ASYMPTOTIC CONSIDERATIONS SHEDDING LIGHT ON INCOMPRESSIBLE SHELL MODELS

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The incompressible singularity found in 3D elasticity when Poisson's ratio approaches 1/2 is not present in classical shell models, nor in the limit models obtained from 3D elasticity when performing an asymptotic analysis with respect to the thickness parameter. However, some specific shell models – such as the 3D-shell model – do retain the incompressible singularity. These observations raise the issue of how adequately shell models can represent incompressible conditions, which this paper aims at investigating. We first perform a combined asymptotic analysis of 3D elasticity with respect to both the thickness parameter and Poisson's ratio and we obtain a commuting property, which is very valuable as a justification of the concept of an "incompressible shell", and substantiates the use of classical shell models with incompressible materials. We then show that the 3D-shell model does not enjoy a similar commuting property; nevertheless we propose a simple modification of this model for which commuting is obtained, hence consistency with incompressibility is recovered. We also illustrate our discussions with some numerical results.

Keywords: Shell models; Incompressibility; Asymptotic analysis.

1. Introduction

As is well-known, the singularity corresponding to material incompressibility – as characterized e.g. in isotropic linear three-dimensional (3D) elasticity by the denominator "$1 - 2\nu$" (ν denoting Poisson's ratio) in the expression of the first Lamé constant λ – is not present in most classical shell models. In particular, this singularity is not seen in the membrane-bending and shear-membrane-bending shell models discussed in Ref.10 – also sometimes called the "Koiter model" and "Naghdi model", respectively, see e.g. Ref.11 – nor in the "basic shell model" that underlies the very widespread family of shell finite element procedures known as "general shell elements", see in particular Refs.8,10. Very clearly indeed, what removes the incompressible singularity from the formulation of these shell models is the use of

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a plane stress assumption that results – via the elimination of the twice-transverse strains – in a strain energy which features the first Lamé constant only in the combined expression (μ denoting the second Lamé constant)

$$\frac{2\lambda\mu}{\lambda + 2\mu}, \tag{1.1}$$

which of course has a finite limit when λ tends to infinity.

We note that the incompressible singularity is also not present in the limit models obtained by an asymptotic analysis performed with 3D elasticity formulations, see Ref.11 and references therein, in which no plane stress assumption is needed. However, one could argue that the primary concern in such an analysis – as, indeed, with classical shell models – lies in the "thinness" of the structure and that incompressibility is considered in a secondary stage only. The following question then very naturally arises: what results if we consider incompressibility first, namely if we use an incompressible (or "almost incompressible") material in a structure that "happens to be thin"? In more mathematical terms, the issue is therefore essentially whether or not we can exchange the asymptotic limits corresponding to incompressibility on the one hand and thinness (i.e. decreasing thickness) on the other hand. Clearly, for the concept of an "incompressible shell" to make any sense we need such a commuting property, which would also provide a justification of the validity of classical shell models in incompressible conditions. This validity is all the more questionable *a priori* as some shell models which do not make use of the plane stress assumption – such as the 3D-shell model discussed in Ref.7 – do retain the incompressible singularity, hence appear to have a rather different asymptotic behavior in the incompressible limit. All these questions are of much concern in various practical applications, such as in the tyre industry where thin layers made of incompressible materials are frequently encountered, as is also the case in biomechanics.

In this paper, we therefore undertake to analyse the interplay of the two above asymptotic behaviors. We first perform such an analysis with a 3D linear elastic formulation and we – indeed – obtain a commuting property, which thus gives a justification of the concept of an "incompressible shell" and of the use of classical shell models in incompressible conditions. By contrast, when considering the 3D-shell model we show that the two limits do not commute in general, in essence because the incompressible constraint is "too strong" for the displacement fields used, namely displacements that satisfy a quadratic kinematical assumption in the transverse direction. We note that the excessive impact of incompressibility on similar shell models was already identified in Ref.2, although not based on asymptotic considerations. We then propose a modified 3D-shell model in order to alleviate the incompressibility constraint and recover the commuting property.

The outline of the paper is as follows. In Section 2, we give the geometrical definitions and notation needed in the paper, and we recall some basic results concerning 3D incompressible elasticity. In Section 3, we analyse the asymptotic behavior of the 3D formulation with respect to both incompressibility and thickness and we establish a commuting property. Next, in Section 4 we consider the 3D-shell model, and also the modified version thereof introduced to recover the commuting property (in addition, some technical results needed in this section are given in an appendix). Section 5 presents some numerical results that illustrate our theoretical discussions. Finally we give some concluding remarks in Section 6.

2. Overview of the three-dimensional elasticity model in curvilinear coordinates

2.1. *Geometry and notation*

Throughout this paper Latin indices $i, j, k, \dots$ are assumed to vary in $\{1, 2, 3\}$ while Greek indices (except for ε and ν) vary in $\{1, 2\}$, and we use the Einstein convention pertaining to implicit summation of repeated indices. In the

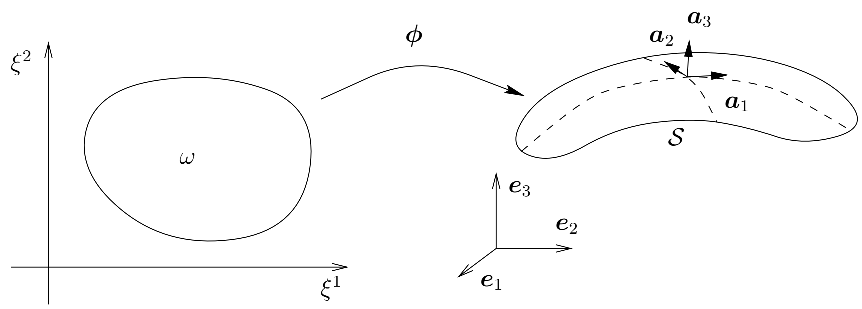

Figure 1: Definition of the midsurface $\mathcal{S}$

three-dimensional (3D) Euclidean space $\mathcal{E}$ equipped with the orthonormal frame $(O; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, we consider the surface $\mathcal{S}$ described by the injective mapping $\boldsymbol{\phi}$ defined over $\overline{\omega}$, the closure of a domain ω of $\mathbb{R}^2$ called the reference domain, see Fig. 1. The mapping $\boldsymbol{\phi}$ is assumed to be "smooth", namely as regular as needed in our analysis, and such that for any point (ξ^1, ξ^2) in $\overline{\omega}$ the vectors

$$\boldsymbol{a}_\alpha = \boldsymbol{\phi}_{,\alpha} = \left(\frac{\partial \boldsymbol{\phi}}{\partial \xi^\alpha}\right), \qquad \alpha = 1, 2, \tag{2.1}$$

form a basis – called the *covariant basis* – of the tangential plane to the surface. In addition, the unit normal vector

$$\boldsymbol{a}_3 = \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{\|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2\|} \tag{2.2}$$

is defined so that the triple $(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ forms a direct basis of $\mathcal{E}$. We also introduce the *contravariant basis* of the tangential plane $(\boldsymbol{a}^1, \boldsymbol{a}^2)$ defined such that $\boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta$ (Kronecker symbol). Then the surface $\mathcal{S}$ is completely characterized (up to a rigid body motion) by two symmetric tensors called the first and second fundamental forms – or alternatively the *metric tensor* and the *curvature tensor*, respectively – whose covariant components $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are defined by

$$a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \qquad b_{\alpha\beta} = \boldsymbol{a}_3 \cdot \boldsymbol{a}_{\alpha,\beta}. \tag{2.3}$$

We will also use the contravariant components of the metric tensor, given by

$$a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta, \tag{2.4}$$

and the surface infinitesimal

$$dS = \sqrt{a}\, d\omega = \sqrt{a}\, d\xi^1 d\xi^2, \tag{2.5}$$

with

$$a = \|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2\|^2 = a_{11}a_{22} - (a_{12})^2. \tag{2.6}$$

The eigenvalues of the curvature tensor are the principal curvatures of $\mathcal{S}$ at any point. The half-sum of the principal curvatures – denoted by H – is called the *mean curvature* while their product K is called the *Gaussian curvature*. Hence, we have

$$H = \frac{1}{2} b^\alpha_\alpha, \qquad K = b^1_1 b^2_2 - b^2_1 b^1_2, \tag{2.7}$$

where the mixed components of the curvature tensor are given by $b^\beta_\alpha = b_{\alpha\lambda} a^{\lambda\beta}$. The 3D geometry of the shell is then defined by the mapping

$$\boldsymbol{\Phi}(\xi^1, \xi^2, \xi^3) = \boldsymbol{\phi}(\xi^1, \xi^2) + \xi^3 \boldsymbol{a}_3(\xi^1, \xi^2), \tag{2.8}$$

for all $\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3) \in \Omega_t$, where Ω_t denotes the 3D reference domain given by

$$\Omega_t = \omega \times \left] -\frac{t}{2}, \frac{t}{2} \right[. \tag{2.9}$$

In this definition, the quantity t represents the thickness of the shell structure, assumed to be constant. Introducing a characteristic length L of $\mathcal{S}$ (for instance its diameter), we also define the dimensionless quantity

$$\varepsilon = \frac{t}{L}. \tag{2.10}$$

We denote by $\mathcal{B}_t$ the volume occupied by the shell body, hence,

$$\mathcal{B}_t = \boldsymbol{\Phi}(\overline{\Omega}_t). \tag{2.11}$$

The 3D covariant basis corresponding to the mapping $\boldsymbol{\Phi}$ is denoted by $(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3)$ and given by

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{\Phi}}{\partial \xi^i}, \ i = 1, 2, 3, \tag{2.12}$$

leading to

$$\boldsymbol{g}_\alpha = \boldsymbol{a}_\alpha + \xi^3 \boldsymbol{a}_{3,\alpha} = (\delta^\gamma_\alpha - \xi^3 b^\gamma_\alpha)\boldsymbol{a}_\gamma, \qquad \boldsymbol{g}_3 = \boldsymbol{a}_3. \tag{2.13}$$

In this basis, the components of the 3D metric tensor are

$$\begin{cases} g_{\alpha\beta} = a_{\alpha\beta} - 2b_{\alpha\beta}\xi^3 + c_{\alpha\beta}(\xi^3)^2, \\ g_{\alpha 3} = 0, \\ g_{33} = 1, \end{cases} \tag{2.14}$$

where $c_{\alpha\beta} = b_{\alpha\lambda} b^\lambda_\beta$ denote the covariant components of the so-called third fundamental form of the surface. The volume measure is expressed as

$$dV = \sqrt{g}\, d\Omega = \sqrt{g}\, d\xi^1 d\xi^2 d\xi^3, \tag{2.15}$$

with

$$g = \big[(\boldsymbol{g}_1 \wedge \boldsymbol{g}_2)\cdot \boldsymbol{g}_3\big]^2 = a\big(1 - 2H\xi^3 + K(\xi^3)^2\big)^2. \tag{2.16}$$

Remark 2.1 *The mapping $\boldsymbol{\phi}$ being bijective from $\overline{\omega}$ to $\mathcal{S}$, a necessary condition for the sets $\overline{\Omega}_t$ and $\mathcal{B}_t$ to be in bijection via $\boldsymbol{\Phi}$ is*

$$1 - 2H(\xi^1,\xi^2)\xi^3 + K(\xi^1,\xi^2)(\xi^3)^2 > 0 \quad \forall \boldsymbol{\xi} \in \overline{\Omega}_t, \tag{2.17}$$

condition that we henceforth assume to be satisfied. ■

We will use the 3D Christoffel symbols given by

$$\bar{\Gamma}^p_{ij} = \boldsymbol{g}^p \cdot \boldsymbol{g}_{i,j}, \tag{2.18}$$

in the covariant differentiation of vector fields, which is denoted and defined by

$$V_{i||j} = V_{i,j} - \bar{\Gamma}^p_{ij} V_p, \tag{2.19}$$

see e.g. Ref.13. Similarly, the surface Christoffel symbols are given by

$$\Gamma^\sigma_{\alpha\beta} = \boldsymbol{a}^\sigma \cdot \boldsymbol{a}_{\alpha,\beta} = \big(\bar{\Gamma}^\sigma_{\alpha\beta}\big)_{\xi^3=0}, \tag{2.20}$$

and used in the covariant differentiation of vectors defined on ω and tangential to the midsurface, such as in

$$v_{\alpha|\beta} = v_{\alpha,\beta} - \Gamma^\sigma_{\alpha\beta} v_\sigma. \tag{2.21}$$

Finally, we will employ the short notation $\|\cdot\|_k$ $(k \in \mathbb{N})$ to denote the $H^k(\omega)$ norm for scalar or tensor fields, and the symbol C to denote a generic constant that may take different values at successive occurrences.

2.2. *Three-dimensional elasticity model and incompressibility*

Let us assume that the body $\mathcal{B}_t$ is fixed on a part of its boundary, corresponding to $\Gamma_0 = \gamma_0 \times]-\frac{t}{2},\frac{t}{2}[\subset \partial\omega \times]-\frac{t}{2},\frac{t}{2}[$ in the reference domain, and is submitted to the distributed loading $\boldsymbol{f}^\varepsilon \in (L^2(\Omega_t))^3$. The variational formulation of isotropic linearized elasticity is posed in the space

$$\mathcal{V}(\Omega_t) = \{\boldsymbol{V} \in (H^1(\Omega_t))^3; \boldsymbol{V} = 0 \text{ on } \Gamma_0\},$$

and reads in curvilinear coordinates, see Ref.11,

$$(\mathcal{P}(\varepsilon,\nu)) \begin{cases} \boldsymbol{U}(\varepsilon,\nu) \in \mathcal{V}(\Omega_t), \\ \displaystyle\int_{\Omega_t} A^{ijkl} e_{ij}(\boldsymbol{U}(\varepsilon,\nu)) e_{kl}(\boldsymbol{V}) \sqrt{g}\, d\Omega = \int_{\Omega_t} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{V} \sqrt{g}\, d\Omega, \quad \forall \boldsymbol{V} \in \mathcal{V}(\Omega_t), \end{cases}$$

where the unknown $\boldsymbol{U}(\varepsilon,\nu)(\boldsymbol{\xi}) = U_i(\varepsilon,\nu)(\boldsymbol{\xi})\boldsymbol{g}^i(\boldsymbol{\xi})$ represents the displacement field and is purposedly indexed by ε and ν to signify that we will specifically analyse the dependence (and asymptotic behavior) of the displacement field with respect to these parameters. In this formulation, the contravariant components of the elasticity tensor are given by

$$A^{ijkl} = \lambda g^{ij} g^{kl} + \mu (g^{ik} g^{jl} + g^{il} g^{jk}), \tag{2.22}$$

where λ and μ denote the Lamé constants that relate to Young's modulus – denoted by E – and Poisson's ratio through

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}, \qquad (-1 < \nu < 1/2). \tag{2.23}$$

In addition, the components of the (linearized) strain tensor are defined by

$$e_{ij}(\boldsymbol{V}) = \frac{1}{2}(V_{i||j} + V_{j||i}). \tag{2.24}$$

We recall that the problem $(\mathcal{P}(\varepsilon,\nu))$ admits a unique solution (Ref.11). In the sequel, we will use this formulation in the following rearranged form

$$(\mathcal{P}(\varepsilon,\nu)) \begin{cases} \boldsymbol{U}(\varepsilon,\nu) \in \mathcal{V}(\Omega_t), \\ \displaystyle\mu \int_{\Omega_t} g^{ijkl} e_{ij}(\boldsymbol{U}(\varepsilon,\nu)) e_{kl}(\boldsymbol{V}) \sqrt{g}\, d\Omega + \lambda \int_{\Omega_t} g^{ij} e_{ij}(\boldsymbol{U}(\varepsilon,\nu)) g^{kl} e_{kl}(\boldsymbol{V}) \sqrt{g}\, d\Omega \\ \displaystyle\qquad\qquad = \int_{\Omega_t} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{V} \sqrt{g}\, d\Omega, \quad \forall \boldsymbol{V} \in \mathcal{V}(\Omega_t), \end{cases}$$

with $g^{ijkl} = g^{ik} g^{jl} + g^{il} g^{jk}$. Of course, this formulation is not well-defined for $\nu = 1/2$, which corresponds to the incompressible limit for which λ tends to infinity. When ν is near $1/2$, it appears that the quantity λ plays the role of a penalization parameter and we have the following classical result (see e.g. Ref.6).

Theorem 2.1 (Limit of $(\mathcal{P}(\varepsilon,\nu))$ as $\nu \to 1/2$) *The solution $\boldsymbol{U}(\varepsilon,\nu)$ of $(\mathcal{P}(\varepsilon,\nu))$ converges, as ν tends to $1/2$, to $\boldsymbol{U}(\varepsilon)$ solution of the variational problem*

$$(\mathcal{P}(\varepsilon)) \begin{cases} \boldsymbol{U}(\varepsilon) \in \mathcal{V}_{\mathcal{I}}(\Omega_t) = \{\boldsymbol{V} \in \mathcal{V}(\Omega_t); g^{ij} e_{ij}(\boldsymbol{V}) = 0\ \}, \\ \displaystyle\frac{E}{3} \int_{\Omega_t} g^{ijkl} e_{ij}(\boldsymbol{U}(\varepsilon)) e_{kl}(\boldsymbol{V}) \sqrt{g}\, d\Omega = \int_{\Omega_t} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{V} \sqrt{g}\, d\Omega, \quad \forall \boldsymbol{V} \in \mathcal{V}_{\mathcal{I}}(\Omega_t), \end{cases}$$

and there exists a constant C (dependent on ε but independent of ν) such that

$$\|\boldsymbol{U}(\varepsilon,\nu)-\boldsymbol{U}(\varepsilon)\|_{H^1(\Omega_t)} \le C(1-2\nu)\|\boldsymbol{U}(\varepsilon)\|_{H^1(\Omega_t)}. \tag{2.25}$$

■

Remark 2.2 *We have $g^{ij}e_{ij}(\boldsymbol{V}) = g^{\alpha\beta}e_{\alpha\beta}(\boldsymbol{V}) + e_{33}(\boldsymbol{V}) = div\, \boldsymbol{V}$.* ■

Remark 2.3 *Clearly, $(\mathcal{P}(\varepsilon))$ admits a unique solution, $\mathcal{V}_{\mathcal{I}}(\Omega_t)$ being a closed subspace of $\mathcal{V}(\Omega_t)$.* ■

3. Analysis of the asymptotic limits and their interplay in the three-dimensional model

3.1. *Analysis of* $\lim\limits_{\nu\to 1/2}\lim\limits_{\varepsilon\to 0}(\mathcal{P}(\varepsilon,\nu))$

In this section, we study the limit of the formulation $(\mathcal{P}(\varepsilon,\nu))$ when making first ε go to zero, and then ν to 1/2. As regards the limit in ε, we summarize some of the results established in Ref.11 where the limit in the parameter ε is sought after performing a scaling in the thickness of the body, which allows to consider a sequence of problems posed over the fixed domain

$$\Omega = \omega \times \left]-\frac{L}{2},\frac{L}{2}\right[. \tag{3.1}$$

To that purpose, we define the operator

$$\pi_t : \boldsymbol{\xi}_\Omega = (\xi^1,\xi^2,\xi) \in \Omega \mapsto \boldsymbol{\xi} = (\xi^1,\xi^2,\xi^3) = (\xi^1,\xi^2,\varepsilon\xi) \in \Omega_t, \tag{3.2}$$

where we recall that ε denotes the dimensionless thickness parameter t/L. In Ω, the field $\boldsymbol{U}(\varepsilon,\nu)(\boldsymbol{\xi})$ then corresponds to $\boldsymbol{U}_\Omega(\varepsilon,\nu)(\boldsymbol{\xi}_\Omega)$. Assuming that the load distribution $\boldsymbol{f}^\varepsilon$ can be written in the form

$$\boldsymbol{f}^\varepsilon(\boldsymbol{\xi}) = \varepsilon^p \boldsymbol{f}_p(\boldsymbol{\xi}_\Omega), \quad \boldsymbol{f}_p \in (L^2(\Omega))^3, \quad p\in\mathbb{N} \text{ (without summation)}, \tag{3.3}$$

the problem $(\mathcal{P}(\varepsilon,\nu))$ becomes

$$(\mathcal{P}_\Omega(\varepsilon,\nu)) \begin{cases} \boldsymbol{U}_\Omega(\varepsilon,\nu) \in \mathcal{V}(\Omega) = \{\boldsymbol{V}\in (H^1(\Omega))^3; \boldsymbol{V}=0 \text{ on } \gamma_0\times]-L/2,L/2[\}, \\ \displaystyle\int_\Omega A^{ijkl}(\varepsilon)e_{ij}(\varepsilon,\boldsymbol{U}_\Omega(\varepsilon,\nu))e_{kl}(\varepsilon,\boldsymbol{V})\sqrt{g(\varepsilon)}\,d\Omega \\ \qquad\qquad = \varepsilon^p \displaystyle\int_\Omega \boldsymbol{f}_p\cdot\boldsymbol{V}\sqrt{g(\varepsilon)}\,d\Omega \quad \forall \boldsymbol{V}\in\mathcal{V}(\Omega), \end{cases}$$

with

$$\begin{cases} e_{\alpha\beta}(\varepsilon,\boldsymbol{V}) = \dfrac{1}{2}(V_{\alpha,\beta}+V_{\beta,\alpha}) - \bar\Gamma^p_{\alpha\beta}(\varepsilon)V_p, \\ e_{\alpha3}(\varepsilon,\boldsymbol{V}) = \dfrac{1}{2}\left(\dfrac{1}{\varepsilon}V_{\alpha,3}+V_{3,\alpha}\right) - \bar\Gamma^\sigma_{\alpha3}(\varepsilon)V_\sigma, \\ e_{33}(\varepsilon,\boldsymbol{V}) = \dfrac{1}{\varepsilon}V_{3,3}, \end{cases} \tag{3.4}$$

and where, for all $\boldsymbol{\xi} = \pi_t \boldsymbol{\xi}_\Omega \in \overline{\Omega}_t$,

$$\begin{cases} \bar{\Gamma}^p_{ij}(\varepsilon)(\boldsymbol{\xi}_\Omega) = \bar{\Gamma}^p_{ij}(\boldsymbol{\xi}), \\ g(\varepsilon)(\boldsymbol{\xi}_\Omega) = g(\boldsymbol{\xi}), \\ A^{ijkl}(\varepsilon)(\boldsymbol{\xi}_\Omega) = A^{ijkl}(\boldsymbol{\xi}). \end{cases} \tag{3.5}$$

We then introduce the space

$$\mathcal{V}(\omega) = \{\boldsymbol{v} \in (H^1(\omega))^3;\ \boldsymbol{v} = 0 \text{ on } \gamma_0\}, \tag{3.6}$$

and the two subspaces

$$\mathcal{V}_0(\omega) = \{\boldsymbol{v} \in \mathcal{V}(\omega);\ \gamma_{\alpha\beta}(\boldsymbol{v}) = 0,\ \alpha, \beta = 1, 2\}, \tag{3.7}$$

and, ∂_ν denoting the normal derivative along the boundary,

$$\mathcal{V}_{\mathcal{F}}(\omega) = \{\boldsymbol{v} \in \mathcal{V}_0(\omega);\ v_3 \in H^2(\omega);\ \partial_\nu v_3 = 0 \text{ on } \gamma_0\ \}, \tag{3.8}$$

where

$$\gamma_{\alpha\beta}(\boldsymbol{v}) = \frac{1}{2}(v_{\alpha|\beta} + v_{\beta|\alpha}) - b_{\alpha\beta} v_3, \tag{3.9}$$

represent the components of the linearized change of metric tensor – also called the membrane strain tensor – associated with the displacement field $\boldsymbol{v} = v_\alpha \boldsymbol{a}^\alpha + v_3 \boldsymbol{a}_3$ defined on the midsurface. We also introduce the quantity

$$\boldsymbol{u}(\varepsilon, \nu) = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \boldsymbol{U}_\Omega(\varepsilon, \nu) d\xi, \tag{3.10}$$

for which convergence results will be obtained. A crucial distinction arises depending on the contents of the spaces $\mathcal{V}_0(\omega)$ and $\mathcal{V}_{\mathcal{F}}(\omega)$. When $\mathcal{V}_0(\omega) = \{\boldsymbol{0}\}$, we define the norm

$$\|\boldsymbol{v}\|_M = \sum_{\alpha,\beta} \left(\|\gamma_{\alpha\beta}(\boldsymbol{v})\|_0^2\right)^{\frac{1}{2}}, \tag{3.11}$$

the space $\mathcal{V}_M(\omega)$ as the completion of $\mathcal{V}(\omega)$ with respect to $\|.\|_M$, and $\big(\mathcal{V}_M(\omega)\big)'$ the corresponding dual space. We then have the following result for $p = 0$ in (3.3).

Theorem 3.1 *Assume that*

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \boldsymbol{f}_0 \, d\xi \in \big(\mathcal{V}_M(\omega)\big)'. \tag{3.12}$$

When ε tends to zero, $\boldsymbol{u}(\varepsilon, \nu)$ converges in $\mathcal{V}_M(\omega)$ to $\boldsymbol{u}^{\mathrm{m}}(\nu)$ solution of

$$(\mathcal{P}_{\mathcal{M}}(\nu)) \begin{cases} \boldsymbol{u}^{\mathrm{m}}(\nu) \in \mathcal{V}_M(\omega), \\ L \displaystyle\int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\alpha\beta}(\boldsymbol{u}^{\mathrm{m}}(\nu)) \gamma_{\sigma\tau}(\boldsymbol{v}) \sqrt{a}\, d\omega \\ \qquad = \displaystyle\int_\omega \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \boldsymbol{f}_0 \, d\xi \right) \cdot \boldsymbol{v} \sqrt{a}\, d\omega, \quad \forall \boldsymbol{v} \in \mathcal{V}_M(\omega), \end{cases}$$

where

$$a^{\alpha\beta\sigma\tau} = \frac{2\lambda\mu}{\lambda+2\mu} a^{\alpha\beta} a^{\sigma\tau} + \mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}). \tag{3.13}$$

■

Proof. See Ref.11, Chapter 5. □

Remark 3.1 *In the detailed analysis the following relations appear*

$$e^0_{\alpha 3} = 0, \qquad e^0_{33} = -\frac{\lambda}{\lambda+2\mu} a^{\alpha\beta} e^0_{\alpha\beta}, \qquad \textit{a.e. in } \Omega, \tag{3.14}$$

where $e^0_{\alpha\beta}, e^0_{33}$ *and* $e^0_{\alpha 3}$ *are defined in (3.22) below. An interpretation of these relations is that the stresses* $\sigma_{i3}(\varepsilon)$ *vanish at the first order in* ε*. In addition, it is by using the second relation that the tensor of components* $a^{\alpha\beta\sigma\tau}$ *arises "in place of"* $A^{ijkl}(\varepsilon)$*, which cancels the singularity for* $\nu = 1/2$*. Note that, indeed,*

$$a^{\alpha\beta\sigma\tau} = \frac{E\nu}{1-\nu^2} a^{\alpha\beta} a^{\sigma\tau} + \frac{E}{2(1+\nu)} (a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}). \tag{3.15}$$

■

By contrast, when $\mathcal{V}_{\mathcal{F}}(\omega)$ is not reduced to zero, denoting for any $\boldsymbol{v} = v_\alpha \boldsymbol{a}^\alpha + v_3 \boldsymbol{a}_3$ with $(v_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$,

$$\rho_{\alpha\beta}(\boldsymbol{v}) = v_{3|\alpha\beta} - b^\sigma_\alpha b_{\sigma\beta} v_3 + b^\sigma_\alpha v_{\sigma|\beta} + b^\tau_\beta v_{\tau|\alpha} + b^\tau_{\beta|\alpha} v_\tau, \tag{3.16}$$

as the components of the linearized change of curvature tensor associated with the displacement field $\boldsymbol{v}$ defined on the midsurface, the asymptotic analysis of Problem $(\mathcal{P}_\Omega(\varepsilon,\nu))$ performed with $p = 2$ leads to the following result.

Theorem 3.2 *When* ε *tends to zero,* $\boldsymbol{u}(\varepsilon,\nu)$ *converges in* $\mathcal{V}(\omega)$ *to* $\boldsymbol{u}^{\mathrm{f}}(\nu)$ *solution of*

$$(\mathcal{P}_{\mathcal{F}}(\nu)) \begin{cases} \boldsymbol{u}^{\mathrm{f}}(\nu) \in \mathcal{V}_{\mathcal{F}}(\omega), \\ \dfrac{L^3}{12} \displaystyle\int_\omega a^{\alpha\beta\sigma\tau} \rho_{\alpha\beta}(\boldsymbol{u}^{\mathrm{f}}(\nu)) \rho_{\sigma\tau}(\boldsymbol{v}) \sqrt{a}\, d\omega \\ \qquad\qquad = \displaystyle\int_\omega \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \boldsymbol{f}_2 \, d\xi \right) \cdot \boldsymbol{v} \sqrt{a}\, d\omega, \quad \forall \boldsymbol{v} \in \mathcal{V}_{\mathcal{F}}(\omega). \end{cases}$$

■

Proof. See Ref.11, Chapter 6. □

Remark 3.2 *In this case the following relations arise in the analysis*

$$e^0_{ij} = 0\ ;\ e^1_{\alpha 3} = 0\ ;\ e^1_{33} = -\frac{\lambda}{\lambda+2\mu} a^{\alpha\beta} e^1_{\alpha\beta} \quad \textit{a.e. in} \quad \Omega, \tag{3.17}$$

with the interpretation that the stresses $\sigma_{i3}(\varepsilon)$ *vanish at the first two orders. In addition, these relations again induce the appearance of the tensor of components* $a^{\alpha\beta\sigma\tau}$. ■

Remark 3.3 *The reader should note that the above results do not cover the case when $\mathcal{V}_{\mathcal{F}}(\omega)$ is reduced to zero while $\mathcal{V}_0(\omega)$ contains non-zero elements. This is what is referred to as a "generalized membrane shell of the second kind" in Ref.11, which also remarks that "it seems that there are no known examples" of this situation. Nevertheless, see Ref.11 for the asymptotic analysis in this case, a variant of that performed in the case when $\mathcal{V}_0(\omega) = \{\mathbf{0}\}$ which itself characterizes a "membrane shell" (for a clamped structure of uniformly elliptic midsurface) or a "generalized membrane shell of the first kind" (for other boundary conditions and/or geometries) in the terminology of this reference. As for the case when $\mathcal{V}_{\mathcal{F}}(\omega)$ contains non-zero elements, it corresponds to "flexural shells" in this same terminology. We also point out that – in the framework of the asymptotic behavior of* shell models *– a quite similar terminology is frequently used: "membrane-dominated shells" when $\mathcal{V}_{\mathcal{F}}(\omega) = \{\mathbf{0}\}$; "bending-dominated shells" otherwise, see in particular Refs.10,15.*
■

In the absence of a singularity for $\nu = 1/2$, obtaining the limit of the above asymptotic shell models when ν tends to $1/2$ is very straightforward. Hence we state the results and omit the proofs. We first consider the case when $\mathcal{V}_0(\omega) = \{\mathbf{0}\}$, namely, that corresponding to $(\mathcal{P}_{\mathcal{M}}(\nu))$.

Theorem 3.3 (Incompressible membrane shell model) *Assume that* (3.12) *holds. Then, when ν tends to $1/2$, $\boldsymbol{u}(\nu)$ converges in $\mathcal{V}_M(\omega)$ to $\boldsymbol{u}^{\mathrm{m}}$ solution of*

$$(\mathcal{P}_{\mathcal{M}}) \begin{cases} \boldsymbol{u}^{\mathrm{m}} \in \mathcal{V}_M(\omega), \\ L \displaystyle\int_\omega a_I^{\alpha\beta\sigma\tau} \gamma_{\alpha\beta}(\boldsymbol{u}^{\mathrm{m}}) \gamma_{\sigma\tau}(\boldsymbol{v}) \sqrt{a}\, d\omega = \int_\omega \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \boldsymbol{f}_0\, d\xi \right) \cdot \boldsymbol{v} \sqrt{a}\, d\omega \quad \forall \boldsymbol{v} \in \mathcal{V}_M(\omega), \end{cases}$$

with

$$a_I^{\alpha\beta\sigma\tau} = \frac{E}{3}(2a^{\alpha\beta}a^{\sigma\tau} + a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}). \tag{3.18}$$

■

Remark 3.4 *The second equation of (3.14) then implies the relation*

$$e_{33}^0 + a^{\alpha\beta} e_{\alpha\beta}^0 = 0, \tag{3.19}$$

which can be interpreted as expressing the incompressibility of the shell body at the first order in ε. ■

Considering the case when $\mathcal{V}_{\mathcal{F}}(\omega)$ contains non-zero elements – namely that corresponding to $(\mathcal{P}_{\mathcal{F}}(\nu))$ – we then have the following result.

Theorem 3.4 (Incompressible flexural shell model) *When ν tends to $1/2$, $\boldsymbol{u}(\nu)$ converges in $\mathcal{V}(\omega)$ to $\boldsymbol{u}^{\mathrm{f}}$ solution of*

$$(\mathcal{P}_{\mathcal{F}}) \begin{cases} \boldsymbol{u}^{\mathrm{f}} \in \mathcal{V}_{\mathcal{F}}(\omega), \\ \dfrac{L^3}{12} \displaystyle\int_\omega a_I^{\alpha\beta\sigma\tau} \rho_{\alpha\beta}(\boldsymbol{u}^{\mathrm{f}}) \rho_{\sigma\tau}(\boldsymbol{v}) \sqrt{a}\, d\omega = \int_\omega \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \boldsymbol{f}_2\, d\xi \right) \cdot \boldsymbol{v} \sqrt{a}\, d\omega, \quad \forall \boldsymbol{v} \in \mathcal{V}_{\mathcal{F}}(\omega). \end{cases}$$

■

Remark 3.5 *The third equation of (3.17) then gives* $e^1_{33} + a^{\alpha\beta} e^1_{\alpha\beta} = 0$ *which – together with* $e^0_{ij} = 0$ *– can be interpreted as expressing the incompressibility of the shell body at the first two orders in* ε. ■

Therefore, the asymptotic behavior of a thin body when the thickness tends first to zero, and then Poisson's ratio to 1/2, is well-identified. In the next section we examine the asymptotic behavior when the incompressible limit is considered first.

3.2. *Analysis of* $\lim\limits_{\varepsilon\to 0}\ \lim\limits_{\nu\to 1/2}\ (\mathcal{P}(\varepsilon,\nu))$

In this section, the starting point is the incompressible limit problem $(\mathcal{P}(\varepsilon))$ – see Theorem 2.1 – and we proceed to analyse the asymptotic behavior of this model when ε tends to zero by a formal asymptotic analysis inspired from that presented in Ref.11, see also Ref.15. Compared to the asymptotic analysis of "usual" thin bodies, however, a direct analysis of Problem $(\mathcal{P}(\varepsilon))$ would require the construction of test functions of $\mathcal{V}_{\mathcal{I}}(\Omega_t)$ whose restriction on ω belongs to $\mathcal{V}_{\mathcal{F}}(\omega)$. In order to circumvent this difficulty, we first transform the problem into the following equivalent mixed formulation – directly written in the fixed domain Ω – see Ref.6.

$$(\mathcal{P}_\Omega(\varepsilon)) \begin{cases} (\boldsymbol{U}_\Omega(\varepsilon), p_\Omega(\varepsilon)) \in \mathcal{V}(\Omega) \times L^2(\Omega), \\ \dfrac{E}{3} \displaystyle\int_\Omega g^{ijkl}(\varepsilon) e_{ij}(\varepsilon, \boldsymbol{U}_\Omega(\varepsilon)) e_{kl}(\varepsilon, \boldsymbol{V}) \sqrt{g(\varepsilon)}\, d\Omega \\ + \displaystyle\int_\Omega p_\Omega(\varepsilon) g^{ij}(\varepsilon) e_{ij}(\varepsilon, \boldsymbol{V}) \sqrt{g(\varepsilon)}\, d\Omega = \varepsilon^p \int_\Omega \boldsymbol{f}_p \cdot \boldsymbol{V} \sqrt{g(\varepsilon)}\, d\Omega, \quad \forall \boldsymbol{V} \in \mathcal{V}(\Omega), \\ \displaystyle\int_\Omega q\, g^{ij}(\varepsilon) e_{ij}(\varepsilon, \boldsymbol{U}_\Omega(\varepsilon)) \sqrt{g(\varepsilon)}\, d\Omega = 0, \quad \forall q \in L^2(\Omega). \end{cases}$$

Then, we assume that there exists an asymptotic expansion of the solution in the form

$$(\boldsymbol{U}_\Omega(\varepsilon), p_\Omega(\varepsilon)) = (\boldsymbol{U}^0, p^0) + \varepsilon(\boldsymbol{U}^1, p^1) + \varepsilon^2(\boldsymbol{U}^2, p^2) + \dots \tag{3.20}$$

This induces an asymptotic expansion of the scaled strains in the form

$$\begin{cases} e_{ij}(\varepsilon, \boldsymbol{U}_\Omega(\varepsilon)) = \varepsilon^{-1} e^{-1}_{ij} + e^0_{ij} + \varepsilon e^1_{ij} + \dots, \\ e_{ij}(\varepsilon, \boldsymbol{V})) = \varepsilon^{-1} e^{-1}_{ij}(\boldsymbol{V}) + e^0_{ij}(\boldsymbol{V}) + \varepsilon e^1_{ij}(\boldsymbol{V}) + \dots, \end{cases} \tag{3.21}$$

where

$$\begin{cases} e^{-1}_{\alpha\beta} = 0, \quad e^{-1}_{\alpha 3} = \dfrac{1}{2} U^0_{\alpha,3}, \quad e^{-1}_{33} = U^0_{3,3}, \\ e^0_{\alpha\beta} = \dfrac{1}{2}(U^0_{\alpha,\beta} + U^0_{\beta,\alpha}) - \Gamma^\sigma_{\alpha\beta} U^0_\sigma - b_{\alpha\beta} U^0_3, \\ e^0_{\alpha 3} = \dfrac{1}{2}(U^0_{3,\alpha} + U^1_{\alpha,3}) + b^\sigma_\alpha U^0_\sigma, \quad e^0_{33} = U^1_{3,3}, \\ e^1_{\alpha\beta} = \dfrac{1}{2}(U^1_{\beta,\alpha} + U^1_{\alpha,\beta}) - \Gamma^\sigma_{\alpha\beta} U^1_\sigma - b_{\alpha\beta} U^1_3 + \xi(b^\sigma_{\beta|\alpha} U^0_\sigma + b^\sigma_\alpha b_{\sigma\beta} U^0_3), \\ e^1_{\alpha 3} = \dfrac{1}{2}(U^1_{3,\alpha} + U^2_{\alpha,3}) + b^\sigma_\alpha U^1_\sigma + \xi b^\tau_\alpha b^\sigma_\tau U^0_\sigma, \quad e^1_{33} = U^2_{3,3}, \end{cases} \tag{3.22}$$

and

$$\begin{cases} e^{-1}_{\alpha\beta}(\boldsymbol{V}) = 0\ ;\ e^{-1}_{\alpha 3}(\boldsymbol{V}) = \dfrac{1}{2}V_{\alpha,3}, \quad e^{-1}_{33}(\boldsymbol{V}) = V_{3,3}, \\ e^{0}_{\alpha\beta}(\boldsymbol{V}) = \dfrac{1}{2}(V_{\alpha,\beta} + V_{\beta,\alpha}) - \Gamma^{\sigma}_{\alpha\beta}V_\sigma - b_{\alpha\beta}V_3, \\ e^{0}_{\alpha 3}(\boldsymbol{V}) = \dfrac{1}{2}V_{3,\alpha} + b^{\sigma}_{\alpha}V_\sigma, \quad e^{0}_{33}(\boldsymbol{V}) = 0, \\ e^{1}_{\alpha\beta}(\boldsymbol{V}) = \xi(b^{\sigma}_{\beta|\alpha}V_\sigma + b^{\sigma}_{\alpha}b_{\sigma\beta}V_3), \quad e^{1}_{\alpha 3}(\boldsymbol{V}) = \xi b^{\tau}_{\alpha}b^{\sigma}_{\tau}V_\sigma, \quad e^{1}_{33}(\boldsymbol{V}) = 0. \end{cases} \tag{3.23}$$

We also have (Ref.11)

$$\begin{cases} g^{ij}(\varepsilon)\sqrt{g(\varepsilon)} = a^{ij}\sqrt{a} + \varepsilon B^{ij,1} + \varepsilon^2 B^{ij,2} + O(\varepsilon^3), \\ g^{ijkl}(\varepsilon)\sqrt{g(\varepsilon)} = g^{ijkl}(0)\sqrt{a} + \varepsilon B^{ijkl,1} + \varepsilon^2 B^{ijkl,2} + O(\varepsilon^3), \end{cases} \tag{3.24}$$

with

$$\begin{cases} g^{\alpha\beta\sigma\tau}(0) = a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}, \quad g^{\alpha 3\beta 3}(0) = a^{\alpha\beta}, \quad g^{3333}(0) = 2, \\ B^{\alpha\beta,1} = 2\xi(b^{\alpha\beta} - Ha^{\alpha\beta})\sqrt{a}, \quad B^{\alpha 3,1} = 0, \quad B^{33,1} = -2H\xi\sqrt{a}, \\ B^{\alpha\beta\sigma\tau,1} = 2\xi\Big[-Hg^{\alpha\beta\sigma\tau}(0) + \Big(a^{\alpha\sigma}b^{\beta\tau} + a^{\beta\tau}b^{\alpha\sigma} + a^{\alpha\tau}b^{\beta\sigma} + a^{\beta\sigma}b^{\alpha\tau}\Big)\Big]\sqrt{a}, \\ B^{\alpha\beta\sigma 3,1} = B^{\alpha\beta 33,1} = B^{\alpha 333,1} = 0, \quad B^{3333,1} = 2B^{33,1}. \end{cases} \tag{3.25}$$

We are now in a position to start the identification of the factors of the successive powers of ε in the asymptotic expansion (3.20). In this procedure, a key distinction arises depending on the contents of the space $\mathcal{V}_0(\omega)$.

3.2.1. *Case* $\mathcal{V}_0(\omega) = \{\mathbf{0}\}$ *(*$p = 0$*)*

In the case when $\mathcal{V}_0(\omega) = \{\mathbf{0}\}$, by inspection the appropriate scaling factor is found to be $p = 0$. The identification of the coefficient of ε^{-2} leads to the equation

$$(\mathcal{P}^{-2}) \qquad \frac{E}{3}\int_\Omega g^{ijkl}(0)e^{-1}_{ij}e^{-1}_{kl}(\boldsymbol{V})\sqrt{a}\,d\Omega = 0, \quad \forall \boldsymbol{V} \in \mathcal{V}(\Omega),$$

which implies, using (3.23),

$$e^{-1}_{i3} = 0 \quad a.e. \text{ in } \Omega, \tag{3.26}$$

hence there exists $\boldsymbol{u}^0 \in \mathcal{V}(\omega)$ such that $\boldsymbol{U}^0(\xi^1,\xi^2,\xi) = \boldsymbol{u}^0(\xi^1,\xi^2)$, see Ref.11. The identification of the coefficient of ε^{-1} then provides

$$(\mathcal{P}^{-1}) \begin{cases} \dfrac{E}{3}\displaystyle\int_\Omega g^{ijkl}(0)e^{0}_{ij}e^{-1}_{kl}(\boldsymbol{V})\sqrt{a}\,d\Omega + \int_\Omega p^0 a^{ij}e^{-1}_{ij}(\boldsymbol{V})\sqrt{a}\,d\Omega = 0, \quad \forall \boldsymbol{V} \in \mathcal{V}(\Omega), \\ \displaystyle\int_\Omega q\, a^{ij}e^{-1}_{ij}\sqrt{a}\,d\Omega = 0, \quad \forall q \in L^2(\Omega). \end{cases}$$

Combined with the identities $e^{-1}_{\alpha\beta}(\boldsymbol{V}) = 0$ and $a^{\alpha 3} = 0$, the first relation leads to

$$e^0_{\alpha 3} = 0, \qquad \frac{2E}{3} e^0_{33} + p^0 = 0 \quad a.e. \text{ in } \Omega, \tag{3.27}$$

whereas the second relation gives $e^{-1}_{33} = 0$, as in (3.26). The identification of the coefficient of ε^0 then implies

$$(\mathcal{P}^0) \begin{cases} \dfrac{E}{3} \displaystyle\int_\Omega \Big[B^{ijkl,1} e^0_{ij} e^{-1}_{kl}(\boldsymbol{V}) + g^{ijkl}(0) \Big(e^1_{ij} e^{-1}_{kl}(\boldsymbol{V}) + e^0_{ij} e^0_{kl}(\boldsymbol{V}) \Big) \sqrt{a} \Big] d\Omega \\ \qquad + \displaystyle\int_\Omega \Big[p^1 a^{ij} e^{-1}_{ij}(\boldsymbol{V}) \sqrt{a} + p^0 \Big(B^{ij,1} e^{-1}_{ij}(\boldsymbol{V}) + a^{ij} e^0_{ij}(\boldsymbol{V}) \sqrt{a} \Big) \Big] d\Omega \\ \qquad\qquad = \displaystyle\int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{V} \sqrt{a}\, d\Omega, \quad \forall \boldsymbol{V} \in \mathcal{V}(\Omega), \\ \displaystyle\int_\Omega q\, a^{ij} e^0_{ij} \sqrt{a}\, d\Omega = 0, \quad \forall q \in L^2(\Omega). \end{cases}$$

The second equation of this system gives

$$a^{\alpha\beta} e^0_{\alpha\beta} + e^0_{33} = 0 \quad a.e. \text{ in } \Omega. \tag{3.28}$$

Furthermore, using (3.27) and the identity $B^{3333,1} = 2B^{33,1}$, the first equation of $(\mathcal{P}^0)$ becomes

$$\begin{aligned} &\frac{E}{3} \int_\Omega \Big[2e^1_{33} e^{-1}_{33}(\boldsymbol{V}) + 4a^{\alpha\sigma} e^1_{\alpha 3} e^{-1}_{\sigma 3}(\boldsymbol{V}) + g^{\alpha\beta\sigma\tau}(0) e^0_{\alpha\beta} e^0_{\sigma\tau}(\boldsymbol{V}) \Big] \sqrt{a}\, d\Omega \\ &\quad + \int_\Omega \Big[p^1 e^{-1}_{33}(\boldsymbol{V}) \sqrt{a} + p^0 B^{\alpha\beta,1} e^{-1}_{\alpha\beta}(\boldsymbol{V}) + p^0 a^{\alpha\beta} e^0_{\alpha\beta}(\boldsymbol{V}) \sqrt{a} \Big] d\Omega \\ &\qquad = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{V} \sqrt{a}\, d\Omega, \quad \forall \boldsymbol{V} \in \mathcal{V}(\Omega). \end{aligned} \tag{3.29}$$

We consider a test function $\boldsymbol{V} = \boldsymbol{v} \in \mathcal{V}(\omega)$ independent of ξ. Taking into account $e^{-1}_{ij}(\boldsymbol{v}) = 0$, the previous relation reduces to

$$\frac{E}{3} \int_\Omega g^{\alpha\beta\sigma\tau}(0) e^0_{\alpha\beta} e^0_{\sigma\tau}(\boldsymbol{v}) \sqrt{a}\, d\Omega + \int_\Omega p^0 a^{\alpha\beta} e^0_{\alpha\beta}(\boldsymbol{v}) \sqrt{a}\, d\Omega = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v} \sqrt{a}\, d\Omega, \quad \forall \boldsymbol{v} \in \mathcal{V}(\omega). \tag{3.30}$$

In this equation, we have

$$e^0_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{u}^0), \qquad e^0_{\alpha\beta}(\boldsymbol{v}) = \gamma_{\alpha\beta}(\boldsymbol{v}), \tag{3.31}$$

and, using the relations (3.27) and (3.28),

$$p^0 = \frac{2E}{3} a^{\alpha\beta} e^0_{\alpha\beta}, \tag{3.32}$$

hence p^0 is independent of ξ. Substituting these expressions in (3.30), we obtain

$$\begin{aligned} &\frac{EL}{3} \int_\omega (g^{\alpha\beta\sigma\tau}(0) + 2a^{\alpha\beta} a^{\sigma\tau}) \gamma_{\alpha\beta}(\boldsymbol{u}^0) \gamma_{\sigma\tau}(\boldsymbol{v}) \sqrt{a}\, d\omega \\ &\qquad = \int_\omega \Big(\int_{-\frac{L}{2}}^{\frac{L}{2}} \boldsymbol{f}_0\, d\xi \Big) \cdot \boldsymbol{v} \sqrt{a}\, d\omega, \quad \forall \boldsymbol{v} \in \mathcal{V}(\omega). \end{aligned}$$

Finally, the identity $\frac{E}{3}(g^{\alpha\beta\sigma\tau}(0)+2a^{\alpha\beta}a^{\sigma\tau})=a_I^{\alpha\beta\sigma\tau}$ allows to conclude that $\boldsymbol{u}^0$ is the solution of the formulation $(\mathcal{P}_\mathcal{M})$ considered in Theorem 3.3.

Remark 3.6 *In addition to the characterization of the solution $\boldsymbol{u}^0$ expressed in $(\mathcal{P}_\mathcal{M})$, we have obtained the identity (3.28) already given and discussed in Section 3.1, Remark 3.4.* ■

3.2.2. *Case $\mathcal{V}_0(\omega)\neq\{\mathbf{0}\}$ ($p=2$)*

When $\mathcal{V}_0(\omega)$ contains non-zero elements, by inspection the appropriate scaling factor is found to be $p=2$. This implies, since $\boldsymbol{u}^0$ satisfies $(\mathcal{P}_\mathcal{M})$ with $\boldsymbol{f}_0=\mathbf{0}$, that $e^0_{\alpha\beta}=\gamma_{\alpha\beta}(\boldsymbol{u}^0)=0$, hence $\boldsymbol{u}^0\in\mathcal{V}_0(\omega)$. Then, recalling (3.32), (3.31) and (3.27), we infer $p^0=0$ and $e^0_{33}=0$, so that we can summarize

$$e^0_{ij}=0,\qquad p^0=0. \tag{3.33}$$

The formulation (3.29) then becomes

$$\frac{E}{3}\int_\Omega\Big[2e^1_{33}e^{-1}_{33}(\boldsymbol{V})+4a^{\alpha\sigma}e^1_{\alpha3}e^{-1}_{\sigma3}(\boldsymbol{V})\Big]\sqrt{a}\,d\Omega+\int_\Omega p^1e^{-1}_{33}(\boldsymbol{V})\sqrt{a}\,d\Omega=0,\quad\forall\boldsymbol{V}\in\mathcal{V}(\Omega), \tag{3.34}$$

leading to

$$e^1_{\alpha3}=0,\qquad p^1+\frac{2E}{3}e^1_{33}=0\quad a.e.\text{ in }\Omega. \tag{3.35}$$

From $e^0_{\alpha3}=0$, we deduce that the function $U^1_{\alpha,3}=-(u^0_{3,\alpha}+2b^\sigma_\alpha u^0_\sigma)$ is independent of ξ. Assuming $\boldsymbol{U}^1\in\mathcal{V}(\Omega)$, there exists $\boldsymbol{u}^1\in\mathcal{V}(\omega)$ such that $U^1_\alpha=u^1_\alpha-\xi(u^0_{3,\alpha}+2b^\sigma_\alpha u^0_\sigma)$ and $U^1_3=u^1_3$ (we recall that $U^1_{3,3}=e^0_{33}=0$). In addition this implies that $\boldsymbol{u}^0$ is in fact in $\mathcal{V}_\mathcal{F}(\omega)$. Then, the cancellation of the coefficient of ε in $(\mathcal{P}_\Omega(\varepsilon))$ leads to

$$(\mathcal{P}^1)\begin{cases}\dfrac{E}{3}\displaystyle\int_\Omega\Big[B^{ijkl,1}e^1_{ij}e^{-1}_{kl}(\boldsymbol{V})+g^{ijkl}(0)\Big(e^2_{ij}e^{-1}_{kl}(\boldsymbol{V})+e^1_{ij}e^0_{kl}(\boldsymbol{V})\Big)\sqrt{a}\Big]d\Omega\\ \quad+\displaystyle\int_\Omega\Big[p^2a^{ij}e^{-1}_{ij}(\boldsymbol{V})\sqrt{a}+p^1\Big(B^{ij,1}e^{-1}_{ij}(\boldsymbol{V})+a^{ij}e^0_{ij}(\boldsymbol{V})\sqrt{a}\Big)\Big]d\Omega=0,\ \forall\boldsymbol{V}\in\mathcal{V}(\Omega),\\ \displaystyle\int_\Omega q\ (a^{\alpha\beta}e^1_{\alpha\beta}+e^1_{33})\sqrt{a}\,d\Omega=0,\quad\forall q\in L^2(\Omega).\end{cases}$$

The second equation of this system gives

$$a^{\alpha\beta}e^1_{\alpha\beta}+e^1_{33}=0\quad a.e.\text{ in }\Omega. \tag{3.36}$$

On the other hand, $e^1_{\alpha\beta}$ has the following remarkable expression

$$e^1_{\alpha\beta}=\gamma_{\alpha\beta}(\boldsymbol{u}^1)-\xi\rho_{\alpha\beta}(\boldsymbol{u}^0). \tag{3.37}$$

Then, using the equality $p^1=\frac{2E}{3}a^{\alpha\beta}e^1_{\alpha\beta}$ obtained from (3.35) and (3.36) in the first equation of $(\mathcal{P}^1)$ and letting $\boldsymbol{V}=\boldsymbol{u}^1\in\mathcal{V}(\omega)$, we obtain

$$\int_\omega a_I^{\alpha\beta\sigma\tau}\gamma_{\alpha\beta}(\boldsymbol{u}^1)\gamma_{\sigma\tau}(\boldsymbol{u}^1)\sqrt{a}\,d\omega=0, \tag{3.38}$$

which shows that $\boldsymbol{u}^1 \in \mathcal{V}_0(\omega)$. Moreover – using (3.22), (3.23), (3.25) and (3.35) – for general test functions this same first equation reduces to

$$\frac{E}{3}\int_\Omega \Big[g^{ijkl}(0)e^1_{ij}e^0_{kl}(\boldsymbol{V}) + 2e^2_{33}e^{-1}_{33}(\boldsymbol{V}) + 4a^{\alpha\sigma}e^2_{\alpha 3}e^{-1}_{\sigma 3}(\boldsymbol{V})\Big]\sqrt{a}\,d\Omega + \int_\Omega \Big[p^2 e^{-1}_{33}(\boldsymbol{V}) + p^1 a^{ij} e^0_{ij}(\boldsymbol{V})\Big]\sqrt{a}\,d\Omega = 0, \quad \forall \boldsymbol{V} \in \mathcal{V}(\Omega). \tag{3.39}$$

Given an arbitrary element $\boldsymbol{\eta}$ in the space $\mathcal{V}_{\mathcal{F}}(\omega)$, let $\boldsymbol{V}(\boldsymbol{\eta})$ be defined by

$$V_\alpha(\boldsymbol{\eta}) = \xi(\eta_{3,\alpha} + 2b^\lambda_\alpha \eta_\lambda), \quad V_3(\boldsymbol{\eta}) = 0. \tag{3.40}$$

With this test function the above equation gives

$$\frac{E}{3}\int_\Omega \Big[g^{ijkl}(0)e^1_{ij}e^0_{kl}(\boldsymbol{V}(\boldsymbol{\eta})) + 4a^{\alpha\sigma}e^2_{\alpha 3}(b^\lambda_\sigma \eta_\lambda + \frac{1}{2}\eta_{3,\sigma})\Big]\sqrt{a}\,d\Omega + \int_\Omega p^1 a^{ij} e^0_{ij}(\boldsymbol{V}(\boldsymbol{\eta}))\sqrt{a} d\Omega = 0, \quad \forall \boldsymbol{\eta} \in \mathcal{V}_{\mathcal{F}}(\omega), \tag{3.41}$$

which will be used below. We conclude the argument by the identification of the coefficient of ε^2 in $(\mathcal{P}_\Omega(\varepsilon))$. From the first variational equation we obtain

$$\begin{aligned}
&\frac{E}{3}\int_\Omega \Big[B^{ijkl,2}e^1_{ij}e^{-1}_{kl}(\boldsymbol{V}) + B^{ijkl,1}\Big(e^2_{ij}e^{-1}_{kl}(\boldsymbol{V}) + e^1_{ij}e^0_{kl}(\boldsymbol{V})\Big)\Big]d\Omega \\
&\quad + \frac{E}{3}\int_\Omega g^{ijkl}(0)\Big[e^3_{ij}e^{-1}_{kl}(\boldsymbol{V}) + e^2_{ij}e^0_{kl}(\boldsymbol{V}) + e^1_{ij}e^1_{kl}(\boldsymbol{V})\Big]\sqrt{a}\,d\Omega \\
&\quad + \int_\Omega \Big[p^3 a^{ij} e^{-1}_{ij}(\boldsymbol{V})\sqrt{a} + p^2\Big(B^{ij,1}e^{-1}_{ij}(\boldsymbol{V}) + a^{ij}e^0_{ij}(\boldsymbol{V})\sqrt{a}\Big)\Big]d\Omega \\
&\quad + \int_\Omega p^1\Big[B^{ij,2}e^{-1}_{ij}(\boldsymbol{V}) + B^{ij,1}e^0_{ij}(\boldsymbol{V}) + a^{ij}e^1_{ij}(\boldsymbol{V})\sqrt{a}\Big]d\Omega \\
&\qquad = \int_\Omega \boldsymbol{f}_2\cdot\boldsymbol{V}\sqrt{a}\,d\Omega, \quad \forall \boldsymbol{V}\in\mathcal{V}(\Omega).
\end{aligned}$$

For $\boldsymbol{V} = \boldsymbol{\eta} \in \mathcal{V}_{\mathcal{F}}(\omega)$, this equation reduces to

$$\frac{E}{3}\int_\Omega \Big[4a^{\alpha\sigma}e^2_{\alpha 3}(b^\lambda_\sigma\eta_\lambda + \frac{1}{2}\eta_{3,\sigma}) + g^{ijkl}(0)e^1_{ij}e^1_{kl}(\boldsymbol{\eta})\Big]\sqrt{a}\,d\Omega + \int_\Omega p^1 a^{ij}e^1_{ij}(\boldsymbol{\eta})\sqrt{a}\,d\Omega = \int_\Omega \boldsymbol{f}_2\cdot\boldsymbol{\eta}\sqrt{a}\,d\Omega, \quad \forall\boldsymbol{\eta}\in\mathcal{V}_{\mathcal{F}}(\omega).$$

Then, subtracting (3.41) we obtain

$$\frac{E}{3}\int_\Omega g^{ijkl}(0)e^1_{kl}\Big(e^1_{ij}(\boldsymbol{\eta}) - e^0_{ij}(\boldsymbol{V}(\boldsymbol{\eta}))\Big)\sqrt{a}\,d\Omega + \int_\Omega p^1 a^{ij}\Big(e^1_{ij}(\boldsymbol{\eta}) - e^0_{ij}(\boldsymbol{V}(\boldsymbol{\eta}))\Big)\sqrt{a}\,d\Omega = \int_\Omega \boldsymbol{f}_2\cdot\boldsymbol{\eta}\sqrt{a}\,d\Omega, \quad \forall\boldsymbol{\eta}\in\mathcal{V}_{\mathcal{F}}(\omega), \tag{3.42}$$

or equivalently, recalling (3.35),

$$\frac{E}{3}\int_\Omega g^{\alpha\beta\sigma\tau}(0)e^1_{\sigma\tau}\Big(e^1_{\alpha\beta}(\boldsymbol{\eta})-e^0_{\alpha\beta}(\boldsymbol{V}(\boldsymbol{\eta}))\Big)\sqrt{a}\,d\Omega + \int_\Omega p^1 a^{\alpha\beta}\Big(e^1_{\alpha\beta}(\boldsymbol{\eta})-e^0_{\alpha\beta}(\boldsymbol{V}(\boldsymbol{\eta}))\Big)\sqrt{a}\,d\Omega = \int_\Omega \boldsymbol{f}_2\cdot\boldsymbol{\eta}\sqrt{a}\,d\Omega, \quad \forall\boldsymbol{\eta}\in\mathcal{V}_{\mathcal{F}}(\omega). \tag{3.43}$$

Using the relations $e^1_{\sigma\tau} = -\xi\rho_{\sigma\tau}(\boldsymbol{u}^0)$ and $e^1_{\alpha\beta}(\boldsymbol{\eta}) - e^0_{\alpha\beta}(\boldsymbol{V}(\boldsymbol{\eta})) = -\xi\rho_{\alpha\beta}(\boldsymbol{\eta})$ for all $\boldsymbol{\eta}\in\mathcal{V}_{\mathcal{F}}(\omega)$, this equation becomes

$$\frac{EL^3}{36}\int_\omega g^{\alpha\beta\sigma\tau}(0)\rho_{\sigma\tau}(\boldsymbol{u}^0)\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,d\omega - \int_\Omega \xi p^1 a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}d\Omega = \int_\omega\Big(\int_{-\frac{L}{2}}^{\frac{L}{2}}\boldsymbol{f}_2\,d\xi\Big)\cdot\boldsymbol{\eta}\sqrt{a}\,d\omega, \quad \forall\boldsymbol{\eta}\in\mathcal{V}_{\mathcal{F}}(\omega). \tag{3.44}$$

Finally, using the relation $p^1 = \frac{2E}{3}a^{\alpha\beta}e^1_{\alpha\beta} = -\frac{2E}{3}\xi a^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{u}^0)$, the equation (3.44) implies

$$\frac{L^3}{12}\int_\omega a_I^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{u}^0)\rho_{\alpha\beta}(\boldsymbol{v})\sqrt{a}\,d\omega = \int_\omega\Big(\int_{-\frac{L}{2}}^{\frac{L}{2}}\boldsymbol{f}_2\,d\xi\Big)\cdot\boldsymbol{v}\sqrt{a}\,d\omega, \quad \forall\boldsymbol{v}\in\mathcal{V}_{\mathcal{F}}(\omega),$$

and since $\boldsymbol{u}^0$ is in $\mathcal{V}_{\mathcal{F}}(\omega)$, we infer that it is the solution of Problem $(\mathcal{P}_{\mathcal{F}})$, see Theorem 3.4.

Remark 3.7 *We assumed in this analysis* $\mathcal{V}_0(\omega)\neq\{\boldsymbol{0}\}$ *whereas the assumption used in Theorem 3.4 was* $\mathcal{V}_{\mathcal{F}}(\omega)\neq\{\boldsymbol{0}\}$*. In fact, if in the above limit model we have* $\mathcal{V}_{\mathcal{F}}(\omega)=\{\boldsymbol{0}\}$ *the solution* $\boldsymbol{u}^0$ *is zero, which contradicts the premises of the formal asymptotic analysis. This is similar to what happens with compressible shells, see Ref.11, and means that the formal analysis cannot account for the behavior of generalized membrane shells of the second kind.* ■

Remark 3.8 *In addition to the characterization of the solution* $\boldsymbol{u}^0$ *expressed in* $(\mathcal{P}_{\mathcal{F}})$*, we have obtained the identity (3.36) already given and discussed in Section 3.1, Remark 3.5.* ■

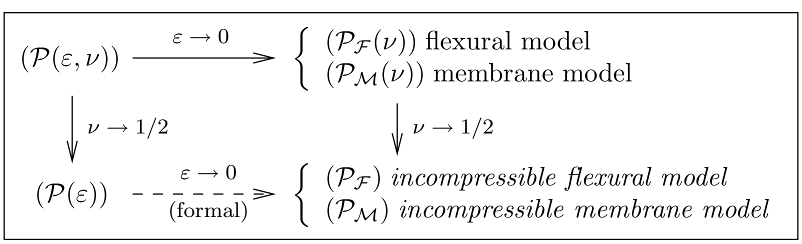

Figure 2: Convergence behaviors for 3D formulation

To conclude this section, we emphasize that the above asymptotic analyses constitute a preliminary justification of the concept of an “incompressible shell”. Indeed, in the 3D variational formulation we can consider the incompressible limit and the shell limit (namely, in the thickness parameter) and obtain a well-posed limit problem which does not depend on the order in which the limits are sought. We summarize these convergence behaviors in the commuting diagram shown in Figure 2. Like for compressible shells, the limit problem obtained is highly dependent on the contents of the spaces $\mathcal{V}_0(\omega)$ and $\mathcal{V}_{\mathcal{F}}(\omega)$, and in fact we can say that the candidate “incompressible shell” models are simply obtained by making “$\nu = 1/2$” in the (compressible) limit shell models. As a consequence, our results also provide a preliminary *justification of the incompressible limit in classical shell models* (valid for finite thicknesses) – see e.g. Refs.4,9,11,15 and their references for examples thereof – for which $\nu = 1/2$ does not correspond to a singularity, since the thickness limit and the incompressible limit clearly commute for these models and give final limit problems “consistent” with the above “incompressible shell” models, see Refs.9,15. Of course, in order to have a more complete justification of these concepts, actual convergence results (i.e., not only a formal analysis) would be needed for the shell limit behavior of an incompressible material.

4. Asymptotic analysis of the “3D-shell model”

Classical shell models are based on the Reissner-Mindlin (or Kirchhoff-Love) kinematical assumption, and on a plane stress assumption, see Refs.4,10,11 and references therein. In fact, as already discussed it is the plane stress assumption that cancels the $\nu = 1/2$ singularity in these models.

For various reasons, shell models based on higher-order kinematical assumptions may sometimes be preferred, see e.g. Refs.5,7. When a quadratic kinematical assumption is used – namely, the displacement profile across the thickness of the structure is assumed to be quadratic – the plane stress assumption is no longer required (in fact, it is then altogether inadequate), see Ref.7. Dispensing with this assumption is a significant advantage, in particular in large strain analysis. However, a consequence is that the incompressible singularity is present in these models, which raises the question of how adequately they can represent incompressible conditions. In the light of the previous section, a relevant criterion to that purpose is whether or not we can exchange the incompressible and shell limits in the asymptotic behavior.

In this section we give the corresponding analysis for the 3D-shell model, a shell model based on a quadratic kinematical assumption presented and analysed in Ref.7 as regards the asymptotic behavior with respect to the thickness parameter.

4.1. *Overview of the 3D-shell model*

Let us summarize the formulation of the 3D-shell model as presented and analyzed in Ref.7. This shell model is obtained by restricting the space $\mathcal{V}(\Omega_t)$ to the following subspace

$$\tilde{\mathcal{V}}(\Omega_t) = \left\{ \boldsymbol{V}(\xi^1,\xi^2,\xi^3) = \boldsymbol{v}(\xi^1,\xi^2)+\xi^3\boldsymbol{\eta}(\xi^1,\xi^2)+(\xi^3)^2\boldsymbol{\rho}(\xi^1,\xi^2);\ (\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) \in (\mathcal{V}(\omega))^3 \right\}.$$

Note that we now use the *unscaled* coordinate ξ^3 that varies in $\left]-t/2, t/2\right[$. In the sequel we identify any element $\boldsymbol{V}$ of $\tilde{\mathcal{V}}(\Omega_t)$ with the corresponding element of $(\mathcal{V}(\omega))^3$, $(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho})$. Hence the variational formulation considered is now

$$(\tilde{\mathcal{P}}(\varepsilon,\nu)) \begin{cases} \tilde{\boldsymbol{U}}(\varepsilon,\nu) = (\tilde{\boldsymbol{u}}(\varepsilon,\nu), \tilde{\boldsymbol{\theta}}(\varepsilon,\nu), \tilde{\boldsymbol{\tau}}(\varepsilon,\nu)) \in (\mathcal{V}(\omega))^3, \\ \displaystyle\int_{\Omega_t} A^{ijkl} e_{ij}(\tilde{\boldsymbol{U}}(\varepsilon,\nu)) e_{kl}(\boldsymbol{V}) \sqrt{g}\, d\Omega = \int_{\Omega_t} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{V} \sqrt{g}\, d\Omega, \quad \forall \boldsymbol{V} \in (\mathcal{V}(\omega))^3, \end{cases}$$

and the components of the strain tensor have the following expressions (see Ref.7)

$$\begin{cases} e_{\alpha\beta}(\boldsymbol{V}) = \gamma_{\alpha\beta}(\boldsymbol{v}) + \xi^3 \chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{\eta}) + (\xi^3)^2 k_{\alpha\beta}(\boldsymbol{\eta},\boldsymbol{\rho}) + (\xi^3)^3 l_{\alpha\beta}(\boldsymbol{\rho}), \\ e_{\alpha 3}(\boldsymbol{V}) = \zeta_\alpha(\boldsymbol{v},\boldsymbol{\eta}) + \xi^3 m_\alpha(\boldsymbol{\eta},\boldsymbol{\rho}) + (\xi^3)^2 n_\alpha(\boldsymbol{\rho}), \\ e_{33}(\boldsymbol{V}) = \delta(\boldsymbol{\eta}) + \xi^3 p(\boldsymbol{\rho}), \end{cases} \tag{4.1}$$

where

$$\begin{cases} \gamma_{\alpha\beta}(\boldsymbol{v}) = \dfrac{1}{2}(v_{\alpha|\beta} + v_{\beta|\alpha}) - b_{\alpha\beta} v_3, \\ \chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{\eta}) = \dfrac{1}{2}(\eta_{\alpha|\beta} + \eta_{\beta|\alpha} - b_\alpha^\lambda v_{\lambda|\beta} - b_\beta^\lambda v_{\lambda|\alpha}) - b_{\alpha\beta}\eta_3 + c_{\alpha\beta} v_3, \\ k_{\alpha\beta}(\boldsymbol{\eta},\boldsymbol{\rho}) = \dfrac{1}{2}(\rho_{\alpha|\beta} + \rho_{\beta|\alpha} - b_\alpha^\lambda \eta_{\lambda|\beta} - b_\beta^\lambda \eta_{\lambda|\alpha}) - b_{\alpha\beta}\rho_3 + c_{\alpha\beta}\eta_3, \\ l_{\alpha\beta}(\boldsymbol{\rho}) = -\dfrac{1}{2}(b_\alpha^\lambda \rho_{\lambda|\beta} + b_\beta^\lambda \rho_{\lambda|\alpha}) + c_{\alpha\beta}\rho_3, \end{cases} \tag{4.2}$$

$$\begin{cases} \zeta_\alpha(\boldsymbol{v},\boldsymbol{\eta}) = \dfrac{1}{2}(\eta_\alpha + b_\alpha^\lambda v_\lambda + v_{3,\alpha}), \\ m_\alpha(\boldsymbol{\eta},\boldsymbol{\rho}) = \dfrac{1}{2}(2\rho_\alpha + \eta_{3,\alpha}), \\ n_\alpha(\boldsymbol{\rho}) = \dfrac{1}{2}(-b_\alpha^\lambda \rho_\lambda + \rho_{3,\alpha}), \end{cases} \tag{4.3}$$

and

$$\delta(\boldsymbol{\eta}) = \eta_3, \quad p(\boldsymbol{\rho}) = 2\rho_3. \tag{4.4}$$

Remark 4.1 *The tensor components $\gamma_{\alpha\beta}$ correspond to the membrane tensor defined in (3.9), while the tensor associated with $\chi_{\alpha\beta}$ is a generalization of the bending strain tensor used in classical shell models when shear deformations are present, see Refs.7,10.* ■

In this case, the space that crucially determines the asymptotic behavior of the shell model (with respect to the thickness parameter) is the subspace of pure bending displacements defined as (see Ref.7)

$$\mathcal{V}_{00}(\omega) = \left\{ (\boldsymbol{v},\boldsymbol{\eta}) \in (\mathcal{V}(\omega))^2;\ \gamma_{\alpha\beta}(\boldsymbol{v}) = 0, \zeta_\alpha(\boldsymbol{v},\boldsymbol{\eta}) = 0, \delta(\boldsymbol{\eta}) = 0 \right\}, \tag{4.5}$$

imposing 6 linear relations to the 6 components $(v_i, \eta_i)_{i=1,3}$. We will say that "pure bending is inhibited" when $\mathcal{V}_{00}(\omega) = \{\mathbf{0}, \mathbf{0}\}$. Whether or not pure bending is inhibited depends on the geometry of the midsurface and on the boundary conditions, see Ref.10 and references therein. Before considering the asymptotic behavior of Problem $(\tilde{\mathcal{P}}(\varepsilon, \nu))$, we introduce the following symmetric bilinear forms

$$A_{\mathrm{m}}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{v}, \boldsymbol{\eta}) = \mu A^d_{\mathrm{m}}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{v}, \boldsymbol{\eta}) + \lambda A^v_{\mathrm{m}}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{v}, \boldsymbol{\eta}), \tag{4.6}$$

with

$$A^d_{\mathrm{m}}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{v}, \boldsymbol{\eta}) = \int_\omega \Big(g^{\alpha\beta\sigma\tau}(0) \gamma_{\alpha\beta}(\boldsymbol{u}) \gamma_{\sigma\tau}(\boldsymbol{v}) + 4 a^{\alpha\beta} \zeta_\alpha(\boldsymbol{u}, \boldsymbol{\theta}) \zeta_\beta(\boldsymbol{v}, \boldsymbol{\eta}) + 2\delta(\boldsymbol{\theta})\delta(\boldsymbol{\eta}) \Big) \sqrt{a}\, d\omega, \tag{4.7}$$

and

$$A^v_{\mathrm{m}}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{v}, \boldsymbol{\eta}) = \int_\omega \Big(a^{\alpha\beta} a^{\sigma\tau} \gamma_{\alpha\beta}(\boldsymbol{u}) \gamma_{\sigma\tau}(\boldsymbol{v}) + a^{\alpha\beta} \big[\gamma_{\alpha\beta}(\boldsymbol{u}) \delta(\boldsymbol{\eta}) + \gamma_{\alpha\beta}(\boldsymbol{v}) \delta(\boldsymbol{\theta}) \big] + \delta(\boldsymbol{\theta}) \delta(\boldsymbol{\eta}) \Big) \sqrt{a}\, d\omega. \tag{4.8}$$

Similarly, we introduce

$$A_{\mathrm{f}}(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\tau}; \boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) = \mu A^d_{\mathrm{f}}(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\tau}; \boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) + \lambda A^v_{\mathrm{f}}(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\tau}; \boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}), \tag{4.9}$$

with

$$A^d_{\mathrm{f}}(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\tau}; \boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) = \frac{L^2}{12} \int_\omega \Big(g^{\alpha\beta\sigma\tau}(0) \chi_{\alpha\beta}(\boldsymbol{u}, \boldsymbol{\theta}) \chi_{\sigma\tau}(\boldsymbol{v}, \boldsymbol{\eta}) + 4 a^{\alpha\beta} m_\alpha(\boldsymbol{\theta}, \boldsymbol{\tau}) m_\beta(\boldsymbol{\eta}, \boldsymbol{\rho}) + 2p(\boldsymbol{\tau}) p(\boldsymbol{\rho}) \Big) \sqrt{a}\, d\omega, \tag{4.10}$$

and

$$A^v_{\mathrm{f}}(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\tau}; \boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) = \frac{L^2}{12} \int_\omega \Big(a^{\alpha\beta} a^{\sigma\tau} \chi_{\alpha\beta}(\boldsymbol{u}, \boldsymbol{\theta}) \chi_{\sigma\tau}(\boldsymbol{v}, \boldsymbol{\eta}) + a^{\alpha\beta} \big[\chi_{\alpha\beta}(\boldsymbol{u}, \boldsymbol{\theta}) p(\boldsymbol{\rho}) + \chi_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{\eta}) p(\boldsymbol{\tau}) \big] + p(\boldsymbol{\tau}) p(\boldsymbol{\rho}) \Big) \sqrt{a}\, d\omega. \tag{4.11}$$

Finally, we assume that the load distribution can be written in the form

$$\boldsymbol{f}^\varepsilon(\boldsymbol{\xi}) = \varepsilon^p \Big(\boldsymbol{f}_p(\xi^1, \xi^2) + \xi^3 \boldsymbol{l}_p(\xi^1, \xi^2, \xi^3) \Big), \tag{4.12}$$

for some $p \in \mathbb{N}$ (without summation on p), and where $\boldsymbol{f}_p$ is in $(L^2(\omega))^3$ while $\boldsymbol{l}_p$ is in $(L^\infty(\Omega_t))^3$ and uniformly bounded with respect to t.

4.2. *Analysis of* $\lim_{\nu \to 1/2} \lim_{\varepsilon \to 0} (\tilde{\mathcal{P}}(\varepsilon, \nu))$

We then start by considering the limit in the parameter ε. Similarly to the 3D case, two very different situations occur depending on the contents of the space $\mathcal{V}_{00}(\omega)$, namely depending on whether or not pure bending is inhibited. When pure bending is inhibited, we introduce the norm (defined on $(\mathcal{V}(\omega))^2$)

$$\|\boldsymbol{v},\boldsymbol{\eta}\|_{\mathrm{m}} = A_{\mathrm{m}}(\boldsymbol{v},\boldsymbol{\eta};\boldsymbol{v},\boldsymbol{\eta})^{\frac{1}{2}}, \tag{4.13}$$

which is equivalent to

$$\Big(\sum_{\alpha,\beta}\|\gamma_{\alpha\beta}(\boldsymbol{v})\|_0^2 + \sum_{\alpha}\|\zeta_\alpha(\boldsymbol{v},\boldsymbol{\eta})\|_0^2 + \|\delta(\boldsymbol{\eta})\|_0^2\Big)^{\frac{1}{2}}. \tag{4.14}$$

We also introduce the space $\mathcal{V}_{\mathrm{m}}(\omega)$ defined as the completion of $(\mathcal{V}(\omega))^2$ with respect to the norm $\|.\|_{\mathrm{m}}$. The convergence behavior of Problem $(\tilde{\mathcal{P}}(\varepsilon,\nu))$ is then obtained in this space with $p=0$ and assuming that $\boldsymbol{f}_0 \in (\mathcal{V}_{\mathrm{m}}(\omega))^{'}$, namely,

$$|\int_\omega \boldsymbol{f}_0\cdot\boldsymbol{v}dS| \leq C\|\boldsymbol{v},\boldsymbol{\eta}\|_{\mathrm{m}}, \quad \forall(\boldsymbol{v},\boldsymbol{\eta})\in\mathcal{V}_{\mathrm{m}}(\omega). \tag{4.15}$$

Theorem 4.1 *Assuming that (4.15) holds, when ε tends to 0 the couple $\big(\tilde{\boldsymbol{u}}(\varepsilon,\nu)+\frac{t^2}{12}\tilde{\boldsymbol{\tau}}(\varepsilon,\nu),\tilde{\boldsymbol{\theta}}(\varepsilon,\nu)\big)$ – obtained from $\tilde{\boldsymbol{U}}(\varepsilon,\nu)$ solution of $(\tilde{\mathcal{P}}(\varepsilon,\nu))$ – converges weakly in $\mathcal{V}_{\mathrm{m}}(\omega)$ to $(\tilde{\boldsymbol{u}}^{\mathrm{m}}(\nu),\tilde{\boldsymbol{\theta}}^{\mathrm{m}}(\nu))$ solution of*

$$(\tilde{\mathcal{P}}_{\mathcal{M}}(\nu))\begin{cases}(\tilde{\boldsymbol{u}}^{\mathrm{m}}(\nu),\tilde{\boldsymbol{\theta}}^{\mathrm{m}}(\nu))\in\mathcal{V}_{\mathrm{m}}(\omega),\\ A_{\mathrm{m}}(\tilde{\boldsymbol{u}}^{\mathrm{m}}(\nu),\tilde{\boldsymbol{\theta}}^{\mathrm{m}}(\nu);\boldsymbol{v},\boldsymbol{\eta}) = \displaystyle\int_\omega \boldsymbol{f}_0\cdot\boldsymbol{v}\sqrt{a}\,d\omega, \quad \forall(\boldsymbol{v},\boldsymbol{\eta})\in\mathcal{V}_{\mathrm{m}}(\omega).\end{cases}$$

■

See Ref.7 for the proof.

Remark 4.2 *Considering arbitrary test functions $\boldsymbol{\eta}$ in $(\tilde{\mathcal{P}}_{\mathcal{M}}(\nu))$, we obtain the relations*

$$\lambda a^{\alpha\beta}\gamma_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{m}}(\nu)) + (\lambda+2\mu)\delta(\tilde{\boldsymbol{\theta}}^{\mathrm{m}}(\nu)) = 0, \qquad \zeta_\alpha(\tilde{\boldsymbol{u}}^{\mathrm{m}}(\nu),\tilde{\boldsymbol{\theta}}^{\mathrm{m}}(\nu)) = 0 \quad \textit{a.e. in } \omega, \tag{4.16}$$

which can be compared with (3.14). ■

By contrast, when pure bending is not inhibited we introduce the norm

$$\|\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}\|_{\mathrm{f}} = \Big(\|\boldsymbol{v}\|_1^2 + \|\underline{\boldsymbol{\eta}}\|_1^2 + \|\eta_3\|_0^2 + \|\rho_3\|_0^2 + \|\underline{\boldsymbol{\rho}} + \frac{1}{2}\underline{\nabla}\eta_3\|_0^2\Big)^{\frac{1}{2}}, \tag{4.17}$$

where $\underline{\boldsymbol{\eta}}$ and $\underline{\boldsymbol{\rho}}$ represent the tangential parts (with respect to the midsurface) of the vectors $\boldsymbol{\eta}$ and $\boldsymbol{\rho}$, and $\underline{\nabla}$ accordingly denotes the surface gradient. This norm is clearly weaker than the H^1-norm $\|\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}\|_1$. Hence, we introduce the space $\mathcal{V}_{\mathrm{f}}(\omega)$

defined as the completion of $(\mathcal{V}(\omega))^3$ for this norm, and also the space $\mathcal{V}_{0\mathrm{f}}(\omega)$ defined as the completion of the space

$$\mathcal{V}_{00}^{\sharp}(\omega) = \{(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in (\mathcal{V}(\omega))^3;\ (\boldsymbol{v}, \boldsymbol{\eta}) \in \mathcal{V}_{00}(\omega)\}, \tag{4.18}$$

for the same norm. Taking $p = 2$ for Problem $(\tilde{\mathcal{P}}(\varepsilon,\nu))$, we then have the following convergence result (Ref.7).

Theorem 4.2 *When ε tends to 0, the solution $\tilde{\boldsymbol{U}}(\varepsilon,\nu)$ of Problem $(\tilde{\mathcal{P}}(\varepsilon,\nu))$ converges weakly in $\mathcal{V}_{\mathrm{f}}(\omega)$ to $\tilde{\boldsymbol{U}}^{\mathrm{f}}(\nu)$ solution of*

$$(\tilde{\mathcal{P}}_{\mathcal{F}}(\nu)) \begin{cases} \tilde{\boldsymbol{U}}^{\mathrm{f}}(\nu) = (\tilde{\boldsymbol{u}}^{\mathrm{f}}(\nu), \tilde{\boldsymbol{\theta}}^{\mathrm{f}}(\nu), \tilde{\boldsymbol{\tau}}^{\mathrm{f}}(\nu)) \in \mathcal{V}_{0\mathrm{f}}(\omega), \\ A_{\mathrm{f}}(\tilde{\boldsymbol{u}}^{\mathrm{f}}(\nu), \tilde{\boldsymbol{\theta}}^{\mathrm{f}}(\nu), \tilde{\boldsymbol{\tau}}^{\mathrm{f}}(\nu); \boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) = \displaystyle\int_\omega \boldsymbol{f}_2 \cdot \boldsymbol{v}\sqrt{a}\, d\omega, \quad \forall (\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \mathcal{V}_{0\mathrm{f}}(\omega). \end{cases}$$

■

Remark 4.3 *Considering arbitrary test functions $\boldsymbol{\rho}$ in $(\tilde{\mathcal{P}}_{\mathcal{F}}(\nu))$, we obtain the relations*

$$\lambda a^{\alpha\beta}\chi_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{f}}(\nu), \tilde{\boldsymbol{\theta}}^{\mathrm{f}}(\nu)) + (\lambda + 2\mu)p(\tilde{\boldsymbol{\tau}}^{\mathrm{f}}(\nu)) = 0, \qquad m_\alpha(\tilde{\boldsymbol{\theta}}^{\mathrm{f}}(\nu), \tilde{\boldsymbol{\tau}}^{\mathrm{f}}(\nu)) = 0 \quad \textit{a.e. in } \omega, \tag{4.19}$$

which can be compared with (3.17). ■

As discussed in Ref.7, we can use (4.16) to eliminate $\tilde{\boldsymbol{\theta}}^{\mathrm{m}}(\nu)$ in the formulation $(\tilde{\mathcal{P}}_{\mathcal{M}}(\nu))$. We then obtain

$$\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{m}}(\nu))\gamma_{\sigma\tau}(\boldsymbol{v})\sqrt{a}\, d\omega = \int_\omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\sqrt{a}\, d\omega, \tag{4.20}$$

to be compared with $(\mathcal{P}_{\mathcal{M}}(\nu))$. Likewise, for non-inhibited pure bending we can use (4.19) to eliminate $\tilde{\boldsymbol{\theta}}^{\mathrm{f}}(\nu)$ and $\tilde{\boldsymbol{\tau}}^{\mathrm{f}}(\nu)$ in the formulation $(\tilde{\mathcal{P}}_{\mathcal{F}}(\nu))$, which leads to

$$\frac{L^2}{12}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{f}}(\nu))\rho_{\sigma\tau}(\boldsymbol{v})\sqrt{a}\, d\omega = \int_\omega \boldsymbol{f}_2 \cdot \boldsymbol{v}\sqrt{a}\, d\omega, \tag{4.21}$$

to be compared with $(\mathcal{P}_{\mathcal{F}}(\nu))$. In addition, the space of pure bending displacements $\mathcal{V}_{00}(\omega)$ is very closely related to the space $\mathcal{V}_{\mathcal{F}}(\omega)$, see Refs.7,10. Hence we may say that the formulations $(\mathcal{P}(\varepsilon,\nu))$ and $(\tilde{\mathcal{P}}(\varepsilon,\nu))$ are asymptotically equivalent (with respect to the parameter ε). As a consequence, at this stage the analysis of the convergence behavior with respect to the parameter ν is similar to that performed in Section 3.1 and very straightforward. Hence we state the results and omit the proofs.

We first consider the case of inhibited pure bending.

Theorem 4.3 *When ν tends to $1/2$, the solution $(\tilde{\boldsymbol{u}}^{\mathrm{m}}(\nu), \tilde{\boldsymbol{\theta}}^{\mathrm{m}}(\nu))$ of $(\tilde{\mathcal{P}}_{\mathcal{M}}(\nu))$ con-*

verges (strongly) in $\mathcal{V}_{\mathrm{m}}(\omega)$ *to* $(\tilde{\boldsymbol{u}}^{\mathrm{m}}, \tilde{\boldsymbol{\theta}}^{\mathrm{m}})$, *solution of*

$$(\tilde{\mathcal{P}}_{\mathcal{M}}) \begin{cases} (\tilde{\boldsymbol{u}}^{\mathrm{m}}, \tilde{\boldsymbol{\theta}}^{\mathrm{m}}) \in \mathcal{V}_{\mathrm{m}}(\omega), \\ \displaystyle\int_\omega \Big(a_I^{\alpha\beta\sigma\tau} \gamma_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{m}}) \gamma_{\sigma\tau}(\boldsymbol{v}) + \frac{4E}{3} \zeta_\alpha(\tilde{\boldsymbol{u}}^{\mathrm{m}}, \tilde{\boldsymbol{\theta}}^{\mathrm{m}}) \zeta_\beta(\boldsymbol{v}, \boldsymbol{\eta}) \Big) \sqrt{a}\, d\omega \\ \qquad\qquad = \displaystyle\int_\omega \boldsymbol{f}_0 \cdot \boldsymbol{v} \sqrt{a}\, d\omega, \quad \forall (\boldsymbol{v}, \boldsymbol{\eta}) \in \mathcal{V}_{\mathrm{m}}(\omega), \\ \delta(\tilde{\boldsymbol{\theta}}^{\mathrm{m}}) = -a^{\alpha\beta} \gamma_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{m}}), \quad \text{a.e. in } \omega. \end{cases}$$

■

Remark 4.4 *As already mentioned, we can eliminate the shear terms from the above variational formulation by considering arbitrary test functions* $\boldsymbol{\eta}$, *which gives*

$$\zeta_\alpha(\tilde{\boldsymbol{u}}^{\mathrm{m}}, \tilde{\boldsymbol{\theta}}^{\mathrm{m}}) = 0, \tag{4.22}$$

which holds in $L^2(\omega)$ *(or a.e. in* ω*). Hence we obtain the analogue of* (4.20) *for* $\nu = 1/2$, *viz.*

$$\int_\omega a_I^{\alpha\beta\sigma\tau} \gamma_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{m}}) \gamma_{\sigma\tau}(\boldsymbol{v}) \sqrt{a}\, d\omega = \int_\omega \boldsymbol{f}_0 \cdot \boldsymbol{v} \sqrt{a}\, d\omega. \tag{4.23}$$

In addition – when $\mathcal{V}_{\mathrm{m}}(\omega)$ *is a distribution space –* (4.22) *provides an explicit expression for the tangential part of* $\tilde{\boldsymbol{\theta}}^{\mathrm{m}}$, *namely,*

$$\tilde{\theta}_\alpha^{\mathrm{m}} = -(\tilde{u}_{3,\alpha}^{\mathrm{m}} + b_\alpha^\lambda \tilde{u}_\lambda^{\mathrm{m}}). \tag{4.24}$$

As for the transverse part, the last equation of $(\tilde{\mathcal{P}}_{\mathcal{M}})$ *always gives an explicit expression for this quantity (since* δ *is in* $L^2(\omega)$ *for any element of* $\mathcal{V}_{\mathrm{m}}(\omega)$*), viz.*

$$\tilde{\theta}_3^{\mathrm{m}} = -a^{\alpha\beta} \gamma_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{m}}), \tag{4.25}$$

which holds in $L^2(\omega)$. ■

When pure bending is not inhibited, the convergence result is as follows.

Theorem 4.4 *When* ν *tends to* $1/2$, *the solution* $\tilde{\boldsymbol{U}}^{\mathrm{f}}(\nu)$ *of* $(\tilde{\mathcal{P}}_{\mathcal{F}}(\nu))$ *converges (strongly) in* $\mathcal{V}_{\mathrm{f}}(\omega)$ *to* $\tilde{\boldsymbol{U}}^{\mathrm{f}}$, *solution of*

$$(\tilde{\mathcal{P}}_{\mathcal{F}}) \begin{cases} \tilde{\boldsymbol{U}}^{\mathrm{f}} = (\tilde{\boldsymbol{u}}^{\mathrm{f}}, \tilde{\boldsymbol{\theta}}^{\mathrm{f}}, \tilde{\boldsymbol{\tau}}^{\mathrm{f}}) \in \mathcal{V}_{0\mathrm{f}}(\omega), \\ \displaystyle\frac{L^2}{12} \int_\omega a_I^{\alpha\beta\sigma\tau} \chi_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{f}}, \tilde{\boldsymbol{\theta}}^{\mathrm{f}}) \chi_{\sigma\tau}(\boldsymbol{v}, \boldsymbol{\eta}) \sqrt{a}\, d\omega = \int_\omega \boldsymbol{f}_2 \cdot \boldsymbol{v} \sqrt{a}\, d\omega, \quad \forall (\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \mathcal{V}_{0\mathrm{f}}(\omega), \\ p(\tilde{\boldsymbol{\tau}}^{\mathrm{f}}) + a^{\alpha\beta} \chi_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{f}}, \tilde{\boldsymbol{\theta}}^{\mathrm{f}}) = 0, \quad m_\alpha(\tilde{\boldsymbol{\theta}}^{\mathrm{f}}, \tilde{\boldsymbol{\tau}}^{\mathrm{f}}) = 0, \quad \text{a.e. in } \omega. \end{cases}$$

■

Remark 4.5 *Recalling that* $\zeta_\alpha(\tilde{\boldsymbol{u}}^{\mathrm{f}}, \tilde{\boldsymbol{\theta}}^{\mathrm{f}}) = 0$ *and* $\delta(\tilde{\boldsymbol{\theta}}^{\mathrm{f}}) = 0$ *are constraints contained in the definition of* $\mathcal{V}_{0\mathrm{f}}(\omega)$, *we can use these equations to eliminate* $\tilde{\boldsymbol{\theta}}^{\mathrm{f}}$ *from the variational formulation, which gives*

$$\frac{L^2}{12} \int_\omega a_I^{\alpha\beta\sigma\tau} \rho_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{f}}) \rho_{\sigma\tau}(\boldsymbol{v}) \sqrt{a}\, d\omega = \int_\omega \boldsymbol{f}_2 \cdot \boldsymbol{v} \sqrt{a}\, d\omega, \tag{4.26}$$

which is (4.21) *written for* $\nu = 1/2$. *In addition, the last line of* $(\tilde{\mathcal{P}}_{\mathcal{F}})$ *provides explicit expressions for* $\tilde{\boldsymbol{\tau}}^{\mathrm{f}}$, *viz.*

$$\tilde{\tau}^{\mathrm{f}}_{\alpha} = -\frac{1}{2}\tilde{\theta}^{\mathrm{f}}_{3,\alpha}, \quad \tilde{\tau}^{\mathrm{f}}_{3} = -\frac{1}{2}a^{\alpha\beta}\chi_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{f}}, \tilde{\boldsymbol{\theta}}^{\mathrm{f}}), \tag{4.27}$$

where the latter $(\tilde{\tau}^{\mathrm{f}}_3)$ *holds in* $L^2(\omega)$, *while the former* $(\tilde{\tau}^{\mathrm{f}}_{\alpha})$ *holds in the distribution sense (note that* $\mathcal{V}_{\mathrm{f}}(\omega)$ *is always a distribution space).* ■

We can obtain some insight regarding the enforcement of the incompressibility constraint in the above limit models by developing the quantity $g^{ij}e_{ij}(\boldsymbol{U})$. Using (4.1) and the following identity (see e.g. Ref.1),

$$g^{\alpha\beta} = \frac{a^{\alpha\beta} + \xi^3 B^{\alpha\beta} + (\xi^3)^2 C^{\alpha\beta}}{(1 - 2H\xi^3 + K(\xi^3)^2)^2}, \tag{4.28}$$

with

$$B^{\alpha\beta} = 2b^{\alpha\beta} - 4Ha^{\alpha\beta}, \quad C^{\alpha\beta} = (4H^2 - K)a^{\alpha\beta} - 2Hb^{\alpha\beta}, \tag{4.29}$$

we obtain for any $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\tau}) \in \tilde{\mathcal{V}}(\Omega_t)$ the expression

$$g^{ij}e_{ij}(\boldsymbol{U}) = \frac{\sum_{k=0}^{5}(\xi^3)^k I_k(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\tau})}{\left(1 - 2H\xi^3 + K(\xi^3)^2\right)^2}, \tag{4.30}$$

with

$$\begin{cases} I_0(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\tau}) = a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{u}) + \delta(\boldsymbol{\theta}), \\ I_1(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\tau}) = a^{\alpha\beta}\chi_{\alpha\beta}(\boldsymbol{u}, \boldsymbol{\theta}) + B^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{u}) + p(\boldsymbol{\tau}) - 4H\delta(\boldsymbol{\theta}), \\ I_2(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\tau}) = a^{\alpha\beta}k_{\alpha\beta}(\boldsymbol{\theta}, \boldsymbol{\tau}) + B^{\alpha\beta}\chi_{\alpha\beta}(\boldsymbol{u}, \boldsymbol{\theta}) + C^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{u}) + (4H^2 + 2K)\delta(\boldsymbol{\theta}) - 4Hp(\boldsymbol{\tau}), \\ I_3(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\tau}) = a^{\alpha\beta}l_{\alpha\beta}(\boldsymbol{\tau}) + B^{\alpha\beta}k_{\alpha\beta}(\boldsymbol{\theta}, \boldsymbol{\tau}) + C^{\alpha\beta}\chi_{\alpha\beta}(\boldsymbol{u}, \boldsymbol{\theta}) + (4H^2 + 2K)p(\boldsymbol{\tau}) - 4HK\delta(\boldsymbol{\theta}), \\ I_4(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\tau}) = B^{\alpha\beta}l_{\alpha\beta}(\boldsymbol{\tau}) + C^{\alpha\beta}k_{\alpha\beta}(\boldsymbol{\theta}, \boldsymbol{\tau}) + K^2\delta(\boldsymbol{\theta}) - 4HKp(\boldsymbol{\tau}), \\ I_5(\boldsymbol{u}, \boldsymbol{\theta}, \boldsymbol{\tau}) = C^{\alpha\beta}l_{\alpha\beta}(\boldsymbol{\tau}) + K^2 p(\boldsymbol{\tau}). \end{cases}$$

In the non-inhibited case, since

$$p(\tilde{\boldsymbol{\tau}}^{\mathrm{f}}) + a^{\alpha\beta}\chi_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{f}}, \tilde{\boldsymbol{\theta}}^{\mathrm{f}}) = 0, \tag{4.31}$$

for the solution of $(\tilde{\mathcal{P}}_{\mathcal{F}})$, using the constraints contained in the definition of $\mathcal{V}_{0\mathrm{f}}(\omega)$ we infer that the first two terms of the expansion of $g^{ij}e_{ij}(\tilde{\boldsymbol{U}}^{\mathrm{f}})$ vanish, hence we can symbolically write "$g^{ij}e_{ij}(\tilde{\boldsymbol{U}}^{\mathrm{f}}) = O((\xi^3)^2)$". By contrast, in the inhibited case only the first term vanish for the limit solution, hence we write "$g^{ij}e_{ij}(\tilde{\boldsymbol{U}}^{\mathrm{m}}) = O(\xi^3)$". In the next section we investigate whether we can exchange the limits henceforth considered.

4.3. *Analysis of* $\lim_{\varepsilon\to 0}\lim_{\nu\to 1/2}(\tilde{\mathcal{P}}(\varepsilon, \nu))$

Considering $(\tilde{\mathcal{P}}(\varepsilon,\nu))$, when ν tends to $1/2$ we have a standard penalized formulation, and the solution $\tilde{\boldsymbol{U}}(\varepsilon,\nu)$ converges to the solution of the following constrained problem.

$$(\tilde{\mathcal{P}}(\varepsilon)) \begin{cases} \tilde{\boldsymbol{U}}(\varepsilon) = (\tilde{\boldsymbol{u}}(\varepsilon), \tilde{\boldsymbol{\theta}}(\varepsilon), \tilde{\boldsymbol{\tau}}(\varepsilon)) \in \mathcal{V}_I(\omega), \\ \dfrac{E}{3} \displaystyle\int_{\Omega_t} g^{ijkl} e_{ij}(\tilde{\boldsymbol{U}}(\varepsilon)) e_{kl}(\boldsymbol{V}) \sqrt{g}\, d\Omega = \int_{\Omega_t} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{V} \sqrt{g}\, d\Omega, \quad \forall \boldsymbol{V} \in \mathcal{V}_I(\omega), \end{cases}$$

where

$$\mathcal{V}_I(\omega) = \big\{ (\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in (\mathcal{V}(\omega))^3;\ I_k(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) = 0,\ k = 0, \ldots, 5 \big\} \tag{4.32}$$

represents, according to (4.30), the subspace of all elements of $(\mathcal{V}(\omega))^3$ that satisfy the incompressibility constraint *exactly*. In this definition, we thus have six constraints acting on the nine components $(v_i, \eta_i, \rho_i)_{i=1,3}$. In fact, these constraints are not independent, as we now show.

Lemma 4.1 *We have*

$$\mathcal{V}_I(\omega) = \big\{ (\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in (\mathcal{V}(\omega))^3;\ I_k(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) = 0,\ k = 0, \ldots, 3 \big\}. \tag{4.33}$$

■

Proof. For brevity we omit the recurring operand $(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho})$ in all expressions of the form $I_k(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho})$ in this proof. We will also use the notation

$$g_v = a^{\alpha\beta} v_{\alpha|\beta}, \quad b_v = b^{\alpha\beta} v_{\alpha|\beta}. \tag{4.34}$$

Then, using the identity

$$K a_{\alpha\beta} - 2H b_{\alpha\beta} + c_{\alpha\beta} = 0, \tag{4.35}$$

we have

$$a^{\alpha\beta} c_{\alpha\beta} = 4H^2 - 2K, \quad b^{\alpha\beta} c_{\alpha\beta} = 2H(4H^2 - 3K), \tag{4.36}$$

and we obtain

$$\begin{cases} a^{\alpha\beta} \gamma_{\alpha\beta}(\boldsymbol{v}) = g_v - 2H v_3, \\ a^{\alpha\beta} \chi_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{\eta}) = g_\eta - b_v - 2H\eta_3 + (4H^2 - 2K) v_3, \\ a^{\alpha\beta} k_{\alpha\beta}(\boldsymbol{\eta}, \boldsymbol{\rho}) = g_\rho - b_\eta - 2H\rho_3 + (4H^2 - 2K)\eta_3, \\ a^{\alpha\beta} l_{\alpha\beta}(\boldsymbol{\rho}) = -b_\rho + (4H^2 - 2K)\rho_3, \\ b^{\alpha\beta} \gamma_{\alpha\beta}(\boldsymbol{v}) = b_v - (4H^2 - 2K) v_3, \\ b^{\alpha\beta} \chi_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{\eta}) = b_\eta + K g_v - 2H b_v - (4H^2 - 2K)\eta_3 + 2H(4H^2 - 3K) v_3, \\ b^{\alpha\beta} k_{\alpha\beta}(\boldsymbol{\eta}, \boldsymbol{\rho}) = b_\rho + K g_\eta - 2H b_\eta - (4H^2 - 2K)\rho_3 + 2H(4H^2 - 3K)\eta_3, \\ b^{\alpha\beta} l_{\alpha\beta}(\boldsymbol{\rho}) = K g_\rho - 2H b_\rho + 2H(4H^2 - 3K)\rho_3. \end{cases} \tag{4.37}$$

With these expressions we can rewrite the constraints so that the independent variables (namely, $(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho})$) appear more explicitly, viz.

$$\begin{cases} I_0 = g_v - 2Hv_3 + \eta_3, \\ I_1 = b_v - (4H^2-2K)v_3 + g_\eta - 2H\eta_3 + 2\rho_3, \\ I_2 = (4H^2+K)g_v - 2Hb_v - 6HKv_3 - 4Hg_\eta + b_\eta \\ \qquad\qquad + 4(2H^2+K)\eta_3 + g_\rho - 10H\rho_3, \\ I_3 = -2HKg_v + Kb_v + 2K^2v_3 + (4H^2+K)g_\eta - 2Hb_\eta \\ \qquad\qquad - 10HK\eta_3 - 4Hg_\rho + b_\rho + 6(2H^2+K)\rho_3, \\ I_4 = -2HKg_\eta + Kb_\eta + 3K^2\eta_3 + (4H^2+K)g_\rho - 2Hb_\rho - 14HK\rho_3, \\ I_5 = K(-2Hg_\rho + b_\rho + 4K\rho_3), \end{cases} \tag{4.38}$$

and we can now verify that

$$\begin{cases} I_4 = (16H^4 - K^2 - 4H^2K)I_0 + 4H(K-2H^2)I_1 + (K-4H^2)I_2 - 2HI_3, \\ I_5 = K(-8H^3I_0 + (4H^2-K)I_1 + 2HI_2 + I_3). \end{cases} \tag{4.39}$$

□

Remark 4.6 *When $HK \neq 0$, $\{I_0 = I_1 = I_4 = I_5 = 0\}$ is a constraint system equivalent to $\{I_0 = I_1 = I_2 = I_3 = 0\}$. When $K = 0$ and $H \neq 0$, we can instead use $\{I_0 = I_1 = I_2 = I_4 = 0\}$. In the planar case ($H = K = 0$), $I_3 = 0$ is automatically satisfied and $\mathcal{V}_I(\omega)$ reduces to*

$$\mathcal{V}_I(\omega) = \{(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) \in (\mathcal{V}(\omega))^3;\ I_0 = g_v + \eta_3 = 0,\ I_1 = g_\eta + 2\rho_3 = 0,\ I_2 = g_\rho = 0\}. \tag{4.40}$$

■

Remark 4.7 *$\mathcal{V}_I(\omega)$ is a closed subspace of $(\mathcal{V}(\omega))^3$, hence $(\tilde{\mathcal{P}}(\varepsilon))$ is always well-posed.* ■

Since the variational problem $(\tilde{\mathcal{P}}(\varepsilon))$ is posed in the constrained subspace $\mathcal{V}_I(\omega)$, in the asymptotic analysis with respect to ε that we aim at performing, the analogue of the subspace $\mathcal{V}^\sharp_{00}(\omega)$ considered in Section 4.2 (see definition in (4.18)) is now

$$\mathcal{V}_{I0}(\omega) = \big\{(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) \in \mathcal{V}_I(\omega);\ \gamma_{\alpha\beta}(\boldsymbol{v}) = 0,\ \zeta_\alpha(\boldsymbol{v},\boldsymbol{\eta}) = 0,\ \delta(\boldsymbol{\eta}) = 0\big\}. \tag{4.41}$$

Here, due to the constraints $\gamma_{\alpha\beta}(\boldsymbol{v}) = 0$ and $\delta(\boldsymbol{\eta}) = 0$, the first incompressibility constraint "$I_0 = 0$" is automatically satisfied, hence there are "only" nine scalar constraints applying on the nine independent components $(v_i, \eta_i, \rho_i)_{i=1,3}$. In the following lemma we give a characterization of this subspace using the following set of simplified constraint operators:

$$\begin{cases} I'_1(\boldsymbol{\eta},\boldsymbol{\rho}) = g_\eta + 2\rho_3, \\ I'_2(\boldsymbol{\eta},\boldsymbol{\rho}) = Hg_\eta + b_\eta + g_\rho, \\ I'_3(\boldsymbol{\eta},\boldsymbol{\rho}) = -2Kg_\eta - 2Hg_\rho + b_\rho, \end{cases} \tag{4.42}$$

these expressions being derived by straightforward computations from (4.38) when taking into account the constraints characterizing pure bending displacements.

Lemma 4.2 *We have*

$$\mathcal{V}_{I0}(\omega) = \big\{(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in (\mathcal{V}(\omega))^3;\ (\boldsymbol{v}, \boldsymbol{\eta}) \in \mathcal{V}_{00}(\omega),\ I'_k(\boldsymbol{\eta}, \boldsymbol{\rho}) = 0,\ k = 1, 2, 3\big\}. \quad (4.43)$$

■

Similarly to the above compressible case, we will say that "incompressible pure bending is inhibited" when

$$(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \mathcal{V}_{I0}(\omega) \Rightarrow (\boldsymbol{v}, \boldsymbol{\eta}) = (\mathbf{0}, \mathbf{0}), \quad (4.44)$$

hence that "incompressible pure bending is not inhibited" when there exist some elements $(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \mathcal{V}_{I0}(\omega)$ with $(\boldsymbol{v}, \boldsymbol{\eta})$ non-zero. Here also, two very different types of asymptotic behavior will arise according to whether or not incompressible pure bending is inhibited, hence we now examine the two cases separately.

4.3.1. *Non-inhibited incompressible pure bending*

In this case, we define $\mathcal{V}_{I\mathrm{f}}(\omega)$ and $\mathcal{V}_{I0\mathrm{f}}(\omega)$ as the completions of $\mathcal{V}_I(\omega)$ and $\mathcal{V}_{I0}(\omega)$, respectively, for the norm $\|.\|_{\mathrm{f}}$. In order to establish the asymptotic behavior, we then need to assume that we have

$$\mathcal{V}_{I0\mathrm{f}}(\omega) = \mathcal{V}_{I\mathrm{f}}(\omega) \cap \mathcal{V}_{0\mathrm{f}}(\omega). \quad (4.45)$$

Note that this amounts to exchanging the orders of completion and intersection, hence we always have

$$\mathcal{V}_{I0\mathrm{f}}(\omega) \subset \mathcal{V}_{I\mathrm{f}}(\omega) \cap \mathcal{V}_{0\mathrm{f}}(\omega), \quad (4.46)$$

whereas the reverse inclusion is – of course – not true for all subspaces. For the particular subspaces considered, we cannot prove this reverse inclusion without making additional assumptions on the surface geometry and on the boundary conditions – although finding counterexamples appears to be extremely difficult also – hence we give some sufficient conditions in the Appendix.

The appropriate scaling for $(\tilde{\mathcal{P}}(\varepsilon))$ is then $p = 2$, and the asymptotic behavior is as follows.

Theorem 4.5 *Assuming that* (4.45) *holds, when* ε *tends to* 0 $(\tilde{\boldsymbol{u}}(\varepsilon), \tilde{\boldsymbol{\theta}}(\varepsilon), \tilde{\boldsymbol{\tau}}(\varepsilon))$ *solution of* $(\tilde{\mathcal{P}}(\varepsilon))$ *converges weakly in* $\mathcal{V}_{\mathrm{f}}(\omega)$ *to* $(\tilde{\boldsymbol{u}}^{\mathrm{f}*}, \tilde{\boldsymbol{\theta}}^{\mathrm{f}*}, \tilde{\boldsymbol{\tau}}^{\mathrm{f}*})$ *solution of*

$$(\tilde{\mathcal{P}}^{\star}_{\mathcal{F}}) \begin{cases} (\tilde{\boldsymbol{u}}^{\mathrm{f}*}, \tilde{\boldsymbol{\theta}}^{\mathrm{f}*}, \tilde{\boldsymbol{\tau}}^{\mathrm{f}*}) \in \mathcal{V}_{I0\mathrm{f}}(\omega), \\ \dfrac{E}{3} A^d_{\mathrm{f}}(\tilde{\boldsymbol{u}}^{\mathrm{f}*}, \tilde{\boldsymbol{\theta}}^{\mathrm{f}*}, \tilde{\boldsymbol{\tau}}^{\mathrm{f}*}; \boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) = \displaystyle\int_\omega \boldsymbol{f}_2 \cdot \boldsymbol{v} \sqrt{a}\, d\omega, \quad \forall (\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \mathcal{V}_{I0\mathrm{f}}(\omega). \end{cases}$$

■

Proof. The proof is analogous to that of Theorem 4.2 above (see Ref.7). In fact, nothing is changed in the required coercivity and boundedness properties when using only the "μ-part" of the bilinear form, and the rest of the argument carries over directly when considering the subspace $\mathcal{V}_I(\omega)$ instead of the whole space, when taking into account the assumption (4.45). □

In order to compare $(\tilde{\mathcal{P}}^\star_{\mathcal{F}})$ and $(\tilde{\mathcal{P}}_{\mathcal{F}})$, we now derive a slightly modified form of the former.

Theorem 4.6 *For any* $(\boldsymbol{u},\boldsymbol{\theta},\boldsymbol{\tau})$ *and* $(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho})$ *in* $\mathcal{V}_{I0\mathrm{f}}(\omega)$*, we have*

$$\frac{E}{3}A^d_{\mathrm{f}}(\boldsymbol{u},\boldsymbol{\theta},\boldsymbol{\tau};\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) = \frac{L^2}{12}\int_\omega\left(a_I^{\alpha\beta\sigma\tau}\chi_{\alpha\beta}(\boldsymbol{u},\boldsymbol{\theta})\chi_{\sigma\tau}(\boldsymbol{v},\boldsymbol{\eta}) + \frac{4E}{3}m_\alpha(\boldsymbol{\theta},\boldsymbol{\tau})m_\beta(\boldsymbol{\eta},\boldsymbol{\rho})\right)\sqrt{a}\,d\omega. \tag{4.47}$$

■

Proof. Take any $(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho})$ in $\mathcal{V}_{I0\mathrm{f}}(\omega)$. The constraint $\gamma_{\alpha\beta}(\boldsymbol{v}) = 0$ implies

$$b_v = (4H^2 - 2K)v_3. \tag{4.48}$$

Recalling that $\eta_3 = 0$ (by definition of $\mathcal{V}_{00}(\omega)$), the second equation of (4.37) gives

$$a^{\alpha\beta}\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{\eta}) = g_\eta, \tag{4.49}$$

which, combined with $I'_1(\boldsymbol{\eta},\boldsymbol{\rho}) = 0$ (recall Lemma 4.2) implies

$$p(\boldsymbol{\rho}) = -a^{\alpha\beta}\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{\eta}). \tag{4.50}$$

Since an analogous identity holds for $(\boldsymbol{u},\boldsymbol{\theta},\boldsymbol{\tau})$, the conclusion directly follows from the definitions of A^d_{f} and $a_I^{\alpha\beta\sigma\tau}$. □

Therefore, since we can also clearly use the bilinear form appearing in the right-hand side of (4.47) to obtain an alternative form of $(\tilde{\mathcal{P}}_{\mathcal{F}})$, we infer that the solutions of $(\tilde{\mathcal{P}}_{\mathcal{F}})$ and $(\tilde{\mathcal{P}}^\star_{\mathcal{F}})$ satisfy the same variational formulation, albeit posed in the spaces $\mathcal{V}_{0\mathrm{f}}$ and $\mathcal{V}_{I0\mathrm{f}}$, respectively. Therefore, since the latter is a subspace of the former, the two solutions coincide if and only if the solution of $(\tilde{\mathcal{P}}_{\mathcal{F}})$ happens to be in $\mathcal{V}_{I0\mathrm{f}}$, namely, if and only if it satisfies the additional constraints (recall Lemma 4.2)

$$I'_k(\tilde{\boldsymbol{\theta}}^{\mathrm{f}},\tilde{\boldsymbol{\tau}}^{\mathrm{f}}) = 0, \quad k = 1,2,3. \tag{4.51}$$

Since $a^{\alpha\beta}\chi_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{f}},\tilde{\boldsymbol{\theta}}^{\mathrm{f}}) = g_{\tilde{\theta}^{\mathrm{f}}}$ (see the above proof), and recalling that the formulation of $(\tilde{\mathcal{P}}_{\mathcal{F}})$ contains $p(\tilde{\boldsymbol{\tau}}^{\mathrm{f}}) + a^{\alpha\beta}\chi_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{f}},\tilde{\boldsymbol{\theta}}^{\mathrm{f}}) = 0$, the first constraint in (4.51) is – indeed – satisfied. Noting that $m_\alpha(\tilde{\boldsymbol{\theta}}^{\mathrm{f}},\tilde{\boldsymbol{\tau}}^{\mathrm{f}}) = 0$ and $\tilde{\theta}^{\mathrm{f}}_3 = 0$ imply $\tilde{\tau}^{\mathrm{f}}_\alpha = 0$, the last two constraints of (4.51) reduce to

$$Hg_{\tilde{\theta}^{\mathrm{f}}} + b_{\tilde{\theta}^{\mathrm{f}}} = 0, \quad Kg_{\tilde{\theta}^{\mathrm{f}}} = 0. \tag{4.52}$$

Clearly – unless of course we have a planar geometry – these constraints are not "likely to be satisfied" for *general* solutions of $(\tilde{\mathcal{P}}_{\mathcal{F}})$, although they may be fulfilled for specific instances of geometry, boundary conditions and loading, as shown in the following example.

Example We consider the surface described by the chart

$$\boldsymbol{\phi}(\xi^1,\xi^2) = \Big(\xi^1, R\sin\frac{\xi^2}{R}, R\cos\frac{\xi^2}{R}\Big), \quad \xi^1 \in]0,L[, \xi^2 \in]0,R\pi/2[, \tag{4.53}$$

namely, a part of a circular cylinder. The structure is clamped on the boundary $\xi^2 = 0$ and submitted to the distributed surface load given by $\boldsymbol{f}_2 = F(\xi^2)\boldsymbol{a}^3$ (independent of ξ^1). Clearly, in this case pure bending is not inhibited. For symmetry reasons, for the solution of $(\tilde{\mathcal{P}}_{\mathcal{F}})$ we have that $\tilde{u}_1^{\mathrm{f}} = 0$ on the line $\xi^1 = L/2$. This implies, using the constraints prevailing in $\mathcal{V}_{0\mathrm{f}}$,

$$\tilde{u}_1^{\mathrm{f}} = \tilde{\theta}_1^{\mathrm{f}} = \tilde{\theta}_3^{\mathrm{f}} = 0, \quad \tilde{u}_{2,1}^{\mathrm{f}} = \tilde{u}_{3,1}^{\mathrm{f}} = \tilde{\theta}_{2,1}^{\mathrm{f}} = 0, \tag{4.54}$$

and we only use test functions that satisfy the corresponding similar identities. In addition, we have

$$\tilde{u}_{2,2}^{\mathrm{f}} = -R^{-1}\tilde{u}_3^{\mathrm{f}}, \quad \tilde{\theta}_{2,2}^{\mathrm{f}} = -(R^{-2}\tilde{u}_3^{\mathrm{f}} + \tilde{u}_{3,22}^{\mathrm{f}}), \tag{4.55}$$

and this – with similar relations for the test functions – allows to obtain the following ODE with $\tilde{u}_3^{\mathrm{f}}$ as the only unknown

$$5R^{-4}\tilde{u}_3^{\mathrm{f}} + 8R^{-2}\tilde{u}_{3,22}^{\mathrm{f}} + 4\tilde{u}_{3,2222}^{\mathrm{f}} = \frac{3F}{E} \quad a.e. \quad in \quad \omega. \tag{4.56}$$

On the other hand, in this case (4.52) reduces to $\tilde{\theta}_{2,2}^{\mathrm{f}} = 0$, which imply when combined with (4.55)

$$\tilde{u}_{3,22}^{\mathrm{f}} = -R^{-2}\tilde{u}_3^{\mathrm{f}}, \tag{4.57}$$

hence, by (4.56),

$$\tilde{u}_3^{\mathrm{f}} = \frac{3F}{R^4E}. \tag{4.58}$$

Due to (4.57) this is only possible when

$$F + R^{-2}F_{,22} = 0, \tag{4.59}$$

which means that we need to enforce a condition on the loading for the solutions of $(\tilde{\mathcal{P}}_{\mathcal{F}})$ and $(\tilde{\mathcal{P}}_{\mathcal{F}}^{\star})$ to coincide.

Remark 4.8 *We point out that the two formulations $(\tilde{\mathcal{P}}_{\mathcal{F}})$ and $(\tilde{\mathcal{P}}_{\mathcal{F}}^{\star})$ are obtained under two different assumptions, namely that $\mathcal{V}_{00}(\omega) \neq \{(\mathbf{0},\mathbf{0})\}$ for $(\tilde{\mathcal{P}}_{\mathcal{F}})$, and for $(\tilde{\mathcal{P}}_{\mathcal{F}}^{\star})$ that there exist some elements $(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) \in \mathcal{V}_{I0}(\omega)$ with $(\boldsymbol{v},\boldsymbol{\eta})$ non-zero. In essence, the equivalence of these two assumptions is conditionned by the integrability of the constraint system $I'_k(\boldsymbol{\eta},\boldsymbol{\rho}) = 0,\ k = 1,2,3$ with respect to $\boldsymbol{\rho}$ (i.e. with $\boldsymbol{\eta}$ given). This integrability holds in particular under the assumptions (H1') or (H2) of the Appendix, see Lemma A.3.* ■

4.3.2. *Inhibited incompressible pure bending*

In this case some particular care is required in the definition of the limit space since the norm in which the convergence occurs does not provide any control on the field $\boldsymbol{\rho}$ of a triple $(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho})$, whereas the constraint system defining $\mathcal{V}_I(\omega)$ does involve $\boldsymbol{\rho}$. We thus define

$$\mathcal{V}_I^\natural(\omega) = \{(\boldsymbol{v},\boldsymbol{\eta}) \in (\mathcal{V}(\omega))^2;\ \exists\boldsymbol{\rho} \text{ such that } (\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) \in \mathcal{V}_I(\omega)\}. \tag{4.60}$$

By definition of inhibited incompressible pure bending, $\|\cdot\|_{\mathrm{m}}$ (as defined in (4.13)) gives a norm in $\mathcal{V}_I^\natural(\omega)$, and we can define $\mathcal{V}_{I\mathrm{m}}(\omega)$ as the completion of $\mathcal{V}_I^\natural(\omega)$ for this norm. Setting $p=0$ for Problem $(\tilde{\mathcal{P}}(\varepsilon))$, we then have the following convergence result.

Theorem 4.7 *Assuming that* (4.15) *holds, when* ε *tends to zero, the couple* $\big(\tilde{\boldsymbol{u}}(\varepsilon)+\frac{t^2}{12}\tilde{\boldsymbol{\tau}}(\varepsilon),\tilde{\boldsymbol{\theta}}(\varepsilon)\big)$ *– obtained from* $\tilde{\boldsymbol{U}}(\varepsilon)=(\tilde{\boldsymbol{u}}(\varepsilon),\tilde{\boldsymbol{\theta}}(\varepsilon),\tilde{\boldsymbol{\tau}}(\varepsilon))$ *solution of* $(\tilde{\mathcal{P}}(\varepsilon))$ *– converges weakly in* $\mathcal{V}_{\mathrm{m}}(\omega)$ *to* $(\tilde{\boldsymbol{u}}^{\mathrm{m}*},\tilde{\boldsymbol{\theta}}^{\mathrm{m}*})$ *solution of*

$$(\tilde{\mathcal{P}}^\star_{\mathcal{M}}) \begin{cases} (\tilde{\boldsymbol{u}}^{\mathrm{m}*},\tilde{\boldsymbol{\theta}}^{\mathrm{m}*}) \in \mathcal{V}_{I\mathrm{m}}(\omega), \\ \dfrac{E}{3}A^d_{\mathrm{m}}(\tilde{\boldsymbol{u}}^{\mathrm{m}*},\tilde{\boldsymbol{\theta}}^{\mathrm{m}*};\boldsymbol{v},\boldsymbol{\eta}) = \displaystyle\int_\omega \boldsymbol{f}_0\cdot\boldsymbol{v}\sqrt{a}\,d\omega, \quad \forall(\boldsymbol{v},\boldsymbol{\eta})\in\mathcal{V}_{I\mathrm{m}}(\omega). \end{cases}$$

■

Proof. Here again, the proof is directly adapted from that of Theorem 4.1, see Ref.7. □

Like in the non-inhibited case, we can derive an alternative form of the bilinear form used in $(\tilde{\mathcal{P}}^\star_{\mathcal{M}})$ in order to be able to compare this problem with $(\tilde{\mathcal{P}}_{\mathcal{M}})$.

Theorem 4.8 *For any* $(\boldsymbol{u},\boldsymbol{\theta})$ *and* $(\boldsymbol{v},\boldsymbol{\eta})$ *in* $\mathcal{V}_{I\mathrm{m}}(\omega)$*, we have*

$$\frac{E}{3}A^d_{\mathrm{m}}(\boldsymbol{u},\boldsymbol{\theta};\boldsymbol{v},\boldsymbol{\eta}) = \int_\omega\left(a_I^{\alpha\beta\sigma\tau}\gamma_{\alpha\beta}(\boldsymbol{u})\gamma_{\sigma\tau}(\boldsymbol{v}) + \frac{4E}{3}\zeta_\alpha(\boldsymbol{u},\boldsymbol{\theta})\zeta_\beta(\boldsymbol{v},\boldsymbol{\eta})\right)\sqrt{a}\,d\omega. \tag{4.61}$$

■

Proof. Straightforward by using $a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{u})+\delta(\boldsymbol{\theta})=0$ (and the similar relation for $(\boldsymbol{v},\boldsymbol{\eta})$), namely the first constraint equation in the definition of $\mathcal{V}_I(\omega)$. □

Therefore, the solutions of the two problems $(\tilde{\mathcal{P}}_{\mathcal{M}})$ and $(\tilde{\mathcal{P}}^\star_{\mathcal{M}})$ satisfy the same variational formulation, posed in the spaces $\mathcal{V}_{\mathrm{m}}(\omega)$ and $\mathcal{V}_{I\mathrm{m}}(\omega)$, respectively. Since $\mathcal{V}_{I\mathrm{m}}(\omega)$ is a subspace of $\mathcal{V}_{\mathrm{m}}(\omega)$ the two solutions coincide if and only if the solution of $(\tilde{\mathcal{P}}_{\mathcal{M}})$ happens to lie in $\mathcal{V}_{I\mathrm{m}}(\omega)$. By definition, this solution $(\tilde{\boldsymbol{u}}^{\mathrm{m}},\tilde{\boldsymbol{\theta}}^{\mathrm{m}})$ is in $\mathcal{V}_{I\mathrm{m}}(\omega)$ if there exists a sequence $(\boldsymbol{u}^n,\boldsymbol{\theta}^n,\boldsymbol{\tau}^n)$ in $\mathcal{V}_I(\omega)$ such that $(\boldsymbol{u}^n,\boldsymbol{\theta}^n)$ tends to $(\tilde{\boldsymbol{u}}^{\mathrm{m}},\tilde{\boldsymbol{\theta}}^{\mathrm{m}})$ for the norm $\|\cdot\|_{\mathrm{m}}$. We now write the constraints acting on $(\boldsymbol{u}^n,\boldsymbol{\theta}^n,\boldsymbol{\tau}^n)$ in the following rearranged form

$$\begin{cases} a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{u}^n) + \delta(\boldsymbol{\theta}^n) = 0, \\ \tau_3^n = -\dfrac{1}{2}\Big(2Ha^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{u}^n) + b^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{u}^n) + g_{\theta^n}\Big), \\ g_{\tau^n} = -\Big(3(2H^2 - K)a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{u}^n) + 3Hb^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{u}^n) + Hg_{\theta^n} + b_{\theta^n}\Big), \\ b_{\tau^n} = -2\Big(H(6H^2 - 5K)a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{u}^n) + (3H^2 - K)b^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{u}^n) \\ \qquad\qquad + (H^2 - K)g_{\theta^n} + Hb_{\theta^n}\Big). \end{cases} \tag{4.62}$$

There is no difficulty with the first constraint equation since it does not contain $\boldsymbol{\tau}^n$ and continuously gives, when taking the limit in n (assuming that we can find the desired sequence),

$$a^{\alpha\beta}\gamma_{\alpha\beta}(\tilde{\boldsymbol{u}}^{\mathrm{m}}) + \delta(\tilde{\boldsymbol{\theta}}^{\mathrm{m}}) = 0, \tag{4.63}$$

which is – indeed – satisfied by the solution of $(\tilde{\mathcal{P}}_{\mathcal{M}})$. Moreover, it is clear that if for any given choice of (smooth, say) couple $(\boldsymbol{u}^n, \boldsymbol{\theta}^n)$ in $\big(\mathcal{V}(\omega)\big)^2$ the last three equations of (4.62) admit (at least) one solution $\boldsymbol{\tau}^n$ in $\mathcal{V}(\omega)$, then they do not give constraints on $(\boldsymbol{u}, \boldsymbol{\theta})$ and in such a case we have

$$\mathcal{V}_{I\mathrm{m}}(\omega) = \{(\boldsymbol{v}, \boldsymbol{\eta}) \in \mathcal{V}_{\mathrm{m}}(\omega);\ a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{v}) + \delta(\boldsymbol{\eta}) = 0\}, \tag{4.64}$$

hence the solution of $(\tilde{\mathcal{P}}_{\mathcal{M}})$ is in $\mathcal{V}_{I\mathrm{m}}(\omega)$ and coincides with the solution of $(\tilde{\mathcal{P}}^{\star}_{\mathcal{M}})$.
Example We consider again the cylindrical surface defined by (4.53). In this coordinate system we have

$$g_{\tau^n} = \tau^n_{1,1} + \tau^n_{2,2}, \quad b_{\tau^n} = -R^{-1}\tau^n_{2,2}. \tag{4.65}$$

Hence, using these equations we obtain that the last three equations of (4.62) are equivalent to

$$\begin{cases} \tau_3^n = \dfrac{R^{-1}}{2}\big(\gamma_{11}(\boldsymbol{u}^n) + 2\gamma_{22}(\boldsymbol{u}^n)\big) - \dfrac{1}{2}\big(\theta^n_{1,1} + \theta^n_{2,2}\big), \\ \tau^n_{1,1} = 0, \\ \tau^n_{2,2} = -\dfrac{3}{2}R^{-2}\big(\gamma_{11}(\boldsymbol{u}^n) + 2\gamma_{22}(\boldsymbol{u}^n)\big) + \dfrac{R^{-1}}{2}\big(\theta^n_{1,1} + 3\theta^n_{2,2}\big). \end{cases} \tag{4.66}$$

Therefore, if the boundary conditions are that, e.g., the structure is clamped on the edges $\xi^1 = 0$ and $\xi^1 = L$ (which implies that pure bending is inhibited, see e.g. Ref.10) this system can easily be solved for $\boldsymbol{\tau}^n$ and the solutions of $(\tilde{\mathcal{P}}_{\mathcal{M}})$ and $(\tilde{\mathcal{P}}^{\star}_{\mathcal{M}})$ coincide.

Remark 4.9 *In the case of a plate the equations in* (4.62) *reduce to*

$$g_{\tau^n} = 0, \quad \tau_3^n = -\frac{1}{2}g_{\theta^n}, \tag{4.67}$$

which admit obvious solutions, hence the solutions of $(\tilde{\mathcal{P}}_{\mathcal{M}})$ *and* $(\tilde{\mathcal{P}}^{\star}_{\mathcal{M}})$ *always coincide.* ■

$$
\begin{array}{ccccc}
(\tilde{\mathcal{P}}(\varepsilon,\nu)) & \xrightarrow[\text{(weak)}]{\varepsilon\to 0} & \left\{\begin{array}{l}(\tilde{\mathcal{P}}_{\mathcal{F}}(\nu)) \Leftrightarrow (\mathcal{P}_{\mathcal{F}}(\nu))\\ (\tilde{\mathcal{P}}_{\mathcal{M}}(\nu)) \Leftrightarrow (\mathcal{P}_{\mathcal{M}}(\nu))\end{array}\right. & & \\
\Big\downarrow \nu \to 1/2 & & & & \Big\downarrow \nu \to 1/2 \\
(\tilde{\mathcal{P}}(\varepsilon)) & \xrightarrow[\text{(weak) } (\dagger)]{\varepsilon\to 0} & \left\{\begin{array}{l}(\tilde{\mathcal{P}}^{\star}_{\mathcal{F}})\\ (\tilde{\mathcal{P}}^{\star}_{\mathcal{M}})\end{array}\right. & \not\Leftrightarrow & \left\{\begin{array}{l}(\tilde{\mathcal{P}}_{\mathcal{F}}) \Leftrightarrow (\mathcal{P}_{\mathcal{F}})\\ (\tilde{\mathcal{P}}_{\mathcal{M}}) \Leftrightarrow (\mathcal{P}_{\mathcal{M}})\end{array}\right.
\end{array}
$$

Figure 3: Convergence behaviors of the 3D-shell model

To conclude this section we summarize the convergence behaviors of the 3D-shell model in the (non-commuting) diagram featured in Figure 3, where the "(†)" symbol refers to Remark 4.8 above.

4.4. *Modified 3D-shell model*

The objective of this section is to propose a modified 3D-shell formulation that circumvents the above difficulties pertaining to the fullfilment of incompressibility constraints. We introduce the variational formulation

$$
(\hat{\mathcal{P}}(\varepsilon,\nu))\left\{\begin{array}{l}
\hat{\boldsymbol{U}}(\varepsilon,\nu) \in (\mathcal{V}(\omega))^3,\\
\displaystyle \mu \int_{\Omega_t} g^{ijkl} e_{ij}(\hat{\boldsymbol{U}}(\varepsilon,\nu)) e_{kl}(\boldsymbol{V}) \sqrt{g}\, d\Omega\\
\displaystyle \qquad\qquad + \lambda \int_{\Omega_t} P_{1,\xi^3}\big(g^{ij} e_{ij}(\hat{\boldsymbol{U}}(\varepsilon,\nu))\big) P_{1,\xi^3}\big(g^{kl} e_{kl}(\boldsymbol{V})\big) \sqrt{g}\, d\Omega\\
\displaystyle \qquad\qquad\qquad = \int_{\Omega_t} \boldsymbol{f}^{\varepsilon}\cdot \boldsymbol{V} \sqrt{g}\, d\Omega, \quad \forall \boldsymbol{V} \in (\mathcal{V}(\omega))^3,
\end{array}\right.
$$

where P_{1,ξ^3} denotes the operator that truncates a function analytic in the ξ^3 variable to its first degree polynomial expansion in ξ^3. Here, recalling (4.30) we have

$$
P_{1,\xi^3}\big(g^{ij} e_{ij}(\boldsymbol{V})\big) = \underbrace{a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{v}) + \delta(\boldsymbol{\eta})}_{=I_0(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho})} + \xi^3 \underbrace{\big(a^{\alpha\beta}\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{\eta}) + 2b^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{v}) + p(\boldsymbol{\rho})\big)}_{=(I_1+4HI_0)(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho})}. \tag{4.68}
$$

Note that this modified variational formulation is well-posed since it features the same coercivity and boundedness properties as $(\tilde{\mathcal{P}}(\varepsilon,\nu))$.

In the incompressible limit ($\nu \to 1/2$), by construction the constraint enforced in the variational formulation $(\hat{\mathcal{P}}(\varepsilon,\nu))$ is that the displacements belong to the subspace

$$
\hat{\mathcal{V}}_I(\omega) = \{(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) \in (\mathcal{V}(\omega))^3;\ I_k(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) = 0,\ k=0,1\}. \tag{4.69}
$$

Note that the constraints acting in this subspace correspond to the first two incompressibility constraints featured in the definition of $\mathcal{V}_I(\omega)$. This modification of the 3D-shell formulation is – indeed – motivated by the fact that the additional constraints "$I_2 = I_3 = 0$" were identified in the above analysis as an obstacle to having the incompressible and thickness limits commute.

With this modified formulation, when considering first the limit in ε the asymptotic behavior is clearly unchanged since the strain terms of degree higher than one (in ξ^3) vanish in the limit, see Ref.7.

When considering first the incompressible limit we obtain in a straightforward manner the following limit constrained problem (and solution)

$$(\hat{\mathcal{P}}(\varepsilon)) \begin{cases} \hat{\boldsymbol{U}}(\varepsilon) = (\hat{\boldsymbol{u}}(\varepsilon), \hat{\boldsymbol{\theta}}(\varepsilon), \hat{\boldsymbol{\tau}}(\varepsilon)) \in \hat{\mathcal{V}}_I(\omega), \\ \dfrac{E}{3} \displaystyle\int_{\Omega_t} g^{ijkl} e_{ij}(\hat{\boldsymbol{U}}(\varepsilon)) e_{kl}(\boldsymbol{V}) \sqrt{g}\, d\Omega = \int_{\Omega_t} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{V} \sqrt{g}\, d\Omega, \quad \forall \boldsymbol{V} \in \hat{\mathcal{V}}_I(\omega). \end{cases}$$

Then, when considering the asymptotic behavior of this solution with respect to ε we are led to introducing the subspace

$$\hat{\mathcal{V}}_{I0}(\omega) = \big\{(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in (\mathcal{V}(\omega))^3;\ (\boldsymbol{v}, \boldsymbol{\eta}) \in \mathcal{V}_{00}(\omega),\ I_1'(\boldsymbol{\eta}, \boldsymbol{\rho}) = 0\big\}, \tag{4.70}$$

and a relaxed form of Condition (4.44), viz.

$$(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \hat{\mathcal{V}}_{I0}(\omega) \Rightarrow (\boldsymbol{v}, \boldsymbol{\eta}) = (\boldsymbol{0}, \boldsymbol{0}). \tag{4.71}$$

Remark 4.10 *The constraint $I_1'(\boldsymbol{\eta}, \boldsymbol{\rho}) = 0$ gives an explicit expression of ρ_3 with respect to $\boldsymbol{\eta}$, hence the subspaces $\hat{\mathcal{V}}_{I0}(\omega)$ and $\mathcal{V}_{00}(\omega)$ are very closely related (in particular all smooth elements of $\mathcal{V}_{00}(\omega)$ allow to construct elements of $\hat{\mathcal{V}}_{I0}(\omega)$) and so are the conditions of inhibited pure bending and* (4.71)*, see also the Appendix regarding the density of regular functions in $\mathcal{V}_{00}(\omega)$.* ■

When Condition (4.71) does not hold – namely, in a "non-inhibited case" – we define the subspaces $\hat{\mathcal{V}}_{I\mathrm{f}}(\omega)$ and $\hat{\mathcal{V}}_{I0\mathrm{f}}(\omega)$ as the completions of $\hat{\mathcal{V}}_I(\omega)$ and $\hat{\mathcal{V}}_{I0}(\omega)$, respectively, for the norm $\|.\|_{\mathrm{f}}$, and we will assume the *ad hoc* commuting property of intersection and completion, namely,

$$\hat{\mathcal{V}}_{I0\mathrm{f}}(\omega) = \hat{\mathcal{V}}_{I\mathrm{f}}(\omega) \cap \mathcal{V}_{0\mathrm{f}}(\omega). \tag{4.72}$$

We also discuss this assumption and provide sufficient conditions in the Appendix. In this case, the appropriate scaling for the asymptotic behavior with respect to ε is then $p = 2$ and we have the following convergence result.

Theorem 4.9 *Assuming that* (4.72) *holds, when ε tends to* 0 $(\hat{\boldsymbol{u}}(\varepsilon), \hat{\boldsymbol{\theta}}(\varepsilon), \hat{\boldsymbol{\tau}}(\varepsilon))$ *solution of $(\hat{\mathcal{P}}(\varepsilon))$ converges weakly in $\mathcal{V}_{\mathrm{f}}(\omega)$ to $(\tilde{\boldsymbol{u}}^{\mathrm{f}}, \tilde{\boldsymbol{\theta}}^{\mathrm{f}}, \tilde{\boldsymbol{\tau}}^{\mathrm{f}})$ solution of $(\tilde{\mathcal{P}}_{\mathcal{F}})$.* ■

Sketch of the proof. First, like in Theorem 4.5 we prove that the sequence converges (weakly in $\mathcal{V}_{\mathrm{f}}(\omega)$) to the solution of a variational problem similar to $(\tilde{\mathcal{P}}^\star_{\mathcal{F}})$ albeit posed in $\hat{\mathcal{V}}_{I0\mathrm{f}}(\omega)$ instead of $\mathcal{V}_{I0\mathrm{f}}(\omega)$. Then, like in Theorem 4.6 we show that (4.47) holds in $\hat{\mathcal{V}}_{I0\mathrm{f}}(\omega)$, hence the limit solution of $(\hat{\mathcal{P}}(\varepsilon))$ coincides with $(\tilde{\boldsymbol{u}}^{\mathrm{f}}, \tilde{\boldsymbol{\theta}}^{\mathrm{f}}, \tilde{\boldsymbol{\tau}}^{\mathrm{f}})$ if and only if the latter lies in $\hat{\mathcal{V}}_{I0\mathrm{f}}(\omega)$. Finally, this property holds because we clearly have

$$\hat{\mathcal{V}}_{I\mathrm{f}}(\omega) = \big\{(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \mathcal{V}_{\mathrm{f}}(\omega);\ I_k(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) = 0,\ k = 0, 1\big\}, \tag{4.73}$$

which implies due to (4.72)

$$\hat{\mathcal{V}}_{I0\mathrm{f}}(\omega) = \big\{(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \mathcal{V}_{0\mathrm{f}}(\omega);\ I_1'(\boldsymbol{\eta}, \boldsymbol{\rho}) = 0\big\}, \tag{4.74}$$

and we already showed that $I_1'(\tilde{\boldsymbol{\theta}}^{\mathrm{f}}, \tilde{\boldsymbol{\tau}}^{\mathrm{f}}) = 0$ in Section 4.3.1. □

We now consider the inhibited case, i.e. we suppose that (4.71) holds. For the scaling $p = 0$ there is then no difficulty in obtaining the following asymptotic result, which we can prove like in Ref.7, using also an equivalence of bilinear forms similar to (4.61).

Theorem 4.10 *Assuming that (4.15) holds, when ε tends to* 0*, the couple* $(\hat{\boldsymbol{u}}(\varepsilon) + \frac{t^2}{12}\hat{\boldsymbol{\tau}}(\varepsilon), \hat{\boldsymbol{\theta}}(\varepsilon))$ *– obtained from* $\hat{\boldsymbol{U}}(\varepsilon) = (\hat{\boldsymbol{u}}(\varepsilon), \hat{\boldsymbol{\theta}}(\varepsilon), \hat{\boldsymbol{\tau}}(\varepsilon))$ *solution of* $(\hat{\mathcal{P}}(\varepsilon))$ *– converges weakly in* $\mathcal{V}_{\mathrm{m}}(\omega)$ *to* $(\tilde{\boldsymbol{u}}^{\mathrm{m}}, \tilde{\boldsymbol{\theta}}^{\mathrm{m}})$ *solution of* $(\tilde{\mathcal{P}}_{\mathcal{M}})$*.* ■

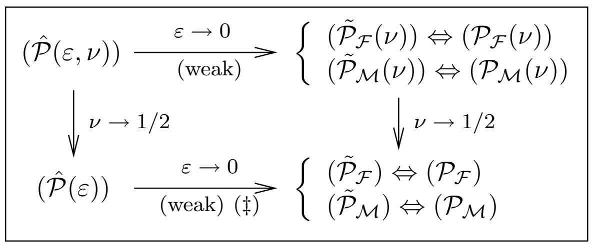

Figure 4: Convergence behaviors of the modified 3D-shell model

We conclude from this section that the formulation $(\hat{\mathcal{P}}(\varepsilon,\nu))$ gives a shell model which enjoys a commuting property for the limits with respect to the parameters ε and ν, with limits when ε tends to 0 *and* ν to 1/2 that correspond to the limits obtained for the 3D elasticity problem (under the assumption (4.72), see also the Appendix). This is summarized in the diagram given in Figure 4, where the "(‡)" symbol refers to Remark 4.10 above.

5. Numerical illustration

The purpose of this section is to illustrate our above discussions by providing some numerical results, in particular using 3D-shell elements. From a numerical point of view we cannot expect to reach the asymptotic limits with respect to the parameters ε and ν, because

- although we use shell elements for which numerical locking is limited for "practical values of the thickness", see in particular Ref.10, uniform convergence of the numerical solution with respect to ε is not to be expected;

- for 3D-shell elements the condition number appears to behave like

$$O\big(h^{-2}\,\varepsilon^{-2}\,(1-2\nu)^{-1}\big),$$

hence we rapidly hit the numerical precision threshold when considering together small values of the thickness and a material near the incompressible limit.

Therefore, we will restrict our computations to a shell of "reasonably small thickness" ($\varepsilon = 10^{-2}$) and investigate the effect of using the modified 3D-shell model on the solutions obtained for various values of ν.

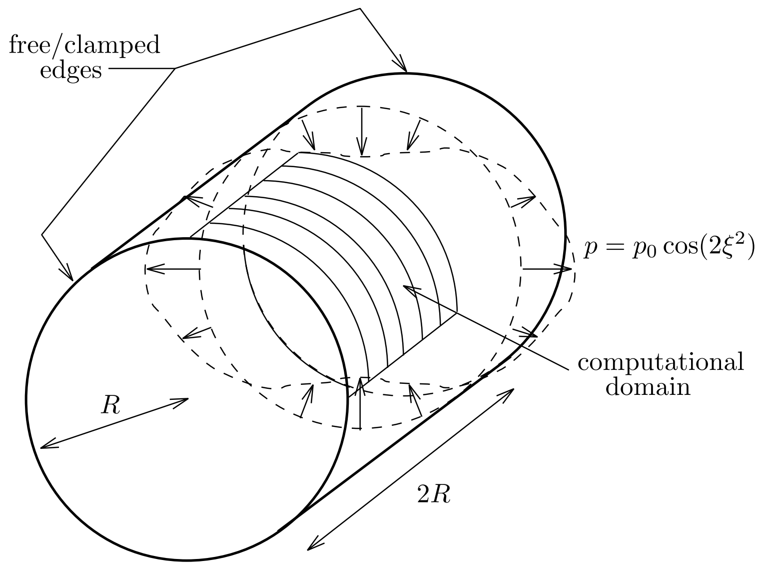

Figure 5: Cylinder loaded by periodic pressure

We then consider the shell structure depicted in Figure 5, namely a cylindrical shell loaded by a periodic pressure. This example was already proposed in Ref.14, which also presented a procedure to obtain numerical solutions of arbitrary precision for classical shell models. Depending on the boundary conditions prescribed on the two ends we can obtain various asymptotic behaviors with respect to ε.

$\nu = 0.4$	$\nu = 0.499$	$\nu = 0.49999$
0.594044	0.535284	0.534628

Table 1: Non-inhibited case - MITC elements - Energy scaled by $\frac{12p_0^2R^3}{E\varepsilon^3}$

We first leave the two ends free in order to obtain a shell with non-inhibited pure bending, see Refs.10,14. Table 1 lists the values of the strain energy obtained with a mesh of 34 by 34 MITC9 elements in the computational domain shown in Figure 5, for various values of ν. These energy values are to be used for comparison purposes in the sequel, although we emphasize that MITC elements correspond to discretizations of the so-called "basic shell model" (see Ref.10) which differs from

N	$\nu = 0.4$	$\nu = 0.499$	$\nu = 0.49999$
4	0.615088	0.551713	0.499036
8	0.617861	0.555567	0.502491
12	0.618107	0.555925	0.502543
16	0.618160	0.555989	0.502013

Table 2: Non-inhibited case - 3D-shell elements - Energy scaled by $\frac{12p_0^2R^3}{E\varepsilon^3}$

N	$\nu = 0.4$	$\nu = 0.499$	$\nu = 0.49999$
4	0.615101	0.552697	0.551724
8	0.617867	0.556560	0.555792
12	0.618127	0.556921	0.556207
16	0.618173	0.556989	0.555899

Table 3: Non-inhibited case - Modified 3D-shell elements - Energy scaled by $\frac{12p_0^2R^3}{E\varepsilon^3}$

the 3D-shell model for finite values of the thickness, see Ref.7. We note, of course, the expected very regular behavior when ν tends to 0.5. In turn, with the corresponding 3D-shell element (also formulated using mixed interpolation procedures, see Ref.9) we obtain the strain energy values given in Table 2, for various meshes of $N \times N$ elements. We note that the energy convergence behavior (with respect to the discretization parameter) does not obviously deteriorate when ν approaches 0.5, which suggests that this asymptotic behavior does not produce a numerical locking phenomenon, although the matter would deserve a much more detailed investigation, of course. However, since the 3D-shell model corresponds to a "more refined" approximation of 3D elasticity than the basic shell model we would expect the converged energy values of the 3D-shell elements to be above those of the MITC elements. This is – indeed – what we observe for $\nu = 0.4$ and 0.499, but not for 0.49999. If we then use the *modified* 3D-shell model presented in Section 4.4 we obtain the values of Table 3, where we observe little change for $\nu = 0.4$ and 0.499, but now the energy behaves as expected also for $\nu = 0.49999$. We mention in passing that very similar numerical solutions are obtained when using, instead of the modified 3D-shell model, the energy expression of the standard 3D-shell model with a 2-point reduced integration strategy across the thickness for the volumetric term, which makes the practical implementation very straightforward.

$\nu = 0.499$	$\nu = 0.49999$
2.52233	2.52304

Table 4: Inhibited case - MITC elements - Energy scaled by $\frac{12p_0^2R^3}{E\varepsilon}$

We perform similar comparisons in a case of inhibited pure bending obtained by clamping the two boundaries. Table 4 gives the "reference solutions" obtained

N	$\nu = 0.499$	$\nu = 0.49999$
4	2.16341	1.82587
8	2.37928	2.32236
12	2.47933	2.46345
16	2.52856	2.52200

Table 5: Inhibited case - 3D-shell elements - Energy scaled by $\frac{12p_0^2R^3}{E\varepsilon}$

N	$\nu = 0.499$	$\nu = 0.49999$
4	2.45439	2.05505
8	2.52031	2.42700
12	2.55803	2.51531
16	2.57804	2.55027

Table 6: Inhibited case - Modified 3D-shell elements - Energy scaled by $\frac{12p_0^2R^3}{E\varepsilon}$

with MITC9 elements ($N = 34$). The energy values obtained with the original and modified 3D-shell models are shown in Tables 5 and 6, respectively. We observe that the modification has a much smaller impact on the converged energy values than in the non-inhibited case, as was expected from the above discussion (note that the models are definitely different for finite thicknesses and ν away from 0.5, so that small differences in the energy values are not significant).

6. Concluding remarks

Although this study of "incompressible shells" should be considered as preliminary in many respects, the results obtained have some valuable implications and lead to interesting observations (and conjectures).

First, we have obtained a preliminary justification of the concept of "incompressible shell" by showing that, in some sense, the limits of $\varepsilon \to 0$ and $\nu \to 0.5$ do commute in the asymptotic behavior of thin elastic structures. In essence, the combined limit is then obtained by setting $\nu = 0.5$ in the membrane and pure-bending models, namely the limits of 3D models and classical shell models when the thickness tends to zero. Hence this also gives a preliminary justification that classical shell models are valid in the incompressible limit.

Then we have shown that the 3D-shell model does not enjoy the same property of commuting limits, which suggests that it is not valid for incompressible materials. In particular, the commuting property is in general not satisfied when pure bending is not inhibited, namely for bending-dominated shell structures. This phenomenon can be compared to numerical locking, since it occurs due to the semi-discretization of the displacements in the transverse direction (quadratic kinematical assumption), as was already recognized in Ref.2. However we can recover the desired commuting property (i.e. "unlock") by slightly modifying the volumetric part of the formulation. These theoretical results have been illustrated (and to some extent confirmed) by numerical solutions, although much more thorough investigations would be required, in particular to analyse the various locking phenomena involved, with their possible mutual interactions.

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Appendix

The result of Theorem A.1 below is needed in the study of the asymptotic behaviour of the "incompressible 3D-shell model" $(\tilde{\mathcal{P}}(\varepsilon))$ and of the "modified incompressible 3D-shell model" $(\hat{\mathcal{P}}(\varepsilon))$ in the bending-dominated cases (see Theorems 4.5 and 4.9).

We assume that the midsurface $\mathcal{S}$ of the shell is regular enough in all parametrizations henceforth considered (i.e., $\mathcal{S} = \boldsymbol{\phi}(\overline{\omega})$ with $\boldsymbol{\phi} \in \mathcal{C}^6(\overline{\omega}; \mathbb{R}^3)$, where ω is a connected, open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary) and that the subset $(\partial\mathcal{S})_0 = \boldsymbol{\phi}(\gamma_0)$ of the boundary of $\mathcal{S}$ is connected and relatively open. We assume in addition that $\mathcal{S}$ satisfies one of the following assumptions:

(H1) The surface $\mathcal{S}$ is hyperbolic and convex in its asymptotic directions.

(H1') The surface $\mathcal{S}$ is hyperbolic and convex in its asymptotic and principal directions and the intersection between $(\partial\mathcal{S})_0$ and any principal curve of $\mathcal{S} \cup \partial\mathcal{S}$ is a connected set.

(H2) The surface $\mathcal{S}$ is parabolic and convex in its asymptotic direction.

We introduce $H^1_{\gamma_0}(\omega) = \{f \in H^1(\omega); f = 0 \text{ on } \gamma_0\}$, $H^2_{\gamma_0}(\omega) = \{f \in H^1(\omega); f = \partial_\nu f = 0 \text{ on } \gamma_0\}$ and we note that $\mathcal{V}(\omega) = (H^1_{\gamma_0}(\omega))^3$.

In what follows, whenever an equation involves generic indices it is understood that this equation holds for *all* such indices (in conjunction with the convention for Latin and Green indices stated in Section 2). For instance, the equation "$v_i \in H^1_{\gamma_0}(\omega)$" means that "$v_i \in H^1_{\gamma_0}(\omega)$ for all $i \in \{1,2,3\}$", the equation "$\gamma_{\alpha\beta}(\boldsymbol{v}) = 0$" means that "$\gamma_{\alpha\beta}(\boldsymbol{v}) = 0$ for all $\alpha, \beta \in \{1,2\}$", and so on.

Let the differential operators R and I be defined by

$$R(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) = (\gamma_{\alpha\beta}(\boldsymbol{v}), \zeta_\alpha(\boldsymbol{v},\boldsymbol{\eta}), \delta(\boldsymbol{\eta})),$$

$$I(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) = g^{ij} e_{ij}(\boldsymbol{v} + \xi^3\boldsymbol{\eta} + (\xi^3)^2\boldsymbol{\rho}),$$

for all $(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) \in (\mathcal{V}(\omega))^3$, so that

$$\begin{aligned}
&\mathcal{V}^\sharp_{00}(\omega) = \{(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) \in (\mathcal{V}(\omega))^3; R(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) = 0\},\\
&\mathcal{V}_I(\omega) = \{(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) \in (\mathcal{V}(\omega))^3; I(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) = 0\},\\
&\hat{\mathcal{V}}_I(\omega) = \{(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) \in (\mathcal{V}(\omega))^3; I_{\mathrm{aff}}(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) = 0\},
\end{aligned}$$

where I_{aff} denotes the affine part of I with respect to ξ^3, namely

$$I_{\mathrm{aff}}(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) = I_0(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}) + \xi^3(I_1 + 4HI_0)(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}),$$

see (4.68). We recall that $\mathcal{V}_{\mathrm{f}}(\omega)$, $\mathcal{V}_{0\mathrm{f}}(\omega)$, $\mathcal{V}_{I\mathrm{f}}(\omega)$, $\hat{\mathcal{V}}_{I\mathrm{f}}(\omega)$, $\mathcal{V}_{I0\mathrm{f}}(\omega)$ and $\hat{\mathcal{V}}_{I0\mathrm{f}}(\omega)$ respectively denote the completions of the spaces $(\mathcal{V}(\omega))^3$, $\mathcal{V}^\sharp_{00}(\omega)$, $\mathcal{V}_I(\omega)$, $\hat{\mathcal{V}}_I(\omega)$, $\mathcal{V}^\sharp_{00}(\omega) \cap \mathcal{V}_I(\omega)$ and $\mathcal{V}^\sharp_{00}(\omega) \cap \hat{\mathcal{V}}_I(\omega)$ with respect to the norm

$$\|(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho})\|_{\mathrm{f}} = (\textstyle\sum_i \|v_i\|_1^2 + \sum_\alpha \|\eta_\alpha\|_1^2 + \|\eta_3\|_0^2 + \|\rho_3\|_0^2 + \sum_\alpha \|\rho_\alpha + \frac{1}{2}\partial_\alpha\eta_3\|_0^2)^{1/2}.$$

Then one can see that

$$\begin{aligned}
\mathcal{V}_{\mathrm{f}}(\omega) = \{(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho}); v_i, \eta_\alpha \in H^1_{\gamma_0}(\omega); \eta_3, \rho_3 \in L^2(\omega); \rho_\alpha \in H^{-1}(\omega),&\\
2\rho_\alpha + \partial_\alpha\eta_3 \in L^2(\omega)\}.&
\end{aligned}$$

The distance from a point x to a subset A of the Euclidean space $\mathbb{R}^k$, $k \geq 1$, is defined by $\mathrm{dist}(x, A) = \inf\{\|x - a\|; a \in A\}$.

The object of this appendix is to prove the following result.

Theorem A.1 *a) Assume that either assumption (H1) or assumption (H2) holds. Then*

$$\hat{\mathcal{V}}_{I0\mathrm{f}}(\omega) = \hat{\mathcal{V}}_{I\mathrm{f}}(\omega) \cap \mathcal{V}_{0\mathrm{f}}(\omega).$$

b) Assume that either assumption (H1') or assumption (H2) holds. Then

$$\mathcal{V}_{I0\mathrm{f}}(\omega) = \mathcal{V}_{I\mathrm{f}}(\omega) \cap \mathcal{V}_{0\mathrm{f}}(\omega).$$

■

The proof of this theorem relies in particular on the following density result:

Lemma A.2 *Assume that either assumption (H1) or assumption (H2) holds. Let* $\boldsymbol{v} \in H^1_{\gamma_0}(\omega) \times H^1_{\gamma_0}(\omega) \times H^2_{\gamma_0}(\omega)$ *be such that* $\gamma_{\alpha\beta}(\boldsymbol{v}) = 0$. *Then there exists a sequence* $\boldsymbol{v}^n \in H^4(\omega) \times H^4(\omega) \times H^3(\omega)$ *such that* $\gamma_{\alpha\beta}(\boldsymbol{v}^n) = 0$, $\boldsymbol{v}^n = 0$ *in a neighborhood of* γ_0, *and* $\boldsymbol{v}^n \to \boldsymbol{v}$ *in* $H^3(\omega) \times H^3(\omega) \times H^2(\omega)$ *as* $n \to \infty$. ■

Proof. We assume that the midsurface of the shell is hyperbolic and convex in its asymptotic directions (the proof is simpler if the midsurface is parabolic and convex in its asymptotic direction). Then there is no loss of generality in assuming that the parameter curves are the asymptotic lines of the surface. In this case, the system $\gamma_{\alpha\beta}(\boldsymbol{v}) = 0$ splits into the equation

$$v_3 = \frac{1}{2b_{12}}(\partial_2 v_1 + \partial_1 v_2 - 2\Gamma^{\sigma}_{12} v_\sigma), \tag{A.1}$$

and the system

$$\begin{cases} \partial_1 v_1 - \Gamma^{\sigma}_{11} v^{\sigma} = 0, \\ \partial_2 v_2 - \Gamma^{\sigma}_{22} v^{\sigma} = 0. \end{cases} \tag{A.2}$$

We can assume, again without loss of generality, that ω is a subset of the set $D =]0,1[\times]0,1[$. If $m, n \in \mathbb{N}$ and X is a subset of an open set $Y \subset \mathbb{R}^n$, the notation $H^m_X(Y)$ designates the space of all (classes of) functions of class H^m over Y that, together with all their derivatives of order $\leq m-1$, vanish on X.

Since $\boldsymbol{v}$ belongs to the space $H^1_{\gamma_0}(\omega) \times H^1_{\gamma_0}(\omega) \times H^2_{\gamma_0}(\omega)$, the above equations imply that $v_\alpha \in H^3_{\gamma_0}(\omega)$ and $\partial_1 v_1, \partial_2 v_2 \in H^3_{\gamma_0}(\omega)$ (we use local coordinates around γ_0 to show that the normal derivatives of v_α vanish on γ_0). Let ω_0 designate the smallest rectangle that contains the set γ_0 and whose edges are parallel to the coordinate axes ($O\xi^1$ and $O\xi^2$). Then the functions v_α are extended by zero in the set ω_0 to the functions (still denoted) $v_\alpha \in H^3(\omega \cup \omega_0)$ that satisfy in addition $\partial_1 v_1, \partial_2 v_2 \in H^3(\omega \cup \omega_0)$. This can be done because $v_\alpha = 0$ in $\omega \cap \omega_0$ (the coordinate axes are the characteristic lines of the system (A.2)). Since the set $\omega \cup \omega_0$ is Lipschitz, the functions $v_\alpha \in H^3(\omega \cup \omega_0)$ can further be extended to functions (still denoted) v_α that belong to the space $H^3(D)$ and satisfy $\partial_1 v_1, \partial_2 v_2 \in H^3(D)$.

Since the set ω is Lipschitz, the functions $\Gamma^{\sigma}_{\alpha\beta} \in \mathcal{C}^4(\overline{\omega})$ can be extended in D to functions (still denoted) $\Gamma^{\sigma}_{\alpha\beta} \in \mathcal{C}^4(\overline{D})$. Let

$$f_1(\xi^1, \xi^2) = e^{\int_0^{\xi^1} \Gamma^1_{11}(s,\xi^2)ds} \text{ and } f_2(\xi^1, \xi^2) = e^{\int_0^{\xi^2} \Gamma^2_{22}(\xi^1,s)ds},$$

and define

$$u_1 = v_1/f_1, \quad u_2 = v_2/f_2, \quad k_1 = (f_2/f_1)\Gamma^2_{11}, \quad k_2 = (f_1/f_2)\Gamma^1_{22},$$
$$h_1 = \frac{\partial_1 v_1 - \Gamma^\sigma_{11} v^\sigma}{f_1}, \quad h_2 = \frac{\partial_2 v_2 - \Gamma^\sigma_{22} v^\sigma}{f_2}.$$

Then $k_\alpha \in \mathcal{C}^4(\overline{D})$, $h_\alpha \in H^3_{\omega\cup\omega_0}(D)$, $u_\alpha \in H^3_{\omega_0}(D)$ and

$$\begin{cases} \partial_1 u_1 = k_1 u_2 + h_1, \\ \partial_2 u_2 = k_2 u_1 + h_2. \end{cases}$$

Let a point $(\xi^1_0, \xi^2_0) \in \omega_0$ be fixed and let $D_1 = \{(\xi^1_0, \xi^2); 0 < \xi^2 < 1\}$ and $D_2 = \{(\xi^1, \xi^2_0); 0 < \xi^1 < 1\}$. Since u_1 and $\partial_1 u_1$ belong to the space $H^3(D)$, one can see that the trace of u_1 on D_1, denoted $u^0_1(\xi^2) = u_1(\xi^1_0, \xi^2)$, belongs to the space $H^3(]0,1[)$ and that there exists a constant $C > 0$ such that

$$\|u^0_1\|_{H^3(]0,1[)} \le C(\|u_1\|_{H^3(D)} + \|\partial_1 u_1\|_{H^3(D)}).$$

In the same way, the trace of u_2 on D_2, denoted $u^0_2(\xi^1) = u_2(\xi^1, \xi^2_0)$, belongs to the space $H^3(]0,1[)$ and there exists a constant $C > 0$ such that

$$\|u^0_2\|_{H^3(]0,1[)} \le C(\|u_2\|_{H^3(D)} + \|\partial_2 u_2\|_{H^3(D)}).$$

Let $V_1 = \{\xi^2; (\xi^1_0, \xi^2) \in D_1 \cap \omega_0\}$ and $V_2 = \{\xi^1; (\xi^1, \xi^2_0) \in D_2 \cap \omega_0\}$. Since $u^0_1 \in H^3_{V_1}(]0,1[)$ and $u^0_2 \in H^3_{V_2}(]0,1[)$, there exist sequences of functions $(u^{0,n}_1)$ and $(u^{0,n}_2)$ in the space $\mathcal{C}^\infty([0,1])$ such that $u^{0,n}_\alpha \to u^0_\alpha$ in $H^3(D)$ and $u^{0,n}_\alpha = 0$ on V^n_α, where

$$V^n_1 = \{\xi^2 \in]0,1[; \mathrm{dist}(\xi^2, V_1) < 1/n\},$$
$$V^n_2 = \{\xi^1 \in]0,1[; \mathrm{dist}(\xi^1, V_2) < 1/n\}.$$

Since the functions h_α belong to the space $H^3_{\omega\cup\omega_0}(D)$, there exist sequences of functions (h^n_α) such that $h^n_\alpha \to h_\alpha$ in $H^3(\omega)$ as $n \to \infty$ and $h^n_\alpha = 0$ on $\omega \cup \omega^n_0$, where $\omega^n_0 = \{(\xi^1, \xi^2) \in D; \mathrm{dist}((\xi^1, \xi^2), \omega_0) < 1/n\}$.

Then one can show that the system

$$\begin{cases} \partial_1 u^n_1 = k_1 u^n_2 + h^n_1 \text{ in } D, \\ \partial_2 u^n_2 = k_2 u^n_1 + h^n_2 \text{ in } D, \\ u^n_1|_{D_1} = u^{0,n}_1, \ u^n_2|_{D_2} = u^{0,n}_2, \end{cases} \tag{A.3}$$

possesses a unique solution $(u^n_1, u^n_2) \in H^4_{\omega^n_0}(\omega) \times H^4_{\omega^n_0}(\omega)$ and that this solution satisfies

$$u^n_\alpha \to u_\alpha \text{ in } H^3(D) \text{ as } n \to \infty.$$

To prove this, let $m \in \{0,1,2,3,4\}$ be fixed and consider the system

$$\begin{cases} \partial_1 w_1 = k_1 w_2 + d_1 \text{ in } D, \\ \partial_2 w_2 = k_2 w_1 + d_2 \text{ in } D, \\ w_1|_{D_1} = \overline{w}^0_1, \ w_2|_{D_2} = \overline{w}^0_2, \end{cases} \tag{A.4}$$

where $k_\alpha \in \mathcal{C}^4(\overline{D})$, $\overline{w}^0_\alpha \in H^m(]0,1[)$ and $d_\alpha \in H^m(D)$. Let $w^0_\alpha = 0$ in D and, for all $j \geq 0$, define recursively

$$\begin{cases} w_1^{j+1}(\xi^1,\xi^2) = \overline{w}_1^0(\xi^2) + \displaystyle\int_{\xi_0^1}^{\xi^1} (k_1 w_2^j + d_1)(s,\xi^2)ds, \\ w_2^{j+1}(\xi^1,\xi^2) = \overline{w}_2^0(\xi^1) + \displaystyle\int_{\xi_0^2}^{\xi^2} (k_2 w_1^j + d_1)(\xi^1,s)ds. \end{cases}$$

Then one can prove that sequences $(w^j_\alpha)_j$ are Cauchy sequences in the space $H^m(D)$ and that there exists a constant $C > 0$ (depending on the $\mathcal{C}^4(\overline{D})$-norms of k_α but independent on j) such that

$$\textstyle\sum_\alpha \|w^j_\alpha\|_{H^m(D)} \leq C \sum_\alpha \|w^1_\alpha\|_{H^m(D)}.$$

Let w_α be the limit of sequence $(w^j_\alpha)_j$ in $H^m(D)$ as $j \to \infty$. Then (w_α) is solution to system (A.4) and satisfies the inequality

$$\textstyle\sum_\alpha \|w_\alpha\|_{H^m(D)} \leq C \sum_\alpha (\|\overline{w}^0_\alpha\|_{H^m(]0,1[)} + \|d_\alpha\|_{H^m(D)}).$$

This shows that system (A.3) possesses a unique solution $(u^n_1, u^n_2) \in H^4(\omega) \times H^4(\omega)$ (by taking $\overline{w}^0_\alpha = u^{0,n}_\alpha$, $d_\alpha = h^n_\alpha$ and $m = 4$ in (A.4)) and that $u^n_\alpha \to u_\alpha$ in $H^3(D)$ as $n \to \infty$ (by taking $\overline{w}^0_\alpha = u^{0,n}_\alpha - u^0_\alpha$, $d_\alpha = h^n_\alpha - h_\alpha$ and $m = 3$ in (A.4)). Moreover, system (A.3) shows that u^n_α vanish in ω^n_0.

Let now $v^n_1 = f_1 u^n_1, v^n_2 = f_2 u^n_2$. Then v^n_α vanish in ω^n_0 and satisfy the system (easily obtained from (A.3)):

$$\begin{cases} \partial_1 v^n_1 - \Gamma^1_{11} v^n_1 - \Gamma^2_{11} v^n_2 = f_1 h^n_1 \text{ in } D, \\ \partial_2 v^n_2 - \Gamma^2_{22} v^n_2 - \Gamma^1_{22} v^n_1 = f_2 h^n_2 \text{ in } D. \end{cases}$$

Since h^n_α vanishes in $\omega \cup \omega^n_0$, the restrictions of v^n_α to ω (still denoted v^n_α) satisfy the system

$$\begin{cases} \partial_1 v^n_1 - \Gamma^1_{11} v^n_1 - \Gamma^2_{11} v^n_2 = 0 \text{ in } \omega, \\ \partial_2 v^n_2 - \Gamma^2_{22} v^n_2 - \Gamma^1_{22} v^n_1 = 0 \text{ in } \omega. \end{cases}$$

Moreover, $v^n_\alpha = 0$ in $\omega \cap \omega^n_0$ and v^n_α converges to v_α in $H^3(\omega)$ as $n \to \infty$. Finally, let

$$v^n_3 = \frac{1}{2b_{12}}(\partial_2 v^n_1 + \partial_1 v^n_2 - 2\Gamma^\sigma_{12} v^n_\sigma).$$

Then $v^n_3 \in H^3(\omega)$, vanishes in $\omega \cap \omega^n_0$, and converges to v_3 in $H^2(\omega)$ as $n \to \infty$. The proof is complete. □

Remark A.1 *Any vector field $\boldsymbol{v} \in H^1_{\gamma_0}(\omega) \times H^1_{\gamma_0}(\omega) \times H^2_{\gamma_0}(\omega)$ that satisfies $\gamma_{\alpha\beta}(\boldsymbol{v}) = 0$ belongs in fact to the space $H^3_{\gamma_0}(\omega) \times H^3_{\gamma_0}(\omega) \times H^2_{\gamma_0}(\omega)$.* ■

Remark A.2 *If the set ω_0 reduces to a segment, one of the sets V_1 or V_2 reduces to a point. The above proof remains valid in this case.* ■

Proof of Theorem A.1. a) It suffices to prove that $\hat{\mathcal{V}}_{I\mathrm{f}}(\omega) \cap \mathcal{V}_{0\mathrm{f}}(\omega) \subset \hat{\mathcal{V}}_{I0\mathrm{f}}(\omega)$, the reverse inclusion being clearly satisfied. Let $(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \hat{\mathcal{V}}_{I\mathrm{f}}(\omega) \cap \mathcal{V}_{0\mathrm{f}}(\omega)$. This implies that $(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \mathcal{V}_{\mathrm{f}}(\omega)$, $R(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) = 0$ and $a^{\alpha\beta}\eta_{\alpha|\beta} + 2\rho_3 = 0$.

First, these relations show that $\boldsymbol{v} \in H^1_{\gamma_0}(\omega) \times H^1_{\gamma_0}(\omega) \times H^2_{\gamma_0}(\omega)$ and $\gamma_{\alpha\beta}(\boldsymbol{v}) = 0$. This allows to apply Lemma A.2, hence to show that there exists a sequence $(\boldsymbol{v}^n) \in H^4(\omega) \times H^4(\omega) \times H^3(\omega)$ vanishing in a neighborhood of γ_0, satisfying $\gamma_{\alpha\beta}(\boldsymbol{v}^n) = 0$, and converging to $\boldsymbol{v}$ in $H^3(\omega) \times H^3(\omega) \times H^2(\omega)$ as $n \to \infty$.

Then, the relation $R(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) = 0$ shows that $\eta_\alpha = -\partial_\alpha v_3 - b^\sigma_\alpha v_\sigma$ and $\eta_3 = 0$. Let $\boldsymbol{\eta}^n = (\eta^n_i)$ be defined by

$$\eta^n_\alpha = -\partial_\alpha v^n_3 - b^\sigma_\alpha v^n_\sigma \text{ and } \eta^n_3 = 0.$$

Clearly, $\boldsymbol{\eta}^n$ is in $(H^2(\omega))^3$ and vanishes in a neighborhood of γ_0, and $\boldsymbol{\eta}^n \to \boldsymbol{\eta}$ in $(H^1(\omega))^3$ as $n \to \infty$.

Next, recall that $a^{\alpha\beta}\eta_{\alpha|\beta} + 2\rho_3 = 0$. Then one can see that the functions defined by $\rho^n_3 = -\frac{1}{2}a^{\alpha\beta}\eta^n_{\alpha|\beta}$ belong to the space $H^1(\omega)$, vanish in a neighborhood of γ_0, and $\rho^n_3 \to \rho_3$ in $L^2(\omega)$ as $n \to \infty$.

Finally, the relation $2\rho_\alpha + \partial_\alpha\eta_3 \in L^2(\omega)$ implies that $\rho_\alpha \in L^2(\omega)$. Therefore there exist sequences (ρ^n_α) in $\mathcal{D}(\omega)$, hence in $H^1_{\gamma_0}(\omega)$, converging to ρ_α in $L^2(\omega)$ as $n \to \infty$.

The relations above show that the sequence $(\boldsymbol{v}^n, \boldsymbol{\eta}^n, \boldsymbol{\rho}^n)$ belongs to the space $\hat{\mathcal{V}}_I(\omega) \cap \mathcal{V}^\sharp_{00}(\omega)$ and converges to $(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho})$ in the $\|\cdot\|_{\mathrm{f}}$-norm.

b) It suffices to prove that $\mathcal{V}_{I\mathrm{f}}(\omega) \cap \mathcal{V}_{0\mathrm{f}}(\omega) \subset \mathcal{V}_{I0\mathrm{f}}(\omega)$. Let $(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \mathcal{V}_{I\mathrm{f}}(\omega) \cap \mathcal{V}_{0\mathrm{f}}(\omega)$ be fixed. Since $\mathcal{V}_I(\omega) \subset \hat{\mathcal{V}}_I(\omega)$, it follows that $(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \hat{\mathcal{V}}_{I\mathrm{f}}(\omega) \cap \mathcal{V}_{0\mathrm{f}}(\omega)$. Therefore, the proof of the first part of Theorem A.1 shows that there exist sequences $(\boldsymbol{v}^n) \in \mathcal{V}(\omega)$ converging to $\boldsymbol{v}$ in $(H^1(\omega))^3$, $(\boldsymbol{\eta}^n) = ((\eta^n_1, \eta^n_2, 0)) \in (H^2(\omega))^3$, with $\eta^n_\alpha = 0$ in a neighborhood of γ_0, converging to $\boldsymbol{\eta}$ in $(H^1(\omega))^3$, and $(\rho^n_3) \in H^1_{\gamma_0}(\omega)$ converging to ρ_3 in $L^2(\omega)$, such that $R(\boldsymbol{v}^n, \boldsymbol{\eta}^n, \boldsymbol{\rho}^n) = 0$ and $a^{\alpha\beta}\eta^n_{\alpha|\beta} + 2\rho^n_3 = 0$.

In view of Lemma 4.2, it suffices to prove that there exist sequences $(\rho^n_\alpha)_{n\in\mathbb{N}} \in H^1_{\gamma_0}(\omega)$, converging to ρ_α in $L^2(\omega)$, such that $g_{\rho^n} = -Hg_{\eta^n} - b_{\eta^n}$ and $b_{\rho^n} = 2(K - H^2)g_{\eta^n} - 2Hb_{\eta^n}$, where we denoted

$$g_\eta = a^{\alpha\beta}\eta_{\alpha|\beta} \text{ and } b_\eta = b^{\alpha\beta}\eta_{\alpha|\beta} \text{ for any } \eta_\alpha \in H^1(\omega).$$

This is the object of the Lemma A.3 below, which thus concludes the proof of Theorem A.1. □

Lemma A.3 *Assume that either assumption (H1') or assumption (H2) holds. Let* $(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \mathcal{V}_{I\mathrm{f}}(\omega) \cap \mathcal{V}_{0\mathrm{f}}(\omega)$ *and let there be given sequences* $\eta^n_\alpha \in H^2(\omega)$ *such that* $\eta^n_\alpha = 0$ *in the set* $\{(\xi^1, \xi^2) \in \omega; \mathrm{dist}((\xi^1, \xi^2), \gamma_0) < 1/n\}$ *and* $\eta^n_\alpha \to \eta_\alpha$ *in* $H^1(\omega)$ *as* $n \to \infty$. *Then there exist sequences* $(\rho^n_\alpha) \in H^1_{\gamma_0}(\omega)$ *such that* $g_{\rho^n} = -Hg_{\eta^n} - b_{\eta^n}$, $b_{\rho^n} = 2(K - H^2)g_{\eta^n} - 2Hb_{\eta^n}$, *and* $\rho^n_\alpha \to \rho_\alpha$ *in* $L^2(\omega)$ *as* $n \to \infty$. ■

Proof. We assume that assumption (H1') holds (the proof is simpler under assumption (H2)). Then there is no loss of generality in assuming that the parameter curves are the principal lines of the surface. Since the set $\overline{\omega}$ is convex in the $O\xi^1$-direction, there exist an interval $[c_L^2, c_R^2] \subset \mathbb{R}$ and two functions $L^1, R^1 : [c_L^2, c_R^2] \to \mathbb{R}$ such that

$$\overline{\omega} = \{(\xi^1, \xi^2); L^1(\xi^2) \leq \xi^1 \leq R^1(\xi^2), c_L^2 \leq \xi^2 \leq c_R^2\}.$$

To simplify the notations, we assume that $[c_L^2, c_R^2] = [0, 1]$. Since γ_0 is connected and its intersection with the lines parallel to $O\xi^1$ is a connected set, there exists an interval $[a^2, b^2] \subset [0, 1]$ such that either $\overline{\gamma_0} = \{(L^1(\xi^2), \xi^2); \xi^2 \in [a^2, b^2]\}$ or $\overline{\gamma_0} = \{(R^1(\xi^2), \xi^2); \xi^2 \in [a^2, b^2]\}$. We will assume the former. In the same way, there exist two functions $L^2, R^2 : [0, 1] \to \mathbb{R}$ and an interval $[a^1, b^1] \subset [0, 1]$ such that

$$\overline{\omega} = \{(\xi^1, \xi^2); 0 \leq \xi^1 \leq 1,\ L^2(\xi^1) \leq \xi^2 \leq R^2(\xi^1)\}$$

and $\overline{\gamma_0} = \{(\xi^1, L^2(\xi^1)); \xi^1 \in [a^1, b^1]\}$.

Let $(\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\rho}) \in \mathcal{V}_{I\mathrm{f}}(\omega) \cap \mathcal{V}_{0\mathrm{f}}(\omega)$. By arguing as in the proof of Lemma 4.2, one can see that the equations $g_\rho = -Hg_\eta - b_\eta$ and $b_\rho = 2(K - H^2)g_\eta - 2Hb_\eta$ are satisfied. Since the parameter curves are the principal lines of the surface, these equations can be written as

$$\begin{cases} a^{11}\rho_{1|1} + a^{22}\rho_{2|2} = -Hg_\eta - b_\eta, \\ b^{11}\rho_{1|1} + b^{22}\rho_{2|2} = 2(K - H^2)g_\eta - 2Hb_\eta. \end{cases}$$

It should be noted that the set ω, the Christoffel symbols and the coefficients of the first two fundamental forms of the surface are not those appearing in the proof of the previous lemma, albeit we use the same notations.

Since $a^{11}b^{22} - a^{22}b^{11} \neq 0$ in ω (due to the assumption that the surface is hyperbolic in (H1')), the previous system can be written as

$$\begin{cases} \partial_1\rho_1 - \Gamma_{11}^\sigma\rho_\sigma = F_1(\boldsymbol{\eta}), \\ \partial_2\rho_2 - \Gamma_{22}^\sigma\rho_\sigma = F_2(\boldsymbol{\eta}), \end{cases} \tag{A.5}$$

with

$$\begin{cases} F_1(\boldsymbol{\eta}) = \dfrac{(2(H^2 - K)a^{22} - Hb^{22})g_\eta + (2Ha^{22} - b^{22})b_\eta}{a^{11}b^{22} - a^{22}b^{11}} \\ F_2(\boldsymbol{\eta}) = \dfrac{(Hb^{11} - 2(H^2 - K)a^{11})g_\eta + (b^{11} - 2Ha^{11})b_\eta}{a^{11}b^{22} - a^{22}b^{11}}. \end{cases} \tag{A.6}$$

It is clear that functions ρ_1 and ρ_2 belong to the space $L^2(\omega)$. Equations (A.5) next imply that $\partial_1\rho_1$ and $\partial_2\rho_2$ also belong to the space $L^2(\omega)$. Now, we extend functions ρ_1 and ρ_2 to functions $\tilde{\rho}_1$ and $\tilde{\rho}_2$ defined over the set $D =]0, 1[\times]0, 1[$. To this end, define the extension operator

$$T_1 : u_1 \in \{v_1 \in \mathcal{C}^0(\overline{\omega}); \partial_1 v_1 \in L^2(\omega)\} \mapsto \tilde{u}_1 \in \{\tilde{v}_1 \in L^2(D); \partial_1\tilde{v}_1 \in L^2(D)\}$$

by letting

$$\tilde{u}_1(\xi^1,\xi^2)=\begin{cases} u_1(\xi^1,\xi^2) \text{ if } (\xi^1,\xi^2)\in\omega,\\ u_1(L^1(\xi^2),\xi^2) \text{ if } 0<\xi^2<1 \text{ and } 0<\xi^1<L^1(\xi^2),\\ u_1(R^1(\xi^2),\xi^2) \text{ if } 0<\xi^2<1 \text{ and } R^1(\xi^2)<\xi^1<1.\end{cases}$$

Then one can see that this operator is linear and that there exists a constant $C>0$ such that

$$\|\tilde{u}_1\|_{L^2(D)}+\|\partial_1\tilde{u}_1\|_{L^2(D)}\le C(\|u_1\|_{L^2(\omega)}+\|\partial_1 u_1\|_{L^2(\omega)}).$$

Consequently, there exists a continuous linear operator

$$\overline{T}_1: u_1\in\{v_1\in L^2(\omega);\partial_1 u_1\in L^2(\omega)\}\mapsto \tilde{u}_1\in\{\tilde{v}_1\in L^2(D);\partial_1\tilde{v}_1\in L^2(D)\}$$

such that $\overline{T}_1(u_1)=T_1(u_1)$ for all $u_1\in\{v_1\in\mathcal{C}^0(\overline{\omega});\partial_1 v_1\in L^2(\omega)\}$.

In the same way, one can define the continuous linear operator

$$\overline{T}_2: u_2\in\{v_2\in L^2(\omega);\partial_2 u_2\in L^2(\omega)\}\mapsto \tilde{u}_2\in\{\tilde{v}_2\in L^2(D);\partial_2\tilde{v}_2\in L^2(D)\}.$$

Let $\tilde{\rho}_1=\overline{T}_1(\rho_1)$ and $\tilde{\rho}_2=\overline{T}_2(\rho_2)$ and define their traces $\overline{\rho}_1=\gamma_1(\tilde{\rho}_1)\in L^2(D_1)$ and $\overline{\rho}_2=\gamma_2(\tilde{\rho}_2)\in L^2(D_2)$, where

$$D_1=\{(a^1,\xi^2);\xi^2\in]0,1[\} \text{ and } D_2=\{(\xi^1,a^2);\xi^1\in]0,1[\},$$

by using the continuous linear operators (of trace)

$$\gamma_1:\tilde{u}_1\in\{\tilde{v}_1\in L^2(D);\partial_1\tilde{v}_1\in L^2(D)\}\mapsto\overline{u}_1\in L^2(D_1)$$

and

$$\gamma_2:\tilde{u}_1\in\{\tilde{v}_1\in L^2(D);\partial_1\tilde{v}_1\in L^2(D)\}\mapsto\overline{u}_2\in L^2(D_2)$$

that extend the usual operators of trace defined over continuous functions. Using the fact that $(\boldsymbol{v},\boldsymbol{\eta},\boldsymbol{\rho})\in\mathcal{V}_{If}(\omega)$, a lengthy calculation shows that $\overline{\rho}_1=0$ a.e. in $V_1=\{(a^1,\xi^2);\xi^2\in]a^2,b^2[\}$ and $\overline{\rho}_2=0$ a.e. in $V_2=\{(\xi^1,a^2);\xi^1\in]a^1,b^1[\}$.

Let functions $\Gamma^\sigma_{\alpha\beta}$ be extended to functions $\tilde{\Gamma}^\sigma_{\alpha\beta}\in\mathcal{C}^1(\overline{D})$. Define $\tilde{F}_1=\partial_1\tilde{\rho}_1-\tilde{\Gamma}^\sigma_{11}\tilde{\rho}_\sigma$ and $\tilde{F}_2=\partial_2\tilde{\rho}_2-\tilde{\Gamma}^\sigma_{22}\tilde{\rho}_\sigma$ and note that these functions belong to the space $L^2(D)$ and satisfy

$$\tilde{F}_1=\tilde{F}_2=0 \text{ a.e. in } \overline{\omega}_0=\{(\xi^1,\xi^2)\in\overline{D};a^1\le\xi^1\le L^1(\xi^2),a^2\le\xi^2\le L^2(\xi^1)\}.$$

Consequently, there exists sequences $(\tilde{F}_\alpha(n))$ in $H^1(D)$ such that $\tilde{F}_\alpha(n)=0$ in $\omega_0^n=\{(\xi^1,\xi^2)\in D;\text{dist}((\xi^1,\xi^2),\overline{\omega}_0)<1/n\}$ and $\tilde{F}_\alpha(n)\to\tilde{F}_\alpha$ in $L^2(D)$ as $n\to\infty$.

Let $\tilde{F}_\alpha(\boldsymbol{\eta}^n)\in H^1(D)$, $\alpha=1,2$, be defined by first extending $F_\alpha(\boldsymbol{\eta}^n)$ with zero in ω_0^n (this can be done since the fields $F_\alpha(\boldsymbol{\eta}^n)$ vanish in a neighborhood of γ_0), then by using a continuous linear operator $E:H^1(\omega\cup\omega_0^n)\to H^1(D)$.

Let $\chi^n \in \mathcal{C}^1(\overline{D})$ be such that $0 \leq \chi^n \leq 1$, $\chi^n = 1$ on ω, and $\chi^n = 0$ on the set $\{(x_1, x_2) \in \overline{D}; \mathrm{dist}((x_1, x_2), \omega) > 1/n\}$. Define $\tilde{F}^n_\alpha = \chi^n \tilde{F}_\alpha(\boldsymbol{\eta}^n) + (1 - \chi^n)\tilde{F}_\alpha(n)$ Then it is clear that $\tilde{F}^n_\alpha \in H^1(D)$, $\tilde{F}^n_\alpha|_\omega = F_\alpha(\boldsymbol{\eta}^n)$, $\tilde{F}^n_\alpha = 0$ in ω^n_0, and $\tilde{F}^n_\alpha \to \tilde{F}_\alpha$ in $L^2(D)$ as $n \to \infty$.

Now, we can define sequences $(\rho^n_\alpha)_n$ satisfying the conditions of Lemma A.3. First, there exist functions $\overline{\rho}^n_1 \in \mathcal{C}^\infty(\overline{D}_1)$ with $\overline{\rho}^n_1 = 0$ on $V^n_1 = \{(a^1, \xi^2) \in D_1; \mathrm{dist}(\xi^2, [a^2, b^2]) < 1/n\}$ and $\overline{\rho}^n_2 \in \mathcal{C}^\infty(\overline{D}_2)$ with $\overline{\rho}^n_2 = 0$ on $V^n_2 = \{(\xi^1, a^2) \in D_2; \mathrm{dist}(\xi^1, [a^1, b^1]) < 1/n\}$ such that $\overline{\rho}^n_1 \to \overline{\rho}_1$ in $L^2(D_1)$ and $\overline{\rho}^n_2 \to \overline{\rho}_2$ in $L^2(D_2)$ as $n \to \infty$. Consider next the system

$$\begin{cases} \partial_1 \tilde{\rho}^n_1 - \tilde{\Gamma}^\sigma_{11} \tilde{\rho}^n_\sigma = \tilde{F}^n_1 \text{ in } L^2(D), \\ \partial_2 \tilde{\rho}^n_2 - \tilde{\Gamma}^\sigma_{22} \tilde{\rho}^n_\sigma = \tilde{F}^n_2 \text{ in } L^2(D), \\ \gamma_1(\tilde{\rho}^n_1) = \overline{\rho}^n_1 \text{ and } \gamma_2(\tilde{\rho}^n_2) = \overline{\rho}^n_2. \end{cases}$$

Since $\tilde{F}^n_1, \tilde{F}^n_2 \in H^1(D)$, one can prove (as in the proof of Lemma A.2) that this system possesses a unique solution $\tilde{\rho}^n_1, \tilde{\rho}^n_2 \in H^1(D)$ and that there exists a constant $C > 0$ such that

$$\sum_\alpha \|\tilde{\rho}^n_\alpha - \tilde{\rho}_\alpha\|_{L^2(D)} \leq C \sum_\alpha (\|\overline{\rho}^n_\alpha - \overline{\rho}_\alpha\|_{L^2(D_\alpha)} + \|\tilde{F}^n_\alpha - \tilde{F}_\alpha\|_{L^2(D)}).$$

Moreover, the fields $\tilde{\rho}^n_\alpha$ vanish on ω^n_0 since $\tilde{F}^n_\alpha = 0$ on ω^n_0 and $\overline{\rho}^n_\alpha = 0$ on V^n_α for $\alpha = 1, 2$.

Then the restrictions $\rho^n_\alpha = \tilde{\rho}^n_\alpha|_\omega$ belong to the space $H^1(\omega)$, vanish on γ_0, and satisfy $\rho^n_\alpha \to \rho_\alpha$ in $L^2(\omega)$ as $n \to \infty$. Moreover they satisfy the system

$$\begin{cases} \partial_1 \rho^n_1 - \Gamma^\sigma_{11} \rho^n_\sigma = F_1(\boldsymbol{\eta}^n) \text{ in } L^2(\omega), \\ \partial_2 \rho^n_2 - \Gamma^\sigma_{22} \rho^n_\sigma = F_2(\boldsymbol{\eta}^n) \text{ in } L^2(\omega), \end{cases}$$

which is equivalent to the system of equations $g_{\rho^n} = -H g_{\eta^n} - b_{\eta^n}$ and $b_{\rho^n} = 2(K - H^2) g_{\eta^n} - 2H b_{\eta^n}$. The proof is complete. □

Remark A.3 *The assumption that the intersection between $(\partial\mathcal{S})_0$ and any principal curve of $\mathcal{S} \cup \partial\mathcal{S}$ is a connected set can be omitted from assumptions (H1') if the sets enclosed within $(\partial\mathcal{S})_0$ and the principal lines of the surface starting from the end-points of $(\partial\mathcal{S})_0$ are contained in a set enclosed within $(\partial\mathcal{S})_0$ and the asymptotic lines of the surface starting from the end-points of $(\partial\mathcal{S})_0$.* ■

Remark A.4 *The proof under assumption (H2) uses the same parametrisation of the surface as that used in the proof of Lemma A.2.* ■