About the monoidal invariance of cohomological dimension of Hopf algebras

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The question I want to discuss is

Question

Let A, B be Hopf algebras such that

$$\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$$

Do we have $\mathrm{cd}(A) = \mathrm{cd}(B)$?

Here:

- $\mathcal{M}^A$ is the tensor category of right A-comodules,
- $\mathrm{cd}(A)$ is the cohomological dimension of A (see below).

We work over an algebraically closed field k.

Cohomological dimension

Classical examples

- If $A = \mathcal{O}(G)$, with G a compact Lie group, then

$$\mathrm{cd}(\mathcal{O}(G)) = \dim(G)$$

- If $A = k\Gamma$, with Γ a discrete group, then $\mathrm{cd}(k\Gamma) = \mathrm{cd}_k(\Gamma)$, the cohomological dimension of Γ with coefficients k.
  - if Γ is finitely generated, then $\mathrm{cd}(k\Gamma) = 1 \iff \Gamma$ has a free subgroup of finite index (Dunwoody's theorem);
  - if Γ is the fundamental group of an aspherical closed manifold of dimension n, then $\mathrm{cd}(k\Gamma) = n$.
  - Let $\Gamma = \langle r, s, a \mid rs = sr, tat^{-1}a = atat^{-1}, sas^{-1} = atat^{-1} \rangle$ (Baumslag). Then $\mathrm{cd}(k\Gamma) = \infty$.
  - If Γ is a finite group, then $\mathrm{cd}(k\Gamma) = 0 \iff |G| \neq 0$ in k, and $\mathrm{cd}(k\Gamma) = \infty$ otherwise.
- If A is a finite-dimensional Hopf algebra, then either $\mathrm{cd}(A) = 0$ (A is semisimple) or $\mathrm{cd}(A) = \infty$.

Cohomological dimension

Let A be an algebra and let M be a (left) A-module.

• M is said to be **projective** if the functor $\mathrm{Hom}_A(M, -)$ is exact. This is equivalent to say that M is a direct summand in a free module.

• A **projective resolution** of M is an exact sequence of A-modules

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

where the P_i's are projective.

• The **projective dimension of** M, $\mathrm{pd}_A(M) \in \mathbb{N} \cup \{\infty\}$, is the smallest possible length (the largest n with $P_n \neq 0$) for a projective resolution of M.

• We have $\mathrm{pd}_A(M) = 0 \iff M$ is projective, so $\mathrm{pd}_A(M)$ measures how far is a module from being projective.

• The **(left) global dimension of** A is defined by

$$\mathrm{l.gldim}(A) = \max \{\mathrm{pd}_A(M),\ M \in \mathcal{M}_A\} \in \mathbb{N} \cup \{\infty\}$$

Cohomological dimension

More generally as soon as we are in an abelian category having enough projective objects (every object is a quotient of a projective), we can define projective dimensions of objects.

When A is a Hopf algebra, we have as well

$$\mathrm{l.gldim}(A) = \mathrm{pd}_A(k_\varepsilon) = \mathrm{cd}(A) = \mathrm{r.gldim}(A)$$

where k_ε denote the trivial A-module, and

$\mathrm{cd}(A)$ is the **Hochschild cohomological dimension of** A

with $\mathrm{cd}(A) = \mathrm{pd}_{{}_A\mathcal{M}_A}(A)$. We simply denote $\mathrm{cd}(A)$ all these numbers.

Cohomological dimension: examples

Example 1. Let $n \geq 2$. Let $A_o(n)$ be the algebra presented by generators $(u_{ij})_{1 \leq i,j \leq n}$ and relations

$$u^t u = I_n = uu^t$$

where u is the matrix $(u_{ij})_{1 \leq i,j \leq n}$. It has a Hopf algebra structure

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \ \varepsilon(u_{ij}) = \delta_{ij}, \ S(u) = u^t$$

This is the coordinate algebra on Wang's free orthogonal quantum group O_n^+. Collins-Härtl-Thom (2008) have shown

$$\mathrm{cd}(A_o(n)) = 3$$

There is a monoidal equivalence $\mathcal{M}^{A_o(n)} \simeq^{\otimes} \mathcal{M}^{\mathcal{O}(SL_q(2))}$ for $n = -q - q^{-1}$, and indeed $\mathrm{cd}(\mathcal{O}(SL_q(2))) = 3$.

Cohomological dimension: examples

Example 1 (continued). More generally, let $E \in \mathrm{GL}_n(k)$, $n \geq 2$, and let $\mathcal{B}(E)$ presented by generators $(u_{ij})_{1\leq i,j\leq n}$ and relations

$$E^{-1}u^tEu = I_n = uE^{-1}u^tE,$$

where u is the matrix $(u_{ij})_{1\leq i,j\leq n}$. It has a Hopf algebra structure defined by $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj},\ \varepsilon(u_{ij}) = \delta_{ij},\ S(u) = E^{-1}u^tE$.
The Hopf algebra $\mathcal{B}(E)$ (Dubois-Violette and Launer, 1990), represents the quantum symmetry group of the bilinear form associated to the matrix E. For a well-chosen $E_q \in \mathrm{GL}_2(k)$ we have $\mathcal{B}(E_q) = \mathcal{O}(\mathrm{SL}_q(2))$.
One has

$$\mathrm{cd}(\mathcal{B}(E)) = 3$$

and we have a monoidal equivalence

$$\mathcal{M}^{\mathcal{B}(E)} \simeq^{\otimes} \mathcal{M}^{\mathcal{O}(\mathrm{SL}_q(2))}$$

for $q \in k^*$ satisfying $\mathrm{tr}(E^{-1}E^t) = -q - q^{-1}$.

Cohomological dimension: examples

Example 2. Let $A_s(n)$ be the algebra presented by generators $(u_{ij})_{1\leq i,j\leq n}$ and relations

$$\sum_k u_{ki} = 1 = \sum_k u_{ik}, \ u_{ik}u_{ij} = \delta_{kj}u_{ij}, \ u_{ki}u_{ji} = \delta_{jk}u_{ji}$$

It has a natural Hopf algebra structure and represents the quantum permutation group S_n^+ (Wang).
For $n \geq 4$, one has

$$\mathrm{cd}(A_s(n)) = 3$$

and a monoidal equivalence $\mathcal{M}^{A_s(n)} \simeq^{\otimes} \mathcal{M}^{\mathcal{O}(PSL_q(2))}$ for $\sqrt{n} = q + q^{-1}$.

In these examples, the monoidal equivalence is important to determine the cohomological dimension, but there are furthermore special types of "equivariant" resolutions that play a role.

Positive answers to our question

Theorem [Wang-Yu-Zhang, 2017]

Let A, B be Hopf algebras such that $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$. If A is twisted Calabi-Yau and B is smooth, then $\mathrm{cd}(A) = \mathrm{cd}(B)$.

Smooth means that the trivial module has a finite resolution by finitely generated projective modules, and twisted Calabi-Yau is a stronger condition (a nice duality between homology and cohomology). In fact they prove that B is twisted Calabi-Yau as well.

Theorem [B, 2016-2018]

Let A, B be Hopf algebras such that $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$. If A, B are cosemisimple and satisfy $S^4 = \mathrm{id}$, then $\mathrm{cd}(A) = \mathrm{cd}(B)$.

Positive answers to our question

The main new result presented in this talk is:

Theorem

Let A, B be Hopf algebras such that $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$. Assume that one of the following conditions hold:

1. A and B are smooth;
2. A, B are cosemisimple and $\mathrm{cd}(A)$, $\mathrm{cd}(B)$ are finite.

Then $\mathrm{cd}(A) = \mathrm{cd}(B)$.

(1) mainly consists in checking that the arguments of Wang-Yu-Zhang still work to get the desired conclusion.

We will focus on explaining the proof of (2).

Strategy: equivariant bimodules

Recall that if R is a right A-comodule algebra (an algebra in the category $\mathcal{M}^A$), the category of R-bimodules inside A-comodules is denoted

$$ {}_R\mathcal{M}_R^A $$

Objects: the A-comodules V with an R-bimodule structure having the Hopf bimodule compatibility conditions ($x \in R$, $v \in V$)

$$ (x\cdot v)_{(0)} \otimes (x\cdot v)_{(1)} = x_{(0)}\cdot v_{(0)} \otimes x_{(1)} v_{(1)}, \ (v\cdot x)_{(0)} \otimes (v\cdot x)_{(1)} = v_{(0)}\cdot x_{(0)} \otimes v_{(1)} x_{(1)} $$

Morphisms: the A-colinear and R-bilinear maps.
The category ${}_R\mathcal{M}_R^A$ is obviously abelian, and the tensor product of bimodules induces a monoidal strucure on it.

Strategy

For a Hopf algebra A, recall (Schauenburg) that it follows from the structure theorem for Hopf modules that the functor

$$ {}_A\mathcal{M} \longrightarrow {}_A\mathcal{M}_A^A, \quad V \longmapsto V \otimes A $$

is a monoidal equivalence, where $V \otimes A$ has the tensor product left A-module structure and the right module and comodule structures are induced by the multiplication and comultiplication of A respectively. Now, starting with a monoidal equivalence $F : \mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$, let $R = F(A)$. This is an algebra in $\mathcal{M}^B$, and F induces an equivalence

$$ {}_A\mathcal{M}_A^A \simeq^{\otimes} {}_R\mathcal{M}_R^B $$

Composing with the previous one, we get an equivalence

$$ {}_A\mathcal{M} \simeq^{\otimes} {}_R\mathcal{M}_R^B $$

sending ${}_\varepsilon k$ to R, and hence $\mathrm{cd}(A) = \mathrm{pd}_{{}_A\mathcal{M}}({}_\varepsilon k) = \mathrm{pd}_{{}_R\mathcal{M}_R^B}(R)$.

Strategy

So, starting from $F : \mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$, we get, for $R = F(A)$,

$$\mathrm{cd}(A) = \mathrm{pd}_{{}_R\mathcal{M}_R^B}(R)$$

Similarly, we have, for $T = F^{-1}(B)$,

$$\mathrm{cd}(B) = \mathrm{pd}_{{}_T\mathcal{M}_T^A}(T)$$

When A, B have bijective antipode, we have $R \simeq T^{\mathrm{op}}$, so $\mathrm{cd}(R) = \mathrm{cd}(T)$.
(here we are with the Hochshild cohomological dimension $\mathrm{cd}(R) = \mathrm{pd}_{{}_R\mathcal{M}_R}(R)$)
So the key question is to compare

$$\mathrm{pd}_{{}_R\mathcal{M}_R^B}(R) \quad \text{and} \quad \mathrm{pd}_{{}_R\mathcal{M}_R}(R) = \mathrm{cd}(R)$$

Remark: at this stage we have not used any assumption on A and B (apart from bijectivity of the antipodes).

Twisted separable functors

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be some categories. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is **twisted separable** if there exist

1. an autoequivalence Θ of the category $\mathcal{D}$;
2. a generating subclass $\mathcal{F} \subset \mathrm{ob}(\mathcal{C})$ (i.e. for every $V \in \mathrm{ob}(\mathcal{C})$, there exists $P \in \mathcal{F}$ and an epimorphism $P \to V$) together with, for any $P \in \mathcal{F}$, an isomorphism $\theta_P : F(P) \to \Theta F(P)$;
3. a natural morphism $\mathbf{M}_{-,-} : \mathrm{Hom}_{\mathcal{D}}(F(-), \Theta F(-)) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that for any $P \in \mathcal{F}$, we have $M_{P,P}(\theta_P) = \mathrm{id}_P$.

The naturality condition above means that for any morphisms $\alpha : V' \to V$, $\beta : W \to W'$ in $\mathcal{C}$ and any morphism $f : F(V) \to \Theta F(W)$ in $\mathcal{D}$, we have

$$\beta \circ \mathbf{M}_{V,W}(f) \circ \alpha = \mathbf{M}_{V',W'}(\Theta F(\beta) \circ f \circ F(\alpha))$$

Twisted separable functors

When $\mathcal{F} = \mathrm{ob}(\mathcal{C})$, $\Theta = \mathrm{id}_{\mathcal{D}}$ and $\theta_P = \mathrm{id}_P$ for any P, we get the notion of **separable functor** by Nastasescu-Van den Bergh-Van Oystaeyen, which provides a convenient setting for various types of generalized Maschke theorems (an exact sequence splits in $\mathcal{C}$ if and only if it splits in $\mathcal{D}$ after applying F).

Basic example of a separable functor: when A is cosemisimple Hopf algebra, the forgetful functor $\mathcal{M}^A \to \mathrm{Vec}_k$. The separability is obtained by averaging with respect to the Haar integral.

Twisted separable functors

Motivation to introduce the present notion of twisted separable functor:

Proposition

Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories having enough projectives and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that the following conditions hold:

1. the functor F is exact and preserves projective objects;
2. the functor F is twisted separable and $\mathcal{F}$, the corresponding class of objects of $\mathcal{C}$, consists of projective objects.

Then for any object V of $\mathcal{C}$ such that $\mathrm{pd}_{\mathcal{C}}(V)$ is finite, we have

$$\mathrm{pd}_{\mathcal{C}}(V) = \mathrm{pd}_{\mathcal{D}}(F(V))$$

Thus, if we know that the forgetful functor $\Omega_R : {}_R\mathcal{M}_R^B \to {}_R\mathcal{M}_R$ satisfies the above conditions and that $\mathrm{pd}_{{}_R\mathcal{M}_R^B}(R)$ is finite, we can conclude that $\mathrm{pd}_{{}_R\mathcal{M}_R^B}(R) = \mathrm{pd}_{{}_R\mathcal{M}_R}(R) = \mathrm{cd}(R)$ (which, in the context of our equivalence $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ will give $\mathrm{cd}(A) = \mathrm{cd}(R)$, as needed).

Twisted separable functors

Let A be a cosemisimple Hopf algebra with Haar integral h. Recall that the Haar integral is not a trace in general, but satisfies a KMS type property, discovered by Woronowicz in the setting of compact quantum groups.

Theorem

There exists a convolution invertible linear map $\psi : A \to k$, called a modular functional on A, satisfying the following conditions:

- $S^2 = \psi * \mathrm{id} * \psi^{-1}$;
- $\sigma := \psi * \mathrm{id} * \psi$ *is an algebra automorphism of A;*
- *we have $h(ab) = h(b\sigma(a))$ for any $a, b \in A$.*

The proof is based on the orthogonality relations.

Twisted separable functors

Let R be right A-comodule over a cosemisimple Hopf algebra A, and let ρ be the automorphism of R defined by $\rho(x) = \psi^{-2}(x_{(1)})x_{(0)}$

Key averaging lemma

Let $V, W \in {}_R\mathcal{M}_R^A$. If $f : V \to W$ is a linear map satisfying

$$f(x \cdot v) = \rho(x) \cdot f(v), \; f(v \cdot x) = f(v) \cdot x$$

for any $v \in V$ and $x \in R$, then $\mathbf{M}_{V,W}(f) : V \to W$ is a morphism in ${}_R\mathcal{M}_R^A$.

Here $\mathbf{M}_{V,W}(f) : V \longrightarrow W$ is the averaging of f defined by

$$v \longmapsto h\left(f(v_{(0)})_{(1)} S(v_{(1)})\right) f(v_{(0)})_{(0)}$$

If G is a compact group, $\mathbf{M}_{V,W}(f) = \int_G \pi_W(g) \circ f \circ \pi_V(g^{-1}) dg$

Twisted separable functors

Now consider

1. the class $\mathcal{F}$ of free objects in ${}_R\mathcal{M}_R^A$, i.e. those of the form

$$R \otimes V \otimes R, \ V \in \mathcal{M}^A$$

with the tensor comodule structure, and bimodule structure by left-right multiplication;

2. the autoequivalence $\Theta : {}_R\mathcal{M}_R \to {}_R\mathcal{M}_R$, $W \mapsto {}_\rho W$ with ${}_\rho W = W$ as vector space and $x \cdot' w \cdot' x = \rho(x) \cdot w \cdot x$, and is trivial on morphisms;
3. for a free object $R \otimes V \otimes R$, the R-bimodule isomorphism $\rho_V = \rho \otimes \mathrm{id}_V \otimes \mathrm{id}_R : R \otimes V \otimes R \to {}_\rho(R \otimes V \otimes R)$.
4. for $V, W \in {}_R\mathcal{M}_R^A$, the averaging map

$$\mathbf{M}_{V,W} : \mathrm{Hom}_A(V, {}_\rho W) \to \mathrm{Hom}_{{}_R\mathcal{M}_R^A}(V, W)$$

from the key averaging lemma.

It follows that the functor $\Omega_R : {}_R\mathcal{M}_R^A \to {}_R\mathcal{M}_R$ is indeed twisted separable.

Twisted separable functors: end of proof

The functor $\Omega_R : {}_R\mathcal{M}_R^A \to {}_R\mathcal{M}_R$ is twisted separable.
Moreover, the class $\mathcal{F}$ consists of projectives (A is cosemisimple), the projectives in ${}_R\mathcal{M}_R^A$ are direct summands of free objects and hence are preserved by Ω_R, which is exact.
Hence we are in the situation of the previous proposition, and as soon as $\mathrm{pd}_{{}_R\mathcal{M}_R^A}(R)$ is finite, we have

$$\mathrm{pd}_{{}_R\mathcal{M}_R^A}(R) = \mathrm{pd}_{{}_R\mathcal{M}_R}(R) = \mathrm{cd}(R)$$

This proves our theorem, as already explained here

Remark: If $S^4 = \mathrm{id}$, $\Omega_R : {}_R\mathcal{M}_R^A \to {}_R\mathcal{M}_R^A$ is separable, and for any comodule algebra

$$\mathrm{pd}_{{}_R\mathcal{M}_R^A}(R) = \mathrm{pd}_{{}_R\mathcal{M}_R}(R) = \mathrm{cd}(R)$$

.

An example

For $n \geq 2$ and $F \in \mathrm{GL}_n(k)$, the universal cosovereign Hopf algebra $H(F)$ is the algebra generated by $(u_{ij})_{1 \leq i,j \leq n}$ and $(v_{ij})_{1 \leq i,j \leq n}$, with relations:

$$uv^t = v^t u = I_n; \quad vFu^tF^{-1} = Fu^tF^{-1}v = I_n,$$

where $u = (u_{ij})$, $v = (v_{ij})$ and I_n is the identity $n \times n$ matrix. The Hopf algebra structure is defined by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad \Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj},$$
$$\varepsilon(u_{ij}) = \varepsilon(v_{ij}) = \delta_{ij}, \quad S(u) = v^t, \quad S(v) = Fu^tF^{-1}.$$

When $F \in \mathrm{GL}_n(\mathbb{C})$ is positive, this is the compact Hopf algebra $A_u(F)$.

An example

A matrix $F \in \mathrm{GL}_n(k)$ is said to be
• an **asymmetry** if there exists $E \in \mathrm{GL}_n(k)$ such that $F = E^t E^{-1}$;
• **normalizable** if $\mathrm{tr}(F) \neq 0$ and $\mathrm{tr}(F^{-1}) \neq 0$ or $\mathrm{tr}(F) = 0 = \mathrm{tr}(F^{-1})$;
• **generic** if it is normalizable and the solutions of the equation $q^2 - \sqrt{\mathrm{tr}(F)\mathrm{tr}(F^{-1})}q + 1 = 0$ are generic, i.e. are not roots of unity of order ≥ 3 (does not depend on the choice of the above square root).

The Hopf algebra $H(F)$ is cosemisimple if and only if F is generic.

Theorem

If F is an asymmetry or F is generic, we have $\mathrm{cd}(H(F)) = 3$.

An example

Theorem

If F is an asymmetry or F is generic, we have $\mathrm{cd}(H(F)) = 3$.

Proof: it was already known that if F is an asymmetry, then $\mathrm{cd}(H(F)) = 3$, and that if F is generic, then $\mathrm{cd}(H(F)) \leq 3$. So suppose that F is generic. Then

$$\mathcal{M}^{H(F)} \simeq^{\otimes} \mathcal{M}^{H(F_q)}$$

for

$$F_q = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \quad q^2 - \sqrt{\mathrm{tr}(F)\mathrm{tr}(F^{-1})}q + 1 = 0$$

F_q is an asymetry, so $\mathrm{cd}(H(F_q)) = 3$, and since we know $\mathrm{cd}(H(F))$ is finite, we can apply our theorem to conclude

$$\mathrm{cd}(H(F)) = \mathrm{cd}(H(F_q)) = 3$$

Other strategy: Gerstenhaber-Schack cohomological dimension

Other strategy to attack our question: use an auxiliary cohomological dimension, the Gerstenhaber-Schack cohomological dimension, based on Yetter-Drinfeld modules. Let A be a Hopf algebra.

Definition

A (right-right) Yetter-Drinfeld module over A is a right A-comodule and right A-module V satisfying the condition, $\forall v \in V$, $\forall a \in A$,

$$(v \leftarrow a)_{(0)} \otimes (v \leftarrow a)_{(1)} = v_{(0)} \leftarrow a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)}$$

$\rightsquigarrow$ category $\mathcal{YD}_A^A$, with $\mathcal{YD}_A^A \simeq^{\otimes} \mathcal{Z}(\mathcal{M}^A) \simeq^{\otimes} \mathcal{Z}(\mathcal{M}_A)$.

The **Gerstenhaber-Schack cohomological dimension of** A is defined by

$$\mathrm{cd}_{\mathrm{GS}}(A) = \max\{n : \ \mathrm{Ext}^n_{\mathcal{YD}_A^A}(k, V) \neq 0 \text{ for some } V \in \mathcal{YD}_A^A\} \in \mathbb{N} \cup \{\infty\}$$

Other strategy: Gerstenhaber-Schack cohomological dimension

We always have $\mathrm{cd}(A) \leq \mathrm{cd}_{\mathrm{GS}}(A)$, and

Theorem (B, 2016)

Let A and B be Hopf algebras such that $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$. Then we have $\max(\mathrm{cd}(A), \mathrm{cd}(B)) \leq \mathrm{cd}_{\mathrm{GS}}(A) = \mathrm{cd}_{\mathrm{GS}}(B)$.

It is therefore important to compare $\mathrm{cd}(A)$ and $\mathrm{cd}_{\mathrm{GS}}(A)$.
When A is cosemisimple, $\mathcal{YD}_A^A$ has enough projective objects, and we also have

$$\mathrm{cd}_{\mathrm{GS}}(A) = \mathrm{pd}_{\mathcal{YD}_A^A}(k)$$

Other strategy: Gerstenhaber-Schack cohomological dimension

Theorem (B, 2016-2018)

Let A be a cosemisimple Hopf algebra. If $S^4 = \mathrm{id}$, then $\mathrm{cd}(A) = \mathrm{cd}_{\mathrm{GS}}(A)$.

The new result is:

Theorem

Let A be a cosemisimple Hopf algebra. If $\mathrm{cd}_{\mathrm{GS}}(A)$ is finite, then $\mathrm{cd}(A) = \mathrm{cd}_{\mathrm{GS}}(A)$.

Keypoint: the forgetful functor $\Omega_A : \mathcal{YD}_A^A \to \mathcal{M}_A$ is twisted separable.

Corollary

Let A and B be cosemisimple Hopf algebras such that $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$. If $\mathrm{cd}_{\mathrm{GS}}(A)$ is finite, then $\mathrm{cd}(A) = \mathrm{cd}(B)$.

Slightly weaker than what we had, but...