Free Wreath Product by the Quantum Permutation Group

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Abstract

Let A be a compact quantum group, let $n \in \mathbb{N}^*$ and let $A_{aut}(X_n)$ be the quantum permutation group on n letters. A free wreath product construction $A *_{\mathrm{w}} A_{aut}(X_n)$ is done. This construction provides new examples of quantum groups, and is useful to describe the quantum automorphism group of the n-times disjoint union of a finite connected graph.

Keywords: Quantum permutation group, Wreath product, Free product, Graph automorphism.

1 Introduction

We discuss a quantum analogue of the following well-known construction. Let $n \in \mathbb{N}^*$ and let G be a subgroup of the permutation group S_n. Let H be an arbitrary group. Then G has a natural action on H^n by automorphisms, and so we may form the semi-direct product $H^n \rtimes G$, known as the wreath product of H by G and denoted $H\mathrm{w}G$. One of the most famous examples of such a construction is the hyperoctaedral group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n = \mathbb{Z}/2\mathbb{Z}\mathrm{w}S_n$, the isometry group of a hypercube in $\mathbb{R}^n$. Wreath products also occur naturally in the study of automorphism groups of finite graphs.

In this paper the classical permutation group S_n is replaced by the quantum permutation group $A_{aut}(X_n)$. This compact quantum group, introduced by S. Wang [11], is the universal compact quantum group acting on n points. It is quite different from the early examples of compact quantum groups constructed by S.L. Woronowicz [12, 14], which where related to the concept of "q-deformation" of a Lie group. The representation theory of the quantum permutation group has been worked out by T. Banica [2]: if $n \geq 4$, the fusion semi-ring of $A_{aut}(X_n)$ is identical to the one of $SO(3)$. The quantum permutation group is a fascinating object from many perspectives. For example we have a paradox from the noncommutative topology viewpoint. On one hand since the C^*-algebra $A_{aut}(X_n)$ is generated by projections, the corresponding quantum space should be totally disconnected. On the other hand since connectedness properties of a compact group may be read off from its fusion semi-ring [6, 5], Banica's classification of representations should imply that the quantum permutation group is connected!

In order to find natural families of quantum subgroups of the quantum permutation group, the quantum automorphism group of a finite graph was introduced in [3]. It turns out that one gets interesting quantum groups when considering the quantum automorphism group of the n-times disjoint union of a finite connected graph. Since the classical automorphism group of such a graph is described by a wreath product, it was natural to think that an analogue of the wreath product should be helpful in the quantum case.

Let A be a compact quantum group. The free wreath product of A by $A_{aut}(X_n)$, denoted $A *_{\mathrm{w}} A_{aut}(X_n)$, is a quotient of the free product C^*-algebra $A^{*n} * A_{aut}(X_n)$ (see Sections 2-3), and so is not a free product, but nor is a tensor product. This construction yields new examples of quantum groups. When $\mathcal{G}$ is a finite connected graph with quantum automorphism group denoted by $A_{aut}(\mathcal{G})$, we have a compact quantum group isomorphism $A_{aut}(\mathcal{G}^{\amalg n}) \cong A_{aut}(\mathcal{G}) *_{\mathrm{w}} A_{aut}(X_n)$, analogous to the classical isomorphism $\mathrm{Aut}(\mathcal{G}^{\amalg n}) \cong \mathrm{Aut}(\mathcal{G}) \mathrm{w} S_n$. In this way we get a simpler presentation for the algebra $A_{aut}(\mathcal{G}^{\amalg n})$.

Our work is organized as follows. Section 2 is devoted to the algebraic construction of the free wreath product. Some new examples of Hopf algebras are constructed, for which the corepresentation theory is studied in special cases. In Section 3, the free wreath product construction is done at the Woronowicz algebra level, using the results of the previous section. The free wreath product is used in section 4 to get a simple description of the quantum automorphism group of the n-times disjoint union of a finite connected graph.

We work over the field of complex numbers. We assume the reader to be familiar with Hopf algebras, Hopf $*$-algebras, CQG algebras and Woronowicz algebras (compact quantum groups). The relevant definitions may be found in the book [7].

2 The Hopf algebra construction

Let $n \in \mathbb{N}^*$ be a positive integer. The main character of this paper is Wang's quantum permutation group [11]. The corresponding Hopf algebra $\mathcal{A}_{aut}(X_n)$, denoted $\mathcal{A}_t(n)$ for simplicity, is the universal (complex) algebra with generators $(x_{ij})_{1 \leq i,j \leq n}$ and satisfying the relations:

$$x_{ij}x_{ik} = \delta_{jk}x_{ij} \quad ; \quad x_{ji}x_{ki} = \delta_{jk}x_{ji} \quad ; \quad \sum_{l=1}^{n} x_{il} = 1 = \sum_{l=1}^{n} x_{li} \quad ; \quad 1 \leq i,j,k \leq n.$$

It is immediate to check that $\mathcal{A}_t(n)$ is a Hopf $*$-algebra, with structure morphisms defined by:

$$x_{ij}^* = x_{ij} \quad ; \quad \Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \quad ; \quad \varepsilon(x_{ij}) = \delta_{ij} \quad ; \quad S(x_{ij}) = x_{ji} \quad ; \quad 1 \leq i,j \leq n.$$

The corepresentation $x = (x_{ij})$ is unitary, and hence it follows from [7], §11, Theorem 27, that $\mathcal{A}_t(n)$ is a CQG algebra.

Now let $\mathcal{A}$ be an arbitrary algebra. We may form the free product $\mathcal{A}^{*n}$, that is the n-times coproduct of $\mathcal{A}$ (in the category of unital algebras). Denote by $\nu_i : \mathcal{A} \longrightarrow \mathcal{A}^{*n}$, $1 \leq i \leq n$, the canonical algebras morphisms. If furthermore $\mathcal{A}$ is a $*$-algebra, then $\mathcal{A}^{*n}$

admits a $*$-algebra structure such that the $\nu_i' s$ are $*$-homomorphisms (there should be no confusion between the "$*$" of a $*$-algebra and the "$*$" of a free product). The first basic observation is that the classical action of the permutation group on $\mathcal{A}^{*n}$ extends to the quantum permutation group.

Proposition 2.1 *Let $\mathcal{A}$ be an algebra. Then the algebra $\mathcal{A}^{*n}$ is, in a natural way, a left $\mathcal{A}_t(n)$-comodule algebra. The coaction $\alpha : \mathcal{A}^{*n} \longrightarrow \mathcal{A}_t(n) \otimes \mathcal{A}^{*n}$ satisfies:*

$$\alpha(\nu_i(a)) = \sum_{k=1}^{n} x_{ik} \otimes \nu_k(a) \ , \quad 1 \leq i \leq n \ , \quad a \in \mathcal{A}.$$

If furthermore $\mathcal{A}$ is a $*$-algebra, the coaction α is a $*$-homomorphism.

Proof. Let us first define, for i with $1 \leq i \leq n$, a linear map $\alpha_i : \mathcal{A} \longrightarrow \mathcal{A}_t(n) \otimes \mathcal{A}^{*n}$, by $\alpha_i(a) = \sum_{k=1}^{n} x_{ik} \otimes \nu_k(a)$. Then $\alpha_i(1) = \sum_k x_{ik} \otimes 1 = 1 \otimes 1$, and

$$\alpha_i(a)\alpha_i(b) = \sum_{k,k'} x_{ik}x_{ik'} \otimes \nu_k(a)\nu_{k'}(b) = \sum_{k} x_{ik} \otimes \nu_k(ab) = \alpha_i(ab) \ , \quad \forall a, b \in \mathcal{A}.$$

Thus each α_i is an algebra morphism, and the universal property of the free product yields an algebra morphism $\alpha : \mathcal{A}^{*n} \longrightarrow \mathcal{A}_t(n) \otimes \mathcal{A}^{*n}$ satisfying the statement of the proposition. It is immediate that α is a coaction and that α is a $*$-homomorphism if $\mathcal{A}$ is a $*$-algebra. $\square$

We now assume $\mathcal{A}$ to be a Hopf algebra. Then the algebra $\mathcal{A}^{*n}$ inherits a natural Hopf algebra structure such that the canonical morphisms $\nu_i : \mathcal{A} \longrightarrow \mathcal{A}^{*n}$ are Hopf algebras morphisms. Using the coaction of Proposition 2.1, we may want to form the semi-direct product $\mathcal{A}^{*n} \rtimes \mathcal{A}_t(n)$, see for example subsection 10.2.6 in [7]. However the third commutativity condition of Definition 8 in [7] does not hold, unless $n \leq 3$ in which case $\mathcal{A}_t(n)$ is the function algebra on the symmetric group. This leads us to a notion of free wreath product.

Definition 2.2 *Let $n \in \mathbb{N}^*$ and let $\mathcal{A}$ be a Hopf algebra. The free wreath product of $\mathcal{A}$ by the quantum permutation group $\mathcal{A}_t(n)$ is the quotient of the algebra $\mathcal{A}^{*n} * \mathcal{A}_t(n)$ by the two-sided ideal generated by the elements:*

$$\nu_k(a)x_{ki} - x_{ki}\nu_k(a) \ , \quad 1 \leq i, k \leq n \ , \quad a \in \mathcal{A}.$$

The corresponding algebra is denoted by $\mathcal{A} *_{\mathrm{w}} \mathcal{A}_t(n)$.

We now state the main result of the section. We use Sweedler's notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$. The class of an element of $\mathcal{A}^{*n} * \mathcal{A}_t(n)$ is still denoted by the same symbol in $\mathcal{A} *_{\mathrm{w}} \mathcal{A}_t(n)$.

Theorem 2.3 *The free wreath product $\mathcal{A}*_{\mathrm{w}}\mathcal{A}_t(n)$ admits a Hopf algebra structure. Let $a \in \mathcal{A}$ and let $i, j \in \{1 \ldots n\}$. The comultiplication Δ satisfies:*

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \quad ; \; \Delta(\nu_i(a)) = \sum_{k=1}^{n} \nu_i \otimes \nu_k(\Delta_{\mathcal{A}}(a))(x_{ik} \otimes 1) = \sum_{k=1}^{n} \nu_i(a_{(1)})x_{ik} \otimes \nu_k(a_{(2)}).$$

The counit ε satisfies $\varepsilon(x_{ij}) = \delta_{ij}$ and $\varepsilon(\nu_i(a)) = \varepsilon_{\mathcal{A}}(a)$. The antipode S satisfies:

$$S(x_{ij}) = x_{ji} \quad ; \; S(\nu_i(a)) = \sum_{k=1}^{n} \nu_k(S_{\mathcal{A}}(a))x_{ki}.$$

If $\mathcal{A}$ is a Hopf $*$-algebra, then so is $\mathcal{A}*_{\mathrm{w}}\mathcal{A}_t(n)$ with $x_{ij}^* = x_{ij}$ and $\nu_i(a)^* = \nu_i(a^*)$. If $\mathcal{A}$ is a CQG algebra, then $\mathcal{A}*_{\mathrm{w}}\mathcal{A}_t(n)$ is also a CQG algebra.

Proof. For simplicity we put $\mathcal{H} = \mathcal{A}*_{\mathrm{w}}\mathcal{A}_t(n)$. For $1 \le i \le n$, define linear maps $\delta_i : \mathcal{A} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ by $\delta_i(a) = \sum_{k=1}^{n} \nu_i(a_{(1)})x_{ik} \otimes \nu_k(a_{(2)})$. It is easy to show that δ_i is an algebra map (see the calculation in the proof of Proposition 2.1). Define now an algebra morphism $\delta_{n+1} : \mathcal{A}_t(n) \longrightarrow \mathcal{H} \otimes \mathcal{H}$ by $\delta_{n+1}(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$. The universal property of the free product yields an algebra morphism $\Delta_0 : \mathcal{A}^{*n} * \mathcal{A}_t(n) \longrightarrow \mathcal{H} \otimes \mathcal{H}$ such that $\Delta_0 \circ \nu_i = \delta_i$, $1 \le i \le n+1$. Then

$$\begin{aligned}\Delta_0(\nu_k(a)x_{ki}) &= \left(\sum_{l=1}^{n} \nu_k(a_{(1)})x_{kl} \otimes \nu_l(a_{(2)})\right)\left(\sum_{r=1}^{n} x_{kr} \otimes x_{ri}\right)\\ &= \sum_{l=1}^{n} \nu_k(a_{(1)})x_{kl} \otimes \nu_l(a_{(2)})x_{li} = \sum_{l=1}^{n} x_{kl}\nu_k(a_{(1)}) \otimes x_{li}\nu_l(a_{(2)})\\ &= \Delta_0(x_{ki}\nu_k(a)).\end{aligned}$$

Hence Δ_0 induces an algebra morphism $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ satisfying the required identity. One has

$$\begin{aligned}(\Delta \otimes \mathrm{id}) \circ \Delta(\nu_i(a)) &= \sum_{k=1}^{n} \Delta(\nu_i(a_{(1)}))\Delta(x_{ik}) \otimes \nu_k(a_{(2)})\\ &= \sum_{k,l,r} \nu_i(a_{(1)})x_{il}x_{ir} \otimes \nu_l(a_{(2)})x_{rk} \otimes \nu_k(a_{(3)})\\ &= \sum_{k,l} \nu_i(a_{(1)})x_{il} \otimes \nu_l(a_{(2)})x_{lk} \otimes \nu_k(a_{(3)}) = (\mathrm{id} \otimes \Delta) \circ \Delta(\nu_i(a)).\end{aligned}$$

Thus Δ is coassociative since the elements $\nu_i(a)$ and x_{ik} generate $\mathcal{H}$ as an algebra. The construction of the counit is left to the reader. Let us now define the antipode of $\mathcal{H}$. For $1 \le i \le n$ let $S_i : \mathcal{A} \longrightarrow \mathcal{H}$ be defined by $S_i(a) = \sum_{k=1}^{n} \nu_k(S_{\mathcal{A}}(a))x_{ki}$. One has $S_i(1) = 1$ and

$$\begin{aligned}S_i(a)S_i(b) &= \sum_{k,l} \nu_k(S_{\mathcal{A}}(a))x_{ki}\nu_l(S_{\mathcal{A}}(b))x_{li} = \sum_{k,l} \nu_k(S_{\mathcal{A}}(a))x_{ki}x_{li}\nu_l(S_{\mathcal{A}}(b))\\ &= \sum_{k} \nu_k(S_{\mathcal{A}}(a))x_{ki}\nu_k(S_{\mathcal{A}}(b)) = \sum_{k} \nu_k(S_{\mathcal{A}}(ba))x_{ki} = S_i(ba).\end{aligned}$$

Thus $S_i : \mathcal{A} \longrightarrow \mathcal{H}^{\mathrm{op}}$ is an algebra morphism. Now let $S_{n+1} : \mathcal{A}_t(n) \longrightarrow \mathcal{H}^{\mathrm{op}}$ be the algebra morphism defined by $S(x_{ij}) = x_{ji}$. The universal property of the free product yields an algebra morphism $S_0 : \mathcal{A}^{*n} * \mathcal{A}_t(n) \longrightarrow \mathcal{H}^{\mathrm{op}}$ such that $S_0 \circ \nu_i = S_i$, $1 \leq i \leq n+1$. We have

$$S_0(\nu_k(a)x_{ki}) = x_{ik}\sum_{l=1}^{n}\nu_l(S_{\mathcal{A}}(a))x_{lk} = \sum_{l=1}^{n}x_{ik}x_{lk}\nu_l(S_{\mathcal{A}}(a)) = x_{ik}\nu_i(S_{\mathcal{A}}(a)) =$$
$$= \sum_{l=1}^{n}\nu_l(S_{\mathcal{A}}(a))x_{lk}x_{ik} = S_0(\nu_k(a))S_0(x_{ki}) = S_0(x_{ki}\nu_k(a)).$$

Thus S_0 induces an algebra morphism $S : \mathcal{H} \longrightarrow \mathcal{H}^{\mathrm{op}}$. One has

$$m \circ (S \otimes \mathrm{id}) \circ \Delta(\nu_i(a)) = \sum_{k,l} x_{ki}\nu_l(S(a_{(2)})x_{li}\nu_k(a_{(2)}) = \sum_{k} x_{ki}\nu_k(S_{\mathcal{A}}(a_{(1)})a_{(2)})$$
$$= \varepsilon_{\mathcal{A}}(a)1 = \varepsilon(\nu_i(a))1,$$

and since the elements $\nu_i(a)$ and x_{ij} generate $\mathcal{H}$ as an algebra, it follows that $m \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \varepsilon$ (u denotes the unit of $\mathcal{H}$). A similar computation shows that $m \circ (\mathrm{id} \otimes S) \circ \Delta = u \circ \varepsilon$: we conclude that $\mathcal{H} = \mathcal{A} *_{\mathrm{w}} \mathcal{A}_t(n)$ is a Hopf algebra.

It is easy to check that if $\mathcal{A}$ is a Hopf $*$-algebra, then so is $\mathcal{H}$ with the $*$-structure announced in the theorem. Now let us assume that $\mathcal{A}$ is a CQG algebra. This means that there exists a family $(u^\lambda)_{\lambda \in \Lambda}$ of unitary corepresentations of $\mathcal{A}$ whose matrix coefficients generate $\mathcal{A}$ as an algebra. Let $u^\lambda = (u^\lambda_{kl})_{1 \leq k,l \leq d_\lambda}$ be such a corepresentation. For $1 \leq i,j \leq n$, $1 \leq k,l \leq d_\lambda$, put $v^\lambda(i,k,j,l) = \nu_i(u^\lambda_{kl})x_{ij}$. Then

$$\Delta(v^\lambda(i,k,j,l)) = \left(\sum_{r=1}^{n}\sum_{s=1}^{d_\lambda}\nu_i(u^\lambda_{ks})x_{ir} \otimes \nu_r(u^\lambda_{sl})\right)\left(\sum_{t=1}^{n} x_{it} \otimes x_{tj}\right) =$$
$$= \sum_{r=1}^{n}\sum_{s=1}^{d_\lambda}\nu_i(u^\lambda_{ks})x_{ir} \otimes \nu_r(u^\lambda_{sl})x_{rj} = \sum_{r,s} v^\lambda(i,k,r,s) \otimes v^\lambda(r,s,j,l)\ ;$$
$$\varepsilon(v^\lambda(i,k,j,l)) = \delta_{ij}\delta_{kl}\ ;$$
$$\sum_{r,s} v^\lambda(i,k,r,s)v^\lambda(j,l,r,s)^* = \sum_{r,s}\nu_i(u^\lambda_{ks})x_{ir}x_{jr}\nu_j(u^{\lambda^*}_{ls}) = \delta_{ij}\sum_{s=1}^{d_\lambda}\nu_i(u^\lambda_{ks}u^{\lambda^*}_{ls}) = \delta_{ij}\delta_{kl}\ ;$$
$$\sum_{r,s} v^\lambda(r,s,i,k)^* v^\lambda(r,s,j,l) = \sum_{r,s} x_{ri}\nu_r(u^{\lambda^*}_{sk})\nu_r(u^\lambda_{sl})x_{rj} = \delta_{kl}\sum_{r} x_{ri}x_{rj} = \delta_{kl}\delta_{ij}.$$

These computations mean that $v^\lambda = (v^\lambda(i,k,j,l))$ is a unitary corepresentation of $\mathcal{H}$. Let $\mathcal{B}$ be the subalgebra of $\mathcal{H}$ by the coefficients of the unitary corepresentations v^λ, $\lambda \in \Lambda$, and $x = (x_{ij})$. Then $\sum_{j=1}^{n} v^\lambda(i,k,j,l) = \nu_i(u^\lambda_{kl})\sum_j x_{ij} = \nu_i(u^\lambda_{kl})$. Thus $\mathcal{B}$ contains the elements $\nu_i(u^\lambda_{kl})$, $1 \leq i \leq n$, $\lambda \in \Lambda$, $1 \leq k,l \leq d_\lambda$ which generate the image of $\mathcal{A}^{*n}$ in $\mathcal{H}$ as an algebra: it follows immediately that $\mathcal{B} = \mathcal{H}$ and hence $\mathcal{H}$ is a CQG algebra. $\square$

Remark 2.4 *It is clear that we may also define the free wreath product by any quantum subgroup of the quantum permutation group, that is a homomorphic quotient of $\mathcal{A}_t(n)$, and that one still has an obvious analogue of Theorem 2.3*

We now examine an example where $\mathcal{A}$ is a group algebra.

Example 2.5 *Let G be a (discrete) group and let $n \in \mathbb{N}^*$. Let $\mathcal{A}_n(G)$ be the universal algebra with generators $a_{ij}(g)$, $1 \le i,j \le n$, $g \in G$, and submitted to the relations ($1 \le i,j,k \le n$; $g,h \in G$):*

$$a_{ij}(g)a_{ik}(h) = \delta_{jk}a_{ij}(gh) \quad ; \quad a_{ji}(g)a_{ki}(h) = \delta_{jk}a_{ji}(gh) \quad ; \quad \sum_{l=1}^{n} a_{il}(1) = 1 = \sum_{l=1}^{n} a_{li}(1).$$

Then $\mathcal{A}_n(G)$ is a CQG algebra, with:

$$a_{ij}(g)^* = a_{ij}(g^{-1}) \ ; \ \Delta(a_{ij}(g)) = \sum_{k=1}^{n} a_{ik}(g) \otimes a_{kj}(g) \ ; \ \varepsilon(a_{ij}(g)) = \delta_{ij} \ ; \ S(a_{ij}(g)) = a_{ji}(g^{-1}).$$

Furthermore $\mathcal{A}_n(G)$ is isomorphic with $\mathbb{C}[G] *_{\mathrm{w}} \mathcal{A}_t(n)$.

Proof. It can be checked directly that $\mathcal{A}_n(G)$ is a CQG algebra. Another way to proceed is to check that the algebras $\mathcal{A}_n(G)$ and $\mathbb{C}[G] *_{\mathrm{w}} \mathcal{A}_t(n)$ are isomorphic, and to transport the Hopf $*$-algebra structure of $\mathbb{C}[G] *_{\mathrm{w}} \mathcal{A}_t(n)$ on $\mathcal{A}_n(G)$. We choose this second possibility.

If $g \in G$, the corresponding element in $\mathbb{C}[G]$ is still denoted by g. For $1 \le i \le n$, let $\phi_i : \mathbb{C}[G] \longrightarrow \mathcal{A}_n(G)$ be the linear map defined by $\phi_i(g) = \sum_{k=1}^{n} a_{ik}(g)$ for $g \in G$. Then $\phi_i(1) = 1$ and

$$\phi_i(g)\phi_i(h) = \sum_{k,l} a_{ik}(g)a_{il}(h) = \sum_{k} a_{ik}(gh) = \phi_i(gh) \ , \ \forall g,h \in G.$$

Hence ϕ_i is an algebra morphism. Let $\phi_{n+1} : \mathcal{A}_t(n) \longrightarrow \mathcal{A}_n(G)$ be the algebra morphism defined by $\phi_{n+1}(x_{ij}) = a_{ij}(1)$. The algebra morphisms $\phi_1, \dots, \phi_{n+1}$ induce an algebra morphism $\phi_0 : \mathbb{C}[G]^{*n} * \mathcal{A}_t(n) \longrightarrow \mathcal{A}_n(G)$. One has

$$\phi_0(\nu_i(g)x_{ij}) = \sum_{k} a_{ik}(g)a_{ij}(1) = a_{ij}(g) = \sum_{k} a_{ij}(1)a_{ik}(g) = \phi_0(x_{ij}\nu_i(g)).$$

Hence ϕ_0 induces an algebra morphism $\phi : \mathbb{C}[G] *_{\mathrm{w}} \mathcal{A}_t(n) \longrightarrow \mathcal{A}_n(G)$. It is then easy to construct an algebra morphism $\psi : \mathcal{A}_n(G) \longrightarrow \mathbb{C}[G] *_{\mathrm{w}} \mathcal{A}_t(n)$ such that $\psi(a_{ij}(g)) = \nu_i(g)x_{ij}$, and it is straightforward to check that ϕ and ψ are mutually inverse isomorphisms. It is also easy to see that the transported Hopf $*$-algebra structure on $\mathcal{A}_n(G)$ is the one announced. □

When G is a finitely generated group, it is clear that $\mathcal{A}_n(G)$ is a finitely generated algebra and hence is a CMQG algebra. The presentation can be improved if G is a cyclic group.

• If $G = \mathbb{Z}$ is the infinite cyclic group, then $\mathcal{A}_n(\mathbb{Z})$ is isomorphic with the universal $*$-algebra generated by elements a_{ij}, $1 \leq i, j \leq n$, and submitted to the relations:

$$a_{ij}^* a_{ij} = a_{ij} a_{ij}^* \ ; \ \sum_l^n a_{li}^* a_{li} = 1 = \sum_l^n a_{il}^* a_{il} \ ; \ a_{ij} a_{ik} = 0 = a_{ij} a_{ik}^* = a_{ij}^* a_{ik} = a_{ij}^* a_{ik}^* \ ;$$
$$a_{ji} a_{ki} = 0 = a_{ji} a_{ki}^* = a_{ji}^* a_{ki} = a_{ji}^* a_{ki}^* \ , 1 \leq i, j, k \leq n, \ j \neq k.$$

The antipode is given by $S(a_{ij}) = a_{ji}^*$.

• If $G = \mathbb{Z}/p\mathbb{Z}$, then $\mathcal{A}_n(\mathbb{Z}/p\mathbb{Z})$ is isomorphic with the universal algebra generated by elements a_{ij}, $1 \leq i, j \leq n$, and submitted to the relations:

$$\sum_{l=1}^n a_{il}^p = 1 = \sum_{l=1}^n a_{li}^p \ ; \ a_{ij} a_{ik} = 0 = a_{ji} a_{ki} \ , \ 1 \leq i, j, k \leq n, \ j \neq k.$$

The antipode satisfies $S(a_{ij}) = a_{ji}^{p-1}$ and we have $a_{ij}^* = a_{ij}^{p-1}$.

We now examine the easiest case, namely $n = 2$. It is immediate that the elements of $\mathcal{A}^{*2}$ commute with those of $\mathcal{A}_t(2) = C(\mathbb{Z}/2\mathbb{Z})$ in $\mathcal{A} *_{\mathrm{w}} \mathcal{A}_t(2)$ and thus as an algebra, we have $\mathcal{A} *_{\mathrm{w}} \mathcal{A}_t(2) = \mathcal{A}^{*2} \otimes C(\mathbb{Z}/2\mathbb{Z})$. The Hopf algebra structure of Theorem 2.3 is just the one of a classical semi-direct product. If h_1 denotes the Haar measure on $\mathcal{A}^{*2}$ (the Haar measure on a free product is described in [10] as the free product of the Haar measures) and h_2 denotes the Haar measure on $C(\mathbb{Z}/2\mathbb{Z})$, it is easy to see that $h = h_1 \otimes h_2$ is the Haar measure on $\mathcal{A} *_{\mathrm{w}} \mathcal{A}_t(2)$.

Let us describe the corepresentation theory of $\mathcal{A}_2(G)$ for any group G. First let us introduce some notation. We consider the group free product $G * G$, with the canonical morphisms still denoted $\nu_1, \nu_2 : G \longrightarrow G * G$. The canonical involutive group automorphism of $G * G$ is denoted by τ, with $\tau \circ \nu_1 = \nu_2$ and $\tau \circ \nu_2 = \nu_1$. An element $x \in G * G$ is the unit element if and only if $\tau(x) = x$.

Proposition 2.6 *Let G be a non-trivial group.*
1) To any element $x \in G * G \setminus \{1\}$ corresponds a two-dimensional irreducible corepresentation v_x of $\mathcal{A}_2(G)$. Two such corepresentations v_x and v_y are isomorphic if and only if $x = y$ or $x = \tau(y)$. There also exists a non-trivial one-dimensional corepresentation d.
2) Any non-trivial irreducible corepresentation of $\mathcal{A}_2(G)$ is isomorphic to one of the corepresentations listed above.
3) One has the following fusion rules ($x, y \in G * G \setminus \{1\}$):

$$v_x \otimes v_y \cong v_{xy} \oplus v_{x\tau(y)} \ \text{if} \ x \neq y^{-1} \ \text{and} \ x \neq \tau(y)^{-1} \ ;$$

$$v_x \otimes v_{x^{-1}} \cong \mathbb{C} \oplus d \oplus v_{x\tau(x)^{-1}} \ ; \ d \otimes d \cong \mathbb{C} \ ; \ v_x \otimes d \cong d \otimes v_x \cong v_x.$$

Proof. We use the algebra identification $\mathcal{A}_2(G) \cong \mathbb{C}[G * G] \otimes C(\mathbb{Z}/2\mathbb{Z})$. The basic tool of the proof is Woronowicz' character theory [13], which we freely use. Let $x \in G * G$, $x \neq 1$ and put

$$A_{11}(x) = x x_{11} \ , \ A_{12}(x) = x x_{12} \ , \ A_{21}(x) = \tau(x) x_{21} \ , \ A_{22}(x) = \tau(x) x_{22}.$$

A straightforward computation shows that $\Delta(A_{ij}(x)) = \sum_k A_{ik}(x) \otimes A_{kj}(x)$ and that $\varepsilon(A_{ij}(x)) = \delta_{ij}$, $1 \leq i,j \leq 2$ (recall that $x_{11} = x_{22}$ and $x_{12} = x_{21}$ in $C(\mathbb{Z}/2\mathbb{Z})$). Let $v_x = (A_{ij}(x))$ be the corresponding matrix corepresentation, with associated character $\chi_x = (x + \tau(x))x_{11}$. Let h be the Haar measure on $\mathcal{A}_2(G)$: $h = h_1 \otimes h_2$, see the notation and remark above. Then

$$h(\chi_x\chi_x^*) = h((x+\tau(x))(x^{-1}+\tau(x^{-1}))x_{11}) = h((2+x\tau(x^{-1})+\tau(x)x^{-1})x_{11}) = 2h(x_{11}) = 1.$$

Hence the corepresentation v_x is irreducible. We have $\chi_{\tau(x)} = (\tau(x) + \tau^2(x))x_{11} = (\tau(x) + x)x_{11} = \chi_x$, hence $v_x \cong v_{\tau(x)}$ Conversely, let $y \in G * G$ be such that the corepresentations v_x and v_y are isomorphic. Then $\chi_x = \chi_y$, i.e. $(x + \tau(x))x_{11} = (y + \tau(y))x_{11}$. This implies that $x + \tau(x) = y + \tau(y)$ in $\mathbb{C}[G * G]$, i.e. $x = y$ or $x = \tau(y)$. Let $d = x_{11} - x_{12}$. Then $\Delta(d) = d \otimes d$, we have a non-trivial one-dimensional corepresentation d with $d \otimes d \cong \mathbb{C}$ since $d^2 = 1$.

Let us now prove part 3). Let $x, y \in G * G \backslash \{1\}$ with $x \neq y^{-1}$ and $x \neq \tau(y)^{-1}$. Then

$$h(\chi_x\chi_y\chi_{xy}^*) = h((x + \tau(x))(y + \tau(y))(y^{-1}x^{-1} + \tau(y)^{-1}\tau(x)^{-1})x_{11}) = 1.$$

(The assumption $x \neq y^{-1}$ and $x \neq \tau(y)^{-1}$ has been used). Thus v_{xy} appears once in the decomposition into irreducibles of $v_x \otimes v_y$. In the same way $h(\chi_x\chi_y\chi_{x\tau(y)}^*) = 1$, and $v_{x\tau(y)}$ appears once in the decomposition of $v_x \otimes v_y$. The corepresentations v_{xy} and $v_{x\tau(y)}$ are not isomorphic if $x, y \neq 1$, and so for obvious dimension reasons we must have

$$v_x \otimes v_y \cong v_{xy} \oplus v_{x\tau(y)}.$$

Now $h(\chi_x\chi_{x^{-1}}) = h((2 + x\tau(x)^{-1} + \tau(x)x^{-1})x_{11}) = 1$: the trivial corepresentation $\mathbb{C}$ appears once in $v_x \otimes v_{x^{-1}}$. Also $h(\chi_x\chi_{x^{-1}}d^*) = h(\chi_x\chi_{x^{-1}}) = 1$: the corepresentation d appears once in $v_x \otimes v_{x^{-1}}$. Then a straightforward computation shows that $h(\chi_x\chi_{x^{-1}}\chi_{x\tau(x)^{-1}}^*) = 1$, and hence the corepresentation $v_{x\tau(x)^{-1}}$ appears once in $v_x \otimes v_{x^{-1}}$. So we have:

$$v_x \otimes v_{x^{-1}} \cong \mathbb{C} \oplus d \oplus v_{x\tau(x)^{-1}}.$$

Finally we have $h(\chi_x\chi_x^*d^*) = h(\chi_x d^*\chi_x^*) = 1$. This shows that $v_x \otimes d \cong v_x \cong d \otimes v_x$.

The family of irreducible corepresentations $\Lambda = \{\mathbb{C}, d, (v_x)_{x \in G*G\backslash\{1\}}\}$ contains the trivial representation, is stable under tensor product and under conjugation, and its coefficients generate linearly $\mathcal{A}_2(G)$. It follows from the orthogonality relations [13] that any irreducible corepresentation of $\mathcal{A}_2(G)$ is isomorphic with one of this family. □

We can see now if the fusion semi-ring of $\mathcal{A}_2(G)$ is identical to the one of a classical group.

Corollary 2.7 *Let G be a non-trivial group. Then the fusion semi-ring of $\mathcal{A}_2(G)$ is commutative if and only if $G \cong \mathbb{Z}/2\mathbb{Z}$. In this case the fusion semi-ring is the same as the one of the orthogonal group $O(2)$.*

Proof. Let us assume that the fusion semi-ring is commutative. Then for all $x, y \in G * G \backslash \{1\}$, we must have $v_x \otimes v_{x^{-1}} \cong v_{x^{-1}} \otimes v_x$. This implies that $v_{x\tau(x)^{-1}} \cong v_{x^{-1}\tau(x)}$, and hence $x\tau(x)^{-1} = x^{-1}\tau(x)$ or $x\tau(x)^{-1} = \tau(x)^{-1}x$. Thus for any element $x \in G * G\backslash\{1\}$, we have the alternative $x^2 = 1$ or $x\tau(x) = \tau(x)x$. Let $g \in G$ with $g \neq 1$. We cannot have $\nu_1(g)\nu_2(g) = \nu_2(g)\nu_1(g)$, so $\nu_1(g^2) = 1$, which implies $g^2 = 1$. If $h \in G$ and $h \neq 1$, then $((\nu_1(g)\nu_2(h))^2 \neq 1$, so $\nu_1(g)\nu_2(h)\nu_2(g)\nu_1(h) = \nu_2(g)\nu_1(h)\nu_1(g)\nu_2(h)$. This implies that $gh = 1$, so $h = g^{-1} = g$. Hence G is the cyclic group of order two.

Conversely, assume that $G = \langle g | g^2 = 1 \rangle$. Let $l : G * G \longrightarrow \mathbb{N}$ be the function which assigns to an element of $G * G$ (written in reduced form) its length. It is immediate to check that $l(x) = l(y)$ if and only if $x = y$ or $x = \tau(y)$. Thus by Proposition 2.6 every irreducible 2-dimensional irreducible corepresentation of $\mathcal{A}_2(G)$ can be labelled as v_i for some $i \in \mathbb{N}^*$. Put $v_0 = \mathbb{C}$. The fusion rules of $\mathcal{A}_2(G)$ now read:

$$v_i \otimes v_j \cong v_{i+j} \oplus v_{|i-j|} \; if \; i \neq j \in \mathbb{N}^* \; ;$$

$$v_i \otimes v_i \cong v_0 \oplus d \oplus v_{2i} \; ; \; d \otimes d \cong v_0 \; ; \; v_i \otimes d \cong d \otimes v_i \cong v_i, \; i \in \mathbb{N}^*.$$

We recognize the fusion rules of the orthogonal group $O(2)$. □

Remark 2.8 *It can even be shown that the category of corepresentations of $\mathcal{A}_2(\mathbb{Z}/2\mathbb{Z})$ and the category of representations of $O(2)$ are monoidally equivalent. In particular the Hopf algebra $\mathcal{A}_2(\mathbb{Z}/2\mathbb{Z})$ is cotriangular.*

3 The compact quantum group construction

We first recall some basic notions. As usual $\otimes$ stands for the minimal C^*-tensor product. There should be no confusion with the algebraic tensor product.

A compact quantum group [13, 15], or Woronowicz algebra, is a unital C^*-algebra A together with a C^*-homomorphism $\Delta : A \longrightarrow A \otimes A$ and a family of unitary matrices $(u^\lambda)_{\lambda\in\Lambda}$, with $u_\lambda \in M_{d_\lambda}(A)$, such that:
(1) The subalgebra $\mathcal{A}$ generated by the entries (u^λ_{ij}) of the matrices $(u^\lambda)_{\lambda\in\Lambda}$ is dense in A.
(2) For $\lambda \in \Lambda$ and $i, j \in \{1, \ldots, d_\lambda\}$, one has $\Delta(u^\lambda_{ij}) = \sum_{k=1}^{d_\lambda} u^\lambda_{ik} \otimes u^\lambda_{kj}$.
(3) For $\lambda \in \Lambda$, the transpose matrix ${}^t u^\lambda$ is invertible.

It follows from [13, 16] that the dense subalgebra $\mathcal{A}$ is uniquely determined, and is a CQG algebra. A Woronowicz algebra is said to be full if A is the enveloping C^*-algebra of $\mathcal{A}$. All the Woronowicz algebras solving universal problems (see e.g. [8, 11, 3]) are full, and more generally the Woronowicz algebras constructed using Woronowicz' Tannaka-Krein duality [14] are full.

Wang's quantum permutation group, denoted $A_{aut}(X_n)$, is the enveloping C^*-algebra of the Hopf $*$-algebra $\mathcal{A}_t(n)$ of the previous section, and is thus a full Woronowicz algebra. The generators of $A_{aut}(X_n)$ are still denoted x_{ij}, $1 \leq i, j \leq n$.

Let A be a C^*-algebra. We consider the free product C^*-algebra (see [1, 9]) A^{*n}, that is the n-times coproduct of A as a C^*-algebra. We still denote by $\nu_i : \mathcal{A} \longrightarrow \mathcal{A}^{*n}$, $1 \leq i \leq n$, the canonical $*$-homomorphisms. Recall from [10] that a free product of Woronowicz algebras is in a natural way, a Woronowicz algebra. The following definition is the C^*-algebra analogue of Definition 2.2.

Definition 3.1 *Let $n \in \mathbb{N}^*$ and let A be a Woronowicz algebra. The free wreath product of A by the quantum permutation group $A_{aut}(X_n)$ is the quotient of the C^*-algebra $A^{*n} * A_{aut}(X_n)$ by the two-sided ideal generated by the elements:*

$$\nu_k(a)x_{ki} - x_{ki}\nu_k(a) \ , \quad 1 \leq i, k \leq n \ , \quad a \in A.$$

The corresponding C^*-algebra is denoted by $A *_{\mathrm{w}} A_{aut}(X_n)$.

Theorem 3.2 *The free wreath product $A *_{\mathrm{w}} A_{aut}(X_n)$ admits a Woronowicz algebra structure. Let $a \in A$ and let $i, j \in \{1 \ldots n\}$. The coproduct Δ satisfies:*

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \quad ; \ \Delta(\nu_i(a)) = \sum_{k=1}^{n} \nu_i \otimes \nu_k(\Delta(a))(x_{ik} \otimes 1).$$

If A is a full Woronowicz algebra, then so is $A *_{\mathrm{w}} A_{aut}(X_n)$.

Proof. We put $H = A *_{\mathrm{w}} A_{aut}(X_n)$. Let $\mathcal{A}$ and $\mathcal{A}_t(n)$ be the dense CQG algebras of A and $A_{aut}(X_n)$ respectively. For $1 \leq i \leq n$, define continuous linear maps $\delta_i : A \longrightarrow H \otimes H$ by $\delta_i(a) = \sum_{k=1}^{n} \nu_i \otimes \nu_k(\Delta(a))(x_{ik} \otimes 1)$. Then $\delta_{i|\mathcal{A}}$ is a $*$-homomorphism (see the proof of Theorem 2.3), and since $\mathcal{A}$ is dense in A, It is clear that δ_i is a $*$-homomorphism. Define now a $*$-homomorphism $\delta_{n+1} : A_{aut}(X_n) \longrightarrow H \otimes H$ by $\delta_{n+1}(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$. The universal property of the C^*-algebra free product yields a $*$-homomorphism $\Delta_0 : A^{*n} * A_{aut}(X_n) \longrightarrow H \otimes H$ such that $\Delta_0 \circ \nu_i = \delta_i$, $1 \leq i \leq n+1$. We know from the proof of Theorem 2.3 that Δ_0 vanishes on the elements $\nu_k(a)x_{ki} - x_{ki}\nu_k(a)$, $1 \leq i, k \leq n$, $a \in \mathcal{A}$, and since $\mathcal{A}$ is dense in A and Δ_0 is continuous, we get a $*$-homomorphism $\Delta : H \longrightarrow H \otimes H$: this is the coproduct announced in the theorem. Consider the canonical $*$-algebra map $\iota : \mathcal{H} = \mathcal{A} *_{\mathrm{w}} \mathcal{A}_t(n) \longrightarrow A *_{\mathrm{w}} A_{aut}(X_n) = H$. It is clear that $\iota(\mathcal{H})$ is dense in H and that ι commutes with the respective coproducts. Furthermore $\mathcal{H}$ is a CQG algebra by theorem 2.3, and hence it immediate that $H = A *_{\mathrm{w}} A_{aut}(X_n)$ is a Woronowicz algebra.

Let us now assume that A is a full Woronowicz algebra. Let us show that if B is a C^*-algebra and $\pi : \mathcal{H} = \mathcal{A} *_{\mathrm{w}} \mathcal{A}_t(n) \longrightarrow B$ is $*$-algebra morphism, then there exists a (unique) $*$-homomorphism $\tilde{\pi} : H = A *_{\mathrm{w}} A_{aut}(X_n) \longrightarrow B$ such that $\tilde{\pi} \circ \iota = \pi$. This will prove that H is the enveloping C^*-algebra of $\iota(\mathcal{H})$ and thus that $H = A *_{\mathrm{w}} A_{aut}(X_n)$ is a full Woronowicz algebra. For $1 \leq i \leq n$, let $\pi_i : \mathcal{A} \longrightarrow B$ be the $*$-algebra morphism defined by the composition

$$\pi_i : \mathcal{A} \xrightarrow{\ \nu_i\ } \mathcal{A}^{*n} \longrightarrow \mathcal{A} *_{\mathrm{w}} \mathcal{A}_t(n) \xrightarrow{\ \pi\ } B.$$

Since A is full, there exists a unique $*$-homomorphism $\hat{\pi}_i : A \longrightarrow B$ such that $\hat{\pi}_{i|\mathcal{A}} = \pi_i$. Let $\hat{\pi}_{n+1}$ be the $*$-homomorphism $A_{aut}(X_n) \longrightarrow B$ such that $\hat{\pi}_{n+1|\mathcal{A}_t(n)} = \pi_{|\mathcal{A}_t(n)}$. The universal property of the free product yields a $*$-homomorphism $\hat{\pi} : A^{*n} * A_{aut}(X_n) \longrightarrow B$. It then immediate from the density of $\mathcal{A}^{*n} * \mathcal{A}_t(n)$ in $A^{*n} * \mathcal{A}_{aut}(X_n)$ that $\hat{\pi}$ induces a $*$-homomorphism $\tilde{\pi} : A *_{\mathrm{w}} A_{aut}(X_n) \longrightarrow B$ such that $\tilde{\pi} \circ \iota = \pi$. $\square$

The examples described in Section 2 may be adapted in an obvious manner to the Woronowicz algebra setting. Let G be discrete group. We define $A_n(G)$ to be the Woronowicz algebra $C^*(G) *_{\mathrm{w}} A_{aut}(X_n)$, where $C^*(G)$ denotes the enveloping C^*-algebra of G. It is clear that $A_n(G)$ is the enveloping C^*-algebra of the CQG algebra $\mathcal{A}_n(G)$ of Section 2.

4 Application to the quantum automorphism group of a finite graph

We use the results of the previous sections to describe the quantum automorphism group of the n-times disjoint union of a finite connected graph. First let us recall some definitions from [3].

In this paper a finite graph $\mathcal{G} = (V, E)$ consists of two finite sets V (set of vertices) and E (set of edges) such that $E \subset V \times V$. The source and target maps $s, t : E \longrightarrow V$ are the restrictions of the first and second projections respectively. So in our conventions, we do not allow a graph to have multiple edges. The quantum automorphism group of a graph $\mathcal{G}$ was defined in [3]. It is still possible to describe the quantum automorphism group of a finite graph with possible multiple edges, using e.g. the general construction of [4]. However some motivation for the construction done in [3] was to describe non-trivial quantum subgroups of Wang's quantum permutation group, so it was quite natural to restrict to graphs for which an automorphism is uniquely determined by its action on the vertices.

Let $\mathcal{G} = (V, E)$ be a finite graph with set of vertices $V = \{1, \dots, m\}$. The quantum automorphism of $\mathcal{G}$ [3], denoted $A_{aut}(\mathcal{G})$, is defined to be the universal C^*-algebra with generators $(X_{ij})_{1 \leq i,j \leq m}$ and relations:
(4.1)

$$X_{ij}^* = X_{ij} \ ; \ X_{ij}X_{ik} = \delta_{jk}X_{ij} \ ; \ X_{ji}X_{ki} = \delta_{jk}X_{ji} \ ; \ \sum_{l=1}^{m} X_{il} = 1 = \sum_{l=1}^{m} X_{li} \ , \ 1 \leq i, j, k \leq m$$

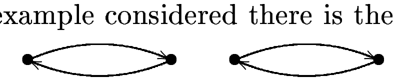
$$X_{s(\gamma)i}X_{t(\gamma)k} = X_{t(\gamma)k}X_{s(\gamma)i} = 0 \tag{4.2}$$

$$X_{is(\gamma)}X_{kt(\gamma)} = X_{kt(\gamma)}X_{is(\gamma)} = 0 \quad , \quad \gamma \in E, \ (i,k) \notin E$$

$$X_{s(\gamma)s(\gamma')}X_{t(\gamma)t(\gamma')} = X_{t(\gamma)t(\gamma')}X_{s(\gamma)s(\gamma')} \quad , \ \gamma, \gamma' \in E \tag{4.3}$$

$$\sum_{\gamma' \in E} X_{s(\gamma')s(\gamma)}X_{t(\gamma')t(\gamma)} = 1 = \sum_{\gamma' \in E} X_{s(\gamma)s(\gamma')}X_{t(\gamma)t(\gamma')} \quad , \quad \gamma \in E \tag{4.4}$$

Then $A_{aut}(\mathcal{G})$ is a full Woronowicz algebra, with coproduct defined by $\Delta(X_{ij}) = \sum_k X_{ik} \otimes X_{kj}$ It is shown in [3] that $A_{aut}(\mathcal{G})$ is the universal compact quantum group acting on the graph $\mathcal{G}$. The non-trivial example considered there is the quantum automorphism group of the following graph:



This quantum group will easily be described using Theorem 4.2 and the results of the previous sections: it is isomorphic with $A_2(\mathbb{Z}/2\mathbb{Z})$.

Let us give a more convenient presentation for $A_{aut}(\mathcal{G})$.

Proposition 4.1 *Let $\mathcal{G} = (V, E)$ be a finite graph with set of vertices $V = \{1, \dots, m\}$. Then $A_{aut}(\mathcal{G})$ is isomorphic with the universal C^*-algebra with generators $(X_{ij})_{1 \le i,j \le m}$ and relations (4.1)-(4.3) and*

$$(4.4)' \qquad \sum_{k,(k,j)\in E} X_{ik} = \sum_{k,(i,k)\in E} X_{kj} \quad ; \quad \sum_{k,(k,j)\in E} X_{ki} = \sum_{k,(i,k)\in E} X_{jk} \quad , \quad 1 \le i, j \le m.$$

Proof. Let A be an algebra with elements $(X_{ij})_{1\le i,j\le m}$ satisfying relations (4.1). Assume that relations (4.4) hold. Let $i, j \in \{1, \dots, m\}$. Then

$$\sum_{\gamma,t(\gamma)=j} X_{is(\gamma)} = \sum_{\gamma,t(\gamma)=j} X_{is(\gamma)}(\sum_{\gamma'\in E} X_{s(\gamma')s(\gamma)}X_{t(\gamma')t(\gamma)}) = \sum_{\gamma,t(\gamma)=j}\sum_{\gamma',s(\gamma')=i} X_{s(\gamma')s(\gamma)}X_{t(\gamma')t(\gamma)}$$
$$= \sum_{\gamma',s(\gamma')=i} (\sum_{\gamma\in E} X_{s(\gamma')s(\gamma)}X_{t(\gamma')t(\gamma)})X_{t(\gamma')j} = \sum_{\gamma,s(\gamma)=i} X_{t(\gamma)j}.$$

In the same way one shows that $\sum_{\gamma,t(\gamma)=j} X_{s(\gamma)i} = \sum_{\gamma,s(\gamma)=i} X_{jt(\gamma)}$. Hence relations (4.4)' hold.

Conversely, assume that relations (4.4)' are fulfilled and let $\gamma \in E$. Then

$$\sum_{\gamma'\in E} X_{s(\gamma')s(\gamma)}X_{t(\gamma')t(\gamma)} = \sum_{i=1}^{m}\sum_{k,(i,k)\in E} X_{is(\gamma)}X_{kt(\gamma)} =$$
$$= \sum_{i=1}^{m} X_{is(\gamma)} \sum_{k,(k,t(\gamma))\in E} X_{ik} = \sum_{i=1}^{m} X_{is(\gamma)} = 1,$$

and

$$\sum_{\gamma'\in E} X_{s(\gamma)s(\gamma')}X_{t(\gamma)t(\gamma')} = \sum_{i=1}^{m}\sum_{k,(i,k)\in E} X_{s(\gamma)i}X_{t(\gamma)k} =$$
$$= \sum_{i=1}^{m} X_{s(\gamma)i} \sum_{k,(k,t(\gamma))\in E} X_{ki} = \sum_{i=1}^{m} X_{s(\gamma)i} = 1.$$

Hence relations (4.4) hold. □

We now begin our study of the quantum automorphism group of the n-times disjoint union of a finite connected graph. Let us recall some basic definitions.

A graph $\mathcal{G} = (V, E)$ is said to be connected if $\forall (i, j) \in V \times V$, there exists a finite sequence of edges $\gamma_1, \dots, \gamma_r$ such that:

- $s(\gamma_1) = i$ or $t(\gamma_1) = i$,
- $s(\gamma_r) = j$ or $t(\gamma_r) = j$,
- For $k \in \{1, \dots, r-1\}$, $s(\gamma_k) = t(\gamma_{k+1})$ or $s(\gamma_k) = s(\gamma_{k+1})$ or $t(\gamma_k) = t(\gamma_{k+1})$ or $t(\gamma_k) = s(\gamma_{k+1})$.

Let $\mathcal{G}_1 = (V_1, E_1)$ and let $\mathcal{G}_2 = (V_2, E_2)$ be two finite graphs. Their disjoint union is defined in the obvious way: $\mathcal{G}_1 \amalg \mathcal{G}_2 = (V_1 \amalg V_2, E_1 \amalg E_2)$.

Let $\mathcal{G} = (V, E)$ be a finite connected graph and let $n \in \mathbb{N}^*$. It is well-known that the groups $\mathrm{Aut}(\mathcal{G}^{\amalg n})$ and $\mathrm{Aut}(\mathcal{G})^n \rtimes S_n = \mathrm{Aut}(\mathcal{G})\mathrm{w}S_n$ are isomorphic, and it is natural to expect that such a kind of isomorphism still holds at the quantum automorphism group level. This is exactly the motivation for our free wreath product construction:

Theorem 4.2 *Let $\mathcal{G}$ be a finite connected graph and let $n \in \mathbb{N}^*$. Then we have a Woronowicz algebras isomorphism:*

$$A_{aut}(\mathcal{G}^{\amalg n}) \cong A_{aut}(\mathcal{G}) *_{\mathrm{w}} A_{aut}(X_n).$$

The proof of Theorem 4.2 will be completed at the end of the section. Let us first note an immediate consequence:

Corollary 4.3 *Let $\mathcal{G}$ be a finite connected graph with non-trivial automorphism group and let $n \in \mathbb{N}^*$ with $n \geq 2$. Then $A_{aut}(\mathcal{G}^{\amalg n})$ is an infinite-dimensional noncommutative and noncocommmutative Woronowicz algebra.* $\square$

As a consequence, we also have a description of the quantum automorphism group of a disjoint union of polygonal graphs. Let $m \in \mathbb{N}^*$ and let $\mathcal{P}_m = (V_m, E_m)$ with $V_m = \{1, \dots, m\}$ and $E_m = \{(1,2), \dots, (m-1, m), (m, 1)\}$ be the polygonal graph with m vertices. It is easy to check that $A_{aut}(\mathcal{P}_m) \cong C^*(\mathbb{Z}/m\mathbb{Z})$. Combining the results of the previous sections and Theorem 4.2, we have $A_{aut}(\mathcal{P}_m^{\amalg n}) \cong A_n(\mathbb{Z}/m\mathbb{Z})$. More generally, if $\mathcal{G}$ is a connected graph with abelian automorphism group G, we have $A_{aut}(\mathcal{G}^{\amalg n}) \cong A_n(G)$.

We now prove some preliminary results for the proof of Theorem 4.2. For this we consider a family of finite connected graphs $\mathcal{G}_1, \dots, \mathcal{G}_n$. For $i \in \{1, \dots, n\}$, the set of vertices (resp. of edges) of $\mathcal{G}_i$ is denoted by V_i (resp. E_i). The next lemmas are basic computations in $A_{aut}(\amalg_{i=1}^n \mathcal{G}_i)$.

Let $k, l \in \{1, \dots, n\}$ and let $i \in V_l$. We define the following element of $A_{aut}(\amalg_{i=1}^n \mathcal{G}_i)$:

$$P_i^{kl} = \sum_{u \in V_k} X_{ui}.$$

Lemma 4.4 *For $k, l \in \{1, \dots, n\}$ and $i, j \in V_l$, one has $P_i^{kl} = P_j^{kl}$. Hence we put $P^{kl} = P_i^{kl}$, $\forall i \in V_l$. For $k, k', l \in \{1, \dots, n\}$, we have*

$$P^{kl} P^{k'l} = \delta_{kk'} P_{kl} \ ; \ \sum_{k=1}^n P^{kl} = 1 \ ; \ (P^{kl})^* = P^{kl} \ ; \ \Delta(P^{kl}) = \sum_{k'=1}^n P^{kk'} \otimes P^{k'l} \ ,$$

and

$$\varepsilon(P^{kl}) = \delta_{kl} \ ; \ P^{lk} P^{lk'} = \delta_{kk'} P^{lk} \quad ; \quad \sum_{k=1}^n P^{lk} = 1.$$

Proof. Let $i \in V_l$. Then by relations (4.1) we have

$$P_i^{kl} P_i^{kl} = \sum_{u,v \in V_k} X_{ui} X_{vi} = P_i^{kl}. \tag{4.5}$$

Let $(i,j) \in E_l$, then by relations (4.2)-(4.3), we have

$$P_i^{kl} P_j^{kl} = \sum_{\gamma \in E_k} X_{s(\gamma)i} X_{t(\gamma)j} = P_j^{kl} P_i^{kl}, \tag{4.6}$$

and thus by relations (4.4) we have

$$\sum_{k=1}^{n} P_i^{kl} P_j^{kl} = 1 = \sum_{k=1}^{n} P_j^{kl} P_i^{kl}. \tag{4.7}$$

Let $(i,j) \in E_l$ and $k, k', l \in \{1, \ldots, n\}$ with $k \neq k'$. Then by relations (4.2) we have

$$P_i^{kl} P_j^{k'l} = \sum_{u \in V_k} \sum_{v \in V_{k'}} X_{ui} X_{vj} = 0 = P_j^{k'l} P_i^{kl}. \tag{4.8}$$

Combining relations (4.7), (4.8) and (4.5), we see that

$$P_j^{kl} = P_i^{kl} P_j^{kl} = P_j^{kl} P_i^{kl} = P_i^{kl} \ , \quad k, l \in \{1, \ldots, n\} \ , \quad (i,j) \in E_l.$$

The graph $\mathcal{G}_l$ is connected, and hence we have $P_i^{kl} = P_j^{kl}$, $\forall i, j \in V_l$. This proves our first claim. The first two relations of the lemma are (4.5,4.8) and (4.7) respectively. Let $k, l \in \{1, \ldots, n\}$ and let $i \in V_l$. Then

$$\Delta(P^{kl}) = \Delta(P_i^{kl}) = \sum_{u \in V_k} \sum_{p=1}^{n} \sum_{v \in V_p} X_{uv} \otimes X_{vi} = \sum_{p=1}^{n} \sum_{v \in V_p} P_v^{kp} \otimes X_{vi} = \sum_{p=1}^{n} P^{kp} \otimes P_i^{pl}.$$

It is obvious that $(P^{kl})^* = P^{kl}$ and $\varepsilon(P^{kl}) = \delta_{kl}$. The last two relations follow from the previous ones and Wang's Theorem 3.1 in [11]. □

We use Lemma 4.4 to prove the following result (we retain the previous notations):

Lemma 4.5 *Let $k, l \in \{1, \ldots, n\}$ and let $i \in V_k$, $i' \notin V_k$, $j, j' \in V_l$. Then we have*

$$X_{ij} X_{i'j'} = X_{i'j'} X_{ij} = 0 = X_{ji} X_{j'i'} = X_{j'i'} X_{ji}.$$

Proof. Let $i \in V_k$ and let $j \in V_l$. Then we have

$$X_{ij} P^{kl} = X_{ij} P_j^{kl} = \sum_{u \in V_k} X_{ij} X_{uj} = X_{ij} = P^{kl} X_{ij}.$$

If $i' \notin V_k$, then

$$X_{i'j} P^{kl} = X_{i'j} P_j^{kl} = \sum_{u \in V_k} X_{i'j} X_{uj} = 0 = P^{kl} X_{i'j}.$$

Hence if $j, j' \in V_l$, $i \in V_k$ and $i' \notin V_k$, one has

$$X_{ij} X_{i'j'} = X_{ij} P^{kl} X_{i'j'} = 0 = X_{i'j'} X_{ij}.$$

The second identity is obtained using the antipode on the dense CQG algebra. □

Proof of Theorem 4.2

Let $\mathcal{G}$ be a finite connected graph with set of vertices $V = \{1, \dots, m\}$. For $k \in \{1, \dots, n\}$, we denote by $\mathcal{G}_k$ the k-th copy of $\mathcal{G}$ in $\mathcal{G}^{\amalg n}$, with set of vertices $V_k = \{k(1), \dots, k(p)\}$. The generators of $A_{aut}(\mathcal{G})$ are denoted by $(u_{ij})_{1 \leq i,j \leq m}$, and the generators of $A_{aut}(X_n)$ are still denoted by $(x_{kl})_{1 \leq k,l \leq n}$. Let $\mathcal{B}$ be the free algebra with generators $X_{k(i)l(j)}$, $1 \leq i,j \leq m$, $1 \leq k,l \leq n$, and define an algebra morphism

$$\Phi_0 : \mathcal{B} \longrightarrow A_{aut}(\mathcal{G}) *_{\mathrm{w}} A_{aut}(X_n) \ , \quad X_{k(i)l(j)} \longmapsto \nu_k(u_{ij}) x_{kl}.$$

We will show that Φ_0 induces a Woronowicz algebra morphism $A_{aut}(\mathcal{G}^{\amalg n}) \longrightarrow A_{aut}(\mathcal{G}) *_{\mathrm{w}} A_{aut}(X_n)$: these are straightforward computations. First let us note that if $\mathcal{B}$ is endowed with the $*$-algebra structure such that $X^*_{k(i)l(j)} = X_{k(i)l(j)}$, then Φ_0 is easily seen to be a $*$-algebra map. Let $k, l, p \in \{1, \dots, n\}$ and $i, j, r \in \{1, \dots, m\}$. Then

- $\Phi_0(X_{k(i)l(j)} X_{k(i)p(r)}) = \nu_k(u_{ij}) x_{kl} \nu_k(u_{ir}) x_{kp} = \nu_k(u_{ij} u_{ir}) x_{kl} x_{kp} = \delta_{l(j)p(r)} \Phi_0(X_{k(i)l(j)})$.
- $\Phi_0(X_{l(j)k(i)} X_{p(r)k(i)}) = \nu_l(u_{ji}) x_{lk} \nu_p(u_{ri}) x_{pk} = \nu_l(u_{ji}) x_{lk} x_{pk} \nu_p(u_{ri}) = \delta_{l(j)p(r)} \Phi_0(X_{l(j)k(i)})$.
- $\Phi_0(\sum_{k=1}^n \sum_{i=1}^m X_{k(i)l(j)}) = \sum_{k=1}^n \sum_{i=1}^m \nu_k(u_{ij}) x_{kl} = \sum_{k=1}^n x_{kl} = 1 = \Phi_0(1)$.
- $\Phi_0(\sum_{l=1}^n \sum_{j=1}^m X_{k(i)l(j)}) = \sum_{l=1}^n \sum_{j=1}^m \nu_k(u_{ij}) x_{kl} = \sum_{l=1}^n x_{kl} = 1 = \Phi_0(1)$.

Let $(i,j) \in E = E(\mathcal{G})$, let $k, l', l'' \in \{1, \dots, n\}$ and $i', i'' \in \{1, \dots, m\}$ with $l' \neq l''$ or $(i', i'') \notin E$. Then:

- $\Phi_0(X_{k(i)l'(i')} X_{k(j)l''(i'')}) = \nu_k(u_{ii'}) x_{kl'} \nu_k(u_{ji''}) x_{kl''} = \nu_k(u_{ii'} u_{ji''}) x_{kl'} x_{kl''} = 0 =$
  $= \Phi_0(X_{k(j)l''(i'')} X_{k(i)l'(i')})$.
- $\Phi_0(X_{l'(i')k(i)} X_{l''(i'')k(j)}) = \nu_{l'}(u_{i'i}) x_{l'k} \nu_{l''k}(u_{i''j}) x_{l''k} = \delta_{l'l''} \nu_{l'}(u_{i'i} u_{i''j}) x_{l'k} = 0 =$
  $= \Phi_0(X_{l''(i'')k(j)} X_{l'(i')k(i)})$.

Let $(i,j), (i',j') \in E$ and let $k, l \in \{1, \dots, n\}$. Then

- $\Phi_0(X_{k(i)l(i')} X_{k(j)l(j')}) = \nu_k(u_{ii'}) x_{kl} \nu_k(u_{jj'}) x_{kl} = \nu_k(u_{ii'} u_{jj'}) x_{kl} =$
  $= \nu_k(u_{jj'}) x_{kl} \nu_k(u_{ii'}) x_{kl} = \Phi_0(X_{k(j)l(j')} X_{k(i)l(i')})$

Let $k, l \in \{1, \dots, n\}$ and $i, j \in \{1, \dots, m\}$. Then relations (4.4)' in $A_{aut}(\mathcal{G}^{\amalg n})$ become

$$\sum_{r,(r,j) \in E} X_{k(i)l(r)} = \sum_{r,(i,r) \in E} X_{k(r)l(j)} \quad ; \quad \sum_{r,(r,j) \in E} X_{l(r)k(i)} = \sum_{r,(i,r) \in E} X_{l(j)k(r)}.$$

Then we have

- $\Phi_0(\sum_{r,(r,j) \in E} X_{k(i)l(r)}) = \sum_{r,(r,j) \in E} \nu_k(u_{ir}) x_{kl} = \sum_{r,(i,r) \in E} \nu_k(u_{rj}) x_{kl} = \Phi_0(\sum_{r,(i,r) \in E} X_{k(r)l(j)})$,

- $\Phi_0(\sum_{r,(r,j)\in E} X_{l(r)k(i)}) = \sum_{r,(r,j)\in E} \nu_l(u_{ri})x_{lk} = \sum_{r,(i,r)\in E} \nu_l(u_{jr})x_{lk} = \Phi_0(\sum_{r,(i,r)\in E} X_{l(j)k(r)})$.

All these computations show that Φ_0 induces a $*$-homomorphism

$$\Phi : A_{aut}(\mathcal{G}^{\amalg n}) \longrightarrow A_{aut}(\mathcal{G})*_{\mathrm{w}} A_{aut}(X_n),\ \Phi(X_{k(i)l(j)}) = \nu_k(u_{ij})x_{kl}.$$

It is easily seen that Φ is a Woronowicz algebra morphism, that is $\Phi\circ\Delta = (\Phi\otimes\Phi)\circ\Delta$.

We have now to construct an inverse for Φ. First by Lemma 4.4 we have a $*$-homomorphism $\pi : A_{aut}(X_n) \longrightarrow A_{aut}(\mathcal{G}^{\amalg n})$ such that for $k,l\in\{1,\dots,n\}$,

$$\pi(x_{kl}) = P^{kl} = \sum_{r=1}^{n} X_{k(r)l(i)},\ \forall i\in\{1,\dots,m\}.$$

Let $\mathcal{C}$ be the free algebra with generators $(u_{ij})_{1\le i,j\le m}$. For $k\in\{1,\dots,n\}$ define an algebra morphism

$$\theta_0^k : \mathcal{C} \longrightarrow A_{aut}(\mathcal{G}^{\amalg n}),\ u_{ij}\longmapsto \sum_{l=1}^{n} X_{k(i)l(j)},\ 1\le i,j\le m.$$

It is immediate that θ_0^k is a $*$-homomorphism, if $\mathcal{C}$ is endowed with the $*$-algebra structure defined by $u_{ij}^* = u_{ij}$. Let $i,j,j'\in\{1,\dots m\}$. Then:

- $\theta_0^k(u_{ij}u_{ij'}) = \sum_{l,l'=1}^{n} X_{k(i)l(j)}X_{k(i)l'(j')} = \sum_{l=1}^{n} X_{k(i)l(j)}X_{k(i)l(j')} = \delta_{jj'}\theta_0^k(u_{ij})$.

Using Lemma 4.5, we have

- $\theta_0^k(u_{ji}u_{j'i}) = \sum_{l,l'=1}^{n} X_{k(j)l(i)}X_{k(j')l'(i)} = \sum_{l=1}^{n} X_{k(j)l(i)}X_{k(j')l(i)} = \delta_{jj'}\theta_0^k(u_{ji})$.

- $\theta_0^k(\sum_{j=1}^{m} u_{ij}) = \sum_{j=1}^{m}\sum_{l=1}^{n} X_{k(i)l(j)} = 1 = \theta_0^k(1)$.

Using Lemma 4.4, we have

- $\theta_0^k(\sum_{j=1}^{m} u_{ji}) = \sum_{j=1}^{m}\sum_{l=1}^{n} X_{k(j)l(i)} = \sum_{l=1}^{n} P^{kl}_{l(i)} = \sum_{l=1}^{n} P^{kl} = 1 = \theta_0^k(1)$.

Let $\gamma\in E$ and let $i,j\in\{1,\dots,m\}$ with $(i,j)\notin E$. Then

- $\theta_0^k(u_{s(\gamma)i}u_{t(\gamma)j}) = \sum_{l,l'=1}^{n} X_{k(s(\gamma))l(i)}X_{k(t(\gamma))l'(j)} = \sum_{l=1}^{n} X_{k(s(\gamma))l(i)}X_{k(t(\gamma))l(j)} =$
  $= 0 = \theta_0^k(u_{t(\gamma)j}u_{s(\gamma)i})$,

and using Lemma 4.5, we have

- $\theta_0^k(u_{is(\gamma)}u_{jt(\gamma)}) = \sum_{l,l'=1}^{n} X_{k(i)l(s(\gamma))}X_{k(j)l'(t(\gamma))} = \sum_{l=1}^{n} X_{k(i)l(s(\gamma))}X_{k(j)l(t(\gamma))} =$
  $= 0 = \theta_0^k(u_{jt(\gamma)}u_{is(\gamma)})$.

Let $\gamma, \gamma' \in E$. Using Lemma 4.5, we have

- $\theta_0^k(u_{s(\gamma)s(\gamma')}u_{t(\gamma)t(\gamma')}) = \sum_{l,l'=1}^{n} X_{k(s(\gamma))l(s(\gamma'))}X_{k(t(\gamma))l'(t(\gamma'))} =$

$$= \sum_{l=1}^{n} X_{k(s(\gamma))l(s(\gamma'))}X_{k(t(\gamma))l(t(\gamma'))} = \sum_{l=1}^{n} X_{k(t(\gamma))l(t(\gamma'))}X_{k(s(\gamma))l(s(\gamma'))} = \theta_0^k(u_{t(\gamma)t(\gamma')}u_{s(\gamma)s(\gamma')}).$$

Let $i, j \in \{1, \ldots, m\}$. We have:

- $\theta_0^k(\sum_{r,(r,j)\in E} u_{ir}) = \sum_{r,(r,j)\in E}\sum_{l=1}^{n} X_{k(i)l(r)} = \sum_{l=1}^{n}\sum_{r,(r,j)\in E} X_{k(i)l(r)} =$
  $= \sum_{l=1}^{n}\sum_{r,(i,r)\in E} X_{k(r)l(j)} = \theta_0^k(\sum_{r,(i,r)\in E} u_{rj})$.

- $\theta_0^k(\sum_{r,(r,j)\in E} u_{ri}) = \sum_{r,(r,j)\in E}\sum_{l=1}^{n} X_{k(r)l(i)} = \sum_{l=1}^{n}\sum_{r,(r,j)\in E} X_{k(r)l(i)} =$
  $= \sum_{l=1}^{n}\sum_{r,(i,r)\in E} X_{k(j)l(r)} = \theta_0^k(\sum_{r,(i,r)\in E} u_{jr})$.

Hence θ_0^k induces a $*$-homomorphism $\theta^k : A_{aut}(\mathcal{G}) \longrightarrow A_{aut}(\mathcal{G}^{\amalg n})$. Using the universal property of the free product, we get a $*$-homomorphism $\Psi_0 : A_{aut}(\mathcal{G})^{*n} * A_{aut}(X_n) \longrightarrow A_{aut}(\mathcal{G}^{\amalg n})$ such that $\Psi_0 \circ \nu_k = \theta^k$ and $\Psi_{0|A_{aut}(X_n)} = \pi$. Let $k, l \in \{1, \ldots, n\}$ and $i, j \in \{1, \ldots m\}$. Then using Lemma 4.4 and Lemma 4.5, we have:

$$\Psi_0(\nu_k(u_{ij})x_{kl}) = \sum_{l'=1}^{n}\sum_{r=1}^{m} X_{k(i)l'(j)}X_{k(r)l(j)} = X_{k(i)l(j)} = \Psi_0(x_{kl}\nu_k(u_{ij})).$$

In this way we get a $*$-homomorphism $\Psi : A_{aut}(\mathcal{G}) *_{\rm w} A_{aut}(X_n) \longrightarrow A_{aut}(\mathcal{G}^{\amalg n})$. It is straightforward to check that Φ and Ψ are mutually inverse isomorphisms: this concludes the proof of Theorem 4.2. $\square$

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