Clermont-Ferrand, Octobre 2008

Intégration numérique géométrique et formalisme des Butcher-séries

Gilles Vilmart
(Rennes & Genève)

Travail commun avec
Philippe Chartier et Ernst Hairer

Plan of the talk

1. Main ideas of the **theory of modified differential equations** for the study of geometric integrators.

   (Backward error analysis)

2. **Modifying (preprocessed) vector field integrators**.

   high-order structure preserving algorithms based on modified differential equations

3. **Case of B-series methods.**

   $\longrightarrow$ A substitution law on B-series.

   $\longrightarrow$ A new Hopf tree algebra (Calaque, Ebrahimi-Fard, Manchon, 2008).

Part 1

Main ideas of the **theory of modified differential equations** for the study of geometric integrators.

(Backward error analysis)

Example

A two-dimensional Hamiltonian system,

$$\begin{aligned}\dot{q} &= p \\ \dot{p} &= -\nabla V(q)\end{aligned}$$

with a quartic potential $V(q) = (q^2 - 1)^2$.

$$\text{Hamiltonian } H(q,p) = \tfrac{1}{2}p^2 + (q^2 - 1)^2.$$

→ animation

Studied recently in the context of the computation of conjugate points for the Martinet case is sub-Riemannian geometry (optimal control).

Monique Chyba, Ernst Hairer & Gilles Vilmart, **The role of sympectic integrators in optimal control**, *Optimal Control, Applications and Methods*, 2008

Backward error analysis

Given a differential equation

$$\dot{y} = f(y), \quad y(0) = y_0$$

and a numerical one-step method

$$y_{n+1} = \Phi_h(y_n)$$

Find a modified differential equation

$$\dot{\widetilde{y}} = \widetilde{f}_h(\widetilde{y}) = f(\widetilde{y}) + hf_2(\widetilde{y}) + h^2 f_3(\widetilde{y}) + h^3 f_4(\widetilde{y}) + \ldots, \quad \widetilde{y}(0) = y_0$$

such that for $t_n = nh$,

$$y_n = \widetilde{y}(t_n)$$

Explanation: backward error analysis

For a symplectic integrator, e.g. the symplectic Euler method

$$\begin{aligned} q_{n+1} &= q_n + hH_p(p_n, q_{n+1}) \\ p_{n+1} &= p_n - hH_q(p_n, q_{n+1}), \end{aligned}$$

it can be shown that the numerical solution is (formally) the exact solution of a modified Hamiltonian system,

$$\begin{aligned} \dot{q} &= \widetilde{H}_p(q,p) \\ \dot{p} &= -\widetilde{H}_q(q,p) \end{aligned}$$

$$\widetilde{H}(q,p) = H(q,p) - hpq(2q^2 - 2) + \frac{h^2}{3}\left(4q^6 - 8q^4 + 4q^2 + 3p^2q^2 - p^2\right) + \dots$$

Here, $\{\widetilde{H}(q,p) = const\}$ is a closed curve. The numerical solution remains exponentially near a closed orbit on exponentially long intervals.

Construction of the modified equation

$$\text{Aim:} \qquad \Phi_{f,h}(y) = \varphi_{\widetilde{f}_h,h}(y).$$

$$\dot{y} = f^{[r]}(y) = f(y) + hf_2(y) + h^2 f_3(y) + \ldots + h^{r-1} f_r(y),$$

The functions $f_j(y)$ are constructed by induction:

$$\Phi_{f,h}(y) = \varphi_{f^{[r]},h}(y) + h^{r+1} f_{r+1}(y) + \mathcal{O}(h^{r+2}).$$

This is possible because

$$\varphi_{f+\varepsilon g,h}(y) = \varphi_{f,h}(y) + h\varepsilon g(y) + \mathcal{O}(h^2 \varepsilon).$$

Hamiltonian system

Theorem (Poincaré, 1899) For all time t, the Jacobian of the exact flow satisfies

$$\varphi_t'^T(y)\,J\,\varphi_t'(y) = J.$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the canonical matrix for Hamiltonian systems.

Definition A numerical integrator $y_{n+1} = \Phi_h(y_n)$ is symplectic if

$$\Phi_h'^T(y)\,J\,\Phi_h'(y) = J.$$

Examples : the exact flow, the symplectic Euler method, Störmer-Verlet, the implicit midpoint rule, Gauss methods...

Property : Hamiltonian systems

Theorem If the integrator $\Phi_{f,h}(y)$ is symplectic for all $f(y) = J^{-1}\nabla H(y)$, then the modified differential equation is also Hamiltonian:

$$\dot{y} = \widetilde{f}(y) = J^{-1}\nabla \widetilde{H}_h(y),$$

with

$$\widetilde{H}_h(y) = H(y) + h\, H_2(y) + h^2 H_3(y) + \dots\ .$$

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with

$$\widetilde{H}_h(y) = H(y) + h\, H_2(y) + h^2 H_3(y) + \ldots\ .$$

Proof: Moser (1968), Benettin & Giorgilli (1994), Tang (1994).

Integrability Lemma : If $f(y) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is differentiable with $f'(y)$ symmetric, then there exists $H(y) : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying

$$f(y) = \nabla H(y).$$

Part 2

Modifying (preprocessed) vector field integrators.

high-order structure preserving algorithms based on modified differential equations

Backward error analysis

Given a differential equation

$$\dot{y} = f(y), \quad y(0) = y_0$$

and a numerical one-step method

$$y_{n+1} = \Phi_h(y_n)$$

Find a modified differential equation

$$\dot{z} = f(z) + hf_2(z) + h^2 f_3(z) + h^3 f_4(z) + \ldots, \quad z(0) = y_0$$

such that for $t_n = nh$,

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Modifying vector field integrators

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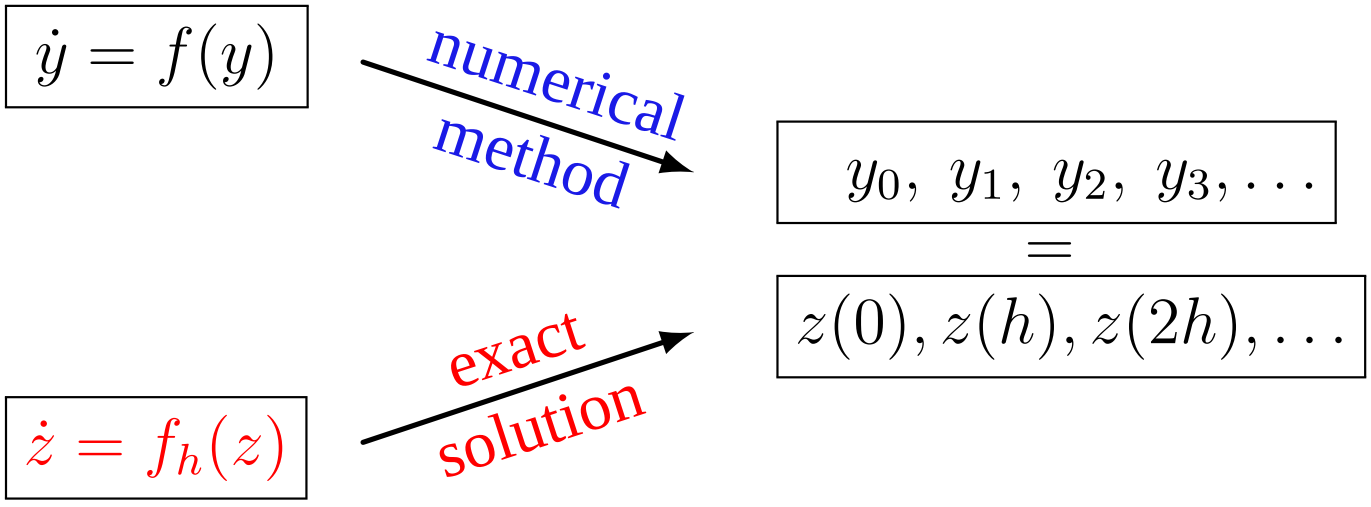
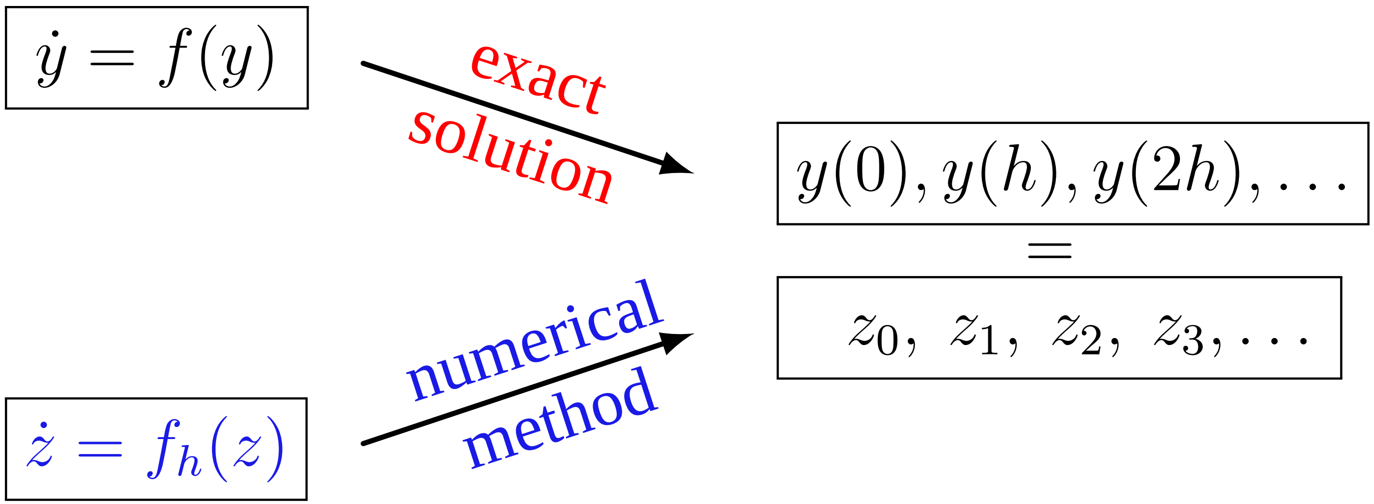
Summary

Backward error analysis



Modifying numerical method



Modifying vector field integrators

Chartier, Hairer & Vilmart, **Numerical integrators based on modified differential equations**, *Math. Comp.*, 2007

Algorithm. Given $y_{n+1} = \Phi_h(y_n)$, consider the truncation

$$\dot{z} = f_h^{[r]}(z) = f(z) + hf_2(z) + \cdots + h^{r-1} f_r(z).$$

Then,

$$z_{n+1} = \Phi_{f_h^{[r]},h}(z_n)$$

defines a numerical method of order r for $\dot{y} = f(y)$.

In general, the geometric properties are maintained
(e.g. symplecticity, symmetry, first integrals preservation. . .).

Related work: Feng Kang, 1986, Feng Kang, Wu, Qin & Wang, 1989, Channel & Scovel, 1990, McLachlan & Zanna, 2005

Example: implicit midpoint rule

For the general differential equation $\dot{y} = f(y)$ we consider

$$y_{n+1} = y_n + hf\Big(\frac{y_n + y_{n+1}}{2}\Big).$$

Modified differential equation for the preprocessed method:

$$\dot{z} = f(z) + h^2 f_3(z) + h^4 f_5(z) + \dots$$

where

$$\begin{aligned} f_3 &= \frac{1}{12}\Big(-f'f'f + \frac{1}{2}f''(f,f)\Big), \\ f_5 &= \frac{1}{120}\Big(f'f'f'f'f - f''(f,f'f'f) + \frac{1}{2}f''(f'f,f'f)\Big) \\ &+ \frac{1}{240}\Big(-\frac{1}{2}f'f'f''(f,f) + f'f''(f,f'f) + \frac{1}{2}f''(f,f''(f,f)) - \frac{1}{2}f^{(3)}(f,f,f'f)\Big) \\ &+ \frac{1}{80}\Big(-\frac{1}{6}f'f^{(3)}(f,f,f) + \frac{1}{24}f^{(4)}(f,f,f,f)\Big). \end{aligned}$$

Part 3

Case of B-series methods.

$\longrightarrow$ A substitution law on B-series.

$\longrightarrow$ A new Hopf tree algebra (Calaque, Ebrahimi-Fard, Manchon, 2008).

B-series

A theory initiated by John C. Butcher in the years 60-70 for the study of Kunge-Kutta methods.

Exact solution

$$\dot{y} = f(y), \quad y(0) = y_0.$$

$$\begin{aligned}
\dot{y} &= f(y), \\
\ddot{y} &= f'(y)\dot{y} = f'(y)f(y), \\
y^{(3)} &= f''(y)(\dot{y}, \dot{y}) + f'(y)\ddot{y} \\
&= f''(y)(f(y), f(y)) + f'(y)f'(y)f(y).
\end{aligned}$$

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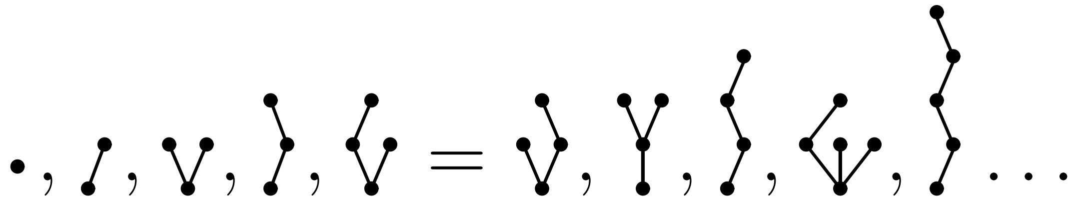
&= f''(y)(f(y), f(y)) + f'(y)f'(y)f(y).
\end{aligned}$$

$$\begin{aligned}
y(h) &= y_0 + hf(y_0) + \frac{h^2}{2} f'(y_0)f(y_0) \\
&\quad + \frac{h^3}{6}\Big(f''(y_0)(f(y_0), f(y_0)) + f'(y_0)f'(y_0)f(y_0)\Big) + \ldots
\end{aligned}$$

Formalism of trees

A theory initiated by John C. Butcher in the years 60-70 for the study of Kunge-Kutta methods.

$\emptyset$ = empty tree
T = Set of trees : 

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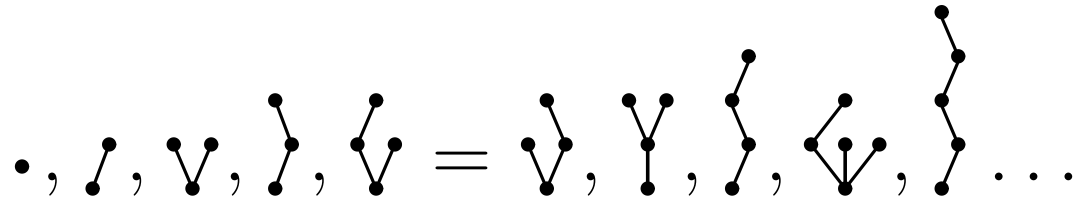
T = Set of trees : 

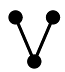
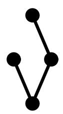
Definition : Elementary differentials (R. H. Merson, 1957)

$$F(\emptyset)(y) = y$$

$$F(\bullet)(y) = f(y)$$

$$F(\tau)(y) = f^{(m)}(y)(F(\tau_1)(y), \ldots, F(\tau_m)(y)),$$
for $\tau = [\tau_1, \ldots, \tau_m]$.

Examples :

$F(\bullet) = f,\ F($  $) = f'f,\ F($  $) = f''(f,f),\ F($  $) = f''(f,f'f).$

B-series

Definition of a B-series (Hairer & Wanner, 1974):
For $a : T \cup \{\emptyset\} \longrightarrow \mathbb{R}$, a B-series is a formal series of the form

$$
\begin{aligned}
B(f, a)(y) &= a(\emptyset)y + h a(\bullet) f(y) + h^2 a([\bullet]) f'(y) f(y) \\
&+ \frac{h^3}{2} a([\bullet,\bullet]) f''(y)(f(y), f(y)) + \ldots
\end{aligned}
$$

Called B-series in honor of John C. Butcher who initiated the theory in the years 60-70.

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Exact solution :
The Taylor series of the exact solution is a B-series

$$y(h) = B(f,e)(y_0).$$

B-series

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$$\begin{aligned} B(f,a)(y) &= a(\emptyset)y + ha(\bullet)f(y) + h^2 a([\bullet])f'(y)f(y) \\ &+ \frac{h^3}{2}a([\bullet,\bullet])f''(y)(f(y),f(y)) + \ldots \end{aligned}$$

Theorem (Butcher) :
All Runge-Kutta methods are B-series methods.

If $y_{n+1} = \Phi_{f,h}(y_n)$ is a Runge-Kutta method, there exists a B-series $B(f,a)(y)$ satisfying (formally) :

$$\Phi_{f,h}(y) = B(f,a)(y).$$

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For $a : T \cup \{\emptyset\} \longrightarrow \mathbb{R}$, a B-series is a formal series of the form

$$\begin{aligned} B(f,a)(y) &= a(\emptyset)y + h a(\bullet) f(y) + h^2 a\left(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\right) f'(y)f(y) \\ &+ \frac{h^3}{2} a\left(\begin{smallmatrix}\bullet\;\;\bullet\\ \vee\\ \bullet\end{smallmatrix}\right) f''(y)(f(y),f(y)) + \ldots \end{aligned}$$

Example: For a one-stage Runge-Kutta method,

$$\begin{array}{c|c} & a_{1,1} \\ \hline & b_1 \end{array},$$

we have

$$a(\tau) = b_1 \; {a_{1,1}}^{|\tau|-1}.$$

Implicit Euler : $a(\tau) = 1$, Midpoint rule : $a(\tau) = \left(\frac{1}{2}\right)^{|\tau|-1}$.

Composition of B-series

Theorem: The Butcher group

For $a(\emptyset) = 1$, inserting $B(f, a)(y)$ into $B(f, b)(\cdot)$ yields a new B-series

$$B(f, b)\Big(B(f, a)(y)\Big) = B(f, a \cdot b)(y).$$

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$$B(f, b)\Big(B(f, a)(y)\Big) = B(f, a \cdot b)(y).$$

$$\begin{aligned}
a \cdot b(\emptyset) &= b(\emptyset), \\
a \cdot b(\bullet) &= b(\emptyset)a(\bullet) + b(\bullet), \\
a \cdot b([\bullet]) &= b(\emptyset)a([\bullet]) + b(\bullet)a(\bullet) + b([\bullet]), \\
a \cdot b([\bullet,\bullet]) &= b(\emptyset)a([\bullet,\bullet]) + b(\bullet)a(\bullet)^2 + 2b([\bullet])a(\bullet) + b([\bullet,\bullet]), \\
a \cdot b([[\bullet]]) &= b(\emptyset)a([[\bullet]]) + b(\bullet)a([\bullet]) + b([\bullet])a(\bullet) + b([[\bullet]]), \\
&\dots
\end{aligned}$$

Connection with Hopf tree algebra

A surprising connection (Arne Dür 1986, Brouder 2000) between Runge-Kutta theory and Hopf tree algebra of Connes & Kreimer in renormalization in quantum field theory.

Let H_{CK} denote the commutative $\mathbb{R}$-algebra of trees,

Examples:

A map $a : T \cup \{\emptyset\} \rightarrow \mathbb{R}$ with $a(\emptyset) = 1$ can be uniquely extended to H_{CK} as algebra morphism.

Connection with Hopf tree algebra

Besides the usual product, the Hopf algebra H_{CK} is a graded algebra equipped with:

- A co-unit $\epsilon : H_{CK} \to \mathbb{R}$. $\quad (\epsilon(\tau) = 1$ if $\tau = \emptyset,\ 0$ else).
- A coproduct $\Delta_{CK} : H_{CK} \to H_{CK} \otimes H_{CK}$.

$$\Delta_{CK}([[\bullet]]) = [[\bullet]] \otimes \emptyset + \bullet^{2} \otimes \bullet + 2\bullet \otimes [\bullet] + \emptyset \otimes [[\bullet]].$$

- An antipode $S_{CK} : H_{CK} \to H_{CK}$.

$$S_{CK}([[\bullet]]) = -[[\bullet]] + 2[\bullet]\bullet - \bullet^{3}.$$

Connection with Hopf tree algebra

- A coproduct $\Delta_{CK} : H_{CK} \to H_{CK} \otimes H_{CK}$.

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- An antipode $S_{CK} : H_{CK} \to H_{CK}$.

$$S_{CK}([\bullet,\bullet]) = -[\bullet,\bullet] + 2[\bullet]\,\bullet - \bullet^3.$$

The Butcher group is the character group of convolution on H_{CK}:

$$a \cdot b = \mu \circ (a \otimes b) \circ \Delta_{CK}$$

$$a \cdot b([\bullet,\bullet]) = a([\bullet,\bullet]) + a(\bullet)^2 b(\bullet) + 2a(\bullet)b([\bullet]) + b([\bullet,\bullet]).$$

$$a^{\cdot -1}([\bullet,\bullet]) = a \circ S_{CK}([\bullet,\bullet]) = -a([\bullet,\bullet]) + 2a([\bullet])a(\bullet) - a(\bullet)^3$$

Connection with Hopf tree algebra

- A coproduct $\Delta_{CK} : H \longrightarrow H \otimes H$.

$$\boxed{\Delta_{CK}\left(\substack{\bullet\\|\\\bullet\\|\\\bullet}\right) = \substack{\bullet\\|\\\bullet\\|\\\bullet} \otimes \emptyset + \bullet^2 \otimes \bullet + 2\bullet \otimes \substack{\bullet\\|\\\bullet} + \emptyset \otimes \substack{\bullet\\|\\\bullet\\|\\\bullet}.}$$

- An antipode $S_{CK} : H \longrightarrow H$.

$$S_{CK}\left(\substack{\bullet\\|\\\bullet\\|\\\bullet}\right) = -\substack{\bullet\\|\\\bullet\\|\\\bullet} + 2\substack{\bullet\\|\\\bullet}\,\bullet - \bullet^3.$$

The Butcher group is the character group of convolution on H:

$$a \cdot b = \mu \circ (a \otimes b) \circ \Delta_{CK}$$

$$\boxed{a \cdot b\left(\substack{\bullet\\|\\\bullet\\|\\\bullet}\right) = a\left(\substack{\bullet\\|\\\bullet\\|\\\bullet}\right) + a(\bullet)^2 b(\bullet) + 2a(\bullet)b\left(\substack{\bullet\\|\\\bullet}\right) + b\left(\substack{\bullet\\|\\\bullet\\|\\\bullet}\right).}$$

$$a^{\cdot -1}\left(\substack{\bullet\\|\\\bullet\\|\\\bullet}\right) = a \circ S_{CK}\left(\substack{\bullet\\|\\\bullet\\|\\\bullet}\right) = -a\left(\substack{\bullet\\|\\\bullet\\|\\\bullet}\right) + 2a\left(\substack{\bullet\\|\\\bullet}\right)a(\bullet) - a(\bullet)^3$$

Substitution law for B-series

Theorem (Chartier, Hairer & Vilmart) For $b(\emptyset) = 0$, the vector field $\frac{1}{h}B(f,b)$ inserted into $B(\cdot,a)(y)$ is still a B-series

$$B\left(\frac{1}{h}B(f,b),a\right)(y) = B(f,b\star a)(y).$$

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$$B\Big(\frac{1}{h}B(f,b), a\Big)(y) = B(f, b \star a)(y).$$

$$\begin{aligned}
&B(f, b\star a)(y)\\
&= a(\emptyset)y + ha(\bullet)\Big(b(\bullet)f(y) + hb(\bullet\!\!-\!\!\bullet)f(y)'f(y) + \dots\Big)\\
&\qquad + h^2 a(\bullet\!\!-\!\!\bullet)\Big(b(\bullet)f(y) + \dots\Big)'\Big(b(\bullet)f(y) + \dots\Big) + \dots\\
&= a(\emptyset)y + ha(\bullet)b(\bullet)f(y) + h^2\Big(a(\bullet)b(\bullet\!\!-\!\!\bullet) + a(\bullet\!\!-\!\!\bullet)b(\bullet)^2\Big)f(y)'f(y) + \dots
\end{aligned}$$

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$$B\Big(\frac{1}{h}B(f,b), a\Big)(y) = B(f, b\star a)(y).$$

$$
\begin{aligned}
(b\star a)(\emptyset) &= a(\emptyset),\\
(b\star a)(\bullet) &= a(\bullet)b(\bullet),\\
(b\star a)([\bullet]) &= a(\bullet)b([\bullet]) + a([\bullet])b(\bullet)^2,\\
(b\star a)([\bullet,\bullet]) &= a(\bullet)b([\bullet,\bullet]) + 2a([\bullet])b(\bullet)b([\bullet]) + a([\bullet,\bullet])b(\bullet)^3,\\
(b\star a)([[\bullet]]) &= a(\bullet)b([[\bullet]]) + 2a([\bullet])b(\bullet)b([\bullet]) + a([[\bullet]])b(\bullet)^3,\\
&\dots
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$$B\Big(\frac{1}{h}B(f,b), a\Big)(y) = B(f, b \star a)(y).$$

Given a numerical integrator $y_{n+1} = B(\cdot, a)(y_n)$, coefficients $b(\tau)$ for the modified differential equations are given by:

- **backward error analysis**:

$$(b \star e)(\tau) = a(\tau).$$

- **preprocessed vector field integrators**:

$$(b \star a)(\tau) = e(\tau).$$

Substitution law $\star$: an explicit formula

$$
\begin{aligned}
b\star a(\emptyset) &= a(\emptyset),\\
b\star a(\bullet) &= a(\bullet)b(\bullet),\\
b\star a([\bullet]) &= a(\bullet)b([\bullet]) + a([\bullet])b(\bullet)^2,\\
b\star a([\bullet\bullet]) &= a(\bullet)b([\bullet\bullet]) + 2a([\bullet])b(\bullet)b([\bullet]) + a([\bullet\bullet])b(\bullet)^3,\\
b\star a([[\bullet]]) &= a(\bullet)b([[\bullet]]) + 2a([\bullet])b(\bullet)b([\bullet]) + a([[\bullet]])b(\bullet)^3,\\
b\star a([\bullet[\bullet]]) &= a(\bullet)b([\bullet[\bullet]]) + a([\bullet])b(\bullet)b([[\bullet]]) + a([\bullet])b([\bullet])^2 + a([\bullet])b(\bullet)b([\bullet\bullet])\\
&\quad +2a([\bullet\bullet])b(\bullet)^2b([\bullet]) + a([[\bullet]])b(\bullet)^2b([\bullet]) + a([\bullet[\bullet]])b(\bullet)^4,
\end{aligned}
$$

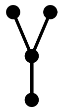
$\dots$

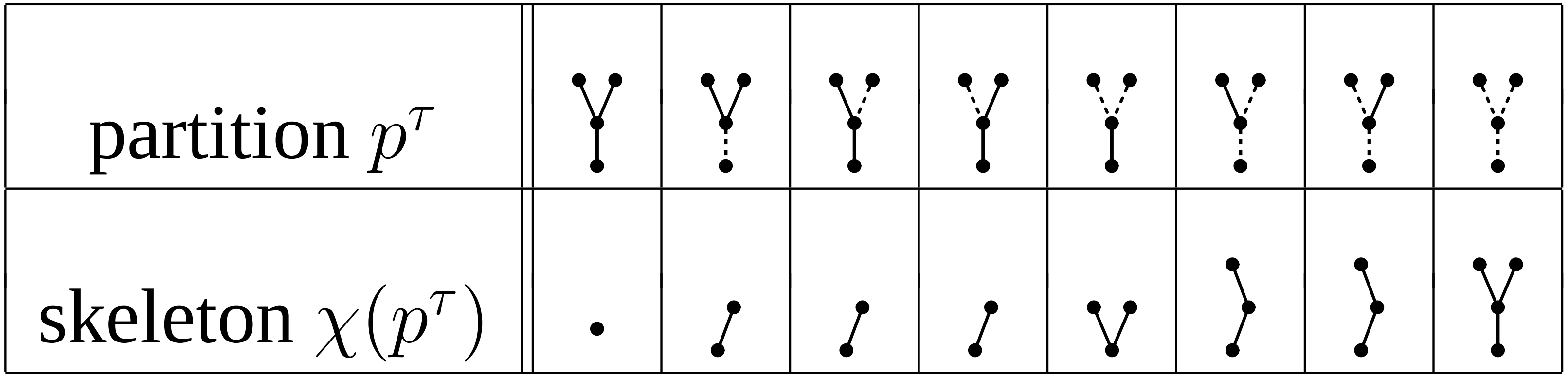
Substitution law $\star$: an explicit formula

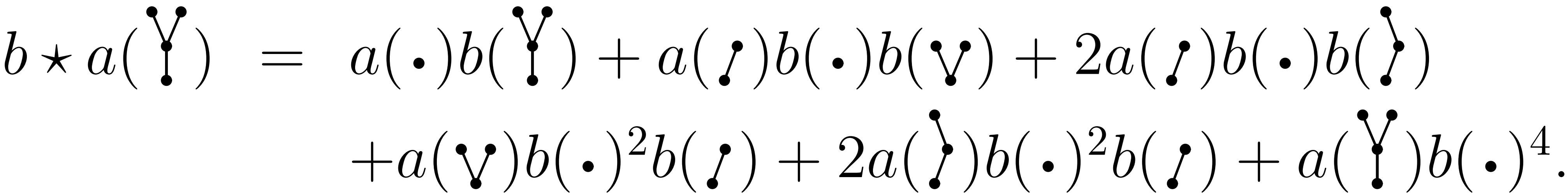
$$\forall \tau \in T,\ b \star a(\tau) = \sum_{p^\tau \in \mathcal{P}(\tau)} a(\chi(p^\tau)) \prod_{\delta \in P(p^\tau)} b(\delta).$$

Substitution law $\star$: an explicit formula

$$\forall \tau \in T,\ b \star a(\tau) = \sum_{p^\tau \in \mathcal{P}(\tau)} a(\chi(p^\tau)) \prod_{\delta \in P(p^\tau)} b(\delta).$$

Example: for tree .





Relationships with the Butcher composition

Let $a, b, \widetilde{b}, c, \widetilde{c} : T \cup \{\emptyset\} \to \mathbb{R}$ be mappings satisfying $a(\emptyset) = 1$ and $b(\emptyset) = \widetilde{b}(\emptyset) = 0$. Then, for all $\lambda, \mu \in \mathbb{R}$:

$$
\begin{aligned}
b \star \delta_{\emptyset} &= \delta_{\emptyset}, \\
b \star \delta_{\bullet} &= \delta_{\bullet} \star b = b, \\
b \star (\lambda c + \mu \widetilde{c}) &= \lambda (b \star c) + \mu (b \star \widetilde{c}), \\
(\widetilde{b} \star b) \star c &= \widetilde{b} \star (b \star c), \\
b \star (a \cdot c) &= (b \star a) \cdot (b \star c), \\
(b \star a)^{\cdot -1} &= b \star a^{-1}, \\
a^{\cdot -1} &= (a - \delta_{\emptyset}) \star (\delta_{\emptyset} + \delta_{\bullet})^{-1},
\end{aligned}
$$

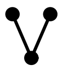
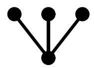
Remark We have $(\delta_{\emptyset} + \delta_{\bullet})^{-1}(\tau) = (-1)^{|\tau|}$.

The ω map and its role

The coefficients b for backward error analysis of a B-series method with coefficients a can be computed in terms of a.
Let ω denote the inverse of $e - \delta_\emptyset$ for $\star$. Then

$$b = (a - \delta_\emptyset) \star \omega.$$

(this is because $b \star e = a$ and $b \star (e - \delta_\emptyset) = a - \delta_\emptyset$).
The $\omega(\tau)$'s can be interpreted as the coefficients of the modified field for backward error analysis of the explicit Euler method.

Examples:
$\omega($$) = B_1 = -\frac{1}{2}$, $\omega($$) = B_2 = \frac{1}{6}$, $\omega($$) = B_3 = 0, \ldots$
where the B_j's are the Bernoulli numbers.

The ω map and its role

The logarithmic map

$$\log : \begin{array}{rcl} \text{Butcher group} & \longrightarrow & \text{Vector fields group (with } \star \text{)} \\ a & \longmapsto & (a - \delta_\emptyset) \star \omega \end{array}$$

establishes a one-to-one correspondence between

- The subgroup of symplectic B-series with the subgroup of Hamiltonian B-series vector fields,
- The subgroup of symmetric B-series with the subgroup of B-series vector fields in even powers of h,
- The subgroup of volume-reserving B-series with the subgroup of divergence-free B-series vector fields.

A new Hopf tree algebra

A Hopf tree algebra has been constructed recently by Calaque, Ebrahimi-Fard & Manchon (2008) with a new coproduct that is closely related to the substitution law:

- A coproduct $\Delta : H_{CEM} \to H_{CEM} \otimes H_{CEM}$.

$$\Delta_{CEM}(\vee) = \vee \otimes \bullet + 2 / \otimes / + \bullet \otimes \vee.$$

- An antipode $S : H_{CEM} \to H_{CEM}$.

$$S(\vee) = -\vee + 2 /^{2}.$$

A new Hopf tree algebra

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$$S(\vee) = -\vee + 2\,\ell^2.$$

The group for the substitution law is the character group of convolution:

$$a \star b = \mu \circ (a \otimes b) \circ \Delta_{CEM}$$

$$a \star b(\vee) = a(\vee) + 2a(\ell)b(\ell) + b(\vee).$$

$$b^{\star -1}(\vee) = -b(\vee) + 2b(\ell)^2$$

It allows them to find alternate algebraic proofs of some previous results.