THE LAZY HOMOLOGY OF A HOPF ALGEBRA

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ABSTRACT. To any Hopf algebra H we associate two commutative Hopf algebras $\mathrm{H}_1^\ell(H)$ and $\mathrm{H}_2^\ell(H)$, which we call the lazy homology Hopf algebras of H. These Hopf algebras are built in such a way that they are "predual" to the lazy cohomology groups based on the so-called lazy cocycles. We establish two universal coefficient theorems relating the lazy cohomology to the lazy homology and we compute the lazy homology of the Sweedler algebra.

KEY WORDS: Hopf algebra, lazy cocycle, (co)homology

INTRODUCTION

One way to construct Galois objects over a Hopf algebra H is to twist the multiplication of H with the help of a 2-cocycle. The Galois objects obtained in this way are called cleft Galois objects. If H is cocommutative, then the 2-cocycles form a group under the convolution product; quotienting this group by an appropriate subgroup of coboundaries, one obtains a cohomology group, which had been constructed by Sweedler in [14]. This cohomology group is in bijection with the set of isomorphism classes of cleft Galois objects. When H is no longer cocommutative, then the convolution product of 2-cocycles is not necessarily a 2-cocycle and the classification of cleft Galois objects is no longer given by a cohomology group (to get around this difficulty, Aljadeff and the second-named author recently introduced the concept of a generic 2-cocycle in [1]).

Now there are 2-cocycles that behave well with respect to the convolution product, namely the so-called *lazy* 2-cocycles, which are the cocycles that commute with the product of H; it was observed by Chen [6] that the convolution product of lazy 2-cocycles is again a lazy 2-cocycle. Quotienting the group of lazy 2-cocycles by certain coboundaries, one obtains a group $\mathrm{H}_\ell^2(H)$, which was introduced by Schauenburg [13]. If H is cocommutative, then all 2-cocycles are lazy and $\mathrm{H}_\ell^2(H)$ coincides with Sweedler's cohomology group mentioned above. In particular, if $H = k[G]$ is a group algebra, then $\mathrm{H}_\ell^2(H)$ coincides with the cohomology group $H^2(G, k^\times)$ of the group G acting trivially on the group $k^\times$ of nonzero elements of the ground field k. This makes it natural to consider $\mathrm{H}_\ell^2(H)$ as an analogue of the Schur multiplier for arbitrary Hopf algebras. The lazy cohomology group $\mathrm{H}_\ell^2(H)$ was systematically investigated by Carnovale and the first-named author in [4]; see also [5, 7, 11]. Lazy cocycles have been used to compute the Brauer group of a Hopf algebra. The group $\mathrm{H}_\ell^2(H)$ also allows to equip the category of projective representations of H with the structure of a crossed π-category in the sense of Turaev [18].

In group cohomology, the group $H^2(G, k^\times)$ is related to the homology groups $H_1(G)$ and $H_2(G)$ of G *via* the so-called *universal coefficient theorem*, which can be formulated as an exact sequence of the form

$$0 \to \mathrm{Ext}^1(H_1(G), k^\times) \to H^2(G, k^\times) \to \mathrm{Hom}(H_2(G), k^\times) \to 0 \,.$$

It is natural to ask whether for an arbitrary Hopf algebra H there exist "lazy homology groups" $\mathrm{H}_1^\ell(H)$ and $\mathrm{H}_2^\ell(H)$ related to the lazy cohomology $\mathrm{H}_\ell^2(H)$ in a similar way and coinciding with Sweedler-type homology groups for cocommutative Hopf algebras.

In this paper we give a positive answer to this question. To each Hopf algebra H we associate two commutative Hopf algebras $\mathrm{H}_1^\ell(H)$ and $\mathrm{H}_2^\ell(H)$, which we call the *lazy homology Hopf*

algebras of H. These Hopf algebras are based on tensors satisfying conditions that are dual to the conditions defining lazy cocycles. When H is cocommutative, then $\mathrm{H}_1^\ell(H)$ and $\mathrm{H}_2^\ell(H)$ are part of an infinite sequence of commutative cocommutative Hopf algebras that appear as the homology of a simplicial object in the category of commutative cocommutative Hopf algebras.

We establish two universal coefficient theorems relating the lazy homology to the lazy cohomology. The first one states that the group $\mathrm{Alg}(\mathrm{H}_1^\ell(H), R)$ of algebra morphisms from $\mathrm{H}_1^\ell(H)$ to a commutative algebra R is isomorphic to the lazy cohomology group $\mathrm{H}_\ell^1(H, R)$ of lazy 1-cocycles with coefficients in R. The second universal coefficient theorem can be expressed as an exact sequence of groups of the form

$$(0.1) \qquad 1 \longrightarrow \mathrm{Ext}^1(H, R) \xrightarrow{\delta_\#} \mathrm{H}_\ell^2(H, R) \xrightarrow{\kappa} \mathrm{Alg}(\mathrm{H}_2^\ell(H), R) \,.$$

The group $\mathrm{Ext}^1(H, R)$ is a Hopf algebra analogue of the usual Ext^1-group. When R coincides with the ground field k and the latter is algebraically closed, then the homomorphism κ in (0.1) is an isomorphism:

$$\mathrm{H}_\ell^2(H) = \mathrm{H}_\ell^2(H, k) \cong \mathrm{Alg}(\mathrm{H}_2^\ell(H), k) \,.$$

We wish to emphasize the fact that the functor Alg (rather than Hom) is necessary to formulate a universal coefficient theorem for lazy cohomology. Indeed, as shown in [9], there are Hopf algebras with nonabelian lazy cohomology, and such nonabelian lazy cohomology groups cannot be realized as duals of ordinary groups. This explains why our lazy homology has to take values in the category of Hopf algebras (rather than groups).

We finally compute the lazy homology of the four-dimensional Sweedler algebra H_4; not surprisingly, our computation agrees with the computation of the lazy cohomology of H_4 performed in [4].

The paper is organized as follows. Section 1 is devoted to various preliminaries. We first recall what lazy cocycles are and what lazy cohomology is; here we consider cocycles with values not only in the ground field, but in an arbitrary commutative algebra. We next define exact sequences of Hopf algebras and investigate the exactness of the induced sequences after application of the functor $\mathrm{Alg}(-, R)$. We also recall Takeuchi's free commutative Hopf algebra generated by a coalgebra, of which we make an extensive use in the sequel.

In Section 2, after recalling the construction of Sweedler's cohomology, we attach a simplicial commutative Hopf algebra $F(\Gamma_*(H))$ to each Hopf algebra H. When H is cocommutative, then $F(\Gamma_*(H))$ is a simplicial object in the category of commutative cocommutative Hopf algebras. Since the latter is abelian, we can take the corresponding homology $\mathrm{H}_*^{\mathrm{Sw}}(H)$, which turns out to be an infinite sequence of commutative cocommutative Hopf algebras. We explicitly compute the low-degree differential in the chain complex associated to the simplicial object $F(\Gamma_*(H))$. The constructions in this section will be a precious guide for the definition of the lazy homology in Sections 4 and 6.

In Section 3 we perform certain basic tensor constructions that allow us to "predualize" the conditions defining lazy cocycles; these constructions will be central in the subsequent sections.

Section 4 starts with the definition of the first lazy homology Hopf algebra $\mathrm{H}_1^\ell(H)$ and the proof of the isomorphism $\mathrm{H}_\ell^1(H, R) \cong \mathrm{Alg}(\mathrm{H}_1^\ell(H), R)$ mentioned above. We next give an alternative definition of $\mathrm{H}_1^\ell(H)$, which allows us to show that if H is the Hopf algebra of functions on a finite group G, then $\mathrm{H}_1^\ell(H)$ is isomorphic to the Hopf algebra of functions on the center of G. We also interpret $\mathrm{H}_1^\ell(H)$ as a kind of homology group, from which we deduce that it coincides with the Sweedler-type homology Hopf algebra $\mathrm{H}_1^{\mathrm{Sw}}(H)$ (introduced in Section 2) when H is cocommutative.

In Section 5 we consider the case when H is a cosemisimple Hopf algebra over an algebraically closed field of characteristic zero, and we compute $\mathrm{H}_1^\ell(H)$ in terms of the "universal abelian grading group" of the tensor category of H-comodules. As an application of our techniques, we recover Müger's construction of the center of a compact group from its representation category.

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In Section 6 we define the second lazy homology Hopf algebra $\mathrm{H}_2^\ell(H)$; it coincides with the Sweedler-type homology Hopf algebra $\mathrm{H}_2^{\mathrm{Sw}}(H)$ when H is cocommutative. The main result of this section is the exact sequence (0.1).

Section 7 is devoted to the computation of the lazy homology Hopf algebras of the Sweedler algebra.

1. Notation and Preliminaries

Throughout the paper, we fix a field k over which all our constructions are defined. In particular, all linear maps are supposed to be k-linear and unadorned tensor products mean tensor products over k.

All algebras that we consider are associative unital k-algebras. The unit of an algebra A will be denoted by 1_A, or 1 if no confusion is possible. All algebra morphisms are supposed to preserve the units. We denote the set of algebra morphisms from A to A' by $\mathrm{Alg}(A, A')$.

All coalgebras considered are coassociative counital k-coalgebras. We denote the coproduct of a coalgebra by Δ and its counit by ε. We shall also make use of a Heyneman-Sweedler-type notation for the image
$$\Delta(x) = x_1 \otimes x_2$$
of an element x of a coalgebra C under the coproduct, and we write
$$\Delta^{(2)}(x) = x_1 \otimes x_2 \otimes x_3$$
for the iterated coproduct $\Delta^{(2)} = (\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta$, and so on.

If C and C' are coalgebras, then $\mathrm{Coalg}(C, C')$ denotes the set of coalgebra morphisms from C to C'. Similarly, if H and H' are Hopf algebras, then $\mathrm{Hopf}(H, H')$ denotes the set of Hopf algebra morphisms from H to H'. In general, we assume that the reader is familiar with coalgebras and Hopf algebras, in particular with [15].

1.1. Convolution monoids.
Let C be a coalgebra with coproduct Δ and let R be an algebra with product μ. Recall that the convolution product of $f, g \in \mathrm{Hom}(C, R)$ is defined by

$$(1.1) \qquad f * g = \mu \circ (f \otimes g) \circ \Delta \in \mathrm{Hom}(C, R).$$

The convolution product endows $\mathrm{Hom}(C, R)$ with a monoid structure whose unit is $\eta \varepsilon$, where $\eta : k \to R$ is the unit of R and $\varepsilon : C \to k$ is the counit of C. The group of convolution invertible elements in $\mathrm{Hom}(C, R)$ is denoted by $\mathrm{Reg}(C, R)$. It is well known that the group $\mathrm{Reg}(C, R)$ is abelian if C is cocommutative and R is commutative.

Suppose in addition that C is an H-module coalgebra and R is an H-module algebra for some Hopf algebra H. Then the subset $\mathrm{Hom}_H(C, R)$ of H-linear maps in $\mathrm{Hom}(C, R)$ is a submonoid of the latter for the convolution product. We denote by $\mathrm{Reg}_H(C, R)$ the group of convolution invertible elements in $\mathrm{Hom}_H(C, R)$. This is a subgroup of $\mathrm{Reg}(C, R)$; it is *abelian* if C is cocommutative and R is commutative.

1.2. Lazy cocycles and cohomology.
Let us recall several notions and notation from [4, 6, 13], with some slight modification. We fix a *commutative* algebra R.

Given a coalgebra C, the subgroup of *lazy elements* of $\mathrm{Reg}(C, R)$ is defined by

$$(1.2) \qquad \mathrm{Reg}_\ell(C, R) = \{\mu \in \mathrm{Reg}(C, R) \mid \mu(x_1) \otimes x_2 = \mu(x_2) \otimes x_1 \text{ for all } x \in C\}.$$

The previous equalities take place in $R \otimes C$.

Let H be a Hopf algebra. The subgroup of elements of $\mu \in \mathrm{Reg}(H, R)$ satisfying $\mu(1) = 1$ is denoted by $\mathrm{Reg}^1(H, R)$, and we set

$$\mathrm{Reg}_\ell^1(H, R) = \mathrm{Reg}^1(H, R) \cap \mathrm{Reg}_\ell(H, R).$$

Definition 1.1. *The first lazy cohomology group of H with coefficients in R is the group*

$$\mathrm{H}_\ell^1(H, R) = \mathrm{Alg}(H, R) \cap \mathrm{Reg}_\ell^1(H, R).$$

The set $\mathrm{reg}_\ell^2(H, R)$ of *lazy elements* in $\mathrm{Reg}(H \otimes H, R)$ consists of all those $\sigma \in \mathrm{Reg}(H \otimes H, R)$ such that

$$(1.3) \qquad \sigma(x_1 \otimes y_1) \otimes x_2 y_2 = \sigma(x_2 \otimes y_2) \otimes x_1 y_1 \in R \otimes H$$

for all $x, y \in H$. The subgroup of normalized elements of $\mathrm{Reg}(H \otimes H, R)$, i.e., satisfying the additional condition

$$\sigma(x \otimes 1) = \varepsilon(x)\, 1_R = \sigma(1 \otimes x)$$

for all $x \in H$, is denoted by $\mathrm{Reg}^2(H, R)$. We set

$$(1.4) \qquad \mathrm{Reg}_\ell^2(H, R) = \mathrm{Reg}^2(H, R) \cap \mathrm{reg}_\ell^2(H, R)\,.$$

A *left 2-cocycle* of H with coefficients in R is an element $\sigma \in \mathrm{Reg}^2(H, R)$ such that

$$(1.5) \qquad \sigma(x_1 \otimes y_1)\, \sigma(x_2 y_2 \otimes z) = \sigma(y_1 \otimes z_1)\, \sigma(x \otimes y_2 z_2)$$

for all $x, y, z \in H$. We denote by $Z^2(H, R)$ the set of such left 2-cocycles and we set

$$(1.6) \qquad Z_\ell^2(H, R) = Z^2(H, R) \cap \mathrm{Reg}_\ell^2(H, R)\,.$$

The set $Z_\ell^2(H, R)$ is called the set of *lazy 2-cocycles with coefficients in R*. Chen [6] was the first to observe that this set is a group under the convolution product.

For $\mu \in \mathrm{Reg}^1(H, R)$, define $\partial(\mu) \in \mathrm{Reg}^2(H, R)$ for all $x, y \in H$ by

$$(1.7) \qquad \partial(\mu)(x \otimes y) = \mu(x_1)\, \mu(y_1)\, \mu^{-1}(x_2 y_2)\,,$$

where μ^{-1} is the convolution inverse of μ. This defines a map

$$\partial : \mathrm{Reg}^1(H, R) \to \mathrm{Reg}^2(H, R)\,.$$

The image $B_\ell^2(H, R)$ of ∂ is a central subgroup of $Z_\ell^2(H, R)$.

Definition 1.2. *The second lazy cohomology group of H with coefficients in R is the quotient-group*

$$\mathrm{H}_\ell^2(H, R) = Z_\ell^2(H, R)/B_\ell^2(H, R)\,.$$

The second lazy cohomology group is not always abelian: indeed an example of a finite group G with nonabelian lazy cohomogy group $\mathrm{H}_\ell^2(k^G, k)$ is given in [9].

When R is the ground field k, then $\mathrm{H}_\ell^i(H, R)$ ($i = 1, 2$) coincides with the group denoted by $H_{\mathrm{L}}^i(H)$ in [4, Def. 1.7]. The above definition of the lazy cohomology with values in an arbitrary commutative algebra R (rather than in the ground field) was already suggested in [4, App. A].

1.3. **Hopf kernels and Hopf quotients.** Following [2, Def. 1.1.5], we say that a Hopf algebra morphism $\pi : H \to B$ is *normal* if

$$(1.8) \qquad \{x \in H \mid \pi(x_1) \otimes x_2 = 1 \otimes x\} = \{x \in H \mid x_1 \otimes \pi(x_2) = x \otimes 1\}\,.$$

If H is cocommutative, then any Hopf algebra morphism $\pi : H \to B$ is normal.

When π is normal, then by [2, Lemma 1.1.4], both sides of (1.8) coincide with

$$(1.9) \qquad \mathrm{HKer}(\pi) = \{x \in H \mid x_1 \otimes \pi(x_2) \otimes x_3 = x_1 \otimes 1 \otimes x_2\}\,.$$

We call $\mathrm{HKer}(\pi)$ the *Hopf kernel* of $\pi : H \to B$. By [2, Lemma 1.1.3], $\mathrm{HKer}(\pi)$ is a Hopf subalgebra of H. Observe that the counit $\varepsilon : H \to k$ is always normal and that $\mathrm{HKer}(\varepsilon) = H$.

Let us illustrate the concept of a Hopf kernel in two special cases. If $u : G \to G'$ is a homomorphism of groups and $k[u] : k[G] \to k[G']$ the induced morphism of Hopf algebras, then

$$(1.10) \qquad \mathrm{HKer}(k[u]) = k[\mathrm{Ker}(u)]\,.$$

When $H = \mathcal{O}(G)$ and $B = \mathcal{O}(N)$, where $N \subset G$ are algebraic groups and k is an algebraically closed field of characteristic zero, then the Hopf kernel of the natural Hopf algebra morphism $\mathcal{O}(G) \to \mathcal{O}(N)$ is isomorphic to $\mathcal{O}(G/\langle N \rangle)$, where $\langle N \rangle$ is the normal algebraic subgroup of G generated by N.

Let $A \subset H$ be a Hopf subalgebra, and let $A^+ = \mathrm{Ker}(\varepsilon : A \to k)$ be the augmentation ideal of A. When $A^+H = HA^+$, we define the *Hopf quotient* $H/\!\!/A$ to be the quotient Hopf algebra H/A^+H:

$$(1.11) \qquad\qquad\qquad H/\!\!/A = H/A^+H \,.$$

If G_0 is a normal subgroup of a group G, then $k[G_0]$ is a Hopf subalgebra of $k[G]$ and $k[G_0]^+k[G] = k[G]k[G_0]^+$. Moreover, the Hopf quotient $k[G]/\!\!/k[G_0]$ is isomorphic to the Hopf algebra of the quotient group G/G_0:

$$(1.12) \qquad\qquad\qquad k[G]/\!\!/k[G_0] \cong k[G/G_0] \,.$$

1.4. **Short exact sequences.** For any Hopf algebra H and any commutative algebra R, the convolution product (1.1) preserves the subset $\mathrm{Alg}(H, R)$ of $\mathrm{Hom}(H, R)$ and turns it into a group, the inverse of each $f \in \mathrm{Alg}(H, R)$ being $f \circ S$, where S is the antipode of H. The group $\mathrm{Alg}(H, R)$ is abelian if H is cocommutative. If $\varphi : H \to H'$ is a morphism of Hopf algebras, then the map

$$\varphi^* : \mathrm{Alg}(H', R) \to \mathrm{Alg}(H, R)$$

defined by $\varphi^*(f) = f \circ \varphi$, where $f \in \mathrm{Alg}(H', R)$, is a homomorphism of groups. In this way, $\mathrm{Alg}(H, R)$ becomes a contravariant functor from the category of Hopf algebras (resp. cocommutative Hopf algebras) to the category of groups (resp. abelian groups).

We have the following obvious fact.

Lemma 1.3. *If $\varphi : H \to H'$ is a surjective morphism of Hopf algebras, then the homomorphism $\varphi^* : \mathrm{Alg}(H', R) \to \mathrm{Alg}(H, R)$ is injective for any commutative algebra R.*

Let

$$(1.13) \qquad\qquad\qquad k \longrightarrow A \overset{\iota}{\longrightarrow} H \overset{\pi}{\longrightarrow} B \longrightarrow k$$

be a sequence of Hopf algebra morphisms. Following [2], we say that the sequence (1.13) is *exact* if ι is injective, π is surjective, and $\mathrm{Ker}(\pi) = \iota(A)^+H$. (In [2] there is an additional condition, which we do not need here; this condition is automatically satisfied if the Hopf algebra H is commutative; see [2, Prop. 1.2.4].) It follows from the third condition that $\pi \circ \iota = \varepsilon\eta$. Note also that if $\iota(A)^+H = H\iota(A)^+$, then B is isomorphic to the Hopf quotient $H/\!\!/\iota(A)$ defined by (1.11).

Proposition 1.4. *Any exact sequence $k \longrightarrow A \overset{\iota}{\longrightarrow} H \overset{\pi}{\longrightarrow} B \longrightarrow k$ of Hopf algebras induces an exact sequence of groups*

$$1 \longrightarrow \mathrm{Alg}(B, R) \overset{\pi^*}{\longrightarrow} \mathrm{Alg}(H, R) \overset{\iota^*}{\longrightarrow} \mathrm{Alg}(A, R)$$

for any commutative algebra R.

It follows under the conditions of the previous proposition that $\mathrm{Alg}(B, R)$ is a *normal* subgroup of $\mathrm{Alg}(H, R)$.

Proof. In view of Lemma 1.3 it is enough to check that $\mathrm{Ker}\,\iota^* = \mathrm{Im}\,\pi^*$. Let $f \in \mathrm{Alg}(H, R)$ be such that $\iota^*(f) = \varepsilon\eta$. This means that $f(i(x)) = \varepsilon(x)\,1$ for all $x \in A$. Therefore, f vanishes on $\iota(A)^+$, hence on $\iota(A)^+H$. Since by assumption, $\mathrm{Ker}(\pi) = \iota(A)^+H$, the map f vanishes on $\mathrm{Ker}(\pi)$. It follows that there is $\bar{f} \in \mathrm{Hom}(B, R)$ such that $f = \bar{f} \circ \pi$. It is easy to check that $\bar{f}$ is an algebra morphism. Hence, $f = \pi^*(\bar{f}) \in \mathrm{Im}\,\pi^*$. $\qquad\square$

There is no reason why the homomorphism ι^* of Proposition 1.4 should be surjective in general. Nevertheless surjectivity holds in the following case.

Proposition 1.5. *If $k \longrightarrow A \overset{\iota}{\longrightarrow} H \overset{\pi}{\longrightarrow} B \longrightarrow k$ is an exact sequence of commutative Hopf algebras and k is algebraically closed, then the sequence*

$$1 \longrightarrow \mathrm{Alg}(B, k) \overset{\pi^*}{\longrightarrow} \mathrm{Alg}(H, k) \overset{\iota^*}{\longrightarrow} \mathrm{Alg}(A, k) \longrightarrow 1$$

is exact.

Proof. In view of Proposition 1.4, it is enough to check the surjectivity of $\iota^* : \mathrm{Alg}(H, k) \to \mathrm{Alg}(A, k)$. We sketch a proof of the surjectivity of ι^* following [19, Chap. 15, Exercise 3 (a)]. We may assume that the Hopf algebras A and H are finitely generated. Let $\chi \in \mathrm{Alg}(A, k)$ and $I = \mathrm{Ker}\,\chi$. Since H is faithfully flat over A by [17, Th. 3.1] (see also [19, Sect. 14.1]), we have $IH \neq H$ by [19, Sect. 13.2]. Therefore there is a maximal ideal J of H such that $IH \subset J$. The field H/J is a finite extension of k; since k is algebraically closed, we have $H/J = k$. Then the natural projection $H \to H/J = k$ is an element of $\mathrm{Alg}(H, k)$ extending χ. $\qquad\square$

1.5. **The free commutative Hopf algebra generated by a coalgebra.** For any coalgebra C we denote by $F(C)$ the *free commutative Hopf algebra* generated by the coalgebra C, as constructed by Takeuchi in [16]: it is the commutative algebra presented by generators $t(x)$ and $t^{-1}(x)$, $x \in C$, subject to the following relations ($x, y \in H$, $\lambda \in k$):

$$t(\lambda x + y) = \lambda t(x) + t(y), \quad t^{-1}(\lambda x + y) = \lambda t^{-1}(x) + t^{-1}(y),$$

and

(1.14) $$t(x_1)\,t^{-1}(x_2) = \varepsilon(x)\,1 = t^{-1}(x_1)\,t(x_2).$$

The coproduct Δ, the counit ε, and the antipode S of $F(C)$ are given for all $x \in C$ by

$$\Delta\big(t(x)\big) = t(x_1) \otimes t(x_2), \quad \Delta\big(t^{-1}(x)\big) = t^{-1}(x_2) \otimes t^{-1}(x_1),$$

(1.15) $$\varepsilon\big(t(x)\big) = \varepsilon\big(t^{-1}(x)\big) = \varepsilon(x),$$

$$S\big(t(x)\big) = t^{-1}(x), \quad S\big(t^{-1}(x)\big) = t(x),$$

and the canonical map $t : C \to F(C)$, $x \mapsto t(x)$, is a coalgebra morphism. By [16], the Hopf algebra $F(C)$ has the following universal property.

Proposition 1.6. *For any commutative Hopf algebra R, the map $f \mapsto f \circ t$ induces an isomorphism*

$$\mathrm{Hopf}(F(C), R) \xrightarrow{\cong} \mathrm{Coalg}(C, R).$$

Thus $C \mapsto F(C)$ defines a functor from the category of coalgebras to the category of commutative Hopf algebras. There is another universal property of $F(C)$ that will be important for us in Section 3 when we "predualize" lazy cohomology.

Proposition 1.7. *For any commutative algebra R, the map $f \mapsto f \circ t$ induces an isomorphism*

$$\mathrm{Alg}(F(C), R) \xrightarrow{\cong} \mathrm{Reg}(C, R).$$

Proof. For $f \in \mathrm{Alg}(F(C), R)$, the map $f \circ t$ is convolution invertible, with convolution inverse given by $(f \circ t)^{-1}(x) = f(t^{-1}(x))$ for all $x \in F(C)$. Therefore the map $\mathrm{Alg}(F(C), R) \to \mathrm{Reg}(C, R)$ is well defined; it is easy to check that it is a homomorphism.

Given $\varphi \in \mathrm{Reg}(C, R)$ with inverse φ^{-1}, it is easy to check that there is a unique algebra morphism $\tilde{\varphi} : F(C) \to R$ such that $\tilde{\varphi}(t(x)) = \varphi(x)$ and $\tilde{\varphi}(t^{-1}(x)) = \varphi^{-1}(x)$ for all $x \in C$. The assignment $\varphi \mapsto \tilde{\varphi}$ defines a map $\mathrm{Reg}(C, R) \to \mathrm{Alg}(F(C), R)$, which is inverse to the previous one. $\qquad\square$

A basic example is the case of the group algebra $H = k[G]$ of a group G: the resulting commutative cocommutative Hopf algebra $F(k[G])$ is isomorphic to the group algebra $k[\mathbb{Z}[G]]$, where $\mathbb{Z}[G]$ is the free (additive) abelian group with basis G. Note that $k[\mathbb{Z}[G]]$ is isomorphic to the Laurent polynomial algebra $k[\,t(g)^{\pm 1} \mid g \in G\,]$.

2. Sweedler's Simplicial Coalgebra

In Section 1.2 we defined the lazy cohomology of a Hopf algebra in a direct way. When the Hopf algebra is cocommutative, the lazy cohomology coincides with the cohomology introduced by Sweedler in [14]. In this section we first recall the construction of Sweedler's cohomology; it is based on a simplicial coalgebra, which exists for any Hopf algebra. Next, starting from the same simplicial coalgebra, to any cocommutative Hopf algebra we associate a chain complex

which yields a Sweedler-type homology theory. Sweedler's cohomology groups are dual in an appropriate sense to these homology groups. The lazy homology defined in Sections 4 and 6 will be obtained from a careful modification of this chain complex.

2.1. **Sweedler's construction.** Let H be a Hopf algebra. The starting point is the following simplicial k-module $C_*(H)$, which was defined by Sweedler in [14, § 2]:

$$\cdots\, C_{n+1} \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} C_n \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots\, C_2 \underset{}{\overset{}{\rightleftarrows}} C_1 \rightleftarrows C_0 \ .$$

Here $C_n = H^{\otimes n+1}$, and the face operators $d_i : H^{\otimes n+1} \to H^{\otimes n}$ and degeneracy operators $s_i : H^{\otimes n+1} \to H^{\otimes n+2}$ $(i = 0, \ldots, n)$ are given by

$$d_i(x^0 \otimes \cdots \otimes x^n) = \begin{cases} x^0 \otimes \cdots \otimes x^i x^{i+1} \otimes \cdots \otimes x^n & \text{for } i = 0, \ldots, n-1\,, \\ x^0 \otimes \cdots \otimes x^{n-1}\, \varepsilon(x^n) & \text{for } i = n\,, \end{cases}$$

and

$$s_i(x^0 \otimes \cdots \otimes x^n) = x^0 \otimes \cdots \otimes x^i \otimes 1 \otimes \cdots \otimes x^n \quad \text{for } i = 0, \ldots, n\,,$$

for all $x^0, \ldots, x^n \in H$. We give each C_n its natural H-module coalgebra structure, where the coalgebra structure is induced from that of H and where the action of H is defined by its left multiplication on the leftmost H-tensorand in C_n. It is a direct verification that the face and degeneracy operators d_i, s_i are H-module coalgebra morphisms. It follows that $C_*(H)$ is a *simplicial H-module coalgebra*. This holds for any Hopf algebra H without any additional assumption, such as cocommutativity.

Let R be an H-module algebra. Applying the contravariant functor $\text{Reg}_H(-, R)$, (defined in Section 1.1) to the simplicial H-module coalgebra $C_*(H)$, we obtain the *cosimplicial group* $\text{Reg}_H(C_*(H), R)$.

If in addition H is cocommutative, then $C_*(H)$ is a simplicial object in the category of *cocommutative H-module coalgebras*. Hence, $\text{Reg}_H(C_*(H), R)$ is a *cosimplicial abelian group* for any *commutative H-module algebra R. Sweedler [14, § 2] defined the *cohomology of H in R* as the cohomology of this cosimplicial abelian group. We henceforth denote Sweedler's cohomology by $\mathrm{H}^*_{\mathrm{Sw}}(H, R)$:

(2.1) $$\mathrm{H}^*_{\mathrm{Sw}}(H, R) = H^*\big(\text{Reg}_H(C_*(H), R)\big)\,.$$

We stress that the groups $\mathrm{H}^*_{\mathrm{Sw}}(H, R)$ are defined only if H is *cocommutative* and R is *commutative*.

Suppose that G is a group and that $H = k[G]$ is the corresponding group algebra with its standard cocommutative Hopf algebra structure. Let R be a commutative H-module algebra. The elements of G act as automorphisms of R and hence act on the subgroup $R^\times$ of invertible elements of R. It follows from [14, § 3] that

$$\mathrm{H}^*_{\mathrm{Sw}}(k[G], R) \cong H^*(G, R^\times)\,,$$

where $H^*(G, R^\times)$ is the group cohomology of G with coefficients in $R^\times$.

2.2. **A Sweedler-type homology theory.** Let us consider again an arbitrary Hopf algebra H. Tensoring the simplicial H-module coalgebra $C_*(H)$ of Section 2.1 degreewise by $k \otimes_H -$, we obtain a new *simplicial coalgebra* $\Gamma_*(H)$:

$$\cdots\, \Gamma_{n+1} \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \Gamma_n \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots\, \Gamma_2 \underset{}{\overset{}{\rightleftarrows}} \Gamma_1 \rightleftarrows \Gamma_0 \ ,$$

where $\Gamma_n = H^{\otimes n}$ if $n > 0$ and $\Gamma_0 = k$, and the face operators $\partial_i : H^{\otimes n} \to H^{\otimes n-1}$ and degeneracy operators $\sigma_i : H^{\otimes n} \to H^{\otimes n+1}$ are given by

(2.2) $$\partial_i(x^1 \otimes \cdots \otimes x^n) = \begin{cases} \varepsilon(x^1)\, x^2 \otimes \cdots \otimes x^n & \text{for } i = 0\,, \\ x^1 \otimes \cdots \otimes x^i x^{i+1} \otimes \cdots \otimes x^n & \text{for } i = 1, \ldots, n-1\,, \\ x^1 \otimes \cdots \otimes x^{n-1}\, \varepsilon(x^n) & \text{for } i = n\,, \end{cases}$$

and

$$\sigma_i(x^1 \otimes \cdots \otimes x^n) = \begin{cases} 1 \otimes x^1 \otimes \cdots \otimes x^n & \text{for } i = 0\,, \\ x^1 \otimes \cdots \otimes x^i \otimes 1 \otimes \cdots \otimes x^n & \text{for } i = 1, \ldots, n\,, \end{cases}$$

for all $x^1, \ldots, x^n \in H$. The functor F from coalgebras to commutative Hopf algebras introduced in Section 1.5 now transforms the simplicial coalgebra $\Gamma_*(H)$ into the *simplicial commutative Hopf algebra $F(\Gamma_*(H))$*.

If in addition H is *cocommutative*, then $\Gamma_*(H)$ is a simplicial cocommutative coalgebra and therefore $F(\Gamma_*(H))$ is a simplicial object in the category of *commutative cocommutative Hopf algebras*. By [17, Cor. 4.16], this category is abelian (recall that $\mathrm{Hopf}(H, H')$ is an abelian group for the convolution if H, H' are commutative cocommutative Hopf algebras) and we can form kernels and quotients in it following (1.9) and (1.11). Therefore we can use the standard methods to form a chain complex out of the simplicial object $F(\Gamma_*(H))$. We pose the following definition.

Definition 2.1. *The Sweedler homology $\mathrm{H}_*^{\mathrm{Sw}}(H)$ of a cocommutative Hopf algebra H is the homology of the chain complex*

$$(2.3) \qquad\qquad \cdots \longrightarrow F(H^{\otimes 3}) \xrightarrow{d_3^{\mathrm{Sw}}} F(H^{\otimes 2}) \xrightarrow{d_2^{\mathrm{Sw}}} F(H) \xrightarrow{d_1^{\mathrm{Sw}}} F(k)$$

associated to the simplicial commutative cocommutative Hopf algebra $F(\Gamma_*(H))$:

$$\mathrm{H}_*^{\mathrm{Sw}}(H) = H_*\big(F(\Gamma_*(H)), d_*^{\mathrm{Sw}}\big)\,.$$

The homology groups $\mathrm{H}_*^{\mathrm{Sw}}(H)$ are commutative cocommutative Hopf algebras.

Let us explicitly compute the low-degree differentials d_1^{Sw}, d_2^{Sw}, d_3^{Sw} in the complex (2.3).

Lemma 2.2. *For $x, y, z \in H$, we have $d_1^{\mathrm{Sw}}\big(t(x)\big) = \varepsilon(x)$ and*

$$d_2^{\mathrm{Sw}}\big(t(x \otimes y)\big) = t(y_1)\, t^{-1}(x_1 y_2)\, t(x_2)\,,$$

$$d_3^{\mathrm{Sw}}\big(t(x \otimes y \otimes z)\big) = t(y_1 \otimes z_1)\, t^{-1}(x_1 y_2 \otimes z_2)\, t(x_2 \otimes y_3 z_3)\, t^{-1}(x_3 \otimes y_4)\,.$$

Since H is assumed to be cocommutative, d_2^{Sw} and d_3^{Sw} can be reformulated as follows:

$$(2.4) \qquad\qquad d_2^{\mathrm{Sw}}\big(t(x \otimes y)\big) = t(x_1)\, t(y_1)\, t^{-1}(x_2 y_2)$$

and

$$(2.5) \qquad d_3^{\mathrm{Sw}}\big(t(x \otimes y \otimes z)\big) = t(y_1 \otimes z_1)\, t(x_1 \otimes y_2 z_2)\, t^{-1}(x_2 y_3 \otimes z_3)\, t^{-1}(x_3 \otimes y_4)\,.$$

Equations (2.4) and (2.5) will be used in the proofs of Propositions 4.12 and 6.3, respectively.

Proof. The map d_1^{Sw} in $\mathrm{Hopf}(F(H), F(k))$ is defined by $d_1^{\mathrm{Sw}} = \partial_0 * \partial_1^{-1}$. Therefore by (2.2),

$$d_1^{\mathrm{Sw}}\big(t(x)\big) = \partial_0(x_1)\, \partial_1^{-1}(x_2) = \varepsilon(x_1)\, \varepsilon(x_2) = \varepsilon(x)\,.$$

The map d_2^{Sw} in $\mathrm{Hopf}(F(H^{\otimes 2}), F(H))$ is given by

$$d_2^{\mathrm{Sw}} = \partial_0 * \partial_1^{-1} * \partial_2\,.$$

By (2.2),

$$\begin{aligned} d_2^{\mathrm{Sw}}\big(t(x \otimes y)\big) &= t\big(\partial_0(x_1 \otimes y_1)\big)\, t^{-1}\big(\partial_1(x_2 \otimes y_2)\big)\, t\big(\partial_2(x_3 \otimes y_3)\big) \\ &= t\big(\varepsilon(x_1)\, y_1\big)\, t^{-1}(x_2 y_2)\, t\big(x_3\, \varepsilon(y_3)\big) \\ &= t(y_1)\, t^{-1}\big(\varepsilon(x_1)\, x_2 y_2\, \varepsilon(y_3)\big)\, t(x_3) \\ &= t(y_1)\, t^{-1}(x_1 y_2)\, t(x_2)\,. \end{aligned}$$

Finally, the map d_3^{Sw} in $\mathrm{Hopf}(F(H^{\otimes 3}), F(H^{\otimes 2}))$ is given by

$$d_3^{\mathrm{Sw}} = \partial_0 * \partial_1^{-1} * \partial_2 * \partial_3^{-1}\,.$$

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Again by (2.2),

$$
\begin{aligned}
d_3^{\mathrm{Sw}}\big(t(x \otimes y \otimes z)\big) &= t\big(\partial_0(x_1 \otimes y_1 \otimes z_1)\big)\, t^{-1}\big(\partial_1(x_2 \otimes y_2 \otimes z_2)\big) \\
&\quad \times t\big(\partial_2(x_3 \otimes y_3 \otimes z_3)\big)\, t^{-1}\big(\partial_3(x_4 \otimes y_4 \otimes z_4)\big) \\
&= t\big(\varepsilon(x_1)\, y_1 \otimes z_1\big)\, t^{-1}(x_2 y_2 \otimes z_2) \\
&\quad \times t(x_3 \otimes y_3 z_3)\, t^{-1}\big(x_4 \otimes y_4\, \varepsilon(z_4)\big) \\
&= t(y_1 \otimes z_1)\, t^{-1}(x_1 y_2 \otimes z_2)\, t(x_2 \otimes y_3 z_3)\, t^{-1}(x_3 \otimes y_4)\,.
\end{aligned}
$$

$\square$

The Sweedler homology of the Hopf algebra of a group is computed as follows.

Proposition 2.3. *For any group G we have*

$$
\mathrm{H}_*^{\mathrm{Sw}}(k[G]) \cong k[H_*(G,\mathbb{Z})]\,,
$$

where $H_*(G,\mathbb{Z})$ is the homology of G with integral coefficients.

Proof. For $H = k[G]$ the simplicial abelian group $C_*(H)$ is the one underlying the unnormal-ized bar k-resolution of G. Tensoring it by $k \otimes_{k[G]} -$, we obtain the simplicial object $\Gamma_*(H)$, whose associated chain complex homology has $H_*(G,k)$ as homology. Applying the functor F, we obtain a simplicial commutative Hopf algebra, whose component in degree n is the group algebra $k[\mathbb{Z}[G^n]]$. We thus obtain the k-linearization of the simplicial complex whose homology is $H_*(G,\mathbb{Z})$. We conclude using (1.10) and (1.12). $\square$

When H is an arbitrary Hopf algebra, we do not know how to use directly the simplicial com-mutative Hopf algebra $F(\Gamma_*(H))$ in order to build up something that looks like a chain complex. In the subsequent sections we shall carefully modify the low-degree terms of $F(\Gamma_*(H))$ in a man-ner that will be suitable for defining what we shall call lazy homology. More precisely, in order to define lazy homology we predualize the laziness conditions at the level of the coalgebras $H^{\otimes q}$ ($q = 1, 2$) by taking appropriate quotients (see Section 3), apply the functor F, and mimic the construction of a complex out of a simplicial object (Sections 4 and 6).

Remark 2.4. Let H be a cocommutative Hopf algebra and R a commutative algebra. We turn R into an H-module algebra by assuming that H acts *trivially* on R via the counit ε. It is easy to check that under these hypotheses there is an isomorphism

$$
\mathrm{Reg}_H(C_*(H), R) \cong \mathrm{Reg}(\Gamma_*(H), R)
$$

of cosimplicial abelian groups. In view of Proposition 1.7, this leads to an isomorphism

$$
\mathrm{Reg}_H(C_*(H), R) \cong \mathrm{Alg}\big(F(\Gamma_*(H)), R\big)
$$

of cosimplicial abelian groups, relating the cochain complex whose cohomology is $\mathrm{H}_{\mathrm{Sw}}^*(H, R)$ and the chain complex (2.3) whose homology is $\mathrm{H}_*^{\mathrm{Sw}}(H)$. From this it is easy to deduce natural homomorphisms

(2.6) $$\mathrm{H}_{\mathrm{Sw}}^i(H, R) \to \mathrm{Alg}(\mathrm{H}_i^{\mathrm{Sw}}(H), R)$$

for all $i \geq 0$. When the ground field k is algebraically closed and $R = k$, one can use Sections 1.3–1.4 and [17] to show that the homomorphisms (2.6) are isomorphisms:

$$
\mathrm{H}_{\mathrm{Sw}}^i(H, k) \cong \mathrm{Alg}(\mathrm{H}_i^{\mathrm{Sw}}(H), k)
$$

for all $i \geq 0$. When $H = k[G]$ is a group algebra, these isomorphisms reduce to the well-known isomorphisms

$$
H^i(G, k^\times) \cong \mathrm{Hom}(H_i(G, \mathbb{Z}), k^\times)
$$

in group cohomology (the group $k^\times$ is divisible under the condition above).

2.3. Addendum: Homology with coefficients. The content of this section will not be needed in the sequel. The construction of the simplicial coalgebra $\Gamma_*(H)$ of Section 2.2 is a special case of the following construction.

Start again from a Hopf algebra H and the associated simplicial coalgebra $C_*(H)$ defined in Section 2.1. Let C be a right H-module coalgebra. In each degree n let us consider the tensor product

$$C \otimes_H C_n(H) = C \otimes_H H^{\otimes n+1} \, ;$$

it is isomorphic to

$$\Gamma_n(H, C) = C \otimes H^{\otimes n}$$

under the map $x^0 \otimes x^1 \otimes \cdots \otimes x^n \mapsto x^0 x^1 \otimes \cdots \otimes x^n$, where $x^0 \in C$ and $x^1, \ldots, x^n \in H$. The face and degeneracy operators of $C_*(H)$ induce operators $\partial_i : C \otimes H^{\otimes n} \to C \otimes H^{\otimes n-1}$ and $\sigma_i : C \otimes H^{\otimes n} \to C \otimes H^{\otimes n+1}$, which are given for $x^0 \in C$ and $x^1, \ldots, x^n \in H$ by

$$\partial_i(x^0 \otimes x^1 \otimes \cdots \otimes x^n) = \begin{cases} x^0 x^1 \otimes x^2 \otimes \cdots \otimes x^n & \text{for } i = 0 \, , \\ x^0 \otimes x^1 \otimes \cdots \otimes x^i x^{i+1} \otimes \cdots \otimes x^n & \text{for } 0 < i < n \, , \\ x^0 \otimes x^1 \otimes \cdots \otimes x^{n-1} \, \varepsilon(x^n) & \text{for } i = n \, , \end{cases}$$

and

$$\sigma_i(x^0 \otimes x^1 \otimes \cdots \otimes x^n) = x^0 \otimes \cdots \otimes x^i \otimes 1 \otimes \cdots \otimes x^n \quad \text{for } i = 0, 1, \ldots, n \, .$$

It is easy to check that these operators are coalgebra morphisms. Therefore, $\Gamma_*(H, C)$ is a simplicial coalgebra. As in Section 2.2, we can apply the functor F to it. The resulting object

$$F\bigl(\Gamma_*(H, C)\bigr)$$

is a simplicial commutative Hopf algebra.

If in addition H and C are cocommutative, then $\Gamma_*(H, C)$ is a simplicial cocommutative coalgebra and therefore $F(\Gamma_*(H, C))$ is a simplicial object in the abelian category of commutative cocommutative Hopf algebras. We define the *Sweedler homology* $\mathrm{H}_*^{\mathrm{Sw}}(H, C)$ *of H with coefficients in the H-module coalgebra C* as the homology of the chain complex associated to $F(\Gamma_*(H, C))$:

$$(2.7) \qquad\qquad \mathrm{H}_*^{\mathrm{Sw}}(H, C) = H_*\bigl(F\bigl(\Gamma_*(H, C)\bigr)\bigr) \, .$$

The resulting homology groups are *commutative cocommutative Hopf algebras* over k.

When $C = k$ is the trivial H-module coalgebra, then we recover the homology groups $\mathrm{H}_*^{\mathrm{Sw}}(H)$ of Definition 2.1:

$$\mathrm{H}_*^{\mathrm{Sw}}(H, k) = \mathrm{H}_*^{\mathrm{Sw}}(H) \, .$$

3. Predualization of Lazy Cocycles

In this section we perform some preliminary basic constructions necessary for defining the lazy homology of a Hopf algebra. The idea is to "predualize" the laziness conditions of Section 1.2.

3.1. The lazy quotient $C^{[1]}$. Given a coalgebra C, we denote by $C^{[1]}$ the quotient of C by the subspace spanned by the elements

$$\varphi(x_1) \, x_2 - \varphi(x_2) \, x_1 \, ,$$

for all $x \in C$ and $\varphi \in C^* = \mathrm{Hom}(C, k)$. The class of $x \in C$ in $C^{[1]}$ will be denoted by $\underline{x}$.

Proposition 3.1. *There is a unique coalgebra structure on $C^{[1]}$ such that the projection $C \to C^{[1]}$ is a coalgebra morphism. Furthermore, we have the following "strong-cocommutativity" identities:*

$$(3.1) \qquad\qquad \underline{x_1} \otimes x_2 = \underline{x_2} \otimes x_1 \in C^{[1]} \otimes C$$

for all $x \in C$.

By definition, the coproduct Δ and the counit ε of $C^{[1]}$ are given for all $x \in C$ by

$$\Delta(\underline{x}) = \underline{x_1} \otimes \underline{x_2} \quad \text{and} \quad \varepsilon(\underline{x}) = \varepsilon(x)$$

for all $x \in C$.

Proof. The coproduct of C induces a linear map $\Delta' : C \to C^{[1]} \otimes C^{[1]}$ defined by $\Delta'(x) = \underline{x_1} \otimes \underline{x_2}$ $(x \in C)$. For $x \in C$ and $\varphi \in C^*$,

$$
\begin{aligned}
\Delta'(\varphi(x_1)\, x_2) &= \varphi(x_1)\, \underline{x_2} \otimes \underline{x_3} = \varphi(x_2)\, \underline{x_1} \otimes \underline{x_3} \\
&= \underline{x_1} \otimes \varphi(x_2)\, \underline{x_3} = \underline{x_1} \otimes \underline{x_2}\, \varphi(x_3) \\
&= \Delta'(\varphi(x_2)\, x_1) \,.
\end{aligned}
$$

Thus Δ' induces a linear map $\Delta : C^{[1]} \to C^{[1]} \otimes C^{[1]}$ such that $\Delta(\underline{x}) = \underline{x_1} \otimes \underline{x_2}$ for all $x \in C$. The map Δ is clearly coassociative. The counit of $C^{[1]}$ is defined similarly and we obtain the desired coalgebra structure on $C^{[1]}$. For $\varphi \in C^*$ and $x \in C$, we have

$$(\mathrm{id} \otimes \varphi)(\underline{x_1} \otimes \underline{x_2}) = \varphi(x_2)\, \underline{x_1} = \varphi(x_1)\, \underline{x_2} = (\mathrm{id} \otimes \varphi)(\underline{x_2} \otimes \underline{x_1}) \,.$$

This yields (3.1). $\qquad\square$

Corollary 3.2. *The coalgebra $C^{[1]}$ is cocommutative.*

Proof. This is a consequence of the second assertion in Proposition 3.1. $\qquad\square$

Definition 3.3. *The coalgebra $C^{[1]}$ is called the lazy quotient of C.*

Observe that the coalgebra $C^{[1]} = C$ if C is cocommutative. The definition of the lazy quotient is motivated by the following result.

Proposition 3.4. *Let C be a coalgebra and R a commutative algebra. For any $\varphi \in \mathrm{Reg}_\ell(C, R)$ there is a unique element $\underline{\varphi} \in \mathrm{Reg}(C^{[1]}, R)$ such that $\underline{\varphi}(\underline{x}) = \varphi(x)$ for all $x \in C$. There are group isomorphisms*

$$\mathrm{Reg}_\ell(C, R) \cong \mathrm{Reg}(C^{[1]}, R) \cong \mathrm{Alg}(F(C^{[1]}), R) \,.$$

Proof. If $\psi \in C^*$ and $x \in H$, then by (1.2),

$$
\begin{aligned}
\varphi\big(\psi(x_1)\, x_2 - \psi(x_2)\, x_1\big) &= \varphi(x_2)\, \psi(x_1) - \varphi(x_1)\, \psi(x_2) \\
&= \varphi(x_2)\, \psi(x_1) - \varphi(x_2)\, \psi(x_1) \\
&= 0 \,.
\end{aligned}
$$

Hence φ induces the announced linear map $\underline{\varphi}$. If $\varphi, \psi \in \mathrm{Reg}_\ell(C, R)$, then

$$\underline{\varphi * \psi} = \underline{\varphi} * \underline{\psi} \,.$$

It follows that $\underline{\varphi}$ is convolution invertible, its inverse being equal to $\underline{\varphi^{-1}}$, where φ^{-1} is the convolution inverse of φ. The map

$$\mathrm{Reg}_\ell(C, R) \to \mathrm{Reg}(C^{[1]}, R)$$

defined by $\varphi \mapsto \underline{\varphi}$ is a homomorphism; it is clearly injective.

For $\varphi \in \mathrm{Reg}(C^{[1]}, R)$, define $\varphi_0 \in \mathrm{Reg}(C, R)$ by $\varphi_0(x) = \varphi(\underline{x})$ for all $x \in C$. Using (3.1), we obtain

$$
\begin{aligned}
\psi\big(\varphi_0(x_1)\, x_2 - \varphi_0(x_2)\, x_1\big) &= \psi\big(\varphi(\underline{x_1})\, x_2 - \varphi(\underline{x_2})\, x_1\big) \\
&= \psi\big(\varphi(\underline{x_2})\, x_1 - \varphi(\underline{x_2})\, x_1\big) = 0
\end{aligned}
$$

for all $\psi \in C^*$ and $x \in C$. It follows that $\varphi_0 \in \mathrm{Reg}_\ell(C, R)$ and, since $\underline{\varphi_0} = \varphi$, we obtain the first isomorphism. The second isomorphism follows from Proposition 1.7. $\qquad\square$

3.2. **The lazy quotient** $H^{[2]}$. Given a Hopf algebra H, we denote by $H^{[2]}$ the quotient of $H \otimes H$ by the subspace spanned by the elements

$$(3.2) \qquad \varphi(x_2 y_2)\, x_1 \otimes y_1 - \varphi(x_1 y_1)\, x_2 \otimes y_2$$

for all $x, y \in H$ and $\varphi \in H^* = \operatorname{Hom}(H, k)$. The class of $x \otimes y \in H \otimes H$ in $H^{[2]}$ will be denoted by $\widetilde{x \otimes y}$.

Proposition 3.5. *There is a unique coalgebra structure on $H^{[2]}$ such that the projection $H \otimes H \to H^{[2]}$ is a coalgebra morphism. Furthermore, we have the "lazy-cocommutativity" identities*

$$(3.3) \qquad \widetilde{x_1 \otimes y_1} \otimes x_2 y_2 = \widetilde{x_2 \otimes y_2} \otimes x_1 y_1 \in H^{[2]} \otimes H$$

for all $x, y \in H$.

By definition, the coproduct Δ and the counit ε of $H^{[2]}$ are given for all $x, y \in H$ by

$$\Delta(\widetilde{x \otimes y}) = \widetilde{x_1 \otimes y_1} \otimes \widetilde{x_2 \otimes y_2} \quad \text{and} \quad \varepsilon(\widetilde{x \otimes y}) = \varepsilon(x)\, \varepsilon(y)\,.$$

Proof. The coproduct of $H \otimes H$ induces a linear map

$$\Delta' : H \otimes H \to H^{[2]} \otimes H^{[2]}$$

defined by $\Delta'(x \otimes y) = \widetilde{x_1 \otimes y_1} \otimes \widetilde{x_2 \otimes y_2}$ $(x, y \in H)$. For $x, y \in H$ and $\varphi \in H^*$,

$$\begin{aligned}
\Delta'\big(\varphi(x_1 y_1)\, x_2 \otimes y_2\big) &= \varphi(x_1 y_1)\, \widetilde{x_2 \otimes y_2} \otimes \widetilde{x_3 \otimes y_3} \\
&= \varphi(x_2 y_2)\, \widetilde{x_1 \otimes y_1} \otimes \widetilde{x_3 \otimes y_3} \\
&= \widetilde{x_1 \otimes y_1} \otimes \varphi(x_2 y_2)\, \widetilde{x_3 \otimes y_3} \\
&= \widetilde{x_1 \otimes y_1} \otimes \varphi(x_3 y_3)\, \widetilde{x_2 \otimes y_2} \\
&= \varphi(x_3 y_3)\, \widetilde{x_1 \otimes y_1} \otimes \widetilde{x_2 \otimes y_2} \\
&= \Delta'\big(\varphi(x_2 y_2)\, x_1 \otimes y_1\big)\,.
\end{aligned}$$

Thus Δ' induces the announced linear map $\Delta : H^{[2]} \to H^{[2]} \otimes H^{[2]}$, which is clearly coassociative. The counit is defined similarly and we obtain the desired coalgebra structure. For $\varphi \in H^*$ and $x, y \in H$,

$$\begin{aligned}
(\operatorname{id} \otimes \varphi)(\widetilde{x_1 \otimes y_1} \otimes x_2 y_2) &= \varphi(x_2 y_2)\, \widetilde{x_1 \otimes y_1} \\
&= \varphi(x_1 y_1)\, \widetilde{x_2 \otimes y_2} \\
&= (\operatorname{id} \otimes \varphi)(\widetilde{x_2 \otimes y_2} \otimes x_1 y_1)\,.
\end{aligned}$$

This yields (3.3). $\qquad \square$

Definition 3.6. *The coalgebra $H^{[2]}$ is called the 2-lazy quotient associated to the Hopf algebra H.*

Observe that $H^{[2]} = H$ if H is cocommutative. Proposition 3.4 has the following counterpart.

Proposition 3.7. *Let H be a Hopf algebra and R a commutative algebra. For any $\sigma \in \operatorname{reg}_\ell^2(H, R)$ there is a unique element $\underline{\sigma} \in \operatorname{Reg}(H^{[2]}, R)$ such that $\underline{\sigma}(\widetilde{x \otimes y}) = \sigma(x \otimes y)$ for all $x, y \in H$. There are group isomorphisms*

$$\operatorname{reg}_\ell^2(H, R) \cong \operatorname{Reg}(H^{[2]}, R) \cong \operatorname{Alg}(F(H^{[2]}), R)\,.$$

Proof. The proof follows the same lines as that of Proposition 3.4. Details are left to the reader. The algebra morphism $\tilde{\underline{\sigma}} : F(H^{[2]}) \to R$ associated to σ is given by

$$\tilde{\underline{\sigma}}\big(t(\widetilde{x \otimes y})\big) = \sigma(x \otimes y)$$

for all $x, y \in H$. $\qquad \square$

4. The First Lazy Homology Hopf Algebra

In this section we define the first lazy homology of a Hopf algebra; it will turn out to be a commutative cocommutative Hopf algebra.

4.1. The Hopf algebra $H_1^\ell(H)$.

To a Hopf algebra H we associate the coalgebra $H^{[1]}$, as defined in Section 3.1, and the commutative Hopf algebra $F(H^{[1]})$, where F is the functor introduced in Section 1.5. We denote by $H_1^\ell(H)$ the quotient of $F(H^{[1]})$ by the ideal generated by all elements of the form

$$t(\underline{xy}) - t(\underline{x})\,t(\underline{y})$$

where $x, y \in H$. The class of an element $a \in F(H^{[1]})$ in $H_1^\ell(H)$ is denoted by $a^\dagger$. By definition,

$$(4.1) \qquad t(\underline{xy})^\dagger = t(\underline{x})^\dagger\, t(\underline{y})^\dagger$$

for all $x, y \in H$.

Lemma 4.1. *For all $x, y \in H$,*

$$t^{-1}(\underline{xy})^\dagger = t^{-1}(\underline{x})^\dagger\, t^{-1}(\underline{y})^\dagger\,.$$

Proof. Using (1.14), (4.1), and the relations satisfied by counits, we obtain

$$
\begin{aligned}
t^{-1}(\underline{xy})^\dagger &= t^{-1}(\underline{x_1 y_1})^\dagger\, \varepsilon(x_2 y_2) = t^{-1}(\underline{x_1 y_1})^\dagger\, \varepsilon(x_2)\, \varepsilon(y_2) \\
&= t^{-1}(\underline{x_1 y_1})^\dagger\, t(\underline{x_2})^\dagger\, t(\underline{y_2})^\dagger\, t^{-1}(\underline{x_3})^\dagger\, t^{-1}(\underline{y_3})^\dagger \\
&= t^{-1}(\underline{x_1 y_1})^\dagger\, t(\underline{x_2 y_2})^\dagger\, t^{-1}(\underline{x_3})^\dagger\, t^{-1}(\underline{y_3})^\dagger \\
&= \varepsilon(x_1 y_1)\, t^{-1}(\underline{x_2})^\dagger\, t^{-1}(\underline{y_2})^\dagger = \varepsilon(x_1)\, \varepsilon(y_1)\, t^{-1}(\underline{x_2})^\dagger\, t^{-1}(\underline{y_2})^\dagger \\
&= t^{-1}(\underline{x})^\dagger\, t^{-1}(\underline{y})^\dagger\,.
\end{aligned}
$$

$\square$

Proposition 4.2. *There is a unique Hopf algebra structure on $H_1^\ell(H)$ such that the natural projection $F(H^{[1]}) \to H_1^\ell(H)$ is a Hopf algebra morphism. The Hopf algebra $H_1^\ell(H)$ is commutative and cocommutative.*

It follows from (1.15) that the coproduct Δ, the counit ε, and the antipode S of $H_1^\ell(H)$ are given for all $x \in H$ by

$$
\begin{aligned}
\Delta\big(t(\underline{x})^\dagger\big) &= t(\underline{x_1})^\dagger \otimes t(\underline{x_2})^\dagger\,, \quad \Delta\big(t^{-1}(\underline{x})^\dagger\big) = t^{-1}(\underline{x_2})^\dagger \otimes t^{-1}(\underline{x_1})^\dagger\,, \\
\varepsilon\big(t(\underline{x})^\dagger\big) &= \varepsilon\big(t^{-1}(\underline{x})^\dagger\big) = \varepsilon(x)\,, \\
S\big(t(\underline{x})^\dagger\big) &= t^{-1}(\underline{x})^\dagger\,, \quad S\big(t^{-1}(\underline{x})^\dagger\big) = t(\underline{x})^\dagger\,.
\end{aligned}
$$

Proof. The coproduct of $F(H^{[1]})$ induces an algebra morphism

$$\Delta' : F(H^{[1]}) \to H_1^\ell(H) \otimes H_1^\ell(H)$$

such that $\Delta'(t(\underline{x})) = t(\underline{x_1})^\dagger \otimes t(\underline{x_2})^\dagger$ and $\Delta'(t^{-1}(\underline{x})) = t^{-1}(\underline{x_2})^\dagger \otimes t^{-1}(\underline{x_1})^\dagger$ for all $x \in H$. Using (4.1), we have

$$
\begin{aligned}
\Delta'\big(t(\underline{xy})\big) &= t(\underline{x_1 y_1})^\dagger \otimes t(\underline{x_2 y_2})^\dagger \\
&= t(\underline{x_1})^\dagger t(\underline{y_1})^\dagger \otimes t(\underline{x_2})^\dagger t(\underline{y_2})^\dagger \\
&= \Delta'\big(t(\underline{x})\big)\, \Delta'\big(t(\underline{y})\big) \\
&= \Delta'\big(t(\underline{x}) t(\underline{y})\big)
\end{aligned}
$$

for all $x, y \in H$. Thus Δ' induces the desired coproduct. The construction of the counit and the antipode are similar and left to the reader (use Lemma 4.1). The Hopf algebra $H_1^\ell(H)$ inherits the commutativity of $F(H^{[1]})$ and the cocommutativity of $H^{[1]}$. $\square$

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Definition 4.3. *The first lazy homology Hopf algebra of H is the commutative cocommutative Hopf algebra* $\mathrm{H}_1^\ell(H)$.

The following result relates $\mathrm{H}_1^\ell(H)$ to the lazy cohomology group $\mathrm{H}_\ell^1(H, R)$ of Definition 1.1.

Theorem 4.4. *For any Hopf algebra H and any commutative algebra R there is a group isomorphism*
$$\mathrm{H}_\ell^1(H, R) \cong \mathrm{Alg}(\mathrm{H}_1^\ell(H), R).$$

Proof. Let $\mu \in \mathrm{H}_\ell^1(H, R)$. Since $\mu \in \mathrm{Reg}_\ell(H, R)$, we know from Propositions 1.7 and 3.4 that μ induces an algebra morphism $\mu^* : F(H^{[1]}) \to R$. The morphism μ^* vanishes on the defining ideal of $\mathrm{H}_1^\ell(H)$, since μ is an algebra morphism. We thus obtain a map
$$\mathrm{H}_\ell^1(H, R) \to \mathrm{Alg}(\mathrm{H}_1^\ell(H), R), \; \mu \mapsto \mu^*,$$
where $\mu^*(t(\underline{x})^\dagger) = \mu(x)$ and $\mu^*(t^{-1}(\underline{x})^\dagger) = \mu(S(x))$ for all $x \in H$. The map $\mu \mapsto \mu^*$ is obviously injective and is a homomorphism.

Any map $\chi \in \mathrm{Alg}(\mathrm{H}_1^\ell(H), R)$ induces a linear map $\chi_0 : H \to R$ defined by $\chi_0(x) = \chi(t(\underline{x})^\dagger)$ for all $x \in H$. By Proposition 3.4 and its proof, χ_0 belongs to $\mathrm{Reg}_\ell(H, R)$. For $x, y \in H$,
$$\chi_0(xy) = \chi(t(\underline{xy})^\dagger) = \chi(t(\underline{x})^\dagger) \chi(t(\underline{y})^\dagger) = \chi_0(x) \chi_0(y);$$
hence, $\chi_0 \in \mathrm{H}_\ell^1(H, R)$. In view of $\chi_0^* = \chi$, this concludes the proof. $\square$

4.2. An alternative description of $\mathrm{H}_1^\ell(H)$. We now give another, more direct, description of $\mathrm{H}_1^\ell(H)$. Let $H^{\langle 1 \rangle}$ be the quotient of H by the two-sided ideal I generated by the elements

$$(4.2) \qquad\qquad \varphi(x_1)\, x_2 - \varphi(x_2)\, x_1,$$

where $x \in H$ and $\varphi \in H^* = \mathrm{Hom}(H, k)$. The canonical projection $H \to H^{\langle 1 \rangle}$ obviously factors through the quotient $H^{[1]}$ defined in Section 3.1.

The proof of the following proposition is similar to that of Proposition 3.1 and is left to the reader.

Proposition 4.5. *There is a unique Hopf algebra structure on $H^{\langle 1 \rangle}$ such that the natural projection $H \to H^{\langle 1 \rangle}$ is a Hopf algebra morphism. The Hopf algebra $H^{\langle 1 \rangle}$ is cocommutative.*

The Hopf algebra $H^{\langle 1 \rangle}$ has the following property.

Proposition 4.6. *For any commutative algebra R, the group $\mathrm{Alg}(H^{\langle 1 \rangle}, R)$ consists of the elements of $\mathrm{Alg}(H, R)$ commuting in $\mathrm{Hom}(H, R)$ with all elements of $\mathrm{Hom}(H, k)\, 1_R$.*

Proof. An element $\psi \in \mathrm{Alg}(H, R)$ factors through $H^{\langle 1 \rangle}$ if and only if it vanishes on the elements (4.2), i.e., if and only if $\psi * \varphi = \varphi * \psi$ for all $\varphi \in H^*$. $\square$

Consider the case where R is the ground field k. It is well known that if k is algebraically closed of characteristic zero and H is commutative, then $\mathrm{Alg}(H, k)$ separates the points of H. We can then sharpen the previous proposition and state that under the previous conditions, $\mathrm{Alg}(H^{\langle 1 \rangle}, k)$ coincides with the center of the group $\mathrm{Alg}(H, k)$.

For any Hopf algebra H, we denote by H_{ab} the quotient of H by the two-sided ideal generated by all commutators $[x, y] = xy - yx$ of elements x, y of H. Since
$$\Delta([x, y]) = [x_1, y_1] \otimes x_2 y_2 + y_1 x_1 \otimes [x_2, y_2],$$
$\varepsilon([x, y]) = 0$, and $S([x, y]) = -[S(x), S(y)]$, the Hopf structure on H induces a commutative Hopf algebra structure on H_{ab}. We call H_{ab} the *abelianization* of H. We are now ready to give the promised alternative description of $\mathrm{H}_1^\ell(H)$.

Proposition 4.7. *For any Hopf algebra H there is a Hopf algebra isomorphism*

$$(4.3) \qquad\qquad \mathrm{H}_1^\ell(H) \cong (H^{\langle 1 \rangle})_{\mathrm{ab}}.$$

Proof. The composition of the canonical projections $H^{[1]} \to H^{\langle 1 \rangle} \to (H^{\langle 1 \rangle})_{\text{ab}}$ induces a surjective coalgebra morphism $H^{[1]} \to (H^{\langle 1 \rangle})_{\text{ab}}$, which by Proposition 1.6 induces a Hopf algebra morphism $p : F(H^{[1]}) \to (H^{\langle 1 \rangle})_{\text{ab}}$. If we denote the class of an element $x \in H$ in $(H^{\langle 1 \rangle})_{\text{ab}}$ by x°, then $p(t(\underline{x})) = x^\circ$ and

$$p\big(t(\underline{xy})\big) = (xy)^\circ = x^\circ\, y^\circ = p\big(t(\underline{x})\big)\, p\big(t(\underline{y})\big) = p\big(t(\underline{x})\, t(\underline{y})\big)$$

for all $x, y \in H$. Hence, p induces a (surjective) Hopf algebra morphism $\mathrm{H}_1^\ell(H) \to (H^{\langle 1 \rangle})_{\text{ab}}$.

An inverse isomorphism is constructed as follows. By (3.1), the Hopf algebra morphism

$$H \to \mathrm{H}_1^\ell(H)\,, \ x \mapsto t(\underline{x})^\dagger$$

induces a Hopf algebra morphism $H^{\langle 1 \rangle} \to \mathrm{H}_1^\ell(H)$. Since by Proposition 4.2 the Hopf algebra $\mathrm{H}_1^\ell(H)$ is commutative, the latter morphism induces a Hopf algebra morphism

$$(H^{\langle 1 \rangle})_{\text{ab}} \to \mathrm{H}_1^\ell(H)\,, \ x^\circ \mapsto t(\underline{x})^\dagger\,,$$

which is the required inverse isomorphism. $\qquad\square$

Corollary 4.8. *If H is a finite-dimensional Hopf algebra, then so is $\mathrm{H}_1^\ell(H)$.*

Proposition 4.7 also allows us to compute the first lazy homology of the Hopf algebra k^G of functions on a finite group G.

Proposition 4.9. *For any finite group G, we have*

$$\mathrm{H}_1^\ell(k^G) \cong k^{Z(G)}\,,$$

where $k^{Z(G)}$ is the Hopf algebra of functions on the center of G.

Proof. For each $g \in G$, let e_g be the function on G that is zero everywhere, except at the point g, where it takes the value 1. The elements $(e_g)_{g \in G}$ form a basis of $H = k^G$. Let $(\psi_g)_{g \in G}$ be the dual basis. Then $H^{\langle 1 \rangle}$ is the quotient of H by the ideal generated by

$$\sum_{a \in G} \psi_{h^{-1}}(e_a)\, e_{a^{-1}g} - \sum_{a \in G} e_{ga^{-1}}\, \psi_{h^{-1}}(e_a)$$

for all $g, h \in G$. The first sum reduces to e_{hg}, while the second sum reduces to e_{gh}. Therefore, $H^{\langle 1 \rangle}$ is the quotient of H by the ideal generated by $w_{g,h} = e_{hg} - e_{gh}$ for all $g, h \in G$. Now, $w_{g,h} = 0$ if g belongs to the center of G. If g is not in the center, then there is h such that $hgh^{-1} \neq g$; for such an element h, we have

$$e_g = e_g^2 = e_g^2 - e_g e_{hgh^{-1}} = e_g w_{g,hgh^{-1}}\,,$$

which shows that the image of e_g in $H^{\langle 1 \rangle}$ is zero. From this one easily deduces that $H^{\langle 1 \rangle} \cong k^{Z(G)}$. The conclusion follows from Proposition 4.7 and the commutativity of H, hence of $H^{\langle 1 \rangle}$. $\qquad\square$

4.3. An interpretation of $\mathrm{H}_1^\ell(H)$ as a homology group. We could have defined $\mathrm{H}_1^\ell(H)$ as the right-hand side of (4.3). Nevertheless, the definition we gave in Section 4.1 will allow us to present $\mathrm{H}_1^\ell(H)$ as a kind of homology group (see Proposition 4.12 below). Such a presentation will be used in the next section to define a lazy homology group $\mathrm{H}_2^\ell(H)$.

We first construct a Hopf algebra morphism generalizing the differential d_2^{Sw} of Lemma 2.2.

Lemma 4.10. *For any Hopf algebra H there is a Hopf algebra morphism*

$$d_2 : F(H^{[2]}) \to F(H^{[1]})$$

such that for all $x, y \in H$,

(4.4) $$d_2\big(t(\widetilde{x \otimes y})\big) = t(\underline{x_1})\, t(\underline{y_1})\, t^{-1}(\underline{x_2 y_2})\,.$$

Moreover,

(4.5) $$d_2(a_1) \otimes a_2 = d_2(a_2) \otimes a_1$$

for all $a \in F(H^{[2]})$.

Proof. Consider the map $d_2' : H \otimes H \to F(H^{[1]})$ given by

$$d_2'(x \otimes y) = t(\underline{x_1})\, t(\underline{y_1})\, t^{-1}(\underline{x_2 y_2})$$

for all $x, y \in H$. Let us show that d_2' is a coalgebra morphism. On one hand, by (1.15), we have

$$\Delta\big(d_2'(x \otimes y)\big) = \Delta\big(t(\underline{x_1})\, t(\underline{y_1})\, t^{-1}(\underline{x_2 y_2})\big)$$
$$= t(\underline{x_1})\, t(\underline{y_1})\, t^{-1}(\underline{x_4 y_4}) \otimes t(\underline{x_2})\, t(\underline{y_2})\, t^{-1}(\underline{x_3 y_3})\,.$$

On the other hand,

$$(d_2' \otimes d_2')\big(\Delta(x \otimes y)\big) = (d_2' \otimes d_2')(x_1 \otimes y_1 \otimes x_2 \otimes y_2)$$
$$= t(\underline{x_1})\, t(\underline{y_1})\, t^{-1}(\underline{x_2 y_2}) \otimes t(\underline{x_3})\, t(\underline{y_3})\, t^{-1}(\underline{x_4 y_4})\,.$$

These two expressions are equal in view of the strong-cocommutativity identities (3.1) of $H^{[1]}$. It is easy to check that $\varepsilon\big(d_2'(x \otimes y)\big) = \varepsilon(x)\,\varepsilon(y)$ for all $x, y \in H$. We have thus shown that d_2' is a coalgebra morphism.

We next claim that d_2' factors through $H^{[2]}$. In view of (3.2) we have to check that

$$d_2'\big(\varphi(x_2 y_2)\, x_1 \otimes y_1 - \varphi(x_1 y_1)\, x_2 \otimes y_2\big) = 0$$

for all $\varphi \in H^*$ and $x, y \in H$. Now the left-hand side is equal to

$$\varphi(x_3 y_3)\, t(\underline{x_1})\, t(\underline{y_1})\, t^{-1}(\underline{x_2 y_2}) - \varphi(x_1 y_1)\, t(\underline{x_2})\, t(\underline{y_2})\, t^{-1}(\underline{x_3 y_3})\,,$$

which vanishes in view of (3.1). By Proposition 1.6, the induced coalgebra morphism $H^{[2]} \to F(H^{[1]})$ induces the desired Hopf algebra morphism $d_2 : F(H^{[2]}) \to F(H^{[1]})$.

Let us now prove (4.5). Since d_2 is a Hopf algebra morphism, it is enough to prove the required identity for $a = t(\widetilde{x \otimes y})$, where $x, y \in H$. We have

$$d_2\big(t(\widetilde{x \otimes y})_1\big) \otimes t(\widetilde{x \otimes y})_2 = d_2\big(t(\widetilde{x_1 \otimes y_1})\big) \otimes t(\widetilde{x_2 \otimes y_2})$$
$$= t(\underline{x_1})\, t(\underline{y_1})\, t^{-1}(\underline{x_2 y_2}) \otimes t(\widetilde{x_3 \otimes y_3})$$
$$= t(\underline{x_1})\, t(\underline{y_1})\, t^{-1}(\underline{x_3 y_3}) \otimes t(\widetilde{x_2 \otimes y_2})$$
$$= t(\underline{x_2})\, t(\underline{y_2})\, t^{-1}(\underline{x_3 y_3}) \otimes t(\widetilde{x_1 \otimes y_1})$$
$$= d_2\big(t(\widetilde{x_2 \otimes y_2})\big) \otimes t(\widetilde{x_1 \otimes y_1})$$
$$= d_2\big(t(\widetilde{x \otimes y})_2\big) \otimes t(\widetilde{x \otimes y})_1\,.$$

We have used the lazy-cocommutativity identities (3.3) in $H^{[2]}$ for the third equality above and the strong-cocommutativity identities (3.1) in $H^{[1]}$ for the fourth equality. $\square$

As a consequence of Lemma 4.10 and Section 1.3, we obtain the following.

Corollary 4.11. *The Hopf algebra morphism d_2 is normal and*

$$\mathrm{HKer}(d_2) = \{a \in F(H^{[2]}) \mid d_2(a_1) \otimes a_2 = 1 \otimes a\} = \{a \in F(H^{[2]}) \mid a_1 \otimes d(a_2) = a \otimes 1\}\,.$$

By Lemma 4.10, the image $\mathrm{Im}(d_2)$ of d_2 is a Hopf subalgebra of the commutative cocommutative Hopf algebra $F(H^{[1]})$, and it is cocommutative. Using the convention of Section 1.3 (see (1.11)), we can define the quotient $F(H^{[1]}) /\!/ \mathrm{Im}(d_2)$ as the quotient of $F(H^{[1]})$ by the ideal generated by the augmentation ideal $\mathrm{Im}(d_2)^+$ of $\mathrm{Im}(d_2)$.

We now express $\mathrm{H}_1^\ell(H)$ in terms of this homology-like quotient.

Proposition 4.12. *For any Hopf algebra H, there is an isomorphism of Hopf algebras*

$$\mathrm{H}_1^\ell(H) \cong F(H^{[1]}) /\!/ \mathrm{Im}(d_2) = \mathrm{HKer}(\varepsilon) /\!/ \mathrm{Im}(d_2)\,.$$

If in addition H is cocommutative, then

$$\mathrm{H}_1^\ell(H) = \mathrm{H}_1^{\mathrm{Sw}}(H)\,.$$

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Proof. In order to prove the first isomorphism, we check that the ideal generated by $\mathrm{Im}(d_2)^+$ coincides with the ideal I of $F(H^{[1]})$ generated by

$$t(\underline{xy}) - t(\underline{x})\,t(\underline{y})\,,$$

where $x, y \in H$. By an easy computation one shows that for $x, y \in H$,

(4.6) $$d_2\big(t(\widetilde{x \otimes y})\big) - \varepsilon\big(t(\widetilde{x \otimes y})\big) = -\big(t(\underline{x_1 y_1}) - t(\underline{x_1})\,t(\underline{y_1})\big)\,t^{-1}(\underline{x_2 y_2})\,.$$

It follows that $d_2(\omega) - \varepsilon(\omega) \in I$ for all $\omega \in F(H^{[2]})$. Hence, $\mathrm{Im}(d_2)^+ \subset I$.

By (1.14), Equation (4.6) implies that

$$\big[d_2\big(t(\widetilde{x_1 \otimes y_1})\big) - \varepsilon\big(t(\widetilde{x_1 \otimes y_1})\big)\big]\,t(\underline{x_2 y_2}) = -\big(t(\underline{xy}) - t(\underline{x})\,t(\underline{y})\big)\,.$$

From this we deduce the converse inclusion $I \subset \mathrm{Im}(d_2)^+$.

The equality $F(H^{[1]})/\mathrm{Im}(d_2) = \mathrm{HKer}(\varepsilon)/\mathrm{Im}(d_2)$ follows from the equality

$$F(H^{[1]}) = \mathrm{HKer}(\varepsilon : F(H^{[1]}) \to k)\,,$$

a general fact observed in Section 1.3.

If H is cocommutative, then $F(H^{[1]}) = F(H)$, $F(H^{[2]}) = F(H \otimes H)$, and by (4.4) and (2.4), the map d_2 coincides with the differential d_2^{Sw} of Lemma 2.2. This yields the second part of the statement. $\square$

When $H = k[G]$ is a group algebra, Propositions 2.3 and 4.12 imply that

(4.7) $$\mathrm{H}_1^\ell(k[G]) = \mathrm{H}_1^{\mathrm{Sw}}(k[G]) \cong k[H_1(G, \mathbb{Z})] = k[G_{\mathrm{ab}}]\,,$$

where G_{ab} is the largest abelian quotient of G.

4.4. **An** Ext^1**-group.** By Lemma 4.10, the image $\mathrm{Im}(d_2)$ of the Hopf algebra morphism $d_2 : F(H^{[2]}) \to F(H^{[1]})$ is a cocommutative Hopf subalgebra of $F(H^{[1]})$. Consider the sequence of Hopf algebra morphisms

$$k \longrightarrow \mathrm{Im}(d_2) \xrightarrow{\ \iota\ } F(H^{[1]}) \xrightarrow{\ \pi\ } \mathrm{H}_1^\ell(H) \longrightarrow k\,,$$

where ι is the embedding of $\mathrm{Im}(d_2)$ into $F(H^{[1]})$ and π is the natural projection defining $\mathrm{H}_1^\ell(H)$. This sequence is exact since ι is injective, π is surjective, and $\mathrm{Ker}(\pi) = \mathrm{Im}(d_2)^+ F(H^{[1]})$ by Proposition 4.12. It follows from Proposition 1.4 that for any commutative algebra R, the sequence of groups

$$0 \longrightarrow \mathrm{Alg}(\mathrm{H}_1^\ell(H), R) \xrightarrow{\ \pi^*\ } \mathrm{Alg}(F(H^{[1]}), R) \xrightarrow{\ \iota^*\ } \mathrm{Alg}(\mathrm{Im}(d_2), R)$$

is exact. Observe that the groups in the previous sequence are all abelian since the Hopf algebras involved are cocommutative.

Mimicking the definition of Ext^1 in standard homological algebra, we pose the following.

Definition 4.13. *The group* $\mathrm{Ext}^1(H, R)$ *is the cokernel of the map* ι^*:

$$\mathrm{Ext}^1(H, R) = \mathrm{Coker}\big(\iota^* : \mathrm{Alg}(F(H^{[1]}), R) \longrightarrow \mathrm{Alg}(\mathrm{Im}(d_2), R)\big)\,.$$

The group $\mathrm{Ext}^1(H, R)$ is *abelian*; it is clearly contravariant in H and covariant in R. The following is an immediate consequence of the definition and of Proposition 1.5.

Proposition 4.14. *If k is algebraically closed, then* $\mathrm{Ext}^1(H, k) = 0$.

Remark 4.15. The corresponding Ext^1-groups in the case of a group G can be computed from the first homology group $H_1(G, \mathbb{Z})$, hence by (4.7) from $\mathrm{H}_1^\ell(H)$ for $H = k[G]$. We may wonder whether for an arbitrary Hopf algebra H the group $\mathrm{Ext}^1(H, R)$ depends only on the lazy cohomology Hopf algebra $\mathrm{H}_1^\ell(H)$. Observe that if $\mathrm{H}_1^\ell(H) \cong k$ is trivial, then $\iota : \mathrm{Im}(d_2) \to F(H^{[1]})$ is an isomorphism and $\mathrm{Ext}^1(H, R) = 0$.

5. The Cosemisimple Case

In this section, assuming that the ground field k is algebraically closed of characteristic zero, we compute the first lazy homology of a cosemisimple Hopf algebra H. More precisely, we show that $\mathrm{H}_1^\ell(H)$ is isomorphic to the group algebra of the universal abelian grading group of the tensor category of H-comodules.

As an application, we recover Müger's description of the center of a compact group [10] as the group of unitary characters of the universal abelian grading group of the representation category. This shows that one can reconstruct the center of such a group from its representation category (or better, from its fusion semiring).

We first fix some notation. Let H be a cosemisimple Hopf algebra and $\mathrm{Irr}(H)$ be the set of isomorphism classes of simple (irreducible) H-comodules. The isomorphism class of a simple H-comodule V is denoted by $[V]$. For each $\lambda \in \mathrm{Irr}(H)$, we fix a simple H-comodule V_λ such that $[V_\lambda] = \lambda$.

For $\lambda, \mu, \nu \in \mathrm{Irr}(H)$, we write $\nu \prec \lambda \otimes \mu$ when V_ν is isomorphic to a subcomodule of $V_\lambda \otimes V_\mu$.

Definition 5.1. *The universal abelian grading group Γ_H of the semisimple tensor category of H-comodules is the quotient of the free (multiplicative) abelian group generated by $\mathrm{Irr}(H)$ modulo the relations*

$$\lambda\mu = \nu \ \text{whenever } \nu \prec \lambda \otimes \mu\,.$$

The class of $\lambda \in \mathrm{Irr}(H)$ in Γ_H is denoted by $|\lambda|$. The universal abelian grading group of a semisimple tensor category has appeared (in various degrees of generality) in several independant papers; see [3, 10, 8, 12].

We have the following result.

Theorem 5.2. *Let H be a cosemisimple Hopf algebra over an algebraically closed field of characteristic zero. Then there is a Hopf algebra isomorphism*

$$\mathrm{H}_1^\ell(H) \cong k[\Gamma_H]\,.$$

The rest of the subsection is essentially devoted to the proof of Theorem 5.2, with an application to centers of compact groups at the very end.

We need some additional notation. For each $\lambda \in \mathrm{Irr}(H)$, we fix a basis $e_1^\lambda, \ldots, e_{d_\lambda}^\lambda$ of V_λ (so that $d_\lambda = \dim(V_\lambda)$) and elements x_{ij}^λ of H ($1 \leq i, j \leq d_\lambda$) such that $\alpha(e_j^\lambda) = \sum_{i=1}^{d_\lambda} e_i^\lambda \otimes x_{ij}^\lambda$, where α stands for the H-coaction on V_λ. The elements $x_{ij}^\lambda \in H$ ($1 \leq i, j \leq d_\lambda$, $\lambda \in \mathrm{Irr}(H)$) are linearly independent and satisfy

$$(5.1) \qquad\qquad \Delta(x_{ij}^\lambda) = \sum_{k=1}^{d_\lambda} x_{ik}^\lambda \otimes x_{kj}^\lambda\,, \qquad \varepsilon(x_{ij}^\lambda) = \delta_{ij}\,,$$

and we have the coalgebra direct sum (Peter-Weyl decomposition)

$$H = \bigoplus_{\lambda \in \mathrm{Irr}(H)} H_\lambda$$

where H_λ is the comatrix subcoalgebra spanned by the elements x_{ij}^λ ($1 \leq i, j \leq d_\lambda$). See e.g. [15, Chapter 14].

We first describe the coalgebra $H^{[1]}$.

Lemma 5.3. *The coalgebra $H^{[1]}$ has a basis consisting of the grouplike elements $x^\lambda = \underline{x_{11}^\lambda}$ ($\lambda \in \mathrm{Irr}(H)$). We have $\underline{x_{ij}^\lambda} = 0$ for all $i \neq j$ and $\underline{x_{ii}^\lambda} = x^\lambda$ for all $i = 1, \ldots, d_\lambda$.*

It follows from this lemma and from (5.1) that each element $x^\lambda \in H^{[1]}$ is grouplike. Thus, $H^{[1]}$ is a *cosemisimple pointed* coalgebra.

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Proof. Let $(\psi_{ij}^\lambda)_{ij}$ be the dual basis of the basis $(x_{ij}^\lambda)_{ij}$. The following relations hold in $H^{[1]}$ for any $\varphi \in H^*$:

$$\sum_k \underline{x_{ik}^\lambda}\, \varphi(x_{kj}^\lambda) = \sum_k \underline{x_{kj}^\lambda}\, \varphi(x_{ik}^\lambda)$$

For $\varphi = \psi_{ij}^\lambda$, this gives $\underline{x_{ii}^\lambda} = \underline{x_{jj}^\lambda}$. For $\varphi = \psi_{ii}^\lambda$, this gives $\underline{x_{ij}^\lambda} = 0$ if $i \neq j$. Hence the elements x^λ, as defined in the statement of the lemma, span $H^{[1]}$. One easily checks that they are linearly independent, using the linear forms $\psi^\lambda \in H^*$ defined by $\psi^\lambda(x_{ij}^\mu) = \delta_{\lambda\mu}\delta_{ij}$. $\qquad\square$

It follows that the Hopf algebra $F(H^{[1]})$ is the group algebra of the free abelian group generated by $\mathrm{Irr}(H)$.

Before giving the proof of Theorem 5.2, we state the following lemma, whose elementary proof is left to the reader.

Lemma 5.4. *Let $g, h_1, \ldots h_n$ be elements of a group G. Assume that in the group algebra $k[G]$ we have*

$$g = \sum_{i=1}^n r_i\, h_i$$

for some positive rational numbers $r_1, \ldots, r_n$. Then $h_1 = \cdots = h_n = g$.

Proof of Theorem 5.2. The character of an element $\lambda \in \mathrm{Irr}(H)$ is the element $\chi^\lambda = \sum_{i=1}^{d_\lambda} x_{ii}^\lambda \in H$. For all $\lambda, \mu \in \mathrm{Irr}(H)$, we have

$$\chi^\lambda \chi^\mu = \sum_{\nu \prec \lambda \otimes \mu} d_{\lambda\mu}^\nu\, \chi^\nu\,,$$

where $d_{\lambda\mu}^\nu = \dim \mathrm{Hom}(V_\nu, V_\lambda \otimes V_\mu)$. By Lemma 5.3, $\underline{\chi}^\lambda = d_\lambda x^\lambda$. This leads to the following computation in $\mathrm{H}_1^\ell(H)$:

$$t(x^\lambda)^\dagger\, t(x^\mu)^\dagger = \frac{1}{d_\lambda d_\mu}\, t(\underline{\chi}^\lambda)^\dagger\, t(\underline{\chi}^\mu)^\dagger = \frac{1}{d_\lambda d_\mu}\, t(\underline{\chi^\lambda \chi^\mu})^\dagger$$

$$= \frac{1}{d_\lambda d_\mu} \sum_{\nu \prec \lambda \otimes \mu} d_{\lambda\mu}^\nu\, t(\underline{\chi}^\nu)^\dagger = \sum_{\nu \prec \lambda \otimes \mu} \frac{d_{\lambda\mu}^\nu}{d_\lambda d_\mu}\, t(\underline{\chi}^\nu)^\dagger$$

$$= \sum_{\nu \prec \lambda \otimes \mu} \frac{d_{\lambda\mu}^\nu d_\nu}{d_\lambda d_\mu}\, t(x^\nu)^\dagger\,.$$

The Hopf algebra $\mathrm{H}_1^\ell(H)$ is a group algebra since it is a quotient of the group algebra $F(H^{[1]})$ and the elements $t(x^\lambda)^\dagger$ are grouplike elements. Hence by Lemma 5.4, if $\nu \prec \lambda \otimes \mu$, then

$$t(x^\lambda)^\dagger\, t(x^\mu)^\dagger = t(x^\nu)^\dagger\,.$$

This shows that there is a Hopf algebra morphism $k[\Gamma_H] \to \mathrm{H}_1^\ell(H)$ given by

$$|\lambda| \mapsto t(x^\lambda)^\dagger\,.$$

This Hopf algebra morphism is clearly surjective, and since it is a Hopf algebra morphism between two group algebras, it is enough to check its injectivity on the grouplike elements. Since the characters of a discrete abelian group separate its points, this is equivalent to show that the induced injective group homomorphism

$$(5.2) \qquad \mathrm{H}_\ell^1(H, k) \cong \mathrm{Alg}(\mathrm{H}_1^\ell(H), k) \longrightarrow \widehat{\Gamma_H} = \mathrm{Hom}(\Gamma_H, k^\times)$$

sending $\varphi \in \mathrm{Alg}(\mathrm{H}_1^\ell(H), k)$ to the element $\varphi' \in \widehat{\Gamma_H}$ defined by

$$\varphi'(|\lambda|) = \varphi(t(x^\lambda)^\dagger)$$

($\lambda \in \mathrm{Irr}(H)$), is surjective. For $\alpha \in \widehat{\Gamma_H}$, consider the linear form α_0 on $H = \bigoplus_{\lambda \in \mathrm{Irr}(H)} H_\lambda$ defined by

$$\alpha_0 = \sum_{\lambda \in \mathrm{Irr}(H)} \alpha(|\lambda|)\, \varepsilon_{|H_\lambda}\,.$$

It is clear that α_0 is an element of $\mathrm{Reg}^1_\ell(H)$. Let us check that α_0 is an algebra morphism. Let $\lambda, \mu \in \mathrm{Irr}(H)$, $1 \le i, j \le d_\lambda$, $1 \le k, l \le d_\mu$. We have $x^\lambda_{ij}\, x^\mu_{kl} = \sum_{\nu \prec \lambda \otimes \mu} z_\nu$ for some elements $z_\nu \in H_\nu$, since by Tannakian reconstruction we have $H_\lambda H_\mu \subset \sum_{\nu \prec \lambda \otimes \mu} H_\nu$. Therefore,

$$
\begin{aligned}
\alpha_0(x^\lambda_{ij} x^\mu_{kl}) &= \alpha_0\Big(\sum_{\nu \prec \lambda \otimes \mu} z_\nu \Big) = \sum_{\nu \prec \lambda \otimes \mu} \alpha(|\nu|)\, \varepsilon(z_\nu) \\
&= \sum_{\nu \prec \lambda \otimes \mu} \alpha(|\lambda||\mu|)\, \varepsilon(z_\nu) = \alpha(|\lambda|)\, \alpha(|\mu|)\, \varepsilon(x^\lambda_{ij} x^\mu_{kl}) \\
&= \alpha_0(x^\lambda_{ij})\, \alpha_0(x^\mu_{kl})\,.
\end{aligned}
$$

Hence, $\alpha_0 \in \mathrm{H}^1_\ell(H, k)$ with $\alpha'_0 = \alpha$. This concludes the proof of Theorem 5.2. $\qquad\square$

Remarks 5.5. (a) We have just proved that the group homomorphism $\mathrm{H}^1_\ell(H, k) \to \widehat{\Gamma_H}$ of (5.2) is an isomorphism. Since $\mathrm{H}^1_\ell(H, k)$ is also isomorphic to the group of monoidal automorphisms of the identity functor of the category of H-comodules, it is possible to prove Theorem 5.2 by using [8, Prop. 3.9] or [12, Prop. 1.3.3].

(b) Theorem 5.2 implies that for any cosemisimple Hopf algebra H over an algebraically closed field of characteristic zero, the first lazy homology Hopf algebra depends only on the fusion semiring of the tensor category of H-comodules.

(c) Let $H = k^G$ be the Hopf algebra of functions on a finite group G. By Proposition 4.9 and Theorem 5.2,

$$\mathrm{H}^\ell_1(H) \cong k^{Z(G)} \cong k[\Gamma_H]\,.$$

Consequently, $\Gamma_H \cong \widehat{Z(G)} = \mathrm{Hom}(Z(G), k^\times)$, a result already proved in [10].

We end this section by showing how to recover Müger's description of the center of a compact group (here the ground field is the field $\mathbb{C}$ of complex numbers).

Corollary 5.6. *The center $Z(G)$ of a compact group G is isomorphic to the group of unitary characters $\widehat{\mathcal{C}(G)} = \mathrm{Hom}(\mathcal{C}(G), \mathbb{S}^1)$ of the chain group $\mathcal{C}(G)$ of G, as defined in [10].*

Proof. Let $\mathcal{R}(G)$ be the Hopf algebra of (complex) representative functions on G; it is cosemisimple. The compact group G is isomorphic to the compact group $\mathrm{Hom}_{*-\mathrm{alg}}(\mathcal{R}(G), \mathbb{C})$ by the Tannaka-Krein theorem, and

$$Z(G) \cong \mathrm{Hom}_{*-\mathrm{alg}}(\mathcal{R}(G), \mathbb{C}) \cap \mathrm{H}^1_\ell(\mathcal{R}(G), \mathbb{C})\,.$$

The chain group of [10] is nothing but $\Gamma_{\mathcal{R}(G)}$, and the $*$-morphisms correspond to unitary characters under the isomorphism $\mathrm{H}^1_\ell(\mathcal{R}(G), \mathbb{C}) \cong \widehat{\Gamma_{\mathcal{R}(G)}}$ of (5.2). This concludes the proof. $\quad\square$

6. The Second Lazy Homology Hopf Algebra

In this section we construct the second lazy homology Hopf algebra of a Hopf algebra H.

6.1. **The Hopf algebra $\mathrm{H}^\ell_2(H)$.** It would be very helpful if we could find a version of the differential d^{Sw}_3 of Lemma 2.2 under the form of a Hopf algebra morphism $F(H \otimes H \otimes H) \to F(H^{[2]})$. Unfortunately this may not be possible for an arbitrary Hopf algebra. Instead we proceed as follows.

Mimicking (2.5), we set

$$(6.1) \qquad d_3(x, y, z) = t(\widetilde{y_1 \otimes z_1})\, t(\widetilde{x_1 \otimes y_2 z_2})\, t^{-1}(\widetilde{x_2 y_3 \otimes z_3})\, t^{-1}(\widetilde{x_3 \otimes y_4}) \in F(H^{[2]})$$

for all x, y, $z \in H$.

Lemma 6.1. *For all $x, y, z \in H$, we have the following identities:*

$$(6.2) \qquad\qquad d_2\big(d_3(x, y, z)\big) = \varepsilon(xyz)\, 1 \,,$$

$$(6.3) \qquad \Delta\big(d_3(x, y, z)\big) = t(\widetilde{y_1 \otimes z_1})\, t(\widetilde{x_1 \otimes y_2 z_2})\, t^{-1}(\widetilde{x_3 y_4 \otimes z_4})\, t^{-1}(\widetilde{x_4 \otimes y_5}) \otimes d_3(x_2, y_3, z_3)\,,$$

$$(6.4) \qquad\qquad (\mathrm{id} \otimes d_2)\big(\Delta(d_3(x, y, z))\big) = d_3(x, y, z) \otimes 1\,.$$

Furthermore, $d_3(x, y, z) \in \mathrm{HKer}(d_2)$ for all $x, y, z \in H$.

Proof. Identity (6.2) is a consequence of the following computation, in which the strong cocommutativity of $H^{[1]}$ is used several times. We have

$$d_2\big(d_3(x, y, z)\big) = d_2(t(\widetilde{y_1 \otimes z_1}))\, d_2(t(\widetilde{x_1 \otimes y_2 z_2}))$$
$$\times\, d_2(t^{-1}(\widetilde{x_2 y_3 \otimes z_3}))\, d_2(t^{-1}(\widetilde{x_3 \otimes y_4}))$$
$$= t(\underline{y_1})\, t(\underline{z_1})\, t^{-1}(\underline{y_2 z_2})\, t(\underline{x_1})\, t(\underline{y_3 z_3})\, t^{-1}(\underline{x_2 y_4 z_4})$$
$$\times\, d_2(t^{-1}(\widetilde{x_3 y_5 \otimes z_5}))\, d_2(t^{-1}(\widetilde{x_4 \otimes y_6}))$$
$$= t(\underline{x_1})\, t(\underline{y_1})\, t(\underline{z_1})\, t^{-1}(\underline{x_2 y_2 z_2})$$
$$\times\, d_2(t^{-1}(\widetilde{x_3 y_3 \otimes z_3}))\, d_2(t^{-1}(\widetilde{x_4 \otimes y_4}))$$
$$= t(\underline{x_1})\, t(\underline{y_1})\, t(\underline{z_1})\, t^{-1}(\underline{x_2 y_2 z_2})\, t^{-1}(\underline{x_4 y_4})\, t^{-1}(\underline{z_4})$$
$$\times\, t(\underline{x_3 y_3 z_3})\, d_2(t^{-1}(\widetilde{x_5 \otimes y_5}))$$
$$= t(\underline{x_1})\, t(\underline{y_1})\, t(\underline{z_1})\, t^{-1}(\underline{z_2})\, t^{-1}(\underline{x_2 y_2})\, d_2(t^{-1}(\widetilde{x_3 \otimes y_3}))$$
$$= \varepsilon(z)\, t(\underline{x_1})\, t(\underline{y_1})\, t^{-1}(\underline{x_2 y_2})\, d_2(t^{-1}(\widetilde{x_3 \otimes y_3}))$$
$$= \varepsilon(z)\, d_2(t(\widetilde{x_1 \otimes y_1})\, t^{-1}(\widetilde{x_2 \otimes y_2}))$$
$$= \varepsilon(xyz)\, 1\,.$$

For Identity (6.3), we use the lazy cocommutativity of $H^{[2]}$. We obtain

$$\Delta\big(d_3(x, y, z)\big) = t(\widetilde{y_1 \otimes z_1})\, t(\widetilde{x_1 \otimes y_3 z_3})\, t^{-1}(\widetilde{x_4 y_6 \otimes z_6})\, t^{-1}(\widetilde{x_6 \otimes y_8})$$
$$\otimes\, t(\widetilde{y_2 \otimes z_2})\, t(\widetilde{x_2 \otimes y_4 z_4})\, t^{-1}(\widetilde{x_3 y_5 \otimes z_5})\, t^{-1}(\widetilde{x_5 \otimes y_7})$$
$$= t(\widetilde{y_1 \otimes z_1})\, t(\widetilde{x_1 \otimes y_2 z_2})\, t^{-1}(\widetilde{x_5 y_7 \otimes z_6})\, t^{-1}(\widetilde{x_6 \otimes y_8})$$
$$\otimes\, t(\widetilde{y_3 \otimes z_3})\, t(\widetilde{x_2 y_4 \otimes z_4})\, t^{-1}(\widetilde{x_3 \otimes y_5 z_5})\, t^{-1}(\widetilde{x_4 \otimes y_6})$$
$$= t(\widetilde{y_1 \otimes z_1})\, t(\widetilde{x_1 \otimes y_2 z_2})$$
$$t^{-1}(\widetilde{x_3 \otimes y_4 z_4})\, t^{-1}(\widetilde{x_4 \otimes y_5}) \otimes d_3(x_2, y_3, z_3)\,.$$

Identity (6.4) follows from (6.2) and (6.3). The final assertion follows from Corollary 4.11. □

It follows from Lemma 6.1 that the ideal $\mathrm{B}_2^{\ell}(H)$ of $\mathrm{HKer}(d_2)$ generated by the elements

$$d_3(x, y, z) - \varepsilon(xyz)1 \quad \text{and} \quad S\big(d_3(x, y, z)\big) - \varepsilon(xyz)$$

for all $x, y, z \in H$, is a Hopf ideal in $\mathrm{HKer}(d_2)$.

Definition 6.2. *The second lazy homology Hopf algebra $\mathrm{H}_2^{\ell}(H)$ of a Hopf algebra H is the quotient Hopf algebra*

$$\mathrm{H}_2^{\ell}(H) = \mathrm{HKer}(d_2)/\mathrm{B}_2^{\ell}(H)\,.$$

Here again we have strongly modeled the definition of $\mathrm{H}_2^{\ell}(H)$ on the simplicial object $F(\Gamma_*(H))$.

Observe that $\mathrm{H}_2^{\ell}(H)$ is a *commutative* Hopf algebra since it is the quotient of a Hopf subalgebra of the commutative Hopf algebra $F(H^{[2]})$.

When H is cocommutative, we have the following isomorphism with the Sweedler homology defined in Section 2.2.

Proposition 6.3. *For any cocommutative Hopf algebra H we have*
$$\mathrm{H}_2^\ell(H) = \mathrm{H}_2^{\mathrm{Sw}}(H)\,.$$

Proof. As observed in the proof of Proposition 4.12, $F(H^{[2]}) = F(H \otimes H)$ and $d_2 = d_2^{\mathrm{Sw}}$. The map d_3 coincides with the differential d_3^{Sw} of Lemma 2.2 in view of (6.1) and (2.5). The conclusion follows. $\qquad\square$

When $H = k[G]$ is a group algebra, Propositions 2.3 and 6.3 imply that
$$\mathrm{H}_2^\ell(k[G]) \cong \mathrm{H}_2^{\mathrm{Sw}}(k[G]) \cong k[H_2(G, \mathbb{Z})]\,.$$

6.2. **A universal coefficient theorem.** We fix a commutative algebra R. Our next aim is to relate the lazy cohomology group $\mathrm{H}_\ell^2(H, R)$ of Definition 1.2 to the above-defined Hopf algebra $\mathrm{H}_2^\ell(H)$.

Recall from (1.6) the group $Z_\ell^2(H, R)$ of normalized lazy 2-cocycles with coefficients in R.

Proposition 6.4. *There is a homomorphism*
$$Z_\ell^2(H, R) \to \mathrm{Alg}(\mathrm{H}_2^\ell(H), R)\,, \ \sigma \mapsto \widehat{\sigma}$$
defined by $\widehat{\sigma}([t(\widetilde{x \otimes y})]) = \sigma(x \otimes y)$ for all $x, y \in H$. It induces a homomorphism
$$\kappa : \mathrm{H}_\ell^2(H, R) \to \mathrm{Alg}(\mathrm{H}_2^\ell(H), R)\,.$$

Proof. Let $\sigma \in Z_\ell^2(H, R)$. Since $\sigma \in \mathrm{reg}_\ell^2(H, R)$, Proposition 3.7 furnishes an element $\underline{\tilde{\sigma}} \in \mathrm{Alg}(F(H^{[2]}), R)$ such that for all $x, y \in H$,
$$\underline{\tilde{\sigma}}\big(t(\widetilde{x \otimes y})\big) = \sigma(x \otimes y) \quad \text{and} \quad \underline{\tilde{\sigma}}\big(t^{-1}(\widetilde{x \otimes y})\big) = \sigma^{-1}(x \otimes y)\,,$$
where σ^{-1} is the convolution inverse of σ. Restricting $\underline{\tilde{\sigma}}$ to $\mathrm{HKer}(d_2)$, we obtain
$$\begin{aligned}
\underline{\tilde{\sigma}}\big(d_3(x, y, z)\big) &= \sigma(y_1 \otimes z_1)\,\sigma(x_1 \otimes y_2 z_2)\,\sigma^{-1}(x_2 y_3 \otimes z_3)\,\sigma^{-1}(x_3 \otimes y_4) \\
&= \sigma(x_1 \otimes y_1)\,\sigma(x_2 y_2 \otimes z_1)\,\sigma^{-1}(x_3 y_3 \otimes z_2)\,\sigma^{-1}(x_4 \otimes y_4) \\
&= \sigma(x_1 \otimes y_1)\,\varepsilon(x_2 y_2 z)\,\sigma^{-1}(x_3 \otimes y_3) \\
&= \varepsilon(z)\,\sigma(x_1 \otimes y_1)\,\sigma^{-1}(x_2 \otimes y_2) \\
&= \varepsilon(xyz)\,,
\end{aligned}$$
for all $x, y, z \in H$. The second equality above is a consequence of (1.5). A similar computation using the fact that σ satisfies (1.3) and (1.5) shows that
$$\underline{\tilde{\sigma}}\big(S(d_3(x, y, z))\big) = \varepsilon(xyz)$$
for all $x, y, z \in H$. Therefore, $\underline{\tilde{\sigma}}$ vanishes on the ideal $\mathrm{B}_2^\ell(H)$; hence, it induces the desired algebra morphism, which we denote by $\widehat{\sigma}$. One checks that the map $\sigma \mapsto \widehat{\sigma}$ defines a homomorphism
$$Z_\ell^2(H, R) \to \mathrm{Alg}(\mathrm{H}_2^\ell(H), R)\,.$$

Let $\mu \in \mathrm{Reg}_\ell^1(H, R)$, so that $\partial(\mu) \in Z_\ell^2(H, R)$. By Propositions 1.7 and 3.4, μ induces the algebra morphism $\tilde{\mu} : F(H^{[1]}) \to R$ given by $\underline{\tilde{\mu}}(t(\underline{x})) = \mu(x)$ and $\underline{\tilde{\mu}}(t^{-1}(\underline{x})) = \mu^{-1}(x)$ for all $x \in H$. We claim that
$$\widetilde{\partial(\mu)} = \tilde{\mu} \circ d_2\,.$$
Indeed, by (1.7) and (4.4), for all $x, y \in H$ we obtain
$$\begin{aligned}
\widetilde{\partial(\mu)}\big(t(\widetilde{x \otimes y})\big) &= \partial(\mu)(x \otimes y) = \mu(x_1)\,\mu(y_1)\,\mu^{-1}(x_2 y_2) \\
&= \underline{\tilde{\mu}}(t(\underline{x_1}))\,\underline{\tilde{\mu}}(t(\underline{y_1}))\,\underline{\tilde{\mu}}(t^{-1}(\underline{x_2 y_2})) \\
&= \tilde{\mu}\big(d_2(t(\widetilde{x \otimes y}))\big)\,.
\end{aligned}$$
By a similar computation, $\widetilde{\partial(\mu)}\big(t^{-1}(\widetilde{x \otimes y})\big) = \tilde{\mu}\big(d_2(t^{-1}(\widetilde{x \otimes y}))\big)$ for all $x, y \in H$. This proves the claim.

If $a \in \mathrm{HKer}(d_2)$, then $d_2(a) = \varepsilon(a)\,1$, so that $\widetilde{\partial(\mu)} = \varepsilon$; hence, we obtain the desired homomorphism $\mathrm{H}^2_\ell(H, R) \to \mathrm{Alg}(\mathrm{H}^\ell_2(H), R)$. $\qquad\square$

For any $f \in \mathrm{Alg}(\mathrm{Im}(d_2), R)$, we set

$$(6.5) \qquad (\delta f)(x \otimes y) = f\Big(d_2\big(t(\widetilde{x \otimes y})\,t^{-1}(\widetilde{1 \otimes 1})\big)\Big) = f\big(t(\underline{x_1})\,t(\underline{y_1})\,t^{-1}(\underline{x_2 y_2})\,t^{-1}(\underline{1})\big)$$

for all $x, y \in H$. This defines an element of $\mathrm{Hom}(H \otimes H, R)$.

Lemma 6.5. *(a) For any $f \in \mathrm{Alg}(\mathrm{Im}(d_2), R)$, the map $\delta f \in \mathrm{Hom}(H \otimes H, R)$ is a lazy 2-cocycle.*
(b) If $f, f' \in \mathrm{Alg}(\mathrm{Im}(d_2), R)$, then $\delta(f * f') = (\delta f) * (\delta f')$.
(c) For any $g \in \mathrm{Alg}(F(H^{[1]}), R)$, we have

$$\delta(\iota^*(g)) = \partial(\check{g})\,,$$

where $\check{g} \in \mathrm{Reg}^1_\ell(H, R)$ is defined by $\check{g}(x) = g(t(\underline{x})\,t^{-1}(\underline{1}))$ $(x \in H)$.

Proof. (a) The map δf is invertible with respect to the convolution product. Indeed, it is easy to check that the map

$$x \otimes y \mapsto f\big(t(\underline{x_1 y_1})\,t^{-1}(\underline{x_2})\,t^{-1}(\underline{y_2})\,t(\underline{1})\big)$$

is an inverse for δf.

Let us verify that δf is normalized. Indeed, for $x \in H$,

$$(\delta f)(x \otimes 1) = f\big(t(\underline{x_1})\,t^{-1}(\underline{x_2})\,t(\underline{1})\,t^{-1}(\underline{1})\big) = \varepsilon(x) f(1) = \varepsilon(x)\,1_R\,.$$

Similarly, $(\delta f)(1 \otimes x) = \varepsilon(x)\,1_R$.

By Proposition 3.1, $t^{\pm 1}(\underline{x_1}) \otimes x_2 = t^{\pm 1}(\underline{x_2}) \otimes x_1$ for all $x \in H$. Therefore,

$$
\begin{aligned}
(\delta f)(x_1 \otimes y_1) \otimes x_2 y_2 &= f\big(t(\underline{x_1})\,t(\underline{y_1})\,t^{-1}(\underline{x_2 y_2})\,t^{-1}(\underline{1})\big) \otimes x_3 y_3 \\
&= f\big(t(\underline{x_1})\,t(\underline{y_1})\,t^{-1}(\underline{x_3 y_3})\,t^{-1}(\underline{1})\big) \otimes x_2 y_2 \\
&= f\big(t(\underline{x_2})\,t(\underline{y_2})\,t^{-1}(\underline{x_3 y_3})\,t^{-1}(\underline{1})\big) \otimes x_1 y_1 \\
&= (\delta f)(x_2 \otimes y_2) \otimes x_1 y_1\,.
\end{aligned}
$$

This shows that δf is lazy.

Let us finally check that δf is a left 2-cocycle. On one hand, for all $x, y, z \in H$,

$$
\begin{aligned}
(\delta f)(x_1, y_1)\,\delta f(x_2 y_2, z) &= f\big(t(\underline{x_1})\,t(\underline{y_1})\,t^{-1}(\underline{x_2 y_2})\,t^{-1}(\underline{1})\big) \\
&\qquad \times f\big(t(\underline{x_3 y_3})\,t(\underline{z_1})\,t^{-1}(\underline{x_4 y_4 z_2})\,t^{-1}(\underline{1})\big) \\
&= f\big(t(\underline{x_1})\,t(\underline{y_1})\,t^{-1}(\underline{x_2 y_2})\,t(\underline{x_3 y_3}) \\
&\qquad \times t(\underline{z_1})\,t^{-1}(\underline{x_4 y_4 z_2})\,t^{-1}(\underline{1})^2\big) \\
&= f\big(t(\underline{x_1})\,t(\underline{y_1})\,t(\underline{z_1})\,t^{-1}(\underline{x_2 y_2 z_2})\,t^{-1}(\underline{1})^2\big)\,.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(\delta f)(y_1, z_1)\,\delta f(x, y_2 z_2) &= f\big(t(\underline{y_1})\,t(\underline{z_1})\,t^{-1}(\underline{y_2 z_2})\,t^{-1}(\underline{1})\big) \\
&\qquad \times f\big(t(\underline{x_1})\,t(\underline{y_3 z_3})\,t^{-1}(\underline{x_2 y_4 z_4})\,t^{-1}(\underline{1})\big) \\
&= f\big(t(\underline{x_1})\,t(\underline{y_1})\,t(\underline{z_1}) \\
&\qquad \times t^{-1}(\underline{y_2 z_2})\,t(\underline{y_3 z_3})\,t^{-1}(\underline{x_2 y_4 z_4})\,t^{-1}(\underline{1})^2\big) \\
&= f\big(t(\underline{x_1})\,t(\underline{y_1})\,t(\underline{z_1})\,t^{-1}(\underline{x_2 y_2 z_2})\,t^{-1}(\underline{1})^2\big)\,,
\end{aligned}
$$

which by the above computation is equal to $(\delta f)(x_1, y_1)\,(\delta f)(x_2 y_2, z)$.

(b) Since d_2 is a coalgebra morphism,

$$
\begin{aligned}
\delta(f * f')(x \otimes y) &= (ff')\big(d_2(t(\widetilde{x \otimes y})\,t^{-1}(\widetilde{1 \otimes 1}))\big) \\
&= f\big(d_2(t(\widetilde{x \otimes y})\,t^{-1}(\widetilde{1 \otimes 1}))_1\big) \\
&\qquad \times f'\big((d_2(t(\widetilde{x \otimes y})\,t^{-1}(\widetilde{1 \otimes 1}))_2\big) \\
&= f\big(d_2(t(\widetilde{x_1 \otimes y_1})\,t^{-1}(\widetilde{1 \otimes 1}))\big) \\
&\qquad \times f'\big((d_2(t(\widetilde{x_2 \otimes y_2})\,t^{-1}(\widetilde{1 \otimes 1})))\big) \\
&= (\delta f) * (\delta f')(x \otimes y)\,.
\end{aligned}
$$

(c) Since g is an algebra morphism on $F(H^{[1]})$,

$$
\begin{aligned}
(\delta(\iota^* g))(x \otimes y) &= g\big(t(\underline{x_1})\,t(\underline{y_1})\,t^{-1}(\underline{x_2 y_2})\,t^{-1}(\underline{1})\big) \\
&= g\big(t(\underline{x_1})\,t^{-1}(\underline{1})\big)\,g\big(t(\underline{y_1})\,t^{-1}(\underline{1})\big)\,g\big(t^{-1}(\underline{x_2 y_2})\,t(\underline{1})\big) \\
&= \check{g}(x_1)\,\check{g}(y_1)\,\check{g}^{-1}(x_2 y_2) \\
&= \partial(\check{g})(x \otimes y)\,.
\end{aligned}
$$

$\square$

We now relate the lazy cohomology group $\mathrm{H}^2_\ell(H, R)$ to the group $\mathrm{Ext}^1(H, R)$ introduced in Section 4.4.

Proposition 6.6. *The map $f \in \mathrm{Alg}(\mathrm{Im}(d_2), R) \mapsto \delta f \in \mathrm{Hom}(H \otimes H, R)$ induces an injective group homomorphism*

$$
\delta_\# : \mathrm{Ext}^1(H, R) \to \mathrm{H}^2_\ell(H, R)\,.
$$

Proof. It follows from the definitions and Lemma 6.5 that δ induces the desired group homomorphism. Let us check that $\delta_\#$ is injective. Let $f \in \mathrm{Alg}(\mathrm{Im}(d_2), R)$ be such that $\delta f = \partial(\mu)$ for some $\mu \in \mathrm{Reg}^1_\ell(H, R)$. This means that for all $x, y \in H$,

$$
(\delta f)(x \otimes y) = \mu(x_1)\,\mu(y_1)\,\mu^{-1}(x_2 y_2) = \mu(x_1)\,\mu(y_1)\,\mu^{-1}(x_2 y_2)\,\mu^{-1}(1)
$$

since μ is normalized. By Propositions 1.7 and 3.4 there is a unique morphism $g \in \mathrm{Alg}(F(H^{[1]}), R)$ such that $g(t(\underline{x})) = \mu(x)$ and $g(t^{-1}(\underline{x})) = \mu^{-1}(x)$ for all $x \in H$. Restricting g to $\mathrm{Im}(d_2)$, we obtain

$$
\begin{aligned}
g\big(d_2(t(\widetilde{x \otimes y})\,t^{-1}(\widetilde{1 \otimes 1}))\big) &= g\big(t(\underline{x_1})\,t(\underline{y_1})\,t^{-1}(\underline{x_2 y_2})\,t^{-1}(\underline{1})\big) \\
&= g(t(\underline{x_1}))\,g(t(\underline{x_2}))\,g(t^{-1}(\underline{x_2 y_2}))\,g(t^{-1}(\underline{1})) \\
&= \mu(x_1)\,\mu(y_1)\,\mu^{-1}(x_2 y_2)\,\mu^{-1}(1) \\
&= (\delta f)(x \otimes y) \\
&= f\big(d_2(t(\widetilde{x \otimes y})\,t^{-1}(\widetilde{1 \otimes 1}))\big)\,.
\end{aligned}
$$

Since $d_2(t^{-1}(\widetilde{1 \otimes 1}))$ is invertible (with inverse $d_2(t(\widetilde{1 \otimes 1}))$), for all $x, y \in H$, we obtain

$$
g\big(d_2(t(\widetilde{x \otimes y}))\big) = f\big(d_2(t(\widetilde{x \otimes y}))\big)\,.
$$

This shows that f is in the image of $\iota^* : \mathrm{Alg}(F(H^{[1]}), R) \to \mathrm{Alg}(\mathrm{Im}(d_2), R)$, hence is zero in $\mathrm{Ext}^1(H, R)$. $\square$

Now we combine the group homomorphisms of Propositions 6.4 and 6.6:

$$
\mathrm{Ext}^1(H, R) \xrightarrow{\delta_\#} \mathrm{H}^2_\ell(H, R) \xrightarrow{\kappa} \mathrm{Alg}(\mathrm{H}^\ell_2(H), R)\,.
$$

We already know that the left arrow $\delta_\#$ is injective.

We can now state a universal coefficient theorem for $\mathrm{H}^2_\ell(H, R)$.

Theorem 6.7. *For any Hopf algebra H and any commutative algebra R, the sequence of groups*

$$1 \longrightarrow \mathrm{Ext}^1(H, R) \xrightarrow{\delta_\#} \mathrm{H}_\ell^2(H, R) \xrightarrow{\kappa} \mathrm{Alg}(\mathrm{H}_2^\ell(H), R)$$

is exact. If in addition the ground field k is algebraically closed, then

$$\mathrm{H}_\ell^2(H, k) \cong \mathrm{Alg}(\mathrm{H}_2^\ell(H), k) \,.$$

Proof. Since $\delta_\#$ is injective, it suffices to prove that $\mathrm{Ker}(\kappa) = \mathrm{Im}(\delta_\#)$. To this end, consider the sequence of Hopf algebra morphisms

(6.6) $$k \longrightarrow \mathrm{HKer}(d_2) \xrightarrow{\ j\ } F(H^{[2]}) \xrightarrow{d_2} \mathrm{Im}(d_2) \longrightarrow k \,,$$

where j is the inclusion of $\mathrm{HKer}(d_2)$ into $F(H^{[2]})$. We claim that this sequence is exact. Indeed, j is injective, d_2 is surjective, and

$$\mathrm{Ker}(d_2) = \mathrm{HKer}(d_2)^+ F(H^{[2]}) \,.$$

The latter equality is a consequence of Corollary 4.11 and of [17, Th. 4.3] (see also the latter's proof). By Proposition 1.4, the exact sequence (6.6) induces the exact sequence of groups

(6.7) $$1 \longrightarrow \mathrm{Alg}(\mathrm{Im}(d_2), R) \xrightarrow{d_2^*} \mathrm{Alg}(F(H^{[2]}), R) \xrightarrow{j^*} \mathrm{Alg}(\mathrm{HKer}(d_2), R) \,.$$

Now consider an element of $\mathrm{H}_\ell^2(H, R)$ and represent it by an element $\sigma \in Z_\ell^2(H, R) \subset \mathrm{reg}_\ell^2(H, R)$. If its image under κ is trivial, then the corresponding algebra morphism $\tilde{\underline{\sigma}} \in \mathrm{Alg}(F(H^{[2]}), R)$ given by

$$\tilde{\underline{\sigma}}\big(t(\widetilde{x \otimes y})\big) = \sigma(x \otimes y)$$

restricts to the trivial element of $\mathrm{Alg}(\mathrm{HKer}(d_2), R)$. By exactness of (6.7), there is $f \in \mathrm{Alg}(\mathrm{Im}(d_2), R)$ such that $\delta f = \tilde{\underline{\sigma}}$. It follows that the element of $\mathrm{H}_\ell^2(H, R)$ represented by σ is in the image of the homomorphism $\delta_\#$ defined in Proposition 6.6.

Assume now that k is algebraically closed. It follows from the exactness of (6.6) and from Proposition 1.5 that the map

$$j^* : \mathrm{Alg}(F(H^{[2]}), k) \to \mathrm{Alg}(\mathrm{HKer}(d_2), k)$$

of (6.7) is surjective. Since by Lemma 1.3, $\mathrm{Alg}(\mathrm{H}_2^\ell(H), k)$ embeds into $\mathrm{Alg}(\mathrm{HKer}(d_2), k)$, the map $\kappa : \mathrm{H}_\ell^2(H, R) \to \mathrm{Alg}(\mathrm{H}_2^\ell(H), R)$ is surjective. The injectivity of κ follows from the first part of the theorem and the vanishing of $\mathrm{Ext}^1(H, k)$, which is a consequence of Proposition 4.14. $\square$

7. Computations for the Sweedler Algebra

In this section k will denote a field of characteristic $\neq 2$. We compute the lazy homology of Sweedler's four-dimensional Hopf algebra.

Recall that the Sweedler algebra is defined by the following presentation:

$$H_4 = k \langle x, g \mid g^2 = 1, \ x^2 = 0, \ xg = -gx \rangle \,.$$

The algebra H_4 has $\{1, g, x, y = xg\}$ as a linear basis. It is a Hopf algebra with coproduct Δ, counit ε, and antipode S given by

$$\Delta(g) = g \otimes g \,, \quad \Delta(x) = 1 \otimes x + x \otimes g \,,$$

$$\varepsilon(g) = 1 \,, \quad \varepsilon(x) = 0 \,, \quad S(g) = g \,, \ S(x) = -y \,.$$

Theorem 7.1. *The lazy homology Hopf algebras of the Sweedler algebra H_4 are given by*

$$\mathrm{H}_1^\ell(H_4) \cong k \quad and \quad \mathrm{H}_2^\ell(H_4) \cong k[X] \,,$$

where X is a primitive element.

The rest of the section is devoted to the proof of this theorem. We start by computing the coalgebra $H_4^{[1]}$, as defined in Section 3.1.

Lemma 7.2. *The coalgebra $H_4^{[1]}$ is one-dimensional, spanned by the grouplike element $\underline{1}$.*

Proof. An easy computation shows that $\underline{1} = \underline{g}$ and $\underline{x} = \underline{y} = 0$ in $H_4^{[1]}$. Since 1 is grouplike, so is $\underline{1}$. $\qquad\square$

It follows from Lemma 7.2 and the definition of $F(H_4^{[1]})$ that there is a Hopf algebra isomorphism

$$f : k[T, T^{-1}] \to F(H_4^{[1]})$$

determined by $f(T^{\pm 1}) = t^{\pm 1}(\underline{1})$. By our first definition of H_1^ℓ (see Section 4.1), the Hopf algebra $\mathrm{H}_1^\ell(H_4)$ is isomorphic to the quotient of $k[T, T^{-1}]$ by the ideal generated by $T^2 - T$, which in view of the invertibility of T is the same as the ideal generated by $T - 1$. We thus obtain the desired isomorphism $\mathrm{H}_1^\ell(H_4) \cong k$.

Let us next determine the Hopf algebra $H_4^{[2]}$, as defined in Section 3.2.

Lemma 7.3. *The coalgebra $H_4^{[2]}$ is five-dimensional with basis $\{h_0, h_1, h_2, h_3, h_4\}$, where*

$$h_0 = \widetilde{1 \otimes 1}, \quad h_1 = \widetilde{x \otimes x}, \quad h_2 = \widetilde{x \otimes y}, \quad h_3 = \widetilde{y \otimes x}, \quad h_4 = \widetilde{y \otimes y},$$

and with coproduct

$$\Delta(h_0) = h_0 \otimes h_0 \quad and \quad \Delta(h_i) = h_0 \otimes h_i + h_i \otimes h_0$$

for $i = 1, 2, 3, 4$.

Proof. By a direct computation one shows that $H_4^{[2]}$ is obtained from $H_4 \otimes H_4$ by killing the eight elements $\widetilde{1 \otimes x}$, $\widetilde{1 \otimes y}$, $\widetilde{g \otimes x}$, $\widetilde{g \otimes y}$, $\widetilde{x \otimes 1}$, $\widetilde{y \otimes 1}$, $\widetilde{x \otimes g}$, $\widetilde{y \otimes g}$, and adding the relations

$$\widetilde{g \otimes 1} = \widetilde{1 \otimes g} = \widetilde{g \otimes g} = h_0.$$

It follows that $H_4^{[2]}$ is spanned by $\{h_0, h_1, h_2, h_3, h_4\}$. The formulas for the coproduct follow easily. $\qquad\square$

Let $B = k[T, T^{-1}, Y_1, Y_2, Y_3, Y_4]$ be the commutative Hopf algebra with coproduct determined by

$$\Delta(T) = T \otimes T \quad and \quad \Delta(Y_i) = T \otimes Y_i + Y_i \otimes T$$

for $i = 1, 2, 3, 4$. Then it follows from Lemma 7.3 that there is a Hopf algebra isomorphism $g : B \to F(H_4^{[2]})$ such that $g(T) = h_0$ and $g(Y_i) = h_i$ for $i = 1, 2, 3, 4$.

Lemma 7.4. *Let $A = k[X_1, X_2, X_3, X_4]$ be the commutative Hopf algebra such that X_1, X_2, X_3, X_4 are primitive. Then there is a commutative diagram of Hopf algebras*

$$
\begin{array}{ccccccccc}
k & \longrightarrow & A & \overset{\nu}{\longrightarrow} & B & \overset{\pi}{\longrightarrow} & k[T, T^{-1}] & \longrightarrow & k \\
& & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\
k & \longrightarrow & \mathrm{HKer}(d_2) & \longrightarrow & F(H_4^{[2]}) & \overset{d_2}{\longrightarrow} & F(H_4^{[1]}) & \longrightarrow & k
\end{array}
$$

in which the horizontal sequences are exact and the vertical morphisms are isomorphisms.

Proof. The Hopf algebra morphism $\nu : A \to B$ is defined by $\nu(X_i) = T^{-1} Y_i$ for $i = 1, 2, 3, 4$; it is clearly injective. The Hopf algebra morphism $\pi : B \to k[T, T^{-1}]$ sends T to itself and Y_i to 0 ($1 \le i \le 4$); it is surjective. It is clear that $\nu(A)^+ B = \mathrm{Ker}(\pi)$, which implies that the top sequence is exact with $\mathrm{HKer}(\pi) = \nu(A)$. The Hopf algebra morphism $d_2 : F(H_4^{[2]}) \to F(H_4^{[1]})$ sends $t(\widetilde{1 \otimes 1})$ to $t(\overline{1})$ and the remaining generators to 0. Therefore d_2 is surjective and the right square is commutative. The left vertical map is constructed by sending each generator X_i to the appropriate element in $\mathrm{HKer}(d_2)$; for instance, X_1 is sent to $t^{-1}(\widetilde{1 \otimes 1}) t(\widetilde{x \otimes x})$. The conclusion follows. $\qquad\square$

By Lemma 7.4 the map d_2 is surjective. It thus follows from Proposition 4.12 that $\mathrm{H}_1^\ell(H_4) \cong F(H_4^{[1]}) /\!/ \operatorname{Im}(d_2) \cong k$, which gives another proof of the first isomorphism in Theorem 7.1.

Let us now compute the values of the trilinear map

$$d_3 : H_4 \times H_4 \times H_4 \to F(H_4^{[2]}) \,.$$

Lemma 7.5. *For all $a, b \in H_4$,*

(7.1)
$$d_3(a, b, 1) = d_3(a, 1, b) = d_3(1, a, b) = \varepsilon(ab) \,,$$

(7.2)
$$d_3(a, g, g) = d_3(g, a, g) = d_3(g, g, a) = \varepsilon(a) \,.$$

If we set

$$a_1 = t^{-1}(\widetilde{1 \otimes 1})t(\widetilde{x \otimes x}) \,, \quad a_2 = t^{-1}(\widetilde{1 \otimes 1})t(\widetilde{x \otimes y}) \,,$$

$$a_3 = t^{-1}(\widetilde{1 \otimes 1})t(\widetilde{y \otimes x}) \,, \quad a_4 = t^{-1}(\widetilde{1 \otimes 1})t(\widetilde{y \otimes y}) \,,$$

then $\varepsilon(a_1) = \varepsilon(a_2) = \varepsilon(a_3) = \varepsilon(a_4) = 0$ and

$$d_3(x, x, g) = -a_1 + a_2 = -d_3(x, y, g) \,,$$

$$d_3(y, x, g) = -a_3 + a_4 = -d_3(y, y, g) \,,$$

$$d_3(y, g, x) = -a_1 - a_4 = d_3(x, g, y) \,,$$

$$d_3(x, g, x) = -a_2 - a_3 = d_3(y, g, y) \,,$$

$$d_3(g, x, x) = a_1 + a_3 = d_3(g, y, x) \,,$$

$$d_3(g, x, y) = a_2 + a_4 = d_3(g, y, y) \,,$$

$$d_3(x, x, x) = d_3(x, y, x) = d_3(y, x, x) = d_3(y, y, x) = 0 \,,$$

$$d_3(x, x, y) = d_3(y, x, y) = d_3(x, y, y) = d_3(y, y, y) = 0 \,.$$

Proof. It follows from Lemma 7.3 that

$$t(\widetilde{a \otimes 1}) = t(\widetilde{1 \otimes a}) = \varepsilon(a)t(\widetilde{1 \otimes 1}) = t(\widetilde{a \otimes g}) = t(\widetilde{g \otimes a}) \,.$$

From this we easily deduce (7.1) and (7.2). The remaining identities are obtained by a brute force computation. For instance,

$$d_3(x, x, g) = t(\widetilde{1 \otimes g})t^{-1}(\widetilde{x \otimes x}) + t(\widetilde{x \otimes y})t^{-1}(\widetilde{g \otimes g})$$
$$= S(a_1) + a_2 = -a_1 + a_2 \,.$$

The other computations are similar. □

Proof of Theorem 7.1. By Lemma 7.4 and Lemma 7.5, the Hopf algebra $\mathrm{H}_2^\ell(H_4)$ is isomorphic to the quotient of $A = k[X_1, X_2, X_3, X_4]$ by the ideal generated by the relations $X_2 = -X_3 = -X_4 = X_1$. It follows that $\mathrm{H}_2^\ell(H_4)$ is isomorphic to the polynomial algebra $k[X]$. □

Using the universal coefficient theorems 4.4 and 6.7, we recover the computation of the lazy cohomology of H_4 performed in [4, Sect. 2].

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