Couches limites, contrôle semi-linéaire, chaussées chauffantes

ARNAUD MÜNCH
Lab. de mathématiques Blaise Pascal - Clermont-Ferrand - France

La Rochelle - June 2022

Symposium AMES-2022 en l'honneur de Philippe Destuynder

Plan

Partie 1: Couches limites (en collaboration avec Youcef Amirat)

Partie 2: Approximation de contrôles exactes pour des EDP semilinéaires

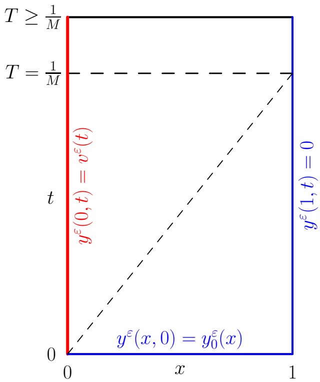
Partie 3: Maintien hors gel des chaussées - une collaboration avec le CEREMA

Part 1: Control and singular PDEs

Amirat, Münch : *Asymptotic analysis of an advection-diffusion equation involving interacting boundary and internal layers*, Mathematical Methods in the Applied Sciences, 2020.

Amirat, Münch : Internal layer intersecting the boundary of a domain in a singular advection-diffusion equation - hal.archives-ouvertes.fr/hal-03657828

Castro, Münch : Singular asymptotic expansion of the exact control for the perturbed wave equation - Asymptotic analysis 2021



Let $T > 0$, $M \in \mathbb{R}^\star$, $\varepsilon > 0$ and $Q_T := (0,1) \times (0,T)$.

$$\boxed{\begin{cases} y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0, & \text{in} \quad Q_T, \\ y^\varepsilon(0,\cdot) = v(t),\ y^\varepsilon(1,\cdot) = 0, & \text{in} \quad (0,T), \\ y^\varepsilon(\cdot,0) = y_0, & \text{in} \quad (0,1). \end{cases}}$$

- Well-posedness:

$$\forall y_0^\varepsilon \in H^{-1}(0,1), v \in L^2(0,T), \quad \exists!\ y^\varepsilon \in L^2(Q_T) \cap C([0,T]; H^{-1}(0,1))$$

- Null controllability property: From [Fursikov'91],

$$\forall T > 0, y_0 \in H^{-1}(0,1), \exists v^\varepsilon \in L^2(0,T) \quad \text{s.t.} \quad y^\varepsilon(\cdot,T) = 0 \quad \text{in } H^{-1}(0,1)$$

- Main issue: Behavior of the controls $v = v(\varepsilon)$ as $\varepsilon \to 0$??

Proposition (Polynomial decay of $\|y^\varepsilon(\cdot,T)\|_{L^2(0,T)}$ for $T = \frac{1}{M}$)

Assume $M > 0$ and $v^\varepsilon \equiv 0$, $y_0^\varepsilon = y_0 \in H^3(0,1)$. For $\varepsilon > 0$ small enough, the free solution y^ε satisfies

$$\left\| y^\varepsilon\left(\cdot, \frac{1}{M}\right)\right\|_{L^2(0,1)} \le c\left(|y_0(0)|\varepsilon^{1/4} + |y_0^{(1)}(0)|\varepsilon^{3/4} + |y_0^{(2)}(0)|\varepsilon^{5/4}\right) + \mathcal{O}(\varepsilon^{3/2}) \quad (1)$$

for some constant $c > 0$, independent of ε.

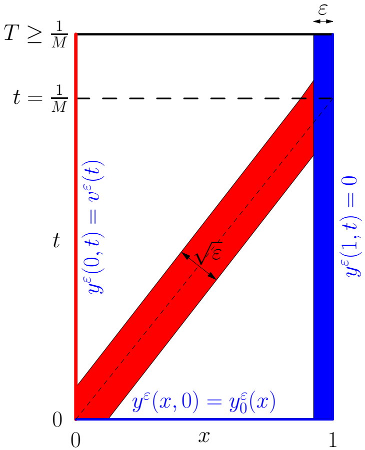
Lemma (Exponential decay of $\|y^\varepsilon(\cdot,T)\|_{L^2(0,T)}$ for $T > \frac{1}{|M|}$)

The free solution (i.e. $v^\varepsilon = 0$) satisfies

$$\|y^\varepsilon(\cdot,t)\|_{L^2(0,1)} \le \|y_0\|_{L^2(0,1)} e^{-\frac{M}{4\varepsilon}}, \quad \forall t > \frac{1}{M}.$$

$\Longrightarrow$ For ε small enough, the cost of approximate controllability is zero (for $T > 1/M$).

$$\begin{cases} y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0, & \text{in} \quad Q_T, \\ y^\varepsilon(0,\cdot) = v(t),\ y^\varepsilon(1,\cdot) = 0, & \text{in} \quad (0,T), \\ y^\varepsilon(\cdot,0) = y_0, & \text{in} \quad (0,1). \end{cases}$$



Occurence of two interacting singular layers of different sizes !

Very few papers dealing with the asymptotic analysis of PDEs involving two interacting singular layers :

• W. Eckhaus, W. and E.M. de Jager, *Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type*, Arch. Rational Mech. Anal., 1966

$$\begin{cases} u^\varepsilon_x(x,y) - \varepsilon\Delta u^\varepsilon(x,y) = 0, & (x,y) \in (0,1)\times(-1,1), \\ u^\varepsilon(0,y) = f(y), \quad u^\varepsilon(x,-1) = u^\varepsilon(x,1) = u^\varepsilon(1,y) = 0, & x \in [0,1], y \in [-1,1], \end{cases} \tag{2}$$

where $f : [-1,1] \to \mathbb{R}$ is a piecewise constant, discontinuous at $y = 0$.

• Larry Bobisud, *Second-order linear parabolic equations with a small parameter*, Arch. Rational Mech. Anal., 1967.

$$\begin{cases} u^\varepsilon_t - \varepsilon u^\varepsilon_{xx} + u^\varepsilon_x + u^\varepsilon = 0, & \text{in} \quad Q_T, \\ u^\varepsilon(0,\cdot) = f, \;\; u^\varepsilon(1,\cdot) = g, & \text{in} \quad (0,T), \\ u^\varepsilon(\cdot,0) = u_0, & \text{in} \quad (0,1). \end{cases} \tag{3}$$

Assuming $u_0 \in C^4([0,1])$ and $f, g \in C^3([0,T])$, obtention of w^ε such that $\|u^\varepsilon - w^\varepsilon\|_{L^\infty(Q_T)} = \mathcal{O}(\sqrt{\varepsilon})$ by the way of a maximum principle.

We take into account the boundary layer on the characteristic and consider three formal asymptotic expansions of y^ε:

– the outer expansion

$$\sum_{k=0}^{m} \varepsilon^k y^k(x,t), \quad (x,t) \in Q_T, \quad x - Mt \neq 0 \implies y_t^k + M y_x^k = y_{xx}^{k-1}$$

– the first inner expansion (on the characteristic $x - Mt = 0$)

$$\sum_{k=0}^{m} \varepsilon^{\frac{k}{2}} W^{k/2}(w,t), \quad w = \frac{x - Mt}{\sqrt{\varepsilon}}, \; t \in (0,T) \implies W_t^{k/2} - W_{ww}^{k/2} = 0$$

– the second inner expansion (at $x = 1$)

$$\sum_{k=0}^{m} \varepsilon^{k/2} Y^{k/2}(z,\tau,t), \quad z = \frac{1-x}{\varepsilon}, \; \tau = \frac{\frac{1}{M} - t}{\sqrt{\varepsilon}} \implies Y_{zz}^{k/2} + M Y_z^{k/2} = Y_t^{(k-2)/2} - Y_\tau^{(k-1)/2}$$

- - y^0 solves the transport eq.:

$$\begin{cases} y^0_t + M y^0_x = 0, & (x,t) \in Q_T \\ y^0(x,0) = y_0, y^0(0,t) = v \end{cases} \Longrightarrow \boxed{y^0(x,t) = \begin{cases} y_0(x - Mt) & x > Mt, \\ v\left(t - \dfrac{x}{M}\right), & x < Mt. \end{cases}}$$

- - W^0 solves the heat eq.:

$$\begin{cases} W^0_t(w,t) - W^0_{ww}(w,t) = 0, & (w,t) \in \mathbb{R} \times (0,T), \\ \lim\limits_{w \to +\infty} W^0(w,t) = y^0((Mt)^+, t) = y_0(0), & t \in (0,T), \\ \lim\limits_{w \to -\infty} W^0(w,t) = y^0((Mt)^-, t) = v(0), & t \in (0,T). \end{cases}$$

$$\Longrightarrow \boxed{W^0(w,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(w-s)^2}{4t}} g_0(s) ds}, \qquad w = \frac{x - Mt}{\sqrt{\varepsilon}}$$

The choice of g_0 also needs to ensure that

$$\lim_{t \to 0} W^0\left(\frac{-a(t)}{\sqrt{\varepsilon}}, t\right) = v(0), \qquad \lim_{x \to 0} W^0\left(\frac{x}{\sqrt{\varepsilon}}, 0\right) = y_0(0)$$

A good choice is

$$g_0^{\varepsilon}(w) = \begin{cases} y_0(0), & w \geq 0, \\ v(0) + (v(0) - y_0(0)) e^{\frac{Mw}{\sqrt{\varepsilon}}}, & w < 0. \end{cases} \tag{4}$$

leading to

$$\boxed{\begin{aligned} W_{\varepsilon}^{0}(w,t) = & \frac{y_0(0) - v(0)}{2} erf\left(\frac{w}{2\sqrt{t}}\right) + \frac{y_0(0) + v(0)}{2} \\ & + \frac{v(0) - y_0(0)}{2} e^{\frac{Mw}{\sqrt{\varepsilon}} + \frac{M^2 t}{\varepsilon}} erfc\left(\frac{w}{2\sqrt{t}} + \frac{M\sqrt{t}}{\sqrt{\varepsilon}}\right). \end{aligned}} \tag{5}$$

- Composite approximation far from $x = 1$

$$p_{\varepsilon}^{0}(x,t) = y^{0}(x,t) + W_{\varepsilon}^{0}(w,t) - y^{0}((Mt)^{\pm}, t)$$

- Y^0 solves the ODE: $z = \frac{1-x}{\varepsilon}$, $\tau = \frac{1/M-t}{\sqrt{\varepsilon}}$

$$\begin{cases} Y^0_{zz}(z,\tau,t) + MY^0_z(z,\tau,t) = 0, & (z,\tau,t) \in \mathbb{R}^+_\star \times \mathbb{R} \times (0,T), \\ Y^0(0,\tau,t) = 0, \quad \lim_{z\to+\infty} Y^0(z,\tau,t) = p^0_\varepsilon(1,t), & (\tau,t) \in \mathbb{R} \times (0,T). \end{cases}$$

$$\boxed{Y^0(z,\tau,t) = p^0_\varepsilon(1,t)\left(1 - e^{-Mz}\right)}, \quad (z,\tau,t) \in \mathbb{R}^+ \times \mathbb{R} \times (0,T).$$

- Composite approximation in $\overline{Q_T}$

$$\boxed{P^0_\varepsilon(x,t) = p^0_\varepsilon(x,t) + Y^0(z,\tau,t) - p^0_\varepsilon(1,t) = p^0_\varepsilon(x,t) - p^0_\varepsilon(1,t)e^{-Mz}}$$

First order approximation (1)

Let

$$P^{\varepsilon} = P^0_{\varepsilon} + \sqrt{\varepsilon}P^{1/2}_{\varepsilon} + \varepsilon P^1_{\varepsilon} + \varepsilon^{3/2}P^{3/2}_{\varepsilon}$$

Theorem (Amirat, M, 20)

Assume $v \in H^3([0,T])$, $y_0 \in H^3([0,1])$. Then $\exists C > 0$ independent of ε s.t.

$$\Big\| y^{\varepsilon}(\cdot,t) - P^{\varepsilon}(\cdot,t) \Big\|_{L^2(0,1))} \le C\big(\varepsilon^{3/2} + \varepsilon^{1/2}e^{-\frac{M^2}{2\varepsilon^{1/2}}t}\big) \quad \forall t \in [0,T]$$

and (assuming $y_0(1) = y_0'(1) = 0$)

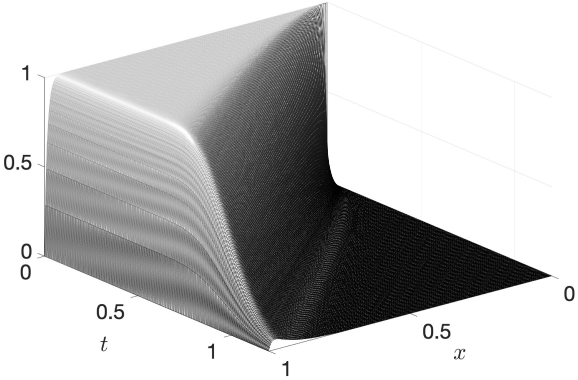
$$\|(y^{\varepsilon} - P^{\varepsilon})_x\|_{L^2(Q_T))} \le C\varepsilon$$

We consider the simple case $v \equiv 0$ and $y_0 \equiv 1$ for which

$$\begin{cases} P^\varepsilon(x,t) = W^0_\varepsilon(w,t) - \Big(W^0_\varepsilon(M\tau,t) + \varepsilon^{1/2} z W^0_{\varepsilon,w}(M\tau,t) + \\ \qquad \varepsilon \dfrac{z^2}{2} W^0_{\varepsilon,ww}(M\tau,t) + \varepsilon^{3/2} \dfrac{z^3}{6} W^0_{\varepsilon,www}(M\tau,t) \Big) e^{-Mz}, \\ w = \dfrac{x-Mt}{\sqrt{\varepsilon}}, \quad M\tau = \dfrac{1-Mt}{\sqrt{\varepsilon}}, \quad z = \dfrac{1-x}{\varepsilon}. \end{cases}$$

with

$$\boxed{\begin{aligned} W^0_\varepsilon(w,t) =& \frac{y_0(0)-v(0)}{2} erf\left(\frac{w}{2\sqrt{t}}\right) + \frac{y_0(0)+v(0)}{2} \\ &+ \frac{v(0)-y_0(0)}{2} e^{\frac{Mw}{\sqrt{\varepsilon}} + \frac{M^2 t}{\varepsilon}} erfc\left(\frac{w}{2\sqrt{t}} + \frac{M\sqrt{t}}{\sqrt{\varepsilon}}\right) \end{aligned}} \tag{6}$$



P^{ε} in $(0,1)\times(0,1.2/M)$; $M=1$, $\varepsilon=10^{-2}$; $v\equiv 0$, $y_0\equiv 1$.

Part 2: Approximation of exact controls for nonlinear PDEs

in collaboration with Jérôme Lemoine (CF), Arthur Bottois (CF) , Trélat (Sorbonne Paris), Ervedoza (Bordeaux), Bhandari (India), Marin-Gayte (Sevilla), Pedregal (Ciudad real), ...

GIVEN some semilinear uniformly exactly controllable PDEs

$$\boxed{\begin{cases} PDE(y,v)=0, \\ y=y(x,t) - \text{state}, \quad v=v(x,t) - \text{control function}, \\ +\text{ initial conditions and boundary conditions} \end{cases}} \tag{7}$$

FIND a sequence $(y_k, v_k)_{k\in\mathbb{N}}$ such that $(y_k, v_k) \to (y, v)$ as $k \to \infty$, with (y, v) a controlled pair for (7) ?

• $\Omega := (0,1)$, $\omega := (\ell_1, \ell_2)$, $0 \le \ell_1 < \ell_2 \le 1$, $T > 0$. $Q_T := \Omega \times (0,T)$, $q_T := \omega \times (0,T)$ and $\Sigma_T := \partial\Omega \times (0,T)$.

$$\boxed{\begin{cases} \partial_{tt} y - \partial_{xx} y + f(y) = v 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases}} \tag{8}$$

• $(u_0, u_1) \in \boldsymbol{V} := H_0^1(\Omega) \times L^2(\Omega)$, $v \in L^2(q_T)$. $f \in C^1(\mathbb{R};\mathbb{R})$;
• $|f(r)| \le C(1+|r|)\ln^2(2+|r|)\ \forall r \in \mathbb{R}$

Theorem

Assume $T > 2\max(\ell_1, 1-\ell_2)$. There exists $\beta > 0$ (only depending on Ω and T) such that, if

$$(\mathbf{H_1}) \quad \limsup_{|r|\to\infty} \frac{|f(r)|}{|r|\ln^2|r|} < \beta$$

then (8) is exactly controllable in time T.

[1]

[1] E. Zuazua, Exact controllability for semilinear wave equations in one space dimension, Ann. Inst. H. Poincaré Anal. Non Linéaire 1993

The proof given in Zuazua'93 is based on a Leray Schauder fixed point argument: Let $\Lambda : L^\infty(Q_T) \to L^\infty(Q_T)$, where $y := \Lambda(\xi)$ is a controlled solution with control function v_ξ of

$$\begin{cases} \partial_{tt} y - \partial_{xx} y + y \dfrac{f(\xi)}{\xi} = v_\xi 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \tag{9}$$

satisfying $(y(\cdot,T), \partial_t y(\cdot,T)) = (0,0)$.

Nice but useless in practice since Λ is not contracting : The Picard iterates $(y_k)_{k\in\mathbb{N}}$

$$\boxed{\begin{cases} y_0 \in L^\infty(Q_T) & \text{given} \\ y_{k+1} = \Lambda(y_k), \ k \geq 0 \end{cases}} \tag{10}$$

may not converge !

We consider the Hilbert space

$$\mathcal{H} := \Big\{(y,v) \in L^2(Q_T) \times L^2(q_T) \;|\; \partial_{tt}y - \partial_{xx}y \in L^2(Q_T),\; y = 0 \text{ on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) \in \boldsymbol{V}\Big\}$$

and the subspace of $\mathcal{H}$ defined by

$$\mathcal{A} := \Big\{(y,v) \in \mathcal{H} \;|\; (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1),\; (y(\cdot,T), \partial_t y(\cdot,T)) = (0,0) \text{ in } \Omega\Big\},$$

We define the least-squares functional $E : \mathcal{A} \to \mathbb{R}$ by

$$\boxed{E(y,v) := \frac{1}{2}\|\partial_{tt}y - \partial_{xx}y + f(y) - v1_\omega\|^2_{L^2(Q_T)}}$$

and consider the nonconvex minimization problem

$$\boxed{\inf_{(y,v)\in\mathcal{A}} E(y,v)} \tag{11}$$

Proposition

$\forall (y, v) \in \mathcal{A}$,

$$\sqrt{E(y,v)} \leq C e^{C\sqrt{\|f'(y)\|_\infty}} \|E'(y,v)\|_{\mathcal{A}_0'}. \tag{12}$$

Consequence:
Any *critical* point $(y, v) \in \mathcal{A}$ of E (i.e., $E'(y, v) = 0$) is a zero of E, and thus is a pair solution of the controllability problem. Moreover:

given any sequence $(y_k, v_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that $\|E'(y_k, v_k)\|_{\mathcal{A}_0'} \underset{k\to+\infty}{\longrightarrow} 0$ and such that $\|f'(y_k)\|_\infty$ is uniformly bounded, we have $E(y_k, v_k) \underset{k\to+\infty}{\longrightarrow} 0$.

A minimizing sequence for E cannot be stuck in a local minimum, even though E fails to be convex (it has multiple zeros).

Assume that $T > 2\max(\ell_1, 1-\ell_2)$. We define the minimizing sequence $(y_k, v_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ by

$$\boxed{\begin{cases} (y_0, v_0) \in \mathcal{A} \\ (y_{k+1}, v_{k+1}) = (y_k, v_k) - \lambda_k (Y_k^1, V_k^1) \qquad \forall k \in \mathbb{N} \\ \lambda_k = \operatorname*{argmin}_{\lambda\in[0,1]} E\big((y_k, v_k) - \lambda(Y_k^1, V_k^1)\big) \end{cases}} \tag{13}$$

where $(Y_k^1, V_k^1) \in \mathcal{A}_0$ is the solution of minimal control norm of

$$\boxed{\begin{cases} \partial_{tt} Y_k^1 - \partial_{xx} Y_k^1 + f'(y_k) Y_k^1 = V_k^1 1_\omega + (\partial_{tt} y_k - \partial_{xx} y_k + f(y_k) - v_k 1_\omega) & \text{in } Q_T, \\ Y_k^1 = 0 & \text{on } \Sigma_T, \\ (Y_k^1(\cdot,0), \partial_t Y_k^1(\cdot,0)) = (0,0) & \text{in } \Omega, \\ (Y_k^1(\cdot,T), \partial_t Y_k^1(\cdot,T)) = (0,0) & \text{in } \Omega, \end{cases}} \tag{14}$$

Strong convergence of the sequences

Given any $p \in (0, 1]$, we set

$$\beta^0(p) := \frac{p^2}{C^2(2p+1)^2} \tag{15}$$

Theorem (Münch, Trélat 2022)

Assume that $T > 2\max(\ell_1, 1-\ell_2)$, that $[f']_p < +\infty$ for some $p \in (0, 1]$, and that there exist $\alpha \geq 0$ and $\beta \in [0, \beta^0(p))$, such that

$$\boxed{|f'(r)| \leq \alpha + \beta \ln^2(1+|r|) \qquad \forall r \in \mathbb{R}.}$$

Then, as $k \to \infty$

- *For any $(y_0, v_0) \in \mathcal{A}$, $(y_k, v_k) \to (y, v)$ a controlled pair for the nonlinear wave eq.*
- $\lambda_k \to 1$.

Moreover, the convergence of these sequences is at least linear, and is at least of order $1+p$ after a finite number of iterations.

[2]

[2] Münch, Trélat: Constructive exact control of semiliniear 1D wave equations, SICON, 2022.

Part 2 - Link with the damped Newton method

Defining $F : \mathcal{A} \to L^2(Q_T)$ by $F(y, v) := (\partial_{tt} y - \partial_{xx} y + f(y) - v 1_\omega)$,

we get

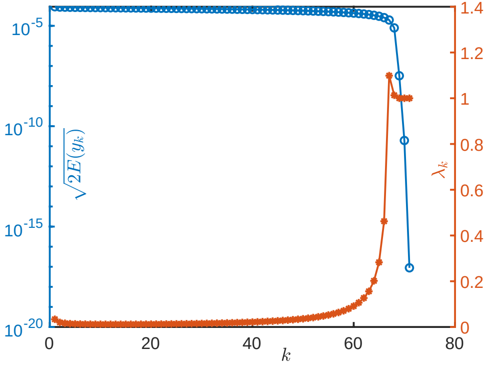
$$E(y, v) = \frac{1}{2} \|F(y, v)\|^2_{L^2(Q_T)}$$

For $\lambda_k = 1$, the least-squares algorithm coincides with the Newton algorithm applied to F (explaining the super-linear convergence property).

Optimizing the parameter $\lambda_k \in [0, 1]$ gives a global convergence property of the algorithm and leads to the so-called damped Newton method applied to F.

Typical behavior of $E(y_k, v_k)$ and λ_k w.r.t. k

The decreases of $E(y_k, f_k)$ is initially slow (order one) far from the solution. λ_k is closed to 0. Then, the decay becomes super-linear and λ_k is closer and closer to one.

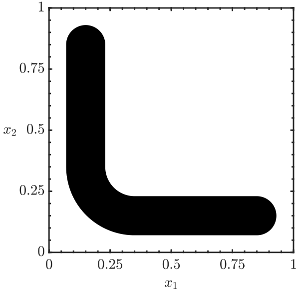


$\Omega = (0,1)^2$ and $T = 3$ and

$$f(r) = -c_f\, r \ln^{1/2}(2 + |r|), \quad \forall r \in \mathbb{R}.$$

The unfavorable situation for which the norm of the uncontrolled corresponding solution grows corresponds to $c_f > 0$. We consider

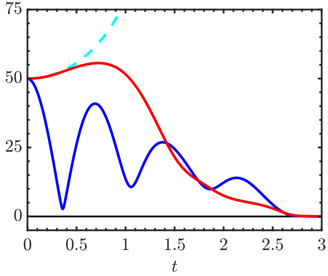
$$(u_0, u_1) = (100 \sin(\pi x_1) \sin(\pi x_2), 0), \qquad (z_0, z_1) = (0, 0)$$



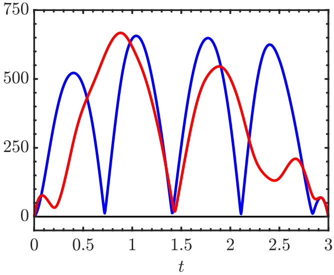
Control domain $\omega \subset \Omega = (0,1)^2$ (black part).

Numerical experiments in the 2d case: $c_f = 10$

$\sharp$iterate k	$\sqrt{2E(y_k, v_k)}$	$\frac{\Vert y_k - y_{k-1}\Vert_{L^2(Q_T)}}{\Vert y_{k-1}\Vert_{L^2(Q_T)}}$	$\frac{\Vert v_k - v_{k-1}\Vert_{L^2_\chi(q_T)}}{\Vert v_{k-1}\Vert_{L^2_\chi(q_T)}}$	$\Vert y_k\Vert_{L^2(Q_T)}$	$\Vert v_k\Vert_{L^2_\chi(q_T)}$	λ_k
0	7.44×10^2	–	–	38.116	732.22	1
1	1.63×10^2	1.79×10^0	9.30×10^{-1}	58.691	667.602	1
2	1.62×10^0	8.42×10^{-2}	1.41×10^{-1}	60.781	642.643	1
3	1.97×10^{-3}	1.21×10^{-3}	4.66×10^{-3}	60.745	643.784	1
4	5.11×10^{-10}	6.43×10^{-7}	2.63×10^{-6}	60.745	643.785	–



(▬) $\Vert y_4(\cdot,t)\Vert_{L^2(\Omega)}$; (▬) $\Vert y_0(\cdot,t)\Vert_{L^2(\Omega)}$; (– –) $\Vert y(\cdot,t;0)\Vert_{L^2(\Omega)}$.



(▬) $\Vert v_4(\cdot,t)\Vert_{L^2_\chi(\omega)}$; (▬) $\Vert v_0(\cdot,t)\Vert_{L^2_\chi(\omega)}$.

Zuazua fixed point operator

$$\begin{cases} \partial_{tt} y_{k+1} - \Delta y_{k+1} + y_{k+1} \dfrac{f(y_k)}{y_k} = v_{k+1} 1_\omega, & \text{in} \quad Q_T, \\ y_{k+1} = 0, & \text{on } \Sigma_T, \\ (y_{k+1}(\cdot,0), \partial_t y_{k+1}(\cdot,0)) = (y_0, y_1), & \text{in } \Omega. \end{cases} \tag{16}$$

$\sharp$iterate k	$\sqrt{2E(y_k, v_k)}$	$\frac{\Vert y_{k+1} - y_k \Vert_{L^2(Q_T)}}{\Vert y_k \Vert_{L^2(Q_T)}}$	$\frac{\Vert v_{k+1} - v_k \Vert_{L^2_\chi(q_T)}}{\Vert v_k \Vert_{L^2_\chi(q_T)}}$	$\Vert y_k \Vert_{L^2(Q_T)}$	$\Vert v_k \Vert_{L^2_\chi(q_T)}$
0	3.72×10^2	1.02×10^0	1.33×10^0	38.116	732.22
1	4.79×10^1	5.85×10^{-2}	1.73×10^{-1}	37.945	562.213
2	2.65×10^0	3.35×10^{-3}	1.55×10^{-2}	36.798	530.787
3	1.54×10^{-1}	3.05×10^{-4}	9.84×10^{-4}	36.812	526.864
4	1.39×10^{-2}	4.70×10^{-5}	8.77×10^{-5}	36.807	527.209
5	2.13×10^{-3}	9.24×10^{-6}	1.81×10^{-5}	36.806	527.221
6	4.20×10^{-4}	1.88×10^{-6}	3.93×10^{-6}	36.806	527.225
7	8.55×10^{-5}	4.07×10^{-7}	8.81×10^{-7}	36.806	527.226
8	1.85×10^{-5}	8.97×10^{-8}	1.99×10^{-7}	36.806	527.226
9	4.08×10^{-6}	–	–	36.806	527.226

Lack of convergence for $|c_f| > 15$

$$\boxed{\begin{cases} y_{tt} - y_{xx} + f(y) = 0 & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(1,\cdot) = v & \text{in } (0,T), \\ (y(\cdot,0), y_t(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases}} \tag{17}$$

• We introduce the operator $\Lambda_s : L^\infty(Q_T) \mapsto L^\infty(Q_T), \Lambda_s(\widehat{y}) = y$ where y solves

$$\begin{cases} y_{tt} - y_{xx} = -f(\widehat{y}) & \text{in } Q_T, \\ y(0,\cdot) = 0, \; y(1,\cdot) = v & \text{in } (0,T), \\ (y(\cdot,0), y_t(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \\ (y(\cdot,T), y_t(\cdot,T)) = (0,0) & \text{in } \Omega, \end{cases} \tag{18}$$

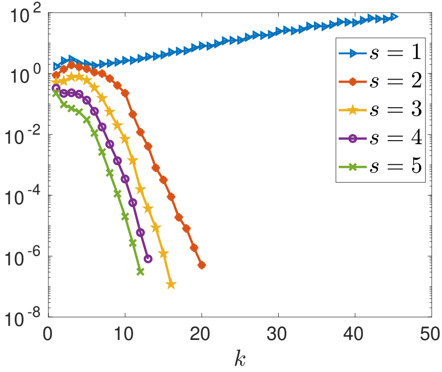
and (y, v) corresponds to the minimizer of a functional J_s

$$J_s(y,v) := s \int_{Q_T} \rho^2(s) y^2 + \int_0^T \eta^{-2} \rho_1^2(s) v^2 \tag{19}$$

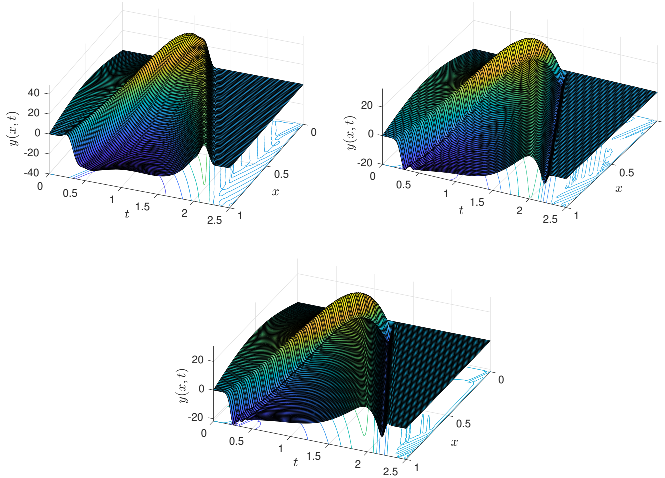
ρ and ρ_1 are some parametrized Carleman weights :

$$\begin{cases} \psi(x,t) = |x - x_0|^2 - \beta\left(t - \dfrac{T}{2}\right)^2 + M_0 \quad \text{in } Q_T, \qquad \beta \in (0,1), \quad x_0 < 0, \\ \lambda > 0, \phi(x,t) = e^{\lambda\psi(x,t)}, \rho(s;x,t) := e^{-s\phi(x,t)}, \quad \rho_1(s;t) = \rho(s;1,t), \quad \forall (x,t) \in Q_T. \end{cases} \tag{20}$$

$$\Omega = (0,1), \quad T = 2.5, \quad , \quad (u_0(x), u_1(x)) = (c_{u_0}\sin(\pi x), 0),$$
$$f(r) = c_f r(1 + \ln^{3/2}(2+|r|)), \quad \forall r \in \mathbb{R}$$



Relative error $\frac{\|\rho(s)y_{k+1} - \rho(s)y_k\|_{L^2(Q_T)}}{\|\rho(s)y_k\|_{L^2(Q_T)}}$ w.r.t. iterations k for $(c_f, c_{u_0}) = (5, 20)$.



Controlled solution y_{k^*} for $c_f = -3$, $c_{u_0} = 10$ and $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; $s \in \{1, 5, 9\}$.

Let $\omega \subset \Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$.

$$\begin{cases} \partial_t y - \Delta y + f(y) = v 1_\omega \quad \text{in} \quad Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = u_0 \text{ in } \Omega, \end{cases} \tag{21}$$

where $u_0 \in L^2(\Omega)$ is the initial state of y and $f \in L^2(q_T)$ is a *control* function such that $y(T, \cdot) = 0$.

Theorem

Let $T > 0$ be given. Assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is locally Lipschitz continuous, satisfies $f(0) = 0$ and

$$(\mathbf{H_0}) \quad |f'(r)| \leq C(1 + |r|^{4+d}) \text{ a.e. in } \mathbb{R}.$$

There exists a $\beta^\star > 0$ such that if

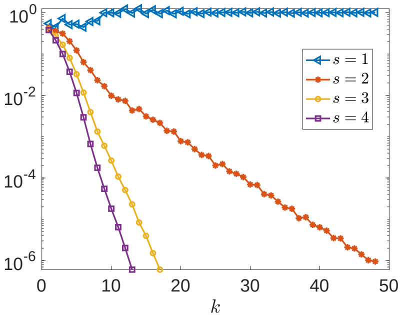
$$(\mathbf{H_1}) \quad \limsup_{|r| \to \infty} \frac{|f(r)|}{|r| \ln_+^{3/2} |r|} \leq \beta^\star$$

then system (21) is globally exactly controllable to 0 at time T with controls in $L^\infty(q_T)$.

[3]

[3] E. Fernández-Cara and E. Zuazua, *Null and approximate controllability for weakly blowing up semilinear,* Ann. Inst. H. Poincaré Anal. Non Linéaire 2000

$$\Omega = (0,1), \quad T = 0.5, \quad ,\omega = (0.2, 0.8) \quad u_0(x) = c_{u_0}\sin(\pi x),$$
$$f(r) = c_f r(1 + \ln^{3/2}(2 + |r|)), \quad \forall r \in \mathbb{R}$$



Relative error $\dfrac{\|\rho_0(s)(y_{k+1} - y_k)\|_{L^2(Q_T)}}{\|\rho_0(s)y_k\|_{L^2(Q_T)}}$ w.r.t. k for $s \in \{1, 2, 3, 4\}$.

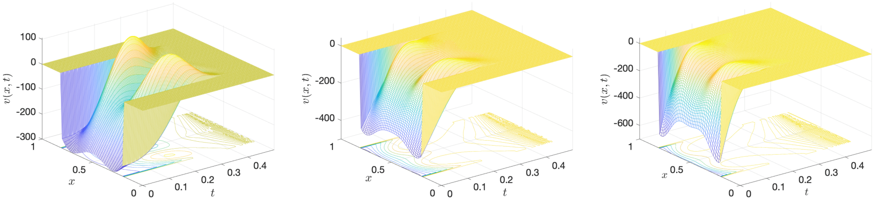


Figure: The control $v_{k^\star}$ in Q_T for $c_{u_0} = 10$, $c_f = -5$ and $s \in \{1, 2, 3\}$.

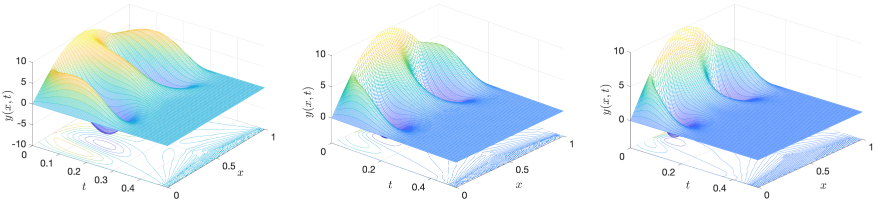


Figure: The controlled solution $y_{k^\star}$ in Q_T for $c_{u_0} = 10$, $c_f = -5$ and $s \in \{1, 2, 3\}$.

Part 3: Maintien hors gel des chaussées

collaboration avec le CEREMA (Centre d'études et d'expertises sur les risques, l'environnement, la mobilité et l'aménagement)

dans le cadre du projet européen "Routes de 5ième génération" :

- Réduction du bruit
- Récupération d'énergie / Panneau solaire
- Utilisation de matériaux recyclable
- Incrustation luminescente interactive
- etc .....

Bernardin, Münch : *Modeling and optimizing a road de-icing device by a nonlinear heating*, M2AN, 2019.

Projet : Chaussées chauffantes et récupératrices d'énergie par circulation d'un fluide caloporteur au sein d'une couche poreuse de la chaussée

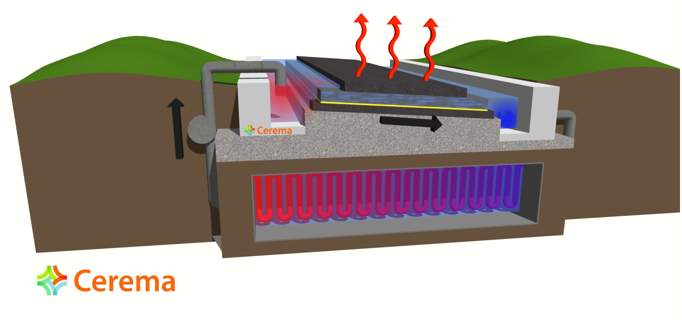


Figure: Schéma du démonstrateur (cas chauffant)

Figure: Le démonstrateur d'Egletons

Figure: Voie de circulation à Egletons

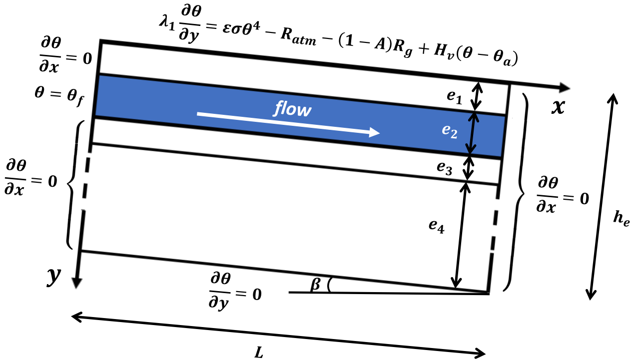


Figure: Schéma transversal de la structure avec condition aux limites: θ_f est la température d'injection du fluide

The road is assumed to have no longitudinal slant and to be infinite in its third dimension. h and L denote the height of the road structure and its length, respectively. The hydraulic regime is assumed stationary with hydraulic parameters independent of temperature T. Denoting by $1 \leq i \leq 4$ the indices of the road layers, the thermo-hydraulic model is as follows. For $0 \leq x \leq L$ and $0 \leq y \leq h$:

$$\boxed{\begin{cases} C_i \dfrac{\partial \theta}{\partial t}(x,y,t) - \lambda_i \Delta\theta(x,y,t) = 0, \quad i \in \{1,3,4\}, \\ C_2 \dfrac{\partial \theta}{\partial t}(x,y,t) + C_f v \dfrac{\partial \theta}{\partial x}(x,y,t) - (\lambda_2 + \phi_2 \lambda_f)\Delta\theta(x,y,t) = 0, \\ v = -K \dfrac{H_2 - H_1}{L}, \end{cases}} \tag{22}$$

where

$(\rho C)_i, \lambda_i, \phi_i$	specific heat, thermal conductivity and porosity of layer i
$(\rho C)_f, \lambda_f$	specific heat, thermal conductivity of the fluid
v	Darcy fluid velocity along x
K	hydraulic conductivity of the porous asphalt
H_1, H_2	hydraulic heads imposed upstream and downstream of fluid circulating in porous draining asphalt layer

Road surface boundary condition expresses the energy balance between road and atmosphere [4]:

$$\boxed{\lambda_1 \frac{\partial \theta}{\partial y}(x,0,t) = \sigma\varepsilon(t)\theta^4(x,0,t) + H_v(t)(\theta(x,0,t) - \theta_a(t)) - R_{atm}(t) - (1 - A(t))R_g(t) + L_f I(t)} \quad (23)$$

ε, A : emissivity and albedo of the road surface,

σ : Stefan-Boltzmann constant (5.67×10^{-8} W/m^{2}K^4),

R_{atm}, R_g : atmospheric and global radiation (W/m^2),

θ_a : air temperature (K),

H_v : convection heat transfer coefficient (W/m^{2}K),

I : snow rate (mm.s^{-1}),

L_f : latent heat of fusion of the ice per kg (J.kg^{-1}).

The convection coefficient is defined by $H_v = Cp_a \times \rho_a(V_{wind} C_d + C_{d_1})$ where the following notations are used :

Cp_a : thermal capacity (J/kg.K) of the air,

V_{wind} : wind velocity (m/s),

ρ_a : density of the air (kg/m^3),

C_d, C_{d_1} : two convection coefficients (-).

[4] Asfour, Bernardin, 2015 : Experimental validation of 2d hydrothermal modelling of porous pavement

$$\begin{cases} \Omega = (0, L) \times (0, h_e), \qquad \Sigma_b = (0, L) \times \{0\}, \\ \Sigma_c = \{0\} \times (e_1, e_1 + e_2), \qquad \Sigma_d = \{L\} \times (e_1, e_1 + e_2), \end{cases} \tag{24}$$

The temperature $\theta = \theta(x, y, t)$ of the four layers road occupying the convex domain Ω satisfies

$$\begin{cases} c(x,y)\theta_t - \mathrm{div}(k(x,y)\nabla\theta) + K\dfrac{H_1(t)}{L}1_{(e_1,e_1+e_2)}(y)\theta_x = 0, & \Omega \times (0,T), \\ k\nabla\theta \cdot \nu = 0, & (\partial\Omega \setminus (\Sigma_b \cup \Sigma_c)) \times (0,T), \\ \theta = \theta_f (= q), & \Sigma_c \times (0,T), \\ k(x,y)\theta_y = \sigma\varepsilon(t)\theta^4 - f_1(t) + f_2(t)\theta, & \Sigma_b \times (0,T), \\ \theta = \theta_0, & \Omega \times \{0\}. \end{cases} \tag{25}$$

θ_f (denoted by q in the sequel) in the temperature of the injected fluid inside the road through the part Σ_c. θ_0 is the initial temperature

We define the heating energy E_h as the energy loss by the coolant between its entrance to the road, at $x = 0$ and its exit at $x = L$, that is:

$$E_h = v\, e_2\, C_f \int_0^T \int_{e_1}^{e_1+e_2} (q(t) - \theta(L, y, t))^+ \, dydt$$

where $v = KH_1/L$

The optimal control problem is then the following:

$$\boxed{\begin{cases} \displaystyle\inf_{q \in H_0^1(0,T)} J(q) := E_h(q) + \frac{\alpha}{2} \|q_t\|^2_{L^2(0,T)} \\ \text{subjected to} \\ \qquad q(0) = q_0 \geq 0, \quad q \geq 0 \text{ in } t \in [0, T], \\ \qquad \theta \geq \underline{\theta} \ \text{ on } \ \Sigma_b \times (0, T), \quad \theta = \theta(q) \text{ solves } (25). \end{cases}} \tag{26}$$

The initial value q_0 of the control q is related to the initial temperature θ_0.

Rk. Under the conditions $q_0 \geq \underline{\theta} \geq 0$ a.e. in Ω and $f_1(t) - f_2(t)\underline{\theta} - \sigma\varepsilon(t)\underline{\theta}^4(t) \geq 0$ for all $t \in (0, T)$ the pb. is well-posed.

$$-k(0)\frac{\partial \theta}{\partial y}(0,t) = f_1(t) - f_2(t)\theta(0,t) - \sigma\varepsilon(t)\theta^4(0,t), \quad t \in (0,T),$$

$$f_1(t) = (1 - A(t))R_g(t) + R_{atm}(t) + H_v(t)\theta_a(t) - \frac{L_f}{3600} I(t),$$

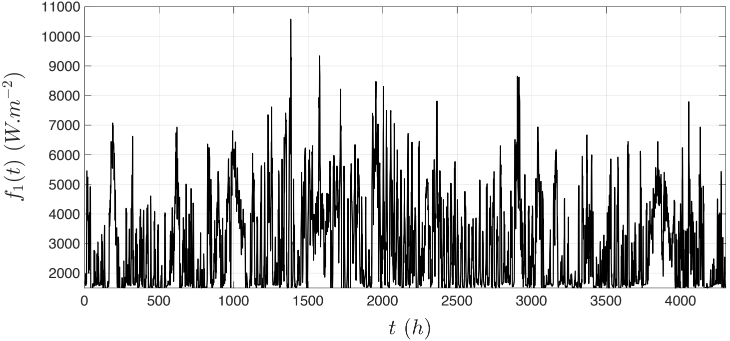


Figure: The function f_1 from data of the french highway A75 in Cantal (1100 m altitude) - October 2009- March 2010

$$J_{\alpha,\epsilon}(q) := \frac{1}{2}\|q\|^2_{L^1(0,T)} + \frac{\alpha}{2}\left(T\|q\|^2_{L^2(0,T)} + \frac{T^3}{4\pi^2}\|q_t\|^2_{L^2(0,T)}\right) + \frac{\epsilon^{-1}}{2}\left\|(\theta_q(0,\cdot) - \underline{\theta})^-\right\|^2_{L^2(0,T)}.$$

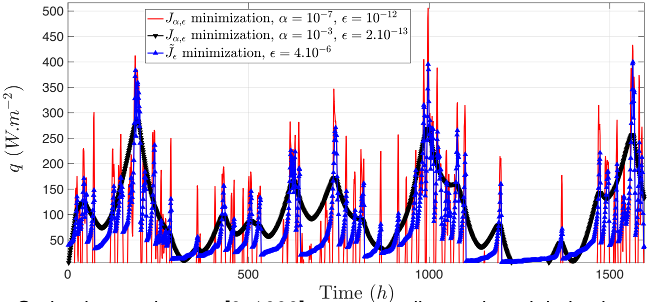


Figure: Optimal controls q on $[0, 1600]$ corresponding to the minimization of $J_{\alpha,\epsilon}$ and $\tilde{J}_{\epsilon}$.

$$J_{\alpha,\epsilon}(q) := \frac{1}{2}\|q\|^2_{L^1(0,T)} + \frac{\alpha}{2}\left(T\|q\|^2_{L^2(0,T)} + \frac{T^3}{4\pi^2}\|q_t\|^2_{L^2(0,T)}\right) + \frac{\epsilon^{-1}}{2}\left\|(\theta_q(0,\cdot) - \underline{\theta})^-\right\|^2_{L^2(0,T)}.$$

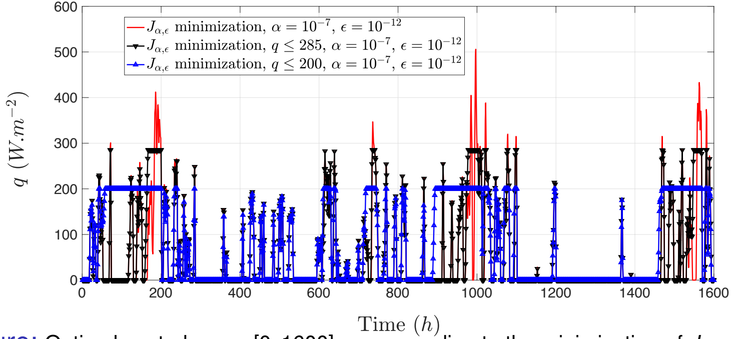


Figure: Optimal controls q on $[0, 1600]$ corresponding to the minimization of $J_{\alpha,\epsilon}$ under the additional constraint $\|q\|_\infty \leq \lambda$ for $\lambda = 200$ and 285.

$$J_{\alpha,\epsilon}(q) := \frac{1}{2}\|q\|^2_{L^1(0,T)} + \frac{\alpha}{2}\left(T\|q\|^2_{L^2(0,T)} + \frac{T^3}{4\pi^2}\|q_t\|^2_{L^2(0,T)}\right) + \frac{\epsilon^{-1}}{2}\left\|(\theta_q(0,\cdot) - \underline{\theta})^-\right\|^2_{L^2(0,T)}.$$

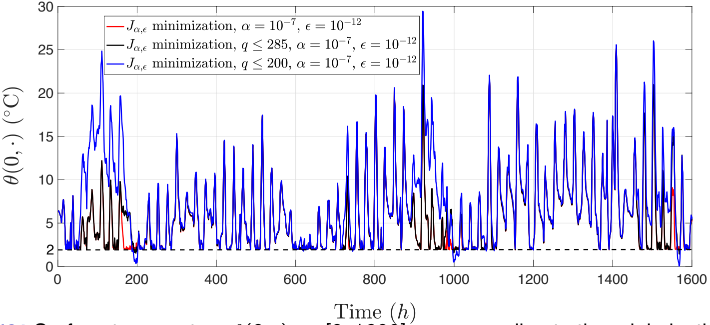


Figure: Surface temperature $\theta(0,\cdot)$ on $[0, 1600]$ corresponding to the minimization of $J_{\alpha,\epsilon}$ for different bounds of $\|q\|_\infty$.

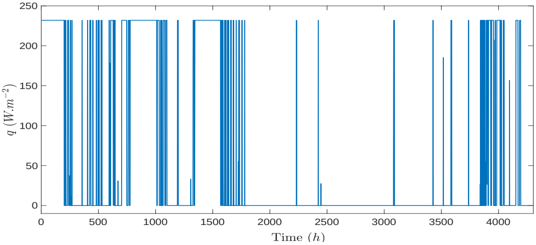


Figure: Optimal bang-bang control q on $[0, T]$ corresponding to $L = 1/4$. $q(t) = \lambda s(t)$ with $\lambda \approx 2.34 \times 10^2$.

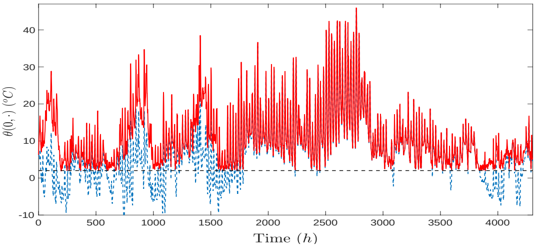


Figure: Temperature $\theta(0, \cdot)$ at the road surface on $[0, T]$ in the controlled (red full line)

• The total energy needed to keep the road surface temperature over 2°C during a winter with snow is about 5.10^8 J $\simeq$ 139 kWh per m^2 of road, with minimal and maximal values per m^2 respectively equal to 124 kWh and 213 kWh.

• The L^∞-norm of the optimal power q ranges in 240-500 W/m^2.

• Some experiments for de-icing obtained by the circulation of a coolant in pipes inserted in the road. equals $100 - 170$ kWh/m^2.

According to the mathematical analysis, if the source q acts on the top of the road ($y_0 = 0$) satisfies the condition $q + f_1(t) - f_2(t)\underline{\theta} - \sigma\varepsilon(t)\underline{\theta}^4 \geq 0$, then the corresponding variable θ_q satisfies $\theta_q(0,t) - \underline{\theta} \geq 0$ for all $t \in (0,T)$. This suggests to consider,the following explicit source

$$\boxed{q(t) = \max\Big(0, -(f_1(t) - f_2(t)\underline{\theta} - \sigma\varepsilon(t)\underline{\theta}^4) + \delta\Big)}$$

for some real $\delta \geq 0$ large enough, dependent of y_0. Table 1 gives the L^1-norm of q and the corresponding value of $\min((\theta(0,\cdot) - \underline{\theta})^-)$ for some values of δ. The value $\delta = 55$ is large enough to satisfy the condition $\theta(0,\cdot) \geq 2^oC$ at the road surface. The corresponding L^1-norm $\|q\|_{L^1(0,T)} \approx 7.52 \times 10^8$ is of the same order as in the previous section.

δ	0	50	54	55
$\|q\|_{L^1(0,T)}$	4.01×10^8	7.2×10^8	7.52×10^8	7.60×10^8
$\|q\|_{L^\infty(0,T)}$	2.72×10^2	3.22×10^2	3.26×10^2	3.27×10^2
$\|(\theta(0,\cdot) - \underline{\theta})^-\|_{L^2(0,T)}$	5.49×10^2	1.29×10^1	1.71	0.
$\|(\theta(0,\cdot) - \underline{\theta})^-\|_{L^\infty(0,T)}$	1.91	1.74×10^{-1}	2.92×10^{-2}	0.

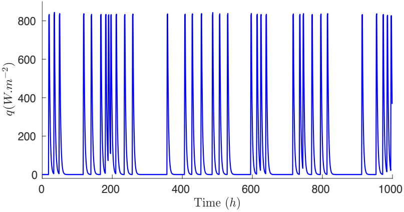
Table: Characteristics of the temperature θ.

LOI DE COMMANDE EXPLICITE (2)

The source from the previous law is active on some period where the value of $\theta(0,\cdot)$ is (significantly) above $\underline{\theta}$. This is due to the large variations of the functions f_1 and f_2. A third way is therefore to consider source term q which depends explicitly on the variable θ_q, for instance as follows:

$$\boxed{q(t) = \begin{cases} 0 & \text{if} \quad \theta(0,t-\delta) \geq \theta_m, \\ 0 & \text{if} \quad \underline{\theta} \leq \theta(0,t-\delta) \leq \theta_m \quad \text{and} \quad \theta'(0,t-\delta) > 0, \\ f(t,\theta)\Big(\theta(0,t-\delta) - \theta_m\Big)^{-} & \text{else} \end{cases}}$$

for some reals $\theta_m > \underline{\theta}$, $\delta \in (0,T)$ and a negative function f which depends only at time t on the temperature $\theta(s)$, $s \in (0,t)$. Figure below depicts the source associated with $\theta_m = 273.15 + 3$, $\delta = 1$ hour and to the corresponding temperature $\theta(0,\cdot)$.



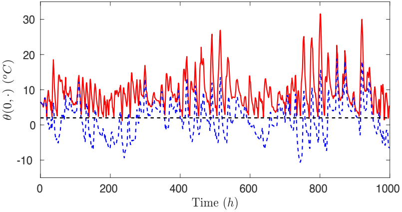


Figure: Source q for $t \in [0, 1000]$ and corresponding temperature $\theta(0,\cdot)$.

Merci pour votre attention !!