Optimal distribution of the internal null control for the one-dimensional heat equation

Arnaud Münch* and Francisco Periago†

October 5, 2009

Abstract

This paper addresses the problem of optimizing the distribution of the support of the internal null control of minimal L^2-norm for the 1-D heat equation. A measure constraint is imposed on the support but no topological assumption such as the number of connected components. Therefore, the problem typically lacks of solution in the class of characteristic functions and needs of relaxation. We show that the relaxed formulation is obtained by replacing the set of characteristic functions by its convex envelope. The proof requires that the observability constant related to the control problem be uniform with respect to the support, property which is obtained by the control transmutation method. The optimality conditions of the relaxed problem as well as the case where the number of connected components is fixed a priori are also discussed. Several numerical experiments complete the study and suggest the ill-posedness of the problem in contrast to the wave situation.

Key words: Heat equation, Optimal design, Relaxation, Optimality conditions, Numerical simulation.

1 Introduction and problem statement

We consider in this work a general optimal design problem in the context of the exact controllability theory. There is by now a large interest in optimal shape design theory [10], specially for dynamical system [9, 18], which consists in optimizing the distributions of materials or the shape of a mechanical structure in order to reach a suitable optimal

*Laboratoire de Mathématiques de Clermont-Ferrand, UMR CNRS 6620, Campus des Cézeaux 63177 Aubière France - arnaud.munch@math.univ-bpclermont.fr. Partially supported by grant ANR-07-JC-183284 (France) and 08720/PI/08 from Fundación Séneca (Agencia de Ciencia y Tecnología de la Región de Murcia (Spain). II PCTRM 2007-10).

†Departamento de Matemática Aplicada y Estadística, ETSI Industriales, Universidad Politécnica de Cartagena, 30202 Cartagena, Spain - f.periago@upct.es. Supported by projects MTM2007-62945 from Ministerio de Educación y Ciencia (Spain) and 08720/PI/08 from Fundación Séneca (Agencia de Ciencia y Tecnología de la Región de Murcia (Spain). II PCTRM 2007-10).

behavior with respect to some initial excitation. On the other hand, since twenty years, a huge literature in the field of control has been devoted to the modeling and the analysis of mechanical systems, stabilized or exactly controlled in time, by some boundary or internal dissipative mechanisms [12]. In order to extend this optimization process, it appears natural to optimize the shape and design of such dissipative mechanisms, distributed on the structure. To our knowledge, this issue have only been analyzed by the authors in [16, 17, 21] in the context of the wave equation. In this work, we consider the different in nature heat equation. Precisely, in the one space dimension, we address the problem of optimizing the distribution of the support of the distributed null control for the heat equation.

Let $\Omega = (0,1)$ and let ω be an open subset of Ω. Given a fixed positive time T and a function $u^0 \in L^2(\Omega)$, the problem of internal null controllability for the heat equation amounts to find a control function $h_\omega = h_\omega(t,x) \in L^2((0,T)\times\Omega)$ such that $\mathrm{supp}(h_\omega) \subset [0,T]\times\overline{\omega}$ and for which the solution u of the system

$$\begin{cases} u_t - u_{xx} = h_\omega, & (t,x) \in]0,T[\times\Omega \\ u|_{\partial\Omega} = 0, & t \in [0,T] \\ u(0,x) = u^0(x), & x \in \Omega \end{cases} \tag{1}$$

satisfies the null controllability condition

$$u(T,x) = 0, \quad x \in \Omega. \tag{2}$$

The first result concerning problem (1)-(2) was obtained by Fattorini and Russell [5] where the null controllability property (2) was proved using moment's theory. We refer to [1] for a more recent presentation and several applications of this approach. Since the pioneering work [5], important progresses have been made during the last two decades. In particular, we mention the contributions by Lebeau-Robbiano [11] based on spectral analysis and by Fursikov-Imanuvilov [7] based on Carleman type inequalities. As it is well-known, the null controllability of system (1) is equivalent to the observability of the solutions of the adjoint system

$$\begin{cases} \varphi_t + \varphi_{xx} = 0, & (t,x) \in]0,T[\times\Omega \\ \varphi|_{\partial\Omega}(t) = 0, & t \in [0,T] \\ \varphi(T,x) = \varphi^T(x), & x \in \Omega. \end{cases} \tag{3}$$

Precisely, the observability inequality

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq C\int_0^T\int_\omega \varphi^2(t,x)\,dxdt \tag{4}$$

holds for all $T > 0$ and $\varphi^T \in L^2(\Omega)$, with a constant C which depends on Ω, ω and T. From (4), it follows that the functional

$$\mathcal{J}_{\mathcal{X}_\omega}(\varphi) = \frac{1}{2}\int_0^T\int_\Omega \mathcal{X}_\omega\varphi^2\,dxdt + \int_\Omega \varphi(0)\,u^0 dx, \tag{5}$$

$\mathcal{X}_\omega$ being the characteristic function of the subset ω, is coercive on the space

$$\begin{aligned} H_{\mathcal{X}_\omega} &= \Big\{ h = \mathcal{X}_\omega \varphi \text{ for some } \varphi \text{ solution of (3) with } \varphi \in L^2\left([0,\tau] : H^2 \cap H_0^1(\Omega)\right) \\ &\qquad \text{for all } 0 < \tau < T \text{ and } \int_0^T \int_\Omega \mathcal{X}_\omega \varphi^2 dxdt < +\infty \Big\}, \end{aligned}$$

endowed with the norm

$$\|h\|_{H_{\mathcal{X}_\omega}} = \left(\int_0^T \int_\Omega \mathcal{X}_\omega \varphi^2 dxdt \right)^{1/2}.$$

We refer to [13, 14] for more details on this space. Since $\mathcal{J}_{\mathcal{X}_\omega}$ is also continuous and strictly convex, there exists a unique minimizer, say $\widehat{\varphi}$ of $\mathcal{J}_{\mathcal{X}_\omega}$ in $H_{\mathcal{X}_\omega}$. Then the function

$$h_\omega = \mathcal{X}_\omega \widehat{\varphi} \tag{6}$$

is a null control for (1)-(2). Moreover, the control given by (6) is the one of minimal $L^2((0,T) \times \omega)$-norm, usually refereed in the literature as the HUM control (where HUM stands for Hilbert Uniqueness Method, see [12]).

The problem we address in this work is to minimize the $L^2((0,T) \times \omega)$-norm of the control (6) in the class of the subsets ω which have a prescribed fixed measure. That is, we look for the best distribution of the support ω of the control h_ω given by (6). Note that this corresponds to a double optimization, since we minimize with respect to ω over the set of HUM controls. Identifying each subset ω with its characteristic function $\mathcal{X}_\omega$, the nonlinear optimal design problem reads as follows:

$$(P) \qquad \inf_{\mathcal{X}_\omega \in \mathcal{U}_L} J(\mathcal{X}_\omega) = \int_0^T \int_\Omega \mathcal{X}_\omega \widehat{\varphi}^2 dxdt$$

where $\widehat{\varphi}$ is the minimizer of (5) and, for some fixed $0 < L < 1$,

$$\mathcal{U}_L = \left\{ \mathcal{X}_\omega \in L^\infty(\Omega; \{0,1\}) : \omega \subset \Omega \text{ is open} \quad \text{and} \quad \int_\Omega \mathcal{X}_\omega(x)\, dx = L|\Omega| \right\},$$

$|\Omega|$ being the Lebesgue measure of Ω.

(P) is a prototype of ill-posed problem in the sense that the infimum of J may be not attained in the class of characteristic functions $\mathcal{U}_L$ (we refer to [9, 18] for some related ill-posed problems). Notice that when we equip the space of admissible designs with the weak-$\star$ topology of $L^\infty(\Omega)$ (which let the functional J be continuous) we lose the compactness of the set $\mathcal{U}_L$. Indeed, $\mathcal{U}_L$ is not closed for that topology.

A way of overcoming this difficulty is by considering a relaxation of (P). That is, we look for another optimization problem (the so-called relaxed problem (RP)) which has a solution and, in addition, the minimum of the relaxed problem coincides with the infimum of (P). In Theorem 2.1, we prove that a relaxation of (P) simply consists of replacing the set $\mathcal{U}_L$ by its convex envelope. The proof is based on the fact that the corresponding observability constant is uniform with respect to $\mathcal{X}_\omega \in \mathcal{U}_L$. In our one-dimensional setting, this uniform property is deduced from the corresponding inequality for the wave equation by the way of the control transmutation method introduced by Miller in [15]. In the rest of the theoretical part, we also characterize the minimizers of the relaxed problem through a first-order optimality condition (Theorem 2.2) and discuss the case where the number N of connected components of ω is fixed *a priori*. This assumption (which makes so sense from a practical point of view) let again the set of admissible designs be compact with respect to weak-$\star$ L^∞ topology. Moreover, the continuity of the cost functional is preserved. Both conditions then lead to the existence of a classical solution $\mathcal{X}_{\omega_N^\star}$. Furthermore, $\{\mathcal{X}_{\omega_N^\star}\}_{N\in\mathbb{N}}$ is a minimizing sequence for the problem (P) (see Theorem 2.3 for detailed statements).

In a second part, we solve numerically the relaxed problem (RP) using a gradient method as in [16, 17, 18]. The numerical simulations provide optimal densities with values strictly in $(0,1)$, and therefore suggest that the original problem (P) is ill-posed. This is in contrast with the observation for the wave equation (see [16, 17]). We finally explain how one can extract a minimizing sequence of characteristics functions for (P) from an optimal density of the relaxed problem.

A few remarks and perspectives conclude this work.

2 Mathematical analysis of (P)

2.1 Relaxation

The obtention of a well-posed relaxation of (P) requires that the observability constant C appearing in (4) - which *a priori* depends on L, T, ω and Ω - be uniformly bounded with respect to the design variable ω. The following result holds :

Lemma 2.1 (Uniform Observability Inequality) *For any solution φ of (3) there exists a positive constant C, which only depends on Ω, L and T, such that*

$$\|\varphi(0)\|^2_{L^2(\Omega)} \le C \int_0^T \int_\Omega \mathcal{X}_\omega \varphi^2 dxdt \quad \textit{for all } \mathcal{X}_\omega \in \mathcal{U}_L. \tag{7}$$

Proof. (7) is proved in [15, Th. 2.3] by using the so-called *control transmutation method*. For the sake of completeness, we briefly indicate the main ingredients of the proof. To begin with, by duality, the constant C in (7) is the same constant appearing in the inequality

$$\|h_\omega\|^2_{L^2((0,T)\times\omega)} \le C \left\|u^0\right\|^2_{L^2(\Omega)}, \tag{8}$$

where h_ω and u^0 are the null control and the initial condition of the controlled system (1)-(2), respectively. Thus, we should prove that (8) holds for a constant C independent of the shape and location of the control region ω. The idea in [15] consists in using the well-known result, initially due to Russell, that controllability for the 1-D wave equation implies controllability for the 1-D heat equation. Indeed, some appropriate integral transforms allow to transmute states and controls of the wave equation into states and controls of the heat equation. Here, the integral transform to be used is a modification of Kannai's formula.

First, let $v = v(t,s)$ be a *controlled fundamental solution* of the heat equation in the segment $]-a,a[$, i.e. $v \in C^0([0,T];\mathcal{M}(]-a,a[))$, where $\mathcal{M}(]-a,a[)$ is the space of Radon measures on $]-a,a[$, solves

$$\begin{cases} v_t - v_{ss} = 0 \quad \text{in} \quad \mathcal{D}'(]0,T[\times]-a,a[) \\ v(0) = \delta \\ v(T) = 0. \end{cases} \tag{9}$$

The existence of such a solution is proved in [15, Prop. 5.2]. In addition, it is shown that there exist constants $A > 0$ and $\alpha > 0$ such that

$$\|v\|_{L^2(]0,T[\times]-a,a[)} \leq A e^{\alpha a^2/T}. \tag{10}$$

Denoting by $\underline{v}(t,s)$ the extension of v to $\mathbb{R}^2$ with zero value outside $]0,T[\times]-a,a[$, it is clear that $\underline{v}$ satisfies (9) and (10).

On the other hand, for any $u^0 \in H_0^1(\Omega)$, there exists a minimal time $\tau = \tau(\Omega\setminus\omega) > 0$, which can be chosen uniformly with respect to ω, and a control function $g \in L^2((0,\tau)\times\Omega)$ such that the solution $y(s,x)$ of

$$\begin{cases} y_{ss} - y_{xx} = \mathcal{X}_\omega g & \text{in} \quad (0,\tau)\times\Omega \\ y|_{\partial\Omega} = 0 & \text{in} \quad [0,\tau] \\ (y(0), y_t(0)) = (u^0, 0) & \text{in} \quad \Omega \end{cases}$$

satisfies the controllability condition $(y(\tau), y_t(\tau)) = (0,0)$ in Ω (see [12]). In addition, similarly to the inequality (8), there exists $B > 0$ such that

$$\|\mathcal{X}_\omega g\|^2_{L^2((0,\tau)\times\omega)} \leq B \left\|u^0\right\|^2_{H_0^1(\Omega)}. \tag{11}$$

Moreover, the constant B can be chosen, in a crucial way for our purpose, independent of the variable ω (we refer to [21, Prop. 2.1] for a direct proof based on Fourier decomposition). Denote by $\underline{y}(s,x)$ and $\underline{g}(s,x)$ the extension of y and g by reflection with respect to $s=0$. Then, $\underline{y}(s,x)$ solves

$$\underline{y}_{ss} - \underline{y}_{xx} = \mathcal{X}_\omega \underline{g} \quad \text{in } \mathcal{D}'(\mathbb{R}\times\Omega), \quad \underline{y} = 0 \text{ on } \mathbb{R}\times\partial\Omega$$

and

$$\left\|\mathcal{X}_\omega \underline{g}\right\|^2_{L^2(\mathbb{R}\times\omega)} \leq \underline{B} \left\|u^0\right\|^2_{H_0^1(\Omega)},$$

with a different constant $\underline{B}$ which depends on the same parameters as B.

Finally, we use $\underline{v}$, with $a = \tau$ in (9), as a kernel to transmute $\underline{y}(s,x)$ and $\underline{g}(s,x)$ into a solution $u(t,x)$ and a control $h(t,x)$ of the system (1), respectively. We define

$$u(t,x) = \int_{\mathbb{R}} \underline{v}(t,s)\,\underline{y}(s,x)\,ds \quad \text{and} \quad h(t,x) = \int_{\mathbb{R}} \underline{v}(t,s)\,\underline{g}(s,x)\,ds.$$

Remark that the time variable s for the wave equation transmutes into the spatial variable for the heat equation. Then, we check that $u(t,x)$ and $h(t,x)$ solve (1)-(2), so that h is control for the heat equation.

Moreover, by the Cauchy-Schwartz inequality with respect to s,

$$\|\mathcal{X}_\omega h\|^2_{L^2((0,T)\times\omega)} \leq \|\underline{v}\|^2_{L^2(\mathbb{R}^2)} \left\|\mathcal{X}_\omega \underline{g}\right\|^2_{L^2(\mathbb{R}\times\omega)} \leq A^2 e^{2\alpha\tau^2/T}\underline{B}\left\|u^0\right\|^2_{H^1_0(\Omega)}.$$

Finally, the regularizing effect of the heat equation leads to a similar inequality, but replacing $\left\|u^0\right\|_{H^1_0(\Omega)}$ by $\left\|u^0\right\|_{L^2(\Omega)}$. ■

We refer the reader to [20] where the control transmutation method is analyzed from the numerical point of view. In particular, the fundamental controlled solution v of (9) in term of infinite series involving the heat kernel is computed.

Let us now consider the space $\overline{\mathcal{U}}_L$

$$\overline{\mathcal{U}}_L = \left\{\theta \in L^\infty(\Omega;[0,1]), \quad \int_\Omega \theta(x)\,dx = L|\Omega|\right\}$$

endowed with the weak-$\star$ topology of $L^\infty(\Omega)$. $\overline{\mathcal{U}}_L$ is the weak-$\star$ closure of $\mathcal{U}_L$ in $L^\infty(\Omega)$.

Let us fix $\theta \in \overline{\mathcal{U}}_L$. The density of $\mathcal{U}_L$ in $\overline{\mathcal{U}}_L$ and the observability inequality (7) imply the following relation

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq C\int_0^T\int_\Omega \theta\varphi^2 dxdt$$

for any solution φ of (3). This implies the coercitivity of the continuous and strictly convex functional

$$\mathcal{J}_\theta(\varphi) = \frac{1}{2}\int_0^T\int_\Omega \theta\varphi^2 dxdt + \int_\Omega \varphi(0)\,u^0 dx, \tag{12}$$

defined on the space

$$\begin{aligned} H_\theta \;\;=\;\; & \big\{h = \theta\varphi \text{ for some } \varphi \text{ solution of (3) with } \varphi \in L^2\left([0,\tau] : H^2 \cap H^1_0(\Omega)\right) \\ & \text{for all } 0<\tau<T \text{ and } \int_0^T\int_\Omega \theta\varphi^2 dxdt < +\infty\Big\}, \end{aligned}$$

which is endowed with the norm

$$\|h\|_{H_\theta} = \left(\int_0^T\int_\Omega \theta\varphi^2 dxdt\right)^{1/2}.$$

Since each element $h \in H_\theta$ is associated with a solution φ of (3), for convenience we will refer to the elements of H_θ as φ solution of (3). The same applies for the elements of $H_{\mathcal{X}_\omega}$. Let us denote by $\widehat{\varphi}_\theta$ the unique minimizer of $\mathcal{J}_\theta$. Proceeding as in the case of characteristic functions it can be proved that the function u, solution of the system

$$\left\{\begin{array}{ll} u_t - u_{xx} = \theta\widehat{\varphi}_\theta, & (t,x) \in \,]0,T[\,\times\Omega \\ u|_{\partial\Omega} = 0, & t \in [0,T] \\ u(0,x) = u^0(x), & x \in \Omega, \end{array}\right. \tag{13}$$

satisfies the null controllability condition (2), i.e., $\theta\widehat{\varphi}_\theta$ is a null control for (13). We then may consider the relaxed problem

$$(RP) \qquad \inf_{\theta\in\overline{\mathcal{U}}_L} \overline{J}(\theta) = \int_0^T\int_\Omega \theta\widehat{\varphi}_\theta{}^2 dxdt \tag{14}$$

where $\widehat{\varphi}_\theta$ is the minimizer of (12). We are now in a position to prove the main result of this section.

Theorem 2.1 *The functional $\overline{J}$ as given by (14) is convex and continuous for the weak-$\star$ topology of $L^\infty(\Omega)$. In particular, there exists $\theta^* \in \overline{\mathcal{U}}_L$ such that*

$$\inf_{\mathcal{X}_\omega\in\mathcal{U}_L} J(\mathcal{X}_\omega) = \min_{\theta\in\overline{\mathcal{U}}_L} \overline{J}(\theta) = \overline{J}(\theta^*).$$

Proof. Let us first prove the convexity of $\overline{J}$. Since $\widehat{\varphi}_\theta$ is the minimizer of (12), it satisfies the Euler-Lagrange equation

$$\int_0^T\int_\Omega \theta\widehat{\varphi}_\theta\psi dxdt + \int_\Omega u^0\psi(0)\,dx = 0 \qquad \text{for all } \psi \text{ solution of (3).} \tag{15}$$

Replacing ψ by $\widehat{\varphi}_\theta$ in (15), a simple computation yields to

$$\begin{aligned} -\frac{1}{2}\overline{J}(\theta) &= -\frac{1}{2}\int_0^T\int_\Omega \theta\widehat{\varphi}_\theta{}^2 dxdt \\ &= \frac{1}{2}\int_0^T\int_\Omega \theta\widehat{\varphi}_\theta{}^2 dxdt + \int_\Omega u^0\widehat{\varphi}_\theta(0)\,dx \\ &= \min_{\varphi\in H_\theta} \mathcal{J}_\theta(\varphi). \end{aligned}$$

This proves that $-\frac{1}{2}\overline{J}(\theta)$ is concave since it is the minimum of affine functions. Hence, $\overline{J}(\theta)$ is convex.

Due to the density of $\mathcal{U}_L$ in $\overline{\mathcal{U}}_L$, to prove the continuity of $\overline{J}(\theta)$ it suffices to show that if $\mathcal{X}_j \in \mathcal{U}_L$ is such that

$$\mathcal{X}_j \rightharpoonup \theta \quad \text{weak-}\star \text{ in } L^\infty(\Omega), \tag{16}$$

then

$$\int_0^T\int_\Omega \mathcal{X}_j\widehat{\varphi}_j^2 dxdt \to \int_0^T\int_\Omega \theta\widehat{\varphi}_\theta^2 dxdt$$

where $\widehat{\varphi}_j$ and $\widehat{\varphi}_\theta$ are the minimizers of $\mathcal{J}_{\mathcal{X}_j}$ and $\mathcal{J}_\theta$, respectively. Since $\widehat{\varphi}_j$ is the unique minimizer of $\mathcal{J}_{\mathcal{X}_j}$, we have $\mathcal{J}_{\mathcal{X}_j}(\widehat{\varphi}_j) \le \mathcal{J}_{\mathcal{X}_j}(0) = 0$. Thus, by the Cauchy-Schwartz inequality and (7), we obtain

$$\begin{aligned} \frac{1}{2}\int_0^T\int_\Omega \mathcal{X}_j\widehat{\varphi}_j^2 dxdt &\le -\int_\Omega u^0\widehat{\varphi}_j(0)\,dx \\ &\le \left\|u^0\right\|_{L^2(\Omega)} \left\|\widehat{\varphi}_j(0)\right\|_{L^2(\Omega)} \\ &\le \left\|u^0\right\|_{L^2(\Omega)} \left[C\int_0^T\int_\Omega \mathcal{X}_j\widehat{\varphi}_j^2 dxdt\right]^{1/2} \end{aligned}$$

and therefore

$$\int_0^T\int_\Omega \mathcal{X}_j\widehat{\varphi}_j^2 dxdt \leq 4C\left\|u^0\right\|_{L^2(\Omega)}^2. \tag{17}$$

By passing to a subsequence, still labeled by the index j,

$$\mathcal{X}_j\widehat{\varphi}_j \rightharpoonup \eta \qquad \text{weakly in } L^2\left((0,T)\times(0,1)\right). \tag{18}$$

Let us now prove that $\eta=\theta\widehat{\varphi}_\theta$, with $\widehat{\varphi}_\theta$ the minimizer of $\mathcal{J}_\theta$ and θ the weak limit of $\mathcal{X}_j$. Since the observability inequality (7) holds for any positive T, in particular, it holds in the interval $[\tau,T]$ for $0<\tau<T$. Applying this estimate to the minimizer $\widehat{\varphi}_j$ we get

$$\left\|\widehat{\varphi}_j\left(\tau\right)\right\|_{L^2(0,1)}^2 \leq C\int_\tau^T\int_\Omega \mathcal{X}_j\widehat{\varphi}_j^2 dxdt, \qquad \forall\tau\in(0,T)$$

with a constant C which only depends on Ω, L and $T-\tau$. Taking (17) into account,

$$\left\|\widehat{\varphi}_j\left(\tau\right)\right\|_{L^2(0,1)}^2 \leq C, \qquad \forall\tau\in(0,T)$$

with a different constant C which is independent of $\mathcal{X}_j$. The regularizing effect of the system (3) implies

$$\left\|\widehat{\varphi}_j\left(\tau\right)\right\|_{H^2\cap H_0^1(0,1)}^2 \leq C, \qquad \forall\tau\in(0,T)$$

for another constant C which depends on the same parameters and which is finite for all $0\leq\tau<T$. By using classical energy estimates for the solutions of (3) and by Aubin-Lions' lemma, up to subsequences (that we still denote by j),

$$\widehat{\varphi}_j\left(t,x\right)\to\overline{\varphi}\left(t,x\right) \quad \text{strongly in } L^2\left((0,\tau)\times\Omega\right), \quad 0<\tau<T \tag{19}$$

and

$$\widehat{\varphi}_j\left(\tau,x\right)\to\overline{\varphi}\left(\tau,x\right) \quad \text{strongly in } L^2\left(\Omega\right), \quad 0\leq\tau<T. \tag{20}$$

Thus, by (16),

$$\mathcal{X}_j\widehat{\varphi}_j \rightharpoonup \theta\overline{\varphi} \qquad \text{in } \mathcal{D}'\left((0,\tau)\times(0,1)\right).$$

From (18), it follows that

$$\eta\left(t,x\right)=\theta\left(x\right)\overline{\varphi}\left(t,x\right) \qquad \text{for all } 0<t<T \text{ and } 0<x<1. \tag{21}$$

In order to identify the function $\overline{\varphi}$, consider a function $\phi\in C^\infty\left((0,\tau)\times(0,1)\right)$ such that

$$\phi\left(t,0\right)=\phi\left(t,1\right)=\phi\left(0,x\right)=0.$$

Multiplying (3) by ϕ and integrating by parts,

$$\int_0^\tau\int_\Omega \widehat{\varphi}_j\left(\phi_t-\phi_{xx}\right)dxdt=\int_\Omega \widehat{\varphi}_j\left(\tau\right)\phi\left(\tau\right)dx.$$

Letting $j\to\infty$ in this expression and taking (19) into account,

$$\int_0^\tau\int_\Omega \overline{\varphi}\left(\phi_t-\phi_{xx}\right)dxdt=\int_\Omega \overline{\varphi}\left(\tau\right)\phi\left(\tau\right)dx.$$

This means that $\overline{\varphi}$ is a solution of the system

$$\begin{cases} \overline{\varphi}_t + \overline{\varphi}_{xx} = 0, & 0 < t < \tau,\ 0 < x < 1 \\ \overline{\varphi}(t,0) = \overline{\varphi}(t,1) = 0, & 0 < t < \tau. \end{cases}$$

On the other hand, if the initial datum φ^T of (3) belongs to $\mathcal{D}(0,1)$, then the corresponding solution $\varphi \in H_{\mathcal{X}_j}$ for all j. Taking limits in the Euler-Lagrange equation associated with the functional $\mathcal{J}_{\mathcal{X}_j}$,

$$\int_0^T \int_\Omega \mathcal{X}_j \widehat{\varphi}_j \varphi dxdt + \int_\Omega u^0 \varphi(0)\, dx = 0, \tag{22}$$

we obtain by (18) and (21),

$$\int_0^T \int_\Omega \theta \overline{\varphi} \varphi dxdt + \int_\Omega u^0 \varphi(0)\, dx = 0.$$

Since this equation characterizes the minimizer $\widehat{\varphi}_\theta$ of $\mathcal{J}_\theta$, we then conclude that $\widehat{\varphi}_\theta = \overline{\varphi}$. Finally, notice that the weak limit $\eta(t,x)$ of the subsequence $\mathcal{X}_j \widehat{\varphi}_j$ is uniquely defined by (21) with $\overline{\varphi} = \widehat{\varphi}_\theta$. This implies that the whole sequence $\mathcal{X}_j \widehat{\varphi}_j$ converges to the same limit.

Finally, replacing φ by $\widehat{\varphi}_j$ in (22) and letting $j \to \infty$ yields

$$\begin{aligned} \lim_{j\to\infty} \int_0^T \int_\Omega \mathcal{X}_j \widehat{\varphi}_j^2 dxdt &= -\lim_{j\to\infty} \int_\Omega u^0 \widehat{\varphi}_j(0)\, dx \\ &= -\int_\Omega u^0 \widehat{\varphi}_\theta(0)\, dx \\ &= \int_0^T \int_\Omega \theta \widehat{\varphi}_\theta^2 dxdt, \end{aligned}$$

where the second equality is a consequence of (20) and the last one is due to (15). This completes the proof. ∎

Theorem 2.1 shows that the formulation (RP), defined in (14) is a well-posed relaxation of the original problem (P). As in the hyperbolic case studied in [21], (RP) is simply obtained from (P) by replacing the set of characteristics functions $\mathcal{U}_L$ by the set of density functions. The situation is in general very different and much more complex when the design variable $\mathcal{X}_\omega$ appears in a differential operator of order greater than zero as is usual in optimal design in conductivity with the divergence operator (we refer to [19] and the references therein).

2.2 First-order optimality condition

In this section, we give a characterization of the minimizers for the relaxed problem (RP). From [3, 10], we recall that the tangent cone $\mathcal{T}'_{\overline{\mathcal{U}}_L}(\theta^*)$ to the set $\overline{\mathcal{U}}_L$ at θ^* in $L^\infty(\Omega)$ is defined as the set of elements $\theta \in L^\infty(\Omega)$ such that for any sequence $t_n \searrow 0$ there exists another sequence $\theta_n \in L^\infty(\Omega)$ such that:

(a) $\theta^* + t_n \theta_n \in \overline{\mathcal{U}}_L$, and

(b) $\theta_n \to \theta$ uniformly, as $n \to \infty$.

THEOREM 2.2 *The functional* $\overline{J}$ *as defined in (14) is Gâteaux differentiable on the set* $\overline{\mathcal{U}}_L$ *and its derivative at* $\theta \in \overline{\mathcal{U}}_L$ *in the admissible direction* $\overline{\theta}$ *is given by*

$$< \overline{J}'(\theta), \overline{\theta} > \;=\; -\int_0^T \int_\Omega \overline{\theta} \widehat{\varphi}_\theta^2 dxdt, \tag{23}$$

$\widehat{\varphi}_\theta$ *being the minimizer of* $\mathcal{J}_\theta$*. In particular,* $\theta^* \in \overline{\mathcal{U}}_L$ *is a minimizer for (RP) if and only if*

$$\int_0^T \int_\Omega \overline{\theta} \widehat{\varphi}_{\theta^*}^2 dxdt \leq 0 \quad \forall\, \overline{\theta} \in \mathcal{T}'_{\overline{\mathcal{U}}_L}(\theta^*). \tag{24}$$

Proof. Let $\overline{\theta} \in L^\infty(\Omega; [0,1])$ be an admissible direction, i.e., for $\varepsilon > 0$ small enough, $\theta + \varepsilon\overline{\theta} \in \overline{\mathcal{U}}_L$. Then, the corresponding minimizers $\widehat{\varphi}_\theta$ and $\widehat{\varphi}_{\theta+\varepsilon\overline{\theta}}$ of $\mathcal{J}_\theta$ and $\mathcal{J}_{\theta+\varepsilon\overline{\theta}}$, respectively, solve the Euler-Lagrange equations

$$\int_0^T \int_\Omega \theta \widehat{\varphi}_\theta \psi dxdt + \int_\Omega u^0 \psi(0)\, dx = 0 \qquad \text{for all } \psi \text{ solution of (3)} \tag{E_θ}$$

and

$$\int_0^T \int_\Omega \left(\theta + \varepsilon\overline{\theta}\right) \widehat{\varphi}_{\theta+\varepsilon\overline{\theta}} \psi dxdt + \int_\Omega u^0 \psi(0)\, dx = 0 \qquad \text{for all } \psi \text{ solution of (3).} \tag{$\mathrm{E}_{\theta+\varepsilon\overline{\theta}}$}$$

Writing first $\psi = \widehat{\varphi}_\theta$ in (E_θ) and $\psi = \widehat{\varphi}_{\theta+\varepsilon\overline{\theta}}$ in ($\mathrm{E}_{\theta+\varepsilon\overline{\theta}}$), and then $\psi = \widehat{\varphi}_{\theta+\varepsilon\overline{\theta}}$ in (E_θ) and $\psi = \widehat{\varphi}_\theta$ in ($\mathrm{E}_{\theta+\varepsilon\overline{\theta}}$) we get

$$\begin{aligned} \frac{\overline{J}\left(\theta + \varepsilon\overline{\theta}\right) - \overline{J}(\theta)}{\varepsilon} &= -\frac{1}{\varepsilon}\left[\int_\Omega \left(\widehat{\varphi}_{\theta+\varepsilon\overline{\theta}}(0) - \widehat{\varphi}_\theta(0)\right) u^0 dx\right] \\ &= -\int_0^T \int_\Omega \overline{\theta} \widehat{\varphi}_{\theta+\varepsilon\overline{\theta}} \widehat{\varphi}_\theta dxdt. \end{aligned}$$

On the other hand, by the mean value theorem for integrals and by Cauchy-Schwartz's inequality, there exists $0 < \tau < T$ such that

$$\begin{aligned} \int_0^T \int_\Omega \left[\widehat{\varphi}_{\theta+\varepsilon\overline{\theta}} - \widehat{\varphi}_\theta\right] \overline{\theta} \widehat{\varphi}_\theta dxdt &= T \int_\Omega \left[\widehat{\varphi}_{\theta+\varepsilon\overline{\theta}}(\tau) - \widehat{\varphi}_\theta(\tau)\right] \overline{\theta} \widehat{\varphi}_\theta(\tau)\, dx \\ &\leq T \left\|\overline{\theta}\right\|_{L^\infty(\Omega)} \left\|\widehat{\varphi}_\theta(\tau)\right\|_{L^2(\Omega)} \left\|\widehat{\varphi}_{\theta+\varepsilon\overline{\theta}}(\tau) - \widehat{\varphi}_\theta(\tau)\right\|_{L^2(\Omega)}. \end{aligned}$$

Taking limits as $\varepsilon \to 0$ in this expression and taking into account that the convergence (20), with $\widehat{\varphi}_j = \widehat{\varphi}_{\theta+\varepsilon\overline{\theta}}$ and $\overline{\varphi} = \widehat{\varphi}_\theta$, holds in our setting, we obtain (23). Therefore, the first-order necessary optimality condition for (RP) translates into (24). Finally, since both $\overline{J}$ and $\overline{\mathcal{U}}_L$ are convex, this condition is also sufficient. ■

We point out that the first order derivative of J does not depend on any adjoint solution. This property is due to the fact that for a fixed density θ, the control given by $\theta\hat{\phi}_\theta$ in $(0,T) \times \Omega$ obtained by duality is the one of minimal L^2-norm.

2.3 A sufficient condition for the existence of classical solutions

As indicated in the introduction, a possibility to guarantee the existence of a classical solution of (P) is by limiting the number of connected components of the admissible designs. Thus, for a fixed $N \in \mathbb{N}^*$ and $L \in (0,1)$, we consider the new set of admissible designs $\mathcal{U}_L^N$ composed of the characteristic functions $\mathcal{X}_\omega$ associated with the open subsets $\omega \subset \Omega$ such that $|\omega| = L\,|\Omega|$ and moreover ω is the union of at most N disjoint intervals of positive Lebesgue measure. Now consider the optimal design problem

$$(P_N) \qquad \inf_{\mathcal{X}_{\omega_N} \in \mathcal{U}_L^N} J_N\left(\mathcal{X}_{\omega_N}\right) = \int_0^T \int_\Omega \mathcal{X}_{\omega_N} \widehat{\varphi}^2 dxdt$$

where as before $\widehat{\varphi}$ is the unique minimizer of (5). Then we have:

Theorem 2.3 *(P_N) is well-posed, that is, there exists $\mathcal{X}_{\omega_N^*} \in \mathcal{U}_L^N$ such that*

$$\inf_{\mathcal{X}_{\omega_N} \in \mathcal{U}_L^N} J_N\left(\mathcal{X}_{\omega_N}\right) = J_N(\mathcal{X}_{\omega_N^*}).$$

Moreover, $\{\mathcal{X}_{\omega_N^*}\}_{N\in\mathbb{N}}$ is a minimizing sequence for (P), i.e.,

$$\lim_{N\to\infty} J(\mathcal{X}_{\omega_N^*}) = \overline{J}(\theta^*), \tag{25}$$

where θ^* is a solution of (RP).

Proof. Clearly, each element $\mathcal{X}_{\omega_N} \in \mathcal{U}_L^N$ may be associated with a vector $(x_1, x_2, \cdots, x_{2N}) \in [0,1]^{2N}$ such that:

$$\begin{cases} \text{(i)} & 0 \le x_1 \le x_2 \le \cdots \le x_{2N} \le 1, \\ \text{(ii)} & \sum_{j=1}^N (x_{2j} - x_{2j-1}) = L, \text{ and} \\ \text{(iii)} & \omega_N = \bigcup_{j=1}^N \left]x_{2j-1}, x_{2j}\right[. \end{cases}$$

Since we identify each subset ω_N with its characteristic function $\mathcal{X}_{\omega_N}$ and as two characteristics functions are equivalent if they are equal almost everywhere, conditions (i)-(iii) actually mean that ω_N is composed of at most N disjoints intervals. Thus, the set of admissible designs $\mathcal{U}_L^N$ is identified with the compact set

$$\mathcal{K}_N = \left\{ (x_1, x_2, \cdots, x_{2N}) \in [0,1]^{2N} \text{ which satisfy (i)-(iii) above} \right\}.$$

Moreover, since the convergence of a sequence in $\mathcal{K}_N$ implies the strong convergence in $L^p(\Omega)$, $1 \le p < \infty$, of the associated sequence of characteristic functions to a characteristic function associated with an element in $\mathcal{U}_L^N$, the same proof as in Theorem 2.1, shows that the map

$$\mathcal{K}_N \ni (x_1, x_2, \cdots, x_{2N}) \mapsto J_N\left(\mathcal{X}_{\omega_N}\right) = \int_0^T \int_\Omega \mathcal{X}_{\omega_N} \widehat{\varphi}^2 dxdt$$

is continuous. Both the continuity of this map and the compactness of $\mathcal{K}_N$ imply the existence of a solution for (P_N).

Now let θ^* be a solution of (RP). Since $\mathcal{U}_L$ is dense in $\overline{\mathcal{U}}_L$, there exist a sequence $\{\mathcal{X}_{\omega_N}\}_{N\in\mathbb{N}}$ such that

$$\mathcal{X}_{\omega_N} \rightharpoonup \theta^* \quad \text{weak-}\star \text{ in } L^\infty(\Omega)\,. \tag{26}$$

What is important here is that the sequence $\{\mathcal{X}_{\omega_N}\}_{N\in\mathbb{N}}$ may be chosen in such a way that ω_N has a finite number of connected components. We refer to the proof of [10, Prop. 7.2.14] for this passage. Without loss of generality we may assume that ω_N has at most N connected components, that is, $\mathcal{X}_{\omega_N} \in \mathcal{U}_L^N$. Then,

$$\overline{J}(\theta^*) \leq \overline{J}\left(\mathcal{X}_{\omega_N^*}\right) = J_N\left(\mathcal{X}_{\omega_N^\star}\right) \leq J_N(\mathcal{X}_{\omega_N}) = J(\mathcal{X}_{\omega_N}) = \overline{J}(\mathcal{X}_{\omega_N})\,.$$

Passing to the limit in this expression and taking into account (26) and the weak-$\star$ in $L^\infty(\Omega)$ continuity of $\overline{J}$ gives (25). ■

3 Numerical analysis of (P)

3.1 Numerical resolution of (RP)

The relaxed problem (RP) is solved using a descent gradient method as done in [16, 17]. In order to take into account the volume constraint $\|\theta\|_{L^1(\Omega)} = L|\Omega|$, we introduce the cost $\underline{J}(\theta,\lambda) = \overline{J}(\theta) + \lambda(\int_\Omega \theta(x)dx - L)$ where $\lambda \in \mathbb{R}$ denotes a Lagrangian multiplier and then minimize $\underline{J}$ over $L^\infty(\Omega;[0,1]) \times \mathbb{R}$. From (23), we deduce that the first variation of $\underline{J}$ with respect to θ is given by

$$< \underline{J}(\theta,\lambda), \overline{\theta} >= \int_\Omega \left(\lambda - \int_0^T \hat{\phi}_\theta^2(t,x)dt\right)\overline{\theta}dx, \qquad \forall \overline{\theta} \in L^\infty(\Omega,[0,1])$$

so that, at each iteration k of the descent algorithm, the density variable is updated as follows :

$$\theta_{k+1} = \theta_k - \eta_k(x)\left(\lambda_k - \int_0^T \hat{\phi}_{\theta_k}^2(t,x)\right), \quad k > 0$$

where the function $\eta_k \in L^\infty(\Omega;\mathbb{R}^+)$ is chosen so as to ensure that $\theta_{k+1}(x) \in [0,1]$ for all $x \in \Omega$. The multiplier λ_k is then explicitly determined in order that $\|\theta_{k+1}\|_{L^1(\Omega)} = L|\Omega|$. We refer to [16, 17] for the details. We point out that each iteration requires the computation of the function $\hat{\varphi}_k$ minimum of the functional $\mathcal{J}_{\theta_k}$ over H_{θ_k}. This corresponds to the numerical resolution of a null controllability problem for the heat equation, the control being given by $\theta_k\hat{\varphi}_k$.

In the case of characteristic functions where the control acts on $\omega \subset \Omega$, the numerical minimization of $\mathcal{J}_{\mathcal{X}}$ - usually performed by a conjugate gradient algorithm - is an ill-posed problem: this is due to the hugeness of the space $H_{\mathcal{X}_\omega}$ which implies that $\mathcal{J}_{\mathcal{X}_\omega}$ is very weakly coercive in L^2. We refer to [4] where this phenomenon was first observed and analyzed and to [6, 20] for recent developments. A simpler method consists in replacing the null controllability requirement (2) by the condition $\|u(T,\cdot)\|_{L^2(\Omega)} \leq \epsilon$ for any $\epsilon > 0$ small enough. The corresponding approximate controllability problem is well-posed and leads to $\hat{\varphi}_\epsilon$ close to $\hat{\varphi}$. Remark that this regularization technique is consistent in our context, because keep unchanged the relaxation procedure. In the very particular case where the control acts on the whole domain (i.e. $\omega = \Omega$), the controllability problem for the heat equation is well-posed (in the sense that ϕ^T is regular and can be approximated numerically with robustness).

In the case of density functions, the situation is slightly better since the control $\theta\hat{\phi}_\theta$ *a priori* may act precisely on the whole domain Ω. This situation occurs when $\theta > 0$ in Ω ((P) is ill-posed in that case and ϵ can be chosen arbitrarily small).

In the sequel, we consider $\epsilon = 10^{-5}$ and use an iterative splitting method, introduced and detailed in [8], section 1.8.8, to ensure precisely that $\|u(T,\cdot)\|_{L^2(\Omega)} = \epsilon$. This gives a meaning to the comparison of two controls with distinct supports, although we observe that the variation of the limit density θ_{lim} (obtained at the convergence of the algorithm) with respect to ϵ is quite low.

3.2 Numerical experiments

In order to have a better control of the heat diffusion during the time interval, we introduce a diffusivity coefficient c lower than one and consider the heat operator $\partial_t - c\partial_{xx}$. In the sequel, we take $c = 1/10$.

We first consider a simple situation where u^0 is the first eigenfunction of the Dirichlet Laplacian: $u^0(x) = \sin(\pi x)$ and take $T = 0.5$, $L = 0.2$. The limit density θ obtained at the convergence of the gradient algorithm is depicted on Figure 1. The corresponding value of the cost is $\overline{J}(\theta_{lim}) \approx 1.112$. We observe that θ_{lim} takes values in $(0,1)$: this indicates that problem (P) is ill-posed in the class of characteristic functions and justifies the relaxation procedure. In particular, the centered solution $\mathcal{X}_{]1/2-L/2,1/2+L/2[}$, which is optimal if we assume that ω is an interval, is not optimal over $\overline{\mathcal{U}}_L$: we obtain $\overline{J}(\mathcal{X}_{]1/2-L/2,1/2+L/2[}) \approx 2.651 > \overline{J}(\theta_{lim})$. We also observe, as in [16, 17], the decreasing of the ratio $\|\hat{\phi}_k(0,\cdot)\|^2_{L^2(\Omega)}/\|\theta_k\hat{\phi}_k\|^2_{L^2((0,T)\times\omega)}$ during the iteration of the algorithm: the reduction of the cost of the control with respect to ω have the effect of improving the observability of u^0. This limit density is obtained starting with the constant density $\theta_0 = L$ over Ω, which is the most natural and *a priori* do not favor any possible local minimum. We observe that other initial functions such as, for instance $\theta_0 = L\Psi^n/\|\Psi^n\|_{L^1(\Omega)}$ with $\Psi^n = \sin(n\pi x)$, lead to different limit densities but provide the same value of the cost $\overline{J}$. As in [18], this suggests that the convex function $\overline{J}$ is not strictly convex.

Let us now illustrate somehow Section 2.3 and associate with θ_{lim} a sequence of characteristic functions $\mathcal{X}_{\omega^M} \in \mathcal{U}_L$ weakly converging toward θ_{lim} and minimizing for J (i.e. $\lim_{M\to\infty} J(\mathcal{X}_{\omega^M}) = \overline{J}(\theta_{lim})$). We proceed as follows having in mind that the density θ_{lim} at the point x represents the volume fraction of *control material*. Let us decompose the interval Ω into $M > 0$ non-empty subintervals such that $\Omega = \cup_{j=1,M}[x_j, x_{j+1}]$. Then, we associate with each interval $[x_j, x_{j+1}]$ the mean value $m_j \in [0,1]$ defined by

$$m_j = \frac{1}{x_{j+1} - x_j} \int_{x_j}^{x_{j+1}} \theta_{lim}(x)dx$$

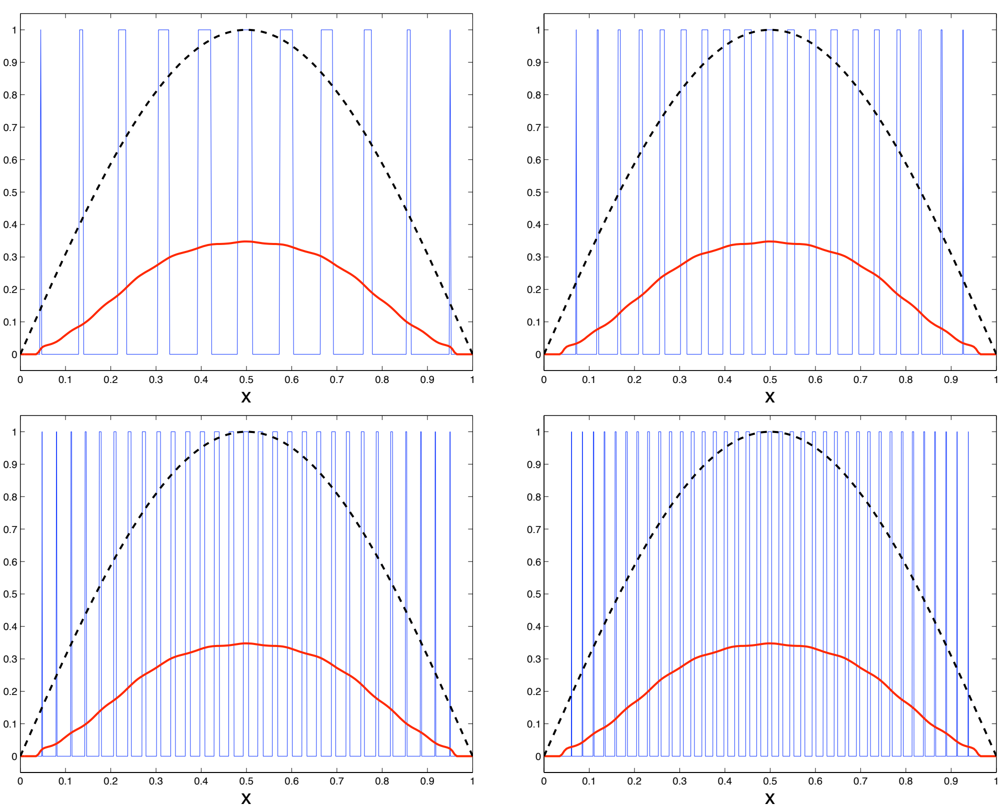

Figure 1: $c = 1/10$, $L = 1/5$, $T = 1/2$, $u^0(x) = \sin(\pi x)$ - Limit density θ and four corresponding characteristic functions of the sequence $\{\mathcal{X}_{\omega^M}\}_{(M>0)}$.

and the division into parts $[x_j,(1-m_j)x_j+m_jx_{j+1}[\cup[(1-m_j)x_j+m_jx_{j+1},x_{j+1}[$. At last, we introduce the function $\mathcal{X}_{\omega^M}$ in $L^\infty(\Omega,\{0,1\})$ by

$$\mathcal{X}_{\omega^M}(x)=\sum_{j=1}^{M}\mathcal{X}_{]x_j,(1-m_j)x_j+m_jx_{j+1}[}(x).$$

We can identify with $\mathcal{X}_{\omega^M}$ the domain $\cup_{j=1}^{M}]x_j,(1-m_j)x_j+m_jx_{j+1}[$ composed of at most M disjoint components. We easily check that $\|\mathcal{X}_{\omega^M}\|_{L^1(\Omega)}=\|\theta_{lim}\|_{L^1(\Omega)}$, for all $M>0$ and that $\mathcal{X}_{\omega^M}\rightharpoonup\theta_{lim}$ weak-$\star$ in $L^\infty(\Omega)$ as $M\to\infty$. In this way, the bi-valued function $\mathcal{X}_{\omega^M}$ takes advantage of the information codified in the optimal density θ_{lim}. Table 1 collects the value of $J(\mathcal{X}_{\omega^M})$ for several values of M and suggests the convergence of $J(\mathcal{X}_{\omega^M})$ towards $\overline{J}(\theta_{lim})$ as M increases.

	$M=10$	$M=20$	$M=30$	$M=40$
$\overline{J}(\mathcal{X}_{\omega^M})$	2.132	1.612	1.381	1.132
$\frac{\overline{J}(\theta^{lim})-\overline{J}(\mathcal{X}_{\omega^M})}{\overline{J}(\theta_{lim})}$	9.17×10^{-1}	4.49×10^{-1}	2.41×10^{-1}	1.79×10^{-2}

Table 1: Value of the cost function $\overline{J}(\mathcal{X}_{\omega^M})$ vs. M.

Figure 2 depicts the limit density obtained at the convergence of the algorithm, obtained for a constant initial datum $u^0(x)=1$ (for which $\overline{J}(\theta_{lim})\approx1.821$), a concentrated datum $u^0(x)=e^{-300(x-0.5)^2}$ on $x=0.5$ ($\overline{J}(\theta_{lim})\approx4.43\times10^{-2}$), a concentrated datum $u^0(x)=e^{-300(x-0.8)^2}$ at $x=0.8$ ($\overline{J}(\theta_{lim})\approx1.665\times10^{-2}$) and a discontinuous datum $u^0(x)=\mathcal{X}_{[1/2,1]}(x)$ ($\overline{J}(\theta_{lim})\approx4.48\times10^{-1}$). Once again, the densities we obtain take values in $(0,1)$, even when the corresponding initial data u^0 are concentrated. This is due to the diffusion of the heat along Ω when time evolves which prevents from a localized support of the control. For small enough diffusion coefficient, the density is mainly concentrated on the support of the initial data u^0. Figure 3 represents the optimal density for $u^0(x)=e^{-300(x-0.8)^2}$ and $c=10^{-2}$ and $c=10^{-3}$ respectively. For an arbitrarily small coefficient c and u^0 such that $|supp(u^0)|\leq L$, we observe that the optimal density θ_{lim} is a characteristic function, with $supp(\theta_{lim})\subset supp(u^0)$.

Finally, we point out that if we exchange the role of u^0 and $u(T,\cdot)$, the result may be different, as a consequence of the irreversibility of the heat operator. The control to trajectory problem consists to drive the solution u of (1) from $u^0=0$ at time $t=0$ to u^T at time T. If uT is a trajectory for the homogeneous heat equation, then u^T is reachable by controls in $L^2((0,T)\times\omega)$. Relaxing the condition $u(T,\cdot)=u^T$ in Ω by the weaker condition $\|u(T,\cdot)-u^T\|_{L^2(\Omega)}\leq\epsilon$, more suitable at the numerical level, one may consider any function $uT\in L^2(\Omega)$. Figure 4 displays the optimal density corresponding to $u_T(x)=e^{-300(x-0.5)^2}$ and $u^T(x)=e^{-3000(x-0.5)^2}$ for $c=10^{-1}$. For the second case, we observe that θ_{lim} is close to a characteristic function. A possible explanation of this phenomenon follows: due to the dissipative property of the solution, the control of minimal L^2-norm mainly acts at the end of the time interval, says on $[T-\delta,T]$ for $0<\delta<<T$. Consequently, when $u^0=0$ in Ω, the controlled solution u is almost zero on $[0,T-\delta]\times\Omega$ independently of the diffusion

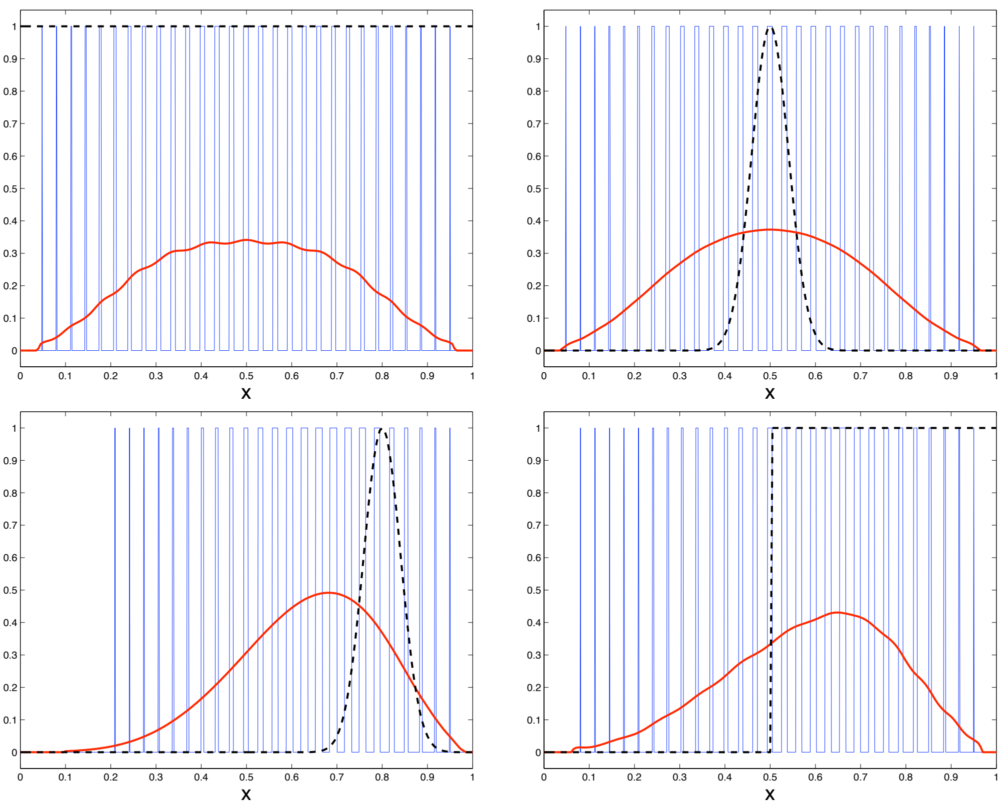

Figure 2: $c = 1/10$ - $T = 1/2$ - Optimal density θ_{lim} and associated characteristic function $\mathcal{X}_{\omega^{30}}$ for $u^0(x) = 1$ (Top Left), $u^0(x) = e^{-300(x-0.5)^2}$, $u^0(x) = e^{-300(x-0.8)^2}$ and $u^0(x) = \mathcal{X}_{[1/2,1[}(x)$.

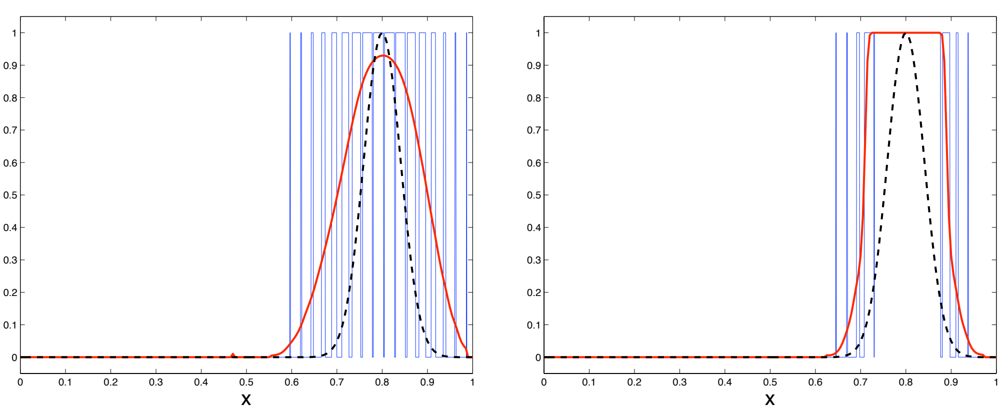

Figure 3: $T = 1/2$ - $u^0(x) = e^{-300(x-0.8)^2}$ - Optimal density θ and corresponding characteristic function $\mathcal{X}_{\omega^{30}}$ for $c = 10^{-2}$ (**Left**) and $c = 10^{-3}$ (**Right**).

coefficient c. Therefore, during the time interval $[T-\delta, T]$, the solution u passes suddenly from almost zero to u^T with a control localized on the support of u^T. As a summary, for $(u^0 \neq 0, u^T \equiv 0)$ the relevant parameter is the coefficient of diffusion c whereas for $(u^0 \equiv 0, u^T \neq 0)$, the relevant one is the support $supp(u^T)$ of the target u^T.

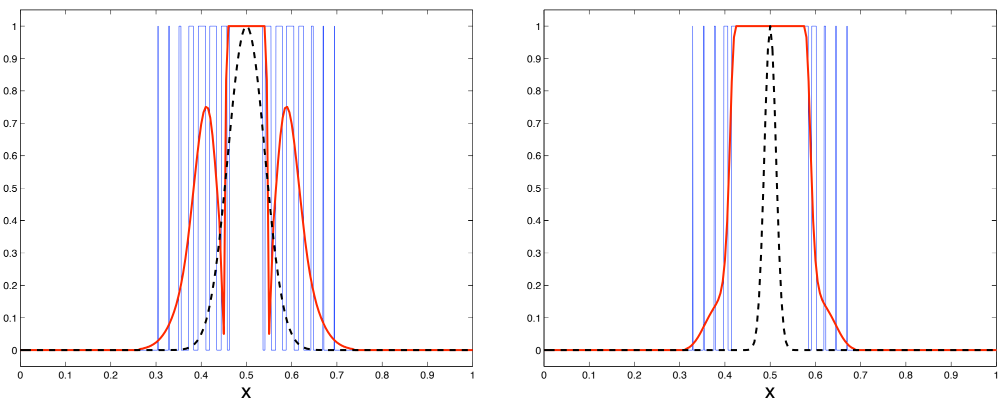

Figure 4: $c = 1/10$ - $T = 1/2$ - Optimal density θ and corresponding characteristic function $\mathcal{X}_{\omega^{30}}$ for $u^T(x) = e^{-300(x-0.5)^2}$ (**Left**) $[\overline{J}(\theta_{lim}) \approx 2.76]$ and $u^T(x) = e^{-3000(x-0.5)^2}$ (**Right**) $[\overline{J}(\theta_{lim}) \approx 18.74]$.

This numerical experiments suggest that the control of minimal L^2-norm which permits to drive the heat solution in the neighborhood of a given target is supported on an arbitrarily large number of disjoints domains. We highlight that this phenomenon is in contrast with the result obtained in [16, 17] for the wave equation. For the wave equation and smooth initial data, the numerical experiments never exhibit ill-posedness of the corresponding optimization problem.

Finally, we point out that other approaches (based on shape derivative, level set method or topological derivative (see [2])) may also be used for solving numerically this type of problems. In particular, those allow us to obtain local minima of the functional J as the union of a finite number of disjoint sub-intervals of Ω. This is particularly of interest in higher space dimension where the reconstruction of a sequence of characteristic function from an optimal density is less straightforward. We refer to [16, 17] for the application of this methodology to the wave equation.

4 Concluding remarks

We have analyzed an optimal shape design problem within the context of null-controllability for the one-dimensional heat equation. Non-existence of classical solutions has been numerically observed, which, in particular, justifies the relaxation process carried out. Since an optimal solution of the relaxed problem represents the local average (weak limit) of a minimizing sequence for the original problem, these minimizing sequences are easily constructed from the optimal relaxed density.

The interest in this type of problems is quite recent and many points remain to be analyzed. It could be interesting to extend the results of this work to the case where the diffusion coefficient depends on the spatial variable as well as the case of Neumann boundary conditions. We also mention the challenging situation where the support ω of the control may evolves in time. Very likely, this work also extend to the N- dimensional case. As indicated in the introduction, the main difficulty arises in the proof of the corresponding uniform observability inequality (7).

The numerical results contrast with those described in [16, 17] and exhibit a profound difference between the wave and the heat equations. In the first case, with smooth data, we always observe the well-posedness of the optimal problem in the class of characteristic functions, while for the second case, we always observed ill-posedness and non classical solutions. It would be interesting to consider the mixed situation and analyze the system

$$\begin{cases} \alpha u_{tt} + \beta u_t - u_{xx} = h_{\omega_{\alpha,\beta}}, & (t,x) \in]0,T[\times\Omega \\ u|_{\partial\Omega} = 0, & t \in [0,T] \\ (u(0,x), u_t(0,x)) = (u^0(x), u^1(x)), & x \in \Omega \end{cases} \tag{27}$$

with respect to the magnitude of the positive and bounded coefficients α and β, assuming $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$. For any β, α positive and $T > 2\mathrm{dist}(\Omega\backslash\omega)\sqrt{\alpha}$, system (27) is null-controllable. When the ratio β/α is small, the term βu_t may be seen as a damping term for the wave equation, the controllability holds uniformly with respect to β in $L^2((0,T)\times\omega)$ and the control $h_{\omega,\alpha,\beta}$ of minimal L^2-norm converges toward the corresponding control $h_{\omega,\alpha,0}$ of the wave equation as $\beta \to 0$. On the other hand, when the ratio β/α is large, the term αu_{tt} may be seen as a singular hyperbolic term for the heat equation. It is proved in [14] that the HUM-control $h_{\omega,\alpha,\beta}$ is uniformly bounded with respect to α and converges toward the control $h_{\omega,0,\beta}$ of the heat equation in $L^2((0,T)\times\omega)$ as $\alpha \to 0$. Therefore, the sensitivity of the optimal support $\omega_{\alpha,\beta}$ of the control $h_{\omega_{\alpha,\beta}}$ with respect to α and β makes sense. We plan to analyze this case in a near future.

References

[1] S. A. Advonin and S. A. Ivanov, *Families of exponentials. The method of moments in controllability problems for distributed parameter systems*, Cambridge Univ. Press, 1995.

[2] G. Allaire, F. Jouve and A.M. Toader, *Shape and topology optimization of the robust compliance via the level set method*, ESAIM: COCV **14**(1), 43-70 (2008).

[3] E. Bednarczuk, M. Pierre, E. Rouy and J. Sokolowski, *Tangent sets in some functional spaces*, J. Math. Anal. Appl. **42** (2000) 871-886.

[4] C. Carthel, R. Glowinski and J.L. Lions, *On exact and approximate Boundary Controllabilities for the heat equation: A numerical approach*, J. Optim. Theory Appl. **82(3)**, (1994) 429-484.

[5] H. Fattorini and D. L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rat. Mech. Anal. **43** (1971) 272-292.

[6] E. Fernandez-Cara and A. Münch, *Numerical approximation of distributed null controls for the heat equation using a mixed elliptic method*, Preprint 09, Université Blaise Pascal.

[7] A.V. Fursikov and O. Yu Imanuvilov, *Controllability of evolution equations*, Lecture Notes Series # **34**, Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, 1996.

[8] R. Glowinski, J.L. Lions, *Exact and approximate controllability for distributed parameter systems*, Acta Numerica (1996) 159-333.

[9] P. Hebrard and A. Henrot, *Optimal shape and position of the actuators for the stabilization of a string*, Systems Control Lett., **48** (2003) 199-209.

[10] A. Henrot and M. Pierre, *Variation and optimisation des formes: une analyse géométrique*, Mathématiques et Applications **48**, Springer, 2005.

[11] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Comm. P.D.E., **20** (1995) 335-356.

[12] J. L. Lions, Contrôlabilité exacte, perturbation et stabilisation des systèmes distribués, Tome 1, Masson, 1988.

[13] A. López, Control y perturbaciones singulares en algunos sistemas parabólicos, Doctoral thesis, Universidad Complutense de Madrid, 1999.

[14] A. López, X. Chang and E. Zuazua, *Null controllability of the heat equation as singular limit of the exact controllability of dissipative wave equations*, J. Math. Pures Appl. **79(8)** (2000) 741-808.

[15] L. Miller, *Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time*, J. Differential Equations **204** (2004) 202-226.

[16] A. Münch, *Optimal location of the support of the control for the 1-D wave equation: numerical investigations*, Comput. Optim. Appl. **42(3)** (2009) 383-412.

[17] A. Münch, Optimal design of the support of the control for the 2-D wave equation: a numerical method, Int. J. Numer. Anal. Model. **5** (2008) 331-351.

[18] A. Münch, P. Pedregal and F. Periago, *Optimal design of the damping set for the stabilization of the wave equation*, J. Differential Equations **231(1)** (2006) 331-358.

[19] A. Münch, P. Pedregal and F. Periago, *Relaxation of an optimal design problem for the heat equation*, J. Math. Pures Appl. **89(3)** (2008) 225-247.

[20] A. Münch and E. Zuazua, *Numerical approximation of trajectory controls for the heat equation through transmutation*, Preprint 09, Université Blaise Pascal.

[21] F. Periago, *Optimal shape and position of the support for the internal exact control of a string*, Systems Control Lett., **58(2)** (2009) 136-140.