EXAMPLES OF INNER LINEAR HOPF ALGEBRAS

NICOLÁS ANDRUSKIEWITSCH AND JULIEN BICHON

Abstract. The notion of inner linear Hopf algebra is a generalization of the notion of discrete linear group. In this paper, we prove two general results that enable us to enlarge the class of Hopf algebras that are known to be inner linear: the first one is a characterization by using the Hopf dual, while the second one is a stability result under extensions. We also discuss the related notion of inner unitary Hopf $*$-algebra.

1. introduction

Throughout the paper, we work over $\mathbb{C}$, the field of complex numbers.

The notion of inner linear Hopf algebra was introduced in [4] as a natural generalization of the notion of discrete linear group. The precise definition is the following one.

Definition 1.1. *A Hopf algebra is said to be inner linear if it contains an ideal of finite codimension that does not contain any non-zero Hopf ideal.*

Indeed, when $H = \mathbb{C}[\Gamma]$ is the group algebra of a discrete group Γ, then H is inner linear if and only if the group Γ is linear in the usual sense, i.e. admits a faithful finite-dimensional linear representation [4].

Also the concept of inner linearity for Hopf algebras generalizes the notion of linear (= finite dimensional) Lie algebra: if $H = U(\mathfrak{g})$ is the enveloping algebra of a Lie algebra $\mathfrak{g}$, then $U(\mathfrak{g})$ is inner linear if and only if $\mathfrak{g}$ is finite-dimensional.

We believe that the problem to know whether a given Hopf algebra is inner linear or not is an important one, since it is a generalization of the celebrated linearity problem for discrete groups.

Several examples were considered in [4], and we continue this study here. We prove two general results that enable us to enlarge the class of Hopf algebras that are known to be inner linear.

(1) We give a reformulation of inner linearity using the Hopf dual. This enables us to show that the Drinfeld-Jimbo quantum algebras attached to semisimple Lie algebras are inner linear if the parameter q is not a root of unity.
(2) We prove a stability result for inner linearity under extensions. This applies to Drinfeld-Jimbo algebras and quantized function algebras at roots of unity, and to the recently introduced half-liberated orthogonal Hopf algebras [5].

The paper is organized as follows. In Section 2 we reformulate the notion of inner linear Hopf algebra by using the Hopf dual, with, as an application, the inner linearity of the Drinfeld-Jimbo quantum algebras $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$ if q is not a root of unity. Section 3 contains some basic results on the possible use of quotient Hopf algebras to show inner linearity, which might be used to show the inner linearity of Hopf algebras having an analogue of the dense big cell of reductive algebraic groups, such as in [17]. In Section 4 we give a stability result for inner linearity under extensions. This applies to quantum algebras $\mathcal{O}_q(G)$ and $U_q(\mathfrak{g})$ at roots of unity, as well as to the half-liberated Hopf algebras $A_o^*(n)$ from [5]. In Section 5 we study the related notion of inner unitary Hopf $*$-algebra, and it is shown that for $q \in \mathbb{R}^*$, $q \neq \pm 1$, and K a connected simply connected simple compact Lie group, the Hopf $*$-algebra $\mathcal{O}_q(K)$ is not inner unitary (while it is inner linear as a Hopf algebra). We also give a Hopf $*$-algebra version of the extension theorem of Section 4.

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We assume that the reader is familiar with the basic notions of Hopf algebras, for which [16] is a convenient reference. Our terminology and notation are the standard ones: in particular, for a Hopf algebra, Δ, ε and S denote the comultiplication, counit and antipode, respectively.

2. Inner linear Hopf algebras and the Hopf dual

In this section we reformulate the notion of linear Hopf algebra by using the Hopf dual, and we apply this reformulation to Drinfeld-Jimbo quantum algebras.

Theorem 2.1. *A Hopf algebra H is inner linear if and only if H^0, the Hopf dual of H, contains a finitely generated Hopf subalgebra that separates the points of H.*

Of course finitely generated Hopf algebra means finitely generated as a *Hopf algebra*. Before proving Theorem 2.1, we need to recall the concept of inner faithful representation of a Hopf algebra [4], which generalizes the notion of faithful representation of a discrete group.

Definition 2.2. *Let H be a Hopf algebra and let A be an algebra. A representation $\pi : H \longrightarrow A$ is said to be inner faithful if* $\mathrm{Ker}(\pi)$ *does not contain any non-zero Hopf ideal.*

It is clear that a Hopf algebra is inner linear if and only it admits a finite-dimensional inner faithful representation.

Proof of theorem 2.1. We first introduce some notation. Let F be the free monoid generated by the set $\mathbb{N}$, with its generators denoted $\alpha_0, \alpha_1, \dots$ and its unit element denoted 1. Let V be a vector space. To any element $g \in F$, we associate a vector space V^g, defined inductively on the length of g as follows. We put $V^1 = k$, $V^{\alpha_k} = V^{*\cdots *(k \text{ times})}$ (so that $V^{\alpha_0} = V, V^{\alpha_1} = V^* \dots$). Now for $g, h \in F$ with $l(g) > 1$ and $l(h) > 1$, we put $V^{gh} = V^g \otimes V^h$. For $g \in F$, we have, if V is finite dimensional, canonical isomorphisms $\mathrm{End}(V^g) \cong \mathrm{End}(V)^g$, where the algebra $\mathrm{End}(V)^g$ is defined as in Section 2 of [4].

Now let $\pi : H \longrightarrow \mathrm{End}(V)$ be a representation, with V finite dimensional, so that V is an H-module. For any $g \in F$, the standard procedure gives an H-module structure on V^g, and we identify, via the algebra isomorphism $\mathrm{End}(V^g) \cong \mathrm{End}(V)^g$, the corresponding algebra map $H \longrightarrow \mathrm{End}(V^g)$ with $\pi^g : H \longrightarrow \mathrm{End}(V)^g$ as defined in Section 2 of [4].

We know, from Proposition 2.2 in [4], that the largest Hopf ideal contained in $\mathrm{Ker}(\pi)$ is $I_\pi = \cap_{g \in F} \mathrm{Ker}(\pi^g)$. Hence π is inner faithful if and only $\cap_{g \in F} \mathrm{Ker}(\pi^g) = (0)$.

Thus if π is inner faithful, we have $\cap_{g \in F} \mathrm{Ker}(\pi^g) = (0)$. The coefficients of the representation π^g belong, by construction, to L, the Hopf subalgebra of H^0 generated by the coefficients of $\pi = \pi^{\alpha_0}$. Hence L separates the points of H.

The proof of the first implication in the theorem follows. If H is inner linear, let $\pi : H \longrightarrow A$ be an inner faithful representation, with A finite-dimensional. We can assume, by using the regular representation of A, that $A = \mathrm{End}(V)$ for some finite-dimensional vector space V. Hence the finitely generated Hopf subalgebra $L \subset H^0$ constructed above separates the points of H, by the previous discussion.

Conversely, assume that we have a finitely generated Hopf subalgebra $L \subset H^0$ that separates the points of H. Then L is generated by the coefficients of a finite dimensional H^0-comodule, corresponding to a finite dimensional H-module V. Let $\pi : H \longrightarrow \mathrm{End}(V)$ be the corresponding algebra map. The elements of L are the coefficients of the H-modules V^g constructed above, hence $\cap_{g \in F} \mathrm{Ker}(\pi^g) = (0)$ since L separates the points of H, and we conclude that π is inner faithful, so that H is inner linear. $\square$

Corollary 2.3. *Let H be a Hopf algebra such that H^0 separates the points of H.*

1. *Assume that H^0 is finitely generated as a Hopf algebra. Then H is inner linear.*
2. *Assume that H is finitely generated as a Hopf algebra. Then H^0 is inner linear.*

Proof. The first asssertion follows from the previous theorem. For the second one, consider the embedding $H \subset (H^0)^0$ (this is indeed an embedding since H^0 separates the points of H). Then H is a finitely generated Hopf subalgebra of $(H^0)^0$, and we conclude by the previous theorem. $\square$

As an application, we get the following result for quantum groups at generic q; the case when q is a root of 1 will be discussed in Section 4.

Theorem 2.4. *Let G be a (complex) connected, simply connected, semisimple algebraic group with Lie algebra $\mathfrak{g}$, and let $q \in \mathbb{C}^*$. If q is not a root of unity, then the Hopf algebras $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$ are inner linear.*

Proof. We know from the representation theory of $U_q(\mathfrak{g})$ (see [13, Lemma 8.3]) that the type I irreducible representations of $U_q(\mathfrak{g})$ separate the points of $U_q(\mathfrak{g})$. Hence the linear span of their coefficients separates the elements of $U_q(\mathfrak{g})$, and is a finitely generated Hopf subalgebra of $U_q(\mathfrak{g})^0$ (actually it is $\mathcal{O}_q(G)$, see [12, 8]). It follows from Theorem 2.1 that $U_q(\mathfrak{g})$ is inner linear. The second part of the previous corollary ensures that $U_q(\mathfrak{g})^0$ is inner linear, and hence so is the Hopf subalgebra $\mathcal{O}_q(G)$. $\square$

Remark 2.5. The proof uses the fact that $U_q(\mathfrak{g})^0$ separates the points of $U_q(\mathfrak{g})$, a deep result in the representation theory of $U_q(\mathfrak{g})$. On the other hand, to prove this separation result, it might be simpler to combine Theorem 2.1 and the inner faithfulness criterion for representations of pointed Hopf algebras in [4] (Theorem 4.1). This last theorem is as follows: a pointed Hopf algebra H is inner linear if and only if there exists a finite-dimensional representation $\pi : H \longrightarrow A$ such that for any group-like $g \in \mathrm{Gr}(H)$, the restriction map $\pi_{|\mathcal{P}_{g,1}(H)} : \mathcal{P}_{g,1}(H) \longrightarrow A$ is injective. A similar idea to prove separation results, using description of skew-primitives, was already used in [9], Section 4.

3. Inner linearity and quotient Hopf algebras

In this section we prove some very basic but useful results on the possible use of quotient Hopf algebras to show inner linearity. We begin with a lemma.

Lemma 3.1. *Let H be a Hopf algebra, let I_1, I_2 be Hopf ideals in H. Let $\rho_k : H/I_k \longrightarrow A$, $k = 1, 2$, be some representations. Consider the representation*

$$\begin{aligned} \rho : H &\longrightarrow A \times B \\ x &\longmapsto (\rho_1 \circ \pi_1(x), \rho_2 \circ \pi_2(x)) \end{aligned}$$

where π_k, $k = 1, 2$ is the canonical projection. Assume that $I_1 \cap I_2$ does not contain any non zero Hopf ideal and that ρ_1 and ρ_2 are inner faithful. Then ρ is inner faithful.

Proof. Let J be a Hopf ideal contained in $\mathrm{Ker}(\rho) = \mathrm{Ker}(\rho_1 \circ \pi_1) \cap \mathrm{Ker}(\rho_2 \circ \pi_2)$. Then for $k = 1, 2$, $\pi_k(J)$ is a Hopf ideal contained in $\mathrm{Ker}(\rho_k)$, and hence $J \subset I_k$ by inner faithfulness of ρ_k. Hence J is a Hopf ideal in $I_1 \cap I_2$, and $J = (0)$, which proves that ρ is inner faithful. $\square$

Corollary 3.2. *Let H be a Hopf algebra, let I_1, I_2 be Hopf ideals in H. Assume that $I_1 \cap I_2$ does not contain any non zero Hopf ideal and that the Hopf algebras H/I_1 and H/I_2 are inner linear. Then H is inner linear.*

Proof. This is a consequence of the lemma, by using finite-dimensional inner faithful representations of H/I_1 and H/I_2. $\square$

Corollary 3.3. *Let H be a Hopf algebra, let I_1, I_2 be Hopf ideals in H. Assume that the algebra map*

$$\begin{aligned} \theta : H &\longrightarrow H/I_1 \otimes H/I_2 \\ x &\longmapsto \pi_1(x_{(1)}) \otimes \pi_2(x_{(2)}) \end{aligned}$$

where π_1, π_2 are the canonical projections, is injective, and that the Hopf algebras H/I_1 and H/I_2 are inner linear. Then H is inner linear.

Proof. Let $J \subset I_1 \cap I_2$ be a Hopf ideal. Then

$$\Delta(J) \subset J \otimes H + H \otimes J \subset (I_1 \cap I_2) \otimes H + H \otimes (I_1 \cap I_2).$$

Hence $\theta(J) = (0)$ and $J = (0)$ by the injectivity of θ, and we are done by the previous result. $\square$

This last result might be used to show the inner linearity of Hopf algebras having an analogue of the dense big cell of reductive algebraic groups, such as in [17]. Indeed for $\mathcal{O}_q(\mathrm{GL}_n(\mathbb{C}))$, the previous result combined with Theorem 8.1.1 in [17] reduces the problem to show the inner linearity of $\mathcal{O}_q(\mathrm{GL}_n(\mathbb{C}))$ to show the inner linearity of the pointed Hopf algebras $\mathcal{O}_q(B)$ and $\mathcal{O}_q(B')$, for which the method of Theorem 4.1 in [4] is available.

4. Stability of inner linearity under extensions

We now study the question of the stability of inner linearity under extensions. At the group level, it is known that linearity is not stable under extensions (see e.g. [11]), but we have the following positive result: If G is a group having a linear normal subgroup H of finite index, then G is linear. We prove a Hopf algebraic analogue of a weak form of this result, and we apply it to two types of Hopf algebras.

4.1. **The general result.**

Here is our more general result on the preservation of inner linearity by extensions.

Theorem 4.1. *Let H be a Hopf algebra and let $A \subset H$ be a normal Hopf subalgebra. Assume that the following conditions hold:*

(1) *A is inner linear and commutative,*
(2) *H is finitely generated as a right A-module.*

Then H is inner linear.

To prove the theorem, we need to recall some facts on exact sequences of Hopf algebras.

First recall that a Hopf subalgebra A of a Hopf algebra H is said to be *normal* if it is stable under both left and right adjoint actions of H on itself, defined, respectively, by

$$\mathrm{ad}_l(x)(y) = x_{(1)} y \mathcal{S}(x_{(2)}), \quad \mathrm{ad}_r(x)(y) = \mathcal{S}(x_{(1)}) y x_{(2)},$$

for all $x, y \in H$. If $A \subseteq H$ is a normal Hopf subalgebra, then the ideal $HA^+ = A^+H$ is a Hopf ideal and the canonical map $H \to \overline{H} := H/HA^+ := H//A$ is a Hopf algebra map.

Now recall that a sequence of Hopf algebra maps

$$k \to A \xrightarrow{i} H \xrightarrow{p} \overline{H} \to k, \tag{4.1}$$

is called *exact* if the following conditions hold:

(1) i is injective and p is surjective,
(2) $p \circ i = \epsilon 1$,
(3) $\mathrm{Ker} p = HA^+$,
(4) $A = H^{\mathrm{co}\, p} = \{h \in H : (\mathrm{id} \otimes p)\Delta(h) = h \otimes 1\}$.

It follows that $i(A)$ is normal Hopf subalgebra of H. Conversely, if we have a sequence (4.1) and H is faithfully flat over A, then (1), (2) and (3) imply (4). See e.g. [1].

We now state several preparatory lemmas for the proof of Theorem 4.1.

Lemma 4.2. *Let H be a Hopf algebra and let $A \subset H$ be a normal Hopf subalgebra. If A is commutative, then $k \to A \to H \to H//A \to k$ is an exact sequence.*

Proof. We know from [3], Proposition 3.12, that H is a faithfully flat as an A-module, and hence we have the announced exact sequence by the previous considerations. ☐

We shall need the following Hopf algebraic version of the 5 lemma.

Lemma 4.3. *Consider a commutative diagram of Hopf algebras*

$$\begin{tikzcd} k \arrow[r] & A \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d, "\theta"] & \overline{H} \arrow[r] \arrow[d, equal] & k \\ k \arrow[r] & A \arrow[r] & H' \arrow[r] & \overline{H} \arrow[r] & k \end{tikzcd} \tag{4.2}$$

where the rows are exact. If A is commutative, then θ is an isomorphism.

Proof. The proof is similar to Corollary 1.15 in [2]: $A \subset H$ and $A \subset H'$ are $\overline{H}$-Galois extensions and since H' is a faithfully flat A-module [3, Proposition 3.12], the $\overline{H}$-colinear A-linear algebra map θ is an isomorphism by Remark 3.11 in [20]. $\square$

Lemma 4.4. *Let $A \subset H$ be a normal and commutative Hopf subalgebra. Let J be a Hopf ideal in H such that $J \cap A = (0)$ and $J \subset A^+H$. Then $J = (0)$.*

Proof. We have, by Lemma 4.2, an exact sequence

$$k \to A \xrightarrow{i} H \xrightarrow{p_H} H//A \to k.$$

Now put $K = H/J$, and let $q : H \to K$ be the canonical surjection. Since $J \cap A = (0)$, we have an injective Hopf algebra map $j : A \to K$ such that $q \circ i = j$, and $j(A)$ is a normal Hopf subalgebra of K. We then have an exact sequence

$$k \to A \xrightarrow{j} K \xrightarrow{p_K} K//A \to k.$$

We now claim that there exists a Hopf algebra isomorphism $\overline{q} : H \to K$ such that the following diagram is commutative

$$\begin{array}{ccccccccc} k & \longrightarrow & A & \longrightarrow & H & \xrightarrow{p_H} & H//A & \longrightarrow & k \\ & & \| & & \downarrow q & & \downarrow \overline{q} & & \\ k & \longrightarrow & A & \longrightarrow & K & \xrightarrow{p_K} & K//A & \longrightarrow & k \end{array}$$

Indeed, we have $p_K \circ q(A^+H) = p_K(j(A)^+H) = 0$, which shows the existence of $\overline{q}$. We also have, since $J \subset A^+H$,

$$\begin{aligned} \mathrm{Ker}(\overline{q}) &= p_H(\mathrm{Ker}(p_K \circ q)) = p_H(q^{-1}(\mathrm{Ker}(p_K)) \\ &= p_H(q^{-1}(j(A)^+K)) = p_H(J + A^+H) \subset p_H(A^+H) = (0) \end{aligned}$$

and hence $\overline{q}$ is an ismorphism. We get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} k & \longrightarrow & A & \longrightarrow & H & \xrightarrow{\overline{q} \circ p_H} & K//A & \longrightarrow & k \\ & & \| & & \downarrow q & & \| & & \\ k & \longrightarrow & A & \longrightarrow & K & \longrightarrow & K//A & \longrightarrow & k \end{array}$$

(the top row is still exact because $\overline{q}$ is an isomorphism) and Lemma 4.3 ensures that q is an isomorphism, hence $J = (0)$. $\square$

Proof of Theorem 4.1. Let $\rho : A \to \mathrm{End}(V)$ be an inner faithful representation, with V finite-dimensional. As usual, we get the induced representation $\tilde{\rho} : H \to \mathrm{End}(H \otimes_A V)$. Since H is finitely generated as an A-module, the vector space $H \otimes_A V$ and the Hopf algebra $H//A$ are finite dimensional. We thus consider the finite-dimensional representation

$$\begin{aligned} \theta : H &\longrightarrow H//A \times \mathrm{End}(H \otimes_A V) \\ x &\longmapsto (p_H(x), \tilde{\rho}(x)) \end{aligned}$$

Let us show that θ is inner faithful. Let $J \subset \mathrm{Ker}(\theta) = A^+H \cap \mathrm{Ker}(\tilde{\rho})$ be a Hopf ideal. Then $J \cap A \subset A$ is a Hopf ideal. It is easy to see, using the faithful flatness of H as a right A-module, that $\mathrm{Ker}(\tilde{\rho}) \cap A \subset \mathrm{Ker}(\rho)$. Thus $J \cap A \subset \mathrm{Ker}(\rho)$ and $J \cap A = (0)$ since ρ is inner faithful. We thus have $J \subset A^+H$ and $J \cap A = (0)$: the previous lemma ensures that $J = (0)$. Hence θ is inner faithful and H is inner linear. $\square$

Question 4.5. *Is true that the induced representation $\tilde{\rho} : H \to \mathrm{End}(H \otimes_A V)$ is inner faithul if ρ is? A positive answer would give a strenghtening of Theorem 4.1, dropping the commutativity assumption on A.*

4.2. **Applications.** Our first application is with quantized function algebras at roots of unity.

Theorem 4.6. *Let G be a connected, simply connected complex semisimple algebraic group with Lie algebra $\mathfrak{g}$, and let q be a root of unity of odd order ℓ, with ℓ prime to 3 if G contains a component of type G_2. The Hopf algebras $\mathcal{O}_q(G)$ and $U_q(\mathfrak{g})$ are inner linear.*

Proof. It is known [10] that $\mathcal{O}_q(G)$ contains a central Hopf subalgebra isomorphic to $\mathcal{O}(G)$, and that $\mathcal{O}_q(G)$ is finitely generated and projective as $\mathcal{O}(G)$-module. The Hopf algebra $\mathcal{O}(G)$ is inner linear [4], and thus Theorem 4.1 gives the result.

Similarly it is known (see e.g. [8]) that $U_q(\mathfrak{g})$ contains an affine central Hopf subalgebra (hence inner linear by [4]) Z_0 such that $U_q(\mathfrak{g})$ is a finitely generated Z_0-module. Hence again Theorem 4.1 gives the result. □

We now turn to the half-liberated orthogonal Hopf algebra. Recall that the half-liberated orthogonal Hopf algebra $A_o^*(n)$, introduced in [5] and further studied in [6], is the algebra presented by generators u_{ij}, $1 \leq i, j \leq n$, submitted to the relations

(1) the matrix $u = (u_{ij})$ is orthogonal,
(2) $u_{ij}u_{kl}u_{pq} = u_{pq}u_{kl}u_{ij}$, $1 \leq i, j, k, l, p, q \leq n$.

It admits a Hopf algebra structure given by the standard formulas

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad S(u_{ij}) = u_{ji}$$

Theorem 4.7. *The Hopf algebra $A_o^*(n)$ is inner linear.*

Proof. Let $A \subset H = A_o^*(n)$ be the subalgebra generated by the elements $u_{ij}u_{kl}$. As remarked in [6], it is a commutative Hopf subalgebra of H. Thus $A \simeq \mathcal{O}(G)$ for a (reductive) algebraic group G (in fact it is shown in [6] that $G = \mathrm{PGL}_n(\mathbb{C})$)) and A is inner linear by [4]. It is easy to check the existence of a Hopf algebra map $\pi : H \to \mathbb{C}[\mathbb{Z}_2]$, $u_{ij} \mapsto \delta_{ij}g$, where $1 \neq g \in \mathbb{Z}_2$, and that $A = H^{\mathrm{co}\,\pi}$. This Hopf algebra map is cocentral ($\pi(x_{(1)}) \otimes x_{(2)} = \pi(x_{(2)}) \otimes x_{(1)}$, for all $x \in H$) and hence A is normal in H (see e.g. Lemma 3.4.2 in [16]). Moreover H is, as an A-module, generated by the elements u_{ij}. Thus H is inner linear by Theorem 4.1. □

5. Inner unitary Hopf $*$-algebras

In this section we discuss the Hopf algebraic analogue of the notion of discrete unitary group. Here we say that a discrete group is unitary if it can be embedded as a subgroup of the group of unitary operators on a finite-dimensional Hilbert space. We work in the framework of Hopf $*$-algebras (see e.g. [14] for the relevant definitions).

Definition 5.1. *A Hopf $*$-algebra H is said to be* ***inner unitary*** *if there exists a $*$-representation $\pi : H \longrightarrow A$ into a finite-dimensional C^*-algebra A such that* $\mathrm{Ker}(\pi)$ *does not contain any non zero Hopf $*$-ideal.*

Of course a group Γ is unitary if and only if the Hopf $*$-algebra $\mathbb{C}[\Gamma]$ is inner unitary.

A possible trouble with the previous natural definition is that it is not clear that an inner unitary Hopf algebra must be inner linear. We do not know if this is true in general, but we shall see that under some mild assumptions (see Proposition 5.4), an inner unitary Hopf $*$-algebra is inner linear. This is true in particular for compact Hopf algebras, i.e. Hopf $*$-algebras arising from compact quantum groups, the class of Hopf $*$-algebras we are most interested in.

We need the following concept.

Definition 5.2. *We say that a Hopf algebra H has a regular antipode if there exists a group-like $a \in H$, an algebra morphism $\Phi : H \longrightarrow \mathbb{C}$ and an integer $m \geq 1$ such that*

$$\forall x \in H,\ S^{2m}(x) = a(\Phi * \mathrm{id}_H * \Phi^{-1}(x))a^{-1}$$

For example, by [7], any co-Frobenius Hopf algebra has a Radford type formula for S^4 and hence has a regular antipode. In particular any cosemisimple Hopf algebra has a regular antipode.

Proposition 5.3. *Let H be a Hopf algebra having a regular antipode. The following assertions are equivalent.*

(1) *H is inner linear.*
(2) *There exists a representation $\pi : H \longrightarrow A$ into a finite-dimensional algebra A such that* $\mathrm{Ker}(\pi)$ *does not contain any non-zero Hopf ideal I such that $S(I) = I$.*

Proof. By the assumption there exists a group-like $a \in H$, an algebra morphism $\Phi : H \longrightarrow \mathbb{C}$ and $m \geq 1$ such that for $x \in H$, one has $S^{2m}(x) = a(\Phi * \mathrm{id}_H * \Phi^{-1}(x))a^{-1}$. Let $\pi : H \longrightarrow A$ be a representation satisfying condition (2), and consider the representation

$$\begin{aligned} \pi' : H &\longrightarrow A \times \mathbb{C}^2 \\ x &\longmapsto (\pi(x), \Phi(x), \Phi^{-1}(x)) \end{aligned}$$

Let $J \subset \mathrm{Ker}(\pi')$ be a Hopf ideal. Then we have

$$\begin{aligned} S^{-1}(J) &= a^{-1}\left(\Phi^{-1} * S^{2m-1} * \Phi)(J)\right) a \\ &\subset a^{-1}\left(\Phi^{-1}(J)S^{2m-1}(H)\Phi(H) + \Phi^{-1}(H)S^{2m-1}(J)\Phi(H) + \Phi^{-1}(H)S^{2m-1}(H)\Phi(J)\right) a \\ &\subset a^{-1}S^{2m-1}(J)a \subset J \end{aligned}$$

since $\Phi(J) = \Phi^{-1}(J) = (0)$. Thus $S(J) = J$ with $J \subset \mathrm{Ker}(\pi)$, and hence $J = (0)$. Thus π' is inner faithful and H is inner linear. $\square$

Proposition 5.4. *Let H be a Hopf $*$-algebra having a regular antipode. If H is inner unitary, then H is inner linear. In particular an inner unitary compact Hopf algebra is inner linear.*

Proof. Let $\pi : H \longrightarrow A$ be a $*$-representation into a finite-dimensional C^*-algebra A such that $\mathrm{Ker}(\pi)$ does not contain any non zero Hopf $*$-ideal. The previous proposition ensures that to show H is inner linear, it is enough to check that $\mathrm{Ker}(\pi)$ does not contain any non-zero Hopf ideal I with $S(I) = I$. So let $I \subset \mathrm{Ker}(\pi)$ be such a Hopf ideal. It is clear that $I + I^*$ is a $*$-bi-ideal contained in $\mathrm{Ker}(\pi)$. Moreover $S(I + I^*) = S(I) + S(I^*) = S(I) + S^{-1}(I)^* = I + I^*$. Hence $I + I^*$ is a Hopf $*$-ideal contained in $\mathrm{Ker}(\pi)$ and $I + I^* = (0) = I$. $\square$

It is well-known that, already at the discrete group level, the converse of this result is not true. For example the group $\Gamma = \langle x, y \mid yxy^{-1} = x^2\rangle$ is linear but is not unitary. For non cocommutative Hopf algebras, we also have the following example.

Proposition 5.5. *Let G be a connected, simply connected and simple complex Lie group and let $K \subset G$ be a maximal compact subgroup. Let $q \in \mathbb{R}^*$, $q \neq \pm 1$. Then the compact Hopf algebra $\mathcal{O}_q(K)$ is inner linear but is not inner unitary.*

Proof. The Hopf algebra underlying $\mathcal{O}_q(K)$ is $\mathcal{O}_q(G)$, hence is inner linear by Theorem 2.4. Assume that $\mathcal{O}_q(K)$ is inner unitary: there exists a $*$-algebra morphism

$$\pi : \mathcal{O}_q(K) \longrightarrow \mathcal{B}(V)$$

where V is a finite-dimensional Hilbert space, such that $\mathrm{Ker}(\pi)$ does not contain any non zero Hopf $*$-ideal. The $\mathcal{O}_q(K)$-module V is semisimple since π is a $*$-algebra map. We know from [15] that the finite-dimensional irreducible Hilbert space representations of $\mathcal{O}_q(K)$ all are one-dimensional, and hence the elements of $\pi(\mathcal{O}_q(K))$ are simultaneously diagonalizable. It follows that $\pi(\mathcal{O}_q(K))$ is a commutative $*$-algebra and that the commutator ideal of $\mathcal{O}_q(K)$, which is a Hopf $*$-ideal, is contained in $\mathrm{Ker}(\pi)$. Hence the commutator ideal is zero and $\mathcal{O}_q(K)$ is commutative: a contradiction (these last arguments are from Proposition 2.13 in [4]). $\square$

Remark 5.6. The Hopf $*$-algebra $\mathcal{O}_{-1}(\mathrm{SU}_2)$ is inner unitary, since the inner faithful representation $\mathcal{O}_{-1}(\mathrm{SU}_2) \to M_2(\mathbb{C}) \otimes \mathbb{C}^4$ constructed in [4], Corollary 6.6, is a $*$-algebra map. Also one can adapt the other constructions in Section 6 of [4] to show that $\mathcal{O}_{-1}(\mathrm{SU}_n)$ is inner linear for any n.

It is now natural to ask if the compact Hopf $*$-algebra $A_o^*(n)$ (whose $*$-structure is defined by $u_{ij}^* = u_{ij}$) is inner unitary. For this we need a Hopf $*$-algebra version of Theorem 4.1.

Theorem 5.7. *Let H be a compact Hopf algebra and let $A \subset H$ be a normal Hopf $*$-subalgebra. Assume that the following conditions hold:*

(1) *A is inner unitary and commutative,*
(2) *H is finitely generated as a right A-module.*

Then H is inner unitary.

Proof. The proof is an adaptation of the proof of Theorem 4.1, using induced representations of C^*-algebras [19]. Similarly to Section 4, H is a faithfully flat A-module since A is commutative, and hence $H^{\mathrm{co}\, p} = A$, where $p : H \longrightarrow H//A$ is the canonical map. Hence using the Haar measure ϕ on the compact Hopf algebra $H//A$, we get a map

$$E = (\mathrm{id} \otimes \phi) \circ (\mathrm{id} \otimes p) \circ \Delta : H \longrightarrow A$$

which is a conditional expectation in the sense of [19] (see e.g. [18]).

Consider now a Hilbert space $*$-representation $\rho : A \to \mathcal{B}(V)$. There is a sesquilinear form on $H \otimes_A V$ such that for $x, y \in H$, $v, w \in V$, we have

$$\langle x \otimes_A v, y \otimes_A w \rangle = \langle \rho(E(y^* x))(v), w \rangle$$

Moreover this is a pre-inner product (Lemma 1.7 in [19]). Killing the norm zero elements, we get a (finite-dimensional) Hilbert space $H \underline{\otimes}_A V$, and an induced $*$-representation (Theorem 1.8 in [19])

$$\begin{aligned} \tilde{\rho} : H &\longrightarrow \mathcal{B}(H \underline{\otimes}_A V) \\ x &\longmapsto \tilde{\rho(x)}, \ \tilde{\rho(x)}(y \underline{\otimes}_A v) = xy \underline{\otimes}_A v \end{aligned}$$

We wave $\mathrm{Ker}(\tilde{\rho}) \cap A \subset \mathrm{Ker}(\rho)$ since the map $V \to H \underline{\otimes}_A V$, $v \mapsto 1 \underline{\otimes}_A v$, is isometric. Then, similarly to the proof of Theorem 4.1, if $\mathrm{Ker}(\rho)$ does not contain any non zero Hopf $*$-ideal, the kernel of the $*$-representation

$$\begin{aligned} \theta : H &\longrightarrow H//A \times \mathcal{B}(H \underline{\otimes}_A V) \\ x &\longmapsto (p(x), \tilde{\rho}(x)) \end{aligned}$$

does not contain any non-zero Hopf $*$-ideal, and hence H is inner unitary. □

Corollary 5.8. *The compact Hopf $*$-algebra $A_o^*(n)$ is inner unitary.*

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Nicolás Andruskiewitsch: Facultad de Matemática, Astronomía y Física, Universidad National de Córdoba. CIEM - CONICET. Medina Allende s/n (5000) Ciudad Universitaria, Córdoba, Argentina
E-mail address: andrus@famaf.unc.edu.ar

Julien Bichon: Clermont Université, Université Blaise Pascal, Laboratoire de Mathématiques CNRS UMR 6620, 63177 Aubière, France
E-mail address: Julien.Bichon@math.univ-bpclermont.fr