COHOMOLOGICAL DIMENSION OF HOPF ALGEBRAS

JULIEN BICHON

Abstract. These are the notes for a series of lectures given at Córdoba University, november 2017. These lectures provide a light and Hopf algebra oriented introduction to homological algebra, with a special emphasis on cohomological dimension.

Contents

Introduction

The cohomological dimension (most often called the global dimension) is a classical and important invariant of an algebra, of homological nature, that powerfully generalizes the usual definition of dimension for an affine algebraic set[1]. In this text, which provides the notes for a series of lectures given at the university of Córdoba (november 2017), we present and discuss cohomological dimension for Hopf algebras, starting from the minimal necessary homological algebra material, and finishing by a number of recent research questions.

These notes are organized as follows. Section 1 provides a short review of Hopf algebra theory, with the basic main definitions, a panel of key examples and a number of results about modules over Hopf algebras. It is hoped that the reader unfamiliar with Hopf algebra theory will find enough material (and enough motivation) to go on with the rest of these notes. In Section 2 we present and discuss in detail projective modules, the basic objects for homological algebra, and define the projective dimension of a module. This already enables us, in Section 3, to define the cohomological dimension of a Hopf algebra. To study and prove properties about cohomological dimension, we need more homological algebra material, that Section 4 provides, with the construction of Ext and Tor. In Section 5 we explain in some detail the meaning of cohomological dimension for Hopf algebras of polynomial functions on affine algebraic groups. In Section 6 we introduce homology and cohomology of Hopf algebras, and we study the case of the function algebra on quantum SL_2 in detail. Section 7 discusses the behaviour of cohomological dimension under exact sequences of Hopf algebras. Section 8 presents Poincaré duality in the setting of Hopf algebras, while Section 9 is a more advanced (and sketchy) one, discussing a recent research question, the possible invariance of cohomological dimension under monoidal equivalence.

Throughout these notes, the base field is the field of complex numbers (which has no effect on the theoretical results, but can have some effect when discussing examples).

1. Hopf algebras

This section is a short review on the theory of Hopf algebras, with the basic definitions, a number of key examples, and some basic results about their modules and comodules.

We warn the reader that the presentation is designed to highlight the facts that we believe to be the most important and useful for the rest of the notes, but does not follow the logical order that one would need for a complete course on the subject.

1.1. Basic definitions and examples.

Definition 1.1. A **Hopf algebra** is an algebra A together with algebra maps
 (1) $\Delta : A \longrightarrow A \otimes A$ (comultiplication)
 (2) $\varepsilon : A \longrightarrow \mathbb{C}$ (counit)
 (3) $S : A \longrightarrow A^{\mathrm{op}}$ (antipode)

satisfying the following axioms:
(a) $(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta$ (Coassociativity)
(b) $(\varepsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A = (\mathrm{id}_A \otimes \varepsilon) \circ \Delta$ (counit axiom)
(c) $m \circ (\mathrm{id}_A \otimes S) \circ \Delta = u \circ \varepsilon = m \circ (S \otimes \mathrm{id}_A) \circ \Delta$ (antipode axiom),
where $m : A \otimes A \to A$ and $u : \mathbb{C} \to A$ are the respective multiplication and unit of A.

The most popular Hopf algebra textbook is [48]. The interested reader will find an historical account of the theory of Hopf algebras in [3].

Example 1.2. Let Γ be a discrete group, and let $\mathbb{C}\Gamma$ be its group algebra with $\mathbb{C}$-basis $\{e_g, g \in \Gamma\}$, multiplication $e_g e_h = e_{gh}$ and unit element e_1. This is a Hopf algebra with, for any $g \in \Gamma$,

$$\Delta(e_g) = e_g \otimes e_g, \quad \varepsilon(e_g) = 1, \quad S(e_g) = e_{g^{-1}}$$

Example 1.3. Let G be an affine algebraic group: G is both an affine algebraic set and a group, and the group law $G \times G \to G$ and inversion map $G \mapsto G$, $x \mapsto x^{-1}$, are morphisms of affine algebraic sets, i.e. are polynomial maps.

[1]There are also other competing natural notions of dimension for an algebra, such as the Gelfand-Kirillov dimension.

Let $\mathcal{O}(G)$ be the algebra of polynomial functions on G. The group structure of G induces a Hopf algebra structure on $\mathcal{O}(G)$, with comultiplication induced by the multiplication $m : G \times G \to G$:

$$\begin{aligned} \Delta : \mathcal{O}(G) &\longrightarrow \mathcal{O}(G \times G) \simeq \mathcal{O}(G) \otimes \mathcal{O}(G) \\ f &\longmapsto f \circ m \longmapsto \Delta(f) \end{aligned}$$

The counit is defined by

$$\begin{aligned} \varepsilon : \mathcal{O}(G) &\longrightarrow \mathbb{C} \\ f &\longmapsto f(1) \end{aligned}$$

and the antipode is induced by the inversion map in G

$$\begin{aligned} S : \mathcal{O}(G) &\longrightarrow \mathcal{O}(G) \\ f &\longmapsto S(f), S(f)(x) = f(x^{-1}) \end{aligned}$$

Example 1.4. Let again G be an affine algebraic group, with G defined as a subgroup of the general linear group $\mathrm{GL}_n(\mathbb{C})$. Let u_{ij}, $1 \leq i, j \leq n$, be the coordinate functions on G: for $g = (g_{ij}) \in G$, $u_{ij}(g) = g_{ij}$. The elements u_{ij} belong to $\mathcal{O}(G)$, and $D = \det((u_{ij})) \in \mathcal{O}(G)$ is an invertible element in $\mathcal{O}(G)$, so that the matrix $(u_{ij}) \in M_n(\mathcal{O}(G))$ is invertible. The algebra $\mathcal{O}(G)$ is generated by the elements u_{ij}, $1 \leq i, j \leq n$, D^{-1}, and we have

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \ \varepsilon(u_{ij}) = \delta_{ij}, \ S(u_{ij}) = (u^{-1})_{ij}$$

where u^{-1} stands for the inverse of the matrix $u = (u_{ij}) \in M_n(\mathcal{O}(G))$.

Example 1.5. Let $\mathfrak{g} = (\mathfrak{g}, [,])$ be a Lie algebra, and let $U(\mathfrak{g})$ be its enveloping algebra: $U(\mathfrak{g}) = T(\mathfrak{g})/([x, y] - xy + yx, x, y \in \mathfrak{g})$. Then $U(\mathfrak{g})$ is a Hopf algebra, with comultiplication, counit and antipode defined, for $x \in \mathfrak{g}$, by

$$\Delta(x) = 1 \otimes x + x \otimes 1, \ \varepsilon(x) = 0, \ S(x) = -x$$

One defines morphisms of Hopf algebras in the obvious way, we get a category, and the constructions of examples 1.2, 1.3 and 1.5 define functors.

There are important groups naturally attached to a Hopf algebra, as well as a Lie algebra.

Definition 1.6. Let A be a Hopf algebra.

(1) An element $a \in A$ is said to be **group-like** if $\Delta(a) = a \otimes a$ and $\varepsilon(a) = 1$. The set of group-like elements in A is denoted $\mathrm{Gr}(A)$, the multiplication of A induces a group structure on $\mathrm{Gr}(A)$, with for $a \in \mathrm{Gr}(A)$, $a^{-1} = S(a)$.

(2) The set of algebra maps $A \longrightarrow \mathbb{C}$, denoted $G(A)$, is a group under the law

$$\phi \cdot \psi := (\phi \otimes \psi) \circ \Delta$$

The unit element is ε and the inverse of $\phi \in G(A)$ is $\phi \circ S$. If A is finitely generated as an algebra, then $G(A)$ is an affine algebraic group.

(3) Let

$$\mathcal{P}(A) = \{x \in A \mid \Delta(x) = 1 \otimes x + x \otimes 1\}$$

An element in $\mathcal{P}(A)$ is called a **primitive element**, and $\mathcal{P}(A)$ is a subspace of A, having a natural Lie algebra structure, whose braket is defined by $[a, b] = ab - ba$.

These constructions allow us to reconstruct the groups and the Lie algebra from the Hopf algebras in examples 1.2, 1.3 and 1.5, and the Hopf algebras arising in this way can be characterized, mainly using properties of their categories of comodules (see the third subsection).

Example 1.7. (1) If Γ is a discrete group, we have a group isomorphism $\Gamma \simeq \mathrm{Gr}(\mathbb{C}\Gamma)$, $x \longmapsto e_x$ (exercise). Therefore the Hopf algebra $\mathbb{C}\Gamma$ completely determines the group Γ. Moreover, a cocommutative (for any $a \in A$, $\Delta(a) = \tau\Delta(a)$, where $\tau : A \otimes A \to A \otimes A$ is the canonical flip) and cosemisimple Hopf algebra A is isomorphic to $\mathbb{C}\Gamma$, for $\Gamma = \mathrm{Gr}(A)$.

(2) Let G be an affine algebraic group. Then $G(\mathcal{O}(G))$ is an affine algebraic group, and

$$\begin{aligned}\iota : G &\longrightarrow G(\mathcal{O}(G)) \\ x &\longmapsto \iota(x), \ \iota(x)(f) = f(x)\end{aligned}$$

is an isomorphim. A commutative and finitely generated Hopf algebra is isomorphic to $\mathcal{O}(G)$, for $G = G(A)$. This is Cartier's theorem, see [48, 67]

(3) Let $\mathfrak{g}$ be a Lie algebra. The map

$$\begin{aligned}\iota : \mathfrak{g} &\longrightarrow \mathcal{P}(U(\mathfrak{g})) \\ x &\longmapsto x,\end{aligned}$$

is a Lie algebra isomorphim, see [48, Chapter 5]). A cocommutative and connected Hopf algebra is isomorphic to $U(\mathfrak{g})$, for $\mathfrak{g} = \mathcal{P}(A)$, again see [48, chapter 5].

To construct more examples of Hopf algebras, in particular examples that are neither commutative or cocommutative, several crossed product like constructions are available. Here is the simplest one.

Example 1.8. Let Γ be a discrete group acting on a Hopf algebra A, via a group morphism $\alpha : \Gamma \to \mathrm{Aut}(A)$ (where $\mathrm{Aut}(A)$ means the group of Hopf algebra automorphisms of A). To this data, we associate, as usual, the crossed product algebra $A \rtimes \mathbb{C}\Gamma$, which has $A \otimes \mathbb{C}\Gamma$ as underlying vector space, and product defined by

$$a \otimes g \cdot b \otimes h = a\alpha_g(b) \otimes gh, \quad a, b \in A, \ g, h \in G$$

Then $A \rtimes \mathbb{C}\Gamma$ has a natural Hopf algebra structure defined by

$$\Delta(a \otimes g) = a_{(1)} \otimes g \otimes a_{(2)} \otimes g, \ \varepsilon(a \otimes g) = \varepsilon(a), \ S(a \otimes g) = \alpha_{g^{-1}}(S(a)) \otimes g^{-1}$$

and A identifies with a Hopf subalgebra of $A \rtimes \mathbb{C}\Gamma$ via $a \mapsto a \otimes 1$.

Example 1.4 exactly paves the way to construct examples by generators and relations, by the following useful result. The proof is left as an exercise.

Lemma 1.9. *Let A be an algebra endowed with algebra maps $\Delta : A \longrightarrow A \otimes A$, $\varepsilon : A \longrightarrow \mathbb{C}$, and $S : A \longrightarrow A^{\mathrm{op}}$. Assume that there exists a matrix $u = (u_{ij}) \in M_n(A)$ such that the following conditions hold:*

(1) $u = (u_{ij})$ is an invertible matrix;
(2) A is generated, as an algebra, by the coefficients of the matrix u;
(3) for any i, j, we have

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \ \varepsilon(u_{ij}) = \delta_{ij}, \ S(u_{ij}) = (u^{-1})_{ij}$$

Then A, endowed with the above structure maps, is a Hopf algebra.

Using this lemma, we can construct the celebrated quantum group $\mathrm{SL}_q(2)$.

Example 1.10 ($\mathrm{SL}(2)$ and its quantum q-deformation $\mathrm{SL}_q(2)$). Let $q \in \mathbb{C}^*$. The algebra $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ is, as an algebra, presented by generators a, b, c, d, submitted to the relations

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad cb = bc,$$
$$ad - da = (q^{-1} - q)bc, \quad ad - q^{-1}bc = 1$$

Letting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

and using Lemma 1.9, we get that $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ is a Hopf algebra with

$$\Delta(u_{ij}) = \sum_{k=1}^{2} u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}$$

and

$$S(a) = d,\ S(b) = -qb,\ S(c) = -q^{-1}c,\ S(d) = a$$

If $q = 1$, then $\mathcal{O}_1(\mathrm{SL}_2(\mathbb{C}))$ is commutative with $G(\mathcal{O}_1(\mathrm{SL}_2(\mathbb{C}))) \simeq \mathrm{SL}_2(\mathbb{C})$ and thus we have $\mathcal{O}_1(\mathrm{SL}_2(\mathbb{C})) \simeq \mathcal{O}(\mathrm{SL}_2(\mathbb{C}))$.

If $q \neq 1$, then $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ is noncocommutative and noncommutative (this is not completely obvious), and is often denoted $\mathcal{O}(\mathrm{SL}_q(2))$, to emphazize the viewpoint that this is the algebra of polynomial functions on the quantum group $\mathrm{SL}_q(2)$. It was introduced in independent fundamental works of Drinfeld [27] and Woronowicz [69]. There are also quantum groups $\mathrm{SL}_q(n)$ and q-deformations for other classical algebraic groups that we do not discuss here, see the textbooks [16, 39].

Example 1.11 (Hopf algebras attached to bilinear forms). Among the many possible generalizations of quantum SL_2, we will be interested in the following one, introduced by Dubois-Violette and Launer [28]. Let $E \in \mathrm{GL}_n(\mathbb{C})$. The algebra $\mathcal{B}(E)$ [28] is presented by generators $(u_{ij})_{1\leq i,j\leq n}$ and relations

$$E^{-1}u^tEu = I_n = uE^{-1}u^tE,$$

where u is the matrix $(u_{ij})_{1\leq i,j\leq n}$. Using Lemma 1.9, we see that $\mathcal{B}(E)$ has a Hopf algebra structure defined by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj},\ \varepsilon(u_{ij}) = \delta_{ij},\ S(u) = E^{-1}u^tE$$

Adding commutation relations, we get the algebra of polynomial functions on the automorphism group of the bilinear form defined by the matrix E, so we consider $\mathcal{B}(E)$ as the Hopf algebra representing the quantum automorphism group of this bilinear form.

For the matrix

$$E_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix}$$

we have $\mathcal{B}(E_q) = \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$, and thus the Hopf algebras $\mathcal{B}(E)$ are generalizations of $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$.

While the algebra $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ share many ring-theoretical properties with $\mathcal{O}(\mathrm{SL}_2(\mathbb{C}))$, this is much less the case for $\mathcal{B}(E)$ in general. For example $\mathcal{B}(E)$ is not Noetherian if $n > 2$. We will see, however, that from the homological algebra viewpoint, $\mathcal{B}(E)$ still has a number of the (pleasant) features of $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$.

Example 1.12 (Free algebras). Let $A = \mathbb{C}\langle x_1, \cdots, x_n\rangle$ be the free algebra on n generators. Then A has a Hopf algebra structure given by

$$\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1,\ \varepsilon(x_i) = 0,\ S(x_i) = -x_i$$

Notice that this is also the universal enveloping algebra of the free Lie algebra on n generators.

Example 1.13 (The Sweedler algebra). The next example mixes those constructed form discrete groups and those constructed from Lie algebras. Let A be the algebra presented by generators x, g submitted to the relations $g^2 = 1$, $x^2 = 0$, $xg = -gx$. Then A is a 4-dimensional algebra, and has a Hopf algebra structure given by

$$\Delta(x) = 1 \otimes x + x \otimes g,\ \Delta(g) = g \otimes g,\ \varepsilon(x) = 0, \varepsilon(g) = 1,\ \ S(x) = -xg,\ S(g) = g$$

This is a noncommutative and noncocommutative Hopf algebra, and a key toy example.

Example 1.14 (S_n and the quantum permutation group S_n^+, [65]). Let $n \geq 1$. Consider the commutative algebra A presented by generators $x_{ij},\ 1 \leq i, j \leq n$, submitted to the relations of permutation matrices $(1 \leq i, j, k \leq n)$

$$\sum_{l=1}^{n} x_{il} = 1 = \sum_{l=1}^{n} x_{li}, \quad x_{ij}x_{ik} = \delta_{jk}x_{ij},\ x_{ji}x_{ki} = \delta_{jk}x_{ji}$$

We have algebra maps

$$\Delta : A \longrightarrow A \otimes A, \quad \varepsilon : A \longrightarrow \mathbb{C}, \quad \text{and} \quad S : A \longrightarrow A$$

defined by $(1 \leq i, j \leq n)$

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) = \delta_{ij}, \quad S(x_{ij}) = x_{ji}$$

that endow A with a Hopf algebra structure. We have $G(A) \simeq S_n$, and hence $A \simeq \mathcal{O}(S_n)$.

The free version "S_n^+" of S_n is obtained by removing the commutativity relations in the above presentation of $\mathcal{O}(S_n)$. More precisely let $A_s(n)$ be the algebra presented by generators x_{ij}, $1 \leq i, j \leq n$, submitted to the relations of permutation matrices $(1 \leq i, j, k \leq n)$

$$\sum_{l=1}^{n} x_{il} = 1 = \sum_{l=1}^{n} x_{li}, \quad x_{ij}x_{ik} = \delta_{jk}x_{ij}, \; x_{ji}x_{ki} = \delta_{jk}x_{ji}$$

We have algebra maps

$$\Delta : A_s(n) \longrightarrow A_s(n) \otimes A_s(n), \quad \varepsilon : A_s(n) \longrightarrow \mathbb{C}, \quad \text{and} \quad S : A_s(n) \longrightarrow A_s(n)^{\mathrm{op}}$$

defined by $(1 \leq i, j \leq n)$

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) = \delta_{ij}, \quad S(x_{ij}) = x_{ji}$$

that endow $A_s(n)$ with a Hopf algebra structure, noncommutative and noncocommutative if $n \geq 4$. We put $A_s(n) = \mathcal{O}(S_n^+)$, and call S_n^+ the quantum permutation group on n points. The quantum permutation group S_n^+ is the largest compact quantum group acting on the classical set formed by n points, whence his name, see [65].

We finish the subsection by presenting the very convenient **Sweedler notation**, which consists of writing, for a Hopf algebra A and $a \in A$,

$$\Delta(a) = a_{(1)} \otimes a_{(2)}$$

With this notation, the Hopf algebra axioms become

$$(\Delta \otimes \mathrm{id}_A)\Delta(a) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)} = (\mathrm{id}_A \otimes \Delta)\Delta(a)$$

$$\varepsilon(a_{(1)})a_{(2)} = a = a_{(1)}\varepsilon(a_{(2)}), \quad S(a_{(1)})a_{(2)} = \varepsilon(a)1 = a_{(1)}S(a_{(2)})$$

1.2. **Modules over a Hopf algebra.** The category $\mathcal{M}_A$ of (right) A-modules has a number of pleasant additional properties and features when A is a Hopf algebra, that we present now.

(1) The trivial A-module $\mathbb{C}_\varepsilon$: this is the A-module whose underlying vector space is $\mathbb{C}$, and whose A-module structure is defined by $1 \cdot a = \varepsilon(a)$.

(2) If M and N are A-modules, then $M \otimes N$ has a natural A-module structure, defined by

$$(x \otimes y) \cdot a = x \cdot a_{(1)} \otimes y \cdot a_{(2)}, \quad \forall x \in M, y \in N, a \in A$$

(3) If M, N are A-modules, then $\mathrm{Hom}(M, N)$ has a natural right A-module structure defined by

$$f \cdot a(x) = f(x \cdot S(a_{(1)})) \cdot a_{(2)}$$

In particular, $M^* = \mathrm{Hom}(V, \mathbb{C})$ has a natural A-module structure, defined by $f \cdot a(x) = f(x \cdot S(a))$.

We leave it to the reader to check that the above receipes indeed define A-module structures. For A-modules M, N, P, it is an immediate verification to check, using the Hopf algebra axioms, that the canonical isomorphisms

$$(M \otimes N) \otimes P \simeq M \otimes (N \otimes P), \quad M \otimes \mathbb{C}_\varepsilon \simeq M \simeq \mathbb{C}_\varepsilon \otimes M$$

are A-linear. This means that $\mathcal{M}_A$ is a tensor subcategory of the category $\mathrm{Vect}(\mathbb{C})$ (see [30]).

If M is a right A-module, we denote

$$M^A = \{x \in M \mid x \cdot a = \varepsilon(a)x\}$$

the subspace of A-invariants.

Lemma 1.15. *We have*

$$\mathrm{Hom}(M,N)^A = \mathrm{Hom}_A(M,N)$$

for any A-modules M, N.

Proof. If $f \in \mathrm{Hom}_A(M,N)$, we have for any $a \in A$ and $x \in M$, $f \cdot a(x) = f(x \cdot S(a_{(1)})) \cdot a_{(2)} = f(x) \cdot S(a_{(1)})a_{(2)} = f(x)\varepsilon(a)$, hence $f \in \mathrm{Hom}(M,N)^A$. Conversely, if $f \in \mathrm{Hom}(M,N)^A$, we have $f(x \cdot S(a_{(1)})) \cdot a_{(2)} = \varepsilon(a)f(x)$ for any $a \in A$, $x \in M$. Hence

$$f(x \cdot a) = f(x \cdot a_{(1)})\varepsilon(a_{(2)}) = f(x \cdot a_{(1)}S(a_{(2)})) \cdot a_{(3)} = f(x) \cdot a$$

and $f \in \mathrm{Hom}_A(M,N)$. □

Lemma 1.16. *Let M be a right A-module.*

(1) The evaluation map

$$e_M : M \otimes M^* \to \mathbb{C}_\varepsilon, \ x \otimes f \mapsto f(x)$$

is A-linear.

(2) If M is finite-dimensional, there exists an A-linear map $\delta_M : \mathbb{C}_\varepsilon \to M^* \otimes M$ such that

$$(e_M \otimes \mathrm{id}_M)(\mathrm{id}_M \otimes \delta_M) = \mathrm{id}_M \quad \text{and} \quad (\mathrm{id}_{M^*} \otimes e_M)(\delta_M \otimes \mathrm{id}_{M^*}) = \mathrm{id}_{M^*}$$

and there are, for any A-modules X and Y, natural isomorphisms

$$\mathrm{Hom}_A(X, Y \otimes M) \simeq \mathrm{Hom}_A(X \otimes M^*, Y)$$

Proof. For $x \in M$, $f \in M^*$, and $a \in A$, we have

$$\begin{aligned} e_M\left((x \otimes f) \cdot a\right) =& e_M\left(x \cdot a_{(1)} \otimes f \cdot a_{(2)}\right) = f \cdot a_{(2)}(x \cdot a_{(1)}) \\ =& f\left(x \cdot a_{(1)}S(a_{(2)})\right) = \varepsilon(a)f(x) = \varepsilon(a)e_M(f \otimes x) \end{aligned}$$

Hence e_M is A-linear. If M is finite-dimensional, let $e_1, \dots, e_n$ be basis of M, with dual basis $e_1^*, \dots, e_n^*$, and define $\delta_M : \mathbb{C}_\varepsilon \to M^* \otimes M$ by $\delta_M(1) = \sum_{i=1}^n e_i^* \otimes e_i$. It is immediate that the above equations are satisfied, and we have to chack that δ_M is A-linear. Consider the linear isomorphism

$$\begin{aligned} F : M^* \otimes M &\longrightarrow \mathrm{End}(M) \\ \psi \otimes x &\longmapsto \widetilde{\psi \otimes x}, \ y \mapsto \psi(y)x \end{aligned}$$

We leave it to the reader to check that F is A-linear, hence $\delta_M = F^{-1}(\mathrm{id}_M) \in (M^* \otimes M)^A$ (Lemma 1.15) is indeed A-linear. To conclude, the announced natural isomorphisms are given by

$$\begin{aligned} \mathrm{Hom}_A(X, Y \otimes M) &\longrightarrow \mathrm{Hom}_A(X \otimes M^*, Y) \\ f &\longmapsto (\mathrm{id}_Y \otimes e_M)(f \otimes \mathrm{id}_{M^*}) \\ (g \otimes \mathrm{id}_M)(\mathrm{id}_X \otimes \delta_M) &\longleftarrow g \end{aligned}$$

□

Proposition 1.17. *Let M be a right A-module. The map*

$$\begin{aligned} M_t \otimes A &\longrightarrow M \otimes A \\ x \otimes a &\longmapsto x \cdot a_{(1)} \otimes a_{(2)} \end{aligned}$$

is an isomorphism of A-modules, where $M_t \otimes A$ is the free A-module whose A-module structure is given by multiplication on the right.

Proof. Denote by ψ be the above map. We have

$$\psi((x \otimes a) \cdot b) = \psi(x \otimes ab) = x \cdot (a_{(1)}b_{(1)}) \otimes a_{(2)}b_{(2)} = (x \cdot a_{(1)}) \cdot b_{(1)}) \otimes a_{(2)}b_{(2)} = \psi(x \otimes a) \cdot b$$

and this shows that ψ is A-linear. Let

$$\begin{aligned} \phi : M \otimes A &\longrightarrow M_t \otimes A \\ x \otimes a &\longmapsto x \cdot S(a_{(1)}) \otimes a_{(2)} \end{aligned}$$

We have

$$\phi\psi(x \otimes a) = \phi(x \cdot a_{(1)} \otimes a_{(2)}) = x \cdot a_{(1)}S(a_{(2)}) \otimes a_{(3)} = x \otimes \varepsilon(a_{(1)})a_{(2)} = x \otimes a$$

and hence $\phi\psi$ is the identity map. One checks similarly that $\psi\phi$ is the identity map. $\square$

1.3. **Comodules.** We now discuss comodules over a Hopf algebra, which correspond to representations of the corresponding algebraic quantum group, and are crucial in analysing its structure.

Definition 1.18. Let A be a Hopf algebra. A (right) A**-comodule** is a vector space V endowed with a linear map $\alpha : V \longrightarrow V \otimes A$ (called coaction) such that the following conditions are satisfied:

(1) $(\alpha \otimes \mathrm{id}_A) \circ \alpha = (\mathrm{id}_V \otimes \Delta) \circ \alpha$;
(2) $(\mathrm{id}_V \otimes \varepsilon) \circ \alpha = \mathrm{id}_V$.

Examples 1.19. (1) The comultiplication $\Delta : A \longrightarrow A \otimes A$ endows A with a right A-comodule structure, called the regular A-comodule.
(2) The one-dimensional A-comodules correspond to the group-like elements of A.
(3) Let Γ be a group and V be a vector space. A $\mathbb{C}\Gamma$-comodule structure on V is the same as a Γ-grading on V, i.e. a direct sum decomposition $V = \oplus_{g \in \Gamma} V_g$.
(4) Let G be an affine algebraic group. An $\mathcal{O}(G)$-comodule structure on a finite-dimensional vector space precisely corresponds to a (polynomial) representation $G \to \mathrm{GL}(V)$, see Proposition 1.21.

One defines morphisms of comodules in a straightforward manner: if A is a Hopf algebra and $V = (V, \alpha_V)$ and $W = (W, \alpha_W)$ are A-comodules, an A-comodule morphism $V \longrightarrow W$ is a linear map $f : V \longrightarrow W$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ \downarrow{\scriptstyle \alpha_V} & & \downarrow{\scriptstyle \alpha_W} \\ V \otimes A & \xrightarrow{f \otimes \mathrm{id}_A} & W \otimes A \end{array}$$

One also says that an A-comodule morphism is an A-colinar map. The set of A-comodule morphisms from V to W is denoted $\mathrm{Hom}^A(V, W)$, this is a linear supspace of $\mathrm{Hom}_{\mathbb{C}}(V, W)$.

The category of A-comodules is denoted $\mathcal{M}^A$. This is an abelian subcategory of $\mathrm{Vect}(\mathbb{C})$, the category of vector spaces, which means that the standard operations in linear algebra such as direct sums, kernels, cokernels can be performed inside this category.

If V is an A-comodule with coaction $\alpha : V \to V \otimes A$, the Sweedler notation is

$$\alpha(v) = v_{(0)} \otimes v_{(1)}$$

and the comodule axioms are

$$(\alpha \otimes \mathrm{id}_A)\alpha(v) = v_{(0)} \otimes v_{(1)} \otimes v_{(2)} = (\mathrm{id}_V \otimes \Delta)\alpha(v), \ (\mathrm{id}_V \otimes \varepsilon)\alpha(v) = v_{(0)}\varepsilon(v_{(1)}) = v$$

A remarkable feature of comodules is that any element in a comodule is contained in a finite-dimensional subcomodule: this is the fundamental theorem of comodules. This shows that the study of comodules essentially reduces to the study of the finite-dimensional ones.

The following definition comes from Example 1.4.

Definition 1.20. Let A be a Hopf algebra and let $u = (u_{ij}) \in M_n(A)$ be a matrix. We say that u is a **multiplicative matrix** if for all $i, j \in \{1, \dots, n\}$, we have

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \ \varepsilon(u_{ij}) = \delta_{ij}$$

Finite-dimensional comodules can be described by means of multiplicative matrices, as shown by the following result, whose verification is an easy exercise.

Proposition 1.21. *Let A be a Hopf algebra and let V be a finite-dimensional vector space.*

(1) Assume that V has an A-comodule structure with coaction $\alpha : V \longrightarrow V \otimes A$. Let $v_1, \dots, v_n$ be a basis of V and let $x = (x_{ij}) \in M_n(A)$ be the matrix such that $\forall i$,

$$\alpha(v_i) = \sum_{j=1}^{n} v_j \otimes x_{ji}$$

Then $x = (x_{ij})$ is a multiplicative matrix.

(2) Conversely, if $x = (x_{ij}) \in M_n(A)$ is a multiplicative matrix, for each basis of V, the above formula defines an A-comodule structure on V.

Similarly to the category of modules, the category of comodules over a Hopf algebra has a tensor category structure, defined as follows.

• If $V = (V, \alpha_V)$, $W = (W, \alpha_W)$ are comodules over A, their tensor product has a natural A-comodule structure defined by

$$V \otimes W \overset{\alpha_V \otimes \alpha_W}{\longrightarrow} V \otimes A \otimes W \otimes A \overset{\mathrm{id} \otimes \tau \otimes \mathrm{id}}{\longrightarrow} V \otimes W \otimes A \otimes A \overset{\mathrm{id} \otimes m}{\longrightarrow} V \otimes W \otimes A$$

The natural associativity isomorphisms $(V \otimes W) \otimes Z \simeq V \otimes (W \otimes Z)$ are morphisms of comodules.

• The trivial comodule $\mathbb{C}$ is defined by $1 \mapsto 1 \otimes 1_A$.

These structures make the category $\mathcal{M}^A$ into a **tensor category**, see [30, 38], and is a tensor subcategory of $\mathrm{Vect}(\mathbb{C})$.

Definition 1.22. A Hopf algebra A is said to be

(1) **cosemisimple** if every A-comodule is semisimple, i.e. every subcomodule of a comodule admits a supplementary comodule (in this case every comodule is a direct sum of simple comodules);
(2) **pointed** if every simple A-comodule is one-dimensional;
(3) **connected** if the trivial comodule is the unique simple comodule.

Examples 1.23. (1) A group algebra is cosemisimple and pointed.
(2) The enveloping algebra of a Lie algebra is connected.
(3) The algebra of polynomial functions on an affine algebraic group is cosemisimple if and only if the group is linearly reductive.
(4) The Hopf algebra $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ is cosemisimple if and only if $q = \pm 1$ or q is not a root of unity (in which case we say that q is generic).

If A is cosemisimple Hopf algebra, the set of isomorphism classes of simple A-comodules together with the decompositions of tensor products of simple comodules into direct sums of simple comodules produces a combinatorial data called the **fusion rules** of A (see e.g. [5]). One of the most exciting results in quantum group theory (in my opinion) is that for q generic, $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ as the same fusion rules as $\mathcal{O}(\mathrm{SL}_2(\mathbb{C}))$.

There is also a stronger relation than the one of having the same fusion rules, the relation of monoidal equivalence.

Definition 1.24. We say that two Hopf algebras A and B are **monoidally equivalent** if there exists a tensor category equivalence $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ (i.e an equivalence of categories that preserves the tensor products up to isomorphism in a coherent way, we refer to [30, 38, 51] for the precise definition).

Among the previous Hopf algebras, let us mention the monoidal equivalences

$$\mathcal{M}^{\mathcal{B}(E)} \simeq^{\otimes} \mathcal{M}^{\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))}, \text{ for } q+q^{-1} = -\mathrm{tr}(E^{-1}E^t), \ [7]$$

and

$$\mathcal{M}^{A_s(n)} \simeq^{\otimes} \mathcal{M}^{\mathcal{O}_q(\mathrm{PSL}_2(\mathbb{C}))}, \text{ for } q+q^{-1} = \sqrt{n}, \ [22, 49]$$

where $\mathcal{O}_q(\mathrm{PSL}_2(\mathbb{C}))$ is defined in Example 7.10.

Notice that $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ and $\mathcal{O}_p(\mathrm{SL}_2(\mathbb{C}))$ are monoidally equivalent only when $p = q^{\pm 1}$, in which case they are isomorphic.

To construct more examples, one can use the notion of Hopf 2-cocycle. Let A be a Hopf algebra. A **2-cocycle** on A (see [26]) is a convolution invertible linear map $\sigma : A \otimes A \longrightarrow k$ satisfying

$$\sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c) = \sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)})$$

and $\sigma(a,1) = \sigma(1,a) = \varepsilon(a)$, for all $a, b, c \in A$.

The deformed Hopf algebra A^σ [26] is defined to be $A^\sigma = A$ as a vector space, the comultiplication and counit are the same as those of A (so that $A^\sigma = A$ as coalgebras), the unit is the one of A, the product is defined by

$$a.b = \sigma(a_{(1)}, b_{(1)})\sigma^{-1}(a_{(3)}, b_{(3)})a_{(2)}b_{(2)}$$

(where σ^{-1} denotes the convolution inverse of σ) and the antipode is defined by

$$S(a) = \sigma(a_{(1)}, S(a_{(2)}))\sigma^{-1}(S(a_{(4)}), a_{(5)})S(a_{(3)})$$

Since $A = A^\sigma$ as coalgebra, the identity functor defines an equivalence of categories $\mathcal{M}^A \simeq \mathcal{M}^{A^\sigma}$ that we can enrich to a monoidal equivalence $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^{A^\sigma}$ using the natural isomorphisms

$$\begin{aligned} V \otimes W &\longrightarrow V \otimes W \\ v \otimes w &\longmapsto \sigma^{-1}(v_{(1)}, w_{(1)})v_{(0)} \otimes w_{(0)} \end{aligned}$$

There are many examples of such deformed Hopf algebras, in particular in the finite-dimensional case. We present an infinite-dimensional example.

Example 1.25. The Hopf algebra $\mathcal{O}_{q,q^{-1}}(\mathrm{GL}_2(\mathbb{C}))$ is the algebra presented by generators a, b, c, d, δ^{-1} subject to the relations

$$ba = qab, \ dc = qcd, \ ca = q^{-1}ac, \ db = q^{-1}bd, \ qcb = q^{-1}bc,$$
$$da = ad, \ (ad - q^{-1}bc)\delta^{-1} = 1 = \delta^{-1}(ad - q^{-1}bc).$$

The Hopf algebra structure on $\mathcal{O}_{q,q^{-1}}(\mathrm{GL}_2(\mathbb{C}))$ is given by the usual formulas

$$\Delta(a) = a \otimes a + b \otimes c, \ \Delta(b) = a \otimes b + b \otimes d, \ \Delta(c) = c \otimes a + d \otimes c, \ \Delta(d) = c \otimes b + d \otimes d,$$
$$\Delta(\delta^{-1}) = \delta^{-1} \otimes \delta^{-1}, \ \varepsilon(a) = \varepsilon(d) = \varepsilon(\delta^{-1}) = 1, \ \varepsilon(b) = \varepsilon(c) = 0,$$
$$S(a) = d\delta^{-1}, \ S(b) = -qb\delta^{-1}, \ S(c) = -q^{-1}c\delta^{-1}, \ S(d) = a\delta^{-1}, \ S(\delta^{-1}) = ad - q^{-1}bc.$$

This Hopf algebra is part of the 2-parameter deformations of $\mathrm{GL}_2(\mathbb{C})$ in [61], and is a 2-cocycle deformation, as above, of $\mathcal{O}(\mathrm{GL}_2(\mathbb{C}))$. See [61].

2. Projective and injective modules

2.1. **Projective modules.** Let A be an algebra and let P be an A-module. The functor $\mathrm{Hom}_A(P, -)$ from A-modules to vector spaces is left exact: if

$$0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$$

is an exact sequence of A-modules (in the usual sense: i is injective, p is surjective, and $\mathrm{Im}(i) = \mathrm{Ker}(p)$), then the sequence

$$0 \to \mathrm{Hom}_A(P, X) \xrightarrow{i\circ -} \mathrm{Hom}_A(P, Y) \xrightarrow{p\circ -} \mathrm{Hom}_A(P, Z)$$

is exact (check this). Projective modules are precisely those for which this functor is exact.

Proposition-Definition 2.1. *A (right) A-module P is said to be* ***projective*** *if one of the equivalent following conditions holds.*

(1) The functor $\mathrm{Hom}_A(P, -)$ *is exact.*
(2) For any surjective A-linear $p : M \to N$ *and any A-linear map* $\phi : P \to N$*, there exists an A-linear map* $\psi : P \to M$ *such that* $p\psi = \phi$*:*

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\psi} & \downarrow{\scriptstyle \phi} & & \\ M & \xrightarrow{p} & N & \longrightarrow & 0 \end{array}$$

(3) Any surjective A-linear map $f : M \to P$ *admits a section, i.e. there exists an A-linear map* $s : P \to M$ *such that* $fs = \mathrm{id}_P$*.*
(4) There exists a free A-module F and an A-module Q such that $F \simeq P \oplus Q$ *as A-modules.*
(5) There exists families $(x_i)_{i\in I}$ *and* $(f_i)_{i\in I}$ *of elements of P and* $\mathrm{Hom}_A(P, A)$ *respectively such that, for any* $x \in P$, $x = \sum_{i\in I} f_i(x)x_i$*.*

Proof. The equivalence between (1) and (2) is done by just writing the definitions. (3) follows from (2), applied to $N = P$ and $\phi = \mathrm{id}_P$. Assume that (3) holds and consider a surjective A-linear map $f : F \to A$ for some free A-module F (obtained by choosing a generating subset of P). One gets an A-linear map $s : P \to F$ such that $fs = \mathrm{id}_P$, and then $F \simeq P \oplus Q$, for $Q = \mathrm{Ker}(sf)$, so (4) holds.

Assume that (4) holds: we have an A-linear isomorphism $F \simeq P \oplus Q$ for F free and another A-module Q. This means that there exists A-linear maps $u_1 : P \to F$, $u_2 : Q \to F$, $q_1 : F \to P$, $q_2 : F \to Q$ such that $q_1u_1 = \mathrm{id}_P$, $q_2u_2 = \mathrm{id}_Q$, $q_2u_1 = 0 = q_1u_2$, $\mathrm{id}_F = u_1p_1 + u_2p_2$. Choosing a basis $(e_i)_{i\in I}$ of F, with dual basis $(e_i^*)_{i\in I}$ ($e_i^* \in \mathrm{Hom}_A(F, A)$), one obtains the announced elements by letting $x_i = q_1(e_i)$ and $f_i = e_i^* u_1$.

Assume finally that (5) holds, and consider a surjective A-linear $p : M \to N$ and an A-linear map $\phi : P \to N$. For $i \in I$, fix z_i in M be such that $p(z_i) = \phi(x_i)$. Define $\psi : P \to M$ by $\psi(x) = \sum_{i\in I} f_i(x)z_i$. It is clear that ψ is A-linear and that $p\psi = \phi$, so (2) holds. □

The proof of the following result is left to the reader.

Proposition 2.2. *If* $M = \oplus_{i\in I} M_i$ *is a direct sum of A-modules , then M is projective if and only if each* M_i *is.*

For a Hopf algebra A, projectivity of the trivial A-module $\mathbb{C}_\varepsilon$ has very strong consequences on the structure of A.

Proposition 2.3. *Let A be a Hopf algebra. The following properties are equivalent.*

(1) The trivial A-module $\mathbb{C}_\varepsilon$ *is projective.*
(2) There exists $t \in A$ *such that* $ta = \varepsilon(a)t$*, for any* $a \in A$*, and* $\varepsilon(t) = 1$*.*
(3) The algebra A is semisimple and finite-dimensional.

Proof. (1) $\Rightarrow$ (2): the counit can be interpreted as a surjective A-linear map $\varepsilon : A \to \mathbb{C}_\varepsilon$. Hence if $\mathbb{C}_\varepsilon$ is projective, the previous proposition furnishes a section to ε, and hence the announced t. An algebra is semisimple precisely when all its modules are projective, so (3) $\Rightarrow$ (1) is trivial.

It remains to prove that (2) $\Rightarrow$ (3). Assume that such a t exists. Given an A-module M, recall from the previous section that M^A denotes the space of A-invariants: $M^A = \{x \in M \mid x \cdot a = \varepsilon(a)x, \forall a \in A\}$. It is not difficult to check that for t as in (2), one has $M^A = M \cdot t$.

Now if M, N are A-modules, endow $\mathrm{Hom}(M, N)$ with the right A-module structure defined by $f \cdot a(x) = f(x \cdot S(a_{(1)})) \cdot a_{(2)}$ (see the previous section). We have seen (Lemma 1.15)that $\mathrm{Hom}_A(M, N) = \mathrm{Hom}(M, N)^A$. So for $f \in \mathrm{Hom}(M, N)$, we have $f \cdot t\ \mathrm{Hom}_A(M, N)$. If $N \subset M$ is a sub-A-module, let $p : M \to N$ be a $\mathbb{C}$-linear map such that $p_{|N} = \mathrm{id}_N$. One sees easily that still $p \cdot t_{|N} = \mathrm{id}_N$, so we have the direct sum of A-modules $M = N \oplus \mathrm{Ker}(p \cdot t)$, and A is indeed semisimple.

To conclude that A is finite-dimensional, we will show that the linear map

$$\begin{aligned} A^* &\longrightarrow A \\ \omega &\longmapsto \omega(t_{(1)})t_{(2)} \end{aligned}$$

is injective (see Lemma 1.2 in [62] for a left-handed version), which will force A to be finite-dimensional.

For $a \in A$, we have

$$\begin{aligned} ta = \varepsilon(a)t \Rightarrow ta_{(1)} \otimes a_{(2)} = t \otimes a \Rightarrow\ t_{(1)}a_{(1)} \otimes t_{(2)}a_{(2)} \otimes a_{(3)} = t_{(1)} \otimes t_{(2)} \otimes a \\ \Rightarrow t_{(1)}a_{(1)} \otimes t_{(2)}a_{(2)}S(a_{(3)}) = t_{(1)} \otimes t_{(2)}S(a) \Rightarrow\ t_{(1)}a \otimes t_{(2)} = t_{(1)} \otimes t_{(2)}S(a) \end{aligned}$$

Hence if ω is in the kernel of the above map, we have $\omega(t_{(1)}a)t_{(2)} = 0$ for any $a \in A$. Writing $\Delta(t) = \sum_{i=1}^m a_i \otimes b_i$ with $b_1, \dots, b_m$ linearly independent, we thus have $\omega(a_i a) = 0$ for any i and any a. Hence $\omega(a_i S(b_i)a) = 0$ for any i, and

$$0 = \sum_{i=1}^m \omega(a_i S(b_i)a) = \omega(t_{(1)}S(t_{(2)})a) = \varepsilon(t)\omega(a) = \omega(a)$$

Hence $\omega = 0$, as needed. □

Proposition 2.4. *Let P be a projective module over a Hopf algebra A, and let X be an A-module. Then the A-module $X \otimes P$ is projective.*

Proof. If $P = A$, we have seen in Proposition 1.17 that $X \otimes A$ is free, hence projective. It follows that if P is free, then so is $X \otimes P$. In general, we have $F \simeq P \oplus Q$ for some free module F, so $X \otimes F \simeq (X \otimes P) \oplus (X \otimes Q)$, hence $X \otimes P$, being a direct summand of a free module, is a projective module. □

2.2. **Projective dimension of a module.** We now define the projective dimension of a module, which measures how far it is from being projective. It is the key step towards the definition of the cohomological dimension of a Hopf algebra in the next section.

Definition 2.5. Let M be an A-module. A **resolution** of M is an exact sequence of A-modules

$$\cdots \to P_{n+1} \overset{\partial_{n+1}}{\to} P_n \overset{\partial_n}{\to} P_{n-1} \cdots \to P_2 \overset{\partial_2}{\to} P_1 \overset{\partial_1}{\to} P_0 \overset{\epsilon}{\to} M \to 0$$

The resolution $P_* \to M$ is said to be

1. **finite** if there exists $n \geq 0$ such that for any $k > n$, $P_k = 0$, the smallest such n being called the **length** of the resolution;
2. **projective** if the P_i's are projective A-modules;
3. **free** is the P_i's are free A-modules.

Of course we make the convention that the 0-module is free.

Proposition 2.6. *Any A-module admits a free (and hence projective) resolution.*

Proof. Let M be an A-module. The construction of a free resolution is done by an obvious step by step procedure: start from a surjective A-linear map $F_0 \to M \to 0$ with F_0 free, apply the same to the kernel of this map to get an exact sequence $F_1 \to F_0 \to M \to 0$, and so on. □

Definition 2.7. The **projective dimension** of a non-zero A-module M is defined to be

$$\mathrm{pd}_A(M) = \min\{n : M \text{ admits a projective resolution of length } n\} \in \mathbb{N} \cup \{\infty\}$$

and we make the convention that the projective dimension of the zero module is zero.

Examples 2.8. (1) An A-module M is projective if and only if $\mathrm{pd}_A(M) = 0$.

(2) Let $A = \mathbb{C}\mathbb{Z} = \mathbb{C}[t, t^{-1}]$ be the group algebra of $\mathbb{Z}$. Then $A^+ = \mathrm{Ker}(\varepsilon)$ is easily seen to be free as an A-module (freely generated by $t-1$), so we have a free resolution of $\mathbb{C}_\varepsilon$

$$0 \to A^+ \to A \xrightarrow{\varepsilon} \mathbb{C}_\varepsilon \to 0$$

and hence $\mathrm{pd}(\mathbb{C}_\varepsilon) \leq 1$. Since A is infinite-dimensional, we have $\mathrm{pd}_A(\mathbb{C}_\varepsilon) > 0$, so $\mathrm{pd}_A(\mathbb{C}_\varepsilon) = 1$. More generally, if $A = \mathbb{C}\mathbb{F}_n$ is the group algebra of the free group on $n \geq 1$ generators, then A^+ is a free A-module, and hence $\mathrm{pd}_A(\mathbb{C}_\varepsilon) = 1$ (see e.g. [68, Chapter 6])

(3) Let $A = \mathbb{C}\langle x_1, \dots, x_n \rangle$ be the free algebra on n generators (Example 1.12). Similarly to the previous example, one has $\mathrm{pd}_A(\mathbb{C}_\varepsilon) = 1$.

(4) Let $A = \mathbb{C}[x]/(x^2)$ and let ε be the unique algebra map $A \to \mathbb{C}$. We have a infinite free resolution

$$\cdots \to A \to A \to \cdots \to A \to A \xrightarrow{\varepsilon} \mathbb{C}_\varepsilon \to 0$$

where each map $A \to A$ is the multiplication by x.

The basic problem to actually compute a projective dimension is that given of length n resolution of M, we know that $\mathrm{pd}_A(M) \leq n$, but it is unclear how to see that this resolution has the smallest possible length. To deal with this question, we will need the homological machinery developed in Section 4.

2.3. **Injective modules.** We now discuss the notion of injective module, a concept dual to that of projective module. We will show that a projective module over a finite-dimensional Hopf algebra is injective, which will have an important consequence about the possible behaviour of cohomological dimension for such Hopf algebras.

Let A be an algebra and let Q be an A-module. The (contravariant) functor $\mathrm{Hom}_A(-, Q)$ from A-modules to vector spaces is left exact: if

$$0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$$

is an exact sequence of A-modules, then the sequence

$$0 \to \mathrm{Hom}_A(Z, Q) \xrightarrow{-\circ p} \mathrm{Hom}_A(Y, Q) \xrightarrow{-\circ i} \mathrm{Hom}_A(X, Q)$$

is exact (check this). Injective modules are precisely those for which this functor is exact.

Proposition-Definition 2.9. *A (right) A-module Q is said to be **injective** if one of the equivalent following conditions holds.*

(1) The functor $\mathrm{Hom}_A(-, Q)$ is exact.

(2) For any injective A-linear $i : M \to N$ and any A-linear map $f : M \to Q$, there exists an A-linear map $\psi : N \to Q$ such that $\psi i = f$:

$$\begin{array}{ccc} 0 \longrightarrow & M & \xrightarrow{\ i\ } N \\ & \big\downarrow f & \swarrow \psi \\ & Q & \end{array}$$

(3) Any injective A-linear map $i : Q \to M$ admits a retraction, i.e. there exists an A-linear map $r : M \to Q$ such that $ri = \mathrm{id}_Q$.

The proof of the equivalence between these conditions is left to the reader, as well as the proof of the following result.

Proposition 2.10. *Let $(Q_i)_{\in I}$ be a family of A-modules. Then $\prod_{i \in I} Q_i$ is injective if and only if $\forall i \in I$, Q_i is injective.*

It is not true however, that $\bigoplus M_i$ is necessarily injective if all the M_i's are (of course in such a counterexample the set I has to be infinite). We will see soon that this is true if A is (right) Noetherian. The proof will use another characterization of injectivity, Baer's criterion.

Proposition 2.11. *An A-module Q is injective if and only if for any right ideal $I \subset A$ and any A-linear map $f : I \to Q$, there exists an A-linear map $\psi : A \to Q$ such that $\psi_{|I} = f$:*

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \longrightarrow & A \\ & & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle \psi} & \\ & & Q & & \end{array}$$

Proof. It is clear that an injective module satisfies the above condition. Conversely assume that Q satisfies this condition, and suppose given a diagram of morphism of A-modules

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{i} & N \\ & & \downarrow{\scriptstyle f} & & \\ & & Q & & \end{array}$$

Let $\mathcal{E}$ be the set of pairs (N_1, ψ_1) where $i(M) \subset N_1 \subset N$ is a submodule and $\psi_1 : N_1 \to Q$ is an A-linear map such that $\psi_1 i = f$. It is clear that $\mathcal{E}$ is non-empty, and order $\mathcal{E}$ by $(N_1, \psi_1) \leq (N_2, \psi_2)$ if $N_1 \subset N_2$ and $(\psi_2)_{|N_1} = \psi_1$. It is not difficult to check that $\mathcal{E}$ is inductively ordered, so by Zorn's Lemma there exists a maximal element (N_0, ψ_0) in $\mathcal{E}$. We have to show that $N_0 = N$.

Otherwise let $x \in N \setminus N_0$, and let $I = \{a \in A \mid x \cdot a \in N_0\}$. It is clear that $I \subset A$ is a right ideal. We get a linear map $g : I \to Q$, $a \mapsto \psi_0(x \cdot a)$. By our assumption there exists an A-linear map $\phi : A \to Q$ extending g. For $y, y' \in N_0$ and $a, a' \in A$ such that $y - y' = x(a - a')$ (so that $a - a' \in I$), we have $\psi_0(y - y') = \psi_0(x \cdot (a - a')) = \phi(a - a')$, and this shows that there is a well-defined A-linear map $\psi : N_0 + x \cdot A \to Q$ such that $\psi(y + x \cdot a) = \psi_0(y) + \phi(a)$, thus extending ψ_0. This contradicts the maximality of (N_0, ψ_0), and hence $N_0 = N$. ☐

Corollary 2.12. *Let A be a (right) Noetherian algebra. If $(Q_\lambda)_{\lambda \in \Lambda}$ is a family of injective A-modules, then so is $\oplus_{\lambda \in \Lambda} Q_\lambda$.*

Proof. We use Baer's criterion. Consider a diagram of A-linear maps

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \longrightarrow & A \\ & & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle \psi} & \\ & & Q & & \end{array}$$

with I an ideal of A. Since A is Noetherian, I is finitely generated and there exists a finite subset $U \subset \Lambda$ such that $f(I) \subset \oplus_{\lambda \in U} Q_\lambda$. By Proposition 2.10, the above finite direct sum is injective, and there exists an A-linear map $\psi : A \to \oplus_{\lambda \in U} Q_\lambda$ such that $\psi_{|I} = f$, and this concludes the proof. ☐

Lemma 2.13. *Let P be a finite-dimensional projective A-module over a finite-dimensional Hopf algebra A. Then P^* is projective as well.*

Proof. Recall that we have, for any A-module Y, natural isomorphisms

$$\mathrm{Hom}_A(P^*, Y) \simeq \mathrm{Hom}_A(\mathbb{C}_\varepsilon, Y \otimes P)$$

Hence the functor $\mathrm{Hom}_A(P^*, -)$ is isomorphic to the functor $\mathrm{Hom}_A(\mathbb{C}_\varepsilon, - \otimes P)$. Let

$$0 \to X \to Y \to Z \to 0$$

be an exact sequence of A-modules. Then the sequence

$$0 \to X \otimes P \to Y \otimes P \to Z \otimes P \to 0$$

is exact as well, and is split since $Z \otimes P$ is projective (Proposition 2.4). It follows that the sequence

$$0 \to \mathrm{Hom}_A(\mathbb{C}_\varepsilon, X \otimes P) \to \mathrm{Hom}_A(\mathbb{C}_\varepsilon, Y \otimes P) \to \mathrm{Hom}_A(\mathbb{C}_\varepsilon, Z \otimes P) \to 0$$

is exact. Hence the functor $\mathrm{Hom}_A(P^*, -)$ is exact, and P^* is projective. ☐

We now can prove the expected result.

Theorem 2.14. *Let P be a projective A-module over a finite-dimensional Hopf algebra A. Then P is injective.*

Proof. First assume that P is a finite-dimensional. We have for any A-module Y, natural isomorphisms

$$\mathrm{Hom}_A(Y, P) \simeq \mathrm{Hom}_A(Y \otimes P^*, \mathbb{C}_\varepsilon)$$

Hence the functor $\mathrm{Hom}_A(-, P)$ is isomorphic to the functor $\mathrm{Hom}_A(- \otimes P^*, \mathbb{C}_\varepsilon)$. Let

$$0 \to X \to Y \to Z \to 0$$

be an exact sequence of A-modules. Then the sequence

$$0 \to X \otimes P^* \to Y \otimes P^* \to Z \otimes P^* \to 0$$

is exact as well, and is split because P^* is projective (Lemma 2.13) and then $Z \otimes P^*$ is projective (Proposition 2.4). Hence the sequence

$$0 \to \mathrm{Hom}_A(X \otimes P^*, \mathbb{C}_\varepsilon) \to \mathrm{Hom}_A(Y \otimes P^*, \mathbb{C}_\varepsilon) \to \mathrm{Hom}_A(Z \otimes P^*, \mathbb{C}_\varepsilon) \to 0$$

is exact, and the functor $\mathrm{Hom}_A(-, P) \simeq \mathrm{Hom}_A(- \otimes P^*, \mathbb{C}_\varepsilon)$ is exact.

Therefore A_A is an injective A-module, and A being Noetherian since finite-dimensional, any free A-module is injective by Corollary 2.12. If now P is projective, there exists a free A-module F with $P \oplus Q \simeq F$, and we have from Proposition 2.10 that P is injective. □

3. Cohomological dimension of a Hopf algebra

We are now ready to define the cohomological dimension of a Hopf algebra, using the trivial module.

Definition 3.1. The **cohomological dimension** of a Hopf algebra A is defined by

$$\mathrm{cd}(A) = \mathrm{pd}_A(\mathbb{C}_\varepsilon) \in \mathbb{N} \cup \{\infty\}$$

We first review, without any proof, the meaning of cohomological dimension for Hopf algebras associated to groups and Lie algebras.

Example 3.2. If Γ is a discrete group, then $\mathrm{cd}(\mathbb{C}\Gamma) = \mathrm{cd}_{\mathbb{C}}(\Gamma)$, the cohomological dimension of Γ with coefficients $\mathbb{C}$, see [18, 13]. We warn the reader that the cohomological dimension of a group, as usually considered [18], is different in general, because it is based on the *integral* group ring $\mathbb{Z}\Gamma$ (this is already clear for finite groups). The cohomological dimension of Γ with coefficients $\mathbb{C}$ rather behaves like virtual cohomological dimension [18]. Here are some examples.

(1) We have $\mathrm{cd}(\mathbb{C}\Gamma) = 0$ if and only if Γ is finite (see Proposition 2.3, recall that the group algebra of a finite group is semisimple).
(2) We have seen in Examples 2.8 that if Γ is free group on n generators, then $\mathrm{cd}(\mathbb{C}\Gamma) = 1$. In fact, if Γ is a finitely generated group, then $\mathrm{cd}(\mathbb{C}\Gamma) = 1$ if and only if Γ contains a free normal subgroup of finite index, see [29, 24, 25], this is Dunwoody's theorem.
(3) If Γ is the fundamental group of an aspherical manifold of dimension n, then $\mathrm{cd}(\mathbb{C}\Gamma) = n$, see [18].

Example 3.3. If $A = \mathbb{C}\langle x_1, \dots, x_n \rangle$ is the free algebra on n generators (Example 1.12), then $\mathrm{cd}(A) = 1$, see the previous section.

Example 3.4. If $A = \mathcal{O}(G)$, the algebra of polynomial functions on an affine algebraic group G, then $\mathrm{cd}(\mathcal{O}(G)) = \dim G$, the usual dimension of G, i.e. the linear dimension of the Lie algebra of G. This is explained is Section 5.

Example 3.5. If $A = U(\mathfrak{g})$, the enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$, then $\mathrm{cd}(A) = \dim(\mathfrak{g})$. See [68].

The case of finite-dimensional Hopf algebras is well understood.

Theorem 3.6. *Let A be a Hopf algebra.*

(1) $\mathrm{cd}(A) = 0$ *if and only if A is finite-dimensional semisimple.*

(2) *If A is finite-dimensional, then* $\mathrm{cd}(A) \in \{0, \infty\}$.

Proof. The first statement is Proposition 2.3. Assume that A is finite-dimensional and $\mathrm{cd}(A) = n > 0$. We thus have a projective resolution

$$0 \to P_n \overset{\partial_n}{\to} P_{n-1} \overset{\partial_{n-1}}{\to} \cdots \to P_1 \overset{\partial_1}{\to} P_0 \overset{\epsilon}{\to} \mathbb{C}_\varepsilon \to 0$$

Since P_n is projective, it is injective by Theorem 2.14, and hence there exists an A-linear map $r : P_{n-1} \to P_n$ such that $r\partial_n = \mathrm{id}_{P_n}$. Hence $P_{n-1} = \partial_n(P_n) \oplus Q$ for a submodule Q, with both $\partial_n(P_n)$ and Q projective, since P_{n-1} is. We get a new projective resolution

$$0 \to Q \overset{\partial_{n-1}}{\to} P_{n-2} \overset{\partial_{n-2}}{\to} \cdots \to P_0 \overset{\epsilon}{\to} \mathbb{C}_\varepsilon \to 0$$

of length $n-1$, so $\mathrm{cd}(A) \leq n-1$, a contradiction. Hence if $\mathrm{cd}(A) > 0$, there does not exist any finite projective resolution of $\mathbb{C}_\varepsilon$, and $\mathrm{cd}(A) = \infty$. ☐

Since the trivial module $\mathbb{C}_\varepsilon$ is a distinguished one, the above definition of the cohomological dimension of a Hopf algebra is perfectly natural, and two isomorphic Hopf algebras have the same cohomological dimension. In fact the following result shows that the cohomological dimension does not even depend on the choice of special module.

Proposition 3.7. *Let A be a Hopf algebra. Then*

$$\mathrm{cd}(A) = \mathrm{Sup}\{\mathrm{pd}_A(M),\ M \in \mathrm{Mod}(A)\}$$

Proof. It is clear that $\mathrm{cd}(A)$ is smaller than the quantity on the right, and to prove the equality, we can assume that $n = \mathrm{cd}(A)$ is finite. We follow the argument in [45]. Consider a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C}_\varepsilon \to 0$$

For an A-module M, tensoring this resolution by M on the right yields a projective resolution

$$0 \to M \otimes P_n \to M \otimes P_{n-1} \to \cdots \to M \otimes P_1 \to M \otimes P_0 \to M \otimes \mathbb{C}_\varepsilon \simeq M \to 0$$

(the terms $M \otimes P_i$ are indeed projective modules by Proposition 2.4) and hence $\mathrm{pd}_A(M) \leq n$, as needed. ☐

Therefore the cohomological dimension of a Hopf algebra coincides with its right global dimension, one of the most classical homological invariants of an algebra, see [68], and only depends on the algebra structure.

4. Homological algebra

We now present the necessary homological algebra background to be able to compute effectively cohomological dimensions.

4.1. **Chain complexes.**

Definition 4.1. A **chain complex** $C_* = (C_*, d_*)$ consists of a sequence of complex vector spaces and linear maps

$$\cdots \to C_{n+2} \overset{d_{n+2}}{\to} C_{n+1} \overset{d_{n+1}}{\to} C_n \to \cdots \to C_2 \overset{d_2}{\to} C_1 \overset{d_1}{\to} C_0 \to 0$$

such that for any $n \geq 0$, we have $d_n d_{n+1} = 0$. The maps d_n are called the differentials of the complex. For $n \geq 0$, the n-th **homology** space of the complex C_* is then defined by

$$H_n(C_*) = \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1})$$

making the convention that $d_0 = 0$.

Remark 4.2. If M is an A-module, a resolution of M

$$\cdots \to P_{n+1} \stackrel{\partial_{n+1}}{\to} P_n \to \cdots \to P_2 \stackrel{\partial_2}{\to} P_1 \stackrel{\partial_1}{\to} P_0 \stackrel{\epsilon}{\to} M \to 0$$

as in Definition 2.5 can be seen as a chain complex with trivial homology. Forgetting M, we get a chain complex P_*

$$\cdots \to P_{n+1} \stackrel{\partial_{n+1}}{\to} P_n \to \cdots \to P_2 \stackrel{\partial_2}{\to} P_1 \stackrel{\partial_1}{\to} P_0 \to 0$$

with $H_0(P_*) \simeq M$ and $H_n(P_*) = 0$ is $n \geq 1$

Definition 4.3. Let $C_* = (C_*, d_*^C)$ and $D_* = (D_*, d_*^D)$ be two chain complexes. A **morphism of complexes** $f : C_* \to D_*$ is a sequence of linear map $f_n : C_n \to D_n$, $n \in \mathbb{N}$, commuting with the differentials:

$$\begin{array}{ccc} C_n & \xrightarrow{f_n} & D_n \\ \downarrow{\scriptstyle d_n^C} & & \downarrow{\scriptstyle d_n^D} \\ C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1} \end{array}$$

The following result is left as an exercise.

Proposition 4.4. *Let $f : C_* \to D_*$ be a morphism of morphism of complexes. Then for any $n \geq 0$, f induces a linear map*

$$\begin{aligned} H_n(f) : H_n(C_*) &\longrightarrow H_n(D_*) \\ [c] &\longmapsto [f_n(c)] \end{aligned}$$

where $[-]$ means the class of the element $c \in \mathrm{Ker}(d_n)$ in $H_n(C_*)$.

Definition 4.5. Let $C_* = (C_*, d_*^C)$ and $D_* = (D_*, d_*^D)$ be some chain complexes and let $f, g : C_* \to D_*$ be some morphisms of complexes. An **homotopy** h from f to g consists of a sequence of linear maps $h_n : C_n \to D_{n+1}$, $n \geq -1$ (with $h_{-1} = 0$), such that $\forall n \geq 0$, we have $d_{n+1}^D h_n + h_{n-1} d_n^C = f_n - g_n$.

$$\begin{array}{ccccc} C_{n+1} & \xrightarrow{d_{n+1}^C} & C_n & \xrightarrow{d_n^C} & C_{n-1} \\ & {\scriptstyle h_n}\swarrow & {\scriptstyle f_n}\downarrow\downarrow{\scriptstyle g_n} & \swarrow{\scriptstyle h_{n-1}} & \\ D_{n+1} & \xrightarrow[d_{n+1}^D]{} & D_n & \xrightarrow[d_n^D]{} & D_{n-1} \end{array}$$

We say that f and g are **homotopic**, and write $f \sim g$, if there exists an homotopy from f to g. We say that f is an **homotopy equivalence** if there exists a morphism of complexes $f' : D_* \to C_*$ such that $ff' \sim \mathrm{id}_{D_*}$ and $f'f \sim \mathrm{id}_{C_*}$.

Proposition 4.6. *Let $C_* = (C_*, d_*^C)$ and $D_* = (D_*, d^D)$ be some chain complexes and let $f, g : C_* \to D_*$ be some morphisms of complexes. If f and g are homotopic, then $H_n(f) = H_n(g)$ for all $n \geq 0$. In particular, if f is an homotopy equivalence, then $H_n(f)$ is, for any $n \geq 0$, an isomorphism $H_n(C_*) \simeq H_n(D_*)$.*

Proof. Let h be an homotopy from f to g, and let $x \in \mathrm{Ker}(d_n^C)$. We have

$$f_n(x) - g_n(x) = d_{n+1}^D(h_n(x)) + h_{n-1}(d_n^C(x)) = d_{n+1}^D(h_n(x))$$

hence $H_n(f)([x]) = [f_n(x)] = [g_n(x)] = H_n(g)([x])$. $\square$

As a first application of homotopies, we get the standard resolution of the trivial module over a Hopf algebra.

Proposition 4.7. *If A is a Hopf algebra, consider the following sequence of A-linear maps (with A acting by right multiplication on the extreme right term of $A^{\otimes n}$):*

$$\cdots \longrightarrow A^{\otimes n+1} \longrightarrow A^{\otimes n} \longrightarrow \cdots \longrightarrow A \otimes A \longrightarrow A \stackrel{\varepsilon}{\longrightarrow} \mathbb{C}_\varepsilon \to 0$$

where each map $A^{\otimes n+1} \to A^{\otimes n}$ is given by

$$a_1 \otimes \cdots \otimes a_{n+1} \mapsto \varepsilon(a_1) a_2 \otimes \cdots \otimes a_{n+1} + \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

Then this is a free resolution of the trivial module $\mathbb{C}_\varepsilon$, called the standard resolution of the standard module.

Proof. We leave it to the reader to check that this is indeed a complex. For $n \geq 0$, let $h_{n+1} : A^{\otimes n+1} \to A^{\otimes n+2}$, $a_1 \otimes \cdots \otimes a_{n+1} \mapsto 1 \otimes a_1 \otimes \cdots \otimes a_{n+1}$, and let $h_0 : \mathbb{C} \to A$, $1 \mapsto 1$. It is an immediate verification to check that this defines an homotopy from the identity to the zero map, so the homology of this complex is trivial, and we indeed have a resolution. ☐

Theorem 4.8. *Let M, N be some A-modules, let $P_* \to M$ be a projective resolution of M and let $f : M \to N$ be an A-linear map. Then for any resolution $Q_* \to N$ of N, there exists a morphism of complexes $\varphi : P_* \to Q_*$ which is a lifting of f, in the sense that $\epsilon' \varphi_0 = f \epsilon$. Moreover, the morphism of complexes φ is unique up to homotopy.*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow 0 \\
& \exists \Big\downarrow \varphi_2 & & \exists \Big\downarrow \varphi_1 & & \exists \Big\downarrow \varphi_0 & & \Big\downarrow f & \\
\cdots \longrightarrow & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\epsilon'} & N & \longrightarrow 0
\end{array}$$

Proof. We begin by showing the existence of φ, by induction.

• Construction of φ_0. We have a diagram

$$\begin{array}{ccc}
& P_0 & \\
& \Big\downarrow f\epsilon & \\
Q_0 \xrightarrow{\epsilon'} & N & \longrightarrow 0
\end{array}$$

hence, since P_0 is projective, there exists $\varphi_0 : P_0 \to Q_0$ such that $\epsilon' \varphi_0 = f \epsilon$.

• Put $\partial_0 = \varepsilon'$, $d_0 = \varepsilon$, and $\varphi_{-1} = f$, and assume now that we have constructed linear maps φ_k, $0 \leq k \leq n$, such that $\partial_k \varphi_k = \varphi_{k-1} d_k$ for $0 \leq k \leq n$, and let us construct φ_{n+1}.

We have $\partial_n \varphi_n d_{n+1} = \varphi_{n-1} d_n d_{n+1} = 0$, hence $\mathrm{Im}(\varphi_n d_{n+1}) \subset \mathrm{Ker}(\partial_n) = \mathrm{Im}(\partial_{n+1})$. Hence we have a diagram

$$\begin{array}{ccc}
& P_{n+1} & \\
& \Big\downarrow \varphi_n d_{n+1} & \\
Q_{n+1} \xrightarrow[\partial_{n+1}]{} & \mathrm{Im}(\partial_{n+1}) & \longrightarrow 0
\end{array}$$

hence, since P_{n+1} is projective, there exists $\varphi_{n+1} : P_{n+1} \to Q_{n+1}$ such that $\partial_{n+1} \varphi_{n+1} = \varphi_n d_{n+1}$.

This proves the existence of φ, and we now have to show uniqueness up to homotopy. Assume that ψ is another lifting of f and put $\theta = \psi - \varphi$. We have to construct linear maps $h_n : P_n \to Q_{n+1}$ such that $\theta_n = \partial_{n+1} h_n + h_{n-1} d_n$ for any $n \geq 0$, where we put $h_{-1} = 0$ (and again $\partial_0 = \varepsilon'$, $d_0 = \varepsilon$). Again we proceed by induction.

• Construction of h_0. We have $\partial_0 \theta_0 = \partial_0 \psi_0 - \partial_0 \varphi_0 = f d_0 - f d_0 = 0$, hence $\mathrm{Im}(\theta_0) \subset \mathrm{Ker}(\partial_0) = \mathrm{Im}(\partial_1)$. We thus have a diagram

$$\begin{array}{ccc}
& P_0 & \\
& \Big\downarrow \theta_0 & \\
Q_1 \xrightarrow[\partial_1]{} & \mathrm{Im}(\partial_1) & \longrightarrow 0
\end{array}$$

and since P_0 is projective, there exists $h_0 : P_0 \to Q_1$ such that $\theta_0 = \partial_1 h_0 = \partial_1 h_0 + h_{-1} d_0$.

• Assume now that we have constructed linear maps $h_k : P_k \to Q_{k+1}$ such that $\theta_k = \partial_{k+1} h_k + h_{k-1} d_k$ for any $0 \leq k \leq n$, and let us construct h_{n+1}.

Consider the linear map $\theta_{n+1} - h_n d_{n+1} : P_{n+1} \to Q_{n+1}$. We have $\partial_{n+1}(\theta_{n+1} - h_n d_{n+1}) = \theta_n d_{n+1} - (\theta_n - h_{n-1} d_n) d_{n+1} = 0$, hence $\mathrm{Im}(\theta_{n+1} - h_n d_{n+1}) \subset \mathrm{Ker}(\partial_{n+1}) = \mathrm{Im}(\partial_{n+2})$. Hence we have a diagram

$$\begin{array}{ccccc} & & P_{n+1} & & \\ & & \downarrow {\scriptstyle \theta_{n+1} - h_n d_{n+1}} & & \\ Q_{n+2} & \xrightarrow[\partial_{n+2}]{} & \mathrm{Im}(\partial_{n+2}) & \longrightarrow & 0 \end{array}$$

and since P_{n+1} is projective, there exists $h_{n+1} : P_{n+1} \to Q_{n+2}$ such that $\theta_{n+1} - h_n d_{n+1} = \partial_{n+2} h_{n+1}$. This finishes the proof. ☐

Corollary 4.9 (Uniqueness of projective resolutions)**.** *Let $P_* \to M$ and $Q_* \to M$ be projective resolutions of an A-module M. Then there exists a morphism of complexes $\varphi : P_* \to Q_*$ which is an homotopy equivalence.*

Proof. Let $\varphi : P_* \to Q_*$ be a lifting of id_M, and let $\psi : Q_* \to P_*$ be a lifting of id_M. Then $\psi\varphi : P_* \to P_*$ is a lifting of id_M, and so is id_{P_*}. The uniqueness of liftings up to homotopy shows that $\psi\varphi \sim \mathrm{id}_{P_*}$, and similarly $\varphi\psi \sim \mathrm{id}_{Q_*}$. ☐

Proposition 4.10. *Let $C_* = (C_*, d_*^C)$ and $D_* = (D_*, d_*^D)$ be some chain complexes of A-modules, and let $F : \mathcal{M}_A \to \mathrm{Vec}(\mathbb{C})$ be a $\mathbb{C}$-linear functor.*

(1) $F(C_*) = (F(C_*), F(d_*^C))$ and $F(D_*) = (F(D_*), F(d_*^D))$ are chain complexes.
(2) If $f : C_* \to D_*$ is a morphism of complexes, then so is $F(f) : F(C_*) \to F(D_*)$.
(3) If $f : C_* \to D_*$ and $g : C_* \to D_*$ are homotopic morphism of complexes, so are $F(f)$ and $F(g)$. If particular, if f is an homotopy equivalence, so is $F(f)$.

Proof. The first assertions are immediate verifications. If h is an homotopy from f to g, it is also immediate that $F(h)$ is an homotopy from $F(f)$ to $F(g)$, and the last assertion follows as well. ☐

4.2. **Cochain complexes.**

Definition 4.11. A **cochain complex** $C_* = (C_*, d_*)$ consists of a sequence of complex vector spaces and linear maps

$$0 \to C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} C_2 \to \cdots \to C_n \xrightarrow{d_n} C_{n+1} \xrightarrow{d_{n+1}} C_{n+2} \to \cdots$$

such that for any $n \geq 0$, we have $d_{n+1} d_n = 0$. The maps d_n are called the differentials of the complex. For $n \geq 0$, the n-th **cohomology** space of the complex C_* is then defined by

$$H^n(C_*) = \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n-1})$$

making the convention that $d_{-1} = 0$.

The definitions and basic results of the previous subsection have easy adaptation to the case of cochain complexes.

Definition 4.12. Let $C_* = (C_*, d_*^C)$ and $D_* = (D_*, d_*^D)$ be two cochain complexes. A **morphism of (cochain) complexes** $f : C_* \to D_*$ is a sequence of linear map $f_n : C_n \to D_n$, $n \in \mathbb{N}$, commuting with the differentials:

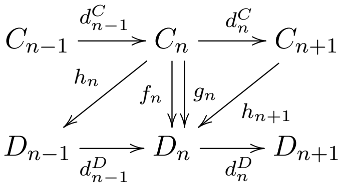
$$\begin{array}{ccc} C_n & \xrightarrow{f_n} & D_n \\ \downarrow {\scriptstyle d_n^C} & & \downarrow {\scriptstyle d_n^D} \\ C_{n+1} & \xrightarrow{f_{n+1}} & D_{n+1} \end{array}$$

Proposition 4.13. *Let $f : C_* \to D_*$ be a morphism of cochain complexes. Then for any $n \geq 0$, f induces a linear map*

$$H^n(f) : H^n(C_*) \longrightarrow H^n(D_*)$$
$$[c] \longmapsto [f_n(c)]$$

where $[-]$ means the class of the element $c \in \mathrm{Ker}(d_n)$ in $H^n(C_*)$.

Definition 4.14. Let $C_* = (C_*, d_*^C)$ and $D_* = (D_*, d_*^D)$ be some cochain complexes and let $f, g : C_* \to D_*$ be some morphisms of complexes. An **homotopy** h from f to g consists of a sequence of linear maps $h_n : C_n \to D_{n-1}$, $n \geq 0$ (with $h_{-1} = 0$), such that $\forall n \geq 0$, we have $d_{n-1}^D h_n + h_{n+1} d_n^C = g_n - f_n$.



We say that f and g are **homotopic**, and write $f \sim g$, if there exists an homotopy from f to g. We say that f is an **homotopy equivalence** if there exists a morphism of complexes $f' : D_* \to C_*$ such that $ff' \sim \mathrm{id}_{D_*}$ and $f'f \sim \mathrm{id}_{C_*}$.

Proposition 4.15. *Let $C_* = (C_*, d_*^C)$ and $D_* = (D_*, d_*^D)$ be some cochain complexes and let $f, g : C_* \to D_*$ be some morphisms of complexes. If f and g are homotopic, then $H^n(f) = H^n(g)$ for all $n \geq 0$. In particular, if f is an homotopy equivalence, then $H^n(f)$ is, for any $n \geq 0$, an isomorphism $H^n(C_*) \simeq H^n(D_*)$.*

Proof. The proof is similar to that of Proposition 4.6. □

Proposition 4.16. *Let $C_* = (C_*, d_*^C)$ and $D_* = (D_*, d_*^D)$ be some chain complexes of A-modules, and let $F : \mathcal{M}_A \to \mathrm{Vec}(\mathbb{C})$ be a contravariant $\mathbb{C}$-linear functor.*

(1) $F(C_*) = (F(C_*), F(d_*^C))$ and $F(D_*) = (F(D_*), F(d_*^D))$ are cochain complexes.
(2) If $f : C_* \to D_*$ is a morphism of complexes, then so is $F(f) : F(D_*) \to F(C_*)$.
(3) If $f : C_* \to D_*$ and $g : C_* \to D_*$ are homotopic morphism of complexes, so are $F(f)$ and $F(g)$. If particular, if f is an homotopy equivalence, so is $F(f)$.

Proof. Similarly to Proposition 4.10, these are immediate verifications. □

4.3. Ext **spaces.** We now provide another interpretation of projective dimension, in terms of certain cohomology spaces, the Ext-spaces.

Theorem-Definition 4.17. *Let M, N be right A-modules. Let $P_* \to M \to 0$ be a projective resolution of M*

$$\cdots \to P_{n+1} \overset{\partial_{n+1}}{\to} P_n \to \cdots \to P_2 \overset{\partial_2}{\to} P_1 \overset{\partial_1}{\to} P_0 \overset{\epsilon}{\to} M \to 0$$

and consider the associated complex $\mathrm{Hom}_A(P_*, N)$

$$0 \to \mathrm{Hom}_A(P_0, N) \overset{-\circ\partial_1}{\longrightarrow} \mathrm{Hom}_A(P_1, N) \overset{-\circ\partial_2}{\longrightarrow} \mathrm{Hom}_A(P_2, N) \overset{-\circ\partial_3}{\longrightarrow} \cdots$$

Then the cohomology spaces $H^*(\mathrm{Hom}_A(P_*, N))$ do not depend on the choice of the projective resolution P_*, and are denoted $\mathrm{Ext}_A^*(M, N)$.

Proof. Let $Q_* \to M \to 0$ be a another projective resolution of M

$$\cdots \to Q_{n+1} \overset{d_{n+1}}{\to} Q_n \to \cdots \to Q_2 \overset{d_2}{\to} Q_1 \overset{d_1}{\to} Q_0 \overset{\epsilon'}{\to} M \to 0$$

Let $\varphi : P_* \to Q_*$ be an homotopy equivalence (Corollary 4.9). It follows from Proposition 4.16, applied to the contravariant functor $\mathrm{Hom}_A(-, N)$, that the cochain complexes

$$0 \to \mathrm{Hom}_A(P_0, N) \overset{-\circ\partial_1}{\longrightarrow} \mathrm{Hom}_A(P_1, N) \overset{-\circ\partial_2}{\longrightarrow} \mathrm{Hom}_A(P_2, N) \overset{-\circ\partial_3}{\longrightarrow} \cdots$$

and

$$0 \to \mathrm{Hom}_A(Q_0, N) \stackrel{-\circ d_1}{\longrightarrow} \mathrm{Hom}_A(Q_1, N) \stackrel{-\circ d_2}{\longrightarrow} \mathrm{Hom}_A(Q_2, N) \stackrel{-\circ d_3}{\longrightarrow} \cdots$$

are homotopy equivalent, and hence have isomorphic cohomologies. ☐

Remark 4.18. We have $\mathrm{Ext}^0_A(M, N) \simeq \mathrm{Hom}_A(M, N)$.

Proof. Let

$$\cdots \to P_{n+1} \stackrel{\partial_{n+1}}{\to} P_n \to \cdots \to P_2 \stackrel{\partial_2}{\to} P_1 \stackrel{\partial_1}{\to} P_0 \stackrel{\epsilon}{\to} M \to 0$$

be a projective resolution of M. By definition, we have

$$\mathrm{Ext}^0_A(M, N) = \{f \in \mathrm{Hom}_A(P_0, N),\ f\partial_1 = 0\} = \{f \in \mathrm{Hom}_A(P_0, N),\ f_{|\mathrm{Ker}(\epsilon)} = 0\}$$

Since $P_0/\mathrm{Ker}(\epsilon) \simeq M$, the result follows. ☐

In fact the equivalences classes of elements in $\mathrm{Ext}^n_A(M, N)$ truly correspond to equivalences classes of exact sequences of A-modules of length $n+2$ starting at N and finishing at M, see [68] again.

The Ext-spaces and the projective dimension are related as follows.

Proposition 4.19. *Let M be an A-module. The following assertions are equivalent.*

(1) $\mathrm{pd}_A(M) \leq n$.
(2) $\mathrm{Ext}^i_A(M, -) = 0$ *for* $i > n$.
(3) $\mathrm{Ext}^{n+1}_A(M, -) = 0$.
(4) For any exact sequence of A-modules $0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with each P_i projective, then K is projective.
(5) For any exact sequence of A-modules $0 \to L \stackrel{i}{\to} P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with each P_i projective, there exists $r \in \mathrm{Hom}_A(P_n, L)$ such that $ri = \mathrm{id}_L$.

Proof. (2) ⇒ (3) is obvious, and so are (4) ⇒ (1) and (1) ⇒ (2), just by writing the definitions. Assume that (3) holds, and let $0 \to K \stackrel{i}{\to} P_{n-1} \stackrel{d_{n-1}}{\to} P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0$ be an exact sequence of A-modules with each P_i projective. Complete this exact sequence to a projective resolution

$$\begin{array}{ccccccccccc}
\longrightarrow \cdots P_{n+2} & \xrightarrow{d_{n+2}} & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & \cdots \longrightarrow P_0 & \longrightarrow M & \longrightarrow 0 \\
 & & & \searrow^{q} & \uparrow{\scriptstyle j} & \searrow^{p} & \uparrow{\scriptstyle i} & & & & \\
 & & & & L & & K & & & &
\end{array}$$

We are going to show that $P_n \simeq K \oplus L$ as A-modules, so that K, being a direct summand of a projective module, will be projective, and this will prove that (3) ⇒ (4).

We have an exact sequence $0 \to L \stackrel{j}{\to} P_n \stackrel{p}{\to} K \to 0$, and hence to show that $P_n \simeq K \oplus L$, it is enough to show that there exists an A-linear map $r : P_n \to L$ such that $rj = \mathrm{id}_L$. Consider $q \in \mathrm{Hom}_A(P_{n+1}, L)$. We have $qd_{n+2} = 0$ since $jqd_{n+2} = d_{n+1}d_{n+2} = 0$ and j is injective. Hence since $\mathrm{Ext}^{n+1}_A(M, L) = 0$, there exists $r \in \mathrm{Hom}_A(P_n, L)$ such that $q = rd_{n+1}$. Then $rjq = q$, and since q is surjective, we have $rj = \mathrm{id}_L$, as needed.

Assume now that (4) holds, and let $0 \to L \stackrel{i}{\to} P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ be exact with each P_i projective. We then have an exact sequence

$$0 \to P_n/\mathrm{Im}(i) \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

By (4), we have that $P_n/\mathrm{Im}(i)$ is a projective A-module, so $P_n \simeq P_n/\mathrm{Im}(i) \oplus L$ as A-modules and (5) follows. The proof of (5) ⇒ (1) is left as an exercise. ☐

Corollary 4.20. *We have, for any A-module M*

$$\mathrm{pd}_A(M) = \sup\{n :\ \mathrm{Ext}^n_A(M, N) \neq 0 \textit{ for some } A\textit{-module } N\}$$

4.4. **Tor spaces.** We finish the section by introducing the Tor-spaces, which will be in use when defining homology spaces of Hopf algebras.

Theorem-Definition 4.21. *Let M be a right A-module and let N be a left A-module. Let $P_* \to M \to 0$ be a projective resolution of M*

$$\cdots \to P_{n+1} \overset{\partial_{n+1}}{\to} P_n \overset{\partial_n}{\to} \cdots \to P_2 \overset{\partial_2}{\to} P_1 \overset{\partial_1}{\to} P_0 \overset{\epsilon}{\to} M \to 0$$

and consider the associated complex $P_* \otimes_A N$

$$\cdots \to P_{n+1} \otimes_A N \overset{\partial_{n+1}\otimes \mathrm{id}_N}{\to} P_n \to \cdots \to P_2 \otimes_A N \overset{\partial_2 \otimes_A \mathrm{id}_N}{\to} P_1 \overset{\partial_1 \otimes_A \mathrm{id}_N}{\to} P_0 \to 0$$

Then the cohomology spaces $H_*(P_* \otimes_A N)$ do not depend on the choice of the projective resolution P_*, and are denoted $\mathrm{Tor}_*^A(M,N)$.

Proof. Let $Q_* \to M \to 0$ be a another projective resolution of M

$$\cdots \to Q_{n+1} \overset{d_{n+1}}{\to} Q_n \to \cdots \to Q_2 \overset{d_2}{\to} Q_1 \overset{d_1}{\to} Q_0 \overset{\epsilon'}{\to} M \to 0$$

Let $\varphi : P_* \to Q_*$ be an homotopy equivalence (Corollary 4.9). Then $\varphi \otimes_A N$ is an homotopy equivalence between the complexes $P_* \otimes_A N$ and $Q_* \otimes_A N$, which thus have isomorphic homologies. ☐

Remark 4.22. We have $\mathrm{Tor}_0^A(M,N) \simeq M \otimes_A N$.

An important fact, that we will only use in the proof of Theorem 8.5 and will not prove, is that $\mathrm{Tor}_*^A(M,N)$ can also be computed using projective resolutions of N: If

$$\cdots \to P_{n+1} \overset{\partial_{n+1}}{\to} P_n \overset{\partial_n}{\to} \cdots \to P_2 \overset{\partial_2}{\to} P_1 \overset{\partial_1}{\to} P_0 \overset{\epsilon}{\to} N \to 0$$

is a projective resolution of N by left A-modules, then $\mathrm{Tor}_*^A(M,N)$ is isomorphic to the homology of the complex $M \otimes_A P_*$. See [68], for example.

5. Example: cohomological dimension of commutative Hopf algebras

In this section we explain the ideas leading to the computation of the cohomological dimension of the coordinate algebra on an affine algebraic group.

5.1. **The Koszul complex.** We first present a general tool. Let A be commutative algebra, and let $\underline{x} = (x_1, \dots, x_r)$ be a sequence of elements of A. The Koszul complex $\mathbf{K}_*(A, \underline{x})$ is defined as follows.

- $\mathbf{K}_0(A, \underline{x}) = A$;
- $\mathbf{K}_1(A, \underline{x}) = E$, the free A-module of rank r, with basis $e_1, \dots, e_r$;
- for $1 \leq p \leq r$, $\mathbf{K}_p(A, \underline{x}) = \Lambda_A^p(E)$, the free A-module with basis $e_{i_1} \wedge \cdots \wedge e_{i_p}$, $i_1 < \cdots < i_p$;
- for $p > r$, $\mathbf{K}_p(A, \underline{x}) = \{0\}$;
- the diffentials are defined by $d_1(e_i) = x_i$ and

$$\begin{aligned} d_p : \mathbf{K}_p(A, \underline{x}) &\longrightarrow \mathbf{K}_{p-1}(A, \underline{x}) \\ e_{i_1} \wedge \cdots \wedge e_{i_p} &\longmapsto \sum_{j=1}^{p} (-1)^{j-1} (e_{i_1} \wedge \cdots \wedge \hat{e_{i_j}} \wedge \cdots \wedge e_{i_p}) x_{i_j} \end{aligned}$$

It is not difficult to check that $(\mathbf{K}_*(A, \underline{x}), d_*)$ is a complex:

$$0 \to \mathbf{K}_r(A, \underline{x}) \overset{d_r}{\to} \mathbf{K}_{r-1}(A, \underline{x}) \overset{d_{r-1}}{\to} \mathbf{K}_{r-2}(A, \underline{x}) \to \cdots \to \mathbf{K}_2(A, \underline{x}) \overset{d_2}{\to} \mathbf{K}_1(A, \underline{x}) \overset{d_1}{\to} \mathbf{K}_0(A, \underline{x}) = A \to 0$$

Definition 5.1. A sequence $\underline{x} = (x_1, \dots, x_r)$ of elements of A is said to be regular if

1. $Ax_1 + \cdots + Ax_r \subsetneq A$;
2. x_1 is not a zero divisor in A ;
3. For any $i \geq 2$, x_i is not a zero divisor in the ring $A/(x_1, \dots, x_{i-1})$.

Theorem 5.2. *If $\underline{x} = (x_1, \ldots, x_r)$ is a regular sequence in A, then the Koszul complex provides a free resolution of the A-module A/I, where $I = Ax_1 + \cdots + Ax_r$:*

$$0 \to \mathbf{K}_r(A, \underline{x}) \stackrel{d_r}{\to} \mathbf{K}_{r-1}(A, \underline{x}) \to \cdots \to \mathbf{K}_2(A, \underline{x}) \stackrel{d_2}{\to} \mathbf{K}_1(A, \underline{x}) \stackrel{d_1}{\to} \mathbf{K}_0(A, \underline{x}) = A \to A/I \to 0$$

Moreover we have

$$\mathrm{Ext}^p_A(A/I, A) \simeq \begin{cases} 0 & \text{if } p \neq r \\ A/I & \text{if } p = r \end{cases}$$

and in particular we have $\mathrm{pd}_A(A/I) = r$.

Proof. This can be found in numerous textbooks, e.g. [42, 47, 68]. The reader will easily treat the case $r = 1, 2, 3$, check that $\mathrm{Ext}^r_A(A/I, A) \simeq A/I$, and deduce the last assertion. □

A first immediate application of Theorem 5.2 is the following result:

Corollary 5.3. *Let A be a commutative Hopf algebra. If the augmentation ideal $I = A^+$ is generated by a regular sequence with r elements, then* $\mathrm{cd}(A) = r$.

Proof. This indeed follows immediately from Theorem 5.2, since $\mathbb{C}_\varepsilon \simeq A/I$. □

As a first illustration, we have:

Example 5.4. The cohomological dimensions of the Hopf algebras

$$\mathcal{O}(\mathbb{C}^n) \simeq \mathbb{C}[X_1, \ldots, X_n], \quad \mathcal{O}(\mathbb{C}^{*n}) \simeq \mathbb{C}\mathbb{Z}^n \simeq \mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}], \quad \mathcal{O}(\mathrm{SL}_2(\mathbb{C}))$$

respectively are n, n, 3.

Proof. For the first two algebras, the sequences $(X_1, \ldots, X_n)$ is regular, while for the third one, the sequence $(b, c, a - 1)$ is regular (all the involved quotients being integral domains). □

5.2. **General case.** We now explain how the construction of the previous subsection enables one to compute the cohomological dimension of the coordinate algebra of an affine algebraic group. We only sketch the main ideas, taking for granted a number of tools from commutative algebra.

5.2.1. *Localization.* Let $S \subset A \setminus \{0\}$ be a multiplicative subset of a commutative ring ($1 \in S$ and $s, t \in S \Rightarrow st \in S$). The localized ring $S^{-1}A$ is the set of pairs $(a, s) \in A \times S$ modulo the equivalence relation

$$(a, s) \sim (b, r) \iff \exists t \in S \text{ such that } tra = tsb$$

It becomes a ring under the laws

$$\frac{a}{s} + \frac{b}{r} = \frac{ar + bs}{sr}, \quad \frac{a}{s}\frac{b}{r} = \frac{ab}{sr}$$

where $\frac{a}{s}$ stands for the class of (a, s) in $S^{-1}A$. There is a morphism of ring

$$A \longrightarrow S^{-1}A, \quad a \longmapsto \frac{a}{1}$$

which is injective if S does not contain any zero-divisor.

The localization of an A-module M is denoted $S^{-1}M$ and is defined to be the set of pair $(x, s) \in M \times S$ modulo the equivalence relation

$$(x, s) \sim (y, r) \iff \exists t \in S \text{ such that } x \cdot tr = y \cdot ts$$

It becomes an $S^{-1}A$-module under the laws

$$\frac{x}{s} + \frac{y}{r} = \frac{x \cdot r + y \cdot s}{sr}, \quad \frac{x}{s}\frac{a}{r} = \frac{x \cdot a}{sr}$$

The construction defines an exact functor from A-modules to $S^{-1}A$-modules.

When $S = A \setminus \mathfrak{m}$ with $\mathfrak{m}$ a maximal ideal of A, the localized ring $S^{-1}A$ is denoted $A_\mathfrak{m}$, while the localized module $S^{-1}M$ is denoted $M_\mathfrak{m}$. The following result will be useful later.

Proposition 5.5. *Let M be an A-module. Then*

$$M = \{0\} \iff M_{\mathfrak{m}} = \{0\} \text{ for any maximal ideal } \mathfrak{m} \subset A$$

5.2.2. *Localization and* Ext. While the previous material was quite basic, we will need a more advanced result. We begin with a preliminary remark.

Let A be an algebra (non necessarily commutative) and M, N be right A-modules. If N is a left B-module for another algebra B such that N is a B-A-bimodule, then the space of right A-linear maps $\mathrm{Hom}_A(M,N)$ carries a natural left B-module structure defined by

$$(b \cdot f)(x) = b \cdot (f(x)$$

This remark ensures that the Ext-spaces

$$\mathrm{Ext}^*_A(M,N)$$

carry a natural left B-module structure. Indeed, if $P_* \to M \to 0$ is a projective resolution of M

$$\cdots \to P_{n+1} \stackrel{\partial_{n+1}}{\to} P_n \to \cdots \to P_2 \stackrel{\partial_2}{\to} P_1 \stackrel{\partial_1}{\to} P_0 \stackrel{\epsilon}{\to} M \to 0$$

the associated complex $\mathrm{Hom}_A(P_*,N)$

$$0 \to \mathrm{Hom}_A(P_0,N) \stackrel{-\circ\partial_1}{\longrightarrow} \mathrm{Hom}_A(P_1,N) \stackrel{-\circ\partial_2}{\longrightarrow} \mathrm{Hom}_A(P_2,N) \stackrel{-\circ\partial_3}{\longrightarrow} \cdots$$

carries a natural left B-module structure, and hence so do the cohomology spaces $\mathrm{Ext}^*_A(M,N) = H^*(\mathrm{Hom}_A(P_*,N))$ (the B-module structure does not depend either on the choice of the projective resolution P_*).

Now assume that A is commutative, and let M, N be some right A-modules. Since A is commutative, the right A-module N can be transformed in a symmetric A-A-bimodule, so that the spaces $\mathrm{Ext}^*_A(M,N)$ carry a left and hence right A-module structure. We can therefore consider localizations of such modules, and we have the following result [68, Proposition 3.3.10]:

Proposition 5.6. *Let A be a commutative Noetherian ring, let M be a finitely generated A-module, let N be any A-module, and let $S \subset A$ be a multiplicative subset. Then we have for any $n \geq 0$*

$$S^{-1}\left(\mathrm{Ext}^*_A(M,N)\right) \simeq \mathrm{Ext}^*_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

Proof. This follows from the combination of two facts.

1. If $P_* \to M \to 0$ is a projective resolution by A-modules, then so is $S^{-1}P_* \to S^{-1}M \to 0$. This is because the localization functor is exact and preserve projective objeects, since it preserves free ones.
2. $S^{-1}\left(\mathrm{Hom}_A(M,N)\right) \simeq \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$.

The details are left to the reader. □

Combined with Proposition 5.5, this gives the following useful result [68, Corollary 3.3.11].

Corollary 5.7. *Let A be a commutative Noetherian ring, let M be a finitely generated A-module, let N be any A-module. Then, for any $n \geq 0$,*

$$\mathrm{Ext}^n_A(M,N) = \{0\} \iff \mathrm{Ext}^n_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{n}}) = \{0\} \text{ for any maximal ideal } \mathfrak{m} \subset A$$

5.2.3. *Local rings.* Recall that local ring is a commutative ring having only one maximal ideal. If A is such a local ring, with maximal ideal $\mathfrak{m}$, then for any A-module M, the quotient $M/\mathfrak{m}M$ carries a natural vector space structure over the field $K = A/\mathfrak{m}$. In particular $\mathfrak{m}/\mathfrak{m}^2$ is vector space over $K = A/\mathfrak{m}$.

The prototypical example of a local ring is the localization A_I of a commutative ring at a maximal ideal I, the unique maximal ideal being $\mathfrak{m} = IA_I$, and the quotient field A_I/IA_I being canonically isomorphic to A/I (check this). Moreover the obvious map gives a linear isomorphism between I/I^2 and $\mathfrak{m}/\mathfrak{m}^2$ (as vector spaces over A/I).

5.2.4. *Krull dimension.* If A is a commutative ring and $\mathfrak{p} \subset A$ is a prime ideal, recall that the height of $\mathfrak{p}$, denoted $\mathrm{ht}(\mathfrak{p})$, is defined to be the maximal length of chains of prime ideals

$$\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n = \mathfrak{p}$$

The Krull dimension of A is then defined by

$$\mathrm{Kdim}(A) = \sup\{\mathrm{ht}(\mathfrak{m}),\ \mathfrak{m} \text{ maximal ideal of } A\} \in \mathbb{N} \cup \{\infty\}$$

If A is local with maximal ideal $\mathfrak{m}$, we then have $\mathrm{Kdim}(A) = \mathrm{ht}(\mathfrak{m})$. Therefore, if $I \subset A$ is a maximal ideal, the map $\mathfrak{p} \mapsto \mathfrak{p}A_I$ being an order-preserving bijection between the set of prime ideals of A contained in I and the set of prime ideals of A_I, we have

$$\mathrm{Kdim}(A) = \sup\{\mathrm{Kdim}(A_I),\ I \text{ maximal ideal of } A\}$$

When $A = \mathcal{O}(G)$ is the coordinate algebra on an affine algebraic group, for any two maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ there exists an algebra automorphism $f : A \to A$ such that $f(\mathfrak{m}) = \mathfrak{n}$, hence have the same height, and we get

$$\mathrm{Kdim}(\mathcal{O}(G)) = \mathrm{Kdim}(\mathcal{O}(G)_I), \text{where } I \text{ denotes the augmentation ideal of } \mathcal{O}(G)$$

5.2.5. *Regular local rings, regular rings.* If A is a Noetherian local ring, then

$$\dim(A) \leq \dim_K(\mathfrak{m}/\mathfrak{m}^2)$$

where the first dim stands for Krull dimension (see e.g. [37, Page 52]). A regular local ring is a Noetherian local ring for which the above equality holds.

It is known that a regular local ring is an integral domain (see [68, Proposition 4.4.5], or [37, Theorem 164]).

We will use the following result (see [68, Corollary 4.6] or [37, Theorem 169]).

Theorem 5.8. *Let A be a regular local ring with maximal ideal $\mathfrak{m}$, and let $x_1, \dots, x_d$ be a sequence of elements of $\mathfrak{m}$ whose classes are a K-basis of $\mathfrak{m}/\mathfrak{m}^2$. Then these elements generate the ideal $\mathfrak{m}$ (by Nakayama's lemma) and form a regular sequence in A.*

A regular ring is a commutative Noetherian ring such that for any maximal ideal I in A, A_I is a regular local ring,

5.2.6. *Dimension of an affine algebraic group.* We finally come back to the situation that is of interest to us: $A = \mathcal{O}(G)$ for an affine algebraic group G. Denote by I the augmentation ideal of A. Let us recall a number of standard definitions and facts.

• The dimension of G is defined to be

$$\dim(G) = \mathrm{Kdim}(\mathcal{O}(G))$$

This is the definition of the dimension of G as an affine algebraic set. If G^0 is the connected component of the neutral element of G, we have $\dim(G) = \dim(G^0)$ since the irreducible components of G all are translate of each other. Since $\mathcal{O}(G^0)$ is an integral domain, we thus have

$$\dim(G) = \mathrm{Kdim}(\mathcal{O}(G)) = \mathrm{Kdim}(\mathcal{O}(G^0)) = \mathrm{trdeg}_{\mathbb{C}}(\mathrm{Fr}(\mathcal{O}(G^0))$$

where $\mathrm{trdeg}_{\mathbb{C}}(\mathrm{Fr}(\mathcal{O}(G^0))$ is the transcendence degree of the fraction field of $\mathcal{O}(G^0)$ (see [47, Theorem 5.6]).

• The Lie algebra of G can be defined as

$$\mathfrak{g} = \mathrm{Der}_\varepsilon(A, \mathbb{C})$$

and we have a linear isomorphism $\mathfrak{g} \simeq (I/I^2)^*$. We thus have, letting $\mathfrak{m} = IA_I$

$$\dim_{\mathbb{C}}(\mathfrak{g}) = \dim_{\mathbb{C}}(I/I^2) = \dim_{\mathbb{C}}(\mathfrak{m}/\mathfrak{m}^2) \geq \mathrm{Kdim}(\mathcal{O}(G)_I) = \mathrm{Kdim}(\mathcal{O}(G)) = \dim(G)$$

The smoothness theorem (see [67]) says that $\dim_{\mathbb{C}}(\mathfrak{g}) = \dim(G)$, and thus $\mathcal{O}(G)_I$ is a regular local ring.

Now put $n = \dim(G)$. We want to show that $n = \mathrm{cd}(\mathcal{O}(G))$. Let us first show that for any A-module M (with $A = \mathcal{O}(G)$), we have

$$\mathrm{Ext}_A^{n+1}(\mathbb{C}_\varepsilon, M) = \{0\}$$

By Corollary 5.7, it is enough to show that for any maximal ideal $\mathfrak{m}$ in A, we have

$$\mathrm{Ext}_{A_\mathfrak{m}}^{n+1}((\mathbb{C}_\varepsilon)_\mathfrak{m}, M_\mathfrak{m}) = \{0\}$$

and since $(\mathbb{C}_\varepsilon)_\mathfrak{m} = \{0\}$ for $\mathfrak{m} \neq I$, we only have to show that

$$\mathrm{Ext}_{A_I}^{n+1}((\mathbb{C}_\varepsilon)_I, M_I) = \mathrm{Ext}_{A_I}^{n+1}(\mathbb{C}_\varepsilon, M_I) = \{0\}$$

Let $x_1, \ldots, x_n \in I$ be such that $\overline{x_1}, \ldots, \overline{x_n}$ is a basis of I/I^2. Then the corresponding elements generate $\mathfrak{m} = IA_I$, and form a basis of $\mathfrak{m}/\mathfrak{m}^2$ and a regular sequence in the regular local ring A_I, by Theorem 5.8. It thus follows from Theorem 5.2 that $\mathrm{Ext}_{A_I}^{n+1}(\mathbb{C}_\varepsilon, M_I) = \{0\}$, According to Proposition 4.19, we have $\mathrm{cd}(A) \leq n$.

If we take now $M = A_A$, we have $M_I = A_I$, and by Theorem 5.2 we have $\mathrm{Ext}_{A_I}^{n}(\mathbb{C}_\varepsilon, A_I) \neq \{0\}$. Hence Corollary 5.7, together with Proposition 4.19, ensure that $\mathrm{cd}(A) \geq n$, as required.

6. Cohomology and homology of a Hopf algebra

6.1. **Generalities.** Let A be Hopf algebra. If M is a right A-module and N a left A-module, the spaces

$$\mathrm{Ext}_A^*(\mathbb{C}_\varepsilon, M) \quad \text{and} \quad \mathrm{Tor}_*^A(\mathbb{C}_\varepsilon, N)$$

serve as cohomology and homology spaces for A. It is thus tempting to denote them $H^*(A, M)$ and $H_*(A, M)$, but we will not do exactly that, since it is contrary to some more usual notations. Indeed, in general, if A is an algebra and M is an A-bimodule, then $H^*(A, M)$ and $H_*(A, M)$ denote usually the Hochschild cohomology and homology of A with coefficients in M. See the appendix for more details.

Definition 6.1 (Cohomology and homology of a Hopf algebra)**.**

(1) The **cohomology of a Hopf algebra A with coefficients in a right A-module** M, denoted $H^*(A, {}_\varepsilon M)$, is defined by

$$H^*(A, {}_\varepsilon M) = \mathrm{Ext}_A^*(\mathbb{C}_\varepsilon, M)$$

(2) The **homology of a Hopf algebra A with coefficients in a left A-module** N, denoted $H_*(A, N_\varepsilon)$, is defined by

$$H_*(A, N_\varepsilon) = \mathrm{Tor}_*^A(\mathbb{C}_\varepsilon, N)$$

We thus have, by Proposition 4.19,

$$\begin{aligned}\mathrm{cd}(A) &= \sup\{n : \ H^n(A, {}_\varepsilon M) \neq 0 \text{ for some } A\text{-module } M\} \\ &= \min\{n : \ H^{n+1}(A, {}_\varepsilon M) = 0 \text{ for any } A\text{-module M}\}\end{aligned}$$

Remark 6.2. Given a right A-module as above, the cohomology $H^*(A, {}_\varepsilon M)$ as above coincides with the Hochschild cohomology $H^*(A, {}_\varepsilon M)$, where ${}_\varepsilon M$ is the A-bimodule having M as underlying right A-module, and trivial left A-module structure given by $a \cdot x = \varepsilon(a)x$. So our notation is consistent with the usual one in the literature. See the appendix.

Remark 6.3. The cohomology of a discrete group Γ is defined similarly as above, but using the *integral* group ring $\mathbb{Z}\Gamma$. Since we cannot seriously impose that the Hopf algebras we are interest in are defined over $\mathbb{Z}$, our definition is *not* a full generalization of ordinary group cohomology, but rather of group cohomology with coefficients into $\mathbb{C}\Gamma$-modules.

Remark 6.4. The cohomology of a Hopf algebra only depends of the underlying augmented algebra.

Using the standard resolution of the trivial module (Proposition 4.7) together with Theorem 4.17, we get, after some identifications, a more concrete definition for cohomology.

Proposition 6.5. *Let A be Hopf algebra and let M be a right A-module. Then the cohomology $H^*(A, {}_\varepsilon M)$ is the cohomology of the complex*

$$0 \longrightarrow \mathrm{Hom}(\mathbb{C}, M) \xrightarrow{\delta} \mathrm{Hom}(A, M) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathrm{Hom}(A^{\otimes n}, M) \xrightarrow{\delta} \mathrm{Hom}(A^{\otimes n+1}, M) \xrightarrow{\delta} \cdots$$

where the differential $\delta\colon \mathrm{Hom}(A^{\otimes n}, M) \longrightarrow \mathrm{Hom}(A^{\otimes n+1}, M)$ is given by

$$\begin{aligned}\delta(f)(a_1 \otimes \cdots \otimes a_{n+1}) =& \varepsilon(a_1) f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}\end{aligned}$$

We thus have in particular

$$H^0(A, {}_\varepsilon M) = M^A = \{x \in M \mid x \cdot a = \varepsilon(a)x, \ \forall a \in A\}$$

and

$$H^1(A, {}_\varepsilon M) = \mathrm{Der}(A, {}_\varepsilon M)/\mathrm{InnDer}(A, {}_\varepsilon M)$$

where $\mathrm{Der}(A, {}_\varepsilon M)$ is the vector space of derivations $d : A \to M$, i.e. $d(ab) = \varepsilon(a)d(b) + d(a) \cdot b$ for any a, b, and $\mathrm{InnDer}(A, {}_\varepsilon M)$ is the subspace of inner derivations, i.e. those of type defined by $d(a) = \varepsilon(a)x - x \cdot a$ for some x in M.

In higher degrees, the concrete description is rarely useful to proceed with concrete computations, and the best is often to search for short or simple resolutions of the trivial module $\mathbb{C}_\varepsilon$.

Example 6.6. Let G be an affine algebraic group. Then

$$H^*(\mathcal{O}(G), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \Lambda^*(\mathfrak{g})$$

where $\mathfrak{g}$ is the (complexification of the) Lie algebra of G. This follows from the HKR (Hochschild-Kostant-Rosenberg) theorem [36], with a few other considerations. See [68, 44] (the reader might also like to read [41])

Example 6.7. (See [19]) Consider the Sweedler algebra A (Example 1.13) with its standard generators x, g. Let

$$P = \mathrm{Span}(1 + g, x + gx), \ Q = \mathrm{Span}(1 - g, x - gx)$$

We have an A-module direct sum $A = P \oplus Q$, so that P and Q are projective A-modules. We have a resolution of the trivial module

$$\cdots \to P_{n+1} \overset{\partial_{n+1}}{\to} P_n \overset{\partial_n}{\to} \cdots \to P_2 \overset{\partial_2}{\to} P_1 \overset{\partial_1}{\to} P_0 \overset{\varepsilon}{\to} \mathbb{C}_\varepsilon \to 0$$

where for $i \geq 0$, $P_{2i} = P$, $P_{2i+1} = Q$ and

$$\begin{aligned}\partial_{2i} : P_{2i} &\longrightarrow P_{2i-1} & \qquad \partial_{2i+1} : P_{2i+1} &\longrightarrow P_{2i} \\ 1 + g \mapsto x - gx,& \ x + gx \mapsto 0 & 1 - g \mapsto x + gx,& \ x - gx \mapsto 0\end{aligned}$$

Computing with this resolution, we get

$$H^p(A, {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \begin{cases} \mathbb{C} & \text{if } p \text{ is even} \\ 0 & \text{if } p \text{ is odd} \end{cases} \quad H^p(A, {}_\varepsilon\mathbb{C}_{-\varepsilon}) \simeq \begin{cases} \mathbb{C} & \text{if } p \text{ is odd} \\ 0 & \text{if } p \text{ is even} \end{cases}$$

where $-\varepsilon$ is the unique algebra map such that $-\varepsilon(g) = -1$.

6.2. **Example : quantum** SL_2**.** In this subsection we study the case of $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ (which has been the subject of numerous papers, see [35, 46, 52], for example) and compute its cohomological dimension.

Theorem 6.8. *We have, for any $q \in \mathbb{C}^*$, $\mathrm{cd}(\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))) = 3$.*

Since $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ looks like $\mathcal{O}(\mathrm{SL}_2(\mathbb{C}))$, the result is not too surprizing, and it seems to natural to try to mimic the Koszul complexes techniques from the previous section to construct a resolution of the trivial object. This is the route followed by Hadfield and Krahmer [35], who present a general method to build Koszul complexes for noncommutative algebras, which works well for $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$. The resolution we present is slightly different from the one in [35]:

Theorem 6.9. *Let $A = \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$. There exists a free resolution of A-modules*

$$0 \to A \xrightarrow{\phi_1} (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes A \xrightarrow{\phi_2} (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes A \xrightarrow{\phi_3} A \xrightarrow{\varepsilon} \mathbb{C}_\varepsilon \to 0$$

and hence $\mathrm{cd}(\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))) \leq 3$.

Proof. For $x \in A$, denoting e_1, e_2 the canonical basis of $\mathbb{C}^2$, the maps ϕ_1, ϕ_2, ϕ_3, are defined by ([8]):

$$\begin{aligned}
\phi_1(x) =& e_1^* \otimes e_1 \otimes ((-q^{-1} + qd)x) + e_1^* \otimes e_2 \otimes (-cx) \\
&+ e_2^* \otimes e_1 \otimes (-bx) + e_2^* \otimes e_2 \otimes ((-q + q^{-1}a)x) \\
\phi_2(e_1^* \otimes e_1 \otimes x) =& e_1^* \otimes e_1 \otimes x + e_2^* \otimes e_1 \otimes (-qbx) + e_2^* \otimes e_2 \otimes ax \\
\phi_2(e_1^* \otimes e_2 \otimes x) =& e_1^* \otimes e_1 \otimes bx + e_1^* \otimes e_2 \otimes (1 - q^{-1}a)x \\
\phi_2(e_2^* \otimes e_1 \otimes x) =& e_2^* \otimes e_1 \otimes (1 - qd)x + e_2^* \otimes e_2 \otimes cx \\
\phi_2(e_2^* \otimes e_2 \otimes x) =& e_1^* \otimes e_1 \otimes dx + e_1^* \otimes e_2 \otimes (-q^{-1}cx) + e_2^* \otimes e_2 \otimes x \\
\phi_3(e_1^* \otimes e_1 \otimes x) =& (a-1)x, \quad \phi_3(e_1^* \otimes e_2 \otimes x) = bx, \\
\phi_3(e_2^* \otimes e_1 \otimes x) =& cx, \quad \phi_3(e_2^* \otimes e_2 \otimes x) = (d-1)x
\end{aligned}$$

One sees easily that these maps define a complex. To see that it is exact, we will frequently use the well-known fact that A and its quotients $A/(b)$, $A/(c)$ and $A/(b,c)$ are integral domains, see e.g. [16, I.1]. The injectivity of ϕ_1 follows from the fact that A is an integral domain and the surjectivity of ϕ_3 is easy. Let $X = \sum_{i,j} e_i^* \otimes e_j \otimes x_{ij} \in \mathrm{Ker}(\phi_3)$. We have

$$X + \phi_2(-e_1^* \otimes e_1 \otimes x_{11}) = e_1^* \otimes e_2 \otimes x_{12} + e_2^* \otimes e_1 \otimes (qbx_{11} + x_{21}) + e_2^* \otimes e_2 \otimes (-ax_{11} + x_{22})$$

and hence to show that $X \in \mathrm{Im}(\phi_2)$, we can assume that $x_{11} = 0$. We have

$$bx_{12} + cx_{21} + (d-1)x_{22} = 0$$

which gives $(d-1)x = 0$ in the integral domain $A/(b,c)$ and thus $x_{22} = b\alpha + c\beta$ for some $\alpha, \beta \in A$. Then we have

$$X + \phi_2(e_1^* \otimes e_2 \otimes qd\alpha - e_2^* \otimes e_1 \otimes \beta - e_2^* \otimes e_2 \otimes b\alpha) = e_1^* \otimes e_2 \otimes x + e_2^* \otimes e_1 \otimes y$$

for some $x, y \in A$, and hence we also can assume that $x_{22} = 0$. Then we have $bx_{12} + cx_{21} = 0$, which gives $bx_{12} = 0$ in the integral domain $A/(c)$, and hence $x_{12} = c\alpha$ for some $\alpha \in A$, and moreover $x_{21} = -b\alpha$. Then we have

$$\phi_2(q^{-1}e_1^* \otimes e_1 \otimes \alpha + e_1^* \otimes e_2 \otimes c\alpha - q^{-1}e_2^* \otimes e_2 \otimes a\alpha) = X$$

and we conclude that $\mathrm{Ker}(\phi_3) = \mathrm{Im}(\phi_2)$.

Let $X = \sum_{i,j} e_i^* \otimes e_j \otimes x_{ij} \in \mathrm{Ker}(\phi_2)$. Then $-qbx_{11} + (1-qd)x_{21} = 0$, hence $(1-qd)x_{21} = 0$ in the integral domain $A/(b)$ and hence $x_{21} = b\alpha$ for some $\alpha \in A$. Hence

$$X + \phi_1(\alpha) = e_1^* \otimes e_1 \otimes (x_{11} + (-q^{-1} + qd)\alpha) + e_1^* \otimes e_2 \otimes (x_{12} - c\alpha) + e_2^* \otimes e_2 \otimes (x_{22} + (-q + q^{-1}a)\alpha)$$

and we can assume that $x_{21} = 0$. But then, using the fact that A is an integral domain, we see that $X = 0$ since $X \in \mathrm{Ker}(\phi_2)$. We conclude that $\mathrm{Ker}(\phi_2) = \mathrm{Im}(\phi_1)$. ☐

Corollary 6.10. *We have*

$$H^p(\mathcal{O}(\mathrm{SL}_2(\mathbb{C})), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \begin{cases} \mathbb{C} & \text{if } p = 0, 3 \\ \mathbb{C}^3 & \text{if } p = 1, 2 \\ 0 & \text{otherwise} \end{cases} \quad H^p(\mathcal{O}_{-1}(\mathrm{SL}_2(\mathbb{C})), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \begin{cases} \mathbb{C} & \text{if } p = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

and for $q \neq \pm 1$,

$$H^p(\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C})), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \begin{cases} \mathbb{C} & \text{if } p = 0, 1 \\ 0 & \text{otherwise} \end{cases} \quad H^p(\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C})), {}_\varepsilon\mathbb{C}_\psi) \simeq \begin{cases} \mathbb{C} & \text{if } p = 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

where $\psi : \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}$ is the algebra map defined by $\psi(a) = q^2$, $\psi(d) = q^{-2}$, $\psi(b) = \psi(c) = 0$.

Proof. Exercise, using the previous resolution. □

Proof of Theorem 6.8. We have $\mathrm{cd}(\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))) \leq 3$ by Theorem 6.9, and $\mathrm{cd}(\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))) \geq 3$ by Corollary 6.10: the result follows. □

Remark 6.11. The fact that for $q \neq \pm 1$, $H^2(\mathcal{O}(\mathrm{SL}_2(\mathbb{C})), {}_\varepsilon\mathbb{C}_\varepsilon) = 0 = H^3(\mathcal{O}(\mathrm{SL}_2(\mathbb{C})), {}_\varepsilon\mathbb{C}_\varepsilon)$, in contrast with the classical case, is known as the dimension drop, see [35] for this question.

For other q-deformations of classical algebraic groups, we refer the reader to [15, 34, 17], and the references therein.

7. Exact sequences of Hopf algebras and cohomological dimension

We now discuss the behaviour of cohomological dimension when passing to Hopf subalgebras and quotient subalgebras. This leads to exact sequence of Hopf algebras.

7.1. **Hopf subalgebras.** Here is the basic result.

Proposition 7.1. *Let $B \subset A$ be a Hopf subalgebra. If A is projective as a right B-module, then $\mathrm{cd}(B) \leq \mathrm{cd}(A)$.*

Proof. Since A is projective as a B-module, the restriction of a projective A-module to a B-module is a projective B-module (check this). The result follows, since the restriction of an A-projective resolution of $\mathbb{C}_\varepsilon$ is a B-projective resolution. □

It is thus crucial, to use the previous result, to know if a Hopf algebra is projective as a module over its Hopf subalgebras. Here is a short review of known fact about this question. In what follows $B \subset A$ is a Hopf subalgebra.

(1) Unfortunately this is not true in general, see [53], but the examples presented there are somewhat pathological.
(2) If A is a group algebra, then A is free over B, so is projective.
(3) If A has bijective antipode and is faithfully flat as a (left or right) B-module, then A is projective as a (left or right) B-module: this is [56, Corollary 1.8]. Here that A is faithfully flat as a right B-module means as usual that the functor ${}_B\mathcal{M} \to {}_A\mathcal{M}$, $M \mapsto A \otimes_B M$ is exact and creates exact sequences (a short sequence that becomes exact after applying this functor was necessarily exact).
(4) If B is commutative and A has bijective antipode, combining (2) with the faithful flatness result in [4] yields that A is projective as a B-module.
(5) If A is cosemisimple, then A is faithfully flat as a B-module [20], so is projective over B by (2).

Therefore there are a number of interesting situations where Hopf algebras are projective over their Hopf subalgebras, and this is expected to be true in any reasonnable situation.

Example 7.2. If Γ is a free abelian group of infinite rank, then combining the proposition and Example 5.4, we see that $\mathrm{cd}(\mathbb{C}\Gamma) = \infty$. Using the group considered by Baumslag in [6] (a finitely presented group whose derived subgroup is free abelian of infinite rank), we thus see that there exist finitely presented groups Γ with $\mathrm{cd}(\mathbb{C}\Gamma) = \infty$.

In order to make the inequality in Proposition 7.1 more precise in some situations, we will consider exact sequences of Hopf algebras.

7.2. **Exact sequences.** We now define exact sequences of Hopf algebras [2]. We begin with the following preliminary notation and results.

- Let $B \subset A$ be a Hopf subalgebra. Let $B^+ = \mathrm{Ker}(\varepsilon) \cap B$ and let B^+A (resp. AB^+) be the right (resp. left) sub-A-module of A generated by B^+. When $B^+A = AB^+$, then this space is a Hopf ideal, and hence the quotient A/B^+A has a Hopf algebra structure such that the canonical map $p : A \to A/B^+A$ is a Hopf algebra map.
- Let $p : A \to L$ be a surjective morphism of Hopf algebras, and let

  $A^{\mathrm{co}L} = \{a \in A : (\mathrm{id} \otimes p)\Delta(a) = a \otimes 1\}, \quad {}^{\mathrm{co}L}A = \{a \in A : (p \otimes \mathrm{id})\Delta(a) = 1 \otimes a\}$

  Both are subalgebras of A, and when $A^{\mathrm{co}L} = {}^{\mathrm{co}L}A$, this is a Hopf subalgebra of A.

Definition 7.3. A sequence of Hopf algebra maps

$$\mathbb{C} \to B \xrightarrow{i} A \xrightarrow{p} L \to \mathbb{C}$$

is said to be **exact** if the following conditions hold.

(1) i injective and p surjective.
(2) $\mathrm{Ker}(p) = Ai(B)^+ = i(B)^+A$.
(3) $i(B) = A^{\mathrm{co}L} = {}^{\mathrm{co}L}A$.

Note that $pi = u\varepsilon$ follows automatically from these axioms.

Example 7.4. If Γ is a discrete group acting on a Hopf algebra A, then

$$\mathbb{C} \to A \xrightarrow{i} A \rtimes \mathbb{C}\Gamma \xrightarrow{\varepsilon \otimes \mathrm{id}} \mathbb{C}\Gamma \to \mathbb{C}$$

is an exact sequence of Hopf algebras.

There are a number of situations where some of the axioms follow from the other ones.

Theorem-Definition 7.5. *Let $\mathbb{C} \to B \xrightarrow{i} A \xrightarrow{p} L \to \mathbb{C}$ be a sequence of Hopf algebra maps with bijective antipodes and with i injective and p surjective. Consider the following conditions.*

(1) A is faithfully flat as a right B-module and $\mathrm{Ker}(p) = Ai(B)^+ = i(B)^+A$.
(2) p is left or right faithfully coflat (this holds if L is cosemisimple) and $i(B) = A^{\mathrm{co}L} = {}^{\mathrm{co}L}A$.

Then if (1) or (2) holds, the sequence is exact. Conversely, if the sequence is exact, then (1) $\iff$ (2). *An exact sequence satisfying (1) or (2) (and hence both) is called* ***strict***.

References for the proof. That the sequence is exact if (1) holds follows from [60, Theorem 1] (see also [48, Proposition 3.4.3], which is of easier access). Similarly if (2) holds, exactness follows from [60, Theorem 2], combined with [50, Remark 1.3]. The equivalence of (1) and (2) under the assumption of exactness is [56, Corollary 1.8]. □

Example 7.6. A sequence $1 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 1$ of morphisms of discrete groups is exact if and only if the corresponding sequence of Hopf algebras $\mathbb{C} \to \mathbb{C}\Gamma_1 \to \mathbb{C}\Gamma_2 \to \mathbb{C}\Gamma_3 \to \mathbb{C}$ is exact.

Example 7.7. A sequence $1 \to N \to G \to H \to 1$ of morphisms of affine algebraic groups is exact if and only if the corresponding sequence of Hopf algebras $\mathbb{C} \to \mathcal{O}(H) \to \mathcal{O}(G) \to \mathcal{O}(N) \to \mathbb{C}$ is exact.

As an example, we consider exact sequence of Hopf algebras of a special type, the cocentral ones. One advantage is that in this restricted setting, we can prove quite simply and directly exactness in all the possible senses of Theorem-Definition 7.5.

A Hopf algebra map $f : A \to B$ is said to be **cocentral** if $f(a_{(1)}) \otimes a_{(2)} = f(a_{(2)}) \otimes a_{(1)}$ for any $a \in A$.

Example 7.8. Let $H \subset G$ be a closed subgroup of classical compact group. Then the restriction map

$$\begin{aligned} p : \mathcal{O}(G) &\longrightarrow \mathcal{O}(H) \\ f &\longmapsto f_{|H} \end{aligned}$$

is cocentral if and only if $H \subset Z(G)$. Indeed, p is cocentral if and only if

$$f(xh) = f(hx), \ \forall f \in \mathcal{O}(G), \ \forall x \in G, \ \forall h \in H$$

and the conclusion follows since $\mathcal{O}(G)$ separates the points of G.

Proposition 7.9. *Let $p : A \to \mathbb{C}\Gamma$ be surjective cocentral morphism of Hopf algebras. Then $A^{\mathrm{co}\mathbb{C}\Gamma} = {}^{\mathrm{co}\mathbb{C}\Gamma}A$, and the sequence*

$$\mathbb{C} \to A^{\mathrm{co}\mathbb{C}\Gamma} \xrightarrow{i} A \xrightarrow{p} \mathbb{C}\Gamma \to \mathbb{C}$$

is strict exact.

Proof. The cocentrality condition clearly ensures that $A^{\mathrm{co}\mathbb{C}\Gamma} = {}^{\mathrm{co}\mathbb{C}\Gamma}A$, so we can use the previous theorem to conclude that the sequence is exact. We give direct proof of exactness in this particular setting. We have to show that letting $B = A^{\mathrm{co}\mathbb{C}\Gamma}$, we have $\mathrm{Ker}(p) = AB^+ = B^+A$.

The map $(\mathrm{id} \otimes p)\Delta : A \to A \otimes \mathbb{C}\Gamma$ endows A with a $\mathbb{C}\Gamma$-comodule structure, so the structure of comodules of a group algebra (Example 3) gives a decomposition $A = \bigoplus_{g\in\Gamma} A_g$ with

$$A_g = \{a \in A \ | a_{(1)} \otimes p(a_{(2)}) = a \otimes g\}$$

and $A_1 = A^{\mathrm{co}\mathbb{C}\Gamma} = B$, $p_{|A_g} = \varepsilon(-)g$, $A_gA_h \subset A_{gh}$, $\Delta(A_g) \subset A_g \otimes A_g$, $S(A_g) \subset A_{g^{-1}}$. For fixed $g, h \in \Gamma$ take $b \in A_{h^{-1}}$ such that $\varepsilon(b) = 1$ (such a b exists by surjectivity of p). Then for any $a \in A_{gh}$ we have

$$a = ab_{(1)}S(b_{(2)}) \in A_gA_h.$$

The same argument shows that if $a \in A_{gh}^+$, then $a \in A_g^+A_h$, hence $A_g^+A_h = A_{gh}^+$. Similarly one checks that $A_{gh}^+ = A_gA_h^+$. Now let $a \in \mathrm{Ker}(p)$, and write $a = \sum_{g\in\Gamma} a_g$, with $a_g \in A_g$. Since $a \in \mathrm{Ker}(p)$, each a_g belongs to $A_g^+ = A_{gg^{-1}g}^+ = A_1^+A_g \in B^+A$, and to $A_g^+ = A_{gg^{-1}g}^+ = A_gA_1^+ \subset AB^+$. This shows that the sequence is indeed exact.

We can show quite directly as well that A is projective as a B-module (hence faithfully flat). We have shown that $A_1 = B = A_gA_{g^{-1}}$ for any $g \in \Gamma$. Hence if we choose $x_i \in A_g$ and $y_i \in A_{g^{-1}}$ such that $\sum_{i=1}^n x_iy_i = 1$, then we can define a right A_1-module map $A_g \to B^n$ by $a \mapsto (y_ia)_{i=1}^n$ and its left inverse $B^n \to A_g$ by $(a_i)_{i=1}^n \mapsto \sum_i x_ia_i$. Hence each A_g is B-projective and so is A. □

Example 7.10. It is an immediate verification that the Hopf algebra map

$$\begin{aligned} \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C})) &\longrightarrow \mathbb{C}\mathbb{Z}_2 \\ u_{ij} &\longmapsto \delta_{ij}g \end{aligned}$$

where g denotes the generator of the cyclic group of order 2, is cocentral. We thus get a cocentral exact sequence

$$\mathbb{C} \to \mathcal{O}_q(\mathrm{PSL}_2(\mathbb{C})) \to \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}\mathbb{Z}_2 \to \mathbb{C}$$

where $\mathcal{O}_q(\mathrm{PSL}_2(\mathbb{C})) = \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))^{\mathrm{co}\mathbb{C}\mathbb{Z}_2}$ is the subalgebra generated by the elements $u_{ij}u_{kl}$, $1 \leq i, j, k, l \leq 2$.

The cohomological dimensions of Hopf algebras involved in a strict exact sequence are related as follows.

Theorem 7.11. *Let $\mathbb{C} \to B \to A \to L \to \mathbb{C}$ be a strict exact sequence of Hopf algebras, with A having bijective antipode. Then $\mathrm{cd}(A) \leq \mathrm{cd}(B) + \mathrm{cd}(L)$, and if L is finite-dimensional semisimple, then $\mathrm{cd}(B) = \mathrm{cd}(A)$.*

This is [10, Proposition 3.2], using Stefan's spectral sequence [58]. We will not use the inequality, and we give an indepent proof of the last equality. We begin with a Lemma.

Lemma 7.12. *Let $\mathbb{C} \to B \to A \xrightarrow{p} L \to \mathbb{C}$ be an exact sequence of Hopf algebras , with L finite-dimensional semisimple. Let $\tau \in L$ be a such that $\tau p(a) = \varepsilon(a)\tau$ for any $a \in A$, with $\varepsilon(\tau) = 1$, and let $t \in A$ be such that $p(t) = \tau$ (Proposition 2.3).*

(1) *Let M be a right A-module, and let $M^B = \{x \in M \mid x \cdot b = \varepsilon(b)x, \forall b \in B\}$ be the space of B-invariants. Then the A-module structure on M induces an L-module structure on M^B with $(M^B)^L = M^A$.*

(2) *Let V, W be right A-modules and let $f : V \to W$ be a B-linear map. Then the linear map $\tilde{f} : V \to W$ defined by $\tilde{f}(v) = f(v \cdot S(t_{(1)})) \cdot t_{(2)}$ is A-linear. If there exists an A-linear map $j : W \to V$ such that $fj = \mathrm{id}_W$, then $\tilde{f}j = \mathrm{id}_W$ as well.*

Proof. (1) For $x \in M^B$ and $b \in B^+$, we have $x \cdot b = 0$. Moreover, for $x \in M^B$, $a \in A$, one easily sees, using that $AB^+ = B^+A$, that $x \cdot a \in M^B$. Hence the formula $x \cdot p(a) = x \cdot a$ provides a well-defined L-module structure on M^B. The last equality is immediate.

(2) Recall that $\mathrm{Hom}(V, W)$ admits a right A-module structure defined by

$$f \cdot a(v) = f(v \cdot S(a_{(1)})) \cdot a_{(2)}$$

and that

$$\mathrm{Hom}_A(V, W) = \mathrm{Hom}(V, W)^A = (\mathrm{Hom}(V, W)^B)^L$$

Recall also that if M is a right L-module over the semisimple algebra L, then $M^L = M \cdot \tau$. Hence, since $f \in \mathrm{Hom}_B(V, W) = \mathrm{Hom}(V, W)^B$, we have $f \cdot \tau \in (\mathrm{Hom}(V, W)^B)^L = \mathrm{Hom}_A(V, W)$. We now have $f \cdot \tau = f \cdot p(t) = f \cdot t$, and it is clear that $f \cdot t$ is the map $\tilde{f}$ in the statement. The last statement is an immediate verification. □

Proof of the equality in Theorem 7.11. We already know that $\mathrm{cd}(B) \leq \mathrm{cd}(A)$, and to prove the equality we can assume that $m = \mathrm{cd}(B)$ is finite. Consider a resolution of the trivial A-module

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C}_\varepsilon$$

by projective A-modules. These are in particular projective as B-modules, so since $m = \mathrm{cd}(B)$, Proposition 4.19 yields an exact sequence of B-modules, and of A-modules

$$0 \to K \xrightarrow{i} P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C}$$

together with $r : P_m \to K$, a B-linear map such that $ri = \mathrm{id}_K$. The previous lemma yields an A-linear map $\tilde{r} : P_m \to K$ such that $\tilde{r}i = \mathrm{id}_K$. We thus obtain, since a direct summand of a projective is projective, a length m resolution of $\mathbb{C}_\varepsilon$ by projective modules over A, and we conclude that $\mathrm{cd}(A) \leq m$, as required. □

Example 7.13. If Γ is a finite group acting on a Hopf algebra A, then $\mathrm{cd}(A \rtimes \mathbb{C}\Gamma) = \mathrm{cd}(A)$.

Example 7.14. If $\mathbb{C} \to A^{\mathrm{co}\mathbb{C}\Gamma} \xrightarrow{i} A \xrightarrow{p} \mathbb{C}\Gamma \to \mathbb{C}$ is a cocentral exact sequence with Γ a finite group, then $\mathrm{cd}(A^{\mathrm{co}\mathbb{C}\Gamma}) = \mathrm{cd}(A)$. In particular $\mathrm{cd}(\mathcal{O}_q(\mathrm{PSL}_2(\mathbb{C}))) = \mathrm{cd}(\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))) = 3$.

8. Homological duality and Poincaré duality Hopf algebras

In this section we present Poincaré duality, a kind of duality between homology and cohomology, for Hopf algebras, following closely the case of groups [18].

Definition 8.1. A Hopf algebra A is said to be **homologically smooth** if the trivial module $\mathbb{C}_\varepsilon$ has a finite resolution by finitely generated projective right A-modules.

For the next definition, recall from Subsection 5.2.2 that if A is an algebra and M, N are right A-modules with N is a left B-module for another algebra B such that N is a B-A-bimodule, then the Ext-spaces $\mathrm{Ext}^*_A(M, N)$ carry a natural left B-module structure.

Definition 8.2. A **homological duality Hopf algebra of dimension** $n \geq 0$ is a Hopf algebra A satisfying the following conditions.

(1) A is homologically smooth.
(2) $\mathrm{Ext}^i_A(\mathbb{C}_\varepsilon, A_A) = (0)$ if $i \neq n$.

When the vector space $\mathrm{Ext}^n_A(\mathbb{C}_\varepsilon, A_A)$ has dimension 1, a homological duality Hopf algebra of dimension $n \geq 0$ is said to be a **Poincaré duality Hopf algebra of dimension** n, and if $\mathrm{Ext}^i_A(\mathbb{C}_\varepsilon, A_A)$ is, as a left A-module, isomorphic to the trivial left A-module ${}_\varepsilon\mathbb{C}$, then A is said to be an **orientable Poincaré duality Hopf algebra of dimension** n.

A Poincaré duality Hopf algebra of dimension n is what is usually called a twisted Calabi-Yau (Hopf) algebra in the literature, the orientable case corresponding to Calabi-Yau algebras. Calabi-Yau algebras were named in [33], one of their remarkable features is that they enjoy a duality between their Hochschild homology and cohomology [63]. We refer the reader to [66] and the references therein. Our treatment follows closely the lines of older works on duality groups [14, 18, 13].

Remark 8.3. The integer n in condition (2) is automatically the cohomological dimension of A, and we have $\mathrm{Ext}^n_A(\mathbb{C}_\varepsilon, A_A) \neq (0)$

Sketch of the proof. This follows from the combination of two facts that we did not (and will not) prove. The first fact is that $\mathrm{cd}(A)$ is the largest integer $m \geq 0$ such that $H^m(A, {}_\varepsilon F) \neq (0)$ for some free A-module F (this is a consequence of the long exact sequence for Ext, see [68] for example). The second fact is that since A is homologically smooth, then $\mathrm{Ext}^i_A(\mathbb{C}_\varepsilon, -)$ commutes with arbitrary direct sums (exercise). Hence the cohomological dimension of A is the largest integer $m \geq 0$ such that $H^m(A, {}_\varepsilon A) = \mathrm{Ext}^m_A(\mathbb{C}_\varepsilon, A_A) \neq (0)$, and this has to be n by the second condition. $\square$

Examples 8.4.
(1) A homological duality Hopf algebra of dimension 0 is precisely a semisimple Hopf algebra.
(2) If Γ is a free group, then $\mathbb{C}\Gamma$ is a homological duality Hopf algebra of dimension 1.
(3) The free algebra on n generators is a homological duality Hopf algebra of dimension 1.
(4) If Γ is a duality group of dimension n over $\mathbb{C}$ in the sense of [13], then $\mathbb{C}\Gamma$ is a homological duality Hopf algebra of dimension n, with Poincaré duality group corresponding to Poincaré duality Hopf algebra, and orientable Poincaré duality group corresponding to orientable Poincaré duality Hopf algebra.
(5) If A is a commutative Hopf algebra such the augmentation ideal $I = A^+$ is generated by a regular sequence with r elements, then A is an orientable Poincaré duality Hopf algebra of dimension r, see Theorem 5.2. In fact, since a Noetherian Hopf algebra is homologically smooth if and only if its cohomological dimension is finite (this can be read off from the considerations in [18, Chapter VIII], see [12, Proposition 3.9]), the reasoning at the end of Section 5 easily enables one to show that a finitely generated commutative Hopf algebra is always a homological duality Hopf algebra.
(6) $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ is a Poincaré duality Hopf algebra of dimension 3 (exercise).

The "homological duality" terminology will be justified by our next result, for which we need the following construction. If A is a Hopf algebra, M is a right A-module, and N is a left A-module, then we denote by $M \otimes' N$ the left A-module whose underlying vector space is $M \otimes N$ and whose A-module structure is defined by

$$a \cdot (x \otimes y) = x \cdot S(a_{(1)}) \otimes a_{(2)} \cdot y$$

Theorem 8.5. *Let A be a homological duality Hopf algebra of dimension $n \geq 0$. Then we have, for any $0 \leq i \leq n$ and any right A-module M, functorial isomorphisms*

$$H^i(A, {}_\varepsilon M) = \mathrm{Ext}^i_A(\mathbb{C}_\varepsilon, M) \simeq \mathrm{Tor}^A_{n-i}(\mathbb{C}_\varepsilon, M \otimes' D) = H_{n-i}(A, (M \otimes' D)_\varepsilon)$$

where D is the left A-module $\mathrm{Ext}^n_A(\mathbb{C}_\varepsilon, A_A)$.

The proof adapts the one developed for duality groups in [18]. We begin with two lemmas.

Lemma 8.6. *Let F, M be a right A-modules and N be a left A-module. We have*

$$F \otimes_A (M \otimes' N) \simeq (M \otimes F) \otimes_A N$$

Proof. The isomorphism is the obvious one: $z \otimes_A x \otimes y \mapsto x \otimes z \otimes_A y$. $\square$

Lemma 8.7. *Let M be a right A-module and N be a left A-module. We have*

$$\mathrm{Tor}_*^A(\mathbb{C}_\varepsilon, M \otimes' N) \simeq \mathrm{Tor}_*^A(M, N)$$

Proof. Start with a free resolution by right A-modules

$$F_* \to \mathbb{C}_\varepsilon \to 0$$

We have, using the first lemma

$$\begin{aligned}\mathrm{Tor}_*^A(\mathbb{C}_\varepsilon, M \otimes' N) &\simeq H^*(F_* \otimes_A (M \otimes' N)) \\ &\simeq H^*((M \otimes F_*) \otimes_A N)\end{aligned}$$

Tensoring $F_* \to \mathbb{C}_\varepsilon \to 0$ by M, we get a resolution $M \otimes F_* \to M \to 0$ of M by free A-modules (Proposition 1.17), so

$$H^*((M \otimes F_*) \otimes_A N) \simeq \mathrm{Tor}_*^A(M, N)$$

which thus proves our claim. $\square$

Proof of Theorem 8.5. Consider a resolution of $\mathbb{C}_\varepsilon$

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C}_\varepsilon \to 0$$

by finitely generated projective right A-modules. Consider the left A-modules

$$\overline{P_i} = \mathrm{Hom}_A(P_i, A_A)$$

and the corresponding complex

$$0 \to \overline{P_0} \to \overline{P_1} \to \cdots \to \overline{P_{n-1}} \to \overline{P_n} \to 0$$

We have $H^*(\overline{P}) = \mathrm{Ext}_A^*(\mathbb{C}_\varepsilon, A_A)$, and our assumption says that

$$0 \to \overline{P_0} \to \overline{P_1} \to \cdots \to \overline{P_{n-1}} \to \overline{P_n} \to \mathrm{Ext}_A^n(\mathbb{C}_\varepsilon, A) \to 0$$

is a resolution of $\mathrm{Ext}_A^n(\mathbb{C}_\varepsilon, A_A)$ by projective left A-modules. We then have, for a right A-module M (see the end of Section 4)

$$\begin{aligned}\mathrm{Tor}_{n-i}^A(M, \mathrm{Ext}_A^n(\mathbb{C}_\varepsilon, A)) &\simeq H^i(M \otimes_A \overline{P}) = H^i(M \otimes_A \mathrm{Hom}_A(P, A)) \\ &\simeq H^i(\mathrm{Hom}_A(P, M)) \text{ (projectivity of each } P_k) \\ &\simeq \mathrm{Ext}_A^i(\mathbb{C}_\varepsilon, M)\end{aligned}$$

The previous lemma concludes the proof. $\square$

Remark 8.8. The converse is true: if A is homologically smooth of dimension n with bijective antipode and if the above homological duality holds (for some D), then (2) in Definition 8.2 holds.

Proof. We have, by assumption, $\mathrm{Ext}_A^i(\mathbb{C}_\varepsilon, A_A) \simeq \mathrm{Tor}_{n-i}^A(\mathbb{C}_\varepsilon, A_A \otimes' D)$. We leave to the reader to check that $A_A \otimes' D$ is a free left A-module, and by the end of Section 4, this proves our claim. $\square$

Remark 8.9. When A is a Poincaré duality Hopf algebra of dimension n, i.e. we have $D = \mathrm{Ext}_A^n(\mathbb{C}_\varepsilon, A_A) \simeq {}_\alpha\mathbb{C}$ for some algebra map $\alpha : A \to \mathbb{C}$, then for any right A-module M, we have

$$\mathrm{Ext}_A^i(\mathbb{C}_\varepsilon, M) \simeq \mathrm{Tor}_{n-i}^A(\mathbb{C}_\varepsilon, {}_\theta M)$$

where θ is the anti-algebra map defined by $\theta(a) = S(a_{(1)})\alpha(a_{(2)})$, and the left A-module structure on M is $a \to x = x \cdot \theta(a)$.

If M is an A-bimodule, we get, for Hochschild cohomology (see the appendix), letting $\sigma = S\theta$

$$H^i(A, M) \simeq H_{d-i}(A, {}_\sigma M)$$

where ${}_\sigma M$ is M as having the A-bimodule structure defined by $a \cdot x \cdot b := \sigma(a) \cdot x \cdot b$.

We refer the reader to [18, 13] for (many) examples of duality groups, to [17] for (again many) purely Hopf algebraic examples, includind the q-deformed algebras of functions on classical algebraic groups, and to [40] for a recent more general framework.

9. Cohomological dimension of monoidally equivalent Hopf algebras

If A and B are Hopf algebras having equivalent categories of modules, it follows from Proposition 3.7 that they have the same cohomological dimension. We are interested here in the dual question, assuming that A and B have equivalent categories of comodules. At this level of generality, not much can be said because this property only takes the coalgebra structure into account (e.g. if A and B are group algebras, this just means that the two groups have the same cardinality), so we have to ask for a stronger condition, that takes the algebra structure into account. The condition that we will consider is that of being monoidally equivalent, see Section 1: recall that two Hopf algebras are said to be monoidally equivalent if their categories of comodules are equivalent as tensor categories. Our basic question is:

Question 9.1. Let A, B be monoidally equivalent Hopf algebras. Is it true that $\mathrm{cd}(A) = \mathrm{cd}(B)$?

Of course a natural related problem is to examine how the cohomologies of two such Hopf algebras are related. This leads us to consider Yetter-Drinfeld modules.

9.1. Yetter-Drinfeld modules.

Let A be a Hopf algebra.

Definition 9.2. a (right-right) **Yetter-Drinfeld module over** A is a right A-comodule and right A-module V satisfying the condition, $\forall v \in V$, $\forall a \in A$,

$$(v \leftarrow a)_{(0)} \otimes (v \leftarrow a)_{(1)} = v_{(0)} \leftarrow a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)}$$

The category of Yetter-Drinfeld modules over A is denoted $\mathcal{YD}_A^A$: the morphisms are the A-linear A-colinear maps. Endowed with the usual tensor product of modules and comodules, it is a tensor category.

An important example of Yetter-Drinfeld module is the right coadjoint Yetter-Drinfeld module A_{coad}: as a right A-module $A_{\mathrm{coad}} = A$ and the right A-comodule structure is defined by

$$\mathrm{ad}_r(a) = a_{(2)} \otimes S(a_{(1)})a_{(3)}, \forall a \in A$$

The following result, which will be of importance for us, generalizes the construction of the right coadjoint comodule.

Proposition 9.3. *Let A be a Hopf algebra and let V be a right A-comodule. Endow $V \otimes A$ with the right A-module structure defined by multiplication on the right. Then the linear map*

$$\begin{aligned} V \otimes A &\longrightarrow V \otimes A \otimes A \\ v \otimes a &\longmapsto v_{(0)} \otimes a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)} \end{aligned}$$

endows $V \otimes A$ with a right A-comodule structure, and with a Yetter-Drinfeld module structure. We denote by $V \boxtimes A$ the resulting Yetter-Drinfeld module, and this construction produces a functor

$$\begin{aligned} L : \mathcal{M}^A &\longrightarrow \mathcal{YD}_A^A \\ V &\longmapsto V \boxtimes A \end{aligned}$$

Proof. This is a direct verification. □

Note that when $V = \mathbb{C}$ is the trivial comodule, then $\mathbb{C} \boxtimes A = A_{\mathrm{coad}}$.

Definition 9.4. Let A be a Hopf algebra. A Yetter-Drinfeld module over A is said to be **free** if it is isomorphic to $V \boxtimes A$ for some right A-comodule V.

Of course a free Yetter-Drinfeld module is free as a right A-module. The terminology is further justified by the following result.

Proposition 9.5. *Let A be a Hopf algebra. The functor $L = - \boxtimes A : \mathcal{M}^A \longrightarrow \mathcal{YD}_A^A$ is left adjoint to the forgetful functor $R : \mathcal{YD}_A^A \longrightarrow \mathcal{M}^A$. In particular if P is a projective object in $\mathcal{M}^A$, then $L(P)$ is a projective object in $\mathcal{YD}_A^A$.*

Here of course a projective object in $\mathcal{YD}_A^A$ is an object X such that the functor $\mathrm{Hom}_{\mathcal{YD}_A^A}(X, -)$ is exact.

Proof. Let $V \in \mathcal{M}^A$ and $X \in \mathcal{YD}_A^A$. It is a direct verification to check that we have a natural isomorphism

$$\begin{aligned} \mathrm{Hom}^A(V, R(X)) &\longrightarrow \mathrm{Hom}_{\mathcal{YD}_A^A}(V \boxtimes A, X) \\ f &\longmapsto \tilde{f},\ \tilde{f}(v \otimes a) = f(v) \leftarrow a \end{aligned}$$

and thus $L = - \boxtimes A$ is left adjoint to the forgetful functor R. We thus have an isomomorphism of functors

$$\mathrm{Hom}^A(P, R(-)) \simeq \mathrm{Hom}_{\mathcal{YD}_A^A}(P \boxtimes A, -)$$

This functor on the left is exact if P is projective (R is exact) and the last assertion, which a general property of adjoint functors, follows. ☐

Definition 9.6. Let V be a Yetter-Drinfeld module over A. Then V is said to be **relative projective** if the functor $\mathrm{Hom}_{\mathcal{YD}_A^A}(V, -)$ transforms exact sequences of Yetter-Drinfeld modules that split as sequences of comodules to exact sequences of vector spaces.

Relative projective Yetter-Drinfeld modules have the following characterization.

Proposition 9.7. *Let P be a Yetter-Drinfeld module over A. The following assertions are equivalent.*

(1) P is relative projective.

(2) Any surjective morphism of Yetter-Drinfeld modules $f : M \to P$ that admits a section which is a map of comodules admits a section which is a morphism of Yetter-Drinfeld modules.

(3) P is a direct summand of a free Yetter-Drinfeld module.

If A is cosemisimple, these conditions are equivalent to P being a projective object of $\mathcal{YD}_A^A$.

Proof. The proof of (1)⇒(2) is similar to the usual one for modules. Assume (2), and consider the surjective Yetter-Drinfeld module morphism $R(P) \boxtimes A \to P$, $x \otimes a \mapsto x \leftarrow a$. The map $P \to R(P) \boxtimes A$, $x \mapsto x \otimes 1$ is an A-colinear section, so by (2) P is indeed, as a Yetter-Drinfeld module, a direct summand of $R(P) \boxtimes A$.

Assume now that P is free, i.e. $P = V \boxtimes A$ for some comodule V, and let

$$0 \to M \xrightarrow{i} N \xrightarrow{p} Q \to 0$$

be an exact sequence of Yetter-Drinfeld modules that splits as a sequence of comodules. The sequence

$$0 \to \mathrm{Hom}_{\mathcal{YD}_A^A}(P, M) \xrightarrow{i-} \mathrm{Hom}_{\mathcal{YD}_A^A}(P, N) \xrightarrow{p-} \mathrm{Hom}_{\mathcal{YD}_A^A}(P, Q)$$

is exact and we have to show the surjectivity of the map on the right. Let $s : Q \to N$ be a morphism of comodules such that $ps = \mathrm{id}_Q$. Let $\varphi \in \mathrm{Hom}_{\mathcal{YD}_A^A}(V \boxtimes A, Q)$, and let $\varphi_0 : V \to Q$ be defined by $\varphi_0(v) = \varphi(v \otimes 1)$: φ_0 is a map of comodules, and so is $s\varphi_0$. Considering now $\widetilde{s\varphi_0} \in \mathrm{Hom}_{\mathcal{YD}_A^A}(V \boxtimes A, N)$, we have $p\widetilde{s\varphi_0} = \varphi$, which gives the expected surjectivity result. Now if $V \boxtimes A \simeq P \oplus M$ as Yetter-Drinfeld modules, then $\mathrm{Hom}_{\mathcal{YD}_A^A}(V \boxtimes A, -) \simeq \mathrm{Hom}_{\mathcal{YD}_A^A}(P, -) \oplus \mathrm{Hom}_{\mathcal{YD}_A^A}(M, -)$, and the usual argument for projective modules works to conclude that P is relative projective.

It is clear that a projective Yetter-Drinfeld module is relative projective, and if A is cosemisimple, a free Yetter-Drinfeld module is a projective object in $\mathcal{YD}_A^A$ (Proposition 9.5), hence a direct summand of a free Yetter-Drinfeld module is projective, and so is a relative projective Yetter-Drinfeld module. ☐

Relative projective Yetter-Drinfeld modules are projective as modules. Our interest for these modules, in view of Question 9.1, comes from the following result.

Theorem 9.8. *Let A and B be monoidally equivalent Hopf algebras, via an equivalence of linear tensor categories $\Theta : \mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$. Then Θ induces an equivalence of linear tensor categories $\widehat{\Theta} : \mathcal{YD}_A^A \simeq^{\otimes} \mathcal{YD}_B^B$ that preserves free (resp. relative projective) Yetter-Drinfeld modules. The functor $\widehat{\Theta}$ associates to any relative projective Yetter-Drinfeld resolution of the trivial Yetter-Drinfeld module in $\mathcal{YD}_A^A$*

$$P_* \to \mathbb{C} \to 0$$

a relative projective Yetter-Drinfeld resolution in $\mathcal{YD}_B^B$

$$\widehat{\Theta}(P_*) \to \mathbb{C} \to 0$$

Proof. Let $R_A : \mathcal{YD}_A^A \longrightarrow \mathcal{M}^A$ and $R_B : \mathcal{YD}_B^B \longrightarrow \mathcal{M}^B$ be the respective forgetful functors with their respective left adjoint $L_A : \mathcal{M}^A \longrightarrow \mathcal{YD}_A^A$ and $L_B : \mathcal{M}^B \longrightarrow \mathcal{YD}_B^B$. The description of $\mathcal{YD}_A^A$ as the weak center of the monoidal category $\mathcal{M}_A$ (this is stated in [55], Appendix, the proof can be done along similar lines as the one for modules over a finite-dimensional Hopf algebra, given in [38], Theorem XIII.5.1) ensures the existence of an equivalence of linear tensor categories $\widehat{\Theta} : \mathcal{YD}_A^A \simeq \mathcal{YD}_B^B$ such that $R_B\widehat{\Theta} \simeq \Theta R_A$ as functors. Denote by Θ^{-1} a quasi-inverse of Θ. Then we have, for any $U \in \mathcal{M}^B$ and $X \in \mathcal{YD}_B^B$, natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{YD}_B^B}(\widehat{\Theta}L_A\Theta^{-1}(U), X) &\simeq \mathrm{Hom}_{\mathcal{YD}_A^A}(L_A\Theta^{-1}(U), \widehat{\Theta}^{-1}(X)) \\
&\simeq \mathrm{Hom}_{\mathcal{M}^A}(\Theta^{-1}(U), R_A\widehat{\Theta}^{-1}(X)) \\
&\simeq \mathrm{Hom}_{\mathcal{M}^A}(\Theta^{-1}(U), \Theta^{-1}R_B(X)) \\
&\simeq \mathrm{Hom}_{\mathcal{M}^B}(U, R_B(X))
\end{aligned}$$

The uniqueness of adjoint functors ensures that $\widehat{\Theta}L_A\Theta^{-1} \simeq L_B$, so that $\widehat{\Theta}L_A \simeq L_B\Theta$, as required: $\widehat{\Theta}$ preserves free Yetter-Drinfeld modules, and preserves the relative projective ones by Proposition 9.7. The last assertion is then immediate. $\square$

Example 9.9. Let $A = \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$. The free resolution of A-modules

$$0 \to A \xrightarrow{\phi_1} (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes A \xrightarrow{\phi_2} (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes A \xrightarrow{\phi_3} A \xrightarrow{\varepsilon} \mathbb{C}_\varepsilon \to 0$$

from Theorem 6.9 is a resolution by free Yetter-Drinfeld modules ($\mathbb{C}^2$ having the canonical structure of comodule over A).

Therefore, for any Hopf algebra B that is monoidally equivalent to $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$, we have $\mathrm{cd}(B) \leq 3$. To use Theorem 9.8 in a more precise way, one needs a more concrete description for the monoidal equivalences involved (using Hopf bi-Galois extensions [54, 55] or cogroupoids [9]), that we do not explain here, see [8]). This enables one to construct the following resolution, using the previous one, see [8].

Theorem 9.10. *Let $E \in \mathrm{GL}_n(\mathbb{C})$, $n \geq 2$, and let V_E be the fundamental n-dimensional $\mathcal{B}(E)$-comodule. There exists an exact sequence of Yetter-Drinfeld modules over $\mathcal{B}(E)$*

$$0 \to \mathbb{C} \boxtimes \mathcal{B}(E) \xrightarrow{\phi_1} (V_E^* \otimes V_E) \boxtimes \mathcal{B}(E) \xrightarrow{\phi_2} (V_E^* \otimes V_E) \boxtimes \mathcal{B}(E) \xrightarrow{\phi_3} \mathbb{C} \boxtimes \mathcal{B}(E) \xrightarrow{\varepsilon} \mathbb{C} \to 0$$

which thus yields a free Yetter-Drinfeld resolution of the trivial Yetter-Drinfeld module over $\mathcal{B}(E)$. Moreover we have $\mathrm{cd}(\mathcal{B}(E)) = 3$.

When $E = I_n$, the resolution was found in [21], using computer calculations. The above resolution can also be used to show (see [8, 64, 70, 66]):

Theorem 9.11. *For any $E \in \mathrm{GL}_n(\mathbb{C})$, $n \geq 2$, the Hopf algebra $\mathcal{B}(E)$ is a Poincaré duality Hopf algebra of dimension* 3.

An interesting partial positive anwer to question 9.1 is given in the recent paper by Wang, Yu and Zhang [66], in the setting of Poincaré duality Hopf algebras:

Theorem 9.12. *Let A, B be monoidally equivalent Hopf algebras. Assume that A is a Poincaré duality Hopf algebra of dimension $n \geq 0$ and that B is homologically smooth. Then* $\mathrm{cd}(A) = \mathrm{cd}(B)$.

Using this result together with the monoidal equivalence mentionned in Section 1 and Theorem 9.8, one can show that $A_s(n)$ (Example 1.14) is, for $n \geq 4$, an orientable Poincaré duality Hopf algebra, see [12].

9.2. **Gerstenhaber-Schack cohomology.** Still motivated by Question 9.1, we consider now Gerstenhaber-Schack cohomology, which is a cohomology for Hopf algebras having coefficients in Yetter-Drinfeld modules. Let A be a Hopf algebra.

(1) If A is cosemisimple, every object in $\mathcal{M}^A$ is projective and it follows from Proposition 9.5 that that any Yetter Drinfeld is a quotient of a projective Yetter-Drinfeld modules ($\mathcal{YD}_A^A$ has enough projective objects), the projective objects are the direct summands of the free ones (Proposition 9.7) and we can define Ext in the category of Yetter-Drinfeld modules exactly as in Section 4, using projective resolutions. We then define the Gerstenhaber-Schack cohomology of A with coefficients in a Yetter-Drinfeld module V by
$$H^*_{\mathrm{GS}}(A, V) = \mathrm{Ext}^*_{\mathcal{YD}_A^A}(\mathbb{C}, V)$$
and the Gerstenhaber-Schack cohomological dimension of A by
$$\mathrm{cd}_{\mathrm{GS}}(A) = \sup\{n : \ H^n_{\mathrm{GS}}(A, V) \neq 0 \text{ for some } V \in \mathcal{YD}_A^A\} \in \mathbb{N} \cup \{\infty\}$$
and similarly to Proposition 4.19, $\mathrm{cd}_{\mathrm{GS}}(A)$ is also the shortest length for a resolution of the trivial Yetter-Drinfeld module by projective Yetter-Drinfeld modules. In the second appendix we present an explicit complex to compute Gerstenhaber-Schack cohomology in the cosemisimple case.

(2) If A is co-Frobenius, which means that $\mathcal{M}^A$ has enough projectives (see [2] for general results and non cosemisimple examples) then again Proposition 9.5 ensures that $\mathcal{YD}_A^A$ has enough projective objects, so we can proceed as in the previous item to define Gerstenhaber-Schack cohomology and Gerstenhaber-Schack cohomological dimension, the main additional difficulty being that we do not have anymore the nice characterization of projective objects from Proposition 9.7.

(3) In general, the basic problem is that the category of Yetter-Drinfeld modules does not always have enough projective objects, so that one cannot define $\mathrm{Ext}^*_{\mathcal{YD}}$ has we did in Section 4. This can be bypassed by the fact that $\mathcal{YD}_A^A$ always has enough injective objects, and one can define Ext by using injective resolutions of the second factor (see [68]). So we define, for a Hopf algebra A and a Yetter-Drinfeld module V,
$$H^*_{\mathrm{GS}}(A, V) = \mathrm{Ext}^*_{\mathcal{YD}_A^A}(\mathbb{C}, V)$$
This is not the original definition of Gerstenhaber-Schack [32], but it coincides with it by [59]. Of course we also put
$$\mathrm{cd}_{\mathrm{GS}}(A) = \sup\{n : \ H^n_{\mathrm{GS}}(A, V) \neq 0 \text{ for some } V \in \mathcal{YD}_A^A\} \in \mathbb{N} \cup \{\infty\}$$

Example 9.13. The resolution in Theorem 9.10 is by free Yetter-Drinfeld modules, so when $\mathcal{B}(E)$ is cosemisimple, one can use it to show that $\mathrm{cd}_{\mathrm{GS}}(\mathcal{B}(E)) = 3$.

It is clear from Theorem 9.8 that if A and B are monoidally equivalent Hopf algebras, then
$$\mathrm{cd}_{\mathrm{GS}}(A) = \mathrm{cd}_{\mathrm{GS}}(B)$$
Hence in view of Question 9.1, a key problem is to compare the two cohomological dimensions. Here is the most general answer known to me at the time of writing these notes [10, 11].

Theorem 9.14. *Let A be a Hopf algebra. We have* $\mathrm{cd}(A) \leq \mathrm{cd}_{\mathrm{GS}}(A)$*, with equality if A is cosemisimple with* $S^4 = \mathrm{id}$.

Proof. We give the proof in the simplified situation where A is cosemisimple and $S^2 = \mathrm{id}$. Since a projective Yetter-Drinfeld module is projective as a module, a resolution of the trivial Yetter-Drinfeld module by projective objects is in particular a resolution of the trivial module by projective modules, so we get that $\mathrm{cd}(A) \leq \mathrm{cd}_{\mathrm{GS}}(A)$ (here we have just used cosemisimplicity).

The proof of the last equality in the statement will use the following lemma.

Lemma 9.15. *Let V, W be Yetter-Drinfeld modules over the cosemisimple Hopf algebra A satisfying $S^2 = \mathrm{id}$, let $i : W \to V$ be an injective morphism of Yetter-Drinfeld modules, and let $r : V \to W$ be an A-linear map such that $ri = \mathrm{id}_W$. Then there exists a morphism of Yetter-Drinfeld modules $\tilde{r} : V \to W$ such that $\tilde{r}i = \mathrm{id}_W$.*

Proof of the lemma. In a dual manner to Proposition 2.3, the cosemisimplicity of A amounts to the existence of a Haar integral on A: a left and right A-colinear map $h : A \to \mathbb{C}$ such that $h(1) = 1$. The orthogonality relations (see [39] for example) ensure that $S^2 = \mathrm{id}$ if and only if h is a trace. So let h be the Haar integral on A. Then for any A-comodules V and W, we have a surjective averaging operator

$$\begin{aligned} M : \mathrm{Hom}(V, W) &\longrightarrow \mathrm{Hom}^A(V, W) \\ f &\longmapsto M(f), \ M(f)(v) = h\left(f(v_{(0)})_{(1)} S(v_{(1)})\right) f(v_{(0)})_{(0)} \end{aligned}$$

with $f \in \mathrm{Hom}^A(V, W)$ if and only if $M(f) = f$. We put $\tilde{r} = M(r)$, and it is straightforward to check that $\tilde{r}i = \mathrm{id}_W$. It remains to check that $\tilde{r}$ is A-linear. We have, using the Yetter-Drinfeld condition and the A-linearity of r,

$$\begin{aligned} \tilde{r}(v \cdot a) &= h\left(r((v \cdot a)_{(0)})_{(1)} S((v \cdot a)_{(1)})\right) r((v \cdot a)_{(0)})_{(0)} \\ &= h\left(r(v_{(0)} \cdot a_{(2)})_{(1)} S(S(a_{(1)}) v_{(1)} a_{(3)})\right) r(v_{(0)} \cdot a_{(2)})_{(0)} \\ &= h\left((r(v_{(0)}) \cdot a_{(2)})_{(1)} S(S(a_{(1)}) v_{(1)} a_{(3)})\right) (r(v_{(0)}) \cdot a_{(2)})_{(0)} \\ &= h\left(S(a_{(2)}) r(v_{(0)})_{(1)} a_{(4)} S(S(a_{(1)}) v_{(1)} a_{(5)})\right) r(v_{(0)})_{(0)} \cdot a_{(3)} \\ &= h\left(S(a_{(2)}) r(v_{(0)})_{(1)} a_{(4)} S(a_{(5)}) S(v_{(1)}) S^2(a_{(1)})\right) r(v_{(0)})_{(0)} \cdot a_{(3)} \\ &= h\left(S(a_{(2)}) r(v_{(0)})_{(1)} S(v_{(1)}) S^2(a_{(1)})\right) r(v_{(0)})_{(0)} \cdot a_{(3)} \end{aligned}$$

Thus, if $S^2 = \mathrm{id}$, we have

$$\begin{aligned} \tilde{r}(v \cdot a) &= h\left(S(a_{(2)}) r(v_{(0)})_{(1)} S(v_{(1)}) S^2(a_{(1)})\right) r(v_{(0)})_{(0)} \cdot a_{(3)} \\ &= h\left(r(v_{(0)})_{(1)} S(v_{(1)})\right) r(v_{(0)})_{(0)} \cdot a \\ &= \tilde{r}(v) \cdot a \end{aligned}$$

and hence $\tilde{r}$ is A-linear. □

Back to the proof of our theorem, to prove our equality we can assume that $m = \mathrm{cd}(A)$ is finite. Consider a resolution of the trivial Yetter-Drinfeld module

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C}$$

by projective Yetter-Drinfeld modules over A. These are in particular projective as A-modules, so since $m = \mathrm{cd}(A)$, Proposition 4.19 yields an exact sequence of Yetter-Drinfeld modules over A

$$0 \to K \xrightarrow{i} P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C}$$

together with $r : P_m \to K$, an A-linear map such that $ri = \mathrm{id}_K$. The previous proposition yields a morphism of Yetter-Drinfeld module $\tilde{r} : P_m \to K$ such that $\tilde{r}i = \mathrm{id}_K$. We thus obtain, since a direct summand of a projective is projective, a length m resolution of $\mathbb{C}$ by projective Yetter-Drinfeld modules over A, and we conclude that $\mathrm{cd}_{\mathrm{GS}}(A) \leq m$, as required. □

Of course, in view of the monoidal invariance of the Gerstenhaber-Schack cohomological dimension, Theorem 9.14 provides a partial positive answer to Question 9.1:

Corollary 9.16. *If A, B are monoidally equivalent Hopf algebras with A cosemisimple and $S^4 = \mathrm{id}$, then $\mathrm{cd}(A) \geq \mathrm{cd}(B)$, with equality if moreover the antipode of B also satisfies $S^4 = \mathrm{id}$.*

Example 9.17. The Hopf algebra $\mathcal{O}_{q,q^{-1}}(\mathrm{GL}_2(\mathbb{C}))$ of Example 1.25 satisfy

$$\mathrm{cd}(\mathcal{O}_{q,q^{-1}}(\mathrm{GL}_2(\mathbb{C})) = \mathrm{cd}_{\mathrm{GS}}(\mathcal{O}_{q,q^{-1}}(\mathrm{GL}_2(\mathbb{C})) = \mathrm{cd}_{\mathrm{GS}}(\mathcal{O}(\mathrm{GL}_2(\mathbb{C})) = \mathrm{cd}(\mathcal{O}(\mathrm{GL}_2(\mathbb{C})) = 4$$

10. An open question

We conclude these notes by a open question, that we believe to be of high interest.

Question 10.1. What are the Hopf algebras of cohomological dimension one ?

Recall from Example 3.2 that Dunwoody's theorem [29] states that a finitely generated discrete group has cohomological dimension one if only if it constains a free subgroup of finite index. So, is there an analogue of this theorem for Hopf algebras ? Notice that it is not difficult to construct examples of noncommutative and noncocommutative Hopf algebras of cohomological dimension one, using free product, crossed product or crossed coproduct constructions. Of course, the ring-theoretic analogues of Bass-Serre techniques [23, 31] should play a role here.

Appendix A. Relation with Hochschild (co)homology

In this appendix we explain the relation between the (co)homology we have considered for Hopf algebras and Hochschild (co)homology.

Let A be an algebra and let M be an A-bimodule.

- The Hochschild cohomology spaces $H^*(A, M)$ are the cohomology spaces of the complex

$$0 \longrightarrow \mathrm{Hom}(\mathbb{C}, M) \xrightarrow{\delta} \mathrm{Hom}(A, M) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathrm{Hom}(A^{\otimes n}, M) \xrightarrow{\delta} \mathrm{Hom}(A^{\otimes n+1}, M) \xrightarrow{\delta} \cdots$$

where the differential $\delta\colon \mathrm{Hom}(A^{\otimes n}, M) \longrightarrow \mathrm{Hom}(A^{\otimes n+1}, M)$ is given by

$$\begin{aligned}\delta(f)(a_1 \otimes \cdots \otimes a_{n+1}) =& a_1 \cdot f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}\end{aligned}$$

- The Hochschild homology spaces $H_*(A, M)$ are the homology spaces of the complex

$$\cdots \longrightarrow M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \cdots \xrightarrow{b} M \otimes A \xrightarrow{b} M \longrightarrow 0$$

where the differential $b\colon M \otimes A^{\otimes n} \longrightarrow M \otimes A^{\otimes n-1}$ is given by

$$\begin{aligned}b(x \otimes a_1 \otimes \cdots \otimes a_n) =& x \cdot a_1 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i x \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ &+ (-1)^n a_n \cdot x \otimes a_1 \otimes \cdots \otimes a_{n-1}\end{aligned}$$

The following result (or variants of it) is proved in many papers.

Proposition A.1. *Let A be a Hopf algebra and let M be an A-bimodule. Define a left A-module structure on M and a right A-module structure on M by*

$$a \rightarrow x = a_{(2)} \cdot x \cdot S(a_{(1)}), \quad x \leftarrow a = S(a_{(1)}) \cdot x \cdot a_{(2)}$$

and denote by M' and M'' the respective corresponding left A-module and right A-module. Then for all $n \in \mathbb{N}$ there exist isomorphisms of vector spaces

$$H_n(A, M) \simeq \mathrm{Tor}_n^A(\mathbb{C}_\varepsilon, M'), \quad H^n(A, M) \simeq \mathrm{Ext}_A^n(\mathbb{C}_\varepsilon, M'')$$

Conversely, if M is a right A-module and N is a left A-module, denote ${}_\varepsilon M$ the and N_ε the A-bimodules whose left and right structures are induced by ε. Then the Hochschild homology and cohomology spaces $H_*(A, N_\varepsilon)$ and $H^*(A, {}_\varepsilon M)$ coincide with the homology and cohomology spaces introduced in Definition 6.1.

Proof. Using the standard resolution of the trivial object, we see that $\mathrm{Tor}_n^A(\mathbb{C}_\varepsilon, M')$ is the homology of the following complex

$$\cdots \longrightarrow A^{\otimes n} \otimes M' \xrightarrow{d} A^{\otimes n-1} \otimes M' \xrightarrow{d} \cdots \xrightarrow{d} A \otimes M' \xrightarrow{d} M' \longrightarrow 0$$

where the differential $d : A^{\otimes n} \otimes M' \longrightarrow A^{\otimes n-1} \otimes M'$ is given by

$$\begin{aligned} d(a_1 \otimes \cdots \otimes a_n \otimes x) =& \varepsilon(a_1)a_2 \otimes \cdots \otimes a_n \otimes x + \sum_{i=1}^{n-1}(-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes x \\ &+ (-1)^n a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n \cdot x \end{aligned}$$

Consider the linear map

$$\begin{aligned} \theta : M \otimes A^{\otimes n} &\longrightarrow A^{\otimes n} \otimes M' \\ x \otimes a_1 \otimes \cdots \otimes a_n &\longmapsto a_{1(2)} \otimes \cdots \otimes a_{n(2)} \otimes x \cdot (a_{1(1)} \cdot \cdots \cdot a_{n(1)}) \end{aligned}$$

It is straightforward to see that θ is an isomorphism with inverse given by

$$\begin{aligned} \theta^{-1} : A^{\otimes n} \otimes M' &\longrightarrow M \otimes A^{\otimes n} \\ a_1 \otimes \cdots \otimes a_n \otimes x &\longmapsto x \cdot S(a_{1(1)} \cdots a_{n(1)}) \otimes (a_{1(2)} \otimes \cdots \otimes \cdots a_{n(2)}) \end{aligned}$$

and that $d \circ \theta = \theta \circ b$. Hence θ induces an isomorphism between the complexes defining $H_*(A, M)$ and $\mathrm{Tor}_*^A(\mathbb{C}_\varepsilon, M')$ and we get the first isomorphism $H_*(A, M) \simeq \mathrm{Tor}_*^A(\mathbb{C}_\varepsilon, M')$. For cohomology, consider the linear map

$$\begin{aligned} \vartheta : \mathrm{Hom}(A^{\otimes n}, M) &\longrightarrow \mathrm{Hom}(A^{\otimes n}, M'') \\ f &\longmapsto \hat{f},\ \hat{f}(a_1 \otimes \cdots \otimes a_n) = S(a_{1(1)} \cdots a_{n(1)}) f(a_{1(2)} \otimes \cdots \otimes a_{n(2)}) \end{aligned}$$

It is easy to see that ϑ is an isomorphism and that $\partial \circ \vartheta = \vartheta \circ \delta$. Hence ϑ induces an isomorphism between the complexes defining $H^*(A, M)$ and $\mathrm{Ext}_A^*(\mathbb{C}_\varepsilon, M'')$ and we get the second isomorphism $H^*(A, M) \simeq \mathrm{Ext}_A^*(\mathbb{C}_\varepsilon, M'')$.

If M is a right A-module and N is a left A-module, then $({}_\varepsilon M)' \simeq M$ and $(N_\varepsilon)'' \simeq N$, so the last assertion follows from the first ones. ☐

Notice that the cohomological dimension of the Hopf algebra A is thus

$$\begin{aligned} \mathrm{cd}(A) &= \sup\{n :\ H^n(A, M) \neq 0 \text{ for some } A\text{-bimodule } M\} \in \mathbb{N} \cup \{\infty\} \\ &= \min\{n :\ H^{n+1}(A, M) = 0 \text{ for for any } A\text{-bimodule } M\} \\ &= \mathrm{pd}_{{}_A\mathcal{M}_A}(A) \end{aligned}$$

where $\mathrm{pd}_{{}_A\mathcal{M}_A}(A)$ is the projective dimension of A in the category of A-bimodules, and coincides with the Hochschild cohomological dimension, defined for any algebra.

Appendix B. An explicit complex for Gerstenhaber-Schack cohomology

In this second appendix we present an explicit complex to compute Gerstenhaber-Schack cohomology in the cosemisimple case.

We begin with the following construction. For any $n \in \mathbb{N}$, we define the comodule $A^{\boxtimes n}$ as follows:

$$A^{\boxtimes 0} = \mathbb{C},\ A^{\boxtimes 1} = \mathbb{C} \boxtimes A = A_{\mathrm{coad}},\ A^{\boxtimes 2} = A^{\boxtimes 1} \boxtimes A,\ \ldots, A^{\boxtimes (n+1)} = A^{\boxtimes n} \boxtimes A, \ldots$$

It is straighforward to check that after the obvious vector space identification of $A^{\boxtimes n}$ with $A^{\otimes n}$, the right A-module structure of $A^{\boxtimes n}$ is given by right multiplication and its comodule structure is given by

$$\begin{aligned} \mathrm{ad}_r^{(n)} : A^{\boxtimes n} &\longrightarrow A^{\boxtimes n} \otimes A \\ a_1 \otimes \cdots \otimes a_n &\longmapsto a_{1(2)} \otimes \cdots \otimes a_{n(2)} \otimes S(a_{1(1)} \cdots a_{n(1)}) a_{1(3)} \cdots a_{n(3)} \end{aligned}$$

Proposition B.1. *Let A be a cosemisimple Hopf algebra and let V be a Yetter-Drinfeld module over A. The Gerstenhaber-Schack cohomology $H^*_{\rm GS}(A,V)$ is the cohomology of the complex*

$$0 \to \mathrm{Hom}^A(\mathbb{C}, V) \xrightarrow{\partial} \mathrm{Hom}^A(A^{\boxtimes 1}, V) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathrm{Hom}^A(A^{\boxtimes n}, V) \xrightarrow{\partial} \mathrm{Hom}^A(A^{\boxtimes n+1}, V) \xrightarrow{\partial} \cdots$$

where the differential $\partial : \mathrm{Hom}^A(A^{\boxtimes n}, V) \longrightarrow \mathrm{Hom}^A(A^{\boxtimes n+1}, V)$ is given by

$$\partial(f)(a_1 \otimes \cdots \otimes a_{n+1}) = \varepsilon(a_1) f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}$$

Proof. One checks that the standard resolution of the trivial module of Proposition 4.7 yields a resolution of the trivial Yetter-Drinfeld module by free (hence projective by cosemisimplicity) Yetter-Drinfled modules

$$\cdots \longrightarrow A^{\boxtimes n+1} \longrightarrow A^{\boxtimes n} \longrightarrow \cdots \longrightarrow A^{\boxtimes 2} \longrightarrow A^{\boxtimes 1} \xrightarrow{\varepsilon} \mathbb{C} \to 0$$

and one thus gets the above complex by applying the functor $\mathrm{Hom}_{\mathcal{YD}^A_A}(-, V)$ to this resolution. □

Notice that the previous complex is a subcomplex of the ones that computes $H^*(A, {}_\varepsilon V)$, hence this yields a map $H^*_{\rm GS}(A,V) \to H^*(A, {}_\varepsilon V)$ which is not injective in general, but is injective under the additional assumption that $S^2 = \mathrm{id}$ (see [10], this is another proof of Theorem 9.14 in this case).

References

[1] N. Andruskiewitsch, J. Cuadra, On the structure of (co-Frobenius) Hopf algebras, *J. Noncommut. Geom.* **7** (2013), no. 1, 83-104.

[2] N. Andruskiewitsch, J. Devoto, Extensions of Hopf algebras, *St. Petersburg Math. J.* **7** (1996), no. 1, 17-52.

[3] N. Andruskiewitsch, W. Ferrer Santos, The beginnings of the theory of Hopf algebras, *Acta Appl. Math.* **108** (2009), no. 1, 3-17.

[4] S. Arkhipov and D. Gaitsgory, Another realization of the category of modules over the small quantum group, *Adv. Math.* **173** (2003), no. 1, 114-143.

[5] T. Banica, Fusion rules for representations of compact quantum groups, *Exposition. Math.* **17** (1999), no. 4, 313-337.

[6] G. Baumslag, A finitely presented metabelian group with a free abelian derived group of infinite rank, *Proc. Amer. Math. Soc.* **35** (1972), 61-62.

[7] J. Bichon, The representation category of the quantum group of a non-degenerate bilinear form, *Comm. Algebra* **31**, No. 10 (2003), 4831-4851.

[8] J. Bichon, Hochschild homology of Hopf algebras and free Yetter-Drinfeld resolutions of the counit, *Compos. Math.* **149** (2013), no. 4, 658-678.

[9] J. Bichon, Hopf-Galois objects and cogroupoids, *Rev. Un. Mat. Argentina* **55** (2014), no. 2, 11-69.

[10] J. Bichon, Gerstenhaber-Schack and Hochschild cohomologies of Hopf algebras, *Doc. Math.* **21** (2016), 955-986.

[11] J. Bichon, Cohomological dimensions of universal cosovereign Hopf algebras, *Publ. Mat.*, to appear, arXiv:1611.02069.

[12] J. Bichon, U. Franz, M. Gerhold, Homological properties of quantum permutation algebras, Preprint arXiv:1704.00589.

[13] R. Bieri, Homological dimension of discrete groups, Queen Mary College Mathematics Notes, 1976.

[14] R. Bieri, B. Eckmann, Groups with homological duality generalizing Poincaré duality, *Invent. Math.* **20** (1973), 103-124.

[15] K.A. Brown, K.R. Goodearl, Homological aspects of Noetherian PI Hopf algebras of irreducible modules and maximal dimension, *J. Algebra* **198** (1997), no. 1, 240-265.

[16] K.A. Brown, K.R Goodearl, Lectures on algebraic quantum groups, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser Verlag, 2002.

[17] K.A. Brown, J.J. Zhang, Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras, *J. Algebra* **320** (2008), no. 5, 1814-1850.

[18] K.S. Brown, Cohomology of groups, Graduate Texts in Mathematics 87, Springer-Verlag, 1982.

[19] S. Burciu, S. Witherspoon, Hochschild cohomology of smash products and rank one Hopf algebras, Proceedings of the XVIth Latin American Algebra Colloquium (Spanish), 153-170, Bibl. Rev. Mat. Iberoamericana, Rev. Mat. Iberoamericana, Madrid, 2007.

[20] A. Chirvasitu, Cosemisimple Hopf algebras are faithfully flat over Hopf subalgebras, *Algebra Number Theory* **8** (2014), no. 5, 1179-1199.
[21] B. Collins, J. Härtel, A. Thom, Homology of free quantum groups. *C. R. Math. Acad. Sci. Paris* **347** (2009), 271-276.
[22] A. De Rijdt, N. Vander Vennet, Actions of monoidally equivalent compact quantum groups and applications to probabilistic boundaries, *Ann. Inst. Fourier* **60** (2010), no. 1, 169-216.
[23] W. Dicks, Mayer-Vietoris presentations over colimits of rings, *Proc. London Math. Soc.* **34** (1977), no. 3, 557-576.
[24] W. Dicks, Groups, trees and projective modules, *Lecture Notes in Mathematics* **790**, Springer, 1980.
[25] W. Dicks, M.J. Dunwoody, Groups acting on graphs. Cambridge Studies in Advanced Mathematics 17, Cambridge University Press, 1989.
[26] Y. Doi, Braided bialgebras and quadratic algebras, *Comm. Algebra* **21**, No.5 (1993), 1731-1749.
[27] V.G. Drinfeld, Quantum groups. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 798-820, Amer. Math. Soc., 1987.
[28] M. Dubois-Violette, G. Launer, The quantum group of a non-degenerate bilinear form, *Phys. Lett. B* **245**, No.2 (1990), 175-177.
[29] M.J. Dunwoody, Accessibility and groups of cohomological dimension one, *Proc. London Math. Soc.* **38** (1979), no. 2, 193-215.
[30] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, Tensor categories. Mathematical Surveys and Monographs 205, American Mathematical Society, Providence, RI, 2015.
[31] P. Fima, A. Freslon, Graphs of quantum groups and K-amenability, *Adv. Math.* **260** (2014), 233-280.
[32] M. Gerstenhaber, S. Schack, Bialgebra cohomology, deformations and quantum groups, *Proc. Nat. Acad. Sci. USA* **87** (1990), no. 1, 78-81.
[33] V. Ginzburg, Calabi-Yau algebras, Preprint arXiv:math/0612139.
[34] K.R. Goodearl, J.J. Zhang, Homological properties of quantized coordinate rings of semisimple groups, *Proc. Lond. Math. Soc.* **94** (2007), no. 3, 647-671.
[35] T. Hadfield, U. Krähmer, Twisted homology of quantum SL(2), *K-Theory* **34** (2005), no. 4, 327-360. Springer, 1963.
[36] G. Hochschild, B. Kostant, A. Rosenberg, Differential forms on regular affine algebras, *Trans. Amer. Math. Soc.* **102** (1962), 383-408.
[37] I. Kaplansky, Commutative rings. Allyn and Bacon. 1970.
[38] C. Kassel, Quantum groups, Graduate Texts in Mathematics 155, Springer, 1995.
[39] A. Klimyk, K. Schmüdgen, Quantum groups and their representations, Texts and Monographs in Physics, Springer, 1997.
[40] N. Kowalzig, , U. Krähmer, Duality and products in algebraic (co)homology theories, *J. Algebra* **323** (2010), no. 7, 2063-2081.
[41] U. Krähmer, Poincaré duality in Hochschild (co)homology. New techniques in Hopf algebras and graded ring theory, 117-125, K. Vlaam. Acad. Belgie Wet. Kunsten (KVAB), Brussels, 2007.
[42] S. Lang, Algebra. Revised third edition. Graduate Texts in Mathematics, 211. Springer-Verlag, New York, 2002.
[43] R.G. Larson, M.E. Sweedler, An associative orthogonal bilinear form for Hopf algebras, *Amer. J. Math.* **91** (1969), 75-94.
[44] J.L. Loday, Cyclic homology. Grundlehren der Mathematischen Wissenschaften 301. Springer-Verlag, Berlin, 1998.
[45] M.E. Lorenz, M. Lorenz, On crossed products of Hopf algebras, *Proc. Amer. Math. Soc.* **123** (1995), no. 1, 33-38.
[46] T. Masuda, Y. Nakagami, J. Watanabe, Noncommutative differential geometry on the quantum SU(2). I. An algebraic viewpoint. *K-Theory* **4** (1990), no. 2, 157-180.
[47] H. Matsumura, Commutative ring theory. Second edition. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, 1989.
[48] S. Montgomery, Hopf algebras and their actions on rings, Amer. Math. Soc. 1993.
[49] C. Mrozinski, Quantum automorphism groups and SO(3)-deformations, *J. Pure Appl. Algebra* **219** (2015), no. 1, 1-32.
[50] E. Müller, H.J. Schneider, Quantum homogeneous spaces with faithfully flat module structures, *Israel J. Math.* **111** (1999), 157-190.
[51] S. Neshveyev, L. Tuset, Compact quantum groups and their representation categories. Cours Spcialiss 20. Société Mathématique de France, Paris, 2013.
[52] M. Rosso, Koszul resolutions and quantum groups, *Nuclear Phys. B Proc. Suppl.* **18B** (1991), 269-276.
[53] P. Schauenburg, Faithful flatness over Hopf subalgebras: counterexamples, *Lecture Notes in Pure and Appl. Math.* **210** (2000), 331-344, Dekker, New York.
[54] P. Schauenburg, *Hopf bigalois extensions*, Comm. Algebra **24**, No. 12 (1996), 3797-3825.
[55] P. Schauenburg, *Hopf-Galois and bi-Galois extensions*, Fields Inst. Comm. **43** (2004), 469-515.

[56] H.J. Schneider, Normal basis and transitivity of crossed products for Hopf algebras, *J. Algebra* **152** (1992), no. 2, 289-312.
[57] H.-J. Schneider, Some remarks on exact sequences of quantum groups, *Comm. Algebra* **21** (1993), no. 9, 3337-3357.
[58] D. Stefan, Hochschild cohomology on Hopf-Galois extensions, *J. Pure Appl. Algebra* **103** (1995), no. 2, 221-233.
[59] R. Taillefer, Injective Hopf bimodules, cohomologies of infinite dimensional Hopf algebras and graded-commutativity of the Yoneda product, *J. Algebra* **276** (2004), no. 1, 259-279.
[60] M. Takeuchi, Relative Hopf modules-equivalences and freeness criteria, *J. Algebra* **60** (1979), 452-471.
[61] M. Takeuchi, Cocycle deformations of coordinate rings of quantum matrices, *J. Algebra* **189** (1997), no. 1, 23-33.
[62] Van Daele, A. The Haar measure on finite quantum groups. Proc. Amer. Math. Soc. 125 (1997), no. 12, 3489-3500.
[63] M. Van den Bergh, A relation between Hochschild homology and cohomology for Gorenstein rings, *Proc. Amer. Math. Soc.* **126** (1998) 1345-1348; Erratum: *Proc. Amer. Math. Soc.* **130** (2002) 2809-2810.
[64] C. Walton, X. Wang, On quantum groups associated to non-Noetherian regular algebras of dimension 2, *Math. Z.* **284** (2016), no. 1-2, 543-574
[65] S. Wang, Quantum symmetry groups of finite spaces, *Comm. Math. Phys.* **195** (1998), no. 1, 195-211.
[66] X. Wang, X. Yu, Y. Zhang, Calabi-Yau property under monoidal Morita-Takeuchi equivalence, *Pacific J. Math.* **290** (2017), no. 2, 481-510.
[67] W.C. Waterhouse, Introduction to affine group schemes. Graduate Texts in Mathematics, 66. Springer-Verlag, New York-Berlin, 1979.
[68] C. Weibel, An Introduction to Homological Algebra, Cambridge University Press, 1994.
[69] S.L. Woronowicz, Twisted SU(2) group. An example of a noncommutative differential calculus, *Publ. Res. Inst. Math. Sci.* **23** (1987), 117-181.
[70] X. Yu, Hopf-Galois objects of Calabi-Yau Hopf algebras, *J. Algebra Appl.* **15** (2016), no. 10, 1650194, 19 pp.

Laboratoire de Mathématiques Blaise Pascal, Université Clermont Auvergne, Complexe universitaire des Cézeaux, 3 place Vasarély 63178 Aubière Cedex, France
E-mail address: julien.bichon@uca.fr