Remarks around a conjecture of Frey and Mazur

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1 The Frey-Mazur Conjecture

Let E be an elliptic curve defined over $\mathbf{Q}$. Given a prime number ℓ, put :

$$\mathcal{A}_E(\ell) = \{ \text{ Elliptic curve } F/\mathbf{Q} \text{ such that } F[\ell] \simeq E[\ell] \text{ as Galois modules}\}/\sim,$$

where the symbol $\sim$ means modulo isomorphism. For any $c > 0$, let

$$\mathcal{A}_{E,c} = \bigcup_{\substack{\ell \geq c \\ \ell \text{ prime}}} \mathcal{A}_E(\ell) = \{F/\mathbf{Q} \text{ s.t. there exists } \ell \geq c \text{ with } F[\ell] \simeq E[\ell] \text{ as Galois modules}\}/\sim .$$

On the other hand, define :

$$\mathcal{B}_E = \{\ell \text{ prime such that there exists } F/\mathbf{Q} \text{ non isogenous to } E \text{ with } F[\ell] \simeq E[\ell] \text{ as Galois modules}\}.$$

Let $X_E(\ell)$ be the twist of the modular curve $X(\ell)$ that parametrizes isomorphism classes of elliptic curves F such that $F[\ell] \simeq E[\ell]$.

Proposition 1.

1. *If $\ell \geq 7$, then $\mathcal{A}_E(\ell)$ is finite.*
2. *The following are equivalent :*
   (a) *There exists a constant c_E depending on E such that $\mathcal{A}_{E,c_E}$ is finite;*
   (b) *The set $\mathcal{B}_E$ is finite.*

Proof. The first part follows from the computation of the genus of $X_E(\ell)$ (which is the same as the one of $X(\ell)$) together with Faltings' theorem on Mordell Conjecture. The proof of (a)⇒(b) uses the following consequence of Faltings' Theorem on Tate Conjecture : if A, B are two elliptic curves over $\mathbf{Q}$ such that $A[\ell] \simeq B[\ell]$ for infinitely many primes ℓ, then A and B are isogenous. The proof of the converse uses Shafarevich Theorem on the finiteness of the number $\mathbf{Q}$-isomorphism classes of rational elliptic curves $\mathbf{Q}$-isogenous to a given one. □

Frey and Mazur have conjectured :

Conjecture 1 (Frey-Mazur)**.** *For any elliptic curve $E/\mathbf{Q}$, the set $\mathcal{B}_E$ is finite.*

We may also conjecture some uniform (stronger) statement.

Conjecture 2 (Uniform Frey-Mazur Conjecture)**.** *There exists an absolute constant $c_{\mathbf{Q}}$ such that for any elliptic curve $E/\mathbf{Q}$, the set $\mathcal{A}_{E,c_{\mathbf{Q}}}$ is finite.*

Remarks.

1. One may replace $\mathbf{Q}$ by any number field K and make the same conjectures over K.
2. Roughly speaking, Frey-Mazur conjecture says that the isogeny class of E is determined by the Galois module $E[\ell]$ for ℓ large enough.
3. Conj. 1 is a consequence of the *abc* conjecture (Mochizuki's Theorem?).
4. In terms of the modular curve $X_E(\ell)$, conj. 1 then says that for ℓ large enough, depending on E, we have $X_E(\ell)(\mathbf{Q}) = \{\text{cusps}\} \cup \{\overline{E}\}$, where $\overline{E}$ denotes the (finite) set of isomorphism classes of rational elliptic curves that are isogenous to E. Conj. 2 moreover asserts that this holds independently of E.
5. The Frey-Mazur conjecture has many interesting consequences in the theory of diophantine equations.

2 Modular formulation of the Frey-Mazur Conjecture

Isogeny classes of elliptic curves over $\mathbf{Q}$ correspond bijectively to weight 2 newforms with rational coefficients (via L-functions). Let us denote by $E \mapsto f_E$ this correspondence. Given a prime ℓ and a newform f (of arbitrary type) let us denote by $\overline{\rho}_{f,\ell}$ *any* mod. ℓ representation attached to f.

Assume now f to be a weight 2 newform with rational coefficients and trivial Nebentype. We define

$$B_f = \{\ell \text{ prime such that there exists } g \neq f \text{ weight 2 newform with rational coefficients with } \overline{\rho}_{g,\ell} \simeq \overline{\rho}_{f,\ell}\}.$$

Then $B_f = B_E$ for any elliptic curve E such that $f_E = f$. Therefore :

Conjecture 3 (Frey-Mazur)**.** *For any weight 2 newform f with rational coefficients and trivial Nebentype, the set B_f is finite.*

3 A modular generalization of the Frey-Mazur Conjecture

Let f be a weight $k \geq 2$ newform. Let us denote by $\mathbf{Q}(f)$ and $\langle f \rangle$ its field of coefficients and Galois conjugacy class respectively. Define

$$B_f = \{\ell \text{ prime such that there exists } g \notin \langle f \rangle \text{ weight } k \text{ newform with } \mathbf{Q}(g) = \mathbf{Q}(f) \text{ with } \overline{\rho}_{g,\ell} \simeq \overline{\rho}_{f,\ell}\}.$$

Conjecture 4 (Generalized Frey-Mazur)**.** *For any weight $k \geq 2$ newform f, the set B_f is finite.*

♣ Try to disprove this conjecture !

We now aim to see whether there is an analogue of Prop. 1 in this context. Define :

$$\mathcal{A}_f(\ell) = \{g \text{ weight } k \text{ newform such that } \mathbf{Q}(g) = \mathbf{Q}(f) \text{ and } \overline{\rho}_{g,\ell} \simeq \overline{\rho}_{f,\ell}\}.$$

♣ Is there a constant $c(f)$ such that if $\ell \geq c(f)$, then $\mathcal{A}_f(\ell)$ is finite ?

[Assuming that such a constant exists, it then makes sense to ask whether $\mathcal{A}_{f,c} = \cup_{\ell \geq c} \mathcal{A}_f(\ell)$ is finite for some $c \geq c(f)$.]

♣ Is there a constant $c(k,d)$ such that if $[\mathbf{Q}(f) : \mathbf{Q}] = d$ and $\ell \geq c(k,d)$, then $\mathcal{A}_f(\ell)$ is finite ?

[Prop. 1 says that $c(2,1)$ exists.]

4 Yet another generalization

Let again E be an elliptic curve over the rationals. For any non-negative integer d and any prime number ℓ put

$$\mathcal{C}_E(\ell,d) = \{\text{Weight 2 newforms } f \text{ with } [\mathbf{Q}(f) : \mathbf{Q}] = d \text{ such that } f \text{ is congruent to } f_E \text{ modulo } \ell\}.$$

The Frey-Mazur conjecture says that $\mathcal{C}_E(\ell,1)$ is finite for ℓ large enough. Therefore one may wonder whether this is true for any $d \geq 1$:

Conjecture 5. *For any elliptic curve $E/\mathbf{Q}$ and any integer $d \geq 1$, there exists a constant $c(E,d)$ such that the set $\mathcal{C}_E(\ell,d)$ is finite whenever ℓ is a prime $> c(E,d)$.*

Playing around with quantifiers one may even conjecture the following statements.

Conjecture 6. *For any elliptic curve $E/\mathbf{Q}$, there exists a constant $c(E)$ such that for any integer $d \geq 1$ and any prime $\ell > c(E)$, the set $\mathcal{C}_E(\ell,d)$ is finite.*

Conjecture 7. *There exists a constant $c_{\mathbf{Q}}$ such that for any elliptic curve $E/\mathbf{Q}$, any integer $d \geq 1$ and any prime $\ell > c_{\mathbf{Q}}$, the set $\mathcal{C}_E(\ell,d)$ is finite.*

♣ Try to disprove these conjectures !

Remark. Ribet and Diamond-Taylor proved that for each $\ell > 3$, the set $\bigcup_{d \geq 1} \mathcal{C}_E(\ell,d)$ is infinite.