Hopf algebras in renormalisation

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1 Renormalisation in physics

Systems in interaction are most common in physics. When parameters (such as mass, electric charge, acceleration, etc.) characterising the system are considered, it is crucial to distinguish between *bare parameters*, which are the values they would take if the interaction were switched off, and the actually observed parameters. Renormalisation can be defined as any procedure able to transform the bare parameters into the actually observed ones (i.e. with interaction taken into account), which will therefore be called *renormalised*. Consider (from [CK1]) the following example: the initial acceleration of a spherical balloon is given by:

$$g = \frac{m_0 - M}{m_0 + \frac{M}{2}} g_0 \tag{1}$$

where $g_0 \simeq 9,81\, m.s^{-2}$ is the gravity acceleration at the surface of the Earth, m_0 is the mass of the balloon, and M is the mass of the volume of the air occupied by it. Note that this acceleration decreases from g_0 to $-2g_0$ when the interaction (represented here by the air mass M) increases from 0 to $+\infty$. The total force $F = mg$ acting on the balloon is the sum of the gravity force $F_0 = m_0 g_0$ and Archimedes' force $-Mg_0$. The bare parameters (i.e. in the absence of air) are thus m_0, F_0, g_0 (mass, force and acceleration respectively), whereas the renormalised parameters are:

$$m = m_0 + \frac{M}{2}, \qquad F = (1 - \frac{M}{m_0})F_0, \qquad g = \frac{m_0 - M}{m_0 + \frac{M}{2}} g_0. \tag{2}$$

In perturbative quantum field theory an extra difficulty arises: the bare parameters are usually infinite! Typically they are given by divergent integrals[2] such as:

$$\int_{\mathbb{R}^4} \frac{1}{1 + \|p\|^2}\, dp. \tag{3}$$

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[2]To be precise, the physical parameters of interest are given by a *series* each term of which is a divergent integral. We do not approach here the question of convergence of this series once each term has been renormalised.

One must then subtract another infinite quantity to the bare parameter to recover the renormalised parameter, which is finite as it can be actually measured! Such a process takes place in two steps:

1) a *regularisation procedure*, which replaces the bare infinite parameter by a function of one variable z which tends to infinity when z tends to some z_0.

2) the *renormalisation procedure* itself, of combinatorial nature, which extracts an appropriate finite part from the function above when z tends to z_0. When this procedure can be carried out, the theory is called *renormalisable*.

There is usually considerable freedom in the choice of a regularisation procedure. Let us mention, among many others, the *cut-off regularisation*, which amounts to consider integrals like (3) over a ball of radius z (with $z_0 = +\infty$), and *dimensional regularisation* which consists, roughly speaking, in "integrating over a space of complex dimension z", with $z_0 = d$, the actual space dimension of the physical situation (for example $d = 4$ for the Minkowski space-time). In this case the function which appears is meromorphic in z with a pole at z_0 ([C],[HV], [Sp]).

The renormalisation procedure is an algorithm of combinatorial nature, the *BPHZ algorithm* (after N. Bogoliubov, O. Parasiuk, K. Hepp and W. Zimmermann). The combinatorial objects involved are *Feynman graphs*: to each graph[3] corresponds (by *Feynman rules*) a quantity to be renormalised, and an integer (the *loop number*) is associated to the graph. The initial data for this algorithm are determined by the choice of a *renormalisation scheme*, which consists in choosing the finite part for the "simplest" quantities, corresponding to graphs with loop number $L = 1$. For example we can simply remove the pole part at z_0 of a meromorphic function and then consider the value at z_0. This is the *minimal subtraction scheme*. The renormalisation of the other quantities involved are then given recursively with respect to the loop number.

2 Connected graded Hopf algebras

The crucial fact first observed by D. Kreimer [K1] is the following: the Feynman graphs involved in the renormalisation procedure are organised in a connected graded Hopf algebra. On these Hopf algebras recursive procedures with respect to the degree can be implemented. In particular the antipode "comes for free" by means of a recursive formula.

[3]together with external momenta. cf. § 4 below.

Let k be a field of characteristic zero. A connected graded bialgebra $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ is a graded unital algebra on k with $\dim \mathcal{H}_0 = 1$ endowed with a compatible co-unit $\varepsilon : \mathcal{H} \to k$ and coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ such that:

$$\Delta(\mathcal{H}_n) \subset \bigoplus_{r+s=n} \mathcal{H}_r \otimes \mathcal{H}_s. \tag{4}$$

For any $x \in \mathcal{H}_n$ we more precisely have, using a variant of Sweedler's notation [Sw]:

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{(x)} x' \otimes x'', \tag{5}$$

where the degrees of x' and x'' are strictly smaller than n. Now let $\mathcal{A}$ be any k-algebra (with multiplication $m_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$), and let $\mathcal{L}(\mathcal{H}, \mathcal{A})$ be the vector space of linear maps from $\mathcal{H}$ to $\mathcal{A}$. The *convolution product* on $\mathcal{L}(\mathcal{H}, \mathcal{A})$ is defined as follows:

$$\varphi * \psi = m_{\mathcal{A}} \circ (\varphi \otimes \psi) \circ \Delta. \tag{6}$$

Convolution product writes then with Sweedler's notation:

$$(\varphi * \psi)(x) = \varphi(x)\psi(1) + \varphi(1)\psi(x) + \sum_{(x)} \varphi(x')\psi(x''). \tag{7}$$

Recall that the antipode S is the inverse of the identity for the convolution product on $\mathcal{L}(\mathcal{H}, \mathcal{H})$. It always exists in a connected graded bialgebra, hence any connected graded bialgebra is a *connected graded Hopf algebra*. The antipode is given by any of the following recursive formulas:

$$S(x) = -x - \sum_{(x)} S(x')x'', \tag{8}$$

$$S(x) = -x - \sum_{(x)} x'S(x''). \tag{9}$$

3 Renormalisation for connected graded Hopf algebras

Let $\mathcal{A}$ be any *commutative* unital algebra over the field k, and suppose that $\mathcal{A}$ splits into two subalgebras:

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+, \tag{10}$$

where the unit $1_{\mathcal{A}}$ belongs to $\mathcal{A}_+$. Such a splitting is called a *renormalisation scheme*. The minimal renormalisation scheme described in paragraph 1 corresponds to the splitting of the $\mathbb{C}$-algebra $\mathcal{A}$ of meromorphic functions in which $\mathcal{A}_-$ is the algebra of polynomials in $(z-z_0)^{-1}$ without constant term (the "pole parts"), and $\mathcal{A}_+$ is the algebra of meromorphic functions which are holomorphic at z_0.

Now let $\mathcal{H}$ be a connected graded Hopf algebra. It is easily seen, thanks to the commutativity of the algebra $\mathcal{A}$, that the set G of unital algebra morphisms from $\mathcal{H}$ to $\mathcal{A}$ is a group for the convolution product, the *group of $\mathcal{A}$-valued characters on $\mathcal{H}$*. The neutral element e is given by $e(1) = 1_{\mathcal{A}}$ and $e(x) = 0$ for $x \in \mathcal{H}_n, n \geq 1$. The inverse is given by composition with the antipode:

$$\chi^{*-1} = \chi \circ S. \tag{11}$$

Now any character admits a *Birkhoff decomposition* which is unique once the renormalisation scheme has been chosen [CK1], [M]: namely for any $\chi \in G$ we have:

$$\chi = \chi_-^{*-1} * \chi_+, \tag{12}$$

where both χ_- and χ_+ belong to G, where $\chi_+(x) \in \mathcal{A}_+$ for any $x \in \mathcal{H}$, and where $\chi_-(x) \in \mathcal{A}_-$ for any $x \in \mathcal{H}_n, n \geq 1$. The components χ_- and χ_+ are given by fairly simple recursive formulas: let us denote by $\pi : \mathcal{A} \rightarrow \mathcal{A}_-$ the projection onto $\mathcal{A}_-$ parallel to $\mathcal{A}_+$. Suppose that $\chi_-(x)$ and $\chi_+(x)$ are known for $x \in \mathcal{H}_k, k \leq n-1$. Define for $x \in \mathcal{H}_n$ *Bogoliubov's preparation map*:

$$\overline{R} : x \longmapsto \chi(x) + \sum_{(x)} \chi_-(x')\chi(x''). \tag{13}$$

The components in the Birkhoff decomposition are then given by:

$$\chi_-(x) = -\pi\left(\overline{R}(x)\right), \tag{14}$$

$$\chi_+(x) = (1-\pi)\left(\overline{R}(x)\right). \tag{15}$$

The component χ_+ is the *renormalised character* whereas $\chi_-(x)$ is the sum of *counterterms* one must add to $\overline{R}(x)$ to get $\chi_+(x)$. In the example of minimal subtraction scheme the *renormalised value* of the character χ at z_0 is the well-defined complex number $\chi_+(z_0)$, whereas $\chi(z_0)$ may not exist[4].

[4]The fact that χ_- and χ_+ are still characters relies on a Rota-Baxter property for the projection π. This aspect is well developed in [EGK1], [EGK2], [EGGV], see also [EG].

4 Feynman graphs

A *Feynman graph* is a (non-planar) graph with a finite number of vertices and edges. An *internal edge* is an edge connected at both ends to a vertex (which can be the same in case of a self-loop), an *external edge* is an edge with one open end, the other end being connected to a vertex. A Feynman graph is called by physicists *vacuum graph, tadpole graph, self-energy graph*, resp. *interaction graph* if its number of external edges is 0,1, 2, resp. > 2. An edge can be of various types according to which elementary particle it represents.
A *one-particle irreducible graph* (in short, 1PI graph) is a connected graph which remains connected when we cut any internal edge. A disconnected graph is said to be *locally 1PI* if any of its connected components is 1PI.

A given perturbative quantum field theory (for example quantum electrodynamics) imposes constraints on the corresponding diagrams: only a few valences for vertices are allowed, for instance. Let V be the vector space freely generated by those diagrams (which we call *admissible*) which are moreover 1PI, and let $\mathcal{H}$ be the free commutative algebra (the symmetric algebra) generated by V. This is a connected graded Hopf algebra[5], with grading given by the loop number: the coproduct of a graph Γ is given by:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\substack{\gamma \text{ locally 1PI proper subgraph of } \Gamma \\ \Gamma/\gamma \text{ admissible}}} \gamma \otimes \Gamma/\gamma. \tag{16}$$

Here Γ/γ stands for the *contracted graph* (or *cograph*), where each connected component of the subgraph is shrinked to a point. The situation is actually less simple than sketched here because Feynman diagrams come together with *exterior structures*, i.e. a vector (called exterior momentum) attached to each external edge. The sum of those vanishes, reflecting the conservation of total momentum in an interaction. It is still possible to organise Feynman diagrams together with exterior structures in a connected graded Hopf algebra ([CK1], [K2]). Feynman rules and dimensional regularisation yield a character of this Hopf algebra with values in the meromorphic functions [C], [HV], [Sp]. In the case of a renormalisable theory the abstract renormalisation algorithm described in § 3 applied to this particular Hopf algebra coincides with the BPHZ algorithm.

[5]This Hopf algebra is *commutative* by construction. Thanks to the Cartier-Milnor-Moore theorem its graded dual is the enveloping algebra of a Lie algebra $\mathfrak{g}$. The Lie structure comes from the pre-Lie operation of insertion of graphs [K2].

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