SPECTRAL MEASURE BLOWUP FOR BASIC HADAMARD SUBFACTORS

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Abstract. We study the subfactor invariants of the deformed Fourier matrices $H = F_M \otimes_Q F_N$, when the parameter matrix $Q \in M_{M \times N}(\mathbb{T})$ is generic. In general, the associated spectral measure decomposes as $\mu = (1 - \frac{1}{N})\delta_0 + \frac{1}{N}\text{law}(A)$, where $A \in C(\mathbb{T}^{MN}, M_M(\mathbb{C}))$ is a certain random matrix ($A(q)$ is the Gram matrix of the rows of $q$). In the case $M = 2$ we deduce from this a true “blowup” result, $\mu = (1 - \frac{1}{N})\delta_0 + \frac{1}{N}\Phi^*\varepsilon$, where $\varepsilon$ is the uniform measure on $\mathbb{Z}_2 \times \mathbb{T}^N$, and $\Phi(e, a) = N + e|\sum_i a_i|$. We conjecture that such decomposition results should hold as well in the non-generic case.

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Introduction

A complex Hadamard matrix is a matrix $H \in M_N(\mathbb{C})$ whose entries are on the unit circle, $|H_{ij}| = 1$, and whose rows are pairwise orthogonal. See [19]. The basic example is the Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$:

$F_N = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & w & w^2 & \ldots & w^{N-1} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
1 & w^{N-1} & w^{2(N-1)} & \ldots & w^{(N-1)^2}
\end{pmatrix}$

Popa discovered in [16] that these matrices parametrize the pairs of orthogonal MASA in the simplest von Neumann algebra, $M_N(\mathbb{C})$. As a consequence, each Hadamard matrix

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$H \in M_N(\mathbb{C})$ produces an irreducible subfactor $M \subset R$ of the Murray-von Neumann hyperfinite factor $R$, having index $[R : M] = N$. The computation of the standard invariant of this subfactor is a key problem. As explained by Jones in [13], the associated planar algebra $P_k = M' \cap M_k$ has a direct combinatorial description in terms of $H$, and the main problem is that of computing the corresponding Poincaré series:

$$f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$$

An alternative approach to these questions is via quantum groups [21], [22]. The idea is that associated to $H \in M_N(\mathbb{C})$ is a certain quantum permutation group $G \subset S_N^+$, and the problem is to compute the spectral measure $\mu \in \mathcal{P}(\mathbb{R}_+)$ of the main character $\chi : G \to \mathbb{C}$. This is the same problem as above, because $f$ is the Stieltjes transform of $\mu$:

$$f(z) = \int_G \frac{1}{1 - z\chi}$$

We refer to [1], [3] for a discussion of these questions, from the quantum permutation group viewpoint, and to [10], [13] for subfactor background on the problem.

Let us summarize the above considerations as:

**Definition.** Associated to a complex Hadamard matrix $H \in M_N(\mathbb{C})$ are a planar algebra $P = (P_k)$ and a quantum group $G \subset S_N^+$. The quantum invariants of $H$ are:

1. The Poincaré series $f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$.
2. The spectral measure $\mu \in \mathcal{P}(\mathbb{R}_+)$ of the main character $\chi : G \to \mathbb{C}$.

These invariants are related by the fact that $f$ is the Stieltjes transform of $\mu$.

In our previous paper [3] we studied these invariants for an important class of complex Hadamard matrices, constructed by Dită in [9]. Given two complex Hadamard matrices $H \in M_M(\mathbb{T})$, $K \in M_N(\mathbb{T})$, the Dită deformation of their tensor product $(H \otimes K)_{ia,jb} = H_{ij}K_{ab}$, with matrix of parameters $Q \in M_{M \times N}(\mathbb{T})$, is by definition $H \otimes_Q K = (Q_{ia}H_{ij}K_{ab})_{ia,jb}$. In the Fourier matrix case, $H = F_M$, $K = F_N$, Burstein proved in [8] that the corresponding subfactors are of Bisch-Haagerup type [7]. Our main result in [3] is the fact that, for $F_M \otimes_Q F_N$ with $Q \in M_{M \times N}(\mathbb{T})$ generic, the associated quantum algebra is a crossed coproduct, $C(G) = C^*(\Gamma_{M,N}) \rtimes C(\mathbb{Z}_M)$, where:

$$\Gamma_{M,N} = \mathbb{Z}^{(M-1)(N-1)} \rtimes \mathbb{Z}_N$$

As pointed out in [3], the above result shows that, in the generic case, the quantum invariants of $F_M \otimes_Q F_N$ are related to the random walks on $\Gamma_{M,N}$. We will investigate here these random walks, by taking some inspiration from the case $N = 2$, where $\Gamma_{M,N}$ is related to the half-liberation procedure in [5], [6], to the counting problem for abelian squares [17] and (at $M = 3$) to the random walks on the honeycomb lattice [20].

We will obtain in this way several concrete results regarding the quantum invariants of $F_M \otimes_Q F_N$, including a purely combinatorial formula for the moments of $\mu$. In the
particular cases $M = 2$ and $N = 2$, which are quite special, we will obtain some improved results. Note that these two particular cases fully cover the index 6 case.

Our main results, which are rather of geometric nature, are as follows:

**Theorem.** Consider the matrix $H = F_M \otimes_Q F_N$, with $Q \in M_{M \times N}(\mathbb{T})$ generic.

1. In general, $\mu = (1 - \frac{1}{N})\delta_0 + \frac{1}{N}\text{law}(A)$, where $A \in C(\mathbb{T}^{MN}, M_M(\mathbb{C}))$ is a certain random matrix $(A(q))$ is the Gram matrix of the rows of $q$).

2. At $M = 2$ we have $\mu = (1 - \frac{1}{N})\delta_0 + \frac{1}{N}\Phi_\ast \varepsilon$, where $\varepsilon$ is the uniform measure on $\mathbb{Z}_2 \times \mathbb{T}^N$, and $\Phi(e, a) = N + e|\sum_i a_i|$.

These formulae are part of the “spectral measure blowup” philosophy, a potentially powerful subfactor concept, that is still in its pioneering stages, however. More precisely, the origins of writings of this type go back to [4], where it was discovered that the spectral measures of the ADE graphs have very simple formulae, when blown up on the circle $\mathbb{T}$. This finding, while elementary and “experimental”, was however inspired by some quite heavy results, namely those of Jones regarding the theta series in [12]. Further details regarding the subfactor meaning of this “circular blowup” phenomenon for ADE graphs came later on, from the work of Evans and Pugh in [11]. Let us also mention that this result was used in [2], for the ADE classification of the quantum subgroups of $S_4^+$. Now back to the above theorem, this is certainly a new, interesting input for this “spectral measure blowup” theory, because the formula there is no longer a small index one, but takes now place in index $MN$, with $M, N \in \mathbb{N}$ arbitrary. There are of course many interesting questions here. Generally speaking, the main question is that of deciding if, given a subfactor $M \subset R$, the associated spectral measure $\mu$, having as moments the numbers $c_k = \dim(P_k)$, has or not a nice blowup on a suitable manifold.

More concretely now, back to the Hadamard matrices, we expect the “blowup” to appear as well in the non-generic case, our conjecture here being as follows:

**Conjecture.** For $H = F_M \otimes_Q F_N$, with $Q \in M_{M \times N}(\mathbb{T})$ arbitrary, the spectral measure blowup should land into a certain arithmetic lattice, canonically associated to $Q$.

This is probably a quite difficult question. The evidence here comes from our previous paper [3], where, using tools from [2], [15], we solved the problem at $M = N = 2$. More precisely, with $Q = (a_{ij})$, $q = ad/bc$ and $n = ord(q^4)$, the formula is as follows, where $\varepsilon_k$ is the uniform measure on the $k$-roots of unity, and $\Phi(q) = (q + q^{-1})^2$:

$$\mu = \frac{1}{2}(\delta_0 + \Phi_\ast \varepsilon_{4n})$$

Finally, we will discuss several questions in relation with the work in [1], [14], [18]. The paper is organized as follows: 1 is a preliminary section, in 2 we discuss representation theory aspects, in 3-4 we work out the combinatorial formula for the invariants, and its geometric interpretation, and in 5 we discuss the case $M = 2$. 
1. Deformed Fourier matrices

A complex Hadamard matrix is a matrix $H \in M_N(\mathbb{C})$ whose entries are on the unit circle, $|H_{ij}| = 1$, and whose rows are pairwise orthogonal. The basic example is the Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$. For examples and motivations, see [19].

Our starting point is the following observation. Let $H_1, \ldots, H_N \in \mathbb{T}^N$ be the rows of $H$, and consider the vectors $H_i/H_j \in \mathbb{T}^N$. We have then:

$$\langle H_i, H_j \rangle = \sum_l H_{il} H_{jl} = \sum_l H_{kl} H_{jl} = \langle H_k, H_j \rangle = \delta_{jk}$$

Similarly, $\langle H_i, H_j \rangle = \delta_{ik}$, so the matrix of rank one projections $P_{ij} = \text{Proj}(H_i H_j)$ is “magic”, in the sense that its entries sum up to 1 on each row and column.

Recall now that $C(S_N^+)$ is the universal $C^*$-algebra generated by the entries of a $N \times N$ magic matrix $u$, with comultiplication $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$, counit $\varepsilon(u_{ij}) = \delta_{ij}$ and antipode $S(u_{ij}) = u_{ji}$. This algebra satisfies the axioms of Woronowicz in [22], and so $S_N^+$ is a compact quantum group, called quantum permutation group. See Wang [21].

**Definition 1.1.** Associated to $H \in M_N(\mathbb{C})$ is the representation $\pi_H(u_{ij}) = \text{Proj}(H_i H_j)$, then is the minimal quantum group $G \subset S_N^+$ producing a factorization of type

$$C(S_N^+) \xrightarrow{\pi_H} M_N(\mathbb{C}) \xrightarrow{C(G)} C(G)$$

and finally is the spectral measure $\mu \in \mathcal{P}(\mathbb{R}_+)$ of the main character $\chi \in C(G)$.

Here we have used the fact, due to Woronowicz [22], that any compact quantum group, and in particular the above quantum group $G$, has a Haar functional. Thus we can indeed speak about the spectral measure of $\chi = \sum_i u_{ii}$, which is by definition given by:

$$\int_\mathbb{R} \varphi(x) d\mu(x) = \int_G \varphi(\chi)$$

The computation of $\mu$ is in general a difficult question, one basic result here being the fact that for $H = F_N$ we have $G = \mathbb{Z}_N$, and so $\mu = (1 - \frac{1}{N})\delta_0 + \frac{1}{N}\delta_N$. We will be interested in what follows in certain affine deformations of $F_N$, constructed by Ditâ in [9]. These deformations appear in the tensor product context, as follows:

**Definition 1.2.** Given two Hadamard matrices $H \in M_M(\mathbb{T}), K \in M_N(\mathbb{T})$, the Ditâ deformation of $(H \otimes K)_{ia,jb} = H_{ij} K_{ab}$, with matrix of parameters $Q \in M_{M \times N}(\mathbb{T})$, is:

$$H \otimes_Q K = (Q_{ia,jb})_{ia,jb}$$

We call a dephased matrix $Q \in M_{M \times N}(\mathbb{T})$ “generic” if its entries are as algebraically independent as possible, in the sense that $\prod_{i,j \geq 1} Q_{ij}^{r_{ij}} = 1$ with $r_{ij} \in \mathbb{Z}$ implies $r_{ij} = 0$. 
Here, and in what follows, we use indices \( i \in \{0, 1, \ldots, M - 1\} \) and \( j \in \{0, 1, \ldots, N - 1\} \), usually taken modulo \( M, N \), as is the most convenient in the Fourier matrix setting.

We can now recall the main result in our previous paper \([3]\). Recall that if \( H \curvearrowright \Gamma \) is a finite group acting by automorphisms on a discrete group, the corresponding crossed product Hopf algebra is \( C^*(\Gamma) \rtimes C(H) = C^*(\Gamma) \otimes C(H) \), with comultiplication given by \( \Delta(r \otimes \delta_k) = \sum_{h \in H} (r \otimes \delta_h) \otimes (h^{-1} \cdot r \otimes \delta_{h^{-1}k}) \), for \( r \in \Gamma, k \in H \).

**Theorem 1.3.** When the parameter matrix \( Q \in M_{M \times N}(\mathbb{T}) \) is generic, the quantum group \( G \subset S^+_{MN} \) associated to the matrix \( H = F_M \otimes_Q F_N \) is given by

\[
C(G) = C^*(\Gamma_{M,N}) \rtimes C(\mathbb{Z}_M)
\]

where \( \Gamma_{M,N} = \langle g_0, \ldots, g_{M-1}\rangle g_0^N = \ldots = g_{M-1}^N = 1, [g_i, \ldots g_i, g_j, \ldots g_{jn}] = 1 \rangle > \), and where the action of \( \mathbb{Z}_M \) on \( \Gamma_{M,N} \) is by cyclic permutation of the generators.

**Proof.** The idea is that for any \( Q \in M_{M \times N}(\mathbb{T}) \) the representation \( u_{ia,jb} \rightarrow P_{ia,jb} \) associated to \( F_M \otimes_Q F_N \) factorizes as follows, where \( w = e^{2\pi i/N} \):

\[
C(S^+_{MN}) \rightarrow C^*(\Gamma_{M,N}) \otimes C(\mathbb{Z}_M) \rightarrow M_{MN}(\mathbb{C})
\]

where \( \frac{1}{N} \sum_c w^{(b-a)c}g_c^i \otimes v_{ij} \rightarrow P_{ia,jb} \)

The point now is that when \( Q \) is generic this factorization is minimal. See \([3]\). \( \square \)

A precise description of the group \( \Gamma_{M,N} \) goes as follows. Given a group acting on another group, \( H \curvearrowright G \), as usual \( G \rtimes H \) denotes the semi-direct product of \( G \) by \( H \), i.e. the set \( G \times H \), with multiplication \( (a,s)(b,t) = (as(b), st) \). Now given a group \( G \), we can make \( Z_N \) act cyclically on \( G_N \), and form the product \( G_N \rtimes Z_N \). Since the elements \( (g, \ldots, g) \) are invariant, we can form as well the product \( (G_N/G) \rtimes Z_N \), and by identifying \( G_N/G \simeq G^{N-1} \) via \((1, g_1, \ldots, g_{N-1}) \rightarrow (g_1, \ldots, g_{N-1}) \) we obtain a product \( G^{N-1} \rtimes Z_N \).

We have the following result, basically from \([3]\):

**Proposition 1.4.** The group \( \Gamma_{M,N} \) has the following properties:

1. \( \Gamma_{M,N} \simeq \mathbb{Z}^{(M-1)(N-1)} \rtimes \mathbb{Z}_N \) via \( g_0 \rightarrow (0, t) \) and \( g_i \rightarrow (a_{i0}, t) \) for \( 1 \leq i \leq M - 1 \), where \( a_{ic} \) and \( t \) are the standard generators of \( \mathbb{Z}^{(M-1)(N-1)} \) and \( \mathbb{Z}_N \).
2. \( \Gamma_{M,N} \subset \mathbb{Z}^{(M-1)N} \rtimes \mathbb{Z}_N \) via \( g_0 \rightarrow (0, t) \) and \( g_i \rightarrow (b_{i0} - b_{i1}, t) \) for \( 1 \leq i \leq M - 1 \), where \( b_{ic} \) and \( t \) are the standard generators of \( \mathbb{Z}^{(M-1)N} \) and \( \mathbb{Z}_N \).

**Proof.** Here (1) is a standard result, explained in detail in \([3]\), and (2) follows from it, after some standard manipulations. \( \square \)

Now let us go back to Theorem 1.3. At \( M = 1 \) we have \( \Gamma_{1,N} = \mathbb{Z}_N \), so \( G = \mathbb{Z}_N \). Also, at \( N = 1 \) we have \( \Gamma_{M,1} = \{1\} \), so \( G = \mathbb{Z}_M \). At \( M = N = 2 \) now, we have \( \Gamma_{2,2} = D_\infty \), and this leads to the quantum group \( O_{2}^{-1} \) from \([2]\). In fact, we have:
Proposition 1.5. The quantum group associated to $F_2 \otimes Q F_2$, with $Q = (a, b, c, d)$, is

$$G = \begin{cases} 
\mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } n = 1 \text{ and } m = 1, 2 \text{ or } m = 4 \\
D_{2n}^{-1} \cdot DC_n^{-1} & \text{if } 1 < n < \infty \text{ and } m \notin 4\mathbb{N} \text{ or } m \in 4\mathbb{N} \\
O_2^{-1} & \text{if } n = \infty 
\end{cases}$$

where $m = \text{ord}(q), n = \text{ord}(q^4)$, with $q = ad/bc$.

Proof. As already explained, the $n = \infty$ assertion follows from Theorem 1.3. Regarding now the $n < \infty$ case, the idea here is that the machinery in [2] leads to the quantum groups $D_{2n}^{-1} \cdot DC_n^{-1}$ constructed by Nikshych in [15], as stated. See [3]. □

As a concrete consequence of Proposition 1.5, we have:

Proposition 1.6. For $F_2 \otimes Q F_2$ with $Q = (a, b, c, d)$, with $q = ad/bc$ and $n = \text{ord}(q^4)$,

$$\mu = \frac{1}{2}(\delta_0 + \Phi \varepsilon_{4n})$$

where $\varepsilon_k$ is the uniform measure on the $k$-roots of unity, and $\Phi(q) = (q + q^{-1})^2$.

Proof. The idea here is that Proposition 1.5 and the ADE tables in [2] show that the associated principal graph is $\tilde{D}_{2n+2}$. On the other hand, according to [2], for $\tilde{D}_{m+2}$ with $m \in \{2, 3, \ldots, \infty\}$ we have $\mu = \frac{1}{2}(\delta_0 + \Phi \varepsilon_{2m})$, and this gives the result. See [3]. □

At $n = \infty$ the corresponding measure $\varepsilon_{\infty}$ is of course the uniform measure on $\mathbb{T}$. Finding an analogue of this result for higher $M, N$ will be our main purpose here.

2. Representation theory

We discuss in this section the representation theory of the quantum group $G$ associated to the matrix $H = F_M \otimes_Q F_N$, in the case where $Q \in M_{M \times N}(\mathbb{T})$ is generic.

Let us first discuss the case of arbitrary crossed coproducts. If $G$ is a quantum group, we denote by $\text{Irr}(G)$ the set of isomorphism classes of irreducible representations of $G$, and by $\text{Irr}_1(G)$ the group of 1-dimensional representations of $G$. We have:

Lemma 2.1. Let $H \curvearrowright \Gamma$ be a finite group acting on a discrete group, and let $G$ be the quantum group given by $C(G) = C^*(\Gamma) \rtimes C(H)$. Then there is a group isomorphism

$$\text{Irr}_1(G) \simeq \Gamma^H \times \text{Irr}_1(H)$$

where $\Gamma^H$ denotes the subgroup of $H$-fixed points in $\Gamma$.

Proof. The elements of $\text{Irr}_1(G)$ correspond to group-like elements in $\mathbb{C}\Gamma \rtimes C(H)$. For $r \in H^\Gamma$ and $\phi$ a 1-dimensional representation of $H$, it is straightforward to check that $r \otimes \phi$ is a group-like element in $\mathbb{C}\Gamma \rtimes C(H)$, and that conversely any group-like element arises in that way, and this proves the assertion. □

When $H$ acts freely on $\Gamma = \{1\}$, we have a more precise result, as follows:
Lemma 2.2. Let $H \curvearrowright \Gamma$ and $C(G) = C^{*}(\Gamma) \times C(H)$ be as above, and assume that $H$ acts freely on $\Gamma - \{1\}$. Then there exists a bijection

$$\text{Irr}(G) \simeq \text{Irr}(H) \times (\Gamma - \{1\})/H$$

where $(\Gamma - \{1\})/H$ is the orbit space for the action of $H$ on $\Gamma - \{1\}$, and the irreducible representations corresponding to elements of $(\Gamma - \{1\})/H$ have dimension $|H|$.

Proof. For $r \in H$, consider the following space:

$$C_r = \text{Span} \left( h^{-1} \cdot r \otimes \delta_{h^{-1}k}, \ h, k \in H \right) = \text{Span} \left( h \cdot r \otimes \delta_k, \ h, k \in H \right)$$

The subspace $C_r$ is a subcoalgebra with $C_1 = C(H)$, and by the freeness assumption, for $r \neq 1$ we have $\dim(C_r) = |H|^2$. If $(r_i)_{i \in I}$ denotes a set of representative of the $H$-orbits in $\Gamma^*$, then we have a direct sum decomposition, as follows:

$$\mathbb{C} \Gamma \rtimes C^{*}(H) = C(H) \oplus (\bigoplus_{i \in I} C_{r_i})$$

For $r \neq 1$, the coalgebra $C_r$ is the coefficient coalgebra of the irreducible corepresentation $V_r = \text{Span}(r \otimes \delta_h, \ h \in H)$ ($V_r$ is indeed irreducible since $\dim(C_r) = \dim(V_r)^2$). Thus from the Peter-Weyl decomposition of $C(H)$ we get the Peter-Weyl decomposition of $C^{*}(\Gamma) \rtimes C(H)$, and this provides the full description of $\text{Irr}(G)$, as in the statement. \qed

We can now state and prove our main result in this section:

Theorem 2.3. Let $G$ be the quantum group associated to $F_M \otimes Q F_N$. Then we have a group isomorphism $\text{Irr}_1(G) \simeq \mathbb{Z}_M$, and if $M$ is prime, there exists an explicit bijection

$$\text{Irr}(G) \simeq \mathbb{Z}_M \rtimes (\Gamma_{M,N} - \{1\})/\mathbb{Z}_M$$

where the elements of $\mathbb{Z}_M$ correspond to 1-dimensional representations, and the elements of $(\Gamma_{M,N} - \{1\})/\mathbb{Z}_M$ correspond to $M$-dimensional irreducible representations.

Proof. We have to prove that $(\Gamma_{M,N})^{\mathbb{Z}_M} = \{1\}$. By Lemma 2.1 and Lemma 2.2 this will prove indeed the first assertion, as well as the second one (because when $M$ is prime, $\mathbb{Z}_M$ has no proper subgroup and hence will act freely on $\Gamma_{M,N} - \{1\}$).

We use a variation on the description of $\Gamma_{M,N}$ given in Proposition 1.4. We identify $\mathbb{Z}^{(M-1)(N-1)}$ with the quotient of $\mathbb{Z}^{MN}$, the free (multiplicative) abelian group on generators $a_{ic}$, $0 \leq i \leq M - 1$, $0 \leq c \leq N - 1$, by the following relations:

$$\prod_{i=0}^{N-1} a_{ic} = 1, \ 0 \leq i \leq M - 1, \ a_{0c} = 1, \ 0 \leq c \leq N - 1$$

The resulting quotient is indeed freely generated by the following elements:

$$a_{ic}, \ 1 \leq i \leq M - 1, \ 0 \leq c \leq N - 2$$
We have an action of $\mathbb{Z}_N = \langle t \rangle$ on $\mathbb{Z}^{(M-1)(N-1)}$, given by $t(a_{ic}) = a_{i,c+1}$. We therefore get a group identification, as follows:

$$\Gamma_{M,N} \rightarrow \mathbb{Z}^{(M-1)(N-1)} \rtimes \mathbb{Z}_N$$

$$g_i \mapsto (a_{i0}, t)$$

Under this identification, the action of $\mathbb{Z}_M = \langle s \rangle$ on the generators $g_i$ corresponds to:

$$s(a_{ic}) = a_{i+1,c}^{-1}, \ s(t) = a_{10}t$$

We have to show that for $w \in \Gamma_{M,N}$, then $s(w) = w \Rightarrow w = 1$. First assume that $w \in \mathbb{Z}^{(M-1)(N-1)}$, $w = \prod_{i=1}^{M-1} \prod_{c=0}^{N-2} a_{ic}^{\lambda_{ic}}, \lambda_{ic} \in \mathbb{Z}$. Then:

$$s(w) = \prod_{i=1}^{M-1} \prod_{c=0}^{M-2} a_{i+1,c}^{\lambda_{ic}}a_{1c}^{-\lambda_{ic}}$$

Thus if $s(w) = w$ then $\lambda_{1,c} = \lambda_{2,c} = \cdots = \lambda_{M-1,c} = -\sum_{i=1}^{M-1} \lambda_{ic}$, for any $c$, so that $w = 1$. Now if $w = ut^l$ with $u = \prod_{i=1}^{M-1} \prod_{c=0}^{M-2} a_{ic}^{\lambda_{ic}} \in \Gamma_{M,N}$ and $1 \leq l \leq N - 1$, we have:

$$s(w) = \prod_{i=1}^{M-1} \prod_{c=0}^{M-2} a_{i+1,c}^{\lambda_{ic}}a_{1c}^{-\lambda_{ic}}a_{10}a_{11} \cdots a_{1,l-1}t^l$$

From this we see that $s(w) \neq w$, which concludes the proof.

3. Invariants, self-duality

We are interested in what follows in the computation of the spectral measure of the main character $\chi \in C(G)$ of the quantum group in Theorem 1.3 above. We have here:

**Proposition 3.1.** For the matrix $F_M \otimes_Q F_N$, with $Q \in M_{M \times N}(\mathbb{T})$ generic, we have

$$\int_G \chi^k = \frac{1}{MN} \# \left\{ i_1, \ldots, i_k \in \mathbb{Z}_M \mid [(i_1, d_1), (i_2, d_2), \ldots, (i_k, d_k)] \right\}$$

where the sets between square brackets are by definition sets with repetition.

**Proof.** According to Theorem 1.3, the main character is given by:

$$\chi = \frac{1}{N} \sum_{iac} g_i^c \otimes v_{ii} = \sum_{ic} g_i^c \otimes v_{ii} = \left( \sum_{ic} g_i^c \right) \otimes \delta_1$$

Now since the Haar functional of $C^*(\Gamma) \rtimes C(H)$ is the tensor product of the Haar functionals of $C^*(\Gamma), C(H)$, this gives the following formula, valid for any $k \geq 1$:

$$\int_G \chi^k = \frac{1}{M} \int_{\hat{\Gamma}_{M,N}} \left( \sum_{ic} g_i^c \right)^k$$
Proposition 3.2. In the generic case, the quantum invariants of the matrices $F_M \otimes_Q F_N$ and $F_N \otimes_Q F_M$ coincide.

It is convenient to use the notation $\frac{1}{1-g_i} = \sum_c g_i^c$. The above formula becomes:

$$\int_G \chi^k = \frac{1}{M} \int_{\hat{\Gamma}_{M,N}} \left( \sum_i \frac{1}{1-g_i} \right)^k$$

We use now the embedding in Proposition 1.4 (2). With the notations there:

$$\frac{1}{1-g_i} = \sum_c (b_{i0} - b_{ic}, t^c)$$

Now by multiplying two such elements, we obtain:

$$\frac{1}{1-g_i} \cdot \frac{1}{1-g_j} = \sum_{cd} (b_{i0} - b_{ic} + b_{jc} - b_{j,c+d}, t^{c+d})$$

In general, the product formula for these elements is:

$$\frac{1}{1-g_{i_1}} \cdots \frac{1}{1-g_{i_k}} = \sum_{c_1 \ldots c_k} \left( \begin{array}{c} b_{i_10} - b_{i_1c_1} + b_{i_2c_1} - b_{i_2,c_1+c_2} \\ b_{i_3,c_1+c_2} - b_{i_3,c_1+c_2+c_3} + \\ \vdots \\ b_{i_k,c_1+\ldots+c_{k-1}} - b_{i_k,c_1+\ldots+c_k} \end{array} , t^{c_1+\ldots+c_k} \right)$$

In terms of the new indices $d_r = c_1 + \ldots + c_r$, this formula becomes:

$$\frac{1}{1-g_{i_1}} \cdots \frac{1}{1-g_{i_k}} = \sum_{d_1 \ldots d_k} \left( \begin{array}{c} b_{i_10} - b_{i_1d_1} + b_{i_2d_1} - b_{i_2d_2} \\ b_{i_3d_2} - b_{i_3d_3} + \\ \vdots \\ b_{i_kd_{k-1}} - b_{i_kd_k} \end{array} , t^{d_k} \right)$$

Now by integrating, we must have $d_k = 0$ on one hand, and on the other hand:

$$[(i_1, 0), (i_2, d_1), \ldots, (i_k, d_{k-1})] = [(i_1, d_1), (i_2, d_2), \ldots, (i_k, d_k)]$$

Equivalently, we must have $d_k = 0$ on one hand, and on the other hand:

$$[(i_1, d_k), (i_2, d_1), \ldots, (i_k, d_{k-1})] = [(i_1, d_1), (i_2, d_2), \ldots, (i_k, d_k)]$$

Thus, by cyclic invariance with respect to $d_k$, we obtain the following formula:

$$\int_{\hat{\Gamma}_{M,N}} \frac{1}{1-g_{i_1}} \cdots \frac{1}{1-g_{i_k}} = \frac{1}{N} \# \left\{ d_1, \ldots, d_k \in \mathbb{Z}_N \left| \left( i_1, d_1, i_2, d_2, \ldots, i_k, d_k \right) \right. \right\}$$

It follows that we have the following moment formula:

$$\int_{\hat{\Gamma}_{M,N}} \left( \sum_i \frac{1}{1-g_i} \right)^k = \frac{1}{N} \# \left\{ i_1, \ldots, i_k \in \mathbb{Z}_M \left| \left( i_1, d_1, i_2, d_2, \ldots, i_k, d_k \right) \right. \right\}$$

Now by dividing by $M$, we obtain the formula in the statement. \qed

A first interesting consequence of Proposition 3.1 is as follows:

**Proposition 3.2.** In the generic case, the quantum invariants of the matrices $F_M \otimes_Q F_N$ and $F_N \otimes_Q F_M$ coincide.
Proof. This follows indeed from Proposition 3.1, because by rotating the $i$ indices, we see that the formula there is symmetric in $M, N$. □

Now let us recall the following basic fact:

**Proposition 3.3.** We have an equivalence $(H \otimes Q)_{\mathbb{T}}K_{\mathbb{T}} \simeq K_{\mathbb{T}} \otimes Q_{\mathbb{T}} H_{\mathbb{T}}$, so in the Fourier matrix case we have an equivalence $(F_{M} \otimes Q_{N} F_{M})_{\mathbb{T}} \simeq (F_{N} \otimes Q_{M} F_{M})_{\mathbb{T}}$.

**Proof.** We have indeed the following computation, where the last manipulation is a Hadamard matrix equivalence, implemented by a certain permutation of the indices:

$$(H \otimes Q)_{\mathbb{T}}K_{\mathbb{T}} \simeq K_{\mathbb{T}} \otimes Q_{\mathbb{T}} H_{\mathbb{T}}$$

As for the second formula, this follows from the fact that $F_{M}, F_{N}$ are symmetric. □

Now by getting back to Proposition 3.2, we have:

**Theorem 3.4.** Let $H = F_{M} \otimes Q_{N}$, with $Q \in M_{M \times N}(\mathbb{C})$ generic, and with $M \neq N$.

1. The quantum groups associated to $H, H^{t}$ have different fusion rules.
2. However, the laws of the main characters of these quantum groups coincide.

**Proof.** Since having the same fusion rules implies having the same Irr$_{1}$ groups, (1) follows from Theorem 2.3. As for (2), this follows from Proposition 3.2 and Proposition 3.3. □

This result is probably quite interesting in connection with the investigations in [1], [14], [18], regarding the duality between the quantum groups associated to $H, H^{t}$.

## 4. Geometric interpretation

In this section we work out a geometric/random matrix interpretation of the formula in Proposition 3.1. First of all, the formula there can be reformulated as follows:

**Proposition 4.1.** For the matrix $F_{M} \otimes Q_{N}$, with $Q \in M_{M \times N}(\mathbb{T})$ generic, we have

$$\int_{G} \chi = \frac{1}{MN} \int_{\mathbb{T}^{MN}} \sum_{i_{1} \cdots i_{k}} \sum_{d_{1} \cdots d_{k}} q_{i_{1}i_{2} \cdots q_{i_{k}d_{k}} \cdots q_{i_{k}d_{k-1}}} dq$$

where the integral on the right is with respect to the uniform measure on $\mathbb{T}^{MN}$.

**Proof.** We use the fact that the polynomial integrals over a torus $\mathbb{T}^{n}$ are given by:

$$\int_{\mathbb{T}^{n}} q_{i_{1} \cdots q_{j_{k}}} dq = \delta_{[i_{1} \cdots i_{k}], [j_{1} \cdots j_{k}]}$$

Here $\delta$ is a Kronecker symbol, and the sets between square brackets at right are as usual sets with repetitions. Now with $n = MN$, we obtain the following formula:

$$\int_{\mathbb{T}^{NM}} q_{i_{1}d_{1} \cdots q_{i_{k}d_{k}}} dq = \delta_{[i_{1}d_{1} \cdots i_{k}d_{k}], [i_{1}d_{1} \cdots i_{k}d_{k-1}]}$$

Now since at right we have exactly the contributions to the integral computed in Proposition 3.1 above, this gives the result. □
A useful version of Proposition 4.1 comes by dephasing, as follows:

**Proposition 4.2.** For $F_M \otimes_Q F_N$, with $Q \in M_{M \times N}(\mathbb{T})$ generic, we have

$$
\int_G \chi^k = \frac{1}{MN} \int_{\mathbb{T}^{(M-1)(N-1)}} \sum_{i_1 \ldots i_k} \sum_{d_1 \ldots d_k} \tilde{r}_{i_1 d_1} \ldots \tilde{r}_{i_k d_k} \, dr
$$

where $\tilde{r} = \left( \begin{smallmatrix} 1 \ 1 \end{smallmatrix} \right)$ is the matrix obtained from $r$ by adding a row and column of 1 entries.

**Proof.** The dephasing is an identification $\mathbb{T} \times \mathbb{T}^{MN} \cong \mathbb{T}^M \times \mathbb{T}^N \times \mathbb{T}^{(M-1)(N-1)}$, given by $cq_{ij} = a_i b_j \tilde{r}_{ij}$. Now if we denote by $\Phi$ the quantity in Proposition 4.1, we have:

$$
\int \Phi(q) dq = \int_{\mathbb{T} \times \mathbb{T}^{MN}} \Phi(cq) d(c, q) = \int_{\mathbb{T}^M \times \mathbb{T}^N \times \mathbb{T}^{(M-1)(N-1)}} \Phi((a_i b_j \tilde{r}_{ij})_{ij}) d(a, b, r) = \int_{\mathbb{T}^{(M-1)(N-1)}} \Phi(\tilde{r}) dr
$$

But this gives the formula in the statement, and we are done. $\square$

Yet another useful reformulation of Proposition 4.1 is as follows:

**Proposition 4.3.** For $F_M \otimes_Q F_N$, with $Q \in M_{M \times N}(\mathbb{T})$ generic, we have

$$
\int_G \chi^k = \frac{1}{MN} \int_{\mathbb{T}^{(M-1)(N-1)}} \sum_{i_1 \ldots i_k} \sum_{d_1 \ldots d_k} q_{i_1 d_1} q_{i_2 d_2} \ldots q_{i_k d_k} \, dr
$$

where $q_{ij} = \frac{a_i d_j}{q_i q_j}$ are the variables appearing in the expansion of $\int_G \chi^2$.

**Proof.** We have indeed the following formula:

$$
\frac{q_{i_1 d_1} q_{i_2 d_2} \ldots q_{i_k d_k}}{q_{i_1 d_k} q_{i_2 d_1} \ldots q_{i_k d_{k-1}}} = q_{i_1 d_1} q_{i_2 d_2} q_{i_3 d_3} \ldots q_{i_{k-1} d_{k-1}} q_{i_k d_k}
$$

Now by summing over $i, d$ and integrating, this gives the formula in the statement. $\square$

Observe that Proposition 4.2 and Proposition 4.3 tell us that the moments of $\chi$ can be expressed in terms of the second moment variables $q_{ij} = \frac{a_i d_j}{q_i q_j}$, and that when defining these variables, we can in addition assume that $q = \tilde{r}$ is a dephased matrix. We will see in the next section that this observation applies particularly well at $M = 2$.

In the general case, Proposition 4.1 is best interpreted as follows:

**Proposition 4.4.** For $F_M \otimes_Q F_N$, with $Q \in M_{M \times N}(\mathbb{T})$ generic, we have

$$
\int_G \chi^k = \frac{1}{MN} \int_{\mathbb{T}^{MN}} \text{Tr}(A(q)^k) dq
$$

where $A(q)_{ij} = < R_i, R_j >$, where $R_1, \ldots, R_M$ are the rows of $q \in \mathbb{T}^{MN} \cong M_{M \times N}(\mathbb{T})$. 
Proof. We use the formula in Proposition 4.1. We have:

\[
\int_G \chi^k = \frac{1}{MN} \int_{\mathbb{T}^{MN}} \sum_{i_1 \ldots i_k} \left( \sum_{d_1} q_{i_1 d_1} \right) \left( \sum_{d_2} q_{i_2 d_2} \right) \ldots \left( \sum_{d_k} q_{i_k d_k} \right) dq
\]

\[
= \frac{1}{MN} \int_{\mathbb{T}^{MN}} < R_{i_1}, R_{i_2} > < R_{i_2}, R_{i_3} > \ldots < R_{i_k}, R_{i_1} >
\]

\[
= \frac{1}{MN} \int_{\mathbb{T}^{MN}} A(q)_{i_1 i_2} A(q)_{i_2 i_3} \ldots A(q)_{i_k i_1}
\]

But this gives the formula in the statement, and we are done. □

We can formulate now our first “spectral measure blowup” result, as follows:

**Theorem 4.5.** For \( F_M \otimes_Q F_N \), with \( Q \in M_{M \times N}(\mathbb{T}) \) generic, we have

\[
\text{law}(\chi) = \left( 1 - \frac{1}{N} \right) \delta_0 + \frac{1}{N} \text{law}(A)
\]

where \( A \in C(\mathbb{T}^{MN}, M_{M}(\mathbb{C})) \) is given by \( A(q) = \text{Gram matrix of the rows of } q \).

**Proof.** This follows from Proposition 4.4. Indeed, in terms of \( tr = \frac{1}{M} Tr \), we have:

\[
\int_G \chi^k = \frac{1}{N} \int_{\mathbb{T}^{MN}} tr(A(q)^k) dq
\]

But this tells us precisely that, up to a rescaling as in the statement, the random variable \( \chi \) follows the same law as the random variable \( A \). □

In general, the moments of the Gram matrix \( A \) are given by a quite complicated formula, and we cannot expect to have a refinement of Theorem 4.5, with \( A \) replaced by a plain (non-matricial) random variable, say over a compact abelian group. However, this kind of simplification does appear at \( M = 2 \), as we will see in the next section.

5. **Toral blowup at \( M=2 \)**

In this section we discuss in detail the cases \( M = 2 \) and \( N = 2 \), with the idea in mind of generalizing Proposition 1.6 at \( n = \infty \). As a first remark, at \( N = 2 \) we have:

**Proposition 5.1.** The groups \( \Gamma_{M,2} \) have the following properties:

1. \( \Gamma_{2,2} \cong D_\infty \).
2. \( \Gamma_{M,2} = \mathbb{Z}_2^M / < abc = cba > \).
3. \( \Gamma_{3,2} \) has the honeycomb lattice as Cayley graph.

**Proof.** All these results are quite well-known, the proof being as follows:

1. At \( M = 2 \) the commutation relations \([ab, cd] = 1\) are automatic, and so we obtain \( \Gamma_{2,2} = \mathbb{Z}_2 \ast \mathbb{Z}_2 \cong D_\infty \), the last isomorphism being well-known.
(2) We have to prove that the commutation relations \([ab, cd] = 1\) are equivalent to the half-commutation relations \(abc = cba\) from [5]. Indeed, assuming that \([ab, cd] = 1\) holds, we have \(ab \cdot ac = ac \cdot ab\), and so \(bac = cab\). Vice versa, assuming that \(abc = cba\) holds, we have \(ab \cdot cd = a \cdot bcd = a \cdot dcb = adc \cdot b = cda \cdot b = cd \cdot ab\).

(3) At \(M = 3\) our group is \(\Gamma_{3,2} = \mathbb{Z}^3_2 < abc >\), whose Cayley graph is the honeycomb lattice (one basic hexagon being \(1 - a - ac - acb - bc - b\)). \(\square\)

We should mention that the above result was extremely useful at some early stages of the present work, when we were exploring various small \(M, N\) situations.

Let us compute now the associated measures, at \(M = 2\). We first have:

**Lemma 5.2.** For \(F_2 \otimes Q F_N\), with \(Q \in M_{2 \times N}(\mathbb{T})\) generic, we have

\[
N \int_G \left( \frac{\chi}{N} \right)^k = \int_{T^N} \sum_{p \geq 0} \frac{k!}{(2p)!} \left| \sum_i a_i \right|^{2p} da
\]

where the integral on the right is with respect to the uniform measure on \(T^N\).

**Proof.** Consider the quantity that appears on the right in Proposition 4.1, namely:

\[
\Phi(q) = \sum_{i_1 \ldots i_k} \prod_{d_1 \ldots d_k} \frac{q_{i_1 d_1} \ldots q_{i_k d_k}}{q_{i_1 d_k} \ldots q_{i_k d_{k-1}}}
\]

As explained in Proposition 4.2 and its proof, we can dephase the matrix \(q \in M_{2 \times N}(\mathbb{T})\) if we want, so in particular we can “half-dephase” it, as follows:

\[
q = \begin{pmatrix} 1 & \ldots & 1 \\
1 & \ldots & a_N \end{pmatrix}
\]

We have to compute the above quantity \(\Phi(q)\) in terms of the numbers \(a_1, \ldots, a_N\). Our claim is that we have the following formula:

\[
\Phi(q) = 2 \sum_{p \geq 0} N^{k-2p} \binom{k}{2p} \left| \sum_i a_i \right|^{2p}
\]

Indeed, the \(2N^p\) contribution will come from \(i = (1 \ldots 1)\) and \(i = (2 \ldots 2)\), then we will have a \(k(k-1)N^{p-2}|\sum_i a_i|^2\) contribution coming from indices of type \(i = (2 \ldots 21 \ldots 1)\), up to cyclic permutations, then a \(2\binom{k}{4} N^{k-4}|\sum_i a_i|^4\) contribution coming from indices of type \(i = (2 \ldots 21 \ldots 12 \ldots 21 \ldots 1)\), and so on. More precisely, in order to find the \(N^{k-2p} |\sum_i a_i|^{2p}\) contribution, we have to count the circular configurations consisting of \(k\) numbers 1, 2, such that the 1 values are arranged into \(p\) non-empty intervals, and the 2 values are arranged into \(p\) non-empty intervals as well. Now by looking at the endpoints of these \(2p\) intervals, we have \(2\binom{k}{2p}\) choices, and this gives the above formula.

Now by integrating, this gives the formula in the statement. \(\square\)
Observe that the integrals in Lemma 5.2 can be computed as follows:

\[
\int_{T^N} |a_1 + \ldots + a_N|^{2p} da = \int_{T^N} \sum_{i_1 \ldots i_p} \sum_{j_1 \ldots j_p} a_{i_1} \ldots a_{i_p} a_{j_1} \ldots a_{j_p} da
\]

\[
= \# \left\{ i_1 \ldots i_p, j_1 \ldots j_p \mid [i_1, \ldots, i_p] = [j_1, \ldots, j_p] \right\}
\]

\[
= \sum_{p=\sum r_i} \left( \begin{array}{c} p \\ r_1, \ldots, r_N \end{array} \right)^2
\]

Regarding the combinatorics of these latter numbers, see [17], [20]. Let us go back now to the “blowup” problematics. First, we have:

**Proposition 5.3.** For \( F_2 \otimes_Q F_N \), with \( Q \in M_{2 \times N}(T) \) generic, we have

\[
\mu = \left( 1 - \frac{1}{N} \right) \delta_0 + \frac{1}{2N} \left( \Psi^+ \varepsilon + \Psi^- \varepsilon \right)
\]

where \( \varepsilon \) is the uniform measure on \( T^N \), and \( \Psi^\pm(a) = N \pm |\sum_i a_i| \).

**Proof.** We use the formula in Lemma 5.2, along with the following identity:

\[
\sum_{p=0}^\infty \left( \begin{array}{c} k \\ 2p \end{array} \right) x^{2p} = \frac{(1+x)^k + (1-x)^k}{2}
\]

By using this identity, Lemma 5.2 reformulates as follows:

\[
N \int_G \left( \frac{\chi}{N} \right)^k = \frac{1}{2} \int_{T^N} \left( 1 + \left| \sum_i a_i \right| \right)^k + \left( 1 - \left| \sum_i a_i \right| \right)^k da
\]

Now by multiplying by \( N^{k-1} \), we obtain the following formula:

\[
\int_G \chi^k = \frac{1}{2N} \int_{T^N} \left( N + \left| \sum_i a_i \right| \right)^k + \left( N - \left| \sum_i a_i \right| \right)^k da
\]

But this gives the formula in the statement, and we are done. \( \Box \)

As an example, at \( N = 1 \) the \( \delta_0 \) term dissapears, \( \varepsilon \) is the uniform measure on \( T \), and \( \Psi^\pm = 0, \Psi^\pm = 2 \). Thus we obtain indeed the spectral measure of \( \mathbb{Z}_2 \), \( \mu = \frac{1}{2}(\delta_0 + \delta_2) \).

At \( N = 2 \) now, \( \mu = \frac{1}{2} \delta_0 + \frac{1}{4}(\Psi^+ \varepsilon + \Psi^- \varepsilon) \). By dephasing, \( a = (1, q) \), we may assume that \( \varepsilon \) is the uniform measure on \( T \), and our blowup maps become \( \Psi^\pm(q) = 2 \pm |1 + q| \).

Now since with \( q = t^4 \) we have \( \{ \Psi^+(q), \Psi^-(q) \} = \{ (t + t^{-1})^2, -(t - t^{-1})^2 \} \), and since with \( t \to it \) we have \( -(t - t^{-1})^2 \to (t + t^{-1})^2 \), we conclude that averaging the measures \( \Psi^\pm \varepsilon \) on \([0,2]\) and \([2,4]\) produces the measure on \([0,4]\) obtained via \( \Phi(t) = (t + t^{-1})^2 \).

Thus we have recovered Proposition 1.6, at \( n = \infty \). These \( N = 2 \) manipulations, however, are quite special and don’t have a higher dimensional analogue, because at \( N \geq 3 \) the quantity \( |\sum_i a_i|^2 = \sum_{ij} \frac{a_i a_j}{a_j} \) does not have a simple writing as a square.
However, we can reduce the maps $\Psi^\pm$ to a single one, as follows:

**Theorem 5.4.** For $F_2 \otimes_Q F_N$, with $Q \in M_{2 \times N}(\mathbb{T})$ generic, we have

$$\mu = \left(1 - \frac{1}{N}\right) \delta_0 + \frac{1}{N} \Phi_* \varepsilon$$

where $\varepsilon$ is the uniform measure on $\mathbb{Z}_2 \times \mathbb{T}^N$, and $\Phi(e, a) = N + e|\sum_i a_i|$.

**Proof.** This is clear indeed from Proposition 5.3 above. □

As explained in the introduction, this result is part of an emerging subfactor machinery, related to [2], [4], [11], [12]. We have as well the following conjecture:

**Conjecture 5.5.** In the non-generic case, the spectral measure blowup should land into a certain arithmetic lattice, canonically associated to $Q \in M_{M \times N}(\mathbb{T})$.

Observe that this conjecture is supported by Proposition 1.6. In order to further comment here, recall from [3] that the parameter $q = \frac{ad}{bc}$ appears by dephasing:

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \simeq \begin{pmatrix} 1 & 1 \\ \frac{c}{a} & \frac{d}{b} \end{pmatrix} \simeq \begin{pmatrix} 1 & 1 \\ 1 & \frac{ad}{bc} \end{pmatrix}$$

On the other hand, the dephasing is given by the action $T_{(A,B)}(H) = AHB^*$ of the group $(K_2 \times K_2)/\mathbb{T}$ on $M_4(\mathbb{T})$, where $K_2 = \mathbb{T} \wr S_2$. Thus the group $<q> \subset \mathbb{T}$ is more or less the “orbit” of the group $<Q> \subset M_4(\mathbb{T})$ under this action. The problem, however, is that Proposition 1.6 involves the group $<q^4> \simeq \mathbb{T}^n$, instead of the group $<q>$ itself.

### References


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