HALF-LIBERATED REAL SPHERES AND THEIR SUBSPACES

JULIEN BICHON

Abstract. We describe the quantum subspaces of Banica-Goswami's half-liberated real-spheres, showing in particular that there is a bijection between the symmetric ones and the conjugation stable closed subspaces of the complex projective spaces.

1. Introduction

Let $n \geq 1$. The half-liberated real sphere $S^{n-1}_{\mathbb{R},*}$ was defined by Banica and Goswami [4] as the quantum space corresponding to the C^*-algebra

$$C(S^{n-1}_{\mathbb{R},*}) = C^*\left(v_1, \dots, v_n \mid \sum_{i=1}^n v_i^2 = 1,\ v_i^* = v_i,\ v_iv_jv_k = v_kv_jv_i,\ 1 \leq i,j,k \leq n\right)$$

It corresponds to a natural quantum homogeneous space over the half-liberated orthogonal quantum group O_n^* introduced by Banica and Speicher in [5]. These quantum spaces and groups, although defined by very simple means by the intriguing half-commutativity relations $abc = cba$ arising from representation-theoretic considerations via Woronowicz' Tannaka-Krein duality [11], turned out to be new in the field, and definitively of interest. See [1, 2] for recent developments and general discussions on non-commutative spheres.

The aim of this paper is to describe the quantum subspaces of $S^{n-1}_{\mathbb{R},*}$: we will show in particular that there is a natural bijection between the symmetric ones (see Section 3 for the definition) and the conjugation stable closed subspaces of the complex projective space $P^{n-1}_{\mathbb{C}}$. The description of all subspaces is also done, but is more technical, and uses representation theory methods, inspired by those used by Podleś [9] in the determination of the quantum subgroups of $SU_{-1}(2) \simeq O_2^*$. It follows from our analysis that the quantum subspaces of $S^{n-1}_{\mathbb{R},*}$ completely can be described by means of classical spaces, a fact already noted in the description of the quantum subgroups of O_n^* in [7]. As in [7], a crossed product model provides the bridge linking the quantum subspaces and the subspaces of an appropriate classical space.

In fact, for the description of all the quantum subspaces, we will work in a more general context, where a quantum space $Z_{\mathbb{R},*}$ is associated to any compact space Z endowed with an appropriate $\mathbb{T} \rtimes \mathbb{Z}_2 \simeq O_2$ continuous action, and where $S^{n-1}_{\mathbb{R},*}$ is obtained from the complex sphere $S^{n-1}_{\mathbb{C}}$. We do not know if our general framework really furnishes new examples of interest, however we think that, as often with abstract settings, it has the merit to clean-up arguments, can ultimately simplify the theory, and could be well-suited for other developments, such as K-theory computations.

The paper is organized as follows. Section 2 consists of preliminaries. In Section 3, we construct a faithful crossed product representation of $C(S^{n-1}_{\mathbb{R},*})$ and use the sign automorphism to construct a $\mathbb{Z}_2$-grading on $C(S^{n-1}_{\mathbb{R},*})$. We then show that there is an explicit bijective correspondence between symmetric quantum subspaces of $S^{n-1}_{\mathbb{R},*}$ (corresponding to $\mathbb{Z}_2$-graded ideals of $C(S^{n-1}_{\mathbb{R},*})$) and conjugation stable closed subspaces of the complex projective space $P^{n-1}_{\mathbb{C}}$. Section 4, in which we work in a slightly more general framework, is devoted to the description of all the quantum subspaces of $S^{n-1}_{\mathbb{R},*}$, in terms of pairs of certain subspaces of the complex sphere $S^{n-1}_{\mathbb{C}}$.

Acknowledgements. I wish to thank Teodor Banica for useful discussions.

2010 *Mathematics Subject Classification.* 46L85.

2. Notations and preliminaries

2.1. **Conventions.** All C^*-algebras are assumed to be unital, as well as all C^*-algebra maps. By a compact space we mean a compact Hausdorff space, unless otherwise specified. The categories of compact spaces and of commutative C^*-algebras are anti-equivalent by the classical Gelfand duality, and we use, in a standard way, the language of quantum spaces: a C^*-algebra A is viewed as the algebra of continuous functions on a (uniquely determined) quantum space X, and we write $A = C(X)$. Quantum subspaces $Y \subset X$ correspond to surjective $*$-algebra maps $C(X) \to C(Y)$, and hence as well to ideals of $C(X)$. The quantum space X is said to be classical (resp. non-classical) if the C^*-algebra $C(X)$ is commutative (resp. non-commutative).

2.2. **Classical spaces.** The real and complex spheres are denoted respectively by $S_{\mathbb{C}}^{n-1}$ and $S_{\mathbb{R}}^{n-1}$, and their C^*-algebras of continuous functions always are endowed with their usual presentations:

$$C(S_{\mathbb{C}}^{n-1}) = C^*\left(z_1, \dots, z_n \mid \sum_{i=1}^n z_i z_i^* = 1,\ z_i z_j = z_j z_i,\ z_i z_j^* = z_j^* z_i,\ 1 \le i, j, \le n\right)$$

where $z_1, \dots, z_n$ are the standard coordinate functions;

$$C(S_{\mathbb{R}}^{n-1}) = C^*\left(x_1, \dots, x_n \mid \sum_{i=1}^n x_i^2 = 1,\ x_i^* = x_i,\ x_i x_j = x_j x_i,\ 1 \le i, j \le n\right)$$

where $x_1, \dots, x_n$ are the standard coordinate functions. The sphere $S_{\mathbb{C}}^0$ is, as usual, denoted $\mathbb{T}$.

The complex projective space $P_{\mathbb{C}}^{n-1}$ is the orbit space $S_{\mathbb{C}}^{n-1}/\mathbb{T}$ for the usual $\mathbb{T}$-action by multiplication. Of course $C(P_{\mathbb{C}}^{n-1})$ is isomorphic to a C^*-subalgebra of $C(S_{\mathbb{C}}^{n-1})$, via the natural identification $C(S_{\mathbb{C}}^{n-1}/\mathbb{T}) \simeq C(S_{\mathbb{C}}^{n-1})^{\mathbb{T}}$, the later C^*-algebra being the C^*-sub-algebra of $C(S_{\mathbb{C}}^{n-1})$ generated by the elements $z_i z_j^*$ (by the Stone-Weierstrass theorem). Moreover, the C^*-algebra $C(P_{\mathbb{C}}^{n-1})$ has the following presentation, communicated to me by T. Banica.

Lemma 2.1. *The C^*-algebra $C(P_{\mathbb{C}}^{n-1})$ has the presentation*

$$C(P_{\mathbb{C}}^{n-1}) \simeq C^*\left(p_{ij}, 1 \le i, j \le n \mid p = p^* = p^2,\ \mathrm{Tr}(p) = 1,\ p_{ij} p_{kl} = p_{kl} p_{ij},\ 1 \le i, j \le n\right)$$

were p denotes the matrix (p_{ij}), and where the element p_{ij} corresponds to the element $z_i z_j^*$.

Proof. Denote by A the C^*-algebra on the right. It is straightforward to check that there exists a $*$-algebra map

$$\begin{aligned} A &\longrightarrow C(P_{\mathbb{C}}^{n-1}) \\ p_{ij} &\longmapsto z_i z_j^* \end{aligned}$$

which is surjective. To show the injectivity, it is enough, by Gelfand duality, to show that the corresponding continuous map

$$\begin{aligned} P_{\mathbb{C}}^{n-1} = S_{\mathbb{C}}^{n-1}/\mathbb{T} &\longrightarrow \{p \in M_n(\mathbb{C}),\ p = p^* = p^2,\ \mathrm{Tr}(p) = 1\} \\ (z_1, \dots, z_n) &\longmapsto (z_i z_j^*) \end{aligned}$$

is surjective, which follows from the structure of rank 1 projections. $\square$

To conclude this section, we introduce a last piece of notation. The complex conjugation induces an order two automorphism of $C(S_{\mathbb{C}}^{n-1})$, that we denote τ, with $\tau(z_i) = z_i^*$. This enables us to form the crossed product $C(S_{\mathbb{C}}^{n-1}) \rtimes \mathbb{Z}_2$, that we use intensively in the rest of the paper.

3. Half-liberated spheres and their symmetric subspaces

Recall from the introduction that the half-liberated real sphere $S^{n-1}_{\mathbb{R},*}$ [4] is the quantum space corresponding to the C^*-algebra

$$C(S^{n-1}_{\mathbb{R},*}) = C^*\left(v_1,\ldots,v_n \mid \sum_{i=1}^n v_i^2 = 1,\ v_i^* = v_i,\ v_iv_jv_k = v_kv_jv_i,\ 1\le i,j,k\le n\right)$$

3.1. $\mathbb{Z}_2$-grading on $C(S^{n-1}_{\mathbb{R},*})$.

Definition 3.1. *The sign automorphism of $C(S^{n-1}_{\mathbb{R},*})$, denoted ν, is the automorphism defined by $\nu(v_i) = -v_i$, for any i.*

The sign automorphism defines a $\mathbb{Z}_2$-grading on the algebra $C(S^{n-1}_{\mathbb{R},*})$:

$$C(S^{n-1}_{\mathbb{R},*}) = C(S^{n-1}_{\mathbb{R},*})_0 \oplus C(S^{n-1}_{\mathbb{R},*})_1$$

where $C(S^{n-1}_{\mathbb{R},*})_0 = \{a \in C(S^{n-1}_{\mathbb{R},*}) \mid \nu(a) = a\}$ and $C(S^{n-1}_{\mathbb{R},*})_1 = \{a \in C(S^{n-1}_{\mathbb{R},*}) \mid \nu(a) = -a\}$. Here of course $C(S^{n-1}_{\mathbb{R},*})_0$ is the fixed point algebra for the $\mathbb{Z}_2$-action on $C(S^{n-1}_{\mathbb{R},*})$ defined by ν. It has the following description.

Lemma 3.2. *The C^*-subalgebra generated by the elements v_iv_j, $1 \le i,j \le n$, is commutative, coincides with $C(S^{n-1}_{\mathbb{R},*})_0$, and we have a $*$-algebra isomorphism*

$$\begin{aligned}\Phi : C(P^{n-1}_{\mathbb{C}}) &\longrightarrow C(S^{n-1}_{\mathbb{R},*})_0\\ z_iz_j^* &\longmapsto v_iv_j\end{aligned}$$

Proof. The commutativity of $C^*(v_iv_j)$, a fundamental and direct observation, is known [6, 4]. It is clear that $v_iv_j \in C(S^{n-1}_{\mathbb{R},*})_0$ for any i,j, hence $C^*(v_iv_j) \subset C(S^{n-1}_{\mathbb{R},*})_0$. Denote by $\mathcal{O}(S^{n-1}_{\mathbb{R},*})$ the dense $*$-subalgebra of $C(S^{n-1}_{\mathbb{R},*})$ generated by the elements v_i. It is clear that $\mathcal{O}(S^{n-1}_{\mathbb{R},*})_0$ consists of the linear span of monomials of even length, hence is generated as an algebra by the elements v_iv_j. Therefore we get the announced result, since $\mathcal{O}(S^{n-1}_{\mathbb{R},*})_0$ is dense in $C(S^{n-1}_{\mathbb{R},*})_0$ (if A is C^*-algebra acted on by a finite group and $\mathcal{A} \subset A$ is a dense $*$-subalgebra, then $\mathcal{A}^G$ is dense in A^G).

The existence of Φ follows from Lemma 2.1, and Φ is surjective by the previous discussion. The injectivity is Theorem 3.3 in [4]. For the sake of completeness, we will present, during the proof of the forthcoming Theorem 3.4, another elementary proof of the injectivity of Φ (the proof in [4] was relying on much more sophisticated diagrammatic quantum group techniques). $\square$

Definition 3.3. *A quantum subspace $X \subset S^{n-1}_{\mathbb{R},*}$ is said to be symmetric if the corresponding ideal I is $\mathbb{Z}_2$-graded, i.e.*

$$I = I_0 \oplus I_1, \text{ with } I_0 = I \cap C(S^{n-1}_{\mathbb{R},*})_0 \text{ and } I_1 = I \cap C(S^{n-1}_{\mathbb{R},*})_1$$

or in other words, if ν induces an automorphism of $C(S^{n-1}_{\mathbb{R},*})/I$.

3.2. Faithful crossed product representation of $C(S^{n-1}_{\mathbb{R},*})$.

We now describe a crossed product model for $C(S^{n-1}_{\mathbb{R},*})$, using the crossed product $C(S^{n-1}_{\mathbb{C}}) \rtimes \mathbb{Z}_2$ associated to the conjugation action on $S^{n-1}_{\mathbb{C}}$, see the previous Section. A related construction was already considered in the quantum group setting in [7].

Theorem 3.4. *There exists an injective $*$-algebra map*

$$\begin{aligned}\pi : C(S^{n-1}_{\mathbb{R},*}) &\longrightarrow C(S^{n-1}_{\mathbb{C}}) \rtimes \mathbb{Z}_2\\ v_i &\longmapsto z_i \otimes \tau\end{aligned}$$

Proof. It is straightforward to construct π, and this is left to the reader. We have $\pi(v_iv_j) = (z_i \otimes \tau)(z_j \otimes \tau) = z_iz_j^* \otimes 1$, hence $\pi\Phi = \mathrm{id} \otimes 1$, and this indeed gives another proof for the injectivity of Φ in Lemma 3.2.

We have to show that π is injective. First recall the following general fact: if A and B are $\mathbb{Z}_2$-graded C^*-algebras and $\pi : A \to B$ is a $*$-algebra map preserving the $\mathbb{Z}_2$-grading, then π is injective if and only if the restriction of π to A_0 is injective. Indeed, assume that $\pi_{|A_0}$ is injective. To show the injectivity of π, we just have to show that $\pi_{|A_1}$ is injective (since π preserves the $\mathbb{Z}_2$-grading). So let $a \in A_1$ with $\pi(a) = 0$. We have $a^*a \in A_0$ and $\pi(a)^*\pi(a) = \pi(a^*a) = 0$, so $a^*a = 0$ and $a = 0$: π is thus injective.

Now note that $C(S_{\mathbb{C}}^{n-1}) \rtimes \mathbb{Z}_2$ is $\mathbb{Z}_2$-graded as well, with grading defined by

$$(C(S_{\mathbb{C}}^{n-1}) \rtimes \mathbb{Z}_2)_0 = C(S_{\mathbb{C}}^{n-1}) \otimes 1, \ (C(S_{\mathbb{C}}^{n-1}) \rtimes \mathbb{Z}_2)_1 = C(S_{\mathbb{C}}^{n-1}) \otimes \tau$$

and that π preserves the respective $\mathbb{Z}_2$-gradings. The previous discussion shows that it is enough to show that the restriction of π to $C(S_{\mathbb{R},*}^{n-1})_0$ is injective, which is immediate since the restriction of π to $C(S_{\mathbb{R},*}^{n-1})_0$ is the injective map $\Phi^{-1} \otimes 1$. □

3.3. **Symmetric subspaces of half-liberated spheres.** Before describing the symmetric subspaces of $S_{\mathbb{R},*}^{n-1}$, we need a last ingredient.

Definition 3.5. *We denote by γ the linear endomorphism of $C(S_{\mathbb{R},*}^{n-1})$ defined by*

$$\gamma(a) = \sum_{i=1}^{n} v_i a v_i$$

The main properties of γ are summarized in the following lemma.

Lemma 3.6. *The endomorphism γ preserves the $\mathbb{Z}_2$-grading of $C(S_{\mathbb{R},*}^{n-1})$, and induces a $*$-algebra automorphism of $C(S_{\mathbb{R},*}^{n-1})_0$. Moreover the following diagram commutes*

$$\begin{array}{ccc} C(P_{\mathbb{C}}^{n-1}) & \xrightarrow{\Phi} & C(S_{\mathbb{R},*}^{n-1})_0 \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\gamma} \\ C(P_{\mathbb{C}}^{n-1}) & \xrightarrow{\Phi} & C(S_{\mathbb{R},*}^{n-1})_0 \end{array}$$

where τ is the automorphism induced by complex conjugation, i.e. $\tau(z_iz_j^*) = z_jz_i^*$. Hence there is a bijective correspondence between γ-stable ideals of $C(S_{\mathbb{R},*}^{n-1})_0$ and conjugation stable closed subsets of $P_{\mathbb{C}}^{n-1}$.

Proof. It is clear that γ preserves the $\mathbb{Z}_2$-grading of $C(S_{\mathbb{R},*}^{n-1})$, that $\gamma(1) = 1$ and that γ commutes with the involution. We have

$$\gamma(v_{i_1}v_{j_1}\cdots v_{i_m}v_{j_m}) = \sum_k v_kv_{i_1}v_{j_1}\cdots v_{i_m}v_{j_m}v_k = \sum_k v_{j_1}v_{i_1}\cdots v_{j_m}v_{i_m}v_kv_k = v_{j_1}v_{i_1}\cdots v_{j_m}v_{i_m}$$

where the formula is shown by induction on m, using the half-commutation relations. This shows that the diagram commutes, and at the same time that γ preserves multiplication, and is an automorphism. The last assertion then follows immediately from the correspondence between conjugation closed subspaces of $P_{\mathbb{C}}^{n-1}$ and τ-stable ideals of $C(P_{\mathbb{C}}^{n-1})$. □

The description of the $\mathbb{Z}_2$-graded ideals of $C(S_{\mathbb{R},*}^{n-1})$ is then as follows.

Theorem 3.7. *We have a bijective correspondence*

$$\begin{aligned} \{\mathbb{Z}_2 - \text{graded ideals of } C(S_{\mathbb{R},*}^{n-1})\} &\longleftrightarrow \{\gamma - \text{stable ideals of } C(S_{\mathbb{R},*}^{n-1})_0\} \\ I &\longmapsto I_0 = I \cap C(S_{\mathbb{R},*}^{n-1})_0 \\ \langle J \rangle = J + C(S_{\mathbb{R},*}^{n-1})_1 J &\longleftarrow J \end{aligned}$$

Proof. For notational simplicity, we put $A = C(S^{n-1}_{\mathbb{R},*})$. Let $I = I_0 + I_1$ be a $\mathbb{Z}_2$-graded ideal of A. It is clear that I_0 is γ-stable since it is an ideal in A, thus the first map is well-defined, we call it $\mathcal{F}$.

Let $J \subset A_0$ be a γ-stable ideal. It is clear that $J + A_1 J$ is a left ideal, and, in order to show that it is as well a right ideal, it is enough to see that $A_1 J = J A_1$. This will follow from the following claim: for $x \in A_0$, we have $v_i x = \gamma(x) v_i$ for any i. This is shown

(1) for elements of type $v_j v_k$, using half commutation,
(2) for polynomials in $v_j v_k$, by induction,
(3) and finally by density for any x.

From this, we have $v_i J = \gamma(J) v_i = J v_i$, and then $A_1 J = \sum_i v_i A_0 J = \sum_i v_i J = \sum_i J v_i = \sum_i J A_0 v_i = J A_1$, as required. Thus $J + A_1 J$ is an ideal, and is $\mathbb{Z}_2$-graded by construction. The second map is well-defined, we call it $\mathcal{G}$.

Let $I = I_0 + I_1$ be a $\mathbb{Z}_2$-graded ideal of A. Then for $x \in I_1$, we have

$$x = \sum_i v_i v_i x \in A_1 I_0, \ x = \sum_i x v_i v_i \in I_0 A_1$$

hence $I_1 = A_1 I_0 = I_0 A_1$, and we have $\mathcal{G}\mathcal{F}(I) = I_0 + A_1 I_0 = I$.

Finally it is clear that $\mathcal{F}\mathcal{G}(J) = J$ for any γ-stable ideal $J \subset A_0$, and this concludes the proof. □

In terms of subspaces, we have the following immediate translation.

Corollary 3.8. *We have a bijection*

$$\{\text{conjugation stable closed subspaces } Y \subset P^{n-1}_{\mathbb{C}}\} \longleftrightarrow \{\text{symmetric quantum subspaces } X \subset S^{n-1}_{\mathbb{R},*}\}$$

The bijection is as follows: let $Y \subset P^{n-1}_{\mathbb{C}}$ be a conjugation stable closed subspace, and let $\mathcal{J}_Y \subset C(P^{n-1}_{\mathbb{C}})$ be the ideal of functions vanishing on Y. Then the corresponding symmetric closed subspace X of $S^{n-1}_{\mathbb{R},*}$ is defined by $C(X) = C(S^{n-1}_{\mathbb{R},*})/\langle \Phi(\mathcal{J}_Y)\rangle$.

The correspondence is easy to use in practice, because Φ and Φ^{-1} are completely explicit, and Φ transforms the monomial $z_{i_1} \cdots z_{i_m} z^*_{j_1} \cdots z^*_{j_m}$ into the monomial $v_{i_1} v_{j_1} \cdots v_{i_m} v_{j_m}$. For example if $f_1, \ldots, f_r$ are polynomials in $z_i z_j^*$ that generate the ideal $\mathcal{J}_Y$, then the corresponding quotient of $C(S^{n-1}_{\mathbb{R},*})$ is $C(S^{n-1}_{\mathbb{R},*})/(\Phi(f_1), \ldots, \Phi(f_r))$.

4. General setup

We now describe all the quantum subspaces of $S^{n-1}_{\mathbb{R},*}$. For this, it will be convenient to work in a more general framework.

We denote by $\mathbb{T} \rtimes \mathbb{Z}_2$ the semi-direct product associated to the conjugation action of $\mathbb{Z}_2$ on $\mathbb{T}$. This is a compact group, isomorphic with the orthogonal group O_2, but the above description will be the more convenient.

General setup. Let Z be a compact space endowed with a continuous action of $\mathbb{T} \rtimes \mathbb{Z}_2$, such that the action is $\mathbb{T}$-free. The $\mathbb{Z}_2$-action on Z corresponds, unless otherwise specified, to the $\mathbb{Z}_2$-action of the second factor of $\mathbb{T} \rtimes \mathbb{Z}_2$, and the corresponding automorphism is denoted τ.

We put $Z_{\mathbb{R}} = Z^{\mathbb{Z}_2} = \{z \in Z \mid \tau(z) = z\}$, $Z_{\text{reg}} = Z \setminus \mathbb{T} Z_{\mathbb{R}}$, and

$$C(Z)^{\mathbb{T}} = \{f \in C(Z), \ f(\omega z) = f(z), \forall \omega \in \mathbb{T}, \forall z \in Z\}$$

$$C_{\mathbb{T}}(Z) = \{f \in C(Z), \ f(\omega z) = \omega f(z), \forall \omega \in \mathbb{T}, \forall z \in Z\}$$

The $\mathbb{Z}_2$-action on Z enables us to form the crossed product $C(Z) \rtimes \mathbb{Z}_2$.

Definition 4.1. *For a compact space Z endowed with a continuous $\mathbb{T} \rtimes \mathbb{Z}_2$-action, such that the action is $\mathbb{T}$-free, we put*

$$C(Z_{\mathbb{R},*}) = \{f_0 \otimes 1 + f_1 \otimes \tau, \ f_0 \in C(Z)^{\mathbb{T}}, \ f_1 \in C_{\mathbb{T}}(Z)\} \subset C(Z) \rtimes \mathbb{Z}_2$$

It is a direct verification to check that $C(Z_{\mathbb{R},*})$ is a C^*-subalgebra of $C(Z) \rtimes \mathbb{Z}_2$.

The following lemma links the present construction to the half-liberated spheres.

Lemma 4.2. *Assume that there exist $f_1, \dots, f_n \in C_{\mathbb{T}}(Z)$ such that*

(1) $C(Z)^{\mathbb{T}} = C^*(f_i f_j^*,\ 1 \leq i,j \leq n)$,

(2) $C_{\mathbb{T}}(Z) = f_1 C(Z)^{\mathbb{T}} + \cdots + f_n C(Z)^{\mathbb{T}}$.

Then $C(Z_{\mathbb{R},*}) = C^*(f_1 \otimes \tau, \dots, f_n \otimes \tau)$, and the elements $f_1 \otimes \tau, \dots, f_n \otimes \tau$ half commute.

Proof. Put $A = C^*(f_1 \otimes \tau, \dots, f_n \otimes \tau)$. We have

$$(f_i \otimes \tau)(f_j \otimes \tau)^* = (f_i \otimes \tau)(\tau(f_j^*) \otimes \tau) = f_i f_j^* \otimes 1 \in A$$

hence by (1) we have $C(Z)^{\mathbb{T}} \otimes 1 \subset A$. For $f \in C_{\mathbb{T}}(Z)$, we have $f = \sum_i g_i f_i$ for some $g_1, \dots, g_n \in C(Z)^{\mathbb{T}}$ by (2), hence

$$f \otimes \tau = \sum_i g_i f_i \otimes \tau = \sum_i (g_i \otimes 1)(f_i \otimes \tau) \in A$$

and this shows that $C(Z_{\mathbb{R},*}) = A$. Moreover

$$(f_i \otimes \tau)(f_j \otimes \tau)(f_k \otimes \tau) = f_i \tau(f_j) f_k \otimes \tau = (f_k \otimes \tau)(f_j \otimes \tau)(f_i \otimes \tau)$$

which concludes the proof. $\square$

Example 4.3. Consider the natural $\mathbb{T} \rtimes \mathbb{Z}_2$-action on $S_{\mathbb{C}}^{n-1}$, where the $\mathbb{T}$-action is by multiplication and the $\mathbb{Z}_2$-action is by conjugation. The $\mathbb{T}$-action is indeed free, we have $(S_{\mathbb{C}}^{n-1})_{\mathbb{R}} = S_{\mathbb{R}}^{n-1}$, and

$$(S_{\mathbb{C}}^{n-1})_{\text{reg}} =: S_{\mathbb{C},\text{reg}}^{n-1} = S_{\mathbb{C}}^{n-1} \setminus \mathbb{T} S_{\mathbb{R}}^{n-1} = \{g = (g_1, \dots, g_n) \in S_{\mathbb{C}}^{n-1} \mid \exists i,j \text{ with } g_i \overline{g_j} \neq g_j \overline{g_j}\}$$

The coordinate functions $z_1, \dots, z_n \in C_{\mathbb{T}}(S_{\mathbb{C}}^{n-1})$ satisfy the conditions of the previous lemma, because $1 = \sum_i z_i z_i^*$, and hence the image of the injective morphism

$$\begin{aligned} \pi : C(S_{\mathbb{R},*}^{n-1}) &\longrightarrow C(S_{\mathbb{C}}^{n-1}) \rtimes \mathbb{Z}_2 \\ v_i &\longmapsto z_i \otimes \tau \end{aligned}$$

of Theorem 3.4 is precisely $C((S_{\mathbb{C}}^{n-1})_{\mathbb{R},*})$. Therefore we identify the two algebras.

The description of the quantum subspaces of $Z_{\mathbb{R},*}$, which has $S_{\mathbb{R},*}^{n-1}$ as a particular case, is as follows.

Theorem 4.4. *There exists a bijection between the set of quantum subspaces $X \subset Z_{\mathbb{R},*}$ and the set of pairs (E, F) where*

(1) *$E \subset Z_{\text{reg}}$ is $\mathbb{T} \rtimes \mathbb{Z}_2$-stable and $E = \overline{E} \cap Z_{\text{reg}}$ (i.e. E is closed in Z_{reg});*

(2) *$F \subset Z_{\mathbb{R}}$ is closed and satisfies $\overline{E} \cap Z_{\mathbb{R}} \subset F$.*

Moreover the quantum subspace X is non-classical if and only if, in the corresponding pair (E, F), we have $E \neq \emptyset$.

The proof is given in the next subsections.

4.1. **Representation theory of $C(Z_{\mathbb{R},*})$.** We now provide the description of the irreducible representations of the C^*-algebra $C(Z_{\mathbb{R},*})$, the main step towards the description of the subspaces of $Z_{\mathbb{R},*}$. It is certainly possible to use general results on crossed products [10] to provide this description, but since everything can be done in a quite direct and elementary manner, we will proceed directly.

The C^*-algebra $C(Z_{\mathbb{R},*})$ is, by definition, a C^*-subalgebra of the crossed product C^*-algebra $C(Z) \rtimes \mathbb{Z}_2$. In particular, it can be seen as a C^*-subalgebra of $M_2(C(Z))$, and hence all its irreducible representations have dimension ≤ 2, and is a 2-subhomogeneous C^*-algebra. The precise description of the irreducible representations of $C(Z_{\mathbb{R},*})$ is given in the following result.

Proposition 4.5. (1) *Any $z \in Z$ defines a representation*

$$\theta_z : C(Z_{\mathbb{R},*}) \longrightarrow M_2(\mathbb{C})$$
$$f_0 \otimes 1 + f_1 \otimes \tau \longmapsto \begin{pmatrix} f_0(z) & f_1(z) \\ f_1(\tau(z)) & f_0(\tau(z)) \end{pmatrix}$$

and any irreducible representation of $C(Z_{\mathbb{R},*})$ is isomorphic to a sub-representation of some θ_z, for some $z \in Z$. The representation θ_z is irreducible if and only if $z \in Z_{\mathrm{reg}}$. Moreover, for $z, x \in Z$, the representations θ_z and θ_x are isomorphic if and only if $(\mathbb{T} \rtimes \mathbb{Z}_2)z = (\mathbb{T} \rtimes \mathbb{Z}_2)x$.

(2) *Any $z \in Z_{\mathbb{R}}$ defines a one-dimensional representation*

$$\phi_z : C(Z_{\mathbb{R},*}) \longrightarrow \mathbb{C}$$
$$f_0 \otimes 1 + f_1 \otimes \tau \longmapsto f_0(z) + f_1(z)$$

and any one-dimensional representation arises in this way. Moreover for $z, y \in Z_{\mathbb{R}}$, we have $\phi_z = \phi_y \iff z = y$.

(3) *If π is an irreducible representation of $C(Z_{\mathbb{R},*})$, then either $\pi \simeq \theta_z$ for some $z \in Z_{\mathrm{reg}}$ or $\pi = \phi_z$ for some $z \in Z_{\mathbb{R}}$.*

Proof. It can be checked directly that θ_z defines a representation of $C(Z_{\mathbb{R},*})$, or by using the standard embedding of the crossed product $C(Z) \rtimes \mathbb{Z}_2$ into $M_2(C(Z))$

$$C(Z_{\mathbb{R},*}) \longrightarrow M_2(C(Z))$$
$$f_0 \otimes 1 + f_1 \otimes \tau \longmapsto \begin{pmatrix} f_0 & f_1 \\ f_1\tau & f_0\tau \end{pmatrix}$$

composed with evaluation at $z \in Z$. Since any irreducible representation of $M_2(C(Z))$ is obtained by evaluation at an element $z \in Z$, we get that any irreducible representation of $C(Z_{\mathbb{R},*})$ is isomorphic to a sub-representation of θ_z for some $z \in Z$, see e.g. [8].

Now assume that $z \in \mathbb{T}Z_{\mathbb{R}}$: $z = \lambda y$ for some $y \in Z_{\mathbb{R}}$. Then $\tau(z) = \tau(\lambda y) = \overline{\lambda}\tau(y) = \overline{\lambda}y = \overline{\lambda}^2 z$, and for $f_0 \otimes 1 + f_1 \otimes \tau \in C(Z_{\mathbb{R},*})$, we have

$$\theta_z(f_0 \otimes 1 + f_1 \otimes \tau) = \begin{pmatrix} f_0(z) & f_1(z) \\ f_1(\tau(z)) & f_0(\tau(z)) \end{pmatrix} = \begin{pmatrix} f_0(z) & f_1(z) \\ \overline{\lambda}^2 f_1(z) & f_0(z) \end{pmatrix}$$

This implies that $\theta_z(C(Z_{\mathbb{R},*}))$ is abelian, and hence θ_z is not irreducible.

Assume that $z \in Z_{\mathrm{reg}}$. To show that θ_z is irreducible, it is enough to show that there exist $f_0 \in C(Z)^{\mathbb{T}}$ and $f_1 \in C_{\mathbb{T}}(Z)$ such that

$$f_0(z) \neq f_0(\tau(z)), \ f_1(z) \neq 0$$

Indeed, we then will have that $\theta_z(f_0 \otimes 1)$ and $\theta_z(f_1 \otimes \tau)$ do not commute, and hence $\theta_z(C(Z_{\mathbb{R},*}))$ is a non-commutative C^*-subalgebra of $M_2(\mathbb{C})$, so both algebras are equal and θ_z is irreducible.

We have, since $z \in Z_{\mathrm{reg}}$, $\mathbb{T}z \neq \mathbb{T}\tau(z)$. Otherwise $z = \lambda\tau(z)$ for some $\lambda \in \mathbb{T}$, and for $\mu \in \mathbb{T}$ such that $\mu^2 = \overline{\lambda}$, we have $z = \overline{\mu}\mu z$, with $\mu z \in Z_{\mathbb{R}}$, a contradiction. Hence by Urysohn's Lemma and the fact that $X/\mathbb{T}$ is Hausdorff, there exists $f_0 \in C(Z)^{\mathbb{T}} \simeq C(Z/\mathbb{T})$ such that $f_0(z) \neq f_0(\tau(z))$, as needed. Finally since the $\mathbb{T}$-action is free, there exists a (continuous) map $f : \mathbb{T}z \to \mathbb{C}$ such that $f(\lambda z) = \lambda$ for any $\lambda \in \mathbb{T}$, that we extend to to a continuous function f on Z (Tietze's extension theorem). Now let $f_1 \in C(Z)$ be defined by

$$f_1(y) = \int_{\mathbb{T}} \lambda^{-1} f(\lambda y) \mathrm{d}\lambda$$

We have $f_1 \in C_{\mathbb{T}}(Z)$ and $f_1(z) = 1$, as needed, and we conclude that θ_z is irreducible.

A finite-dimensional representation is determined by its character, and the character of the representation θ_z is given by $\chi_z(f_0 \otimes 1 + f_1 \otimes \tau) = f_0(z) + f_0(\tau(z))$.

Let $z, x \in Z$ be such that $(\mathbb{T} \rtimes \mathbb{Z}_2)z \neq (\mathbb{T} \rtimes \mathbb{Z}_2)x$. Then there exists $f_0 \in C(Z)^{\mathbb{T}\rtimes\mathbb{Z}_2}$ such that $f_0(z) = 1 = f_0(\tau(z))$ and $f_0(x) = 0 = f_0(\tau(x))$. We have $f_0 \in C(Z)^{\mathbb{T}}$ and $\chi_z(f) = 2$, $\chi_x(f) = 0$, hence the representations θ_z and θ_x are not isomorphic.

If $(\mathbb{T} \rtimes \mathbb{Z}_2)z = (\mathbb{T} \rtimes \mathbb{Z}_2)x$, then either $z = \lambda x$ or $z = \lambda\tau(x)$ for some $\lambda \in \mathbb{T}$. From this we see that $f(z) + f(\tau(z)) = f(x) + f(\tau(x))$ for any $f \in C(Z)^{\mathbb{T}}$, and hence $\chi_z = \chi_x$, which shows that θ_z and θ_x are isomorphic, and concludes the proof of (1).

For $z \in Z_{\mathbb{R}}$, it is a direct verification to check that ϕ_z above defines a $*$-algebra map $C(Z_{\mathbb{R},*}) \to \mathbb{C}$. Now let $\psi : C(Z_{\mathbb{R},*}) \to \mathbb{C}$ be a $*$-algebra map. The representation defined by ψ is isomorphic to a sub-representation of θ_z for some $z \in Z$, and with $z \notin Z_{\mathrm{reg}}$ because θ_z is not irreducible. Hence $z = \lambda y$ for $y \in Z_{\mathbb{R}}$. We then have, for $f_0 \otimes 1 + f_1 \otimes \tau \in C(Z_{\mathbb{R},*})$,

$$\theta_z(f_0 \otimes 1 + f_1 \otimes \tau) = \begin{pmatrix} f_0(y) & \lambda f_1(y) \\ \overline{\lambda} f_1(y) & f_0(y) \end{pmatrix}$$

and from this we see that the lines generated by $(1, \overline{\lambda})$ and $(1, -\overline{\lambda})$ both are stable under $C(Z_{\mathbb{R},*})$, so that $\theta_z \simeq \phi_y \oplus \phi_{-y}$ (this can also be seen using characters), and finally $\psi = \phi_y$ or $\psi = \phi_{-y}$.

Let $y, z \in Z_{\mathbb{R}}$ be such that $\phi_y = \phi_z$. Then for any $f_0 \in C(Z)^{\mathbb{T}}$, we have $f_0(y) = f_0(z)$, hence $\mathbb{T}y = \mathbb{T}z$, hence $y = \pm z$. As before, there exists $f_1 \in C_{\mathbb{T}}(Z)$ such that $f_1(\lambda z) = \lambda$ for any $\lambda \in \mathbb{T}$. Then $\phi_z(f_1 \otimes \tau) = f_1(\tau(z)) = f_1(z) = 1 = \pm f_1(y) = \pm\phi_y(f_1 \otimes \tau)$, hence $z = y$.

This proves (2), and (3) follows from the combination of (1) and (2). $\square$

Corollary 4.6. *The C^*-algebra $C(Z_{\mathbb{R},*})$ is non-commutative if and only if $Z_{\mathrm{reg}} \neq \emptyset$.*

Proof. This follows from the proposition, because a C^*-algebra is non-commutative if and only if it has an irreducible representation of dimension > 1. $\square$

4.2. **Closed subspaces of $\widehat{C(Z_{\mathbb{R},*})}$.** We now discuss $\widehat{C(Z_{\mathbb{R},*})}$, the spectrum of $C(Z_{\mathbb{R},*})$, endowed with its usual topology, see [8] (since all the irreducible representations of A are finite-dimensional, we know [8] that the topological spaces $\widehat{A}$ and $\mathrm{Prim}(A)$ are canonically homeomorphic).

For $E \subset Z_{\mathrm{reg}}$ and $F \subset Z_{\mathbb{R}}$, we put

$$M(E, F) = (\{\theta_z,\ z \in E\} \cup \{\phi_z, z \in F\})/\sim\ \subset\ \widehat{C(Z_{\mathbb{R},*})}$$

where of course $\sim$ means that we identify isomorphic representations. The previous proposition ensures that any subset of $\widehat{C(Z_{\mathbb{R},*})}$ is of the form $M(E, F)$ for $E \subset Z_{\mathrm{reg}}$, a $\mathbb{T} \rtimes \mathbb{Z}_2$-stable subspace, and $F \subset Z_{\mathbb{R}}$. The next result describes the closed subsets of $\widehat{C(Z_{\mathbb{R},*})}$.

Proposition 4.7. *Let $E \subset Z_{\mathrm{reg}}$ be a $\mathbb{T} \rtimes \mathbb{Z}_2$-stable subspace, and let $F \subset Z_{\mathbb{R}}$. Then we have*

$$\overline{M(E, F)} = M(\overline{E} \cap Z_{\mathrm{reg}}, \overline{F} \cup (\overline{E} \cap Z_{\mathbb{R}}))$$

In particular there exists a bijection between closed subsets of $\widehat{C(Z_{\mathbb{R},*})}$ and pairs (E, F) with

1. *$E \subset Z_{\mathrm{reg}}$ is $\mathbb{T} \rtimes \mathbb{Z}_2$-stable and $E = \overline{E} \cap Z_{\mathrm{reg}}$;*
2. *$F \subset Z_{\mathbb{R}}$ is closed and satisfies $\overline{E} \cap Z_{\mathbb{R}} \subset F$.*

Proof. Put $A = C(Z_{\mathbb{R},*})$. For $S \subset \widehat{A}$, we have

$$\overline{S} = \{\pi \in \widehat{A} \mid \bigcap_{\rho \in S} \mathrm{Ker}(\rho) \subset \mathrm{Ker}(\pi)\}$$

For E, F as in the statement of the proposition, we have

$$\overline{M(E, F)} = \overline{M(E, \emptyset) \cup M(\emptyset, F)} = \overline{M(E, \emptyset)} \cup \overline{M(\emptyset, F)}$$

Hence we can study the two pieces separately. The bijective map

$$\begin{aligned} Z_{\mathbb{R}} &\longrightarrow \widehat{A}_1 \\ z &\longmapsto \phi_z \end{aligned}$$

where $\widehat{A}_1$ consists of the set of 1-dimensional representations, is clearly continuous, and since $Z_{\mathbb{R}}$ and $\widehat{A}_1$ both are compact, this is an homeomorphism. Hence it sends the closure of a subset in $Z_{\mathbb{R}}$ to the closure of the image in $\widehat{A}_1$, and hence to the closure of the image in $\widehat{A}$, since $\widehat{A}_1$ is closed in $\widehat{A}$. Thus $\overline{M(\emptyset, F)} = M(\emptyset, \overline{F})$. Consider now the following two claims:

$$\text{For } y \in Z_{\mathbb{R}}, \ \bigcap_{z\in E} \mathrm{Ker}(\theta_z) \subset \mathrm{Ker}(\phi_y) \iff y \in \overline{E} \cap Z_{\mathbb{R}} \quad (*)$$

$$\text{For } y \in Z_{\mathrm{reg}}, \ \bigcap_{z\in E} \mathrm{Ker}(\theta_z) \subset \mathrm{Ker}(\theta_y) \iff y \in \overline{E} \cap Z_{\mathrm{reg}} \quad (**)$$

Once these claims are proved, we will indeed have

$$\overline{M(E, \emptyset)} = M(\overline{E} \cap Z_{\mathrm{reg}}, \overline{E} \cap Z_{\mathbb{R}})$$

as required. We begin with $(*)$. Let $y \in Z_{\mathbb{R}}$. Assume first that $y \in \overline{E} \cap Z_{\mathbb{R}}$. Let $f_0 \otimes 1 + f_1 \otimes \tau \in \cap_{z\in E}\mathrm{Ker}(\theta_z)$. Then f_0 and f_1 vanish on E, and hence on $\overline{E}$, and $f_0 \otimes 1 + f_1 \otimes \tau \in \mathrm{Ker}(\phi_y)$. Thus $\cap_{z\in E}\mathrm{Ker}(\theta_z) \subset \mathrm{Ker}(\phi_y)$. Conversely assume that $y \notin \overline{E} \cap Z_{\mathbb{R}}$. Then $\mathbb{T}y \cap \overline{E} = \emptyset$ since E is $\mathbb{T}$-stable and there exists $f_0 \in C(Z)^{\mathbb{T}}$ such that $f_0(y) = 1$ and $f_0(\overline{E}) = 0$. We then have $f_0 \otimes 1 \in \cap_{z\in E}\mathrm{Ker}(\theta_z)$ while $f_0 \otimes 1 \notin \mathrm{Ker}(\phi_y)$, and $(*)$ is proved.

Now let $y \in Z_{\mathrm{reg}}$. Assume first that $y \in \overline{E} \cap Z_{\mathrm{reg}}$. Let $f_0 \otimes 1 + f_1 \otimes \tau \in \cap_{z\in E}\mathrm{Ker}(\theta_z)$. Then f_0 and f_1 vanish on E, and hence on $\overline{E}$, and $f_0 \otimes 1 + f_1 \otimes \tau \in \mathrm{Ker}(\theta_y)$. Thus $\cap_{z\in E}\mathrm{Ker}(\theta_z) \subset \mathrm{Ker}(\theta_y)$. Conversely assume that $y \notin \overline{E} \cap Z_{\mathrm{reg}}$. Then $(\mathbb{T} \rtimes \mathbb{Z}_2)y \cap \overline{E} = \emptyset$ since E is $\mathbb{T} \rtimes \mathbb{Z}_2$-stable and there exists $f_0 \in C(Z)^{\mathbb{T}\rtimes\mathbb{Z}_2}$ such that $f_0(y) = 1$ and $f_0(\overline{E}) = 0$. We then have $f_0 \otimes 1 \in \cap_{z\in E}\mathrm{Ker}(\theta_z)$ while $f_0 \otimes 1 \notin \mathrm{Ker}(\theta_y)$, and $(**)$ is proved.

For $E, E' \subset Z_{\mathrm{reg}}$, and $F, F' \subset Z_{\mathbb{R}}$, then by Proposition 4.5 we have $\mathcal{M}(E, F) = \mathcal{M}(E', F') \Rightarrow F = F'$ and $E = E'$ if E and E' are $\mathbb{T} \rtimes \mathbb{Z}_2$-stable, hence the last assertion follows from the first one. □

We are now ready to prove the Theorem 4.4.

Proof of Theorem 4.4. Recall that we have to show that there is a bijection between the set of quantum subspaces $X \subset Z_{\mathbb{R},*}$ and the set of pairs (E, F) where

(1) $E \subset Z_{\mathrm{reg}}$ is $\mathbb{T} \rtimes \mathbb{Z}_2$-stable and $E = \overline{E} \cap Z_{\mathrm{reg}}$ (i.e. E is closed in Z_{reg});
(2) $F \subset Z_{\mathbb{R}}$ is closed and satisfies $\overline{E} \cap Z_{\mathbb{R}} \subset F$.

By Proposition 4.7 the set of such pairs (E, F) is in bijection with the set of closed subsets of $\widehat{C(Z_{\mathbb{R},*})}$, hence the result follows from the standard correspondence between quotients of a C^*-algebra and closed subsets of its spectrum [8]. More precisely to a pair (E, F) as in the statement is associated the C^*-algebra $C(Z_{\mathbb{R},*})/(\cap_{\pi\in M(E,F)}\mathrm{Ker}(\pi))$, and conversely, to a quotient C^*-algebra map $q : C(Z_{\mathbb{R},*}) \to B$, is associated the pair (E, F) such that $\{\pi \in \widehat{C(Z_{\mathbb{R},*})} \mid \mathrm{Ker}(q) \subset \mathrm{Ker}(\pi)\} = M(E, F)$, i.e.

$$E = \{z \in Z_{\mathrm{reg}} \mid \mathrm{Ker}(q) \subset \mathrm{Ker}(\theta_z)\}, \quad F = \{z \in Z_{\mathbb{R}} \mid \mathrm{Ker}(q) \subset \mathrm{Ker}(\phi_z)\}$$

The C^*-algebra B is non-commutative if and only if it has an irreducible representation of dimension > 1, if and only if the corresponding E is non-empty. □

Remark 4.8. It follows from the above considerations that plenty of intermediate quantum subspaces $S^{n-1}_{\mathbb{R}} \subset X \subset S^{n-1}_{\mathbb{R},*}$ exist for $n \geq 2$. Indeed, for any $m \geq 1$, there exists $S^{n-1}_{\mathbb{R}} \subset X \subset S^{n-1}_{\mathbb{R},*}$ such that $C(X)$ has precisely m isomorphism classes of irreducible representations of dimension 2. This follows from the fact that the orbit space $S^{n-1}_{\mathbb{C},\mathrm{reg}}/(\mathbb{T} \rtimes \mathbb{Z}_2)$ is infinite, and hence if we pick $E \subset S^{n-1}_{\mathbb{C},\mathrm{reg}}$ (closed and $\mathbb{T} \rtimes \mathbb{Z}_2$-stable) such that $E/(\mathbb{T} \rtimes \mathbb{Z}_2)$ has m elements, the quantum space corresponding to the pair $(E, S^{n-1}_{\mathbb{R}})$ satisfies to the above property.

This is in contrast with the situation for quantum groups, where there are no intermediate quantum subgroup $O_n \subset G \subset O_n^*$ [3]. The absence of such intermediate quantum subgroups probably can be roughly explained by some strong rigidity arising from the associated algebraic structure (Lie and Hopf algebras) used in [3]. It certainly would be interesting and useful to find axioms on half-liberated spaces that would ensure that the "no intermediate subobjects" situation in the quantum group case still holds.

4.3. **Symmetric subspaces.** It is now natural to wonder about the link between the description of the symmetric subspaces of $S^{n-1}_{\mathbb{R},*}$ in Section 3 and the one that should arise from Theorem 4.4. We first note that there exists also a notion of symmetric subspace in the general framework. Indeed, define an automorphism σ of $C(Z_{\mathbb{R},*})$ by $\sigma(f_0 \otimes 1 + f_1 \otimes \tau) = f_0 \otimes 1 - f_1 \otimes \tau$. We say that a subspace $X \subset Z_{\mathbb{R},*}$ is symmetric if $\sigma(X) = X$, or in other words, if σ induces an automorphism on the corresponding quotient of $C(Z_{\mathbb{R},*})$. When $Z = S^{n-1}_{\mathbb{C}}$, the automorphism σ is the sign automorphism ν of Section 3.

Lemma 4.9. *Let $X \subset Z_{\mathbb{R},*}$ be a quantum subspace. Then X is symmetric if and only if in the corresponding pair (E, F) of Theorem 4.4, we have $F = -F$.*

Proof. We retain the notation of Proposition 4.5. It is straightforward to check that for $z \in Z_{\mathrm{reg}}$, we have $\theta_z \sigma = \theta_{-z} \simeq \theta_z$ and that for $z \in Z_{\mathbb{R}}$, we have $\phi_z \sigma = \phi_{-z}$.

Now if $X \subset Z_{\mathbb{R},*}$ is a quantum subspace, then saying that X is symmetric precisely means that for the corresponding pair (E, F), we have that the corresponding ideal $\cap_{\pi \in M(E,F)} \mathrm{Ker}(\pi)$ is σ-stable. We have

$$\begin{aligned}\sigma(\cap_{\pi \in M(E,F)} \mathrm{Ker}(\pi)) &= \sigma((\cap_{z \in E} \mathrm{Ker}(\theta_z)) \cap \sigma(\cap_{z \in F} \mathrm{Ker}(\phi_z)) \\ &= \big(\cap_{z \in E} \mathrm{Ker}(\theta_z \sigma^{-1})\big) \cap \big(\cap_{z \in E} \mathrm{Ker}(\phi_z \sigma^{-1})\big) = (\cap_{z \in E} \mathrm{Ker}(\theta_z)) \cap (\cap_{z \in F} \mathrm{Ker}(\phi_{-z})) \\ &= \cap_{\pi \in M(E,-F)} \mathrm{Ker}(\pi)\end{aligned}$$

Hence X is symmetric if and only if $F = -F$. $\square$

Theorem 4.10. *There exists a bijection*

$$\{\text{symmetric quantum subspaces } X \subset Z_{\mathbb{R},*}\} \longleftrightarrow \{\mathbb{T} \rtimes \mathbb{Z}_2\text{−stable closed subspaces } Y \subset Z\}$$

Moreover the subspace $X \subset Z_{\mathbb{R},*}$ is non-classical if and only if, for the corresponding $Y \subset Z$, we have $Y_{\mathrm{reg}} = Y \cap Z_{\mathrm{reg}} \neq \emptyset$.

Proof. To a pair (E, F) as in Theorem 4.4 and satisfying $F = -F$, we associate the closed $\mathbb{T} \rtimes \mathbb{Z}_2$-stable subset $Y = E \cup \mathbb{T}F$. Conversely, if Y is $\mathbb{T} \rtimes \mathbb{Z}_2$-stable, then $(Y_{\mathrm{reg}}, Y_{\mathbb{R}})$ is a pair as above. The two maps are inverse bijections (in particular because $(E \cup \mathbb{T}F)_{\mathbb{R}} = \pm F$). $\square$

Of course the above Theorem reproves the first part of Corollary 3.8, since closed $\mathbb{T} \rtimes \mathbb{Z}_2$-stable subspaces of $S^{n-1}_{\mathbb{C}}$ correspond to conjugation stable closed subspaces of $P^{n-1}_{\mathbb{C}}$.

To finish, we provide another explicit description of the bijection in the proof of Theorem 4.10. If $Y \subset Z$ is a closed $\mathbb{T} \rtimes \mathbb{Z}_2$−stable subspace, then we may form the C^*-algebra $C(Y_{\mathbb{R},*})$ as before, and the restriction of functions yields a $*$-algebra map $C(Z_{\mathbb{R},*}) \to C(Y_{\mathbb{R},*})$, which is easily seen to be surjective thanks to the standard extension theorems. We thus get a quantum subspace $Y_{\mathbb{R},*} \subset Z_{\mathbb{R},*}$.

Theorem 4.11. *The map*

$$\begin{aligned}\{\mathbb{T} \rtimes \mathbb{Z}_2\text{−stable closed subsets } Y \subset Z\} &\longrightarrow \{\text{symmetric quantum subspaces } X \subset Z_{\mathbb{R},*}\} \\ Y &\longmapsto Y_{\mathbb{R},*}\end{aligned}$$

is a bijection.

Proof. Let $q : C(Z_{\mathbb{R},*}) \to C(Y_{\mathbb{R},*})$ the surjective $*$-algebra map associated to the restriction of functions. It is a direct verification to check that

$$\mathrm{Ker}(q) = (\cap_{y \in Y_{\mathrm{reg}}} \mathrm{Ker}(\theta_y)) \cap (\cap_{y \in Y_{\mathbb{R}}} \mathrm{Ker}(\phi_y)) = \bigcap_{\pi \in M(Y_{\mathrm{reg}}, Y_{\mathbb{R}})} \mathrm{Ker}(\pi)$$

Hence, in terms of pairs (E, F) corresponding to closed subsets of $\widehat{C(Z_{\mathbb{R},*})}$, our map associates the pair $(Y_{\mathrm{reg}}, Y_{\mathbb{R}})$ to Y. This map is, as already discussed, a bijection. $\square$

As a last comment to close the paper, we would like to mention that it is possible to get the result for the description of the quantum subgroups of O_n^* from [7] using Theorem 4.4, with some more work, i.e. studying when the ideal corresponding to a pair (E, F) is a Hopf ideal. As pointed out in the introduction, that was the original approach of Podleś [9] in the determination of the quantum subgroups of $SU_{-1}(2) \simeq O_2^*$.

References

[1] T. Banica, Liberations and twists of real and complex spheres, *J. Geom. Phys.* **96** (2015), 1-25.

[2] T. Banica, Half-liberated manifolds, and their quantum isometries, arXiv:1505.00646.

[3] T. Banica, J. Bichon, B. Collins, S. Curran, A maximality result for orthogonal quantum groups, *Comm. Algebra* **41** (2013), no. 2, 656-665.

[4] T. Banica, D. Goswami, Quantum isometries and noncommutative spheres, *Comm. Math. Phys.* **298** (2010), no. 2, 343-356.

[5] T. Banica, R. Speicher, Liberation of orthogonal Lie groups, *Adv. Math.* **222** (2009), no.4, 1461-1501.

[6] T. Banica, R. Vergnioux, Invariants of the half-liberated orthogonal group, *Ann. Inst. Fourier* **60** (2010), no. 6, 2137-2164.

[7] J. Bichon, M. Dubois-Violette, Half-commutative orthogonal Hopf algebras, *Pacific J. Math.* **263** (2013), no. 1, 13-28.

[8] J. Dixmier, C^*-algebras, North-Holland Mathematical Library 15. North-Holland Publishing Co. (1977).

[9] P. Podleś, Symmetries of quantum spaces. Subgroups and quotient spaces of quantum $SU(2)$ and $SO(3)$ groups, *Comm. Math. Phys.* **170** (1995), no. 1, 1-20.

[10] D. P. Williams, Crossed products of C^*-algebras. Mathematical Surveys and Monographs, 134. American Mathematical Society, Providence, RI, 2007.

[11] S.L. Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups, *Invent. Math.* **93** (1988), no. 1, 35-76.

Laboratoire de Mathématiques, Université Blaise Pascal, Campus des cézeaux, 3 place Vasarely, 63178 Aubière Cedex, France

E-mail address: Julien.Bichon@math.univ-bpclermont.fr