n$ if $n \leq 3$,

2. $A_s(n + m) \rightarrow A_s(n) \ast A_s(m)$, so $\dim A_s(n) = \infty$ if $n \geq 4$.

Hence the symmetric group $S_n$ has an infinite quantum analogue if $n \geq 4$!

Banica has shown that the fusion rules of $A_s(n)$ are the same as those of $PGL_2$ (1999, when $k = \mathbb{C}$).
Early examples of quantum permutation algebras

1. $O(O_{-1}(n))$ (corresponding to the quantum automorphism group of the hypercube in $\mathbb{R}^n$).

2. $(kA_5)^\sigma$ (so that $A_5$ has a quantum analogue acting faithfully on 4 points).

3. The Kac-Paljutkin algebra of dimension 8 (as well as other series of Hopf algebras studied by Masuoka).

4. Some 2-cocycle deformations of $kS_n$.

Several of these examples were unexpected at first sight.

So it becomes natural to wonder if there are lots of quantum permutation algebras. A basic obstruction to being a quantum permutation algebra is the following one:

If $H$ is a quantum permutation algebra, then $\text{Hom}_{k-\text{alg}}(H, k)$ is finite and $S^2 = \text{id}_H$. So if $H$ is a finite-dimensional quantum permutation algebra, then $H$ is semisimple.
So a reasonable question is:

Is any (finite dimensional) semisimple Hopf algebra a quantum permutation algebra?

In other words, in view of the universal property of \( A_s(n) = \mathcal{O}(S_n^+) \), is there a Cayley theorem for finite quantum groups?

Naturally this leads to other more specific questions.

Is the class of finite quantum permutation algebras stable under

1. duality?
2. extensions?
3. 2-cocycle deformations?
Extensions and quantum permutation algebras

We now wish to study the stability of the class of quantum permutation algebras under extensions.
If \( \Gamma \) is a finite group, the algebras \( k^\Gamma \) and \( k\Gamma \) are quantum permutation algebras.

**Theorem**

Let \( H \) be a Hopf algebra that fits into an exact sequence

\[
k \rightarrow k^\Gamma \rightarrow H \rightarrow kF \rightarrow k
\]

for some finite groups \( \Gamma, F \). Assume that one of the following conditions holds:

1. \( k^\Gamma \) is central in \( H \);
2. the sequence is split (\( H = k^\Gamma \# kF \)) and \( F \) is generated by its \( \Gamma \)-stable abelian subgroups;

Then \( H \) is a quantum permutation algebra.
Idea of proof: we observe that $H$ is a quantum permutation algebra if and only if $H$ is generated by its commutative (right) coideal subalgebras. So we find a family of such coideal subalgebras. □

By using the theorem together with various classification results (Masuoka, Natale, Kashina, Etingof-Nikshych-Ostrik) we get

**Corollary**

Let $H$ be a semisimple Hopf algebra. Then $H$ is a quantum permutation algebra if one the following holds:

1. $\dim H = p^3$, with $p$ prime;
2. $\dim H = 2q^2$, with $q$ prime;
3. $\dim H = pq^2$, with $p > q$ prime;
4. $\dim H = pqr$, with $p, q, r$ distinct primes;
5. $\dim H = 16$.

In particular if $\dim H \leq 23$, then $H$ is a quantum permutation algebra.
Theorem

The Hopf algebras $k^{C_4} \# kS_3$, $k^{C_5} \# kS_4$, $k^{C_5} \# kA_4$ (respectively associated to the group exact factorizations $S_4 = S_3 C_4$, $S_5 = S_4 C_5$, $A_5 = A_4 C_5$) are not quantum permutation algebras.

Thus there exists a semisimple Hopf algebra of dimension 24 that is not a quantum permutation algebra.

Corollary

The class of quantum permutation algebras is not stable under extensions, duality or 2-cocycle deformations.

Indeed, $H = k^{C_4} \# kS_3$ is not a quantum permutation algebra, while $H^* = k^{S_3} \# kC_4$ is a quantum permutation algebra by the first theorem. Moreover $D(H)^* \cong (D(S_4)^*)^\sigma$ for some 2-cocycle $\sigma$ (Beggs-Gould-Majid). The first theorem ensures that $D(S_4)^*$ is a quantum permutation algebra, while $D(H)^*$ is not (because $D(H)^* \twoheadrightarrow H$). □
Sketch of the proof of the theorem

We have to see that $H = k\Gamma \# kF$ is not generated by its commutative (right) coideal subalgebras. It is not easy to have the full list of these coideal subalgebras, so instead we use the following observations:

**Lemma**

If $\pi : H \to kF$ is a surjective Hopf algebra map and if there exits a proper subgroup $F' \subsetneq F$ such that $\pi(R) \subset kF'$ for any commutative (right) coideal subalgebra $R \subset H$, then $H$ is not a quantum permutation algebra.

**Lemma**

Let $H = k\Gamma \# kF$ and $\pi = \epsilon \otimes \text{id} : H \to kF$. Let $R \subset H$ be a commutative right coideal subalgebra. Then $\pi(R) = kT$, where $T$ is an abelian subgroup of $F$, and we have:

(i) If $k\Gamma \subseteq R$, then $T$ acts trivially on $\Gamma$ via $\triangleleft$.

(ii) If $k\Gamma \cap R = k1$, then $T$ is stable under the action $\triangleright$ of $\Gamma$. 

Now assume that $H = k^{C_5} \# k^{S_4}$ (exact factorization $S_5 = S_4 C_5$ and actions : $C_5 \leftarrow C_5 \times S_4 \rightarrow S_4$).

If $R$ is a commutative right coideal subalgebra of $H$, then $R \cap k^{C_5}$ is a right coideal subalgebra of $k^{C_5}$, hence a Hopf subalgebra of $k^{C_5}$ and thus $\dim(R \cap k^{C_5})$ divides 5. We are in the situation of the previous lemma: we have $\pi(R) = kT$ where $T$ is an abelian subgroup of $S_4$ and either $T$ acts trivially on $C_5$ via $\triangleleft$ or $T$ is stable under the action $\triangleright$ of $C_5$.

The only subgroup of $S_4$ that acts trivially on $C_5$ is $\{1\}$, and the only abelian subgroups of $S_4$ that are stable under the action $\triangleright$ of $C_5$ are contained in $\langle (1324) \rangle = F'$. Thus $\pi(R) \subset kF'$, and we conclude by the first lemma. □

**Question**

What is the smallest dimension that a self dual non quantum permutation algebra can have?
Some quantum permutation algebras obtained by 2-cocycle deformations

We have seen that the class of quantum permutation algebras is not stable under 2-cocycle deformations. We wish to show however that large classes of quantum permutation algebras can be constructed in this way.

Let $\Gamma$ be an abelian group and let $\sigma \in Z^2(\Gamma, k^*)$. The character group $\hat{\Gamma}$ acts faithfully on the twisted group algebra $k_\sigma \Gamma$ by $\chi.g = \chi(g)g$ ($\chi \in \hat{\Gamma}$, $g \in \Gamma$), hence $\hat{\Gamma} \subset \text{Aut}(k_\sigma \Gamma)$.

**Theorem**

Let $\Gamma$ be a finite abelian group and let $\sigma \in Z^2(\Gamma, k^*)$. Let $G$ be a linear algebraic group with $\hat{\Gamma} \subset G \subset \text{Aut}(k_\sigma \Gamma)$. Then $\sigma$ induces a 2-cocycle $\sigma'$ on $\mathcal{O}(G)$ such that $\mathcal{O}(G)^{\sigma'}$ is a quantum permutation algebra (non commutative if the only subgroup of $\hat{\Gamma}$ that is normal in $G$ is $\{1\}$ and if $k_\sigma \Gamma$ is non commutative).
Examples: \( \hat{\Gamma} = C_2^n \subset G \subset O_n(k) \subset \text{Aut}(C\ell_n(k)) \)
\( \hat{\Gamma} = C_n \times C_n \subset G \subset \text{PGL}_n(k) = \text{Aut}(M_n(k)) \)

**Question**

If \( G \) is a finite group and \( \sigma \) is a 2-cocycle on \( k^G \), is \( (k^G)^\sigma \) a quantum permutation algebra?