

# Lazy cohomology: an analogue of the Schur multiplier for arbitrary Hopf algebras

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## Abstract

We propose a detailed systematic study of a group  $H_L^2(A)$  associated, by elementary means of lazy 2-cocycles, to any Hopf algebra  $A$ . This group was introduced by Schauenburg in order to generalize G.I. Kac's exact sequence. We study the various interplays of lazy cohomology in Hopf algebra theory: Galois and biGalois objects, Brauer groups and projective representations. We obtain a Kac-Schauenburg-type sequence for double crossed products of possibly infinite-dimensional Hopf algebras. Finally the explicit computation of  $H_L^2(A)$  for monomial Hopf algebras and for a class of cotriangular Hopf algebras is performed.

**Key words:** Hopf 2-cocycle, Galois objects, biGalois objects.

## Introduction

In 1968 Sweedler defined in [32] the cohomology for a cocommutative Hopf algebra  $H$  with coefficients in a module algebra  $A$ , relating it to the Brauer group of a field and to well known cohomology theories such as Lie algebra cohomology, group cohomology, Galois cohomology. For instance, he showed that when the algebra  $A$  is commutative, there is a bijection between the second cohomology group and the equivalence classes of  $H$ -cleft extensions of  $A$ .

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After this important achievement, several results have been obtained in the direction of providing a cohomological interpretation of many natural constructions arising in Hopf algebra theory in the non-cocommutative case. For instance, the theory of cleft Hopf-Galois extensions can be described in terms of 2-cocycles up to coboundaries ([5], [12], [13]). However, the words “cocycle” and “coboundary” do not really correspond to a given complex. They are used just because they satisfy equations that look like Sweedler’s conditions and coincide with it in the cocommutative case.

The basic problem with Hopf 2-cocycles is that the convolution product of two 2-cocycles is no longer a 2-cocycle in general. This problem disappears when we deal with what we call lazy 2-cocycles, i.e. those cocycles that are convolution commuting with the product of  $A$ . This fact was already observed in [10]. Considering only lazy cocycles one can associate a group  $H_L^2(A)$  to any Hopf algebra  $A$  with  $H_L^2(k[G]) = H^2(G, k)$  in the case of a group algebra, so it might reasonably be seen as an analogue of the Schur multiplier for Hopf algebras. The group  $H_L^2(A)$  was introduced by Schauenburg in [30] (under the notation  $H_c^2(A)$ ) in his generalization of G. I. Kac’s exact sequence [18]. The Kac exact sequence is a useful tool for computing Hopf algebras extensions by group algebras.

In this paper we propose a detailed systematic study of the group  $H_L^2(A)$ , that we call the (second) lazy cohomology group of  $A$ . Apart from the Kac-Schauenburg exact sequence, of which we give a generalization in the case of double crossed product Hopf algebras, to possibly infinite-dimensional Hopf algebras, the motivation for this study has several origins, one of which is the study of the biGalois group of  $A$ . This group, denoted by  $\text{BiGal}(A)$ , might be defined as the group of isomorphism classes of linear monoidal auto-equivalences of the category of  $A$ -comodules. According to Schauenburg [28], this group might also be described as the set of isomorphism classes of  $A$ - $A$ -biGalois extensions endowed with the cotensor product (we assume for simplicity in this introduction that the base ring is a field). Schauenburg’s description is certainly the most efficient one for concrete computations of  $\text{BiGal}(A)$ . The simplest example is when  $A = k[G]$  is a group algebra and we have  $\text{BiGal}(k[G]) = \text{Aut}(G) \rtimes H^2(G, k)$ . However it is in general a difficult task to give an explicit description of the biGalois group, even when  $\text{Gal}(A)$ , the set of isomorphism classes of right  $A$ -Galois objects, has been determined. For example Schauenburg [29] has described the set  $\text{Gal}(A \otimes B)$  for a Hopf algebra tensor product using  $\text{Gal}(A)$  and  $\text{Gal}(B)$  but a similar description at the biGalois group level seems to be still unknown. Schauenburg’s computation of the biGalois groups of the generalized Taft algebras  $H_{N,m}$  with  $m$  grouplike also shows that the complete description of the biGalois group is a more delicate problem, in general, than the description of the Galois objects. Some simplifications of Schauenburg’s formula for  $H_{N,m}$ , at least from a theoretical viewpoint, are given in [4] but there is certainly still more to be understood.

The group  $H_L^2(A)$  is studied in this perspective. It may be realized as a normal subgroup of  $\text{BiGal}(A)$ , hence it might serve as a first approximation to understand the structure of  $\text{BiGal}(A)$ . From the monoidal categories viewpoint, it is the subgroup of  $\text{BiGal}(A)$  consisting of isomorphism classes of linear monoidal auto-equivalences of the category of  $A$ -comodules that are isomorphic, as functors, to the identity functor. The technical simplification is that once  $\text{Gal}(A)$  has been determined,  $H_L^2(A)$  is much easier to describe than  $\text{BiGal}(A)$ . A group morphism from  $\text{CoOut}(A) \times H_L^2(A)$  to  $\text{BiGal}(A)$  is constructed (this morphism is an isomorphism when  $A$  is cocommutative [28]), and the corresponding exact sequence is described. Coming back to Hopf tensor products, a Schur-Yamazaki type formula, derived from the generalized Kac-Schauenburg exact sequence, is given, describing  $H_L^2(A \otimes B)$  from  $H_L^2(A)$ ,  $H_L^2(B)$  and the group of central  $A \otimes B$  pairings. This formula generalizes the classical one (see e.g. [19]) describing the second cohomology group of a direct product of groups.

Another motivation for the study of lazy cohomology comes from the theory of Brauer groups of a Hopf algebra. Although a systematic description was not given yet, lazy cocycles were used extensively for the computation of the Brauer groups  $BC(k, H, r)$  (see [6] for the construction of  $BC$ ) of the families of coquasitriangular Hopf algebras  $E(n)$  and  $H_\nu$  (see [7] where the results in [35] are generalized, [8], [9]). According to the known examples, we expect that the lazy cohomology group will occur as a subquotient of the Brauer group of any cotriangular Hopf algebra.

Finally, although there exists a nice cohomology theory for Hopf algebras, with coefficients into Hopf bimodules (see [33] and the references therein), there is not in this framework a cohomology with coefficients in the multiplicative group of the field, which is used for example in projective representation theory of groups. Our group  $H_L^2(A)$  enables us to construct a monoidal  $H_L^2(A)$ -category of projective representations of  $A$ . Such categorical structures were considered by Turaev in homotopy quantum field theory ([34]).

This paper is organized as follows. The first Section is devoted to the direct elementary construction of  $H_L^2(A)$  using lazy 2-cocycles, and to other preliminary constructions. Section 2 contains the first few direct computations with explicit cocycles for Sweedler's Hopf algebra  $H_4$  and for  $E(n)$ . The setup of Section 3 is the very classical link between Hopf-Galois extensions and 2-cocycles: the Galois objects corresponding to lazy cocycles are characterized by a special symmetry property. The lazy cohomology group is then embedded as a normal subgroup in the group of BiGalois objects and it is shown to be cocycle twist-invariant. In Section 4 we formulate and prove a Kac-Schauenburg-type sequence for double crossed product Hopf algebras, generalizing it to the possibly infinite-dimensional case. We derive from it a Schur-Yamazaki-type formula so that the description of the lazy cohomology for the Drinfeld double of a (finite-dimensional) Hopf algebra is given as a corollary. Section 5 deals with the connection between lazy cohomology and universal  $R$ -forms

for coquasitriangular Hopf algebras with applications to the Brauer group of a Hopf algebra. In Section 6 we introduce a monoidal category of projective representations for Hopf algebras. This category has the structure of monoidal  $H_L^2(A)$ -category over the base field  $k$ . The last two sections deal with examples: in Section 7 the lazy cohomology group of a monomial Hopf algebra is computed. Section 8 contains the description of lazy cohomology for those finite-dimensional cotriangular Hopf algebras for which the linear action of the group datum in Andruskiewitsch, Etingof and Gelaki's terminology is faithful. An Appendix deals with relations between lazy cocycles and general crossed systems.

## Notation and conventions

Unless otherwise stated  $k$  is a commutative ring. Unadorned tensor products will be over  $k$ . The group of invertible elements in  $k$  will be denoted by  $k^\cdot$ . For any pair of  $k$ -modules  $V$  and  $W$ , the usual flip map  $V \otimes W \rightarrow W \otimes V$  interchanging the tensorands will be denoted by  $\tau$ . For a Hopf algebra over  $k$  the product will be denoted by  $m$ , the coproduct by  $\Delta$  and the antipode by  $S$ . We adopt a Sweedler's like notation e.g.:  $\Delta(a) = a_1 \otimes a_2$ .

## 1 Lazy cocycles

Let  $A$  be a Hopf algebra. In this section we give the detailed elementary direct construction, using lazy cocycles, of the group  $H_L^2(A)$  and some further preliminary constructions.

We first recall some classical notions. The set of convolution invertible linear maps  $\mu : A \rightarrow k$  satisfying  $\mu(1) = 1$  is denoted by  $\text{Reg}^1(A)$ . Similarly the set of convolution invertible linear maps  $\sigma : A \otimes A \rightarrow k$  satisfying  $\sigma(a, 1) = \varepsilon(a) = \sigma(1, a)$ , for all  $a \in A$ , is denoted by  $\text{Reg}^2(A)$ . It is clear that both  $\text{Reg}^1(A)$  and  $\text{Reg}^2(A)$  are groups under the convolution product.

**Definition 1.1** *An element  $\mu \in \text{Reg}^1(A)$  is said to be **lazy** if  $\mu * \text{id}_A = \text{id}_A * \mu$ . The set of lazy elements of  $\text{Reg}^1(A)$ , denoted  $\text{Reg}_L^1(A)$ , is a central subgroup of  $\text{Reg}^1(A)$ .*

*An element  $\sigma \in \text{Reg}^2(A)$  is said to be **lazy** if  $\sigma * m = m * \sigma$ , which is to say that*

$$\sigma(a_1, b_1)a_2b_2 = \sigma(a_2, b_2)a_1b_1, \quad \forall a, b \in A. \quad (1.1)$$

*The set of lazy elements of  $\text{Reg}^2(A)$ , denoted  $\text{Reg}_L^2(A)$ , is a subgroup of  $\text{Reg}^2(A)$ .*

If  $A$  is cocommutative then  $\text{Reg}_L^q(A) = \text{Reg}^q(A)$  for  $q = 1, 2$ .

We recall that a **left 2-cocycle** over  $A$  is an element  $\sigma \in \text{Reg}^2(A)$  such that

$$\sigma(a_1, b_1)\sigma(a_2b_2, c) = \sigma(b_1, c_1)\sigma(a, b_2c_2), \quad \forall a, b, c \in A.$$

We shall denote by  $Z^2(A)$  the set of left 2-cocycles and by  $Z_L^2(A)$  the set  $Z^2(A) \cap \text{Reg}_L^2(A)$  of **lazy 2-cocycles**.

Similarly a **right 2-cocycle** over  $A$  is an element  $\sigma \in \text{Reg}^2(A)$  such that

$$\sigma(a_1 b_1, c) \sigma(a_2, b_2) = \sigma(a, b_1 c_1) \sigma(b_2, c_2), \quad \forall a, b, c \in A.$$

It is well known that  $\sigma \in \text{Reg}^2(A)$  is a left 2-cocycle if and only if  $\sigma^{-1}$  is a right 2-cocycle.

The basic problem with left (or right) 2-cocycles is that the convolution product of two left 2-cocycles is no longer a left 2-cocycle in general. This problem disappears when we deal with lazy 2-cocycles.

**Lemma 1.2** *Let  $\sigma, \omega \in Z^2(A)$ .*

1. *If  $\omega \in Z_L^2(A)$ , then  $\sigma * \omega \in Z^2(A)$ .*
2. *If  $\sigma \in Z_L^2(A)$  then  $\sigma$  is a right 2-cocycle and  $\sigma^{-1}$  is a left 2-cocycle.*

*In particular,  $Z_L^2(A)$  is a subgroup of  $\text{Reg}_L^2(A)$ .*

**Proof:** The proof is by straightforward computation, both statements were already observed in [10, Pages 227-228].  $\square$

The terminology *lazy* for 2-cocycles is motivated by the fact that a lazy 2-cocycle does not alter the Hopf algebra  $A$  through Doi's twisting procedure, i.e., if  $\sigma$  is a lazy 2-cocycle,  $\text{id}_A$  is a Hopf algebra isomorphism between  $A$  and the Hopf algebra  ${}_\sigma A_{\sigma^{-1}}$  that has the underlying coalgebra as  $A$  and product given by

$$a \cdot_\sigma b = \sigma(a_1, b_1) a_2 b_2 \sigma^{-1}(a_3, b_3). \quad (1.2)$$

Similarly, given  $\gamma \in \text{Reg}^1(A)$ , the measuring  $\text{ad}(\gamma): A \rightarrow \text{End}(A)$  given by  $\text{ad}(\gamma)(a) = \gamma^{-1}(a_1) a_2 \gamma(a_3)$  is trivial if and only if  $\gamma$  is lazy.

**Example 1.3** It is well known that if  $A$  is coquasitriangular with universal  $r$ -form  $r$ , then  $r$  is a left 2-cocycle for  $A$ . The form  $r$  is lazy if and only if  $A$  is commutative.

**Example 1.4** Let  $A$  be a coquasitriangular Hopf algebra and let  $r, s$  lie in the set of coquasitriangular structures  $\mathcal{U}$  of  $A$ . Then  $(r \circ \tau) * s \in Z_L^2(A)$ . Indeed for  $a, b, c \in A$  we have:

$$\begin{aligned} & r(a_1, b_1) s(b_2, a_2) r(b_3 a_3, c_1) s(c_2, b_4 a_4) \\ &= r(a_1, b_1) r(a_2 b_2, c_1) s(b_3, a_3) s(c_2, b_4 a_4). \end{aligned}$$

Since universal  $r$ -forms are 2-cocycles the above is equal to:

$$\begin{aligned} & r(b_1, c_1) r(a_1, b_2 c_2) s(c_3, b_3) s(c_4 b_4, a_2) \\ &= r(b_1, c_1) r(a_1, c_3 b_3) s(c_2, b_2) s(c_4 b_4, a_2) \\ &= (r\tau * s)(c_1, b_1) (r\tau * s)(c_2 b_2, a) \end{aligned}$$

where we used again that  $s \in \mathcal{U}$ . Hence  $(r \circ \tau) * s$  is a 2-cocycle. It is not hard to see that  $(r \circ \tau) * s$  satisfies condition (1.1). It follows that  $Z_L^2(A)$  is nontrivial for coquasitriangular Hopf algebras which are not cotriangular.

**Remark 1.5** By part 1 of Lemma 1.2, if  $\sigma$  is any 2-cocycle and  $\omega \in Z_L^2(A)$  then  $\sigma * \omega$  is a 2-cocycle. Similarly if  $\sigma$  is a right 2-cocycle and  $\omega \in Z_L^2(A)$  then  $\omega * \sigma$  is a right 2-cocycle. Hence  $Z_L^2(A)$  acts on the right on  $Z^2(A)$  by  $\sigma \mapsto \sigma * \omega$ , as it was observed in [10, Page 228]. More generally, one can define a right action of  $Z_L^2(A)$  on the set of general crossed systems. This will be done in the Appendix.

The group  $H_L^2(A)$  will be defined as a quotient group of  $Z_L^2(A)$ . For this we need a “differential” connecting  $\text{Reg}_L^1(A)$  and  $\text{Reg}_L^2(A)$ . In order to do so, we simply have to mimic what happens in the cocommutative case (see [32]).

The map

$$\partial : \text{Reg}^1(A) \longrightarrow \text{Reg}^2(A)$$

is defined by  $\partial(\mu) = (\mu \otimes \mu) * (\mu^{-1} \circ m)$  for  $\mu \in \text{Reg}^2(A)$ , that is:  $\partial(\mu)(a, b) = \mu(a_1)\mu(b_1)\mu^{-1}(a_2b_2)$  for  $a, b \in A$ . Some basic properties of the operator  $\partial$  are given in the following lemma.

**Lemma 1.6** *Let  $\mu, \phi \in \text{Reg}^1(A)$ .*

1.  $\partial(\mu) = \varepsilon \otimes \varepsilon$  if and only if  $\mu \in \text{Alg}(A, k)$ .
2. If  $\mu \in \text{Reg}_L^1(A)$ , then  $\partial(\mu) \in \text{Reg}_L^2(A)$ .
3.  $\partial(\mu * \phi) = (\mu \otimes \mu) * \partial(\phi) * (\mu^{-1} \circ m)$ .
4. If  $\mu \in \text{Reg}_L^1(A)$ , then  $\partial(\mu * \phi) = \partial(\phi) * \partial(\mu)$ .
5. If  $\mu \in \text{Reg}_L^1(A)$ , then for  $\sigma \in \text{Reg}_L^2(A)$ , we have  $\partial(\mu) * \sigma = \sigma * \partial(\mu)$ .
6. If  $\partial(\phi) \in \text{Reg}_L^2(A)$ , then  $\partial(\mu * \phi) = \partial(\mu) * \partial(\phi)$ .
7.  $\partial(\mu)$  is a left 2-cocycle.

*In particular the map  $\partial$  induces a group morphism  $\text{Reg}_L^1(A) \rightarrow Z_L^2(A)$  with image contained in the center of  $Z_L^2(A)$ .*

**Proof:** The proof of all statements follows by direct computation and we leave it to the reader. □

Lemma 1.6 leads to the following definition:

**Definition 1.7** Let  $A$  be a Hopf algebra. The lazy cohomology groups  $H_L^1(A)$  and  $H_L^2(A)$  are defined in the following way:

$$H_L^1(A) := \text{Ker}(\partial_{|\text{Reg}_L^1(A)}) = \{\mu \in \text{Alg}(A, k) \mid \mu * \text{id}_A = \text{id}_A * \mu\},$$

$$H_L^2(A) := Z_L^2(A)/B_L^2(A),$$

where  $B_L^2(A)$  is the central subgroup  $\partial(\text{Reg}_L^1(A))$  of  $Z_L^2(A)$ .

Let us observe that, according to Lemma 1.6, lazy cocycles belonging to the same class in  $H_L^2(A)$  are cohomologous in the the sense of [12].

The group  $H_L^1(A)$  is obviously commutative. Although all the examples of  $H_L^2(A)$  computed here are commutative, we see no reason why this group should be commutative in general.

**Remark 1.8** Schauenburg has defined more generally the lazy cohomology group, denoted by  $\mathcal{H}_c^2(H)$  where  $c$  stands for central, of a coquasibialgebra  $H$  in [30, Section 6]. This construction is in fact very natural in view of the monoidal category interpretation of lazy cohomology as given further in Remark 3.12.

**Remark 1.9** We could also define an abelian analogue of the Schur multiplier in the following way. Let us say that a 2-cocycle  $\sigma$  is absolutely central if

$$\sigma(a_1, b_1)a_2 \otimes b_2 = \sigma(a_2, b_2)a_1 \otimes b_1, \quad \forall a, b \in A.$$

The absolutely central 2-cocycles clearly form a central subgroup of  $\text{Reg}_L^2(A)$ , denoted by  $Z_{\text{ab}}^2(A)$ . We do not necessarily have  $\partial(\text{Reg}_L^1(A)) \subset Z_{\text{ab}}^2(A)$  but we still can define an abelian group  $H_{\text{ab}}^2(A)$  as the subgroup of  $H_L^2(A)$  generated by the classes of absolutely central 2-cocycles. We focus on the present  $H_L^2(A)$  for several reasons: absolute centrality seems to be very restrictive and already for Sweedler's Hopf algebra the set of absolutely central 2-cocycles is trivial; we wanted our analogue of the Schur multiplier to be the biggest possible as a subgroup of  $\text{BiGal}(A)$ ; there is a nice property corresponding to the notion of lazy cocycle at the Hopf-Galois extension level (see Section 3) while we have not found such a property for absolutely central 2-cocycles; lazy cocycles turned out to be a useful tool in the computation of the Brauer group of a coquasitriangular Hopf algebra. From the known examples (see [7], [8] and [9]) we expect that lazy cohomology would occur as a subquotient of the Brauer group of any cotriangular Hopf algebra.

**Remark 1.10** It is possible to generalize the notion of laziness to higher degree cochains and to adopt a suitable form of Sweedler's operators  $D^q$ . In particular, one may view the lazy 2-cocycles as those cochains  $\omega$  for which a suitable variation of  $D^2$  gives  $D^2\omega = \varepsilon^{\otimes 3}$ . However, even if for degree 3 we still have the inclusion  $B_L^3(A) \subset Z_L^3(A)$ , at a first sight there seems to be no natural group structure on  $Z_L^3(A)$  nor on the quotient  $H_L^3(A)$ .

We have seen that  $\partial(\text{Reg}_L^1(A)) \subset Z_L^2(A)$ . However, it might happen that if  $\gamma \in \text{Reg}^1(A)$  and  $\gamma \notin \text{Reg}_L^1(A)$  still  $\partial(\gamma) \in Z_L^2(A)$ . We shall analyze such elements  $\gamma$ .

Let  $\gamma \in \text{Reg}^1(A)$ , let  $\sigma \in Z^2(A)$ , and let  $\sigma^\gamma$  be the 2-cocycle, cohomologous to  $\sigma$  in the usual sense, defined by:  $\sigma^\gamma := (\gamma \otimes \gamma) * \sigma * (\gamma^{-1} \circ m)$ . It is well known that cohomologous cocycles yield isomorphic Hopf algebra twists  ${}_\sigma A_{\sigma^{-1}}$  and  ${}_{\sigma^\gamma} A_{(\sigma^\gamma)^{-1}}$  with isomorphism  $\text{ad}(\gamma): {}_\sigma A_{\sigma^{-1}} \rightarrow {}_{\sigma^\gamma} A_{(\sigma^\gamma)^{-1}}$  given by  $\gamma^{-1} * \text{id}_A * \gamma$ . In particular, if  $\sigma = \varepsilon \otimes \varepsilon$  then  $\text{ad}(\gamma)$  is an Hopf algebra isomorphism  $A \rightarrow {}_{\partial(\gamma)} A_{(\partial(\gamma))^{-1}}$ . Therefore, if  $\partial(\gamma) \in Z_L^2(A)$ ,  $\text{ad}(\gamma) \in \text{Aut}_{\text{Hopf}}(A)$ , the group of Hopf algebra automorphisms of  $A$ . In fact, condition (1.1) for  $\partial(\gamma)$  is equivalent to the requirement that  $\text{ad}(\gamma)$  is an Hopf algebra automorphism. We have proved the following

**Lemma 1.11** *A coboundary  $\partial(\gamma) \in \text{Reg}_L^2(A)$  if and only if  $\text{ad}(\gamma)$  is an Hopf algebra automorphism.  $\square$*

We call an Hopf automorphism of type  $\text{ad}(\gamma)$  a **cointernal** automorphism. We shall denote the set of cointernal automorphisms of  $A$  by  $\text{CoInt}(A)$ .

**Lemma 1.12** *The following properties hold:*

1.  $\text{ad}: \text{Reg}^1(A) \rightarrow \text{Aut}_{\text{coalg}}(A)$  is a group morphism with kernel  $\text{Reg}_L^1(A)$ .

2. The set

$$\begin{aligned} \text{Reg}_{aL}^1(A) : &= \{\gamma \in \text{Reg}^1(A) \mid \partial(\gamma) \in \text{Reg}_L^2(A)\} \\ &= \{\gamma \in \text{Reg}^1(A) \mid \text{ad}(\gamma) \in \text{CoInt}(A)\} \\ &= \text{ad}^{-1}(\text{Aut}_{\text{Hopf}}(A)) \end{aligned}$$

is a subgroup of  $\text{Reg}^1(A)$ . An element of  $\text{Reg}_{aL}^1(A)$  is called almost lazy.

3.  $\text{CoInt}(A)$  is a normal subgroup of  $\text{Aut}_{\text{Hopf}}(A)$ .

4. The normal subgroup  $\text{CoInn}(A)$  of  $\text{Aut}_{\text{Hopf}}(A)$  given by:

$$\text{CoInn}(A) = \{f \in \text{Aut}_{\text{Hopf}}(A) \mid \exists \phi \in \text{Alg}(A, k) \text{ with } f = (\phi \circ S) * \text{id}_A * \phi\}$$

is contained in  $\text{CoInt}(A)$ .

5.  $\partial: \text{Reg}_{aL}^1(A) \rightarrow Z_L^2(A)$  is a group morphism and its kernel is  $\text{Alg}(A, k)$ .

**Proof:** It is well known that  $\text{ad}(\gamma * \theta) = \text{ad}(\gamma) \circ \text{ad}(\theta)$  for every  $\gamma$  and  $\theta \in \text{Reg}^1(A)$ . This implies most of the statements. Statement 5 follows from part 6 of Lemma 1.6.  $\square$

The knowledge of  $\partial(\text{Reg}_{aL}^1(A)) = B^2(A) \cap Z_L^2(A)$  helps us to detect when  $\text{CoInn}(A) = \text{CoInt}(A)$ .



**Lemma 1.13** *With notation as before,  $B^2(A) \cap \text{Reg}_L^2(A) = B_L^2(A)$  if and only if  $\text{CoInn}(A) = \text{CoInt}(A)$ .*

**Proof:** Let us suppose that  $B^2(A) \cap \text{Reg}_L^2(A) = B_L^2(A)$ . For every  $\gamma \in \text{Reg}_{aL}^1(A)$  there exists  $\chi \in \text{Reg}_L^1(A) = \text{Ker}(\text{ad})$  such that  $\partial(\gamma) = \partial(\chi)$ . Then  $\partial(\gamma * \chi^{-1}) = 1$ , i.e.,  $\gamma * \chi^{-1}$  is an algebra morphism  $A \rightarrow k$ . Then,

$$\text{ad}(\gamma) = \text{ad}(\gamma) \circ \text{ad}(\chi)^{-1} = \text{ad}(\gamma * \chi^{-1}) \in \text{CoInn}(A).$$

Let us now suppose that  $\text{CoInn}(A) = \text{CoInt}(A)$  and let  $\sigma$  be any element of  $\partial(\text{Reg}_{aL}^1(A))$ . Then  $\sigma = \partial(\gamma)$  for some  $\gamma \in \text{Reg}_{aL}^1(A)$  and  $\text{ad}(\gamma) = \text{ad}(\chi)$  for some algebra map  $A \rightarrow k$  so that  $\partial(\chi^{-1}) = \partial(\chi \circ S) = \varepsilon \otimes \varepsilon$  and  $\chi \circ S \in \text{Reg}_{aL}^1(A)$ .

Therefore,  $\text{ad}(\gamma * (\chi \circ S)) = \text{id}_A$  so that  $\gamma * (\chi \circ S) \in \text{Reg}_L^1(A)$  and

$$\sigma = \partial(\gamma) = \partial(\gamma) * \partial(\chi \circ S) = \partial(\gamma * (\chi \circ S)) \in B_L^2(A).$$

□

We define the following group:

**Definition 1.14** *Let  $A$  be a Hopf algebra. The group  $\text{CoOut}^-(A)$  is*

$$\text{CoOut}^-(A) := \text{Reg}_{aL}^1(A) / \text{ad}^{-1}(\text{CoInn}(A)) \cong \text{CoInt}(A) / \text{CoInn}(A).$$

By the second equality  $\text{CoOut}^-(A)$  can be viewed as a subgroup of  $\text{CoOut}(A) = \text{Aut}_{\text{Hopf}}(A) / \text{CoInn}(A)$ , the group of co-outer automorphisms of  $A$ . In Example 7.5 we shall consider a Hopf algebra for which  $\text{CoOut}^-(A)$  is not trivial.

We shall relate the groups  $\text{CoOut}^-(A)$  and  $H_L^2(A)$ . In order to do that, we define an action of  $\text{CoOut}(A)$  on  $H_L^2(A)$ .

Let  $\sigma \in \text{Reg}^2(A)$  and  $\alpha \in \text{Aut}_{\text{Hopf}}(A)$ . We put  $\sigma \leftarrow \alpha := \sigma \circ (\alpha \otimes \alpha)$ . It is clear that  $\sigma \leftarrow \alpha \in \text{Reg}^2(A)$ . The basic properties of this construction, recorded in the following lemma, enable us to construct a right action by automorphisms of  $\text{CoOut}(A)$  on  $H_L^2(A)$ .

**Lemma 1.15** *Let  $\alpha \in \text{Aut}_{\text{Hopf}}(A)$  and let  $\sigma \in \text{Reg}^2(A)$ .*

1. *If  $\sigma \in \text{Reg}_L^2(A)$ , then  $\sigma \leftarrow \alpha \in \text{Reg}_L^2(A)$ .*
2. *If  $\sigma$  is a left cocycle, then so is  $\sigma \leftarrow \alpha$ .*
3. *Let  $\omega \in \text{Reg}^2(A)$ . Then  $(\sigma * \omega) \leftarrow \alpha = (\sigma \leftarrow \alpha) * (\omega \leftarrow \alpha)$ .*
4. *Let  $\beta \in \text{Aut}_{\text{Hopf}}(A)$ . Then  $\sigma \leftarrow (\alpha \circ \beta) = (\sigma \leftarrow \alpha) \leftarrow \beta$ .*
5. *If  $\mu \in \text{Reg}_L^1(A)$ , then  $\mu \circ \alpha \in \text{Reg}_L^1(A)$ .*

6. For  $\mu \in \text{Reg}^1(A)$ , we have  $\partial(\mu) \leftarrow \alpha = \partial(\mu \circ \alpha)$ .

7. If  $\alpha \in \text{CoInn}(A)$  and  $\sigma \in \text{Reg}_L^2(A)$ , then  $\sigma \leftarrow \alpha = \sigma$ .

Hence the formula

$$\begin{aligned} H_L^2(A) \times \text{CoOut}(A) &\longrightarrow H_L^2(A) \\ (\overline{\sigma}, \overline{\alpha}) &\longmapsto \overline{\sigma \leftarrow \alpha} \end{aligned}$$

defines a right action by automorphisms of  $\text{CoOut}(A)$  on  $H_L^2(A)$ .

**Proof:** The proof of all statements follows by direct computation and we leave it to the reader.  $\square$

We can thus consider the semi-direct product group  $\text{CoOut}(A) \ltimes H_L^2(A)$ .

**Lemma 1.16** *The map*

$$\begin{aligned} i_0 : \text{Reg}_{aL}^1(A) &\longrightarrow \text{CoOut}(A) \times H_L^2(A) \\ \mu &\longmapsto (\overline{\text{ad}(\mu)}, \overline{\partial(\mu^{-1})}) \end{aligned}$$

induces an injective group morphism

$$\begin{aligned} \iota : \text{CoOut}^-(A) &\longrightarrow \text{CoOut}(A) \ltimes H_L^2(A) \\ \overline{\mu} &\longmapsto (\overline{\text{ad}(\mu)}, \overline{\partial(\mu^{-1})}). \end{aligned}$$

**Proof:** The map  $i_0$  is a group morphism if  $\partial(\lambda^{-1})\partial(\mu^{-1}) = (\partial(\mu^{-1}) \leftarrow \text{ad}(\lambda)) * \partial(\lambda^{-1})$  for every  $\mu, \lambda \in \text{Reg}_{aL}^1(A)$ . By Lemma 1.15,  $\partial(\mu^{-1}) \leftarrow \text{ad}(\lambda) = \partial(\mu^{-1} \circ \text{ad}(\lambda))$ . By Lemma 1.12 and Lemma 1.6,  $\partial(\mu^{-1} \circ \text{ad}(\lambda)) * \partial(\lambda^{-1}) = \partial((\mu^{-1} \circ \text{ad}(\lambda)) * \lambda^{-1})$ . Using that  $(\mu^{-1} \circ \text{ad}(\lambda)) * \lambda^{-1} = \lambda^{-1} * \mu^{-1}$ , we have that  $i_0$  is a group morphism.

An element  $\gamma$  lies in the kernel of  $i_0$  if and only if  $\gamma * \mu \in \text{Alg}(A, k)$  for some  $\mu \in \text{Reg}_L^1(A)$ . Therefore,  $\text{Ker}(i_0) = \text{ad}^{-1}(\text{CoInn}(A))$  and the statement is proved.  $\square$

## 2 Examples

This section contains, as first illustrative examples, the computation of  $H_L^2$  for Sweedler's Hopf algebra  $H_4$  and for  $E(n)$ . The computation here is based on the knowledge of explicit cocycles. These examples will be generalized in the last sections using more theoretical methods. Although  $H_4$  coincides with  $E(1)$ , we deal with it separately to have an easily manageable example throughout the paper. In this Section we assume that 2 is invertible in  $k$ .

**Example 2.1** Let  $H_4$  be Sweedler Hopf algebra, with generators  $g$  and  $x$ , relations

$$g^2 = 1, \quad x^2 = 0, \quad gx + xg = 0$$

and coproduct

$$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g.$$

It follows from centrality and self-duality of  $H_4$  that  $\text{Reg}_L^1(H_4)$  is trivial.

It is well known that  $\text{Aut}_{\text{Hopf}}(H_4) \cong k$  where  $t \in k$  corresponds to the automorphism  $\alpha_t$  satisfying  $g \mapsto g$ ,  $x \mapsto tx$ . Let  $\gamma \in \text{Reg}_{aL}^1(H_4)$ . Then  $\text{ad}(\gamma) = \alpha_t$  for some  $t \in k$ . The relations  $\text{ad}(\gamma)(x) = tx$  and  $\text{ad}(\gamma)(gx) = t gx$  imply that  $\gamma(x) = \gamma(gx) = 0$  and  $t = \gamma(g)\gamma(1)^{-1} = \gamma(1)\gamma^{-1}(g) = \pm 1$ . Hence, either  $t = 1$  and  $\gamma = \varepsilon$ , or  $t = -1$  and  $\gamma = (1^* - g^*)$ . The map  $1^* - g^*$  is an algebra morphism so  $\text{CoInn}(H_4) = \text{CoInt}(H_4) \cong \text{Reg}_L^1(A) \cong \mathbb{Z}_2$ . By Lemma 1.13

$$B_L^2(H_4) = \text{Reg}_L^2(H_4) \cap \partial(\text{Reg}_L^1(H_4)) = \partial(\text{Reg}_L^1(H_4)) = \{\varepsilon^{\otimes 2}\}.$$

By the classification in [24, Table] the lazy 2-cocycles are exactly those appearing in [7], i.e., those  $\sigma_t$  for  $t \in k$  such that:

$$\sigma_t(g, g) = 1,$$

$$\sigma_t(g, x) = \sigma_t(x, g) = \sigma_t(g, gx) = \sigma_t(gx, g) = 0$$

$$\sigma_t(x, x) = \sigma_t(gx, x) = -\sigma_t(x, gx) = -\sigma_t(gx, gx) = \frac{t}{2}.$$

It is straightforward to check that  $Z_L^2(H_4) \cong k$ . Therefore  $H_L^2(H_4) \cong k$ .

**Example 2.2** Let  $n$  be a positive integer and let  $E(n)$  be the Hopf algebra generated by  $c$  and  $x_i$  for  $1 \leq i \leq n$  with relations

$$c^2 = 1; \quad cx_i + x_i c = 0; \quad x_i x_j + x_j x_i = 0; \quad x_i^2 = 0$$

and coproduct

$$\Delta(c) = c \otimes c; \quad \Delta(x_i) = 1 \otimes x_i + x_i \otimes c.$$

The map  $\Phi: E(n) \rightarrow E(n)^*$  with  $\Phi(1) = \varepsilon$ ,  $\Phi(c) = 1^* - c^*$ ,  $\Phi(x_i) = x_i^* + (cx_i)^*$  and  $\Phi(cx_i) = x_i^* - (cx_i)^*$  for every  $i$  defines an Hopf algebra isomorphism. For a subset  $P = \{p_1, \dots, p_l\} \subset \{1, \dots, n\}$  we put  $|P| = l$  and  $x_P = x_{p_1} \cdots x_{p_l}$ . If  $n$  is odd, the centre of  $E(n)$  is the span of 1 and  $x_P$  for  $|P|$  even; if  $n$  is even the centre of  $E(n)$  is the span of 1,  $x_P$  for  $|P|$  even and  $cx_1 \cdots x_n$ .

It is not hard to check that  $\Phi(x_P) = (x_P)^* + (cx_P)^*$  and that  $\Phi(cx_1 \cdots x_n) = (x_1 \cdots x_n)^* - (cx_1 \cdots x_n)^*$ . Hence,  $\text{Reg}_L^1(E(n))$  consists of the linear combinations of  $\varepsilon$  (with coefficient 1) and elements of the above form.

By [26, Lemma 1]  $\text{Aut}_{\text{Hopf}}(E(n)) \cong \text{GL}_n(k)$ . The automorphisms of  $E(n)$  act trivially on  $c$  and as linear maps on  $x_1, \dots, x_n$ : for  $M \in \text{GL}_n(k)$  the corresponding

automorphism  $\alpha_M$  is such that  $\alpha_M(c) = c$  and  $\alpha_M(x_i) = \sum_j m_{ij}x_j$  for every  $i = 1, \dots, n$ . Let  $\gamma \in \text{Reg}_{aL}^1(E(n))$ . Then  $\text{ad}(\gamma) = \alpha_M$  for some  $M \in \text{GL}_n(k)$ . The computation of  $\text{ad}(\gamma)(x_i)$ ,  $\text{ad}(\gamma)(cx_i)$  and the relation  $\text{ad}(\gamma)(cx_i) = (\text{ad}(\gamma)(c))(\text{ad}(\gamma)(x_i))$  imply that  $\gamma(cx_i) = \gamma(x_i) = 0$  for every  $i$  and that  $M = \pm I_n$ . If  $M = I_n$ ,  $\gamma \in \text{Reg}_L^1(E(n))$ ; if  $M = -I_n$ ,  $\gamma \in (1^* - c^*) * \text{Reg}_L^1(E(n))$ . Since  $(1^* - c^*)$  is an algebra morphism,  $\text{CoInn}(E(n)) = \text{CoInt}(E(n)) \cong \mathbb{Z}_2$  and  $Z^2(E(n)) \cap \text{Reg}_L^2(E(n)) = B_L^2(E(n))$ .

Left 2-cocycles up to usual cohomology are classified in [27]. These classes are parametrized by 2-cocycles satisfying some recurrence relations and

$$\begin{aligned}\sigma(c, c) &= \alpha \in k; & \sigma(c, x_i) &= 0; \\ \sigma(x_i, c) &= \gamma_i; & \sigma(x_i, x_j) &= m_{ij}\end{aligned}$$

where  $M_{ij} \in T_n(k)$ , the vector space of lower-triangular  $n \times n$  matrices with entries in  $k$ . It was proved in [9, Lemma 3.2, Lemma 3.3] that these particular cocycles are lazy if and only if  $\alpha = 1$  and  $\gamma_i = 0$  for every  $i$  and that  $\alpha = 1$  and  $\gamma_i = 0$  for every  $i$  is also a necessary condition for all 2-cocycles to be lazy. Let  $\theta \in Z_L^2(E(n))$  and let  $l_{ij} = \theta(x_i, x_j)$ . If we consider the lazy cochain  $\gamma_\theta = \varepsilon + \sum_{i < j} l_{ij} \Phi(x_i x_j)$  it is not hard to verify that the lazy 2-cocycle  $\theta * \partial(\gamma_\theta)(x_i, x_j) = 0$  if  $i < j$ , while  $\theta * \partial(\gamma_\theta)(x_i, x_j) = \theta(x_i, x_j) + \theta(x_j, x_i)$  if  $i > j$  and  $\theta * \partial(\gamma_\theta)(x_i, x_i) = \theta(x_i, x_i)$ . In particular, if  $\theta$  is one of the cocycles constructed in [27] and it is lazy, then  $\gamma_\sigma = \varepsilon$ . The map

$$\begin{aligned}\Psi: Z_L^2(E(n)) &\longrightarrow T_n(k) \\ \sigma &\mapsto M_{ij} := (\sigma * \partial(\gamma_\sigma)(x_i, x_j))\end{aligned}$$

is surjective by [9, Lemma 3.3]. Besides, if  $\sigma, \omega \in Z_L^2(E(n))$  then

$$\sigma * \partial(\gamma_\sigma) * \omega * \partial(\gamma_\omega) = (\sigma * \omega) * \partial(\gamma_\sigma) * \partial(\gamma_\omega).$$

One has

$$\sigma * \omega(x_i, x_j) = \sigma(x_i, x_j) + \omega(x_i, x_j)$$

and

$$\partial(\gamma_\sigma) * \partial(\gamma_\omega)(x_i, x_j) = \partial(\gamma_\sigma)(x_i, x_j) + \partial(\gamma_\omega)(x_i, x_j)$$

and therefore  $\Psi$  is a group morphism  $\Psi: Z_L^2(E(n)) \rightarrow T_n(k)$ .

It is not hard to check that  $\theta \in \text{Ker}(\Psi)$  if and only if  $M = (\theta(x_i, x_j))$  is a skew-symmetric matrix, so that that  $\Psi(\partial(\gamma)) = 0$  for every  $\partial(\gamma) \in B_L^2(E(n))$ .

On the other hand, if  $\theta \in \text{Ker}(\Psi)$  then the 2-cocycle  $\omega := \theta * \partial(\gamma_\theta)$  coincides with  $\varepsilon \otimes \varepsilon$  when restricted to the two-fold tensor product of the span of  $1, c$  and  $x_1, \dots, x_n$ . Hence, the cleft extension  $k \sharp_\omega E(n)$  is generated by  $C, X_1, \dots, X_n$  with relations

$$C^2 = 1; \quad X_i X_j + X_j X_i = 0; \quad C X_i + X_i C = 0.$$

Since  $k\sharp_\omega E(n)$  is isomorphic as an  $E(n)$ -module algebra to  $E(n)$ , by [12, Theorem 2.2]  $\omega$  is a (usual) coboundary, and it is a lazy 2-cochain because  $\theta$  and  $\partial(\gamma_\theta)$  are so. Hence,  $\omega \in B^2(E(n)) \cap \text{Reg}_L^2(E(n)) = B_L^2(E(n))$  so  $\theta \in B_L^2(E(n))$ . Therefore  $\Psi$  defines a group isomorphism  $H_L^2(E(n)) \cong T_n(k)$ . If we denote by  $S^2(k^n)$  the group of  $n \times n$  symmetric matrices with entries in  $k$  we have  $H_L^2(E(n)) \cong S^2(k^n)$ .

This isomorphism will be generalized in Section 8 to a wider class of cotriangular Hopf algebras.

### 3 Lazy Galois objects

It is classical to associate Hopf-Galois extensions to 2-cocycles. We characterize the Galois objects associated to lazy 2-cocycles as Galois objects enjoying a special symmetry property: we will call them lazy Galois objects.

Let us first recall a few facts concerning Hopf-Galois extensions ([25]). Let  $A$  be a Hopf algebra. A **right  $A$ -Galois extension (of  $k$ )** is a non-zero right  $A$ -comodule algebra  $Z$  with  $Z^{\text{co}A} = k$  such that the linear map  $\kappa_r$  defined by the composition

$$\kappa_r : Z \otimes Z \xrightarrow{1_Z \otimes \rho} Z \otimes Z \otimes A \xrightarrow{m_Z \otimes 1_A} Z \otimes A$$

where  $\rho$  is the coaction of  $A$  and  $m_Z$  is the multiplication of  $Z$ , is bijective. We also say that a right  $A$ -Galois extension (of  $k$ ) is an  **$A$ -Galois object**. A morphism of  $A$ -Galois objects is an  $A$ -colinear algebra morphism. It is known that any morphism of  $A$ -Galois extensions that are faithfully flat as  $k$ -modules is an isomorphism ([31, Remark 3.11]). The set of isomorphism classes of  $k$ -faithfully flat  $A$ -Galois objects is denoted by  $\text{Gal}(A)$ . For example, when  $A = k[G]$  is a group algebra, we have  $\text{Gal}(k[G]) \cong H^2(G, k)$ . However  $\text{Gal}(A)$  does not carry a natural group structure in general.

Galois objects are classically associated with 2-cocycles as follows. Let  $\sigma$  be a left 2-cocycle. The right  $A$ -comodule algebra  ${}_\sigma A = k\sharp_\sigma A$  is defined in the following way. As a right  $A$ -comodule  ${}_\sigma A = A$  and the product of  ${}_\sigma A$  is defined to be

$$a_\sigma b = \sigma(a_1, b_1) a_2 b_2, \quad a, b \in A.$$

The right  $A$ -Galois objects constructed from 2-cocycles are characterized as the ones with the **normal basis property**, i.e. those that are isomorphic to  $A$  as  $A$ -comodules. The Galois objects with the normal basis property are also characterized as the **cleft** ones, see [25] for details and proofs of these statements. For future use, let us recall how a left 2-cocycle is constructed from a right Galois object having the normal basis property. Let  $Z$  be right  $A$ -Galois object with a right  $A$ -colinear isomorphism  $\psi : A \rightarrow Z$  with  $\psi(1) = 1$ . Then  $\sigma$  defined by  $\sigma(a, b) = \varepsilon(\psi^{-1}(\psi(a)\psi(b)))$ ,  $\forall a, b \in A$ , is a left 2-cocycle such that  $\psi : {}_\sigma A \rightarrow Z$  is an  $A$ -comodule algebra isomorphism.

Let us now introduce an additional property for Galois objects, that will be shown to be the Galois translation of laziness for 2-cocycles.

**Definition 3.1** *A right  $A$ -Galois object  $Z$  is said to be **lazy** if there exists a right  $A$ -colinear isomorphism  $\psi : A \rightarrow Z$  such that  $\psi(1) = 1$  and such that the morphism*

$$\beta_\psi := (\psi^{-1} \otimes \psi) \circ \rho : Z \rightarrow A \otimes Z$$

*is an algebra morphism. Such a map  $\psi$  is called a **symmetry morphism** for  $Z$ .*

The condition  $\psi(1) = 1$  is not a restriction. Given an isomorphism  $\theta : Z \rightarrow M$  of right  $A$ -Galois objects,  $Z$  is lazy with symmetry morphism  $\psi_Z$  if and only if  $M$  is lazy with symmetry morphism  $\psi_M = \theta \circ \psi_Z$ . The subset of  $\text{Gal}(A)$  consisting of isomorphism classes of lazy right  $A$ -Galois objects will be denoted by  $\text{Gal}^L(A)$ .

**Proposition 3.2** *Let  $Z$  be a right  $A$ -Galois object. Then the following assertions are equivalent.*

1.  $Z$  is a lazy right  $A$ -Galois object.
2. There exists  $\sigma \in Z_L^2(A)$  such that  ${}_\sigma A \cong Z$  as right  $A$ -comodule algebras.

**Proof:**  $1 \Rightarrow 2$ . Since  $Z$  is lazy, it is cleft and we can assume that  $Z = {}_\omega A$  for a 2-cocycle  $\omega$ . Let  $\psi : A \rightarrow {}_\omega A$  be a symmetry morphism. Since  $\psi$  is right  $A$ -colinear, there exists  $\mu = \varepsilon \circ \psi \in A^*$  such that  $\psi(a) = \mu(a_1)a_2$ ,  $\forall a \in A$ . The map  $\mu \in \text{Reg}^1(A)$  with inverse  $\varepsilon \circ \psi^{-1}$ . Then  $\beta_\psi(a) = \mu^{-1}(a_1)a_2 \otimes \mu(a_3)a_4$ . Let  $\sigma = \omega^\mu = (\mu \otimes \mu) * \omega * (\mu^{-1} \circ m)$ . By [12, Theorem 2.2]  $\sigma$  is a 2-cocycle and  $\psi$  induces a right  $A$ -comodule algebra isomorphism  ${}_\sigma A \rightarrow {}_\omega A$ . Let us check that  $\sigma$  is lazy. For  $a, b \in A$  we have

$$\beta_\psi(a)\beta_\psi(b) = \mu^{-1}(a_1)\mu^{-1}(b_1)\mu(a_3)\mu(b_3)\omega(a_4, b_4)a_2b_2 \otimes a_5b_5$$

and

$$\beta(a_\omega \cdot b) = \omega(a_1, b_1)\mu^{-1}(a_2b_2)a_3b_3\mu(a_4b_4) \otimes a_5b_5.$$

Hence, since  $\beta_\psi$  is an algebra morphism, we have

$$(\mu^{-1} \otimes \mu^{-1}) * m * (\mu \otimes \mu) * \omega = \omega * (\mu^{-1} \circ m) * m * (\mu \circ m).$$

This exactly means that  $m * \sigma = \sigma * m$  and thus  $\sigma$  is lazy.

$2 \Rightarrow 1$ . We can assume that  $Z = {}_\sigma A$ . Taking  $\psi = \text{id}_A$  a direct computation shows that  $\beta_\psi = \Delta : {}_\sigma A \rightarrow A \otimes {}_\sigma A$  is an algebra morphism because  $\sigma$  is lazy.  $\square$

**Remark 3.3** It is possible to introduce a notion of lazy Hopf-Galois extension for general  $A$ -Galois extensions  $R \subset Z$ , corresponding to a notion of lazy crossed system. Since this is not needed for the strict study of the lazy cohomology group, it will be done in the appendix.

When  $A$  is a  $k$ -flat Hopf algebra and hence  $k$ -faithfully flat,  $\sigma \mapsto {}_\sigma A$  defines a surjective map  $Z_L^2(A) \longrightarrow \text{Gal}^L(A)$ . By [12, Theorem 2.2] this map induces a map  $H_L^2(A)$  to  $\text{Gal}^L(A)$ , which is not injective in general. This leads to consider biGalois objects.

Similarly to the right case, a **left  $A$ -Galois object** is a non-zero left  $A$ -comodule algebra  $Z$  with  ${}^{\text{co}A}Z = k$  such that the linear map  $\kappa_l$  defined by the composition

$$\kappa_l : Z \otimes Z \xrightarrow{\beta \otimes 1_Z} A \otimes Z \otimes Z \xrightarrow{1_A \otimes m_Z} A \otimes Z$$

where  $\beta$  is the coaction of  $A$  and  $m_Z$  is the multiplication of  $Z$ , is bijective.

Let  $A$  and  $B$  be Hopf algebras. An algebra  $Z$  is said to be an  **$A$ - $B$ -biGalois object** (cfr. [28]) if  $Z$  is both a left  $A$ -Galois extension and a right  $B$ -Galois extension, and if  $Z$  is an  $A$ - $B$ -bicomodule.

Left Galois objects are related to 2-cocycles as follows. Let  $\sigma$  be a right 2-cocycle. The left  $A$ -comodule algebra  $A_\sigma$  is defined in the following way. As a left  $A$ -comodule  $A_\sigma = A$  and the product of  $A_\sigma$  is defined to be

$$a \cdot_\sigma b = a_1 b_1 \sigma(a_2, b_2), \quad a, b \in A.$$

When  $\sigma \in Z_L^2(A)$  the algebra  ${}_\sigma A$ , endowed with  $\Delta$  as left and right  $A$ -comodule structure, is a left  $A$ -Galois object, and is an  $A$ -biGalois object. This is straightforward and it can be seen as an immediate consequence of [28, Theorem 3.9]. We shall denote the corresponding biGalois object by  $A(\sigma)$ . More generally, we have the following result.

**Proposition 3.4** *Let  $Z$  be a lazy right  $A$ -Galois object. Then for any symmetry morphism  $\psi$ , the map  $\beta_\psi : Z \longrightarrow A \otimes Z$  endows  $Z$  with a left  $A$ -comodule algebra structure for which  $Z$  is  $A$ -biGalois.*

**Proof:** We already know that  $\beta_\psi$  is an algebra morphism. Besides:

$$\begin{aligned} (\text{id}_A \otimes \beta_\psi) \circ \beta_\psi &= (\text{id}_A \otimes \psi^{-1} \otimes \psi) \circ (\text{id}_A \otimes \rho) \circ \beta_\psi \\ &= (\text{id}_A \otimes \text{id}_A \otimes \psi) \circ (\text{id}_A \otimes \Delta) \circ (\text{id}_A \otimes \psi^{-1}) \circ \beta_\psi \\ &= (\text{id}_A \otimes \text{id}_A \otimes \psi) \circ (\text{id}_A \otimes \Delta) \circ (\psi^{-1} \otimes \text{id}_A) \circ \rho \\ &= (\text{id}_A \otimes \text{id}_A \otimes \psi) \circ (\text{id}_A \otimes \Delta) \circ \Delta \circ \psi^{-1} \\ &= (\Delta \otimes \psi) \circ \Delta \circ \psi^{-1} \\ &= (\Delta \otimes \text{id}_Z) \circ \beta_\psi \end{aligned}$$

and

$$(\varepsilon \otimes \text{id}_A) \circ (\psi^{-1} \otimes \psi) \circ \rho = (\varepsilon \otimes \psi) \circ \Delta \circ \psi^{-1} = \text{id}_Z$$

hence  $\beta_\psi$  defines a left comodule structure on  $Z$ . Besides  $Z$  is an  $A$ -bicomodule because

$$\begin{aligned} (\beta_\psi \otimes \text{id}_A) \circ \rho &= (\psi^{-1} \otimes \psi \otimes \text{id}_A) \circ (\rho \otimes \text{id}_A) \circ \rho \\ &= (\psi^{-1} \otimes \psi \otimes \text{id}_A) \circ (\text{id}_Z \otimes \Delta) \circ \rho \\ &= (\psi^{-1} \otimes \rho) \circ (\text{id}_Z \otimes \psi) \circ \rho \\ &= (\text{id}_A \otimes \rho) \circ \beta_\psi. \end{aligned}$$

Note also that  $\beta_\psi \circ \psi = (\text{id}_A \otimes \psi) \circ \Delta$ , hence  $\psi$  is also a left  $A$ -colinear isomorphism. There remains to check that  $Z$  is left  $A$ -Galois for the coaction  $\beta_\psi$ .

If  $f: Z \rightarrow T$  is a right  $A$ -comodule algebra isomorphism from  $Z$  to a lazy right  $A$ -Galois object  $T$  for which the statement holds, then  $\varphi = f \circ \psi: A \rightarrow T$  is a symmetry morphism and  $\beta_\varphi$  endows  $T$  with a left comodule algebra morphism. A direct computation shows that  $f$  is also left  $A$ -colinear so  $Z$  is left  $A$ -Galois because  $T$  is so.

Thus, by Proposition 3.2 we can assume that  $Z = {}_\sigma A$  for  $\sigma \in Z_L^2(A)$ . Let  $\psi: A \rightarrow {}_\sigma A$  be a symmetry morphism. As in the proof of Proposition 3.2  $\psi(a) = \mu(a_1)a_2$  for every  $a \in A$ , with  $\mu = \varepsilon \circ \psi$  and  $\beta_\psi(a) = \mu^{-1}(a_1)a_2 \otimes \mu(a_3)a_4$ . Similarly to the proof of Proposition 3.2 we have

$$(\mu^{-1} \otimes \mu^{-1}) * m * (\mu \otimes \mu) * \sigma = \sigma * (\mu^{-1} \circ m) * m * (\mu \circ m),$$

and since  $\sigma \in Z_L^2(A)$ , we see that  $\partial(\mu) * \sigma \in Z_L^2(A)$ . By standard theory  $\psi: \partial(\mu) {}_\sigma A \rightarrow {}_\sigma A$  is an algebra morphism and it is left colinear (for  $\Delta$  as a left coaction on  $\partial(\mu) {}_\sigma A$ ). We conclude that  ${}_\sigma A$  is left  $A$ -Galois for the coaction  $\beta_\psi$ .  $\square$

**Definition 3.5** *Let  $Z$  be an  $A$ -biGalois object. We say that  $Z$  has **the binormal basis property** if  $Z \cong A$  as an  $A$ -bicomodule. We also say that an  $A$ -biGalois object with the binormal basis property is **bicleft**.*

We characterize now the biGalois objects arising from lazy 2-cocycles.

**Proposition 3.6** *Let  $Z$  be an  $A$ -biGalois object. Then the following assertions are equivalent:*

1.  $Z$  has the binormal basis property.
2. There exists  $\sigma \in Z_L^2(A)$  such that  $A(\sigma) \cong Z$  as  $A$ -bicomodule algebras.

*When this occurs,  $Z$  is a lazy right  $A$ -Galois object.*

**Proof:**  $1 \Rightarrow 2$ . Let  $\psi: A \rightarrow Z$  be an  $A$ -bilinear isomorphism. We can assume without loss of generality that  $\psi(1) = 1$ . Let  $\sigma: A \otimes A \rightarrow k$  be defined by  $\sigma(a, b) = \varepsilon(\psi^{-1}(\psi(a)\psi(b)))$ . Since  $\psi$  is right  $A$ -colinear and  $Z$  is right  $A$ -Galois, it is well known that  $\sigma$  is a left 2-cocycle and that  $\psi: {}_\sigma A \rightarrow Z$  is a right  $A$ -comodule algebra isomorphism. Similarly since  $\psi$  is left  $A$ -colinear and  $Z$  is left  $A$ -Galois, it is well known that  $\sigma$  is a right 2-cocycle and that  $\psi: A_\sigma \rightarrow Z$  is a left  $A$ -comodule algebra isomorphism. Hence

$$\psi(a)\psi(b) = \psi(\sigma(a_1, b_1)a_2b_2) = \psi(\sigma(a_2, b_2)a_1b_1).$$

Since  $\psi$  is bijective we conclude that  $\sigma \in Z_L^2(A)$ . It is then clear that  $\psi: A(\sigma) \rightarrow Z$  is an  $A$ -bicomodule algebra isomorphism.



2  $\Rightarrow$  1. This follows because  $A(\sigma) = A$  as an  $A$ -bicomodule.  $\square$

Let us denote the set of isomorphism classes of  $k$ -faithfully flat  $A$ - $B$ -biGalois objects by  $\text{BiGal}(A, B)$  (a morphism being an  $A$ - $B$ -bicomodule morphism), and let  $\text{BiGal}(A) = \text{BiGal}(A, A)$ . The isomorphism class of a faithfully flat  $A$ -biGalois object  $Z$  is denoted by  $[Z]$  in  $\text{BiGal}(A, B)$ .

We assume now that  $A$  is a  $k$ -flat Hopf algebra. Then  $\text{BiGal}(A)$  inherits a natural group structure: this is the **biGalois group** of  $A$ , defined by Schauenburg [28]. The group law is induced by the cotensor product: if  $V$  is a right  $A$ -comodule and  $W$  is a left  $A$ -comodule their cotensor product over  $A$ , denoted by  $V \square_A W$ , is defined to be the kernel of the linear map  $\rho_V \otimes \text{id}_W - \text{id}_V \otimes \rho_W : V \otimes W \longrightarrow V \otimes A \otimes W$ . In particular when  $V$  and  $W$  are  $A$ -bicomodules, their cotensor product will be so. The faithful flatness assumption ensures that we indeed have an associative law. We denote by  $\text{Bicleft}(A)$  the subset of  $\text{BiGal}(A)$  consisting of isomorphism classes of bicleft biGalois objects. We have the following result.

**Proposition 3.7** *Let  $A$  be a  $k$ -flat Hopf algebra. Then  $\text{Bicleft}(A)$  is a normal subgroup of  $\text{BiGal}(A)$ .*

**Proof:** The neutral element for the cotensor product  $[A]$  lies in  $\text{Bicleft}(A)$  which is stable for this group law. By Proposition 3.6, for any  $[Z] \in \text{Bicleft}(A)$ , we have  $[Z] = [A(\sigma)]$  for some lazy 2-cocycle  $\sigma$ . Since  $A(\sigma^{-1}) \square_A A(\sigma) \cong A$  we have  $[A(\sigma^{-1})] = [A(\sigma)]^{-1}$  so  $\text{Bicleft}(A)$  is stable under inverses and it is a subgroup of  $\text{BiGal}(A)$ . Now let  $[Z] \in \text{BiGal}(A)$  and  $[T] \in \text{Bicleft}(A)$ . We denote by  $Z^{-1}$  an  $A$ -biGalois object such that  $[Z^{-1}] = [Z]^{-1}$ . Then we have, as  $A$ -bicomodules,

$$Z \square_A T \square_A Z^{-1} \cong Z \square_A A \square_A Z^{-1} \cong Z \square_A Z^{-1} \cong A$$

and hence  $\text{Bicleft}(A)$  is normal in  $\text{BiGal}(A)$ .  $\square$

Most of the work has been done now to prove the following result.

**Theorem 3.8** *Let  $A$  be a  $k$ -flat Hopf algebra. Then we have a group isomorphism*

$$H_L^2(A) \cong \text{Bicleft}(A).$$

*In particular we may identify  $H_L^2(A)$  with a normal subgroup of  $\text{BiGal}(A)$ .*

**Proof:** By Proposition 3.6 the map

$$\begin{aligned} f : Z_L^2(A) &\longrightarrow \text{Bicleft}(A) \\ \sigma &\longmapsto [A(\sigma)] \end{aligned}$$

is surjective. Using the coproduct  $\Delta : A \longrightarrow A \otimes A$ , we see by direct computation that for  $\sigma, \omega \in Z_L^2(A)$  we have  $A(\sigma) \square_A A(\omega) \cong A(\sigma * \omega)$  and hence our map  $f$  is

a morphism of groups. A lazy 2-cocycle  $\sigma$  lies in  $\text{Ker}(f)$  if and only if there exists a biGalois isomorphism  $\phi: A \rightarrow A(\sigma)$ . Since  $\phi$  is left and right colinear, we have  $\varepsilon(\phi(a_1))a_2 = \phi(a) = a_1\varepsilon(\phi(a_2))$ . Since  $\phi$  is an algebra morphism,  $\sigma = \partial(\varepsilon \circ \phi^{-1})$ , hence  $\text{Ker}(f) \subset B_L^2(A)$ . Viceversa, if  $\sigma = \partial(\gamma)$  with  $\gamma \in \text{Reg}_L^1(A)$ , then  $a \mapsto \gamma(a_1)a_2$  gives an isomorphism  $A(\sigma) \rightarrow A$ .  $\square$

**Corollary 3.9** *Let  $A$  and  $B$  be  $k$ -flat Hopf algebras. If there exists a  $k$ -faithfully flat  $A$ - $B$ -biGalois object, then we have a group isomorphism*

$$H_L^2(A) \cong H_L^2(B).$$

**Proof:** Let  $Z$  be an  $A$ - $B$ -biGalois object with  $Z^{-1}$  its inverse in the Harrison groupoid  $\mathcal{H}$  defined by Schauenburg in [28, §4], so that  $Z^{-1}$  is a  $B$ - $A$ -biGalois object. Using the composition law for biGalois objects (in [28, §4]) we have a map

$$\begin{aligned} \text{BiGal}(A) &\longrightarrow \text{BiGal}(B) \\ [T] &\longmapsto [Z^{-1} \square_A T \square_A Z] \end{aligned}$$

that is clearly a group isomorphism. It is also clear that this map induces an isomorphism between the groups of the corresponding bicleft objects (same proof as in the proof of Proposition 3.7), so we have our result by the previous theorem.  $\square$

**Remark 3.10** The above result is a monoidal co-Morita invariance type result. Indeed for two  $k$ -flat Hopf algebras, there exists a faithfully flat  $A$ - $B$  biGalois object if and only if the linear monoidal categories  $\text{Comod}(A)$  and  $\text{Comod}(B)$  are equivalent ([28, Corollary 5.7]).

**Remark 3.11** It follows from Corollary 3.9 that the second lazy cohomology group is invariant under Doi's twists when  $A$  is  $k$ -flat. This fact holds in general and it can be directly seen. Indeed, if  $\omega$  is a 2-cocycle for  $A$ , then  $\sigma$  is a lazy 2-cocycle for the twisted Hopf algebra  ${}_{\omega}A_{\omega^{-1}}$  if and only if  $\omega^{-1} * \sigma * \omega$  is a lazy cocycle for  $A$  ([10, Proposition 1.8]). By centrality the assignment  $\sigma \mapsto \omega^{-1} * \sigma * \omega$  induces a group isomorphism  $H_L^2({}_{\omega}A_{\omega^{-1}}) \rightarrow H_L^2(A)$ .

**Remark 3.12** Let us assume that  $A$  is flat. According to the results of [28], the group  $\text{BiGal}(A)$  is isomorphic to the group of isomorphism classes of  $k$ -linear monoidal auto-equivalences of  $\text{Comod}(A)$ . The subgroup  $\text{Bicleft}(A)$  is then, in this setting, identified with the group of isomorphism classes of  $k$ -linear monoidal auto-equivalences of  $\text{Comod}(A)$  that are isomorphic, as linear functors, with the identity functor. This might be checked directly at the lazy cocycle level. With this interpretation of lazy cohomology, since the category of comodules over a coquasibialgebra is a monoidal category, we recover a natural way to define the lazy cohomology groups for coquasibialgebras as done by Schauenburg in [30].

For a cocommutative ( $k$ -flat) Hopf algebra  $A$ , there is a group isomorphism  $\text{Aut}_{\text{Hopf}}(A) \times \text{Gal}(A) \cong \text{BiGal}(A)$  [28, §5]. A group morphism between these two groups still exists in general and it induces the following exact sequence.

**Theorem 3.13** *Let  $A$  be a  $k$ -flat Hopf algebra. There is a group exact sequence*

$$1 \longrightarrow \text{CoOut}^-(A) \longrightarrow \text{CoOut}(A) \times H_L^2(A) \longrightarrow \text{BiGal}(A).$$

**Proof:** To any pair  $(\alpha, \sigma) \in \text{Aut}_{\text{Hopf}}(A) \times Z_L^2(A)$ , we associate the  $A$ -biGalois object  ${}^\alpha A(\sigma)$ , which is  $A(\sigma)$  as a right  $A$ -comodule algebra with left coaction  $\rho = (\alpha \otimes \text{id}_A) \circ \Delta$ . This assignment induces a well-defined map

$$\begin{aligned} \Upsilon : \text{CoOut}(A) \times H_L^2(A) &\longrightarrow \text{BiGal}(A) \\ (\bar{\alpha}, \bar{\sigma}) &\longmapsto [{}^\alpha A(\sigma)]. \end{aligned}$$

Indeed, if  $\mu \in \text{Reg}_L^1(A)$  and  $\phi \in \text{Alg}(A, k)$ , the map  $\phi^{-1} * \mu * \text{id}_A$  is an isomorphism of biGalois objects between the  $A$ -bicomodule algebras  ${}^{\alpha \circ \text{ad}(\phi)} A(\sigma \partial(\mu))$  and  ${}^\alpha A(\sigma)$ .

Let  $(\alpha, \sigma), (\beta, \omega) \in \text{Aut}_{\text{Hopf}}(A) \times Z_L^2(A)$ . Let  $\gamma : A \rightarrow A \otimes A$  be defined by  $\gamma(a) = \beta(a_1) \otimes a_2$ . It is straightforward to verify that  $\gamma$  induces an  $A$ -bicomodule algebra map  ${}^{\alpha \circ \beta} A((\sigma \leftarrow \beta) * \omega) \rightarrow {}^\alpha A(\sigma) \square_A {}^\beta A(\omega)$  and hence an isomorphism. Thus  $\Upsilon$  is a morphism of groups. Let  $(\bar{\alpha}, \bar{\sigma}) \in \text{Ker}(\Upsilon)$ : there exists an  $A$ -bicomodule algebra isomorphism  $f : {}^\alpha A(\sigma) \rightarrow A$ . Since  $f$  is right  $A$ -colinear, there exists  $\mu \in \text{Reg}^1(A)$  such that  $f = \mu * \text{id}_A$ . Since  $f$  is left colinear, we have  $\mu * \text{id}_A = \alpha * \mu$  and hence  $\alpha = \text{ad}(\mu^{-1})$ . Since  $f$  is an algebra map, we have  $\partial(\mu) = \sigma \in Z_L^2(A)$ . Therefore  $(\bar{\alpha}, \bar{\sigma}) = \iota(\overline{\mu^{-1}})$  where  $\iota$  is as in Lemma 1.16. Therefore  $\text{Ker}(\Upsilon) \subset \iota(\text{CoOut}^-(A))$ . Viceversa, if  $\theta \in \text{Reg}_{aL}^1(A)$ , then the map  $\theta^{-1} * \text{id}_A$  is a bicomodule algebra isomorphism  ${}^{\text{ad}(\theta)} A(\partial(\theta^{-1})) \rightarrow A$ , whence the statement.  $\square$

**Example 3.14** Let  $A$  be Sweedler's Hopf algebra  $H_4$ . By the results in Example 2.2,  $\text{Ker}(\iota)$  is trivial and  $\text{CoOut}(H_4) \times H_L^2(H_4) \cong k/(\pm 1) \times k$ . By [29, Theorem 5]  $\text{BiGal}(H_4) \cong k \times k$  so the map  $\Upsilon$  is not surjective and  $\text{CoOut}(H_4) \times H_L^2(H_4)$  is a normal subgroup of  $\text{BiGal}(H_4)$ . Their quotient is  $k/(k)^2 \cong H^2(\mathbb{Z}_2, k)$ . The subgroup  $k$  corresponding to  $(1, t)$  in [29, Theorem 5] corresponds to  $H_L^2(H_4)$  while the subgroup  $\text{CoOut}(H_4)$  corresponds to the elements in  $\text{BiGal}(H_4)$  of the form  $(a^2, 0)$  for  $a \in k$ . Let us observe that  $\text{CoOut}(A) \times H_L^2(A)$  appeared in [36, §4] as a subgroup of the Brauer group  $BQ(k, H_4)$ .

## 4 The Kac-Schauenburg exact sequence

This section is devoted to the construction of a Kac-Schauenburg-type exact sequence for a double crossed Hopf algebra of possibly infinite-dimensional Hopf algebras. As a consequence we will derive a Schur-Yamazaki type formula, as well as

the description of the lazy cohomology of the Drinfeld double of (finite-dimensional) Hopf algebras.

Recall [20] that a matched pair  $(B, A, \rightharpoonup, \leftharpoonup)$  consists of two Hopf algebras  $B$  and  $A$  together with linear maps  $\rightharpoonup: A \otimes B \rightarrow B$  and  $\leftharpoonup: A \otimes B \rightarrow A$  making  $B$  into a left  $A$ -module coalgebra and  $A$  into a right  $B$ -module coalgebra respectively, and satisfying the following conditions

$$\begin{aligned} (aa') \leftharpoonup b &= (a \leftharpoonup (a'_1 \rightharpoonup b_1))(a'_2 \leftharpoonup b_2), & 1 \leftharpoonup b &= \varepsilon_B(b)1, \\ a \rightharpoonup (bb') &= (a_1 \rightharpoonup b_1)((a_2 \leftharpoonup b_2) \rightharpoonup b'), & a \rightharpoonup 1 &= \varepsilon_A(a)1, \\ (a_1 \leftharpoonup b_1) \otimes (a_2 \rightharpoonup b_2) &= (a_2 \leftharpoonup b_2) \otimes (a_1 \rightharpoonup b_1). \end{aligned}$$

One associates a Hopf algebra  $B \bowtie A$  to a matched pair  $(B, A, \rightharpoonup, \leftharpoonup)$  in the following way: as a coalgebra  $B \bowtie A$  is the tensor product coalgebra  $B \otimes A$ , the product (with unit  $1 \otimes 1$ ) is defined by

$$(b \otimes a)(b' \otimes a') = b(a_1 \rightharpoonup b'_1) \otimes (a_2 \leftharpoonup b'_2)a',$$

and the antipode is defined by

$$S(a \otimes b) = (1 \otimes S_B(b))(S_A(a) \otimes 1).$$

The Hopf algebra  $B \bowtie A$  is said to be a double crossed product Hopf algebra.

**Definition 4.1** *Let  $(B, A, \rightharpoonup, \leftharpoonup)$  be a matched pair and consider the associated Hopf algebra  $B \bowtie A$ . A **central  $B \bowtie A$ -pairing** is a convolution invertible linear map  $\beta: B \otimes A \rightarrow k$  satisfying the following conditions:*

$$\begin{aligned} \beta(bb', a) &= \beta(b_1, a_1)\beta(b', a_2 \leftharpoonup b_2), & \beta(b, aa') &= \beta(b_1, a'_1)\beta(a'_2 \rightharpoonup b_2, a) \\ \beta(b_2, a_2)a_1 \leftharpoonup b_1 &= \beta(b_1, a_1)a_2 \leftharpoonup b_2, & \beta(b_1, a_1)a_2 \rightharpoonup b_2 &= \beta(b_2, a_2)a_1 \rightharpoonup b_1 \\ \beta(1, a) &= \varepsilon_A(a), & \beta(1, b) &= \varepsilon_B(b). \end{aligned}$$

The set of central  $B \bowtie A$ -pairings is denoted by  $\mathcal{ZP}(B \bowtie A)$ .

It is straightforward to check that  $\mathcal{ZP}(B \bowtie A)$ , endowed with the convolution product, is a group.

Here is our version of the Kac-Schauenburg exact sequence. A brief comparison with [30, Theorem 6.5.1] will be given after the proof.

**Theorem 4.2** *Let  $(B, A, \rightharpoonup, \leftharpoonup)$  be a matched pair of Hopf algebras and consider the associated double crossed product Hopf algebra  $B \bowtie A$ . Then we have a group exact sequence:*

$$\begin{array}{ccccccc} 1 & \longrightarrow & H_L^1(B \bowtie A) & \xrightarrow{\text{res}} & H_L^1(B) \times H_L^1(A) & \xrightarrow{\Lambda} & \mathcal{ZP}(B \bowtie A) \\ & & \xrightarrow{\Sigma} & H_L^2(B \bowtie A) & \xrightarrow{\text{res}} & H_L^2(B) \times H_L^2(A) & \end{array}$$

Since the Hopf algebras  $A$  and  $B$  are identified with Hopf subalgebras of  $B \rtimes A$ , we get well-defined restriction maps, which clearly are group morphisms. Also it is clear that the first one is injective. We now have to construct the maps  $\Lambda$  and  $\Sigma$ .

**Lemma 4.3** *Let  $(\phi_B, \phi_A) \in H_L^1(B) \times H_L^1(A)$ . Define  $\Lambda(\phi_A, \phi_B) : B \rtimes A \longrightarrow k$  by*

$$\Lambda(\phi_B, \phi_A)(b, a) = \phi_A^{-1}(a_1)\phi_B^{-1}(b_1)\phi_B(a_2 \rightarrow b_2)\phi_A(a_3 \leftarrow b_3).$$

*Then  $\Lambda(\phi_A, \phi_B) \in \mathcal{ZP}(B \rtimes A)$ , and this defines a group morphism such that the sequence*

$$H_L^1(B \rtimes A) \xrightarrow{\text{res}} H_L^1(B) \times H_L^1(A) \xrightarrow{\Lambda} \mathcal{ZP}(B \rtimes A)$$

*is exact.*

**Proof:** It is not difficult to check, using the conditions of a matched pair and the laziness of  $\phi_B$  and  $\phi_A$ , that  $\Lambda(\phi_B, \phi_A) \in \mathcal{ZP}(B \rtimes A)$  and that  $\Lambda$  is a group morphism. For  $\phi \in \text{Reg}_L^1(B \rtimes A)$ , we have  $\phi \in H_L^1(B \rtimes A)$  if and only if the restrictions to  $A$  and  $B$  belong to  $H_L^1(A)$  and  $H_L^1(B)$ , respectively, and

$$\begin{aligned} \phi(b \otimes a) &= \phi(b \otimes 1)\phi(1 \otimes a) = \phi((1 \otimes a)(b \otimes 1)) \\ &= \phi(a_1 \rightarrow b_1 \otimes a_2 \leftarrow b_2) = \phi(a_1 \rightarrow b_1 \otimes 1)\phi(1 \otimes a_2 \leftarrow b_2). \end{aligned}$$

Hence we see that for  $\phi \in H_L^1(B \rtimes A)$ , then  $\Lambda(\text{res}(\phi)) = \varepsilon_B \otimes \varepsilon_A$ . Conversely, if  $(\phi_B, \phi_A) \in \text{Ker}(\Lambda)$  then  $\phi = \phi_B \otimes \phi_A \in H_L^1(B \rtimes A)$  and  $(\phi_B, \phi_A) = \text{res}(\phi)$ .  $\square$

**Lemma 4.4** *Let  $\beta \in \mathcal{ZP}(B \rtimes A)$ . Let  $\sigma_\beta : B \otimes A \otimes B \otimes A \longrightarrow k$  be defined by*

$$\sigma_\beta(b \otimes a, b' \otimes a') = \beta(b', a)\varepsilon_B(b)\varepsilon_A(a).$$

*Then  $\sigma_\beta \in Z_L^2(B \rtimes A)$  and this assignment determines a group morphism*

$$\begin{aligned} \Sigma : \mathcal{ZP}(B \rtimes A) &\longrightarrow H_L^2(B \rtimes A) \\ \beta &\longmapsto \overline{\sigma_\beta} \end{aligned}$$

**Proof:** It is easy to see, using that  $\beta \in \mathcal{ZP}(B \rtimes A)$ , that  $\sigma_\beta \in \text{Reg}_L^2(B \rtimes A)$ . Let us check that  $\sigma_\beta$  is a (left) 2-cocycle. We have

$$\begin{aligned} &\sigma_\beta(b_1 \otimes a_1, b'_1 \otimes a'_1)\sigma_\beta((b_2 \otimes a_2)(b'_2 \otimes a'_2), b'' \otimes a'') \\ &= \sigma_\beta(b_1 \otimes a_1, b'_1 \otimes a'_1)\sigma_\beta((b_2[a_2 \rightarrow b'_2] \otimes [a_3 \leftarrow b'_3]a'_2), b'' \otimes a'') \\ &= \varepsilon_B(b)\varepsilon_A(a'')\beta(b'_1, a_1)\beta(b'', [a_2 \leftarrow b'_2]a') \\ &= \varepsilon_B(b)\varepsilon_A(a'')\beta(b'_1, a_1)\beta(b''_1, a'_1)\beta(a'_2 \rightarrow b''_2, a_2 \leftarrow b'_2) \\ &= \varepsilon_B(b)\varepsilon_A(a'')\beta(b''_1, a'_1)\beta(b'[a'_2 \rightarrow b''_2], a) \\ &= \sigma_\beta(b'_1 \otimes a'_1, b''_1 \otimes a''_1)\sigma_\beta(b \otimes a, b'_2[a'_2 \rightarrow b''_2] \otimes [a'_3 \leftarrow b''_3]a''_2) \\ &= \sigma_\beta(b'_1 \otimes a'_1, b''_1 \otimes a''_1)\sigma_\beta(b \otimes a, (b'_2 \otimes a'_2)(b''_2 \otimes a''_2)), \end{aligned}$$

where the properties of the central  $B \rtimes A$ -pairing  $\beta$  have been used. Hence  $\sigma_\beta \in Z_L^2(B \rtimes A)$  and it is immediate that the induced map  $\Sigma$  is a group morphism.  $\square$

**Lemma 4.5** *The sequence*

$$H_L^1(B) \times H_L^1(A) \xrightarrow{\Lambda} \mathcal{ZP}(B \rtimes A) \xrightarrow{\Sigma} H_L^2(B \rtimes A)$$

*is exact.*

**Proof:** Let  $(\phi_B, \phi_A) \in H_L^1(B) \times H_L^1(A)$ , and let  $\mu = \phi_B^{-1} \otimes \phi_A^{-1}$ . Clearly  $\mu \in \text{Reg}_L^1(B \rtimes A)$ , and a direct computation shows that  $\Sigma(\Lambda(\phi_B, \phi_A)) = \overline{\partial(\mu)}$ . Hence  $\text{Im}(\Lambda) \subset \text{Ker}(\Sigma)$ .

Conversely, let  $\beta \in \text{Ker}(\Sigma)$ . Then there exists  $\mu \in \text{Reg}_L^1(B \rtimes A)$  such that  $\sigma_\beta = \partial(\mu)$ . Let  $\phi_B : B \rightarrow k$  be defined by  $\phi_B(b) = \mu^{-1}(b \otimes 1)$ , and similarly, let  $\phi_A : A \rightarrow k$  be defined by  $\phi_A(a) = \mu^{-1}(1 \otimes a)$ . Since  $\mu \in \text{Reg}_L^1(B \rtimes A)$ , we have  $\phi_B \in \text{Reg}_L^1(B)$  and  $\phi_A \in \text{Reg}_L^1(A)$ . Computing  $\sigma_\beta(b \otimes 1, b' \otimes 1)$  and  $\sigma_\beta(1 \otimes a, 1 \otimes a')$ , we see that  $\phi_B \in H_L^1(B)$  and  $\phi_A \in H_L^1(A)$ . Computing  $\sigma_\beta(b \otimes 1, 1 \otimes a)$ , we find  $\mu^{-1}(b \otimes a) = \mu^{-1}(b \otimes 1)\mu^{-1}(1 \otimes a)$ . Then we have

$$\begin{aligned} \beta(b, a) &= \sigma_\beta(1 \otimes a, b \otimes 1) = \mu(1 \otimes a_1)\mu(b_1 \otimes 1)\mu^{-1}(a_2 \rightarrow b_2 \otimes a_3 \leftarrow b_3) \\ &= \phi_A^{-1}(a_1)\phi_B^{-1}(b_1)\phi_B(a_2 \rightarrow b_2)\phi_A(a_3 \leftarrow b_3), \end{aligned}$$

which proves that  $\beta = \Lambda(\phi_B, \phi_A)$  and finishes the proof of the lemma.  $\square$

There remains to check the exactness of our sequence at  $H_L^2(B \rtimes A)$ . The following lemma, taken from [30], will be useful:

**Lemma 4.6** ([30, Lemma 6.2.7]) *Let  $\sigma \in Z_L^2(B \rtimes A)$ . Then there exists  $\sigma' \in Z_L^2(B \rtimes A)$  having the same class as  $\sigma$  in  $H_L^2(B \rtimes A)$  and satisfying:*

$$\sigma'(b \otimes 1, b' \otimes a') = \sigma(b \otimes 1, b' \otimes 1)\varepsilon_A(a'), \text{ and } \sigma'(b \otimes a, 1 \otimes a') = \sigma(1 \otimes a, 1 \otimes a')\varepsilon_B(b).$$

**Proof:** Let  $\mu : B \otimes A \rightarrow k$  be defined by  $\mu(b \otimes a) = \sigma(b \otimes 1, 1 \otimes a)$ . It is straightforward to check that  $\mu \in \text{Reg}_L^1(B \rtimes A)$  and that  $\sigma' = \sigma * \partial(\mu)$  satisfies the above conditions.  $\square$

**Lemma 4.7** *The sequence*

$$\mathcal{ZP}(B \rtimes A) \xrightarrow{\Sigma} H_L^2(B \rtimes A) \xrightarrow{\text{res}} H_L^2(B) \times H_L^2(A)$$

*is exact.*

**Proof:** It is clear from the definitions that  $\text{Im}(\Sigma) \subset \text{Ker}(\text{res})$ . Conversely let  $\sigma \in Z_L^2(B \rtimes A)$  be such that  $\bar{\sigma} \in \text{Ker}(\text{res})$ . Then there exists  $\mu_1 \in \text{Reg}_L^1(B)$  and  $\mu_2 \in \text{Reg}_L^1(A)$  such that

$$\sigma(b \otimes 1, b' \otimes 1) = \mu_1(b_1)\mu_1(b'_1)\mu_1^{-1}(b_2b'_2) \text{ and } \sigma(1 \otimes a, 1 \otimes a') = \mu_2(a_1)\mu_2(a'_1)\mu_2^{-1}(a_2a'_2).$$

Define  $\mu : B \otimes A \longrightarrow k$  by  $\mu(b \otimes a) = \mu_1(b)\mu_2(a)$ . Clearly  $\mu \in \text{Reg}_L^1(B \rtimes A)$ . A direct computation gives

$$\sigma * \partial(\mu^{-1})(b \otimes 1, b' \otimes 1) = \varepsilon_B(bb') \quad \text{and} \quad \sigma * \partial(\mu^{-1})(1 \otimes a, 1 \otimes a') = \varepsilon_A(aa').$$

Therefore we can assume, without changing the class of  $\sigma$  in  $H_L^2(B \rtimes A)$ , that

$$\sigma(b \otimes 1, b' \otimes 1) = \varepsilon_B(bb') \quad \text{and} \quad \sigma(1 \otimes a, 1 \otimes a') = \varepsilon_A(aa').$$

By Lemma 4.6 we can assume that

$$\sigma(b \otimes 1, b' \otimes a') = \varepsilon_B(bb')\varepsilon_A(a') \quad \text{and} \quad \sigma(b \otimes a, 1 \otimes a') = \varepsilon_B(b)\varepsilon_A(aa').$$

Then using the fact that  $\sigma$  is a 2-cocycle, we have

$$\begin{aligned} \sigma(b \otimes a, b' \otimes a') &= \varepsilon_B(b_1)\varepsilon_A(a_1)\sigma(b_2 \otimes a_2, b' \otimes a') \\ &= \sigma(b_1 \otimes 1, 1 \otimes a_1)\sigma(b_2 \otimes a_2, b' \otimes a') \\ &= \sigma(1 \otimes a_1, b'_1 \otimes a'_1)\sigma(b \otimes 1, (1 \otimes a_2)(b'_2 \otimes a'_2)) \\ &= \varepsilon_B(b)\sigma(1 \otimes a, b' \otimes a') \\ &= \varepsilon_B(b)\sigma(1 \otimes a, b'_2 \otimes a'_2)\sigma(b'_1 \otimes 1, 1 \otimes a'_1) \\ &= \varepsilon_B(b)\sigma(1 \otimes a_1, b'_1 \otimes 1)\sigma((1 \otimes a_2)(b'_2 \otimes 1), 1 \otimes a') \\ &= \varepsilon_B(b)\varepsilon_A(a')\sigma(1 \otimes a, b' \otimes 1). \end{aligned}$$

We will show that  $\beta : B \otimes A \longrightarrow k$  defined by  $\beta(b, a) = \sigma(1 \otimes a, b \otimes 1)$  belongs to  $\mathcal{ZP}(B \rtimes A)$ . It is clear that  $\beta$  is convolution invertible. We have

$$\begin{aligned} \beta(bb', a) &= \sigma(1 \otimes a, bb' \otimes 1) = \sigma(1 \otimes a, (b_1 \otimes 1)(b'_1 \otimes 1))\sigma(b_2 \otimes 1, b'_2 \otimes 1) \\ &= \sigma(1 \otimes a_1, b_1 \otimes 1)\sigma(a_2 \rightharpoonup b_2 \otimes a_3 \leftarrow b_3, b' \otimes 1) = \beta(b_1, a_1)\beta(b', a_2 \leftarrow b_2). \end{aligned}$$

Using that  $\sigma$  is lazy for  $B \rtimes A$  and the counits, one sees easily that  $\beta$  satisfies the last two conditions defining central  $B \rtimes A$ -pairings. Finally one checks that  $\beta(b, aa') = \beta(b_1, a'_1)\beta(a'_2 \rightharpoonup b_2, a)$  similarly to the case of the first condition, using centrality of  $\beta$ .  $\square$

Combining Lemma 4.3, 4.5 and 4.7 together concludes the proof of Theorem 4.2.  $\square$

Let us briefly compare Theorem 4.2 with Schauenburg's version of G.I. Kac's exact sequence [30, Theorem 6.5.1]. For this we assume that  $A$  is finite-dimensional. Under this assumption the group  $\mathcal{ZP}(B \rtimes A)$  is isomorphic to the automorphism group  $\text{Aut}_{\text{ext}}(A^* \# B)$  of the Hopf algebra extension corresponding to the bismash product of the dual Singer pair of the original matched pair. Therefore in this case we have the same exact sequence.

We now examine the case of a matched pair  $(B, A, \rightharpoonup, \leftarrow)$  with  $\rightharpoonup$  and  $\leftarrow$  trivial. In this case the double crossed product  $B \rtimes A$  is the tensor product Hopf algebra  $B \otimes A$  and the group  $\mathcal{ZP}(B \otimes A)$  is abelian group. Examining the Kac-Schauenburg exact sequence, we get the following Schur-Yamazaki type formula, generalizing the classical one in group cohomology:

**Theorem 4.8** *Let  $A$  and  $B$  be Hopf algebras. Then we have a group isomorphism*

$$H_L^2(A \otimes B) \cong H_L^2(A) \times H_L^2(B) \times \mathcal{ZP}(A \otimes B).$$

**Proof:** First note that since  $\rightarrow$  and  $\leftarrow$  are trivial, then the map  $\Lambda$  is trivial, and by the exactness at  $\mathcal{ZP}(B \otimes A)$  we see that  $\Sigma$  is injective. Now for  $(\sigma_1, \sigma_2) \in Z_L^2(A) \times Z_L^2(B)$ , it is straightforward to check that  $\sigma : B \otimes A \rightarrow k$  defined by  $\sigma(b \otimes a, b' \otimes a') = \sigma_1(b, b')\sigma_2(a, a')$  is a lazy 2-cocycle. This defines a group morphism  $j : H_L^2(A) \times H_L^2(B) \rightarrow H_L^2(B \otimes A)$  such that  $\text{res} \circ j = \text{id}$ . Thus by Theorem 4.2 we have a split exact sequence:

$$1 \longrightarrow \mathcal{ZP}(B \otimes A) \xrightarrow{\Sigma} H_L^2(B \otimes A) \xrightarrow{\text{res}} H_L^2(B) \times H_L^2(A) \longrightarrow 1$$

There just remains to be remarked that, since both actions are trivial,  $\Sigma$  maps  $\mathcal{ZP}(B \otimes A)$  into a central subgroup of  $H_L^2(B \otimes A)$ : our split exact sequence is central and hence we have the announced isomorphism.  $\square$

We assume for the rest of the section that  $k$  is a field. We shall give an application of the previous formula to the computation of the second lazy cohomology group of a Drinfeld double.

First let us begin with a more convenient description of the group of central  $A \otimes B$ -pairings. Let  $A$  and  $B$  be Hopf algebras, and let  $Z(A)$  denote the center of  $A$ . The set of Hopf algebra morphisms  $f : A \rightarrow B$  satisfying

$$f(A) \subset Z(B) \quad \text{and} \quad f(a_1) \otimes a_2 = f(a_2) \otimes a_1, \quad \forall a \in A, \quad (4.3)$$

will be denoted by  $\mathcal{L}(A, B)$ . A direct computation shows that  $\mathcal{L}(A, B)$  is a group under the convolution product (the inverse  $f \circ S$  of  $f$  is a Hopf algebra morphism because of (4.3)). We put  $\mathcal{L}(A) := \mathcal{L}(A, A)$ .

**Lemma 4.9** *Let  $A$  and  $B$  be Hopf algebras. Assume that  $B^o$ , the dual Hopf algebra of  $B$ , separates the points of  $B$ . Then the groups  $\mathcal{ZP}(A \otimes B)$  and  $\mathcal{L}(A, B^o)$  are isomorphic.*

**Proof:** For any pairing  $\beta : A \otimes B \rightarrow k$  we define the Hopf algebra map  $f_\beta : A \rightarrow B^o$  by  $f_\beta(a)(b) := \beta(a, b)$ . It is well known that this establishes a bijective correspondence between pairings  $A \otimes B \rightarrow k$  and Hopf algebra maps  $A \rightarrow B^o$ . Let  $a \in A$  and  $b \in B$ . Since  $B^o$  separates the points of  $B$ , we have

$$\begin{aligned} \beta(a, b_1)b_2 = \beta(a, b_2)b_1 &\iff \forall \phi \in B^o, \phi(\beta(a, b_1)b_2) = \phi(\beta(a, b_2)b_1) \\ &\iff \forall \phi \in B^o, f_\beta(a) * \phi(b) = \phi * f_\beta(a)(b). \end{aligned}$$

Hence  $\beta(a, b_1)b_2 = \beta(a, b_2)b_1$  for any  $a \in A$  and  $b \in B$  if and only if  $f_\beta(A) \subset Z(B^o)$ . One also checks directly that  $\beta(a_1, b)a_2 = \beta(a_2, b)a_1$  for any  $a \in A$  and  $b \in B$  if and only if  $f_\beta(a_1) \otimes a_2 = f_\beta(a_2) \otimes a_1$ , for any  $a \in A$ . Thus we have a bijection  $\beta \mapsto f_\beta$  between  $\mathcal{ZP}(A \otimes B)$  and  $\mathcal{L}(A, B^o)$ . It is not hard to verify that this bijection is a group morphism.  $\square$



**Remark 4.10** It is also possible to give a Hopf algebra morphism interpretation of general central  $B \bowtie A$ -pairings.

Let  $A$  be a finite-dimensional Hopf algebra and let  $D(A)$  be its Drinfeld double. It is well known that  $D(A)$  is a cocycle twist of  $A \otimes (A^*)^{\text{cop}}$  ([14, Proposition 2.2, Remark 2.3]). Combining the invariance of the second lazy cohomology group under twisting (Corollary 3.9), the Schur-Yamazaki formula and Lemma 4.9 we get the following result.

**Corollary 4.11** *Let  $A$  be a finite-dimensional Hopf algebra and let  $D(A)$  be its Drinfeld double. We have a group isomorphism:*

$$H_L^2(D(A)) \cong H_L^2(A) \times H_L^2(A^*) \times \mathcal{L}(A).$$

□

**Example 4.12** Let  $H_4$  be Sweedler's Hopf algebra. Its centre is  $k$ . If  $f \in \mathcal{L}(H_4)$ , then  $f \in \text{Alg}(H_4, k)$  hence  $f = \varepsilon$  or  $f = 1^* - g^*$  with notation as in Example 2.2. Condition (4.3) is not verified for  $f = 1^* - g^*$  and  $a = x$ , so  $\mathcal{L}(H_4)$  is trivial and by self-duality of  $H_4$  it follows that  $H_L^2(D(H_4)) \cong k \times k$ .

## 5 Action on universal $r$ -forms

Let us suppose that  $A$  is coquasitriangular with universal  $r$ -form  $r$ . It is well known ([23, Page 61]) that if  $\sigma$  is a left 2-cocycle and if  $\tau$  is the usual flip operator then  $(\sigma\tau) * r * \sigma^{-1} = r_\sigma$  is a universal  $r$ -form for the twisted Hopf algebra  ${}_\sigma A_{\sigma^{-1}}$ . In particular, if  $\sigma$  is a lazy cocycle then  $r_\sigma$  is again a universal  $r$ -form for  $A$  and this defines a left action of  $Z_L^2(A)$  on the set  $\mathcal{U}$  of universal  $r$ -forms of  $A$ .

**Lemma 5.1** *Let  $A$  be a coquasitriangular Hopf algebra. Then*

1. *The action of  $Z_L^2(A)$  on  $\mathcal{U}$  factors through an action of  $H_L^2(A)$ .*
2. *The right action  $\leftarrow$  of  $\text{Aut}_{\text{Hopf}}(A)$  on  $\mathcal{U}$  induces an action of  $\text{CoOut}(A)$  on  $\mathcal{U}$ .*
3. *The actions of  $H_L^2(A)$  and of  $\text{CoOut}(A)$  on  $\mathcal{U}$  combine to a right action of  $\text{CoOut}(A) \rtimes H_L^2(A)$  given by  $r \leftarrow (\bar{\alpha}, \bar{\sigma}) = \sigma^{-1} \tau * (r \leftarrow \bar{\alpha}) * \sigma$ .*
4. *The kernel of this action contains  $\text{CoOut}(A)^-$ .*

**Proof:** 1. The group  $B_L^2(A)$  acts trivially on  $\mathcal{U}$ . Indeed, let  $\gamma \in \text{Reg}_L^1(A)$ , let  $r \in \mathcal{U}$  and let  $a, b \in A$ . Then

$$\begin{aligned} r_{\partial(\gamma)}(a, b) &= \partial(\gamma)(b_1, a_1)r(a_2, b_2)(\partial(\gamma))^{-1}(a_3, b_3) \\ &= \gamma(a_1)\gamma(b_1)\gamma^{-1}(b_2a_2)r(a_3, b_3)\gamma(a_4b_4)\gamma^{-1}(a_5)\gamma^{-1}(b_5) \\ &= \gamma(a_1)\gamma(b_1)r(a_2, b_2)\gamma^{-1}(a_3)\gamma^{-1}(b_3) = r(a, b). \end{aligned}$$

2. It is well known that the right action  $\leftarrow$  of  $\text{Aut}_{\text{Hopf}}(A)$  on  $\text{Reg}^2(A)$  stabilizes  $\mathcal{U}$  if  $A$  is coquasitriangular. If  $\text{ad}(\gamma) \in \text{CoInn}(A)$  we may assume that  $\gamma$  is an algebra morphism  $A \rightarrow k$ . Then, for every  $a, b \in A$ :

$$\begin{aligned} r(\text{ad}(\gamma)(a), \text{ad}(\gamma)(b)) &= \gamma^{-1}(a_1)\gamma^{-1}(b_1)r(a_2, b_2)\gamma(a_3b_3) \\ &= \gamma^{-1}(b_1)\gamma^{-1}(a_1)r(a_3, b_3)\gamma(b_2a_2) \\ &= r(a, b). \end{aligned}$$

3. It is not hard to check that  $r \leftarrow (1, \bar{\sigma})(\bar{\alpha}, 1) = r \leftarrow (\bar{\alpha}, \overline{\sigma \leftarrow \bar{\alpha}})$ .

4. The group  $\text{CoOut}(A)^-$  is represented by pairs  $(\text{ad}(\mu), \partial(\mu)^{-1})$ . For every  $a, b \in A$  we have:

$$\begin{aligned} (r \leftarrow (\text{ad}(\mu), \partial(\mu)^{-1}))(a, b) &= \partial(\mu^{-1})^{-1}(b_1, a_1)r(\text{ad}(\mu)(a_2), \text{ad}(\mu)(b_2))\partial(\mu)^{-1}(a_3, b_3) \\ &= \mu^{-1}(b_1a_1)\mu(a_2)\mu(b_2)\mu^{-1}(a_3)\mu^{-1}(b_3)r(a_4, b_4)\mu(b_5a_5) \\ &= r(a_1, b_1)\mu^{-1}(a_2b_2)\mu(a_3b_3) = r(a, b). \end{aligned}$$

□

Let  $\mathcal{G}$  denote the semi-direct product of  $H_L^2(A)$  and  $\text{Aut}_{\text{Hopf}}(A)$ . Then  $\mathcal{G}$  acts again on  $\mathcal{U}$  and the  $\mathcal{G}$ -orbits on  $\mathcal{U}$  coincide with the  $\text{CoOut}(A) \times H_L^2(A)$ -orbits on  $\mathcal{U}$ .

Let  $(A, r)$  be a coquasitriangular Hopf algebra. We recall that the Brauer group  $BC(k, A, r)$  is the Brauer group of the braided monoidal category of right  $A$ -comodules, where the braiding is given on  $M \otimes N$  by  $\psi_{MN}(y \otimes n) = n_1 \otimes y_1 r(n_2 \otimes y_2)$  if the comodule structures on  $M$  and  $N$  are given by  $\rho_M(y) = y_1 \otimes y_2$  and  $\rho_N(n) = n_1 \otimes n_2$ , respectively.

In this new language, [7, Proposition 3.1] becomes:

**Theorem 5.2** *The Brauer group  $BC(k, A, r)$  is constant on the  $H_L^2(A)$ -orbits of  $\mathcal{U}$ .* □

**Corollary 5.3** *The Brauer group  $BC(k, A, r)$  is constant on the  $\mathcal{G}$ -orbits of  $\mathcal{U}$ . The  $\mathcal{G}$ -orbits on  $\mathcal{U}$  coincide with the  $\text{CoOut}(A) \times H_L^2(A)$ -orbits on  $\mathcal{U}$ .* □

It is an interesting problem to understand the orbits under the  $\mathcal{G}$ -action. Let us recall that if  $(A, r)$  is cotriangular then  $(A, s)$  will be cotriangular for every  $s \in r \leftarrow \mathcal{G}$ . One may wonder in which cases the  $\mathcal{G}$ -action on cotriangular structures is transitive. For instance, if  $A = k[\mathbb{Z}_2]$  there are 2 distinct universal  $r$ -forms on  $A$ :  $\varepsilon \otimes \varepsilon$  and  $\frac{1}{2}(1^* \otimes 1^* + g^* \otimes 1^* + g^* \otimes 1^* - g^* \otimes g^*)$  and they do not lie in the same orbit because every cocycle for  $k[\mathbb{Z}_2]$  is a symmetric form. On the other hand we have:

**Example 5.4** Let  $E(n)$  and  $k$  be as in Example 2.2. By [26] the quasitriangular structures for  $E(n)$  (and, dually, the coquasitriangular structures) are parametrized

by  $n \times n$  matrices with entries in  $k$  and the  $\text{Aut}_{\text{Hopf}}(E(n))$ -orbits correspond to congruence classes of matrices. By [9, Theorem 3.9] the  $\mathcal{G}$ -orbits on  $\mathcal{U}$  are parametrized by congruence classes of skew-symmetric  $n \times n$  matrices with coefficients in  $k$ , i.e., by the matrices of the form  $J_l = \begin{pmatrix} 0_l & I_l & 0 \\ -I_l & 0_l & 0 \\ 0 & 0 & 0_{n-2l} \end{pmatrix}$  for  $0 \leq l \leq \lfloor \frac{n}{2} \rfloor$ . All triangular  $r$ -forms lie in the orbit represented by  $J_0 = 0$ , and they correspond to all symmetric matrices.

## 6 A monoidal category of projective representations for Hopf algebras

The Schur multiplier is the traditional companion of the theory of projective representations of groups. For Hopf algebras, there is still a naive notion of a projective representation, that is a representation of the Galois object corresponding to a 2-cocycle. However since the convolution product of 2-cocycles is no longer a cocycle in general, there is no nice monoidal structure on the category of such general projective representations. However, when we restrict ourselves to lazy 2-cocycles, we are able to construct a monoidal  $G$ -category (with  $G = H_L^2(A)$ ) of projective representations of  $A$ . Such categorical structures were considered by Turaev in the setting of homotopy quantum field theory [34]. The base ring  $k$  is a field in this section.

Let us first recall some notions introduced in [34]. We assume familiarity with monoidal categories. First a category  $\mathcal{C}$  is said to be  $k$ -**additive** if all the Hom's in  $\mathcal{C}$  are  $k$ -modules and the composition of morphisms is bilinear over  $k$ .

A  $k$ -additive category is said to be  $\Lambda$ -**split** for some set  $\Lambda$  if there exists a family  $(\mathcal{C}_\lambda)_{\lambda \in \Lambda}$  of  $k$ -additive full subcategories of  $\mathcal{C}$  such that each object belongs to  $\mathcal{C}_\lambda$  for some  $\lambda \in \Lambda$ , and such that for any pair of objects  $V \in \mathcal{C}_\lambda$  and  $W \in \mathcal{C}_\nu$  with  $\lambda \neq \nu$ , then  $\text{Hom}_{\mathcal{C}}(V, W) = \{0\}$  (in particular any non-zero object belongs to  $\mathcal{C}_\lambda$  for a unique  $\lambda \in \Lambda$ ). The family of full subcategories  $(\mathcal{C}_\lambda)_{\lambda \in \Lambda}$  is said to be a  $\Lambda$ -**splitting** of  $\mathcal{C}$ .

A  $k$ -**additive monoidal category** is a monoidal category which is  $k$ -additive as a category, and such that the tensor product is bilinear over  $k$ .

Let  $G$  be a group. A **monoidal  $G$ -category over  $k$**  is a  $k$ -additive monoidal category with left duality  $\mathcal{C}$  which is  $G$ -split and such that

- (i) If  $V \in \mathcal{C}_\lambda$  and  $W \in \mathcal{C}_\nu$ , then  $V \otimes W \in \mathcal{C}_{\lambda\nu}$ ,
- (ii) If  $V \in \mathcal{C}_\lambda$ , then  $V^* \in \mathcal{C}_{\lambda^{-1}}$ .

Note that  $\mathbf{1}$ , the monoidal unit of  $\mathcal{C}$ , necessarily belongs to  $\mathcal{C}_1$ , which is itself a  $k$ -additive monoidal category with left duality.

We are going to construct, for a Hopf algebra  $A$ , a monoidal  $H_L^2(A)$ -category over  $k$ , consisting of projective representations of  $A$ .

**Definition 6.1** Let  $A$  be a Hopf algebra. A (finite-dimensional) **projective representation of  $A$**  consists of a triplet  $(\sigma, V, \pi_V)$  where  $\sigma \in Z_L^2(A)$  is a lazy 2-cocycle,  $V$  is a finite dimensional vector space and  $\pi_V : A(\sigma) \longrightarrow \text{End}(V)$  is an algebra morphism.

Let  $X = (\sigma, V, \pi_V)$  and  $Y = (\omega, W, \pi_W)$  be some projective representations of  $A$ . A **basic morphism**  $f : X \longrightarrow Y$  is a linear map  $f : V \rightarrow W$  such that there exists  $\mu \in \text{Reg}_L^1(A)$  satisfying

$$f \circ \pi_V(a) = \mu(a_1)\pi_W(a_2) \circ f, \quad \forall a \in A.$$

A **morphism**  $X \longrightarrow Y$  is a linear combination of basic morphisms  $X \longrightarrow Y$ .

It is easy to check that the composition of two basic morphisms is again a basic morphism. Therefore, extending the composition by bilinearity to all morphisms, we obtain a **category of projective representations of  $A$** , which we denote by  $\mathbb{P}(A)$ . Clearly  $\mathbb{P}(A)$  is a  $k$ -additive category. We are going to show that  $\mathbb{P}(A)$  has a structure of monoidal  $H_L^2(A)$ -category over  $k$ . The basic tool for proving the  $H_L^2(A)$ -splitting and for proving the existence of left duals is the generalized antipode of the Hopf-Galois system associated with a cocycle ([3]).

**Lemma 6.2** Let  $\sigma$  be a left 2-cocycle on a Hopf algebra  $A$ . Then the linear map  $\phi_\sigma : {}_\sigma A \longrightarrow A_{\sigma^{-1}}$  defined by  $\phi_\sigma(a) = \sigma(a_1, S(a_2))S(a_3)$ , is an algebra anti-morphism. Futhermore we have, for  $a \in A$ ,

$$\phi_\sigma(a_1)_{\cdot\sigma^{-1}} a_2 = \varepsilon(a)1 = a_{1\cdot\sigma^{-1}} \phi_\sigma(a_2).$$

**Proof:** Let  $a, b \in A$ . Since  $\sigma$  is a left 2-cocycle, we get

$$\begin{aligned} & \sigma(a_1, b_1)\sigma(a_2b_2, S(a_3b_3)) \\ &= \sigma(b_1, S(a_3b_4))\sigma(a_1, b_2S(a_2b_3)) \\ &= \sigma(b_1, S(a_3b_2))\sigma(a_1, S(a_2)) \\ &= \sigma^{-1}(S(b_5), S(a_4))\sigma(b_1, S(b_4))\sigma(b_2S(b_3), S(a_3))\sigma(a_1, S(a_2)) \\ &= \sigma(b_1, S(b_2))\sigma(a_1, S(a_2))\sigma^{-1}(S(b_3), S(a_3)). \end{aligned}$$

Using this computation, it is immediate to see that  $\phi$  is an algebra anti-morphism. The last statement is proved in the proof of [3, Proposition 2.1] (In fact since  $\phi_\sigma$  is the antipode of the Hopf-Galois system associated with  $\sigma$ , the anti-multiplicativity of  $\phi_\sigma$  also follows from [3, Corollary 1.10]).  $\square$

**Lemma 6.3** Let  $X = (\sigma, V, \pi_V)$  and  $Y = (\omega, W, \pi_W)$  be some projective representations of  $A$  and let  $f : X \longrightarrow Y$  be a non-zero basic morphism. Then  $\bar{\sigma} = \bar{\omega}$  in  $H_L^2(A)$ .

**Proof:** Let  $\mu \in \text{Reg}_L^1(A)$  satisfying  $f \circ \pi_V(a) = \mu(a_1)\pi_W(a_2) \circ f, \forall a \in A$ . Let  $a, b \in A$ . We have, using Lemma 6.2 and the laziness of  $\sigma$  and  $\mu$ ,

$$\begin{aligned}
\sigma(a_1, b_1)\mu(a_2b_2)f &= \sigma(a_1, b_1)\mu(a_2b_2)\varepsilon(a_3b_3)f \\
&= \sigma(a_1, b_1)\mu(a_2b_2)\pi_W(\phi_{\omega^{-1}}(a_3b_3)) \circ \pi_W(a_4b_4) \circ f \\
&= \pi_W(\phi_{\omega^{-1}}(a_1b_1))\sigma(a_2, b_2)\mu(a_3b_3)\pi_W(a_4b_4) \circ f \\
&= \pi_W(\phi_{\omega^{-1}}(a_1b_1))\sigma(a_2, b_2)f \circ \pi_V(a_3b_3) \\
&= \pi_W(\phi_{\omega^{-1}}(a_1b_1)) \circ f \circ \pi_V(a_2) \circ \pi_V(b_2) \\
&= \pi_W(\phi_{\omega^{-1}}(a_1b_1))\mu(a_2)\mu(b_2)\pi_W(a_3) \circ \pi_W(b_3) \circ f \\
&= \pi_W(\phi_{\omega^{-1}}(a_1b_1))\mu(a_2)\mu(b_2)\omega(a_3, b_3)\pi_W(a_4b_4) \circ f \\
&= \mu(a_1)\mu(b_1)\omega(a_2, b_2)\pi_W(\phi_{\omega^{-1}}(a_3b_3))\pi_W(a_4b_4) \circ f \\
&= \mu(a_1)\mu(b_1)\omega(a_2, b_2)f.
\end{aligned}$$

Hence, since  $f$  is non-zero, we have  $\sigma = \partial(\mu)\omega$ .  $\square$

Let  $x \in H_L^2(A)$ . We define  $\mathbb{P}_x(A)$  to be the full subcategory of  $\mathbb{P}(A)$  consisting of projective representations  $(\sigma, V, \pi_V)$  with  $\bar{\sigma} = x$ . It is clear from the above lemma that the  $k$ -additive category  $\mathbb{P}(A)$  is  $H_L^2(A)$ -split, with  $(\mathbb{P}_x(A))_{x \in H_L^2(A)}$  as  $H_L^2(A)$ -splitting.

Let us now endow  $\mathbb{P}(A)$  with a monoidal structure. Let  $X = (\sigma, V, \pi_V)$  and  $Y = (\omega, W, \pi_W)$  be some projective representations of  $A$ . Since  $\Delta : A(\sigma * \omega) \rightarrow A(\sigma) \otimes A(\omega)$  is an algebra morphism, it follows that  $\pi_{V \otimes W} = (\pi_V \otimes \pi_W) \circ \Delta : A(\sigma * \omega) \rightarrow \text{End}(V \otimes W)$ , is an algebra morphism. Thus we have defined a new projective representation of  $A$ :

$$X \otimes Y := (\sigma * \omega, V \otimes W, \pi_{V \otimes W}).$$

It is easy to check that the tensor product of two basic morphism is still a basic morphism, and hence the tensor product of two morphisms is still a morphism. In this way  $(\mathbb{P}(A), \otimes, \mathbf{1})$ , endowed with the obvious associativity constraints and with  $\mathbf{1} = (\varepsilon \otimes \varepsilon, k, \varepsilon)$  as monoidal unit, is a  $k$ -additive monoidal category. Let us now check that every object has a left dual.

**Lemma 6.4** *Let  $X = (\sigma, V, \pi_V)$  be a projective representation of  $A$ . Let  $\pi_{V^*} : A \rightarrow \text{End}(V^*)$  be defined by  $\pi_{V^*}(a) = {}^t\pi_V(\phi_{\sigma^{-1}}(a))$ . Then*

$$X^* := (\sigma^{-1}, V^*, \pi_{V^*})$$

*is a projective representation of  $A$ , and is a left dual for  $X$ .*

**Proof:** It follows from Lemma 6.2, that  $\pi_{V^*}$  is an algebra morphism and hence  $X^*$  is a projective representation. Let  $e : V^* \otimes V \rightarrow k$  be the evaluation map. Let

$f \in V^*$  and  $v \in V$ . Then, by Lemma 6.2, we have for  $a \in A$ ,

$$\begin{aligned} e \circ \pi_{V^* \otimes V}(a)(f \otimes v) &= e((f \circ \pi_V(\phi_{\sigma^{-1}}(a_1))) \otimes \pi_V(a_2)(v)) \\ &= f \circ \pi_V(\phi_{\sigma^{-1}}(a_1)) \circ \pi_V(a_2)(v) \\ &= f \circ \pi_V(\phi_{\sigma^{-1}}(a_1) \cdot_{\sigma} a_2)(v) \\ &= \varepsilon(a)f(v) = \varepsilon(a)e(f \otimes v). \end{aligned}$$

This shows that  $e : X^* \otimes X \rightarrow \mathbf{1}$  is a morphism of projective representations. One shows similarly that the coevaluation map  $\delta : k \rightarrow V \otimes V^*$  is a morphism  $\mathbf{1} \rightarrow X \otimes X^*$ . Thus the triplet  $(X^*, e, \delta)$  is a left dual for  $X$  in  $\mathbb{P}(A)$ .  $\square$

Let  $X$  and  $Y$  be some projective representations of  $A$ . It is clear that if  $X \in \mathbb{P}_x(A)$  and  $Y \in \mathbb{P}_y(A)$ , then  $X \otimes Y \in \mathbb{P}_{xy}(A)$  and  $X^* \in \mathbb{P}_{x^{-1}}(A)$ . Summarizing the results of the section, we have proved:

**Theorem 6.5** *Let  $A$  be a Hopf algebra. Then  $\mathbb{P}(A)$  is a monoidal  $H_L^2(A)$ -category over  $k$ .*  $\square$

## 7 Examples: monomial Hopf algebras

In this section  $k$  is a field containing all primitive roots of unity. We compute the second lazy cohomology group for the monomial Hopf algebras.

Recall ([11]) that a **group datum** (over  $k$ ) is a quadruplet  $\mathbb{G} = (G, g, \chi, \mu)$  consisting of a finite group  $G$ , a central element  $g \in G$ , a character  $\chi : G \rightarrow k$  with  $\chi(g) \neq 1$  and an element  $\mu \in k$  such that  $\mu = 0$  if  $o(g) = o(\chi(g))$ , and that if  $\mu \neq 0$ , then  $\chi^{o(\chi(g))} = 1$ .

Let  $\mathbb{G} = (G, g, \chi, \mu)$  be a group datum. A Hopf algebra  $A(\mathbb{G})$  is associated with  $\mathbb{G}$  in [11]. We will slightly change the conventions of [11] for the formula defining the coproduct but this will not change the whole set of isomorphism classes. As an algebra  $A(\mathbb{G})$  is the quotient of the free product algebra  $k[x] * k[G]$  by the two-sided ideal generated by the relations

$$xh = \chi(h)hx, \quad \forall h \in G, \quad x^d = \mu(1 - g^d), \quad \text{where } d = o(\chi(g)).$$

The Hopf algebra structure of  $A(\mathbb{G})$  is defined by:

$$\begin{aligned} \Delta(x) &= 1 \otimes x + x \otimes g, \quad \varepsilon(x) = 0, \quad S(x) = -xg^{-1}, \\ \Delta(h) &= h \otimes h, \quad \varepsilon(h) = 1, \quad S(h) = h^{-1}, \quad \forall h \in G. \end{aligned}$$

Using the diamond lemma [2], it is not difficult to see that the set  $\{hx^i, 0 \leq i \leq d-1, h \in G\}$  is a linear basis of  $A(\mathbb{G})$ , and hence  $\dim_k(A(\mathbb{G})) = |G|d$ .

The Hopf algebras  $A(\mathbb{G})$  have been shown in [11] to be exactly the monomial non semisimple Hopf algebras: see [11] for the precise concept of a monomial Hopf algebra.

We need the notion of type I group datum introduced in [4]. A **type I group datum**  $\mathbb{G} = (G, g, \chi, \mu)$  is a group datum with  $\mu = 0$ ,  $d = o(\chi(g)) = o(g)$  and  $\chi^d = 1$ . In this case we simply write  $\mathbb{G} = (G, g, \chi)$ . We will not need the other types of group data. We can state now the main result of the section.

**Theorem 7.1** *Let  $\mathbb{G} = (G, g, \chi, \mu)$  be a group datum. Then we have the following description for  $H_L^2(A(\mathbb{G}))$ .*

- *If  $\mathbb{G}$  is of type I, then  $H_L^2(A(\mathbb{G})) \cong H^2(G/\langle g \rangle, k) \times k$ .*
- *If  $\mathbb{G}$  is not of type I, then  $H_L^2(A(\mathbb{G})) \cong H^2(G/\langle g \rangle, k)$ .*

The proof will use the description of the biGalois objects given in [4]. Before going into the heart of the proof, we need some cohomological preliminaries. Let us first recall the definition, introduced in [4], of the modified second cohomology group of a group.

Let  $G$  be a group and let  $g \in G$  be a central element. We put

$$Z_g^2(G, k) = \{\sigma \in Z^2(G, k), \sigma(g, h) = \sigma(h, g), \forall h \in G\} \quad \text{and}$$

$$B_g^2(G, k) = \{\partial(\mu), \mu : G \rightarrow k, \mu(g) = 1 = \mu(1)\}.$$

For  $g_1, g_2 \in Z(G)$ , it is clear that  $B_{g_2}^2(G, k)$  is a subgroup of  $Z_{g_1}^2(G, k)$  and we define

$$H_{g_1, g_2}^2(G, k) = Z_{g_1}^2(G, k) / B_{g_2}^2(G, k).$$

We have  $H_{1,1}^2(G, k) = H^2(G, k)$ . Let us introduce another group now. We define  $Z_{L,g}^2(G, k)$  to be the subset of  $Z_g^2(G, k)$  consisting of elements  $\sigma$  such that there exists  $\mu : G \rightarrow k$  satisfying  $\mu(g) = 1 = \mu(1)$  and  $\sigma(g, h) = \mu(h)\mu(gh)^{-1}$ ,  $\forall h \in G$ . It is clear that  $Z_{L,g}^2(G, k)$  is a subgroup of  $Z_g^2(G, k)$  containing  $B_g^2(G, k)$ . So we put

$$L_g(G, k) = Z_{L,g}^2(G, k) / B_g^2(G, k).$$

This subgroup of  $H_{g,g}^2(G, k)$  will appear naturally in the study of the bicleft biGalois objects. In fact we have:

**Lemma 7.2** *We have a group isomorphism  $L_g(G, k) \cong H^2(G/\langle g \rangle, k)$ .*

**Proof:** Let  $\pi : G \rightarrow G/\langle g \rangle$  be the canonical surjection. Let  $\sigma \in Z^2(G/\langle g \rangle, k)$ : it is clear that  $\sigma \circ (\pi \times \pi) \in Z_{L,g}^2(G, k)$  and that if  $\sigma$  is a coboundary, then  $\sigma \circ (\pi \times \pi)$  belongs to  $B_g^2(G, k)$ . Therefore we get a group morphism

$$\begin{aligned} \theta : H^2(G/\langle g \rangle, k) &\longrightarrow L_g(G, k) \\ \bar{\sigma} &\longmapsto \overline{\sigma \circ (\pi \times \pi)}. \end{aligned}$$

Let us show that  $\theta$  is an isomorphism.

**Claim 1.**  $\theta$  is injective.

*Proof of Claim 1.* Let  $\sigma \in Z^2(G/\langle g \rangle, k)$  be such that there exists  $\phi : G \rightarrow k$  such that  $\phi(g) = 1 = \phi(1)$  and  $\sigma \circ (\pi \times \pi) = \partial(\phi)$ . Computing  $\sigma(\pi \times \pi)(g, h) = \sigma(\pi \times \pi)(1, h) = 1$  we see that  $\phi(hg) = \phi(h)$  for any  $h \in G$ . Hence there exists  $\phi' : G/\langle g \rangle \rightarrow k$  such that  $\phi = \phi' \circ \pi$  and then  $\sigma \circ (\pi \times \pi) = \partial(\phi' \circ \pi)$ . We conclude that  $\sigma$  is a coboundary and that  $\theta$  is injective.

**Claim 2.** Let  $\mu : G \rightarrow k$  with  $\mu(1) = 1$  be such that  $\partial(\mu) \in Z_{L,g}^2(G, k)$ . Then  $\overline{\partial(\mu)} \in \text{Im}(\theta)$ .

*Proof of Claim 2.* Let  $\omega \in Z_{L,g}^2(G, k)$ . Then one can find  $\omega' \in Z_{L,g}^2(G, k)$  having the same class as  $\omega$  in  $L_g(G, k)$  such that  $\omega'(g, h) = \omega'(h, g) = 1$ . Hence we can assume that  $\mu(g h) = \mu(g)\mu(h)$ ,  $\forall h \in G$ . With this assumption the restriction of  $\mu$  to  $\langle g \rangle$  is a character.

Now let us fix  $s : G/\langle g \rangle \rightarrow G$  a section of  $\pi$  with  $s(1) = 1$ . We have  $G = \langle g \rangle s(G/\langle g \rangle)$  and we define a map  $\gamma : G \rightarrow k$  by  $\gamma(g^i s(X)) = \mu(s(X))$  for  $i \in \mathbb{Z}$  and  $X \in G/\langle g \rangle$ . We have  $\gamma(g) = 1$ . Now consider the function  $f_s : G/\langle g \rangle \times G/\langle g \rangle \rightarrow \langle g \rangle$  defined by  $f_s(X, Y) = s(X)s(Y)s(XY)^{-1}$ . Then  $\mu \circ f_s$  is a 2-cocycle (in fact the image of the character  $\mu|_{\langle g \rangle}$  by the transgression map) and we put  $\sigma = (\mu \circ f_s)^{-1}$ . A straightforward computation shows that  $\partial(\mu) = (\partial(\gamma)) * (\sigma \circ (\pi \times \pi))$ , and hence  $\overline{\partial(\mu)} = \theta(\overline{\sigma})$ .

**Claim 3.**  $\theta$  is surjective.

*Proof of Claim 3.* Let  $\sigma \in Z_{L,g}^2(G, k)$ . As in the proof of Claim 2, we can assume that  $\sigma(g, h) = \sigma(h, g) = 1$  for any  $h \in G$ . Then the restriction of  $\sigma$  to  $\langle g \rangle$  is trivial and the group pairing  $G \times \langle g \rangle \rightarrow k$ ,  $(h, g^i) \mapsto \sigma(h, g^i)\sigma(g^i, h)^{-1}$ , is trivial. Hence we can use the exact sequence of Iwahori and Matsumoto (see [19, Theorem 2.2.7]): there exists  $\omega \in Z^2(G/\langle g \rangle, k)$  and  $\mu : G \rightarrow k$  with  $\mu(1) = 1$  such that  $\sigma = (\omega \circ (\pi \times \pi)) * \partial(\mu)$ . Then  $\partial(\mu) \in Z_{L,g}^2(G, k)$  and, by Claim 2,  $\overline{\sigma} \in \text{Im}(\theta)$ .

The proof of the Lemma is now complete.  $\square$

Let  $\mathbb{G} = (G, g, \chi, 0)$  be a group datum. Let us recall now the description of the  $A(\mathbb{G})$ -biGalois objects. Let  $\sigma \in Z^2(G, k)$ , let  $u \in \text{Aut}_g(G)$  (i.e.  $u(g) = g$ ) and let  $a \in k$ . Assume that the triplet  $(\sigma, u, a)$  satisfies the following compatibility conditions:

$$\chi \circ u(h) = \sigma(g, h)^{-1} \sigma(h, g) \chi(h), \quad \forall h \in G \quad (7.4)$$

$$a = 0 \text{ if } \mathbb{G} \text{ is not of type I} \quad (7.5)$$

Let the algebra  $A_{\sigma,a}^u(\mathbb{G})$  be the algebra presented by generators  $X, (T_h)_{h \in G}$  with defining relations,  $\forall h, h_1, h_2 \in G$ :

$$T_{h_1} T_{h_2} = \sigma(h_1, h_2) T_{h_1 h_2}, \quad T_1 = 1, \quad X T_h = \chi(h) T_h X, \quad X^d = a T_{g^d}.$$



It is shown in [4] that  $A_{\sigma,a}^u(\mathbb{G})$  is an  $A(\mathbb{G})$ -biGalois object, with respective right and left coactions  $\rho : A_{\sigma,a}^u(\mathbb{G}) \longrightarrow A_{\sigma,a}^u(\mathbb{G}) \otimes A(\mathbb{G})$  and  $\beta : A_{\sigma,a}^u(\mathbb{G}) \longrightarrow A(\mathbb{G}) \otimes A_{\sigma,a}(\mathbb{G})$  defined by

$$\begin{aligned}\rho(X) &= 1 \otimes x + X \otimes g, & \rho(T_h) &= T_h \otimes h, & \forall h \in G, \\ \beta(X) &= 1 \otimes X + x \otimes T_g, & \beta(T_h) &= u(h) \otimes T_h, & \forall h \in G.\end{aligned}$$

Every  $A(\mathbb{G})$ -biGalois object is isomorphic to one of the form  $A_{\sigma,a}^u(\mathbb{G})$  for some triplet  $(\sigma, u, a)$  satisfying (7.4) and (7.5).

**Lemma 7.3** *Let  $(\sigma, u, a)$  be a triplet as above. Then the  $A(\mathbb{G})$ -biGalois object  $A_{\sigma,a}^u(\mathbb{G})$  is bicleft if and only if  $\sigma \in Z_{L,g}^2(G, k')$  and  $u = \text{id}_G$ .*

**Proof:** Let  $f : A_{\sigma,a}^u(\mathbb{G}) \longrightarrow A(\mathbb{G})$  be a bilinear isomorphism. We can assume that  $f(1) = 1$ . By [4, Proposition 2.3] there is a right  $A(\mathbb{G})$ -colinear isomorphism  $\Phi : A(\mathbb{G}) \longrightarrow A_{\sigma,a}^u(\mathbb{G})$ ,  $hx^i \longmapsto T_h X^i$ . Hence, since  $f \circ \Phi$  is a right  $A(\mathbb{G})$ -colinear automorphism of  $A(\mathbb{G})$ , there exists  $\mu \in \text{Reg}^1(A(\mathbb{G}))$  such that  $f \circ \Phi = \mu * \text{id}$ . Thus we have, for  $h \in G$  and  $0 \leq i \leq d-1$

$$f(T_h X^i) = \sum_{l=0}^i \binom{i}{l}_q \mu(hx^{i-l}) h g^{i-l} x^l,$$

where  $q = \chi(g)$  and we have used the  $q$ -binomial coefficients. In particular  $f(T_h) = \mu(h)h$  for  $h \in G$ , and since  $f$  is left colinear, we find that  $u = \text{id}_G$ . By condition (7.4)  $\sigma \in Z_g^2(G, k')$ . We also have  $f(T_h X) = \mu(hx)hg + \mu(h)hx$  and using again the left colinearity of  $f$ , we find that  $\sigma(h, g) = \mu(h)\mu(hg)^{-1}$  and therefore  $\mu(g) = 1$ , which means that  $\sigma \in Z_{L,g}^2(G, k')$ . Conversely, assume that  $\sigma \in Z_{L,g}^2(G, k')$  and that  $u = \text{id}_G$ . As for claim 2 in the proof of Lemma 7.2 we can assume, without changing the class of  $\sigma$  in  $H_{g,g}^2(G, k')$  and hence without changing the isomorphism class of the  $A(\mathbb{G})$ -bicomodule algebra  $A_{\sigma,a}^u(\mathbb{G})$  ([4, Proposition 3.4]), that  $\sigma(g, h) = \sigma(h, g) = 1$ ,  $\forall h \in G$ . Then we define a linear isomorphism  $f : A_{\sigma,a}^u(\mathbb{G}) \longrightarrow A(\mathbb{G})$  by  $f(T_h X^i) = hx^i$ , for  $h \in G$  and  $0 \leq i \leq d-1$ . One can check that  $f$  is a bilinear isomorphism using  $\sigma(g, h) = \sigma(h, g) = 1$ ,  $\forall h \in G$  for the left colinearity.  $\square$

**Proof of Theorem 7.1.** Let  $\mathbb{G} = (G, g, \chi, \mu)$  be a group datum. Assume first that  $\mathbb{G}$  is a type I group datum, i.e. that  $\mu = 0$ , that  $o(\chi(g)) = d = o(g)$  and that  $\chi^d = 1$ . Then by Lemma 7.3 we have a map

$$\begin{aligned}\Psi_0 : Z_{L,g}^2(G, k') \times k &\longrightarrow \text{Bicleft}(A(\mathbb{G})) \\ (\sigma, a) &\longmapsto A_{\sigma,a}^{\text{id}}(\mathbb{G}).\end{aligned}$$

For  $\sigma \in Z_{L,g}^2(G, k')$ , we have  $\sigma(g, g) \dots \sigma(g, g^{d-1}) = 1$ , so by [4, Proposition 3.5]  $\Psi_0$  is a group morphism. Then  $\Psi_0$  induces an injective group morphism  $\Psi : L_g(G, k') \times k \longrightarrow \text{Bicleft}(A)$  by [4, Proposition 3.4]. Now let  $Z$  be an  $A(\mathbb{G})$ -biGalois object: by

[4, Proposition 3.8] there exists a triplet  $(\sigma, u, a)$  as above such that  $[Z] = [A_{\sigma,a}^u(\mathbb{G})]$ . If  $Z$  is bicleft, we have  $\sigma \in Z_{L,g}^2(G, k)$  and  $u = \text{id}_G$  by Lemma 7.3 and therefore  $\Psi$  is surjective and it is an isomorphism. Then Lemma 7.2 concludes the proof in the type I case.

Assume now that  $\mathbb{G} = (G, g, \chi, 0)$  is not of type I. Then the proof, using the results in [4, Section 3], is essentially the same as the one of the type I case. This is left to the reader.

Finally if  $\mathbb{G} = (G, g, \chi, \mu)$  with  $\mu \neq 0$ , then by [4, Corollary 3.18], there exists an  $A(\mathbb{G})$ - $A(\mathbb{G}_{\text{red}})$ -biGalois object for  $\mathbb{G}_{\text{red}} = (G, g, \chi, 0)$  so the statement follows from the previous case and Corollary 3.9  $\square$

**Example 7.4** Recall ([4]) that a cyclic datum is a datum  $(d, n, N, \alpha, q)$  where  $d, n, N > 1$  are integers,  $\alpha \in \mathbb{N}$  and  $q \in k$  is a root of unity, satisfying:

$$d|n|N, \quad \alpha | \frac{N}{n}, \quad \text{GCD}(\alpha, d) = 1, \quad o(q) = \frac{Nd}{\alpha n}.$$

To any cyclic datum  $(d, n, N, \alpha, q)$  we associate a group datum

$$\mathbb{C}[d, n, N, \alpha, q] := (C_N = \langle z \mid z^N = 1 \rangle, g = z^{\frac{N}{n}}, \chi_q, 0)$$

where  $\chi_q$  is the character defined by  $\chi_q(z) = q$ . The associated Hopf algebra  $A(d, n, N, \alpha, q) = A(\mathbb{C}[d, n, N, \alpha, q])$  is then the algebra presented by generators  $z, x$  submitted to the relations

$$x^d = 0, \quad z^N = 1, \quad xz = qzx.$$

The coproduct is defined by  $\Delta(z) = z \otimes z$  and  $\Delta(x) = 1 \otimes x + x \otimes z^{\frac{N}{n}}$ . Using [4, Lemma 4.4] and Theorem 7.1 we find that

$$H_L^2(A(d, n, N, \alpha, q)) \cong \begin{cases} k & \text{if } d = n = N, \\ (k \cdot / (k \cdot)^{\frac{N}{n}}) \times k & \text{if } d = n < N, \\ & \text{GCD}(\frac{N}{n}, n) = 1 \text{ and } \alpha = \frac{N}{n}, \\ k \cdot / (k \cdot)^{\frac{N}{n}} & \text{otherwise.} \end{cases}$$

As a particular case, for the Taft algebras  $H_{N,q} = A(\mathbb{C}[N, N, N, 1, q])$ , we get  $H_L^2(H_{N,q}) \cong k$ .

**Example 7.5** Let

$$G = \langle a, b, g \mid a^2 = 1 = b^2 = g^4, \quad ag = ga, \quad bg = gb, \quad ab = bag^2 \rangle.$$

$G$  is a non-abelian group of order 16 with  $g \in Z(G)$  and  $o(g) = 4$ . Let  $\chi: G \rightarrow k$  be the character defined by  $\chi(a) = \chi(b) = 1, \chi(g) = -1$ . We consider the group datum

$\mathbb{G} = (G, g, \chi)$ . For any  $\phi \in \text{Alg}(A(\mathbb{G}), k)$  we have  $\text{ad}(\phi)(x) = \pm x$ . Let  $\mu_0: G \rightarrow k$  be defined by:  $\mu_0(a^\alpha b^\beta g^\gamma) = (\sqrt{-1})^\gamma$ . Now let  $\mu \in \text{Reg}^1(A(\mathbb{G}))$  be defined by:  $\mu(x^i h) = \delta_{i,0} \mu_0(h)$  for  $h \in G$ . Since  $\mu_0(gh) = \mu_0(g)\mu_0(h)$  for every  $h \in G$  it follows that  $\text{ad}(\mu)$  is a Hopf algebra automorphism of  $A(\mathbb{G})$  with  $\text{ad}(\mu)(x) = \sqrt{-1}x$ . Therefore  $\text{ad}(\mu)$  is not coinner, so  $\text{CoInt}(A(\mathbb{G})) \neq \text{CoInn}(A(\mathbb{G}))$  which means that  $\text{CoOut}^-(A(\mathbb{G}))$  is non-trivial.

## 8 Lazy cohomology for some cotriangular Hopf algebras

In this section the base field will be  $\mathbb{C}$ . We shall start recalling notation and results in [1], [15], [16] and [17]. For terminology we mainly refer to these papers.

Let  $\mathcal{A}$  be a finite-dimensional Hopf superalgebra, with a grouplike element  $g$  such that  $g x g^{-1} = (-1)^{\deg(x)} x$  for every homogeneous element in  $\mathcal{A}$ . Then we can apply bosonization (see [23, §9.4], [1, §3.1]) obtaining a finite-dimensional Hopf algebra  $A$ . The Hopf algebra  $A$  is equal to  $\mathcal{A}$  as an algebra but with coproduct given, for homogeneous elements, by:  $\Delta_A(h) = g^{\deg(h_2)} h_1 \otimes (-1)^{(\deg(h_2)(1+\deg(h)))} h_2$  if  $\Delta_{\mathcal{A}}(h) = h_1 \otimes h_2$ . The Hopf superalgebra  $\mathcal{A}$  is triangular with (necessarily even)  $R$ -matrix  $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1 \in \mathcal{A}_0 \otimes \mathcal{A}_0 + \mathcal{A}_1 \otimes \mathcal{A}_1$  if and only if  $A$  is triangular with  $R$ -matrix  $R = (\mathcal{R}_0 + (1 \otimes g)\mathcal{R}_1)R_g$ , where  $R_g = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)$ .

Let  $\mathcal{A}$  be a finite-dimensional cocommutative Hopf superalgebra over  $\mathbb{C}$ . By [21, Theorem 3.3]  $\mathcal{A} = \mathbb{C}[G] \ltimes \wedge W$  where  $W$  is the (purely odd) space of primitive elements and  $G$  is the group of grouplikes acting on  $W$ , hence on  $\wedge W$ , by conjugation.

If  $G$  contains an element such that  $g^2 = 1$  and such that  $g x g^{-1} = (-1)^{\deg(x)} x$ , the procedure above described yields a triangular Hopf algebra  $A$  with  $R$ -matrix  $R_g$ .

By [1, Proposition 3.4.1] if  $r \in S^2(W)$ , the symmetric algebra of  $W$  (viewed in  $W \otimes W$ ), then  $\mathcal{J} = e^{r/2} \in \mathcal{A} \otimes \mathcal{A}$  satisfies

$$(\Delta \otimes \text{id}_A)(\mathcal{J})\mathcal{J}_{12} = (\text{id}_A \otimes \Delta)(\mathcal{J})\mathcal{J}_{23} \text{ and } (\varepsilon \otimes \text{id}_A)(\mathcal{J}) = (\text{id}_A \otimes \varepsilon)(\mathcal{J}) = 1, \quad (8.6)$$

i.e, it is a **Drinfeld twist** for  $\mathcal{A}$ . If we write  $\mathcal{J} = \mathcal{J}_0 + \mathcal{J}_1$  with  $\mathcal{J}_i \in \mathcal{A}_i \otimes \mathcal{A}_i$  for  $i = 0, 1$ , then  $J = \mathcal{J}_0 - (g \otimes 1)\mathcal{J}_1$  is a Drinfeld twist for  $A$ , i.e.,  $J$  is a right 2-cocycle for  $A^*$ . It is not hard to check that if we take  $r \in S^2(W)^G$ , the invariants under the  $G$ -action, then  $J$  commutes with  $\Delta(a)$  for every  $a \in A$ , i.e,  $J$  is a lazy cocycle for  $A^*$ . We shall call a lazy cocycle for  $A^*$  also a **lazy twist** for  $A$ . The dual version of cohomology of cocycles is **gauge equivalence**: two Drinfeld twists  $J$  and  $F$  for a Hopf algebra  $H$  are said to be **gauge equivalent** if  $F = \Delta(x)J(x^{-1} \otimes x^{-1})$  for some  $x \in H$  (see [15], [23] for details). In particular, two gauge equivalent Drinfeld twists  $F$  and  $J$  are cohomologous in lazy cohomology for  $H^*$  if and only if the element  $x$  can be chosen to be central in  $H$ .

By [1], [16] the triangular Hopf algebra  $A$ , with  $R$ -matrix  $R_g$  is the key model of finite-dimensional triangular Hopf algebras over  $\mathbb{C}$ . Indeed, all other such Hopf

algebras are obtained from a Hopf algebra of this type twisting the coproduct. In other words, for every finite-dimensional triangular Hopf algebra  $H$  there exists a Drinfeld twist  $J$  such that  $H^J \cong A$  for some  $G$  and  $W$ , where  $H^J$  has the same underlying algebra as  $H$  and  $\Delta_{H^J}(h) = J^{-1}\Delta_H(h)J$  for every  $h \in H$ . The  $R$ -matrix of  $H^J$  is  $(\tau J)^{-1}R_gJ$ . Therefore,  $A^*$  is the key model of finite-dimensional cotriangular Hopf algebras over  $\mathbb{C}$  and we have a method for the construction of special lazy 2-cocycles. More precisely,

**Lemma 8.1** *Let  $A = (\mathbb{C}[G] \ltimes \wedge W)^*$  with  $g \in A$  and with coproduct as before. Then the assignment  $r \mapsto \mathcal{J} = e^r \mapsto J$  defines an injective group morphism  $\Sigma: S^2(W)^G \rightarrow H_L^2(A)$ .*

**Proof:** If  $\mathcal{J} = e^m$  and  $\mathcal{J}' = e^{m'}$  in the corresponding Hopf superalgebra, then  $J = 1 \otimes 1 - (g \otimes 1)m + \dots$  and  $J' = 1 \otimes 1 - (g \otimes 1)m' + \dots$  so  $J * J' = 1 \otimes 1 - (g \otimes 1)(m + m') + \dots$  and the assignment gives a group morphism  $S^2(W)^G \rightarrow Z_L^2(A)$ . Combined with the standard projection we have a group morphism  $\Sigma: S^2(W)^G \rightarrow H_L^2(A)$ . If for some  $r \in S^2(W)^G$  we had  $\Sigma(r) = (z^{-1} \otimes z^{-1})\Delta(z)$  for some central  $z \in A^*$ , then the  $R$ -matrix of  $A^*$  obtained through  $\Sigma(r)$  would be  $\tau(\Delta z^{-1})R_g\Delta(z) = R_g$  because  $R_g$  is an  $R$ -matrix. The corresponding  $R$ -matrix in the Hopf superalgebra  $\mathcal{A}^*$  is  $e^{2r} = R_g^2 = 1 \otimes 1$ . Hence  $r = 0$  and  $\Sigma$  is injective.  $\square$

Example 2.2 shows that for the family  $E(n)$  this construction exhausts all the lazy 2-cohomology classes. This holds in a more general framework.

Let  $(A, R)$  be a finite-dimensional cotriangular Hopf algebra. Then there exists a cocycle  $\omega$ , a group  $G$  acting on a vector space  $W$ , and a central element  $g \in G \subset A^*$  acting as  $-1$  on  $W$  and such that  $g^2 = 1$  for which  $(\omega A_{\omega^{-1}})^* \cong \mathbb{C}[G] \ltimes \wedge W$  and the  $r$ -form corresponding to  $R$  under twist is  $R_g := \frac{1}{2}(\varepsilon \otimes \varepsilon + \varepsilon \otimes g + g \otimes \varepsilon - g \otimes g)$ . In particular, if  $g = \varepsilon$  then  $W$  is trivial and  $R_g = \varepsilon \otimes \varepsilon$ . The data  $G$  and  $W$  are unique up to isomorphism and the twist is unique up to gauge equivalence. By abuse of language, we will also say that  $G$  and  $W$  are data associated to  $A$ .

Let  $A$  be a cotriangular Hopf algebra with associated data  $G$  and  $W$ . We shall denote by  $\{w_1, \dots, w_n\}$  a fixed basis of  $W$  and we shall denote by  $\rho: G \rightarrow \text{GL}_n(\mathbb{C})$  the group morphism given by:  $g^{-1}w_i g = \sum_j \rho(g)_{ij}w_j$  for every  $i$ .

**Theorem 8.2** *Let  $A$  be a finite-dimensional cotriangular Hopf algebra, with associated data  $G$  and  $W$ . If the representation  $\rho$  of  $G$  on  $W$  is faithful then  $H_L^2(A) \cong S^2(W)^G$ .*

**Proof:** By Corollary 3.9 it is enough to prove the result when  $A^* = \mathbb{C}[G] \ltimes \wedge W$  because the hypothesis on  $\rho$  still holds if we twist the coproduct of  $A^*$ . We shall use the terminology of Drinfeld twists rather than the terminology of 2-cocycles. Let  $F$  be a lazy twist for  $A^*$  and let  $\pi$  be the Hopf algebra projection of  $A^*$  onto  $\mathbb{C}[G]$ . Then  $\bar{F} = (\pi \otimes \pi)(F)$  is a lazy twist for  $\mathbb{C}[G]$  and, since  $\mathbb{C}[G]$  is also a

sub Hopf algebra of  $A^*$ , it is a twist for  $A^*$ . Since  $F$  commutes with  $g \otimes g$ , it is even in the  $\mathbb{Z}_2$ -gradation induced by the action of  $g$ . Then, if  $\text{Rad}(A^*)$  denotes the Jacobson radical of  $A^*$ , i.e., the ideal generated by the  $w_j$ 's,  $F = \overline{F} +$  terms in  $(\text{Rad}(A^*))^2 \otimes A^* + A^* \otimes (\text{Rad}(A^*))^2 + \text{Rad}(A^*) \otimes \text{Rad}(A^*)$ . The elements of the form  $hw_{i_1} \cdots w_{i_m}$  with  $1 \leq i_1 < \cdots < i_m \leq n$  and  $h \in G$  form a basis for  $A^*$ . Looking at the expression of  $F\Delta(w_i) = \Delta(w_i)F$  as a linear combination of the corresponding basis of  $A^* \otimes A^*$  we see that also  $\overline{F}$  commutes with  $\Delta(w_i)$  for every  $i$  (hence it is a lazy twist for  $A^*$ ). This implies that if  $\overline{F} = \sum_{s,h \in G} f_{sh}s \otimes h$  then

$$\sum_{s,h \in G} f_{sh}sg \otimes hw_i = \sum_{l,p \in G} \sum_j f_{lp}lu \otimes p\rho(p)_{ij}w_j$$

and

$$\sum_{s,h \in G} f_{sh}sw_i \otimes h = \sum_{l,p \in G} \sum_j f_{lp}l\rho(l)_{ij}w_j \otimes p.$$

The coefficient of  $gs \otimes hw_j$  in the first equality is:  $\delta_{ij}f_{sh} = f_{sh}\rho(h)_{ij}$  so if  $h$  is such that  $f_{sh} \neq 0$  for some  $s \in G$ , then  $h = 1$  because  $\rho$  is injective. With a similar computation from the second equality we get that  $\overline{F} = 1 \otimes 1$ .

Let us consider  $K = ((A^*)^F)$  and its corresponding Hopf superalgebra  $\mathcal{K} = ((A^*)^{\mathcal{F}})$ . The minimal part of  $\mathcal{K}$  satisfies the conditions of [1, Lemma 5.3.2], hence its  $R$ -matrix is of the form  $\mathcal{R}_{\mathcal{K}} = e^r$  for  $r \in S^2(W)$ . Since the corresponding  $R$ -matrix  $\mathcal{R}_{\mathcal{K}}$  commutes with  $\Delta(w)$  for every  $w$  in  $W$ ,  $r \in S^2(W)^G$ . By the classification in [16]  $\mathcal{K}$  is isomorphic, as a triangular Hopf superalgebra, to  $(A^*)^{\mathcal{J}}$  where  $\mathcal{J} = e^{r/2}$ . It follows from the classification in [15] that the corresponding twist  $J$  of  $A^*$  is gauge equivalent to  $F$ , i.e.,  $F = \Delta(z) * J * (z^{-1} \otimes z^{-1}) = J * \Delta(z) * (z^{-1} \otimes z^{-1})$  for  $z \in A^*$  with  $\Delta(z) * (z^{-1} \otimes z^{-1})$  a coboundary for  $A$  commuting with  $\Delta(A^*)$ . Since  $(\pi \otimes \pi)(J) = 1 \otimes 1$ , the projection of  $z$  is grouplike, so  $\pi(z) = h$  for some  $h \in G$ . If we replace  $z$  by  $zh^{-1}$  we see that  $F = J * \Delta(zh^{-1}) * (hz^{-1} \otimes hz^{-1})$  with  $\nu = zh^{-1} - 1$  nilpotent.  $\Delta(1 + \nu) * ((1 + \nu)^{-1} \otimes (1 + \nu)^{-1})$  centralizes  $\Delta(A^*)$  if and only if  $l \mapsto (1 + \nu)^{-1}l(1 + \nu)$  is a coalgebra map in  $A^*$ . But then for  $h \in G$ ,  $(1 + \nu)^{-1}h(1 + \nu)$  is grouplike if and only if it coincides with  $h$  and for  $w \in W$ ,  $(1 + \nu)^{-1}w(1 + \nu)$  is  $(g, 1)$  skew-primitive if and only if it coincides with  $w$ . Hence  $1 + \nu$  is central in  $A^*$  so  $\Delta(1 + \nu) * ((1 + \nu)^{-1} \otimes (1 + \nu)^{-1}) \in B_L^2(A)$ . So if  $F$  is a lazy 2-cocycle, then  $F$  is cohomologous to  $J$  with  $\mathcal{J} = e^m$  and  $m \in S^2(W)^G$ . It follows that the map  $\Sigma$  in Lemma 8.1 is surjective, whence the proof.  $\square$

Let us observe that in this case  $H_L^2(A)$  coincides with the Hochschild cohomology of  $A^*$ , which is computed in [16, Lemma 3.2].

**Remark 8.3** Let us observe that a Hopf algebra  $A$  isomorphic to  $\mathbb{C}[G] \ltimes \wedge W$  with  $g$  as before and with faithful  $G$ -action is in general not self dual, even for  $G$  abelian. Indeed, the intersection of the centre of  $A$  with its grouplikes is trivial, while the intersection of the grouplikes of its dual with the centre is  $\text{Alg}(\mathbb{C}[G/\langle g \rangle], k)$ .

**Proposition 8.4** *Let  $A = (\mathbb{C}[G] \ltimes \wedge W)^*$ . Suppose that there exists  $g \in G$  central acting as  $-1$  on  $W$  and such that  $g^2 = 1$  and suppose that the  $G$ -action on  $W$  is faithful. Then  $\text{CoInn}(A) = \text{CoInt}(A)$ .*

**Proof:** An invertible element  $x$  in  $A^* = \mathbb{C}[G] \ltimes \wedge W$  is in  $\text{Reg}_{aL}^1(A)$  if and only if conjugation by  $x$  in  $A^*$  is a coalgebra morphism. If  $x$  is central, there is nothing to prove. If  $x$  is not central, then  $x = \pi(x) + y = \pi(x)(1 + \nu)$  for some  $\nu, y \in \text{Rad}(A^*)$ , with  $\pi(x) \in \mathbb{C}[G]$  invertible. Conjugation by  $\pi(x)$  in  $\mathbb{C}[G]$  is a coalgebra map. Let  $x = x_0 + x_1$  in the  $\mathbb{Z}_2$ -gradation induced by conjugation by  $g$ . Then  $x^{-1}gx = h \in G$  and  $x_0 = x - x_1$  is invertible because all elements in the radical of  $A^*$  are nilpotent. Hence,  $x_0h + x_1h = gx_0 + gx_1 = x_0g - x_1g$  so  $x_0hg = x_0$  obtaining that  $h = g$  and  $x_1 = 0$ . In particular,  $x = x_0$  is even and conjugation by  $x$  maps each  $w_i$  to a linear combination of the  $w_j$ 's:  $x^{-1}w_ix = (1 + \nu)^{-1}\pi(x)^{-1}w_i\pi(x)(1 + \nu) = \sum_j t_{ij}w_j$  and

$$\pi(x)(1 + \nu) \sum_j t_{ij}w_j = w_i\pi(x)(1 + \nu).$$

Looking at linear combinations of elements of the basis of  $A^*$  we see that this implies that conjugation by  $\pi(x)$  gives already a coalgebra morphism. Then  $w_i\pi(x) = \sum_j t_{ij}\pi(x)w_j$ . Putting  $\pi(x) = \sum_{h \in G} c_h h$ , we have

$$\sum_{h \in G} c_h w_i h = \sum_j \sum_{v \in G} t_{ij} c_v v w_j = \sum_{j,l} \sum_{v \in G} t_{ij} c_v \rho(v^{-1})_{jl} w_l v.$$

As in the proof of Theorem 8.2, we have that  $\delta_{il}c_h = c_h \sum_j t_{ij}\rho(h^{-1})_{jl}$  for every  $i, l = 1, \dots, n$  and every  $h \in G$ . Then if  $c_h, c_v \neq 0$ ,  $\rho(h) = \rho(v) = T = (t_{ij})$  and by faithfulness of  $\rho$ ,  $h = v$  and  $\pi(x) = c_v v$  for some  $v$ . Up to rescaling of  $x$ , conjugation by  $\pi(x)$  gives an element of  $\text{CoInn}(A)$ , so we might as well assume that  $x = 1 + \nu$ . Again, using the basis of  $A^*$  we see that conjugation by  $1 + \nu$  can be a coalgebra morphism if and only if  $\nu$  is central, hence the statement.  $\square$

**Theorem 8.5** *Let  $A$  be a complex, finite-dimensional cotriangular Hopf algebra, with associated data  $G$  and  $W$  and with central element  $g \in A^*$ . If  $G = G' \times \langle g \rangle$  and the representation of  $G'$  on  $W$  is trivial then  $H_L^2(A) \cong S^2(W) \times H_L^2((\mathbb{C}[G'])^*)$ .*

**Proof:** As before it is enough to show the statement for  $A = (\mathbb{C}[G] \ltimes \wedge W)^*$ . If the representation of  $G'$  on  $W$  is trivial then  $A^* \cong \mathbb{C}[G'] \otimes (\mathbb{C}[g] \ltimes \wedge W)$ , so  $A \cong (\mathbb{C}[G']^*)^* \otimes E(n)$  where  $n = \dim W$ . By Theorem 4.8

$$H_L^2(A) \cong H_L^2(\mathbb{C}[G']^*) \times H_L^2(E(n)) \times \mathcal{ZP}((\mathbb{C}[G']^*)^*, E(n)).$$

By Example 2.2, it is enough to show that  $\mathcal{ZP}(\mathbb{C}[G']^*, E(n))$  is trivial. Let  $B$  be any Hopf algebra and let  $\beta$  be a central pairing between  $B$  and  $E(n)$ . We have, for  $b \in B$  and  $w \in W$  and for  $a = 0, 1$ :

$$\beta(b, g^a w) g^a + \beta(b, g^{a+1}) g^a w = \beta(b, g^a) g^a w + \beta(b, g^a w) g^{a+1}.$$

Therefore  $\beta(b, g^a w) = 0$  and  $\beta(b, g^a) = \beta(b, g^{a+1})$  for every  $b \in B$ , every  $a = 0, 1$  and every  $w \in W$ . Since  $\beta(b, 1) = \varepsilon(b)$ , we see that  $\beta(b, l) = \varepsilon(b)\varepsilon(l)$  for every  $b \in B$  and every  $l \in \mathbb{C}[\mathbb{Z}_2] \cdot W$ . Since  $\beta(b, cc') = \beta(b_1, c)\beta(b_2, c')$ , the group  $\mathcal{ZP}(B, E(n))$  is always trivial and we have the statement.  $\square$

## 9 Appendix: Laziness and general Hopf-Galois extensions

In this Appendix we study the relations between lazy cocycles and general crossed systems.

The actions in Remark 1.5 extend to all crossed systems and are not defined only on cocycles with values in  $k$ . We recall that a crossed system over a  $k$ -algebra  $R$  is a pair  $(\dashv, \sigma)$  where  $\dashv: A \rightarrow \text{End}(R)$  is a measuring of  $A$  on  $R$ ,  $\sigma$  is a convolution invertible linear map  $A \otimes A \rightarrow R$  and they satisfy the relations:

$$\begin{aligned}\sigma(a, 1) &= \sigma(1, a) = \varepsilon(a)1 \\ (a_1 \dashv b_1 \dashv x)\sigma(a_2, b_2) &= \sigma(a_1, b_1)(a_2 b_2 \dashv x) \\ \sigma(a_1, b_1)\sigma(a_2 b_2, c) &= (a_1 \dashv \sigma(b_1, c_1))\sigma(a_2, b_2 c_2)\end{aligned}$$

for every  $x \in R$  and for every  $a, b, c \in A$ .

**Proposition 9.1** *The group  $Z_L^2(A)$  acts by convolution on the right on the set of crossed systems over  $R$  corresponding to a fixed measuring  $\dashv$ .*

**Proof:** Let  $(\dashv, \sigma)$  be a crossed system of  $A$  on a  $k$ -algebra  $R$  and let  $\omega$  be a lazy 2-cocycle. Let then  $a, b \in A$  and  $x \in R$ . We have:

$$\begin{aligned}(a_1 \dashv (b_1 \dashv x))(\sigma * \omega)(a_2, b_2) &= (a_1 \dashv (b_1 \dashv x))\sigma(a_2, b_2)\omega(a_3, b_3) \\ &= \sigma(a_1, b_1)(a_2 b_2 \dashv x)\omega(a_3, b_3) \\ &= \sigma(a_1, b_1)\omega(a_2, b_2)(a_3 b_3 \dashv x) \\ &= (\sigma * \omega)(a_1, b_1)(a_2 b_2 \dashv x).\end{aligned}$$

The other relation is proved similarly. Hence  $(\dashv, \sigma * \omega)$  is again a crossed system corresponding to the measuring  $\dashv$ . Then it is clear that  $(\dashv, \sigma) \rightarrow (\dashv, \sigma * \omega)$  defines a right action of  $Z_L^2(A)$ .  $\square$

**Example 9.2** Let  $H_4$  be Sweedler's Hopf algebra. The action of  $Z_L^2(A)$  on the set of all crossed systems, parametrized as in [24, Table] by  $t$ -uples  $(\alpha, \delta, u, a, b, s)$  is as follows. If  $\sigma$  corresponds to  $(\alpha, \delta, u, a, b, s)$  and  $\sigma_d$  is as in Example 2.1, then  $\sigma * \sigma_d$  will be the cocycle corresponding to  $(\alpha, \delta, u, a + \frac{d}{2}1, b, s)$  so the orbits are parametrized by the  $t$ -uples  $(\alpha, \delta, u, 0, b, s)$ .

After having the notion of a lazy cocycle, we should expect to have a notion of lazy crossed system. The route for such a definition is clearly indicated by the straightforward generalization of lazy Galois objects.

Let  $Z$  be a right  $A$ -comodule algebra. Recall that  $R \subset Z$  is said to be a **right  $A$ -Galois extension** if  $Z^{\text{co}A} = R$  and if the linear map  $\kappa_r$  defined by the composition

$$\kappa_r : Z \otimes Z \xrightarrow{1_Z \otimes \rho} Z \otimes Z \otimes A \xrightarrow{m_Z \otimes 1_A} Z \otimes A$$

induces an isomorphism  $Z \otimes_R Z \cong Z \otimes A$ .

**Definition 9.3** *A right  $A$ -Galois extension  $R \subset Z$  is said to be **lazy** if there exists a left  $R$ -linear right  $A$ -colinear isomorphism  $\psi : R \otimes A \longrightarrow Z$  such that  $\psi(1 \otimes 1) = 1$  and such that the morphism*

$$\beta_\psi := (\text{id}_A \otimes m_Z) \circ (\tau \otimes \text{id}_Z) \circ (\psi^{-1} \otimes \psi|_A) \circ \rho : Z \longrightarrow A \otimes Z$$

is an algebra morphism. Such a map  $\psi$  is called a **symmetry morphism** for  $R \subset Z$ .

We have the following corresponding definition at the crossed system level.

**Definition 9.4** *A crossed system  $(\rightharpoonup, \sigma)$  over  $R$  is said to be **lazy** if*

$$a_1 b_1 \otimes ((a_2 \rightharpoonup y) \sigma(a_3, b_2)) = a_3 b_2 \otimes ((a_1 \rightharpoonup y) \sigma(a_2, b_1)), \quad \forall a, b \in A, \forall y \in R.$$

We have the following generalization of Proposition 3.2. The proof is completely similar and is left to the reader.

**Proposition 9.5** *Let  $R \subset Z$  be a right  $A$ -Galois extension. Then the following assertions are equivalent.*

1.  $R \subset Z$  is a lazy right  $A$ -Galois extension.
2. There exists a lazy crossed system  $(\rightharpoonup, \sigma)$  over  $R$  such that  $R \sharp_\sigma A \cong Z$  as right  $A$ -comodule algebras.

Let  $(\rightharpoonup, \sigma)$  be a lazy crossed system over  $R$ . Then the map  $\beta : R \sharp_\sigma A \longrightarrow A \otimes R \sharp_\sigma A$ ,  $x \sharp a \longmapsto a_1 \otimes x \sharp a_2$ , is an algebra morphism (this is  $2 \Rightarrow 1$ ) in the proof of Proposition 9.5). In fact  $\beta$  endows  $R \sharp_\sigma A$  with a left  $A$ -comodule algebra structure and  $R \subset Z$  is a left  $A$ -Galois extension. This leads to consider general biGalois extension, a notion which seems not to have been studied before, although the definition requires no imagination.

**Definition 9.6** *Let  $A$  and  $B$  be some Hopf algebras. Let  $Z$  be an  $A$ - $B$ -bicomodule algebra. We say that  $R \subset Z$  is an  $A$ - $B$ -biGalois extension if  $R \subset Z$  is a left  $A$ -Galois extension and is a right  $A$ -Galois extension. An  $A$ - $A$ -biGalois extension  $R \subset Z$  is said to be **bicleft** if there exists a left  $R$ -linear bilinear isomorphism  $R \otimes A \cong Z$ .*



It is clear that if  $(\dashv, \sigma)$  is a lazy crossed system over  $R$ , then  $R \subset R\sharp_{\sigma}A$  is a bicleft  $A$ - $A$ -biGalois object. Similarly to Proposition 3.6, we have the following result.

**Proposition 9.7** *Let  $R \subset Z$  be an  $A$ - $A$ -biGalois extension. Then the following assertions are equivalent:*

1.  $R \subset Z$  is bicleft.
2. There exists a lazy crossed system  $(\dashv, \sigma)$  over  $R$  such that  $R\sharp_{\sigma}A \cong Z$  as  $A$ - $A$ -bicomodule algebras.

We conclude with a few words concerning possible generalizations of the second lazy cohomology group. We would have liked to be able to compose lazy crossed systems  $(\dashv, \sigma)$  over  $R$  with  $\dashv$  fixed, in order to have a cohomology with coefficients in possibly non-commutative algebras. However we have found that for doing this, one needs to require that  $\dashv$  is trivial and  $R$  is commutative. In this case we can define in a straightforward manner groups  $H_L^2(A, R)$ , which just turns out to be  $H_L^2(R \otimes A)$  when one works in the category of  $R$ -algebras.

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