HALF-COMMUTATIVE ORTHOGONAL HOPF ALGEBRAS

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ABSTRACT. A half-commutative orthogonal Hopf algebra is a Hopf $*$-algebra generated by the self-adjoint coefficients of an orthogonal matrix corepresentation $v = (v_{ij})$ that half commute in the sense that $abc = cba$ for any $a, b, c \in \{v_{ij}\}$. The first non-trivial such Hopf algebras were discovered by Banica and Speicher. We propose a general procedure, based on a crossed product construction, that associates to a self-transpose compact subgroup $G \subset U_n$ a half-commutative orthogonal Hopf algebra $\mathcal{A}_*(G)$. It is shown that any half-commutative orthogonal Hopf algebra arises in this way. The fusion rules of $\mathcal{A}_*(G)$ are expressed in term of those of G.

1. INTRODUCTION

The half-liberated orthogonal quantum group O_n^* were recently discovered by Banica and Speicher [7]. These are compact quantum groups in the sense of Woronowicz [22], and the corresponding Hopf $*$-algebra $A_o^*(n)$ is the universal $*$-algebra presented by self-adjoint generators v_{ij} submitted to the relations making $v = (v_{ij})$ an orthogonal matrix and to the half-commutation relations

$$abc = cba, \ a, b, c \in \{v_{ij}\}$$

The half-commutation relations arose, via Tannaka duality, from a deep study of certain tensor subcategories of the category of partitions, see [7]. More examples of Hopf algebras with generators satisfying the half-commutation relations were given in [4], and the classification of "easy" orthogonal Hopf algebras (which means that the tensor category of corepresentations is spanned by partitions) with generators satisfying the half-commutation relations was very recently done in [21].

The representation theory of O_n^* was discussed in [8], where strong links with the representation theory of the unitary group U_n were found. It followed that the fusion rules of O_n^* are non-commutative if $n \geq 3$. Moreover a matrix model $A_o^*(n) \hookrightarrow M_2(\mathcal{R}(U_n))$ was found in [5].

The aim of this paper is to continue these works by a general study of what we call half-commutative orthogonal Hopf algebras: Hopf $*$-algebras generated by the self-adjoint coefficients of an orthogonal matrix corepresentation $v = (v_{ij})$ whose coefficients satisfy the previous half-commutation relations. Our main results are as follows.

(1) To any self-transpose compact subgroup $G \subset U_n$ we associate a half-commutative orthogonal Hopf algebra $\mathcal{A}_*(G)$, with $\mathcal{A}_*(U_n) \simeq A_o^*(n)$. The Hopf algebra $\mathcal{A}_*(G)$ is a Hopf $*$-subalgebra of the crossed product $\mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ of $\mathcal{R}(G)$ is induced by the transposition.

(2) Conversely we show that any noncommutative half-commutative orthogonal Hopf algebra arises from the previous construction for some compact group $G \subset U_n$.

(3) We show that the fusion rules of $\mathcal{A}_*(G)$ can be described in terms of those of G.

Therefore it follows from our study that quantum groups arising from half-commutative orthogonal Hopf algebras are objects that are very close from classical groups. This was suggested by the representation theory results from [8], by the matrix model found in the "easy" case in [5] and by the results of [3] where it was shown that the quantum group inclusion $O_n \subset O_n^*$ is maximal. The techniques from [3], and especially the short five lemma for cosemisimple Hopf algebras, are used in essential way here. The use of versions of the five lemma for Hopf algebras was initiated in [2].

The paper is organized as follows. In Section 2 we fix some notation and recall the necessary background. In Section 3 we formally introduce half-commutative orthogonal Hopf algebras, and recall the early examples from [7, 4]. Section 4 is devoted to our main construction, which associates to a self-transpose compact subgroup $G \subset U_n$ a half-commutative orthogonal Hopf algebra $\mathcal{A}_*(G)$, and we show that any half-commutative orthogonal Hopf algebra arises in this way. At the end of the section we use our construction to propose a possible orthogonal half-liberation of the unitary group U_n. In Section 5 we describe the fusion rules of $\mathcal{A}_*(G)$ in terms of those of G.

We assume that the reader is familiar with Hopf algebras [15], Hopf $*$-algebras and with the algebraic approach (via algebras of representative functions) to compact quantum groups [12, 14].

2. PRELIMINARIES

2.1. **Classical groups.** We first fix some notation. As usual, the group of complex $n \times n$ unitary matrices is denoted by U_n, while O_n denotes the group of real orthogonal matrices. We denote by $\mathbb{T}$ the subgroup of U_n consisting of scalar matrices, and by PU_n the quotient group $U_n/\mathbb{T}$.

We shall need the following notions.

Definition 2.1. *Let $G \subset U_n$ be a compact subgroup.*

(1) *We say that G is self-transpose if $\forall g \in G$, we have $g^t \in G$.*
(2) *We say that G is non-real if $G \not\subset O_n$, i.e. if there exists $g \in G$ with $g_{ij} \notin \mathbb{R}$, for some i, j.*
(3) *We say that G is doubly non-real if there exists $g \in G$ with $g_{ij}\overline{g_{kl}} \notin \mathbb{R}$, for some i, j, k, l.*

Note that the subgroup $\tilde{O}_n = \mathbb{T}O_n \subset U_n$ (considered in [3]) is non-real but is not doubly non-real.

2.2. **Orthogonal and unitary Hopf algebras.** In this subsection we recall some definitions on the algebraic approach to compact quantum groups. We work at the level of Hopf $*$-algebras of representative functions. The following simple key definition arose from Woronowicz' work [22].

Definition 2.2. *A unitary Hopf algebra is a $*$-algebra A which is generated by elements $\{u_{ij}|1 \leq i, j \leq n\}$ such that the matrices $u = (u_{ij})$ and $\overline{u} = (u_{ij}^*)$ are unitaries, and such that:*

(1) *There is a $*$-algebra map $\Delta : A \to A \otimes A$ such that $\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$.*
(2) *There is a $*$-algebra map $\varepsilon : A \to \mathbb{C}$ such that $\varepsilon(u_{ij}) = \delta_{ij}$.*
(3) *There is a $*$-algebra map $S : A \to A^{op}$ such that $S(u_{ij}) = u_{ji}^*$.*

If $u_{ij} = u_{ij}^*$ for $1 \leq i, j \leq n$, we say that A is an orthogonal Hopf algebra.

It follows that Δ, ε, S satisfy the usual Hopf $*$-algebra axioms and that $u = (u_{ij})$ is a matrix corepresentation of A. Note that the definition forces that a unitary Hopf algebra is of Kac type, i.e. $S^2 = \mathrm{id}$. The motivating examples of unitary (resp. orthogonal) Hopf algebra is $A = \mathcal{R}(G)$, the algebra of representative functions on a compact subgroup $G \subset U_n$ (resp. $G \subset O_n$). Here the standard generators u_{ij} are the coordinate functions which take a matrix to its (i, j)-entry.

In fact every commutative unitary Hopf algebra is of the form $\mathcal{R}(G)$ for some unique compact group $G \subset U_n$ defined by $G = \mathrm{Hom}_{*-alg}(A, \mathbb{C})$ (this the Hopf algebra version of the Tannaka-Krein theorem). This motivates the notation "$A = \mathcal{R}(G)$" for any unitary (resp. orthogonal) Hopf algebra, where G is a *unitary (resp. orthogonal) compact quantum group*.

The universal examples of unitary and orthogonal Hopf algebras are as follows [18].

Definition 2.3. *The universal unitary Hopf algebra $A_u(n)$ is the universal $*$-algebra generated by elements $\{u_{ij}|1 \leq i, j \leq n\}$ such that the matrices $u = (u_{ij})$ and $\overline{u} = (u_{ij}^*)$ in $M_n(A_u(n))$ are unitaries.*

The universal orthogonal Hopf algebra $A_o(n)$ is the universal $*$-algebra generated by self-adjoint elements $\{u_{ij}|1 \leq i,j \leq n\}$ such that the matrix $u = (u_{ij})_{1 \leq i,j \leq n}$ in $M_n(A_o(n))$ is orthogonal.

The existence of the Hopf $*$-algebra structural morphisms follows from the universal properties of $A_u(n)$ and $A_o(n)$. As discussed above, we use the notations $A_u(n) = \mathcal{R}(U_n^+)$ and $A_o(n) = \mathcal{R}(O_n^+)$, where U_n^+ is the *free unitary quantum group* and O_n^+ is the *free orthogonal quantum group*.

The Hopf $*$-algebra $A_u(n)$ was introduced by Wang [18], while the Hopf algebra $A_o(n)$ was defined first in [13] under the notation $\mathcal{A}(I_n)$, and was then defined independently in [18] in the compact quantum group framework.

2.3. **Exact sequences of Hopf algebras.** In this subsection we recall some facts on exact sequences of Hopf algebras.

Definition 2.4. *A sequence of Hopf algebra maps*

$$\mathbb{C} \to B \xrightarrow{i} A \xrightarrow{p} L \to \mathbb{C}$$

is called pre-exact if i is injective, p is surjective and $i(B) = A^{cop}$, where:

$$A^{cop} = \{a \in A | (id \otimes p)\Delta(a) = a \otimes 1\}$$

A pre-exact sequence as in Definition 2.4 is said to be exact [1] if in addition we have $i(B)^+ A = \ker(p) = Ai(B)^+$, where $i(B)^+ = i(B) \cap \ker(\varepsilon)$. For the kind of sequences to be considered in this paper, pre-exactness is actually equivalent to exactness.

The following lemma, that we record for future use, is Proposition 3.2 in [3].

Lemma 2.5. *Let A be an orthogonal Hopf algebra with generators u_{ij}. Assume that we have surjective Hopf algebra map $p : A \to \mathbb{C}\mathbb{Z}_2$, $u_{ij} \to \delta_{ij}g$, where $< g >= \mathbb{Z}_2$. Let $P_u A$ be the subalgebra generated by the elements $u_{ij}u_{kl}$ with the inclusion $i : P_u A \subset A$. Then the sequence*

$$\mathbb{C} \to P_u A \xrightarrow{i} A \xrightarrow{p} \mathbb{C}\mathbb{Z}_2 \to \mathbb{C}$$

is pre-exact.

Exact sequences of compact groups induce exact sequences of Hopf algebras. In particular if $G \subset U_n$ is a compact subgroup, we have an exact sequence of compact groups

$$1 \to G \cap \mathbb{T} \to G \to G/G \cap \mathbb{T} \to 1$$

that induces an exact sequence of Hopf algebras

$$\mathbb{C} \to \mathcal{R}(G/G \cap \mathbb{T}) \to \mathcal{R}(G) \to \mathcal{R}(G \cap \mathbb{T}) \to \mathbb{C}$$

We will use the following probably well-known lemma. We sketch a proof for the sake of completeness.

Lemma 2.6. *Let $G \subset U_n$ be a compact subgroup. Then $\mathcal{R}(G/G \cap \mathbb{T})$ is the subalgebra of $\mathcal{R}(G)$ generated by the elements $u_{ij}u_{kl}^*$, $i,j,k,l \in \{1,\ldots,n\}$. Moreover, if $G = U_n$, then $\mathcal{R}(PU_n) = \mathcal{R}(U_n/\mathbb{T})$ is isomorphic with the commutative $*$-algebra presented by generators $w_{ij,kl}$, $1 \leq i,j,k,l \leq n$ and submitted to the relations*

$$\sum_{j=1}^n w_{ik,jj} = \delta_{ik} = \sum_{j=1}^n w_{jj,ik}, \ w_{ij,kl}^* = w_{ji,lk}$$

$$\sum_{k,l=1}^n w_{ij,kl}w_{pq,kl}^* = \delta_{ip}\delta_{jq}$$

The isomorphism is given by $w_{ij,kl} \longmapsto u_{ik}u_{jl}^*$.

Proof. Let $p : \mathcal{R}(G) \longrightarrow \mathcal{R}(G \cap \mathbb{T})$ be the restriction map. It is clear $\mathrm{Ker}(p)$ is generated as a $*$-ideal by the elements u_{ij}, $i \neq j$, and $u_{ii} - u_{jj}$. Let B be the subalgebra generated by the elements $u_{ij}u_{kl}^*$. Then B is a Hopf $*$-subalgebra of $\mathcal{R}(G)$ and it is clear that $B \subset \mathcal{R}(G)^{cop}$. To prove the reverse inclusion we form the Hopf algebra quotient $\mathcal{R}(G)//B = \mathcal{R}(G)/B^+\mathcal{R}(G)$ and denote by $\rho : \mathcal{R}(G) \longrightarrow \mathcal{R}(G)//B$ the canonical projection. It is not difficult to see that in $\mathcal{R}(G)//B$ we have $\rho(u_{ij}) = 0$ if $i \neq j$ and $\rho(u_{ii}) = \rho(u_{jj})$ for any i, j. Hence there exists a Hopf $*$-algebra map $p' : \mathcal{R}(G/\mathbb{T}) \longrightarrow \mathcal{R}(G)//B$ such that $p' \circ p = \rho$. It follows that $\mathcal{R}(G)^{\mathrm{cop}} \subset \mathcal{R}(G)^{\mathrm{cop}}$. But since our algebras are commutative, $\mathcal{R}(G)$ is a faithfully flat B-module and hence by [17] (see also [1]) we have $\mathcal{R}(G)^{\mathrm{co}\rho} = B$, and hence $\mathcal{R}(G/G \cap \mathbb{T}) = \mathcal{R}(G)^{\mathrm{cop}} = B$.

The last assertion is just the reformulation of the standard fact that PU_n is the automorphism group of the $*$-algebra $M_n(\mathbb{C})$ (see e.g. [20]). $\qquad\square$

3. Half-commutative Hopf algebras

We now formally introduce half-commutative orthogonal Hopf algebras. Of course the definition of half-commutativity can be given in a general context, as follows. It was first formalized, in a probabilistic context, in [6].

Definition 3.1. *Let A be an algebra. We say that a family $(a_i)_{i \in I}$ of elements of A half-commute if $abc = cba$ for any $a, b, c \in \{a_i, i \in I\}$. The algebra A is said to be half-commutative if it has a family of generators that half-commute.*

At a Hopf algebra level, a reasonnable definition seems to be the following one.

Definition 3.2. *A half-commutative Hopf algebra is a Hopf algebra A generated by the coefficients of a matrix corepresentation $v = (v_{ij})$ whose coefficients half-commute.*

We will not study half-commutative Hopf algebras in this generality. A reason for this is that it is unclear if the half-commutativity relations outside of the orthogonal case are the natural ones in the categorical framework of [7]. Thus we will restrict to the following special case.

Definition 3.3. *A half-commutative orthogonal Hopf algebra is a Hopf $*$-algebra A generated by the self-adjoint coefficients of an orthogonal matrix corepresentation $v = (v_{ij})$ whose coefficients half-commute.*

The first example is the universal one, defined in [7].

Definition 3.4. *The half-liberated orthogonal Hopf algebra $A_o^*(n)$ is the universal $*$-algebra generated by self-adjoint elements $\{v_{ij} | 1 \leq i, j \leq n\}$ which half-commute and such that the matrix $v = (v_{ij})_{1 \leq i,j \leq n}$ in $M_n(A_o^*(n))$ is orthogonal.*

The existence of the Hopf algebra structural morphisms follows from the universal property of $A_o^*(n)$, and hence $A_o^*(n)$ is a half-commutative orthogonal Hopf algebra. We use the notation $A_o^*(n) = \mathcal{R}(O_n^*)$, where O_n^* is the *half-liberated orthogonal quantum group*. Note that we have $\mathcal{R}(O_n^+) \twoheadrightarrow \mathcal{R}(O_n^*) \twoheadrightarrow \mathcal{R}(O_n)$, i.e. $O_n \subset O_n^* \subset O_n^+$. At $n = 2$ we have $O_2^* = O_2^+$, but for $n \geq 3$ these inclusions are strict.

Another example of half-commutative orthogonal Hopf algebra is the following one, taken from [4].

Definition 3.5. *The half-liberated hyperoctaedral Hopf algebra $A_h^*(n)$ is the universal $*$-algebra generated by self-adjoint elements $\{v_{ij} | 1 \leq i, j \leq n\}$ which half-commute, such that $v_{ij}v_{ik} = 0 = v_{ki}v_{ji}$ for $k \neq j$, and such that the matrix $v = (v_{ij})_{1 \leq i,j \leq n}$ in $M_n(A_o^*(n))$ is orthogonal.*

Again the existence of the Hopf algebra structural morphisms follows from the universal property of $A_h^*(n)$, and hence $A_h^*(n)$ is a half-commutative orthogonal Hopf algebra. See [4] and [21] for further examples.

The following lemma will be an important ingredient in the proof of the structure theorem of half-commutative orthogonal Hopf algebras.

Lemma 3.6. *Let A be a half-commutative orthogonal Hopf algebra generated by the self-adjoint coefficients of an orthogonal matrix corepresentation $v = (v_{ij})$ whose coefficients half-commute. Then $P_v A$ is a commutative Hopf $*$-subalgebra of A. If moreover A is noncommutative then there exists a Hopf $*$-algebra map $p : A \longrightarrow \mathbb{C}\mathbb{Z}_2$ such that for any i, j, $p(v_{ij}) = \delta_{ij}s$, where $\langle s \rangle = \mathbb{Z}_2$, that induces a pre-exact sequence*

$$\mathbb{C} \to P_v A \xrightarrow{i} A \xrightarrow{p} \mathbb{C}\mathbb{Z}_2 \to \mathbb{C}$$

Proof. The key observation that $P_v A$ is commutative is Proposition 3.2 in [8]. It is clear that $P_v A$ is a normal Hopf $*$-subalgebra of A, and hence we can form the Hopf $*$-algebra quotient $A//P_v A = A/A(P_v A)^+$, with $p : A \longrightarrow A//P_v A$ the canonical surjection. It is not difficult to see that in $A//P_v A$ we have $p(v_{ij}) = 0$ if $i \neq j$, $p(v_{ii}) = p(v_{jj})$ for any i, j and if we put $g = p(v_{ii})$, $g^2 = 1$. So we have to prove that $g \neq 1$. If $g = 1$, then $A//P_v A$ is trivial and $p = \varepsilon$. We know from [11] that A is faithfully flat as a $P_v A$-module (since orthogonal Hopf algebras are cosemisimple), and hence by [16], we have $A^{\mathrm{cop}} = P_v A$. So if $g = 1$ we have $A^{\mathrm{cop}} = P_v A = A$ and A is commutative. Thus if A is noncommutative we have $g \neq 1$, the map p satisfies the conditions in the statement and we have the announced exact sequence (Lemma 2.5). $\qquad\square$

Remark 3.7. The previous exact sequence is cocentral. Thus it is possible, in principle, to classify the finite-dimensional half-commutative orthogonal Hopf algebras according to the scheme used in [10]. The classification data will involve in particular pairs (Γ, ω) formed by a finite subgroup $\Gamma \subset PU_n$ and a cocycle $\omega \in H^2(\Gamma, \mathbb{Z}_2)$, see [10] for details.

4. The main construction

In this section we perform our main construction that associates to any self-transpose compact subgroup $G \subset U_n$ a half-commutative orthogonal Hopf algebra $\mathcal{A}_*(G)$ and we show any half-commutative orthogonal Hopf algebra arises in this way.

We begin with a well-known lemma. We give a proof for the sake of completeness.

Lemma 4.1. *Let $G \subset U_n$ be a compact subgroup, and denote by u_{ij} the coordinate functions on G. The following assertions are equivalent.*

(1) *G is self-transpose.*
(2) *There exists a unique involutive Hopf $*$-algebra automorphism $s : \mathcal{R}(G) \longrightarrow \mathcal{R}(G)$ such that $s(u_{ij}) = u_{ij}^*$.*

Moreover if G is self-transpose the automorphism is non-trivial if and only G is non-real.

Proof. Assume that G is self-transpose. Then we have an involutive compact group automorphism

$$\sigma : G \longrightarrow G$$
$$g \longmapsto (g^t)^{-1} = \overline{g}$$

which induces an involutive Hopf $*$-algebra automorphism $s : \mathcal{R}(G) \longrightarrow \mathcal{R}(G)$ such that $s(u_{ij}) = u_{ij}^*$. Uniqueness is obvious since the elements u_{ij} generate $\mathcal{R}(G)$ as a $*$-algebra. Conversely, the existence of s will ensure the existence of the automorphism σ since $G \simeq \mathrm{Hom}_{*-alg}(\mathcal{R}(G), \mathbb{C})$, and hence G will be self-transpose. The last assertion is immediate. $\qquad\square$

Definition 4.2. *Let $G \subset U_n$ be a self-transpose non-real compact subgroup. We denote by $\mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2$ the crossed product Hopf $*$-algebra associated to the involutive Hopf $*$-algebra automorphism s of Lemma 4.1.*

Recall that the Hopf $*$-algebra structure of $\mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2$ is defined as follows (see e.g. [14]).

(1) As a coalgebra, $\mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2 = \mathcal{R}(G) \otimes \mathbb{C}\mathbb{Z}_2$.
(2) We have $(f \otimes s^i) \cdot (g \otimes s^j) = f s^i(g) \otimes s^{i+j}$, for any $f, g \in \mathcal{R}(G)$ and $i, j \in \{0, 1\}$.
(3) We have $(f \otimes s^i)^* = s^i(f)^* \otimes s^i$ for any $f \in \mathcal{R}(G)$ and $i \in \{0, 1\}$.

(4) The antipode is given by $S(u_{ij} \otimes 1) = u_{ji}^* \otimes 1$, $S(u_{ij} \otimes s) = u_{ji} \otimes s$ (in short $S(f \otimes s^i) = s^i(S(f)) \otimes s^i$ for any $f \in \mathcal{R}(G)$ and $i \in \{0, 1\}$).

For notational simplicity we denote, for $f \in \mathcal{R}(G)$, the respective elements $f \otimes 1$ and $f \otimes s$ of $\mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2$ by f and fs.

Definition 4.3. *Let $G \subset U_n$ be a self-transpose compact subgroup. We denote by $\mathcal{A}_*(G)$ the subalgebra of $\mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2$ generated by the elements $u_{ij}s$, $i, j \in \{1, \ldots, n\}$.*

Proposition 4.4. *Let $G \subset U_n$ be a self-transpose compact subgroup. Then $\mathcal{A}_*(G)$ is a Hopf $*$-subalgebra of $\mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2$, and there exists a surjective Hopf $*$-algebra morphism*

$$\pi : A_o^*(n) \longrightarrow \mathcal{A}_*(G)$$
$$v_{ij} \longmapsto u_{ij}s$$

Hence $\mathcal{A}_*(G)$ is a half-commutative orthogonal Hopf algebra, and is noncommutative if and only if G is doubly non-real.

Proof. We have $(u_{ij}s)^* = su_{ij}^* = u_{ij}s$ and hence the elements $u_{ij}s$ are self-adjoint and generate a $*$-subalgebra. Moreover, using the coproduct and antipode formula, it is immediate to check that $\Delta(u_{ij}s) = \sum_k u_{ik}s \otimes u_{kj}s$ and $S(u_{ij}s) = u_{ji}s$, and hence $\mathcal{A}_*(G)$ is an orthogonal Hopf $*$-subalgebra of $\mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2$. We have

$$u_{ij}su_{kl}su_{pq}s = u_{ij}u_{kl}^*u_{pq}s = u_{pq}u_{kl}^*u_{ij}s = u_{pq}su_{kl}su_{ij}s$$

Hence the coefficients of the orthogonal matrix $(u_{ij}s)$ half-commute, and we get our Hopf $*$-algebra map $\pi : A_o^*(n) \longrightarrow \mathcal{A}_*(G)$. The algebra $\mathcal{A}_*(G)$ is commutative if and only if the elements $u_{ij}s$ pairwise commute. We have $u_{ij}su_{kl}s = u_{ij}u_{kl}^*$, so $\mathcal{A}_*(G)$ is noncommutative if and only if there exist i, j, k, l with $u_{ij}u_{kl}^* \neq u_{kl}u_{ij}^*$, which precisely means that G is doubly non-real. $\square$

The Hopf $*$-algebra $\mathcal{A}_*(G)$ is part of a natural pre-exact sequence.

Proposition 4.5. *Let $G \subset U_n$ be a self-transpose compact subgroup. Then there exists a Hopf $*$-algebra embedding $\mathcal{R}(G/G \cap \mathbb{T}) \hookrightarrow \mathcal{A}_*(G)$ and a pre-exact sequence*

$$\mathbb{C} \to \mathcal{R}(G/G \cap \mathbb{T}) \xrightarrow{j} \mathcal{A}_*(G) \xrightarrow{q} \mathbb{C}\mathbb{Z}_2 \to \mathbb{C}$$

Proof. The map q is defined as the restriction to $\mathcal{A}_*(G)$ of the Hopf $*$-algebra map $\varepsilon \otimes \mathrm{id} : \mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2 \to \mathbb{C}\mathbb{Z}_2$. Hence we have $q(u_{ij}s) = \delta_{ij}s$. Let B be the subalgebra of $\mathcal{A}_*(G)$ generated by the elements $u_{ij}su_{kl}s = u_{ij}u_{kl}^*$. It is clear that $B = \mathcal{A}_*(G)^{coq}$, and hence we have a pre-exact sequence

$$\mathbb{C} \to B \xrightarrow{j} \mathcal{A}_*(G) \xrightarrow{q} \mathbb{C}\mathbb{Z}_2 \to \mathbb{C}$$

Consider now the injective Hopf algebra map $\nu : \mathcal{R}(G) \hookrightarrow \mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2$, $f \mapsto f \otimes 1$. Since $\mathcal{R}(G/G \cap \mathbb{T}) = \mathcal{R}(G)^{G \cap \mathbb{T}}$ is the subalgebra generated by the elements $u_{ij}u_{kl}^*$ (Lemma 2.6), we have $\nu(\mathcal{R}(G/\mathbb{T})) = B$, and we get our pre-exact sequence. $\square$

We will prove (Theorem 4.7) that a noncommutative half-commutative orthogonal Hopf algebra is isomorphic to $\mathcal{A}_*(G)$ for some compact group $G \subset U_n$. Before this we first prove that the morphism in Proposition 4.4 is an isomorphism $A_o^*(n) \simeq \mathcal{A}_*(U_n)$. This can be seen as a consequence of the forthcoming Theorem 4.7, but the proof is less technical while it already well enlights the main ideas.

Theorem 4.6. *We have a Hopf $*$-algebra isomorphism $A_o^*(n) \simeq \mathcal{A}_*(U_n)$.*

Proof. Let $\pi : A_o^*(n) \longrightarrow \mathcal{A}_*(U_n)$ be the Hopf $*$-algebra map from Proposition 4.4, defined by $\pi(v_{ij}) = u_{ij}s$. It induces a commutative diagram of Hopf algebra maps with pre-exact rows

$$
\begin{array}{ccccccccc}
\mathbb{C} & \longrightarrow & P_v A_o^*(n) & \xrightarrow{i} & A_o^*(n) & \xrightarrow{p} & \mathbb{C}\mathbb{Z}_2 & \longrightarrow & \mathbb{C} \\
& & \downarrow{\scriptstyle \pi_|} & & \downarrow{\scriptstyle \pi} & & \| & & \\
\mathbb{C} & \longrightarrow & \mathcal{R}(PU_n) & \xrightarrow{j} & \mathcal{A}_*(U_n) & \xrightarrow{q} & \mathbb{C}\mathbb{Z}_2 & \longrightarrow & \mathbb{C}
\end{array}
$$

where the sequence on the top row is the one of Lemma 3.6 and the sequence on the lower row is the one of Proposition 4.5. The standard presentation of $\mathcal{R}(PU_n)$ (Lemma 2.6) ensures the existence of a $*$-algebra map $\mathcal{R}(PU_n) \longrightarrow P_v A_o^*(n)$, $u_{ij}u_{kl}^* \mapsto v_{ij}v_{kl}$ which is clearly an inverse isomorphism for $\pi_|$. Thus we can invoke the short five lemma from [3] (Theorem 3.4) to conclude that π is an isomorphism. $\qquad\square$

Note that a precursor for the previous isomorphism $A_o^*(n) \simeq \mathcal{A}_*(U_n)$ was the matrix model $A_o^*(n) \hookrightarrow M_2(\mathcal{R}(U_n))$ found in [5], Section 8.

Theorem 4.7. *Let A be a noncommutative half-commutative orthogonal Hopf algebra. Then there exists a self-transpose doubly non-real compact group G with $\mathbb{T} \subset G \subset U_n$ such that $A \simeq \mathcal{A}_*(G)$.*

Proof. Let A be a noncommutative half-commutative orthogonal Hopf algebra. The proof is divided into two steps.

Step 1. In this preliminary step, we first write a convenient presentation for A. By Lemma 3.6 there exist surjective Hopf $*$-algebra maps

$$A_o^*(n) \xrightarrow{f} A \xrightarrow{p} \mathbb{C}\mathbb{Z}_2$$

with $pf(v_{ij}) = \delta_{ij}s$. We denote by V the comodule over $A_o^*(n)$ corresponding to the matrix $v = (v_{ij}) \in M_n(A_o^*(n))$, with its standard basis $e_1, \ldots, e_n$. To any linear map $\underline{\lambda} : \mathbb{C} \to V^{\otimes m}$,

$$\underline{\lambda}(1) = \sum_{i_1,\ldots,i_m} \lambda(i_1,\ldots,i_m)e_{i_1} \otimes \cdots \otimes e_{i_m}$$

we associate families $X(\underline{\lambda})$ and $X'(\underline{\lambda})$ of elements of $A_o^*(n)$ defined by

$$X(\underline{\lambda}) = \{ \sum_{j_1,\ldots,j_m} v_{i_1j_1} \cdots v_{i_mj_m}\lambda(j_1,\ldots,j_m) - \lambda(i_1,\ldots,i_m)1, \ i_1,\ldots,i_m \in \{1,\ldots,n\}\}$$

$$X'(\underline{\lambda}) = \{ \sum_{j_1,\ldots,j_m} v_{j_mi_m} \cdots v_{j_1i_1}\lambda(j_1,\ldots,j_m) - \lambda(i_1,\ldots,i_m)1, \ i_1,\ldots,i_m \in \{1,\ldots,n\}\}$$

These elements generate a $*$-ideal in $A_o^*(n)$, which is in fact a Hopf $*$-ideal, that we denote by $I_{\underline{\lambda}}$. We also view V as an A-comodule via f, and the map $\underline{\lambda}$ is a morphism of A-comodules if and only if $f(I_{\underline{\lambda}}) = 0$. Now given a family $\mathcal{C}$ of linear maps $\mathbb{C} \to V^{\otimes m}$, $m \in \mathbb{N}$, we denote by $I_{\mathcal{C}}$ the Hopf $*$-ideal of $A_o^*(n)$ generated by all the elements of $X(\underline{\lambda})$ and $X'(\underline{\lambda})$, $\underline{\lambda} \in \mathcal{C}$. It follows from Woronowicz Tannaka-Krein duality [23] that f induces an isomorphism $A_o^*(n)/I_{\mathcal{C}} \simeq A$ for a suitable set $\mathcal{C}$ of morphisms of A-comodules (typically $\mathcal{C}$ is a family of morphisms that generate the tensor category of corepresentations of A).

Step 2. We now construct a compact group G with $\mathbb{T} \subset G \subset U_n$. We start with a presentation $A_o^*(n)/I_{\mathcal{C}} \simeq A$ as in Step 1. Note that the existence of the map $p : A \longrightarrow \mathbb{C}\mathbb{Z}_2$ implies that for $\underline{\lambda} : \mathbb{C} \longrightarrow V^{\otimes m}$, if $\underline{\lambda} \neq 0$ and $\underline{\lambda} \in \mathcal{C}$, then m is even (evaluate p on the elements of $X(\underline{\lambda})$). We associate to $\underline{\lambda} : \mathbb{C} \longrightarrow V^{\otimes 2m} \in \mathcal{C}$ the following families of elements in $\mathcal{R}(U_n)$

$$X_1(\underline{\lambda}) = \{ \sum_{j_1,\ldots,j_{2m}} u_{i_1j_1}u_{i_2j_2}^* \cdots u_{i_{2m-1}j_{2m-1}}u_{i_{2m}j_{2m}}^*\lambda(j_1,\ldots,j_{2m}) - \lambda(i_1,\ldots,i_{2m})1,$$
$$i_1,\ldots,i_{2m} \in \{1,\ldots,n\}\}$$

$$X_1'(\underline{\lambda}) = \{ \sum_{j_1,\ldots,j_{2m}} u_{j_1i_1}^*u_{j_2i_2} \cdots u_{j_{2m-1}i_{2m-1}}^*u_{j_{2m}i_{2m}}\lambda(j_1,\ldots,j_{2m}) - \lambda(i_1,\ldots,i_{2m})1,$$
$$i_1,\ldots,i_{2m} \in \{1,\ldots,n\}\}$$

$$X_2(\underline{\lambda}) = \{ \sum_{j_1,\ldots,j_{2m}} u_{i_1j_1}^*u_{i_2j_2} \cdots u_{i_{2m-1}j_{2m-1}}^*u_{i_{2m}j_{2m}}\lambda(j_1,\ldots,j_{2m}) - \lambda(i_1,\ldots,i_{2m})1,$$
$$i_1,\ldots,i_{2m} \in \{1,\ldots,n\}\}$$

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$$X_2'(\underline{\lambda}) = \{ \sum_{j_1,\ldots,j_{2m}} u_{j_1 i_1} u_{j_2 i_2}^* \cdots u_{j_{2m-1} i_{2m-1}} u_{j_{2m} i_{2m}}^* \lambda(j_1,\ldots,j_{2m}) - \lambda(i_1,\ldots,i_{2m})1,$$

$$i_1,\ldots,i_{2m} \in \{1,\ldots,n\}\}$$

Now denote by $J_\mathcal{C}$ the $*$-ideal of $\mathcal{R}(U_n)$ generated by the elements of $X_1(\underline{\lambda})$, $X_1'(\underline{\lambda})$, $X_2(\underline{\lambda})$ and $X_2'(\underline{\lambda})$ for all the elements $\underline{\lambda} \in \mathcal{C}$. In fact $J_\mathcal{C}$ is a Hopf $*$-ideal and we define G to be the compact group $G \subset U_n$ such that $\mathcal{R}(G) \simeq \mathcal{R}(U_n)/J_\mathcal{C}$. The existence of a Hopf $*$-algebra map $\rho : \mathcal{R}(G) \longrightarrow \mathbb{C}\mathbb{Z}$, $u_{ij} \longmapsto \delta_{ij} t$, where t denotes a generator of $\mathbb{Z}$, is straightforward, and thus $\mathbb{T} \subset G$. Also it is easy to chek the existence of a Hopf $*$-algebra map $\mathcal{R}(G) \longrightarrow \mathcal{R}(G)$, $u_{ij} \longmapsto u_{ij}^*$, and this show that G is self-transpose. We have, by Proposition 4.4, a Hopf $*$-algebra map $\pi : A_o^*(n) \longrightarrow \mathcal{A}_*(G)$, $v_{ij} \longmapsto u_{ij}s$. It is a direct verification to check that π vanishes on $I_\mathcal{C}$, so induces a Hopf $*$-algebra map $\overline{\pi} : A \longrightarrow \mathcal{A}_*(G)$. We still denote by v_{ij} the element $f(v_{ij})$ in A. We get a commutative diagram with pre-exact rows

$$\begin{array}{ccccccccc} \mathbb{C} & \longrightarrow & P_v A & \overset{i}{\longrightarrow} & A & \overset{p}{\longrightarrow} & \mathbb{C}\mathbb{Z}_2 & \longrightarrow & \mathbb{C} \\ & & \downarrow{\overline{\pi}_|} & & \downarrow{\overline{\pi}} & & \| & & \\ \mathbb{C} & \longrightarrow & \mathcal{R}(G/\mathbb{T}) & \overset{j}{\longrightarrow} & \mathcal{A}_*(G) & \overset{q}{\longrightarrow} & \mathbb{C}\mathbb{Z}_2 & \longrightarrow & \mathbb{C} \end{array}$$

where the sequence on the top row is the one of Lemma 3.6 and the sequence on the lower row is the one of Proposition 4.5. To prove that $\overline{\pi}$ is an isomorphism, we just have, by the short five-lemma for cosemisimple Hopf algebra [3], to prove that $\overline{\pi}_| : P_v A \longrightarrow \mathcal{R}(G/\mathbb{T})$ is an isomorphism. Let $J_\mathcal{C}'$ be the $*$-ideal of $\mathcal{R}(PU_n)$ generated by the elements of $X_1(\underline{\lambda})$, $X_1'(\underline{\lambda})$, $X_2(\underline{\lambda})$ and $X_2(\underline{\lambda})$ for all the elements $\underline{\lambda} \in \mathcal{C}$. It is clear, using the $\mathbb{Z}$-grading on $\mathcal{R}(G)$ induced by the inclusion $\mathbb{T} \subset G$ and the fact that $J_\mathcal{C}$ is generated by elements of degree zero, that $J_\mathcal{C}' = J_\mathcal{C} \cap \mathcal{R}(PU_n)$, so $\mathcal{R}(G/\mathbb{T}) \simeq \mathcal{R}(PU_n)/J_\mathcal{C}'$. But then the natural $*$-algebra map $\mathcal{R}(PU_n) \longrightarrow P_v A$ (Lemma 2.6) vanishes on $J_\mathcal{C}'$, and hence induces a $*$-algebra map $\mathcal{R}(G/\mathbb{T}) \longrightarrow P_v A$, which is an inverse for $\overline{\pi}_|$. Hence $\overline{\pi}$ is an isomorphism, and the algebra A being noncommutative, it follows from Proposition 4.4 that G is doubly non-real. $\qquad\square$

Note that the proof of Theorem 4.7 also provides a method to find the compact group G from the half-commutative orthogonal Hopf algebra A.

Example 4.8. On can check, by following the proof of Theorem 4.7, that the hyperoctaedral Hopf algebra $A_h^*(n)$ is isomorphic to $\mathcal{A}_*(K_n)$, where K_n is the subgroup of U_n formed by matrices having exactly one non-zero element on each column and line (with $K_n \simeq \mathbb{T}^n \rtimes S_n$).

Remark 4.9. Let $H \subset G \subset U_n$ be self-transpose compact subgroups. The inclusion $H \subset G$ induces a surjective Hopf $*$-algebra map $\mathcal{A}_*(G) \to \mathcal{A}_*(H)$, compatible with the exact sequence in Proposition 4.5. Thus if the inclusion $H \subset G$ induces an isomorphism $H/H \cap \mathbb{T} \simeq G/G \cap \mathbb{T}$, the short five lemma ensures that $\mathcal{A}_*(G) \simeq \mathcal{A}_*(H)$. In particular $\mathcal{A}_*(U_n) \simeq \mathcal{A}_*(SU_n)$.

We now propose a tentative orthogonal half-liberation for the unitary group. In fact another possible half-liberation of U_n has already been proposed in [9], using the symbol $A_u^*(n)$. We shall use the notation $A_u^{**}(n)$ for the object we construct, which is different from the one in [9].

Example 4.10. Let $A_u^{**}(n)$ be the quotient of $A_u(n)$ by the ideal generated by the elements

$$abc - cba, \ a,b,c, \in \{u_{ij}, u_{ij}^*\}$$

Then $A_u^{**}(n)$ is isomorphic with $\mathcal{A}_*(U_{2,n})$, where $U_{2,n}$ is the subgroup of U_{2n} consisting of unitary matrices of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \ A, B \in M_n(\mathbb{C})$$

and hence is a half-commutative orthogonal Hopf algebra,

Proof. Let $\omega \in \mathbb{C}$ be a primitive 4th root of unity. We start with the probably well-known surjective Hopf $*$-algebra map

$$A_o(2n) \longrightarrow A_u(n)$$

$$x_{i,j}, x_{n+i,n+j} \longmapsto \frac{u_{ij} + u_{ij}^*}{2}, \ i, j \in \{1, \ldots, n\}$$

$$x_{n+i,j} \longmapsto \frac{u_{ij} - u_{ij}^*}{2\omega}, \ i, j \in \{1, \ldots, n\}$$

$$x_{i,n+j} \longmapsto \frac{u_{ij}^* - u_{ij}}{2\omega}, \ i, j \in \{1, \ldots, n\}$$

where $x_{i,j}$ denote the standard generators of $A_o(2n)$. It is clear that it induces a surjective Hopf $*$-algebra map $A_o^*(2n) \longrightarrow A_u^{**}(n)$, and hence $A_u^{**}(n)$ is a half-commutative orthogonal Hopf algebra.

Let J be the ideal of $A_o^*(2n)$ generated by the elements

$$v_{i,j} - v_{n+i,n+j}, \ v_{n+i,j} + v_{i,n+j}, \ i, j \in \{1, \ldots, n\}$$

(where $v_{i,j}$ denotes the class of x_{ij} in $A_o^*(n)$). Then J is a Hopf $*$-ideal in $A_o^*(2n)$ and the previous Hopf $*$-algebra map induces an isomorphism $A_o^*(2n)/J \simeq A_u^{**}(n)$ (the inverse sends u_{ij} to $x_{ij} + \omega x_{n+i,j}$). Now having the presentation $A_o^*(2n)/J \simeq A_u^{**}(n)$, the proof of Theorem 4.7 yields $A_u^{**}(n) \simeq \mathcal{A}_*(U_{2,n})$. $\qquad\square$

5. Representation theory

In this section we describe the fusion rules of $\mathcal{A}_*(G)$ for any compact group G (as usual by fusion rules we mean the set of isomorphism classes of simple comodules together with the decomposition of tensor products of simple comodules into simple constituents). Thanks to Theorem 4.7, this gives a description of the fusion rules of any half-commutative orthogonal Hopf algebra.

If A is a cosemisimple Hopf algebra, we denote by $\mathrm{Irr}(A)$ the set of simple (irreducible) comodules over A. If $A = \mathcal{R}(G)$ for some compact group, then $\mathrm{Irr}(\mathcal{R}(G)) = \mathrm{Irr}(G)$, the set of isomorphism classes of irreducible representations of G. By a slight abuse of notation, for a simple A-comodule V, we write $V \in \mathrm{Irr}(A)$.

Let $G \subset U_n$ be a self-transpose compact subgroup. Recall that the transposition induces an involutive compact group automorphism

$$\sigma : G \longrightarrow G$$

$$g \longmapsto (g^t)^{-1} = \bar{g}$$

For $V \in \mathrm{Irr}(G)$, we denote by V^σ the (irreducible) representation of G induced by the composition with σ. If U is the fundamental n-dimensional representation of G, then $U^\sigma \simeq \overline{U}$.

We begin by recalling the description of the fusion rules for the crossed product $\mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2$. This is certainly well-known (see e.g. [19], Theorem 3.7)).

Proposition 5.1. *Let $G \subset U_n$ be a self-transpose compact subgroup. Then there is a bijection*

$$\mathrm{Irr}(\mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2) \simeq \mathrm{Irr}(G) \amalg \mathrm{Irr}(G)$$

More precisely, if $X \in \mathrm{Irr}(\mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2)$, then there exists a unique $V \in \mathrm{Irr}(G)$ with either $X \simeq V$ or $X \simeq V \otimes s$. For $V, W \in \mathrm{Irr}(G)$, we have

$$V \otimes (W \otimes s) \simeq (V \otimes W) \otimes s, \ (V \otimes s) \otimes W \simeq (V \otimes W^\sigma) \otimes s, \ (V \otimes s) \otimes (W \otimes s) \simeq V \otimes W^\sigma$$

Proof. The description of the simple comodules follows in a straightforward manner from the fact that $\mathcal{R}(G) \rtimes \mathbb{C}\mathbb{Z}_2 = \mathcal{R}(G) \otimes \mathbb{C}\mathbb{Z}_2$ as coalgebras. The tensor product decompositions are obtained by using character theory, see [22] or [14]. $\qquad\square$

Remark 5.2. If $G \subset U_n$ is connected and has a maximal torus T of G contained in $\mathbb{T}^n$, it follows from highest weight theory that $V^\sigma \simeq \overline{V}$ for any $V \in \mathrm{Irr}(G)$. We do not know if this is still true without these assumptions.

To express the fusion rules of $\mathcal{A}_*(G)$, we need more notation. Let $G \subset U_n$ be a compact subgroup, and denote by U the fundamental n-dimensional representation of G. For $m \in \mathbb{Z}$, we put

$$\mathrm{Irr}(G)_{[m]} = \{V \in \mathrm{Irr}(G),\ V \subset U^{\otimes m} \otimes (U \otimes \overline{U})^{\otimes l},\ \text{for some } l \in \mathbb{N}\}$$

where $U^{\otimes 0} = \mathbb{C}$ and for $m < 0$ $U^{\otimes m} = \overline{U}^{\otimes -m}$.

Now if $V \in \mathrm{Irr}(G)_{[0]}$, then $V \in \mathrm{Irr}(G/G \cap \mathbb{T})$ (see Lemma 2.6), and since $\mathcal{R}(G/G \cap \mathbb{T}) \subset \mathcal{A}_*(G)$, we get an element in $\mathrm{Irr}(\mathcal{A}_*(G))$, still denoted V.

If $V \in \mathrm{Irr}(G)_{[1]}$, then $V \subset U \otimes (U \otimes \overline{U})^{\otimes l}$, for some $l \in \mathbb{N}$, and hence the coefficients of $V \otimes s$ belong to $\mathcal{A}_*(G)$. Thus we get an element of $\mathrm{Irr}(\mathcal{A}_*(G))$, denoted Vs.

Corollary 5.3. *Let $G \subset U_n$ be a self-transpose compact subgroup. Then the map*

$$\mathrm{Irr}(G)_{[0]} \amalg \mathrm{Irr}(G)_{[1]} \longrightarrow \mathrm{Irr}(\mathcal{A}_*(G))$$

$$V \longmapsto \begin{cases} V & \text{if } V \in \mathrm{Irr}(G)_{[0]} \\ Vs & \text{if } V \in \mathrm{Irr}(G)_{[1]} \end{cases}$$

is a bijection. Moreover, for $V \in \mathrm{Irr}(G)_{[0]}$, $W, W' \in \mathrm{Irr}(G)_{[1]}$, we have

$$V \otimes Ws \simeq (V \otimes W)s,\ Ws \otimes V \simeq (W \otimes V^\sigma)s,\ Ws \otimes W's \simeq W \otimes W'^\sigma,\ \overline{Ws} \simeq \overline{W}^\sigma s$$

Proof. The existence of the map follows from the discussion before the corollary, while injectivity comes from Proposition 5.1. Note that for $V \in \mathrm{Irr}(G)_{[m]}$, $V' \in \mathrm{Irr}(G)_{[m']}$, the simple constituents of $V \otimes V'$ all belong to $\mathrm{Irr}(G)_{[m+m']}$, and that $V^\sigma \in \mathrm{Irr}(G)_{[-m]}$. So the isomorphisms in the statement (that all come from the isomorphisms of Proposition 5.1) yield decompositions into simple $\mathcal{A}_*(G)$-comodules. Thus we have a family of simple $\mathcal{A}_*(G)$-comodules, stable under decompositions of tensor products and conjugation, and that contains the fundamental comodule Us: we conclude (e.g. from the orthogonality relations [22], [14]) that we have all the simple comodules. $\qquad \square$

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