An algebraic theory of order of integration schemes

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Composition integration schemes based on Euler's method

The system of ODEs

Consider a smooth system of autonomous ODEs

$$\dot{y} = f(y), \quad f : \mathbb{R}^d \to \mathbb{R}^d. \tag{1}$$

A one-step integrator $\psi_h : \mathbb{R}^d \to \mathbb{R}^d$ gives, for a given initial value $y(t_0) = y_0$, the numerical solution

$$y(t_{k+1}) \approx y_{k+1} = \psi_h(y_k), \quad k = 0, 1, 2, \ldots$$

for the time grid $t_k = t_0 + kh$.
Euler method: $\psi_h(y) = y + h\, f(y)$. Local error:

$$\psi_h(y(t)) = y(t+h) + \mathcal{O}(h^2) \quad \text{as} \quad h \to 0.$$

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A more precise integrator can be obtained from $\chi_h(y) = y + hf(y)$

$$\psi_h(y) = \chi_{h/2} \circ \chi_{h/2}^{-1}(y)$$

In that case $\psi_h(y(t)) = y(t+h) + \mathcal{O}(h^3)$. So that it is of order 2.

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The system of ODEs

Consider a smooth system of autonomous ODEs

$$\dot{y} = f(y), \quad f : \mathbb{R}^d \to \mathbb{R}^d. \tag{2}$$

A one-step integrator $\psi_h : \mathbb{R}^d \to \mathbb{R}^d$ gives, for a given initial value $y(t_0) = y_0$, the numerical solution

$$y(t_{k+1}) \approx y_{k+1} = \psi_h(y_k), \quad k = 0, 1, 2, \ldots$$

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An integrator of order 4 from $\chi_h(y) = y + hf(y)$

$$\psi_h \;=\; \chi_{a_6 h} \circ \chi^{-1}_{a_5 h} \circ \chi_{a_4 h} \circ \chi^{-1}_{a_3 h} \circ \chi_{a_2 h} \circ \chi^{-1}_{a_1 h}.$$

where

$$a_1 = -\frac{193}{396} \quad a_2 = \frac{97}{132} \quad a_3 = \frac{89}{66} \quad a_4 = \frac{25}{198} \quad a_5 = \frac{1}{4} \quad a_6 = \frac{5}{4}$$

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$$\dot{y} = f(y), \quad f : \mathbb{R}^d \to \mathbb{R}^d, \tag{3}$$

and explicit Euler $\chi_h(y) = y + h\, f(y)$, which for any solution $y(t)$ of (3) gives

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Composition integration schemes based on Euler's method

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We can define for each $(a_1, \ldots, a_{2m}) \in \mathbb{R}^{2m}$, a new integrator

Composition integration schemes based on Euler's method

$$\psi_h \quad = \quad \chi_{a_{2m}h} \circ \chi^{-1}_{a_{2s-1}h} \circ \cdots \circ \chi_{a_2 h} \circ \chi^{-1}_{a_1 h}. \tag{4}$$

Conditions on $(a_1, \ldots, a_{2m})$ for $\psi_h(y(t)) = y(t+h) + \mathcal{O}(h^{n+1})$?

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Conditions on $(a_1, \ldots, a_{2m})$ for $\psi_h(y(t)) = y(t+h) + \mathcal{O}(h^{n+1})$?
For arbitrary $\chi_h(y) = y + h\, f(y) + \mathcal{O}(h^2)$, more order conditions?

Example

$$\mathcal{S} = \{e\} \cup \bigcup_{m \geq 1} \mathbb{R}^{2m}, \quad \lambda \cdot (a_1, \ldots, a_{2m}) = (\lambda a_1, \ldots, \lambda a_{2m}),$$

$$(a_1, \ldots, a_{2m}) \circ (a_{2m+1}, \ldots, a_{2(m+k)}) = (a_1, \ldots, a_{2(m+k)}).$$

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Definition

We say that $(\mathcal{S}, \circ, e, \nu)$ is a scaled semigroup (resp. scaled group) if $(\mathcal{S}, \circ, e)$ is a semigroup (resp. group) with neutral element e and

$$\begin{array}{rcl} \nu : \mathbb{R} \times \mathcal{S} & \rightarrow & \mathcal{S} \\ (\lambda, s) & \mapsto & \lambda \cdot s \end{array}$$

is a map satisfying that, for all $s, s' \in \mathcal{S}$, $\lambda, \mu \in \mathbb{R}$,

- $1 \cdot s = s$ and $0 \cdot s = e$,
- $\lambda \cdot (\mu \cdot s) = (\lambda\mu) \circ s$,
- $\lambda \circ (s \circ s') = (\lambda \cdot s) \circ (\lambda \cdot s')$ and $\lambda \cdot e = e$.

Definition

A map $\theta : \mathcal{S} \to \hat{\mathcal{S}}$ is a morphism of scaled semigroups if it is a morphism of semigroups satisfying that $\lambda \cdot \theta(s) = \theta(\lambda \cdot s)$ for all $\lambda \in \mathbb{R}$ and $s \in \mathcal{S}$.

Let $\mathcal{A}$ be an associative algebra with unity $1_{\mathcal{A}}$, and consider

$$\begin{aligned} \mathcal{A}[[h]] &= \left\{ \sum_{n=0}^{\infty} h^n A_n \ : \ \forall n \geq 0, \quad A_n \in \mathcal{A} \right\}, \\ G(\mathcal{A}) &= \left\{ 1_{\mathcal{A}} + \sum_{n=1}^{\infty} h^n A_n \ : \ \forall n \geq 1, \quad A_n \in \mathcal{A} \right\}, \end{aligned}$$

where h is an indeterminate variable. Clearly, $\mathcal{A}[[h]]$ has an algebra structure, and $G(\mathcal{A}) \subset \mathcal{A}[[h]]$ is a scaled group with

$$\lambda \cdot \left(1_{\mathcal{A}} + \sum_{n=1}^{\infty} h^n A_n \right) = 1_{\mathcal{A}} + \sum_{n=1}^{\infty} h^n \lambda^n A_n.$$

Example

For each $n \geq 1$ and each $s = (a_1, \ldots, a_{2m})$, consider the linear differential operator $\theta_n(s)$ that gives a smooth function $\theta_n(s)[g]$ for each $g \in \mathrm{C}^\infty(\mathbb{R}^d; \mathbb{R})$ as follows:

$$\theta_n(s)[g](y) = \frac{1}{n!} \frac{d^n}{dh^n} \, g(\psi_h(y))|_{h=0} \, , \tag{5}$$

so that formally,

$$g(\psi_h(y)) = \theta(s)[g](y), \quad \text{where} \quad \theta(s) = I + \sum_{n \geq 1} h^n \theta_n(s),$$

where I represents the identity operator. Here, $\mathcal{C} = \mathrm{C}^\infty(\mathbb{R}^d; \mathbb{R})$ is a commutative algebra, $\mathcal{A} = \mathrm{End}_{\mathbb{R}} \mathcal{C}$ is an associative algebra with unity I, and $\theta : \mathcal{S} \to G(\mathcal{A})$ is a morphism of scaled semigroups.

Example (cont.)

Given $f : \mathbb{R}^d \to \mathbb{R}^d$, let us consider $X_n \in \mathcal{A} = \mathrm{End}_{\mathbb{R}}\mathcal{C}$ is such that, for $g \in \mathcal{C} = \mathrm{C}^\infty(\mathbb{R}^d; \mathbb{R})$ and $y \in \mathbb{R}^d$,

$$X_n[g](y) = \frac{1}{n!}\frac{d^n}{dh^n}\, g(\chi_h(y))|_{h=0} = \frac{1}{n!} g^{(n)}(y)(f(y), \ldots, f(y)).$$

We have that $g(\psi_h(y)) = \theta(s)[g](y)$, where

$$\theta(s) = X(a_1 h)^{-1} X(a_2 h) \cdots X(a_{2m-1} h)^{-1} X(a_{2m} h),$$

with $X(ah) = I + \sum_{n\geq 1} a^n\, h^n X_n$.

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with $X(ah) = I + \sum_{n\geq 1} a^n\, h^n X_n$.

For any solution $y(t)$ of the ODE system,

$$g(y(t+h)) = g(y) + \sum_{n\geq 1}\frac{h^n}{n!}F^n[g](y(t)) = \exp(h\,F)[g](y(t)),$$

where $F[g](y) = g'(y)f(y)$. (as expected, $X_1 = F$).

Let us denote in addition $L(\mathcal{A}) = h\mathcal{A}[[h]]$. The exponential and the logarithm

$$\exp : L(\mathcal{A}) \longrightarrow G(\mathcal{A}), \quad \log : G(\mathcal{A}) \longrightarrow L(\mathcal{A})$$

are reciprocal bijections defined in the usual way.
We are interested in morphisms of scaled semigroups of the form

$$\begin{array}{rcl} \theta : \mathcal{S} & \rightarrow & G(\mathcal{A}) \\ s & \mapsto & 1_{\mathcal{A}} + \sum_{n \geq 1} h^n \theta_n(s). \end{array}$$

Definition

We write $\theta(s) \stackrel{(n)}{\equiv} \theta(s')$ if $\theta_k(s) = \theta_k(s')$ for $k = 1, \ldots, n$.

We want to characterize $\theta(s) \stackrel{(n)}{\equiv} \theta(s')$ in terms of functions on $\mathcal{S}$.

Example (Composition based on Euler's method for $\dot{y} = y$)

$\mathcal{S} = \{e\} \cup \bigcup_{m\geq 1} \mathbb{R}^{2m}$, $\mathcal{A} = \mathbb{R}$ and

$$\theta(a_1, \dots, a_{2m}) = 1 + \sum_{n\geq 1} h^n \theta_n(a_1, \dots, a_{2m}) = \prod_{j=1}^{m} \frac{1 + a_{2j-1}h}{1 + a_{2j}h}.$$

Consider the logarithm

$$\begin{aligned} \log(\theta(a_1, \dots, a_{2m})) &= \sum_{j=1}^{m} \log(1 + a_{2j-1}h) - \log(1 + a_{2j}h) \\ &= \sum_{k\geq 1} \frac{(-1)^{k+1}}{k} h^n u_k(a_1, \dots, a_{2m}), \end{aligned}$$

where $u_k(a_1, \dots, a_{2m}) = \sum_{j=1}^{2m} (-1)^j a_j^k$.

Thus, $\theta(s) \overset{(n)}{\equiv} \theta(s')$ if and only if $u_k(s) = u_k(s')$ for $1 \leq k \leq n$.

Consider the (commutative) algebra $\mathbb{R}^{\mathcal{S}}$ of functions $u : \mathcal{S} \to \mathbb{R}$, with unity $\mathbb{1} \in \mathbb{R}^{\mathcal{S}}$ (i.e., defined as $\mathbb{1}(s) = 1$ for all $s \in \mathcal{S}$).

Given a morphism $\theta : \mathcal{S} \to G(\mathcal{A})$, a linear form $\gamma \in \mathcal{A}^*$ and $n \geq 1$, consider the function $u_{\theta,n,\gamma} \in \mathbb{R}^{\mathcal{S}}$ defined by

$$u_{\theta,n,\gamma}(s) = \gamma(\theta_n(s)).$$

Observe that $u_{\theta,n,\gamma}(\lambda \cdot s) = \lambda^n \, u_{\theta,n,\gamma}(s)$.

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Definition

Given $u \in \mathbb{R}^{\mathcal{S}}$, we say that u is homogeneous of degree $|u| = n$ if

$$\forall (\lambda, s) \in \mathbb{R} \times \mathcal{S}, \quad u(\lambda \cdot s) = \lambda^n u(s).$$

Convention: $0^0 = 1$. In particular, if $|u| = 0$, then $u(s) = u(\lambda \cdot s) = u(0 \cdot s) = u(e) = u(e)\mathbb{1}(s)$, and thus $u = u(e)\mathbb{1}$.

Given a morphism $\theta : \mathcal{S} \to G(\mathcal{A})$, consider the subalgebra $\mathcal{H}^\theta \subset \mathbb{R}^{\mathcal{S}}$ generated by

$$\{u_{\theta,n,\gamma} \ : \ n \geq 1, \quad \gamma \in \mathcal{A}^*\},$$

and denote $\mathcal{H}^\theta_n = \{u \in \mathcal{H}^\theta \ : \ |u| = n\}$. (In particular, $\mathcal{H}^\theta_0 = \mathbb{R}\mathbb{1}$.)

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and denote $\mathcal{H}^\theta_n = \{u \in \mathcal{H}^\theta \ : \ |u| = n\}$. (In particular, $\mathcal{H}^\theta_0 = \mathbb{R}\mathbb{1}$.) Clearly, $\mathcal{H}^\theta = \bigoplus_{n \geq 0} \mathcal{H}^\theta_n$. Obviously, given $s, s' \in \mathcal{S}$,

$$\theta(s) \overset{(n)}{\equiv} \theta(s') \quad \Longleftrightarrow \quad \forall u \in \bigoplus_{k \leq n} \mathcal{H}^\theta_k, \quad u(s) = u(s').$$

If the subspace of $\mathcal{A}$ spanned by the range of θ_n is finite dimensional, then $\mathcal{H}^\theta_n$ is finite dimensional.

Definition

Let $\mathcal{S}$ be a scaled semigroup and $\mathcal{A}$ an algebra, we say that a morphism of scaled semigroups $\theta : \mathcal{S} \to G(\mathcal{A})$ is of finite type if each $\mathcal{H}^\theta_n$ is finite dimensional.

Theorem

If each $\mathcal{H}_n^\theta$ is finite dimensional, then given $u \in \mathcal{H}_n^\theta$ $(n \geq 0)$, there exist $m \geq 1$ and $v_1, w_1, \ldots, v_m, w_m$ with $|v_j| + |w_j| = n$ such that

$$\forall (s, s') \in \mathcal{S} \times \mathcal{S}, \quad u(s \circ s') = \sum_{j=1}^{m} v_j(s) w_j(s').$$

Given a subspace V of $\mathbb{R}^{\mathcal{S}}$, we make the standard identification of $V \otimes V$ with a subspace of $\mathbb{R}^{\mathcal{S} \times \mathcal{S}}$. That is, given $u_i, v_i \in V$, $\lambda_i \in \mathbb{R}$

$$\forall (s, s') \in \mathcal{G} \times \mathcal{G}, \quad (\textstyle\sum_i \lambda_i \, u_i \otimes v_i)(s, s') = \sum_i \lambda_i \, u_i(s) v_i(s').$$

Definition

Given $u \in \mathbb{R}^{\mathcal{S}}$, we define $\Delta u \in \mathbb{R}^{\mathcal{S} \times \mathcal{S}}$, as

$$\Delta u(s, s') = u(s \circ s'), \quad \text{for} \quad s, s' \in \mathcal{S}.$$

According to previous theorem, if for given $\theta : \mathcal{S} \rightarrow G(\mathcal{A})$ each $\mathcal{H}_n^\theta$ is finite dimensional, then $\Delta \mathcal{H}^\theta \subset \mathcal{H}^\theta \otimes \mathcal{H}^\theta$.

Furthermore, the semigroup structure of $\mathcal{S}$ together with $\mathcal{H}_0^\theta = \mathbb{R}\mathbb{1}$ implies that $\Delta\mathbb{1} = \mathbb{1} \otimes \mathbb{1}$ and for each $u \in \mathcal{H}_n^\theta$ with $n \geq 1$

$$\Delta u - u \otimes \mathbb{1} - \mathbb{1} \otimes u \in \bigoplus_{k=1}^{n-1} \mathcal{H}_k^\theta \otimes \mathcal{H}_{n-k}^\theta .$$

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Definition (Representative funcions of a scaled semigroup)

Given a scaled semigroup $\mathcal{S}$, we define $H(\mathcal{S}) = \bigoplus_{n\geq 0} H(\mathcal{S})_n$, where

$$\begin{aligned} H(\mathcal{S})_0 &= \{u \in \mathbb{R}^{\mathcal{S}} \ : \ |u| = 0\} = \mathbb{R}\mathbb{1}, \quad \text{and for} \quad n \geq 1, \\ H(\mathcal{S})_n &= \Big\{u \in \mathbb{R}^{\mathcal{S}} \ : \ |u| = n, \\ &\qquad \Delta u - u \otimes \mathbb{1} - \mathbb{1} \otimes u \in \bigoplus_{0\leq k<n} H(\mathcal{S})_k \otimes H(\mathcal{S})_{n-k} \Big\}. \end{aligned}$$

We say that u is a representative function of $\mathcal{S}$ if $u \in H(\mathcal{S})$.

Some immediate results:

- The scaled semigroup structure of $\mathcal{S}$ gives a connected graded Hopf algebra structure to $H(\mathcal{S})$.
- For each $\theta : \mathcal{S} \to G(\mathcal{A})$, $\mathcal{H}^\theta$ is a Hopf subalgebra of $H(\mathcal{S})$.

Lemma

Given $u \in \mathbb{R}^{\mathcal{S}}$, $u \in H(\mathcal{S})_n$ $(n \geq 1)$ if and only if there exists an algebra $\mathcal{A}$, a morphism of scaled semigroups $\theta : \mathcal{S} \to G(\mathcal{A})$ of finite type and a linear form $\gamma \in \mathcal{A}^*$ such that

$$\forall s \in \mathcal{S}, \quad u(s) = \gamma(\theta_n(s)).$$

Example (The group of composition integration schemes)

Consider again the scaled semigroup

$$\mathcal{S} = \{e\} \cup \bigcup_{m\geq 1} \mathbb{R}^{2m}, \quad \lambda \cdot (a_1, \ldots, a_{2m}) = (\lambda a_1, \ldots, \lambda a_{2m}),$$

$$(a_1, \ldots, a_{2m}) \circ (a_{2m+1}, \ldots, a_{2(m+k)}) = (a_1, \ldots, a_{2(m+k)}),$$

and let $\sim$ be the finest equivalence relation satisfying that $(a, a) \sim e$ and $(a_1, \ldots, a_{j-1}, b, b, a_j, \ldots, a_{2m}) \sim (a_1, \ldots, a_{2m})$. Clearly, $\mathcal{G}_c = \mathcal{S}/\sim$ has a scaled group structure inherited from the scaled semigroup structure of $\mathcal{S}$. Each element in $\psi = \mathcal{G}_c \backslash \{e\}$ can be uniquely written as

$$\psi = \chi(a_1)^{-1} \circ \chi(a_2) \circ \cdots \circ \chi(a_{2m-1})^{-1} \circ \chi(a_{2m})$$

where $a_{j-1} \neq a_j$ $(2 \leq j \leq 2m)$ and $\chi(a)$ represented by $(0, a) \in \mathcal{S}$.

Example (cont.)

For an arbitrary algebra $\mathcal{A}$ and any morphism of scaled groups

$$\begin{array}{rcl} \theta : \mathcal{G}_c & \rightarrow & G(\mathcal{A}) \\ s & \mapsto & 1_{\mathcal{A}} + \sum_{n \geq 1} h^n \theta_n(s), \end{array}$$

with the notation $X_n = \theta_n(\chi(1)) \in \mathcal{A}$, one necessarily has for each $\psi = \chi(a_1)^{-1} \circ \chi(a_2) \circ \cdots \circ \chi(a_{2m-1})^{-1} \circ \chi(a_{2m}) \in \mathcal{G}_c$ that

$$\theta(\psi) = X(a_1 h)^{-1} X(a_2 h) \cdots X(a_{2m-1} h)^{-1} X(a_{2m} h),$$

where $X(ah) = 1_{\mathcal{A}} + \sum_{n \geq 1} h^n a^n X_n$ and

$$X(ah)^{-1} = 1_{\mathcal{A}} + \sum_{n \geq 1} a^n h^n \sum_{j_1 + \cdots + j_r = n} (-1)^r X_{j_1} \cdots X_{j_r}.$$

Example (cont.)

That shows that, for each $\psi \in \mathcal{G}_c$,

$$\theta(\psi) = 1_{\mathcal{A}} + \sum_{n \geq 1} h^n \sum_{j_1 + \cdots + j_r = n} u_{j_1 \cdots j_r}(\psi) \, X_{j_1} \cdots X_{j_r},$$

for some $u_{j_1 \cdots j_r} \in \mathbb{R}^{\mathcal{G}_c}$ with $|u_{j_1 \cdots j_r}| = j_1 + \cdots + j_r$.

Example (cont.)

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for some $u_{j_1\cdots j_r} \in \mathbb{R}^{\mathcal{G}_c}$ with $|u_{j_1\cdots j_r}| = j_1 + \cdots + j_r$. Actually,

$$\begin{aligned}
u_i(a_1,\ldots,a_{2m}) &= \sum_{1\leq j\leq 2m} (-1)^j\, a_j^i, \\
u_{i_1 i_2}(a_1,\ldots,a_{2m}) &= \sum_{1\leq j_1\leq j_2^*\leq j_2\leq 2m} (-1)^{j_1+j_2}\, a_{j_2}^{i_2} a_{j_1}^{i_1}, \\
u_{i_1 i_2 i_3}(a_1,\ldots,a_{2m}) &= \sum_{1\leq j_1\leq j_2^*\leq j_2\leq j_3^*\leq j_3\leq 2m} (-1)^{j_1+j_2+j_3}\, a_{j_3}^{i_3} a_{j_2}^{i_2} a_{j_1}^{i_1},
\end{aligned}$$

and so on. Notation: $j^* = j-1$ if j is even, and $j^* = j$ if j is odd. From previous lemma, $H(\mathcal{G}_c)$ is spanned by the functions $u_{j_1\cdots j_r}$.

For a given $\mathcal{S}$, one is not always interested in characterizing

$$\theta(s) \overset{(n)}{\equiv} \theta(s') \tag{6}$$

for all posible morphisms $\theta : \mathcal{S} \to G(\mathcal{A})$. Recall that a characterization of (6) for one particular morphism θ is obtained with the Hopf subalgebra $\mathcal{H}^{\theta} \subset H(\mathcal{S})$ (instead of the whole $H(\mathcal{S})$).

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Theorem

Consider a scaled group $\mathcal{G}$ and a family $\{\theta^j : \mathcal{G} \to G(\mathcal{A}^j)\}_{j\in\mathcal{J}}$ of morphisms of scaled semigroups. Let $\mathcal{H}$ be a subalgebra of $\mathcal{H}(\mathcal{G})$ with finite dimensional $\mathcal{H}_n = \{u \in \mathcal{H} \ : \ |u| = n\}$ $(n \geq 1)$. The following statement holds for arbitrary $s, s' \in \mathcal{G}$ and $n \geq 1$

$$\forall u \in \bigoplus_{0\leq k\leq n} \mathcal{H}_k, \quad u(s) = u(s') \quad \Longleftrightarrow \quad \forall j \in \mathcal{J}, \quad \theta^j(s) \overset{(n)}{\equiv} \theta^j(s'),$$

iff $\mathcal{H}$ is the Hopf subalgebra of $H(\mathcal{G})$ generated by $\bigcup_{j\in\mathcal{J}} \mathcal{H}^{\theta^j}$.

Definition

We say that $(\mathcal{G}, \mathcal{H})$ is a group of abstract integration schemes if $\mathcal{G}$ is a scaled subgroup and $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ is a graded Hopf subalgebra of $H(\mathcal{G})$ satisfying the following:

- Each $\mathcal{H}_n$ is finite dimensional.
- $\mathcal{H}$ separates the elements in $\mathcal{G}$, i.e., $\forall (s, s') \in \mathcal{G}$, $\exists u \in H$ such that $u(s) \neq u(s')$.

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As an algebra $\mathcal{H}$ is freely generated (as a consequence of Milnor and Moore theorem). If $\mathcal{H} \neq H(\mathcal{G})$, then the functions in $\mathcal{H}$ characterize $\theta(s) \overset{(n)}{\equiv} \theta(s')$ for a strict subclass of morphisms $\theta : \mathcal{G} \to G(\mathcal{A})$ of finite type (precisely, the morphisms θ such that $\mathcal{H}^\theta \subset \mathcal{H}$).

Theorem ($\mathcal{G}$ dense in $\overline{\mathcal{G}}$)

Let $(\mathcal{G}, \mathcal{H})$ be a group of abstract integration schemes, and let $\overline{\mathcal{G}}$ denote the group of characters of $\mathcal{H}$. For each $\alpha \in \overline{\mathcal{G}}$ and each $n \geq 1$, there exists $\psi \in \mathcal{G}$ such that

$$\forall u \in \bigoplus_{k \leq n} \mathcal{H}_k, \quad u(\psi) = \alpha(u). \tag{7}$$

Let $\mathcal{T}$ be a set of homogeneous functions on $\mathcal{G}$ that freely generate the algebra $\mathcal{H}$, then

$$u(\psi) = \alpha(u), \quad \forall u \in \mathcal{T} \text{ with } |u| \leq n,$$

provides necessary and sufficient independent conditions for (7).

- The coalgebra structure of $\mathcal{H}$ endows its linear dual $\mathcal{H}^*$ with an algebra structure. ($\{\nu_\lambda\}$ induce $\{\bar{\nu}_\lambda\}$).
- The subset $\overline{\mathcal{G}} \subset \mathcal{H}^*$ of algebra maps $\alpha : \mathcal{H} \to \mathbb{R}$ is a group (the group of characters). It is a scaled group with $\bar{\nu}_\lambda(\alpha) = \lambda \cdot \alpha$

$$\lambda \cdot \alpha(u) = \lambda^n \alpha(u) \quad \forall u \in \mathcal{H}_n.$$

- The map $\pi : \mathcal{G} \to \overline{\mathcal{G}}$ such that $\forall \psi \in \mathcal{G}$, $\pi(\psi)$ is defined by $\pi(\psi)(u) = u(\psi)$ is a monomorphism of (scaled) groups. So that $\mathcal{G}$ can be seen as a scaled subgroup of $\overline{\mathcal{G}}$.
- There is a subset $\mathfrak{g} \subset \mathcal{H}^*$ that is a Lie algebra under the bracket $[\alpha, \beta] = \alpha\beta - \beta\alpha$ (the Lie algebra of infinitesimal characters), such that $\exp : \mathfrak{g} \to \overline{\mathcal{G}}$ is a bijection.

Example (cont.)

Consider $\mathcal{H} = H(\mathcal{G}_c)$. Given an algebra $\mathcal{A}$ (for instance, $\mathcal{A} = \mathrm{End}_{\mathbb{R}}\mathrm{C}^\infty(\mathbb{R}^d;\mathbb{R})$) and $\theta : \mathcal{G}_c \to G(\mathcal{H})$, we define for each $\alpha \in \mathcal{H}^*$ the algebra morphism $\bar{\theta} : \mathcal{H}^* \to \mathcal{A}[[h]]$ as

$$\bar{\theta}(\alpha) = \alpha(\mathbb{1})\, 1_{\mathcal{A}} + \sum_{n\geq 1} h^n \sum_{j_1+\cdots+j_r=n} \alpha(u_{j_1\cdots j_r})\, X_{j_1}\cdots X_{j_r}.$$

Given $g_1, g_2 \in \mathrm{C}^\infty(\mathbb{R}^d;\mathbb{R})$, if $\alpha \in \overline{\mathcal{G}}$, then

$$\bar{\theta}(\alpha)[g_1 g_2] = \bar{\theta}(\alpha)[g_1]\,\bar{\theta}(\alpha)[g_2].$$

And if $\alpha \in \mathfrak{g}$, then

$$\bar{\theta}(\alpha)[g_1 g_2] = g_1\,\bar{\theta}(\alpha)[g_2] + g_2\bar{\theta}(\alpha)[g_1].$$

Theorem

Let $(\mathcal{G}, \mathcal{H})$ be a group of abstract integration schemes, and let $\theta : \mathcal{G} \to G(\mathcal{A})$ (with $\mathcal{A}$ certain algebra) be a morphism of scaled groups such that $\mathcal{H}^{\theta} \subset \mathcal{H}$. Then, there exists a unique algebra morphism $\bar{\theta} : \mathcal{H}^* \to \mathcal{A}[[h]]$ such the $\bar{\theta}(\pi(\psi)) = \theta(\psi)$. When restricted to $\overline{\mathcal{G}}$, it is a morphism $\bar{\theta} : \overline{\mathcal{G}} \to G(\mathcal{A})$ of scaled groups.

Observations:

- In applications to numerical analysis, there is typically a distinguished element $\alpha \in \overline{\mathcal{G}}$ such that

$$\bar{\theta}(\alpha) = 1_{\mathcal{A}} + \sum_{n \geq 1} \bar{\theta}_n(\alpha)\, h^n$$

  represents the exact solution to be approximated.
- Backward error analysis: For each $\psi \in \mathcal{G}$, $\theta(\psi) = \exp(\bar{\theta}(\beta))$, where $\beta = \log(\pi(\psi)) \in \mathfrak{g}$.

Theorem

Let $\mathcal{G}$ be a scaled group, and let $\mathcal{H}$ be a subalgebra of $\mathcal{H}(\mathcal{G})$ with finite dimensional $\mathcal{H}_n = \{u \in \mathcal{H} \ : \ |u| = n\}$ $(n \geq 1)$. Assume that the following statement holds for arbitrary $s, s' \in \mathcal{G}$ and $n \geq 1$:

$$\forall u \in \bigoplus_{0 \leq k \leq n} \mathcal{H}_k, \quad u(s) = u(s') \quad \Longrightarrow \quad \theta(s) \overset{(n)}{\equiv} \theta(s')$$

for arbitrary algebras $\mathcal{A}$ and arbitrary morphisms of scaled groups $\theta : \mathcal{G} \to G(\mathcal{A})$. Then $\mathcal{H} = H(\mathcal{G})$.

Theorem

Let $\mathcal{S}$ be a scaled semigroup. For arbitrary $s, s' \in \mathcal{S}$ and $n \geq 1$, the following two statements are equivalent:

- $\forall u \in \bigoplus_{0 \leq k \leq n} H(\mathcal{S})_k, \quad u(s) = u(s')$.
- $\theta(s) \overset{(n)}{\equiv} \theta(s')$ *for arbitrary algebras $\mathcal{A}$ and arbitrary morphisms of scaled semigroups $\theta : \mathcal{S} \to G(\mathcal{A})$ of finite type.*

Example (cont.)

$H(\mathcal{G}_c)$ is isomorphic to the quasi-shuffle Hopf algebra of Hoffman, thus is freely generated by the functions $u_{j_1\cdots j_r}$ indexed by the set of Lyndon words $j_1\cdots j_r$ on the alphabet $\{1,2,3,\ldots\}$

$$\mathcal{L} = \{u_{j_1\cdots j_r} \ : \ (j_1\cdots j_k) < (j_{k+1}\cdots j_r) \text{ for each } 1 \le k < r\}$$

The first sets $\mathcal{L}_n = \{u_{j_1\cdots j_r} \in \mathcal{L} \ : |u_{j_1\cdots j_r}| = j_1 + \cdots + j_r = n\}$ are

$$\begin{aligned}\mathcal{L}_1 &= \{u_1\}, \quad \mathcal{L}_2 = \{u_2\}, \quad \mathcal{L}_3 = \{u_{12}, u_3\}, \quad \mathcal{L}_4 = \{u_{112}, u_{13}, u_4\},\\ \mathcal{L}_5 &= \{u_{1112}, u_{113}, u_{122}, u_{14}, u_{23}, u_5\}.\end{aligned}$$

Theorem Given $\psi, \psi' \in \mathcal{G}_c$, $\theta(\psi) \overset{(n)}{\equiv} \theta(\psi')$ for arbitrary algebras $\mathcal{A}$ and any morphism $\theta : \mathcal{S} \to G(\mathcal{A})$, if and only if

$$\forall u \in \bigcup_{k\ge 1}^{n} L_k, \quad u(\psi) = u(\psi').$$