On quantization of quadratic Poisson structures

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Abstract : Any classical r-matrix on the Lie algebra of linear operators on a real vector space V gives rise to a quadratic Poisson structure on V which admits a deformation quantization stemming from the construction of V. Drinfel'd [Dr], [Gr]. We exhibit in this article an example of quadratic Poisson structure which does not arise this way.

I. Introduction

let V be a finite-dimensional real vector space. The linear action of the Lie group $Gl(V)$ on V induces by differentiation a Lie algebra isomorphism from $\mathfrak{g} = \mathfrak{gl}(V)$ to the Lie algebra of linear vector fields on V. Given a basis $(e_1, \ldots, e_n)$ and then identifying $\mathfrak{gl}(V)$ with the Lie algebra of real $n \times n$ matrices the isomorphism is given by :

$$J(E_{ij}) = x_i \partial_j,$$

where E_{ij} is the matrix with entries all vanishing except one equal to 1 on the i^{th} line and j^{th} column.

There is a unique way to extend the Lie bracket of $\mathfrak{g}$ to a graded Lie bracket, called the *Schouten bracket* on the shifted exterior algebra $\Lambda(\mathfrak{g})[1]$ in a way compatible with the exterior product. The shift means that elements of $\Lambda^k(\mathfrak{g})$ are of degree $k-1$, and then the Schouten bracket maps $\Lambda^k(\mathfrak{g}) \times \Lambda^l(\mathfrak{g})$ to $\Lambda^{k+l-1}(\mathfrak{g})$. The exterior algebra $\Lambda(\mathfrak{g})$ inherits then a structure of Gerstenhaber algebra (cf. for example [V], introduction).

The space $T^{\mathrm{poly}}(V)$ of polyvector fields on M is also endowed with a Gerstenhaber algebra structure, with Schouten bracket extending Lie bracket of vector fields [V]. The subalgebra (for exterior product) generated by linear vector fields is a Gerstenhaber subalgebra $\widetilde{\Lambda}(V)$ of $T^{\mathrm{poly}}(V)$. The isomorphism J extends to a surjective Gerstenhaber algebra morphism :

$$J^\bullet : \Lambda^\bullet(\mathfrak{g}) \longmapsto \widetilde{\Lambda}^\bullet(V).$$

Map J^k has nontrivial kernel for $k \geq 2$ as long as V has dimension ≥ 2 : for example we have :

$$J^2(E_{ij} \wedge E_{kj}) = 0.$$

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A *classical r-matrix* on $\mathfrak{g}$ is by definition an element r of $\mathfrak{g}\wedge\mathfrak{g}$ such that $[r,r]=0$. According to the discussion above the bivector field $J^2(r)$ defines then a quadratic Poisson structure on V. A natural question arises then : can one recover this way any quadratic Poisson structure on V? It was claimed true in [BR] but Z.H. Liu and P. Xu discovered that the authors' argument was not correct [LX]. They brought up a positive answer in the two-dimensional case ([LX] prop. 2.1) : namely the general quadratic Poisson structure $(ax_1^2+2bx_1x_2+cx_2^2)\partial_1\wedge\partial_2$ is equal to $J^2(r)$ with :

$$r=\begin{pmatrix} b & -a \\ c & -b \end{pmatrix}\wedge\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We give here a *negative* answer to this question in general : after a somewhat lengthy but elementary computation we show in § III that bivector field $(x_1^2+x_2x_3)\partial_2\wedge\partial_3$ on $\mathbb{R}^3$ is a counterexample to this conjecture : it is outside the image of the set of r-matrices by J^2.

We recall in § II the construction by V.G. Drinfel'd of a translation-invariant deformation quantization on any Lie group G once given a classical r-matrix on the Lie algebra $\mathfrak{g}$ [Dr], [T]. The problem reduces to the case when r is non-degenerate, and the deformation quantization is then obtained by suitable restriction and transportation of Baker-Campbell-Hausdorff deformation quantization ([Ka]) of the dual $\widetilde{\mathfrak{g}}^*$ of the central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ defined by r. The construction works moreover for any Kontsevich-type star product [ABM] on $\widetilde{\mathfrak{g}}^*$. For $\mathfrak{g}=\mathfrak{gl}(V)$, such a star product on $\widetilde{\mathfrak{g}}^*$ gives almost immediately through this construction a deformation quantization of quadratic Poisson structure $J^2(r)$ on V.

Let us mention that the existence of a deformation quantization for any quadratic Poisson structure has been proved by Omori, Maeda and Yoshioka [OMY], a few years before the proof by Kontsevich of the existence of a deformation quantization for any Poisson structure on any manifold [K], [CFT]. Explicit computations for all quadratic Poisson structures in dimension 3 (then including our counterexample as well) have been performed by El Galiou and Tihami [ET], by a case-by-case method based on the classification of Dufour and Haraki [DH].

II. Quantization of Poisson structures coming from r-matrices

Let $\mathfrak{g}$ be a Lie algebra, and let $r\in\mathfrak{g}\wedge\mathfrak{g}$ a classical r-matrix. It defines an antisymmetric operator :

$$\widetilde{r}:\mathfrak{g}^*\longrightarrow\mathfrak{g}.$$

Classical Yang-Baxter equation $[r,r]=0$ is equivalent to :

$$<\xi,\,[\widetilde{r}(\eta),\,\widetilde{r}(\zeta)]>+<\eta,\,[\widetilde{r}(\zeta),\,\widetilde{r}(\xi)]>+<\zeta,\,[\widetilde{r}(\xi),\,\widetilde{r}(\eta)]>=0 \qquad (*)$$

for any $\xi,\eta,\zeta\in\mathfrak{g}^*$. The r-matrix r defines a left translation-invariant Poisson structure on any Lie group G with Lie algebra $\mathfrak{g}$.

II.1. A central extension

We can firstly suppose r nondegenerate, i.e. that $\widetilde{r}$ is inversible with inverse $\widetilde{\omega}$, where ω belongs to $\mathfrak{g}^* \wedge \mathfrak{g}^*$. Classical Yang-Baxter equation is in this case equivalent to :

$$< \widetilde{\omega}X,\, [Y,Z] > + < \widetilde{\omega}Y,\, [Z,X] > + < \widetilde{\omega}Z,\, [X,Y] >= 0$$

for any $X, Y, Z \in \mathfrak{g}$, i.e is equivalent to the fact that ω is a 2-cocycle with values in the trivial representation. Let $\widetilde{\mathfrak{g}}$ the central extension of $\mathfrak{g}$ by this cocycle, defined by $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ with bracket :

$$[X + \alpha, Y + \beta] = [X, Y] + < \omega, X \wedge Y > .$$

The cocycle condition on ω is equivalent to de Jacobi identity for this bracket. Let $X_0 = (0, 1) \in \widetilde{\mathfrak{g}}$, and let $\mathcal{H}$ the hyperplane in $\widetilde{\mathfrak{g}}^*$ defined by :

$$\mathcal{H} = \{\xi \in \widetilde{\mathfrak{g}}^* / < \xi, X_0 >= 1\}.$$

It is the symplectic leaf through the point ξ_0 defined by $< \xi_0, \mathfrak{g} >= 0$ and $< \xi_0, X_0 >= 1$. It is then the coadjoint orbit $\mathrm{Ad}^*\, \widetilde{G}.\xi_0$ for any Lie group $\widetilde{G}$ with Lie algebra $\widetilde{\mathfrak{g}}$.

II.2. Kontsevich star products (after [ABM])

The linear Poisson manifold $\widetilde{\mathfrak{g}}^*$ admits a whole bunch of equivalent $\mathrm{Ad}^*\, \widetilde{G}$-invariant deformation quantizations which can be built from the enveloping algebra $\mathcal{U}(\widetilde{\mathfrak{g}})$, for example the Baker-Campbell-Hausdorff quantization or the Kontsevich quantization [K], [ABM], [Ka], [Di]. The baker-Campbell-Hausdorff quantization is given by the following integral formula [ABM] :

$$(u \underset{BCH}{\#} v)(\xi) = \iint_{\mathfrak{g}\times\mathfrak{g}} \mathcal{F}^{-1}u(x)\mathcal{F}^{-1}v(y)e^{\frac{i<\xi,\, x \underset{\hbar}{\cdot} y>}{\hbar}}\, dx\, dy,$$

where the inverse Fourier transform is given by :

$$\mathcal{F}^{-1}u(x) = (2\pi)^{-n} \int_{\mathfrak{g}^*} u(\eta)e^{i<x,\eta>}d\eta,$$

and $x \underset{\hbar}{\cdot} y$ stands for the Baker-Campbell-Hausdorff expansion :

$$x + y + \frac{\hbar}{2}[x, y] + \frac{\hbar^2}{12}([x, [x, y]] + [y, [y, x]]) + \cdots .$$

The Lebesgue measure $d\eta$ on $\mathfrak{g}^*$ is normalized so that it is the dual mesure of Lebesgue measure dx on $\mathfrak{g}$. The quantizations we can consider here are the ones called “Kontsevich star products” in [ABM]. They are all equivalent to the BCH quantization. The equivalence is a formal series of differential operators with constant coefficients on $\mathfrak{g}^*$ precisely given by a formal series of G-invariant polynomials on $\mathfrak{g}$ of the following form :

$$F(x) = 1 + \sum_{k\geq 1} \hbar^{2k} \sum_{c\geq 1} \sum_{(s_1,\ldots,s_c)\in S^c_{2k}} a_{s_1,\ldots,s_c} \operatorname{Tr}(\operatorname{ad} x)^{s_1} \cdots \operatorname{Tr}(\operatorname{ad} x)^{s_c},$$

where S_{2k}^c stands for those $(s_1, \ldots, s_c)$ in $\mathbb{N}^c$ such that $s_1 + \cdots + s_c = 2k$, $s_1 \leq s_2 \leq \cdots \leq s_c$ and $s_j \neq 1$. The star product obtained this way admits the following integral form :

$$(u\#v)(\xi) = \iint_{\mathfrak{g}\times\mathfrak{g}} \mathcal{F}^{-1}u(x)\mathcal{F}^{-1}v(y)\frac{F(-ix)F(-iy)}{F\big(-i(x \underset{\hbar}{\cdot} y)\big)} e^{i<\xi,\, x \underset{\hbar}{\cdot} y>} \, dx\, dy.$$

II.3. Quantization of left-invariant Poisson structures

It is easy to derive from the fact that X_0 is central that any of the deformation quantizations defined above does define by restriction a deformation quantization of $\mathcal{H}$. Let G be the subgroup of $\widetilde{G}$ with Lie algebra $\mathfrak{g}$. We clearly have :

$$\mathrm{Ad}^*\, G.\xi_0 = \mathrm{Ad}^*\, \widetilde{G}.\xi_0 = \mathcal{H}.$$

It is moreover easy to check that the stabilizer of ξ_0 in $\widetilde{G}$ is the one-dimensional subgroup with Lie algebra generated by X_0. It is a simple consequence of the nondegeneracy of the alternate bilinear form ω. The dimension of G is the equal to the dimension of $\mathcal{H}$. The map :

$$\begin{aligned} \varphi : G' &\longrightarrow \mathcal{H} \\ g &\longmapsto \mathrm{Ad}^*\, g.\xi_0 \end{aligned}$$

is then a local G-equivariant diffeomorphism near the identity (with left translation on the left-hand side and coadjoint action on the right-hand side). We can then transport any deformation quantization of $\mathcal{H}$ and get a left translation-invariant deformation quantization of a neighbourhood of the identity in G. It extends by translation invariance to the whole group G, as well as to any Lie group G' locally isomorphic to G.

The deformation quantization on G can be written :

$$u\#v = \sum_{k\geq 0} \hbar^k C_k(u,v),$$

where the C_k's are left-invariant bidifferential operators on G. There exists then an element $F = \sum \hbar^k F_k$ in $\big(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})\big)[[\hbar]]$ such that :

$$u\#v(g) = F(u \otimes v)(g,g). \tag{$**$}$$

Let us now fix a basis $x_1, \ldots, x_n$ of $\mathfrak{g}$, and consider elements of $\mathcal{U}(\mathfrak{g})$ as polynomials $F(x) = F(x_1, \ldots, x_n)$ of the n noncommuting variables $x_1, \ldots, x_n$, which satisfy the relations :

$$x_i x_j - x_j x_i = [x_i, x_j] = \sum_k c_{ij}^k x_k.$$

Introducing a second identical set of noncommuting variables $y = (y_1, \ldots, y_n)$ commuting with th $x_j'^{}s$ we can write any element $A \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ as $A(x,y)$. The element F defined above can then be written $F(x,y)$ as a formal series with coefficients $F_k(x,y)$.

Proposition II.1.
The formal series $F = F(x, y) \in \big(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})\big)[[\hbar]]$ *above verifies :*

1) $F_0(x, y) = 0.$
2) $F_1(x, y) = \frac{1}{2} \sum_{i,j} r_{ij} x_i y_j.$
3) $F_k(x, 0) = F_k(0, y) = 0$ *for* $k \geq 1$.
4) $F(x + y, z)F(x, y) = F(x, y + z)F(y, z).$

Conversely any $F(x, y)$ *endowed with those 4 properties defines by formula* $(**)$ *a left translation deformation quantization of* G.

Proof. It is well-known : see for example [Dr], [T] : first condition comes from the fact that $C_0(u, v)$ is the ordinary product uv. Second property comes from the expression of left-invariant Poisson bracket on G defined from the r-matrix, third property expresses the fact that $1\#u = u\#1 = u$, and last property is an expression of the associativity of star product $\#$. Let us elaborate a bit on that last point : any element X_j of the basis corresponds to polynomial expression $G(x) = x_j$. The Leibniz rule :

$$X_j.(\varphi\psi) = (X_j.\varphi)\psi + \varphi.(X_j.\psi)$$

can be written as :

$$G(x) \circ m = m \circ G(x + y),$$

where $m : C^\infty(G \times G) \to C^\infty(G)$ stands for multiplication, here the restriction to the diagonal. The formula above extends to any polynomial expression G representing any element of the enveloping algebra. We have then :

$$\begin{aligned}
(u\#v)\#w &= \big(m \circ F(u \otimes v)\big)\#w \\
&= m \circ F\Big(\big(m \circ F(u \otimes v)\big) \otimes w\Big) \\
&= m \circ F \circ (m \otimes I) \circ (F \otimes I)(u \otimes v \otimes w) \\
&= m \circ F(x, z) \circ (m \otimes I) \circ F(x, y)(u \otimes v \otimes w) \\
&= m \circ (m \otimes I) \circ F(x + y, z)F(x, y)(u \otimes v \otimes w).
\end{aligned}$$

Similarly we have :

$$u\#(v\#w) = m \circ (I \otimes m) \circ F(x, y + z)F(y, z)(u \otimes v \otimes w).$$

The associativity condition for product $\#$ is then equivalent to property 4) of the proposition.

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let us now look at the case when r is degenerate. Then the image $\mathfrak{g}_0$ of $\widetilde{r}$ is a subspace strictly contained in $\mathfrak{g}$. By skew-symmetry $\mathfrak{g}_0$ is also the orthogonal of the kernel of $\widetilde{r}$, and classical Yang-Baxter equation $[r,r]=0$ ensures thanks to $(*)$ that $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$. We get this way a nondegenerate $r_0 \in \mathfrak{g}_0 \wedge \mathfrak{g}_0$ such that $[r_0,r_0]=0$. Applying the procedure above we get an $F=\sum \hbar^k F_k$ in $\big(\mathcal{U}(\mathfrak{g}_0)\otimes\mathcal{U}(\mathfrak{g}_0)\big)[[\hbar]]$ which can be seen as an element of $\big(\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})\big)[[\hbar]]$.

II.4. A class of easily quantizable Poisson structures

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $F(x,y)$ a formal series in $\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})[[\hbar]]$ satisfying properties 1-4 of proposition II.1 (for example that one constructed from an r-matrix along the lines above). Let M be any differentiable manifold endowed with an action of G. The differentiation of this action induces a Lie algebra morphism from $\mathfrak{g}$ to the vector fields on M, which extends to an algebra morphism from $\mathcal{U}(\mathfrak{g})$ to the algebra of differential operators on M. Similarly it induces an algebra morphism from $\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$ to the algebra of differential operators on $M\times M$. The formal series of bidifferential operators defined by the formula :

$$* = m\circ F(x,y)$$

(where $m : C^\infty(M\times M)\to C^\infty(M)$ stands for ordinary multiplication of functions on M) defines then a star product on M, the associated Poisson bivector being defined by $F_1(x,y)-F_1(y,x)$. The proof of this fact goes the same way as that of proposition II.1. It is easily seen that if $F(x,y)$ comes from a classical r-matrix $r\in\mathfrak{g}\wedge\mathfrak{g}$ then the Poisson structure on M is $J^2(r)$ where $J^\bullet$ is the Gerstenhaber algebra morphism from $\Lambda(\mathfrak{g})$ to multivector fields on M extending the action of $\mathfrak{g}$.

We will be interested in the sequel by the following particular situation : the manifold M is a vector space V, the action of G is linear, and there is a classical r-matrix r on $\mathfrak{g}$. We can as in the introduction view J as a Lie algebra morphism from $\mathfrak{g}$ to the space of linear vector fields on V, and extend J to a morphism $J^\bullet$ of Gerstenhaber algebras from $\Lambda(\mathfrak{g})$ to $\widetilde{\Lambda}(V)$. In particular $J^2(r)$ defines a quadratic Poisson structure on V, and formula just above gives a quantization of this particular quadratic Poisson structure.

III. Quadratic Poisson structures and r-matrices

III.1. Some definitions

We keep the notations of the introduction. The Gerstenhaber algebra $\widetilde{\Lambda}(V)$ can be written as :

$$\widetilde{\Lambda}(V)=\bigoplus_{n\geq 0}\big(S^n(V)\otimes\Lambda^n(V)\big)[1]=\bigoplus_{n\geq 0}\widetilde{\Lambda}^n(V)[1].$$

A quadratic Poisson structure on V can be defined as a bivector field Λ in $\widetilde{\Lambda}^2(V)$ such that :

$$[\Lambda,\,\Lambda]=0.$$

Let Λ be an element of $\widetilde{\Lambda}^2(V)$, and let r an element of $\Lambda^2(\mathfrak{g})$ such that $J^2(r)=\Lambda$. It is then obvious that $[\Lambda,\Lambda]=0$ if and only if $J^3([r,r])=0$. If $n\geq 2$ then J^2 and J^3 have nontrivial kernels : Precisely we have :

$$\dim\ker J^2 = \frac{n^2(n^2-1)}{4} \qquad \text{and} \qquad \dim\ker J^3 = \frac{n^2(n^2-1)(5n^2-8)}{36}.$$

III.2. A counterexample in dimension 3

With the notations of § I, an element of $\mathfrak{g}\wedge\mathfrak{g}$ can be written as :

$$r = \sum_{i,j,k,l=1}^{n} r_{ik}^{jl} E_{ij}\wedge E_{kl}.$$

Whe shall need for further calculations the following result :

Proposition III.1.
Let $r = \sum_{i,j,k,l=1}^{n} r_{ik}^{jl} E_{ij}\wedge E_{kl}$ *be an element of* $\mathfrak{g}\wedge\mathfrak{g}$ *then* $[r,r]=0$ *if and only if for any* $i,j,k,l,m,p \in \{1,\dots,n\}$ *such that* $(i,j)<(k,l)<(m,p)$ *according to lexicographical order we have :*

$$\sum_{d=1}^{n} r_{ik}^{dl} r_{dm}^{jp} - r_{mk}^{dl} r_{di}^{pj} + r_{km}^{dp} r_{di}^{lj} - r_{im}^{dp} r_{dk}^{jl} + r_{mi}^{dj} r_{dk}^{pl} - r_{ki}^{dj} r_{dm}^{lp} = 0.$$

Proof. This proposition is a direct consequence of formula :

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$$

and the following lemma :

Lemma III.1.
Let $\mathfrak{h}$ *be a finite-dimensional Lie algebra and let* $X_1,\dots,X_N$ *be a basis of* $\mathfrak{h}$.
If $r = \sum_{I,J=1}^{N} r^{IJ} X_I\wedge X_J$ *is an element of* $\mathfrak{h}\wedge\mathfrak{h}$ *(* $r^{IJ}=-r^{JI}$*) then*

$$[r,r] = 4\sum_{I,J,K,L=1}^{N} r^{IJ} r^{KL} [X_I, X_K]\wedge X_J\wedge X_L.$$

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Remark : We can directly show proposition III.1 using relation $(*)$ of beginning of § II applied to elements of the dual basis of $X_1,\dots,X_n$.

Proposition III.2.
The Poisson structure Λ on $\mathbb{R}^3$ given by

$$\Lambda = (x_1^2 + \alpha x_2 x_3)\partial_2 \wedge \partial_3$$

with $\alpha \neq 0$ is not the image of a classical r-matrix by J^2.

Proof. An element $r = \sum_{i,j,k,l=1}^{n} r_{ik}^{jl} E_{ij} \wedge E_{kl}$ of $\mathfrak{g} \wedge \mathfrak{g}$ is parametrized by 36 coefficients r_{ik}^{jl}. It has image Λ if and only if the following 18 equations are satisfied :

$$\begin{array}{llll}
r_{11}^{12} = r_{11}^{13} = r_{22}^{12} = r_{22}^{13} = r_{22}^{23} = r_{33}^{12} = r_{33}^{13} = r_{33}^{23} = 0, & & & r_{11}^{23} = 1 \\
r_{12}^{12} = r_{12}^{21}, & r_{12}^{13} = r_{12}^{31}, & r_{13}^{12} = r_{13}^{21} & \\
r_{13}^{13} = r_{13}^{31}, & r_{12}^{23} = r_{12}^{32}, & r_{13}^{23} = r_{13}^{32} & \\
r_{23}^{12} = r_{23}^{21}, & r_{23}^{13} = r_{23}^{31}, & r_{23}^{32} = r_{23}^{23} - \alpha &
\end{array}$$

To lighten writing we rename the 18 remaining unknowns as follows :

$$\begin{array}{llllll}
r_{12}^{11} = a, & r_{12}^{12} = b, & r_{12}^{13} = c, & r_{13}^{11} = d, & r_{13}^{12} = e, & r_{13}^{13} = f, \\
r_{12}^{22} = g, & r_{12}^{23} = h, & r_{13}^{22} = i, & r_{13}^{23} = j, & r_{12}^{33} = k, & r_{13}^{33} = l, \\
r_{23}^{11} = m, & r_{23}^{12} = n, & r_{23}^{13} = p, & r_{23}^{22} = q, & r_{23}^{23} = r, & r_{23}^{33} = s.
\end{array}$$

The unknown r must not be confused with the classical r-matrix on the whole. The context will not lead to any confusion. We have then :

$$\begin{array}{lll}
r_{11}^{23} = 1, & & \\
r_{12}^{21} = b, & r_{13}^{21} = e, & r_{12}^{31} = c, \\
r_{12}^{32} = h, & r_{13}^{31} = f, & r_{13}^{32} = j, \\
r_{23}^{21} = n, & r_{23}^{31} = p, & r_{23}^{32} = r - \alpha,
\end{array}$$

and other r_{ik}^{jl} are equal to 0.

If r is such an element the equation $[r, r] = 0$ develops according to Proposition III.2. into a system of 84 equations involving our 18 unknowns $a, b, \ldots, s$, given by the vanishing of the 84 coefficients of elements of the basis $E_{ij} \wedge E_{kl} \wedge E_{mn}$ of $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$. The 84 equations reduce to 66 thanks to the fact that we already have $[\Lambda, \Lambda] = 0$. But we shall only consider 20 of them, which will be sufficient for exhibiting the counterexample :

Let us order the E_{ij}'s lexicographically from first to 9th, rename them accordingly ($A_1 = E_{11}, A_2 = E_{12}, \ldots, A_9 = E_{33}$), and labelize by (x, y, z) the equation obtained by the vanishing of the coefficient of $A_x \wedge A_y \wedge A_z$. We shall consider precisely the following equations :

$(1,2,5)\ ci-eh+n=0$ $(1,2,9)\ eh-ci+d=0$
$(1,3,5)\ cj-ek-a=0$ $(1,3,7)\ ce+f^2-ja-ld+m=0$
$(1,3,9)\ ek-cj+p=0$ $(1,4,5)\ mh-nc=0$
$(1,4,6)\ mk-pc=0$ $(1,5,6)\ \alpha c+nk-ph=0$
$(1,7,8)\ en-im=0$ $(1,7,9)\ ep-jm=0$
$(1,8,9)\ \alpha e+ip-jn=0$ $(2,3,9)\ -3f+r+ik-jh=0$
$(2,4,5)\ ag-b^2+nh-qc=0$ $(2,5,6)\ 2\alpha h+bh-rh+qk-cg=0$
$(2,8,9)\ ej+ir+\alpha i-jq-fi=0$ $(3,8,9)\ is+2\alpha j+el-fj-jr=0$
$(4,5,7)\ pn-rm-bm+na=0$ $(5,6,9)\ -nk-rs+hp+s(r-\alpha)=0$
$(5,8,9)\ nj-ip+\alpha q=0$ $(6,8,9)\ ln-pj+qs-(r-\alpha)^2=0.$

Consider the following two sums :

$$\begin{aligned}(1,2,5)+(1,2,9): &\quad n+d=0\\ (1,3,5)+(1,3,9): &\quad p-a=0.\end{aligned}$$

Hence $n=-d$ and $p=a$. We will discuss the four cases $a=d=0$, $a=0$ and $d\neq 0$, $a\neq 0$ and $d=0$, $a\neq 0$ and $d\neq 0$.

First case : $a=d=0$. Then looking successively at the following equations we get :

$$\begin{array}{lll}(5,8,9)\Longrightarrow q=0 & (6,8,9)\Longrightarrow r=\alpha & (1,5,6)\Longrightarrow c=0\\ (1,8,9)\Longrightarrow e=0 & (2,4,5)\Longrightarrow b=0 & (2,5,6)\Longrightarrow h=0\\ (4,5,7)\Longrightarrow m=0 & (5,6,9)\Longrightarrow s=0 & (1,3,7)\Longrightarrow f=0\\ (3,8,9)\Longrightarrow j=0 & (2,8,9)\Longrightarrow i=0 & (2,3,9)\Longrightarrow \alpha=0,\end{array}$$

hence a contradiction to the hypothesis $\alpha\neq 0$.
Second case : $a=0$ and $d\neq 0$ (hence $n\neq 0$).

$$(1,4,6)\Longrightarrow mk=0.$$

First subcase : $m=0$. Then :

$$(1,4,5)\Longrightarrow c=0 \qquad (1,7,8)\Longrightarrow e=0 \qquad (1,2,5)\Longrightarrow n=0,$$

hence a contradiction.
Second subcase : $m\neq 0$, hence $k=0$.

$$(1,5,6)\Longrightarrow c=0 \qquad (1,4,5)\Longrightarrow h=0 \qquad (1,2,5)\Longrightarrow n=0,$$

hence a contradiction again.
Third case : $a\neq 0$ and $d=0$ (hence $p\neq 0$).

$$(1,4,5)\Longrightarrow mh=0.$$

First subcase : $m = 0$. Then :

$$(1,4,6) \Longrightarrow c = 0 \qquad (1,7,9) \Longrightarrow e = 0 \qquad (1,3,5) \Longrightarrow a = 0,$$

hence a contradiction.
Second subcase : $m \neq 0$, hence $h = 0$.

$$(1,5,6) \Longrightarrow c = 0 \qquad (1,4,6) \Longrightarrow k = 0 \qquad (1,3,5) \Longrightarrow a = 0,$$

hence a contradiction again.
Fourth case : $a \neq 0$ and $d \neq 0$.
First subcase : $m = 0$.

$$(1,4,5) \Longrightarrow c = 0 \qquad (1,7,8) \Longrightarrow e = 0 \qquad (1,3,5) \Longrightarrow a = 0,$$

contradiction.
Second subcase : $m \neq 0$.

$$(1,4,6) \Longrightarrow k = \frac{a}{m}c \qquad (1,7,9) \Longrightarrow j = \frac{a}{m}e \qquad (1,3,5) \Longrightarrow a = 0,$$

contradiction.
This proves proposition III.2.

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III.3. Cartan-type quadratic Poisson structures

Recall from [DH] that the curl of a Poisson stucture $\Lambda = \sum_{i,j} \Lambda^{ij}\partial_i \wedge \partial_j$ is defined by :

$$\operatorname{rot}\Lambda = \sum_{i,j} \partial_j \Lambda^{ij}.\partial_i.$$

It is a linear vector field (and hence can be viewed as an $n \times n$ matrix) when Λ is quadratic. A quadratic Poisson structure is *of Cartan type* if it can be written for some choice of coordinates as :

$$\Lambda = \sum_{i,j=1}^{n} c_{ij} x_i x_j \partial_i \wedge \partial_j$$

with $c_{ji} = -c_{ij}$. J.P. Dufour and A. Haraki proved the following result :

Theorem III.3 (Dufour - Haraki).
Any quadratic Poisson structure the curl of which has eigenvalues λ_i such that $\lambda_i + \lambda_j \neq \lambda_r + \lambda_s$ for any (i,j,r,s) with $r \neq s$ and $\{i,j\} \neq \{r,s\}$ is of Cartan type.

Such a Cartan-type Poisson structure is image by $J^{(2)}$, of a classical r-matrix, namely :

$$r = \sum_{i,j} c_{ij} E_{ii} \wedge E_{jj}.$$

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