EXACT BOUNDARY CONTROLLABILITY OF A SYSTEM OF MIXED ORDER WITH ESSENTIAL SPECTRUM

FARID AMMAR KHODJA *, KARINE MAUFFREY †, AND ARNAUD MÜNCH ‡

Abstract. We address in this work the exact boundary controllability of a linear hyperbolic system of the form $u'' + Au = 0$ with $u = (u_1, u_2)^T$ posed in $(0,T) \times (0,1)^2$. A denotes a self-adjoint operator of mixed order, that usually appears in the modelization of linear elastic membrane shell. The operator A possesses an essential spectrum which prevents the exact controllability to hold uniformly with respect to the initial data $\left(u^0, u^1\right)$. We show that the exact controllability holds by a one Dirichlet control acting on the first variable u_1 for any initial data $\left(u^0, u^1\right)$ generated by the eigenfunctions corresponding to the discrete part of the spectrum $\sigma(A)$. The proof relies on a suitable observability inequality obtained by the way of a full spectral analysis and the adaptation of an Ingham type inequality for the Laplacian in two space dimension. This work provides a non trivial example of system controlled by a number of controls strictly lower than the number of components. Some numerical experiments illustrate our study.

Key words. boundary controllability, essential spectrum, Ingham inequality, mixed order operator

AMS subject classifications. 35L20, 93B05, 93B07, 93B60, 93C20

1. Introduction - Problem statement.

Let $\Omega = (0,1)^2$ and Γ be the part of $\partial\Omega$ defined by $\Gamma = \{(x,y) \in \partial\Omega, xy = 0\}$. Let T be a positive real number and a, α be two real numbers such that $a > \alpha^2 > 0$ and $\sqrt{a - \alpha^2}/\pi \notin \mathbb{N}^*$. We analyze in this work the exact boundary controllability of the following system in $u = (u_1, u_2)^T$

$$\begin{cases} u_1'' = \Delta u_1 + \alpha \partial_x u_2 & \text{in } Q_T = \Omega \times (0,T) \\ u_2'' = -\alpha \partial_x u_1 - a u_2 & \text{in } Q_T \\ u_1 = v\, 1_\Gamma & \text{on } \Sigma_T = \partial\Omega \times (0,T) \\ (u(\cdot,0), u'(\cdot,0)) = \left(u^0, u^1\right) & \text{in } \Omega. \end{cases} \tag{1.1}$$

We set $H = L^2(\Omega) \times L^2(\Omega)$ and $H_{1/2} = H_0^1(\Omega) \times L^2(\Omega)$. Let $H_{-1/2}$ denotes the dual of $H_{1/2}$ with respect to the pivot space H. System (1.1) is said exactly controllable at time $T > 0$ if for any initial data $\left(u^0, u^1\right) \in H \times H_{-1/2}$ and any target $\left(u_T^0, u_T^1\right) \in H \times H_{-\frac{1}{2}}$ there exists a control function v in a suitable space, such that the unique solution $u = (u_1, u_2)^T$ of system (1.1) satisfies

$$(u(\cdot,T), u'(\cdot,T)) = \left(u_T^0, u_T^1\right) \quad \text{in } \Omega.$$

We point out that the variable u_2 in system (1.1) is free of any condition on the boundary $\partial\Omega$. In particular, system (1.1) provides a non trivial example for which the number of controls is strictly lower than the number of components.

*Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon cedex, France (farid.ammar-khodja@univ-fcomte.fr).

†Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, Université de Franche-Comte, 16 route de Gray, 25030 Besançon cedex, France (karine.mauffrey@univ-fcomte.fr).

‡Laboratoire de Mathématiques de Clermont-Ferrand, UMR CNRS 6620, Université Blaise Pascal, Campus des Cézeaux, 63177 Aubière cedex, France (arnaud.munch@math.univ-bpclermont.fr).
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As it is usual, the controllability issue is equivalent to an observability inequality that we will rigorously prove to be:

$$\left\|\left(\Phi^0,\Phi^1\right)\right\|^2_{H_{1/2}\times H}\leq C\int_0^T\int_\Gamma\left(\frac{\partial\varphi}{\partial\nu}+\alpha\psi\nu_1\right)^2 d\sigma dt, \tag{1.2}$$

($\nu=(\nu_1,\nu_2)$ denotes the unit outward normal to Γ) for any initial data $\left(\Phi^0,\Phi^1\right)$ belonging to $H_{1/2}\times H$, for the homogeneous adjoint system in $\Phi=(\varphi,\psi)^T$:

$$\begin{cases}\varphi''=\Delta\varphi+\alpha\partial_x\psi & \text{in } Q_T\\ \psi''=-\alpha\partial_x\varphi-a\psi & \text{in } Q_T\\ \varphi=0 & \text{on } \Sigma_T\\ (\Phi(\cdot,0),\Phi'(\cdot,0))=\left(\Phi^0,\Phi^1\right) & \text{in } \Omega.\end{cases} \tag{1.3}$$

The observation zone Γ is defined so that the triplet (Ω,Γ,T) satisfies the geometric optic condition. The natural operator we want to consider to transform (1.3) into a second order differential equation is the operator $A_0=\begin{pmatrix}-\Delta & -\alpha\partial_x\\ \alpha\partial_x & a\end{pmatrix}$ in $L^2(\Omega)\times L^2(\Omega)$ with domain $D(A_0)=\left(H^2(\Omega)\cap H^1_0(\Omega)\right)\times D(\partial_x)$. Here ∂_x is considered as an unbounded operator in $L^2(\Omega)$. We can prove that this operator A_0 is not closed in $L^2(\Omega)\times L^2(\Omega)$. Therefore we have to consider its closure A which is defined by : $A:D(A)\subset L^2(\Omega)\times L^2(\Omega)\to L^2(\Omega)\times L^2(\Omega)$,

$$A\begin{pmatrix}\varphi\\ \psi\end{pmatrix}=\begin{pmatrix}-\Delta\left(\varphi+\alpha\Delta^{-1}\partial_x\psi\right)\\ \alpha\partial_x\varphi+a\psi\end{pmatrix}$$

and $D(A)=\left\{(\varphi,\psi)^T\in H^1_0(\Omega)\times L^2(\Omega)/\varphi+\alpha\Delta^{-1}\partial_x\psi\in H^2(\Omega)\right\}$ (see [1]).

The originality and difficulty of the - apparently simple - system (1.3) are related to the fact that A is a mixed order operator, and therefore, possesses a non-empty essential spectrum $\sigma_{ess}(A)$, as shown in [6] in a general situation. As a consequence, the observability does not hold uniformly with respect to the data $\left(\Phi^0,\Phi^1\right)$. Precisely, in [10] the authors exhibit Weil sequences, associated with some elements of $\sigma_{ess}(A)$ for which the observability inequality (1.2) is not true. The observability is therefore only expected, roughly speaking, in the orthogonal of some space related to the essential spectrum. To our knowledge, the only way used up to now to address this kind of problem is based on spectral analysis and Ingham type approach which allows to prove the observability for the discrete part of the spectrum, provided some spectral gap conditions (see [11]). In that framework, the existing literature mainly concerns the controllability of dynamical systems modeling the vibrations of elastic membrane shell, where precisely mixed order and self-adjoint operators appear (we refer to [19] for a detailed spectral analysis). We also mention [8, 9] where the controllability of an hemi-spherical cap is studied using a nonharmonic spectral analysis. The analysis, reduced to the one space dimension by axial symmetry, exhibits the loss of uniform observability due to the essential spectrum composed of a single positive element. A similar study is performed in [2] for a nonuniform elliptic operator A for which $0\in\sigma_{ess}(A)$. We also refer to the chapter 5 of [12] for results based on some recent extensions of Ingham type inequalities. For systems of this kind, the uniform partial controllability, which consists to drive to rest only a restricted number of components is proved in [13]. The observability is obtained by a so-called *spectral compensation* argument, remarking that the bad behavior of the part of the spectrum which accumulates to $\sigma_{ess}(A)$ is somehow compensated by the suitable gap of the discrete part. In a different context, we also mention [3, 16, 18] for the controllability of systems with spectral accumulation point.

The significant novelty of this present work with respect to the literature mentioned above is that it concerns the two space dimension. The proof of a positive observability result for the discrete part of the spectrum, much less straightforward (than for the one-dimensional situation), is obtained by the adaptation of a recent Ingham type theorem due to Mehrenberger [17] that allows to prove the observability for the wave equation in any space dimension. An other difficulty in the study of the controllability for system (1.1) is that the associated control operator is not bounded from the space of controls $L^2(\Gamma \times (0,T))$ to the state space X_{-1} (the dual of $X_1 = H_{1/2} \times H$ with respect to the pivot space $X = H \times H_{-1/2}$).

The paper is organized as follows. In Section 2, we first state the spectral properties of the operator A. Explicit computations show that the set of the eigenvalues of A is composed of three parts: $\{\lambda^-_{p,q}\}_{p,q\geq 1} \cup \{\lambda^+_{p,q}\}_{p,q\geq 1} \cup \{a\}$, where $\{\lambda^-_{p,q}\}_{p,q\geq 1}$ is a bounded sequence which accumulates on the full interval $[a-\alpha^2, a] = \sigma_{ess}(A)$ and $\{\lambda^+_{p,q}\}_{p,q\geq 1}$ - the set of isolated eigenvalues of A of finite multiplicity - is an unbounded sequence such that $\lambda^+_{p,q} \sim (p^2+q^2)\pi^2$ for p, q large. In Section 3, we establish the well-posedness of the boundary value problem (1.1) for v and any initial data (u^0, u^1) in suitable spaces. We divide Section 4 into two parts. The first part is devoted to the analysis of the adjoint system and to the formulation of the observability inequality as (1.2). In the second part, we prove the observability for any initial data (Φ^0, Φ^1) belonging to $H^+_{1/2} \times H^+$, the space spanned by the eigenfunctions $\{e^+_{p,q}\}_{p,q\geq 1}$ associated with the eigenvalues $\{\lambda^+_{p,q}\}_{p,q\geq 1}$. The keypoint is that the sequence $\{\lambda^+_{p,q}\}_{p,q\geq 1}$ enjoys gap properties similar to those of the eigenvalues of $-\Delta$ with Dirichlet boundary values, used in [17]. On the contrary, Section 5 exhibits the lack of observability in spaces related to the essential spectrum. In particular, by numerical approximation, we check that the corresponding observability constant $C^-(T)$ is not bounded uniformly with respect to (Φ^0, Φ^1). We also discuss the uniform observability with respect to the coupling parameter α (as $\alpha \to 0$). Section 6 concludes this work with some remarks and open problems.

2. The operator A.

We consider the operator A defined by : $A : D(A) \subset L^2(\Omega) \times L^2(\Omega) = H \to H$,

$$
\begin{gathered}
A \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} -\Delta\left(\varphi + \alpha \Delta^{-1} \partial_x \psi\right) \\ \alpha \partial_x \varphi + a\psi \end{pmatrix}, \\
D(A) = \left\{ (\varphi, \psi)^T \in H^1_0(\Omega) \times L^2(\Omega) / \varphi + \alpha \Delta^{-1} \partial_x \psi \in H^2(\Omega) \right\}.
\end{gathered} \tag{2.1}
$$

It is well-known and easy to check that the eigenvalues and normalized eigenfunctions of $-\Delta$ with domain $H^2(\Omega) \cap H^1_0(\Omega)$ are respectively given by $\mu_{pq} = \left(p^2+q^2\right)\pi^2$ and $\varphi_{pq}(x,y) = 2\sin(p\pi x)\sin(q\pi y)$ for (p,q) in $\mathbb{N}^* \times \mathbb{N}^*$ and $(x,y) \in \Omega = (0,1)^2$. We introduce the following notations: for $p, q \geq 1$

$$\begin{aligned}
&\lambda^{\pm}_{p,q} = \tfrac{1}{2}\left(\mu_{pq} + a \pm \sqrt{\left(\mu_{pq} - a\right)^2 + 4\alpha^2 p^2 \pi^2}\right),\\
&\psi_{pq}(x,y) = 2\cos(p\pi x)\sin(q\pi y),\\
&e^{\pm}_{p,q} = \left(\frac{(\lambda^{\pm}_{p,q} - a)}{\sqrt{\left(\lambda^{\pm}_{p,q} - a\right)^2 + \alpha^2 p^2 \pi^2}}\varphi_{pq}, \frac{\alpha p \pi}{\sqrt{\left(\lambda^{\pm}_{p,q} - a\right)^2 + \alpha^2 p^2 \pi^2}}\psi_{pq}\right)^T,\\
&e_q(x,y) = \left(0, \sqrt{2}\sin(q\pi y)\right)^T.
\end{aligned}$$

Lemma 2.1.

1. *For all $(p,q) \in \mathbb{N}^* \times \mathbb{N}^*$, we have:* $Ae^{\pm}_{p,q} = \lambda^{\pm}_{p,q} e^{\pm}_{p,q}, \quad Ae_q = ae_q$.
   $\operatorname{Ker}(A - \lambda I) = \{0\}, \ \forall \lambda \notin \{a\} \cup \{\lambda^{+}_{p,q}\}_{p,q\geq 1} \cup \{\lambda^{-}_{p,q}\}_{p,q\geq 1}$.
   In other words, the sets of eigenvalues and associated eigenfunctions of A are respectively $\{a\} \cup \{\lambda^{+}_{p,q}\}_{p,q\geq 1} \cup \{\lambda^{-}_{p,q}\}_{p,q\geq 1}$ and $\{e_q\}_{q\geq 1} \cup \left\{e^{+}_{p,q}\right\}_{p,q\geq 1} \cup \left\{e^{-}_{p,q}\right\}_{p,q\geq 1}$.
2. *We have $\lambda^{+}_{p,q} \underset{\|(p,q)\| \to +\infty}{\sim} \mu_{pq}$.*

Proof. Let λ be an eigenvalue of A and $u = (u_1, u_2)^T$ be an associated eigenvector. Then u is a non-zero solution of the system

$$\begin{cases} (\lambda - a)\, u_2 = \alpha \partial_x u_1 \\ \Delta\left(u_1 + \alpha \Delta^{-1} \partial_x u_2\right) + \lambda u_1 = 0 \\ u_{1|\partial\Omega} = 0. \end{cases} \tag{2.2}$$

Assuming first that $\lambda = a$, system (2.2) implies that $\partial_x u_1 = 0$, $\Delta\left(u_1 + \alpha\Delta^{-1}\partial_x u_2\right) = 0$ in Ω together with the boundary condition $u_{1|\partial\Omega} = 0$. This implies that $u_1 = 0$ and $\partial_x u_2 = 0$ in Ω. Consequently, we can write $u_2(x,y) = f(y)$ with $f \in L^2(0,1)$. It follows that $\lambda = a$ is an eigenvalue of A and the associated eigenspace is $\{(x,y) \mapsto (0, f(y))^T : f \in L^2(0,1)\}$. Now, if $\lambda \neq a$, then system (2.2) may be written as

$$\begin{cases} u_2 = \frac{\alpha}{\lambda - a} \partial_x u_1 \\ \left(1 + \frac{\alpha^2}{\lambda - a}\right) \partial_{xx} u_1 + \partial_{yy} u_1 + \lambda u_1 = 0 \\ u_{1|\partial\Omega} = 0. \end{cases} \tag{2.3}$$

We can look for u_1 in the form $u_1 = \sum_{p,q\in\mathbb{N}^*} u_{p,q}\varphi_{pq}$ with $\sum_{p,q\in\mathbb{N}^*} (u_{p,q})^2 < +\infty$. u_1 is a solution of (2.3) if and only if for all $(p,q) \in \mathbb{N}^* \times \mathbb{N}^*$,

$$\left(-p^2\pi^2\left(1 + \frac{\alpha^2}{\lambda - a}\right) - q^2\pi^2 + \lambda\right) u_{p,q} = 0. \tag{2.4}$$

It follows from (2.4) that λ satisfies the eigenvalue equation

$$\lambda^2 - \left(a + \left(p^2 + q^2\right)\pi^2\right)\lambda + \left(\left(a - \alpha^2\right)p^2 + aq^2\right)\pi^2 = 0. \tag{2.5}$$

We conclude that $\lambda = \frac{1}{2}\left(a + \left(p^2+q^2\right)\pi^2 \pm \sqrt{\left(a - \left(p^2+q^2\right)\pi^2\right)^2 + 4\alpha^2 p^2 \pi^2}\right) = \lambda^{\pm}_{p,q}$. It is easily seen that $Ae^{\pm}_{p,q} = \lambda^{\pm}_{p,q} e^{\pm}_{p,q}$. This completes the proof of item 1. Item 2 is easy to check. ∎

We refer to [5] for the definition of the essential spectrum of A, $\sigma_{ess}(A)$. As a consequence of the results in [6], we have: $\sigma_{ess}(A) = \left[a-\alpha^2, a\right]$. On the other hand, we can check that the set of accumulation points of the sequence $\left(\lambda^-_{p,q}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$ is $\sigma_{ess}(A) = [a-\alpha^2, a]$.

REMARK 2.2.

1. *The asymptotic behavior of* λ^+_{pq} *in Lemma 2.1 is in agreement with [7] where it is shown that the asymptotic behavior of* $\sigma(A)$ *is related to the spectrum of the principal part of* A*, in our case* $-\Delta$.
2. *We check that*

$$\lambda^-_{p,q} \leq a < \lambda^+_{1,1} \leq \lambda^+_{p,q}, \quad \forall (p,q) \in \mathbb{N}^* \times \mathbb{N}^*.$$

Thus all the $\lambda^+_{p,q}$ *are isolated eigenvalues (of finite multiplicity).*
Similarly, simple computations lead to:

$$\forall p \geq 1, \ \left(\lambda^-_{p,q} \in [a-\alpha^2, a] \Leftrightarrow q > \sqrt{a-\alpha^2}/\pi\right)$$

Therefore, if $\sqrt{a-\alpha^2}/\pi < 1$*, then* $\overline{\left(\lambda^-_{p,q}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}} = \sigma_{ess}(A)$ *and* $\sigma_{ess}(A)$ *is bounded away from* $\left(\lambda^+_{p,q}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$.
Figures 2.1 and 2.2 illustrate these two points.

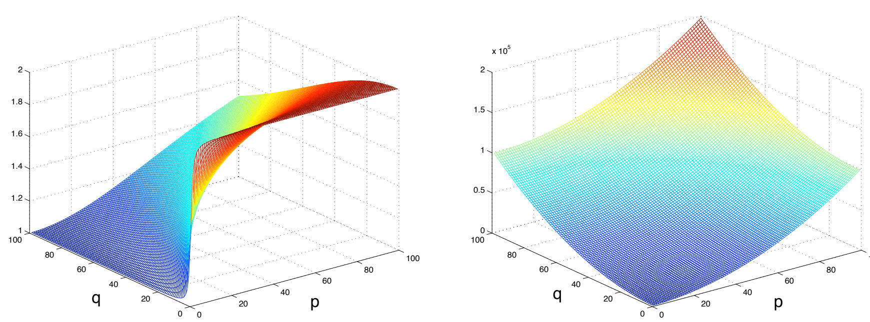

FIG. 2.1. *Graphs of* $\left(\lambda^-_{p,q}\right)_{(p,q)\in[1,100]^2}$ ***(Left)*** *and of* $\left(\lambda^+_{p,q}\right)_{(p,q)\in[1,100]^2}$ ***(Right)*** *for* $\alpha = 1$ *and* $a = 2$.

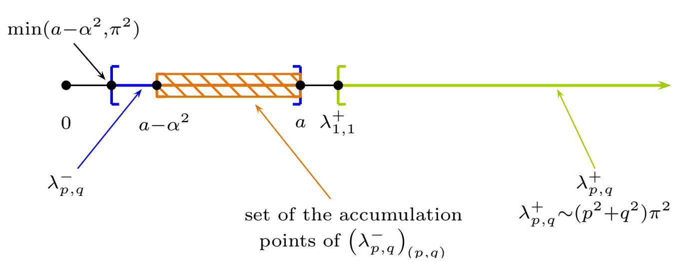

FIG. 2.2. *Distribution of* $\sigma(A)$ *along* $\mathbb{R}$.

3. Using the fact that the families $\{\varphi_{pq}\}_{p\geq 1, q\geq 1}$ and $\{\psi_{pq}\}_{p\geq 0, q\geq 1}$ are two Hilbert bases of $L^2(\Omega)$, we can easily check that the family $\mathcal{B} = \{e^+_{p,q}\}_{p\geq 1, q\geq 1} \cup \{e^-_{p,q}\}_{p\geq 1, q\geq 1} \cup \{e_q\}_{q\geq 1}$ forms a Hilbert basis of $H = L^2(\Omega)\times L^2(\Omega)$.

The operator A defined in (2.1) is self-adjoint in H, and positive if, as we have assumed it, $a > \alpha^2$. This last point comes from the formula:

$$\langle Au, u\rangle_H = \int_\Omega \left[(\partial_x u_1 + \alpha u_2)^2 + \left(a-\alpha^2\right) u_2^2 + \left(\partial_y u_1\right)^2\right] dx\, dy,$$

where $u = (u_1, u_2)^T \in D(A)$.

For any $\delta \in \mathbb{R}_+$, we recall that the operator A^δ is defined by

$$A^\delta = \sum_{p,q\geq 1} \left(\lambda^+_{p,q}\right)^\delta \left\langle \cdot, e^+_{p,q}\right\rangle_H e^+_{p,q} + \sum_{p,q\geq 1} \left(\lambda^-_{p,q}\right)^\delta \left\langle \cdot, e^-_{p,q}\right\rangle_H e^-_{p,q} + a^\delta \sum_{q\geq 1} \left\langle \cdot, e_q\right\rangle_H e_q,$$

$$D\left(A^\delta\right) = \left\{\phi \in H, \sum_{p,q\geq 1}\left(\lambda^+_{p,q}\right)^{2\delta}\left\langle \phi, e^+_{p,q}\right\rangle_H^2 + \sum_{p,q\geq 1}\left(\lambda^-_{p,q}\right)^{2\delta}\left\langle \phi, e^-_{p,q}\right\rangle_H^2 + a^{2\delta}\sum_{q\geq 1}\left\langle \phi, e_q\right\rangle_H^2 < \infty\right\}.$$

Since $\left(\lambda^-_{p,q}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$ is bounded, $D\left(A^\delta\right) = \left\{\phi \in H, \sum_{p,q\geq 1} \left(\lambda^+_{p,q}\right)^{2\delta}\left\langle \phi, e^+_{p,q}\right\rangle_H^2 < \infty\right\}$.

In the sequel, we will set

$$H_\delta = D\left(A^\delta\right),\ \delta \geq 0.$$

The operator A is a bounded operator from $D(A)$, equipped with the graph norm, to H. It is well-known that A can be extended to a bounded operator from H to $D(A)'$, the dual space of $D(A)$ with respect to the pivot space H. We continue to denote this extension by A and, thus, A can be seen as an unbounded self-adjoint operator on $D(A)'$ with domain H. A is also a unitary operator from $D(A)$ to H and from H to $D(A)'$. We will set in the sequel:

$$H_{-1} = D(A)'.$$

The other extension of A we will use later is the following. A can also be extended as a unitary operator from $D(A^{1/2})$ equipped with the graph norm to $D(A^{1/2})'$, the dual space of $D(A^{1/2})$ with respect to the pivot space H. We will set:

$$H_{1/2} = D\big(A^{1/2}\big),\ H_{-1/2} = D\big(A^{1/2}\big)'.$$

For details on these extensions, see for instance [20].

The last notations we will need in the next sections are the following:

$$H^\pm = \operatorname{span}\left(\left\{e^\pm_{p,q},\ p,q\geq 1\right\}\right),\ H^a = \operatorname{span}\left(\left\{e_q,\ q\geq 1\right\}\right)$$

and for $\delta \in \mathbb{R}$,

$$H^\pm_\delta = H_\delta \cap H^\pm,\ H^a_\delta = H_\delta \cap H^a.$$

3. Well-posedness of the controlled system. This section is devoted to the study of existence and uniqueness of solution for system (1.1).

3.1. A Dirichlet map. In this part, we introduce and analyze the Dirichlet map $\mathcal{D} : D(\mathcal{D}) \subset L^2(\Gamma) \to H$ corresponding to system (1.1) defined as follows: $D(\mathcal{D}) = \left\{v \in L^2(\Gamma), \mathcal{D}v \in H\right\}$ and for $v \in L^2(\Gamma)$, we denote by $\mathcal{D}v$ the solution $\theta = (\theta_1, \theta_2)^T$ of the abstract elliptic problem

$$\begin{cases} \Lambda\theta = 0 & \text{in } \Omega \\ \theta_1 = v\, 1_\Gamma & \text{on } \partial\Omega \end{cases}, \qquad \Lambda = \begin{pmatrix} -\Delta & -\alpha\partial_x \\ \alpha\partial_x & a \end{pmatrix}. \tag{3.1}$$

The map $\mathcal{D}$ satisfies the following continuity property:

PROPOSITION 3.1. *For every $\epsilon \in (1/2, 1]$, $\mathcal{D} \in \mathcal{L}(H^\epsilon(\Gamma), H)$. Moreover for every $v \in D(\mathcal{D})$ we have*

$$\left\langle \mathcal{D}v, e^{\pm}_{p,q} \right\rangle_H = \frac{p\pi\left(\lambda^{\pm}_{p,q} - a + \alpha^2\right)}{\lambda^{\pm}_{p,q}\sqrt{\left(\lambda^{\pm}_{p,q} - a\right)^2 + \alpha^2p^2\pi^2}} v_{2,q} + \frac{q\pi\left(\lambda^{\pm}_{p,q} - a\right)}{\lambda^{\pm}_{p,q}\sqrt{\left(\lambda^{\pm}_{p,q} - a\right)^2 + \alpha^2p^2\pi^2}} v_{1,p} \tag{3.2}$$

$$\left\langle \mathcal{D}v, e_q \right\rangle_H = \frac{\alpha}{a\sqrt{2}} v_{2,q} \tag{3.3}$$

with $v_{1,p} = 2\int_0^1 v(x,0)\sin(p\pi x)\,dx$ *and* $v_{2,q} = 2\int_0^1 v(0,y)\sin(q\pi y)\,dy$.

Proof. Let $\epsilon \in (1/2, 1]$ and $v \in H^\epsilon(\Gamma)$. Suppose that (3.1) has a solution $\theta \in H$. Using integrations by parts, we have

$$0 = \left\langle \Lambda\theta, e^{\pm}_{p,q} \right\rangle_H = \left\langle \theta, Ae^{\pm}_{p,q} \right\rangle_H + \int_\Gamma \left(\frac{\partial \varphi^{\pm}_{p,q}}{\partial \nu} + \alpha\psi^{\pm}_{p,q}\nu_1 \right) v \, d\sigma dt$$

where $e^{\pm}_{p,q} = \left(\varphi^{\pm}_{p,q}, \psi^{\pm}_{p,q}\right)^T$. Since $Ae^{\pm}_{p,q} = \lambda^{\pm}_{p,q} e^{\pm}_{p,q}$, this yields

$$\left\langle \theta, e^{\pm}_{p,q} \right\rangle_H = \frac{p\pi\left(\lambda^{\pm}_{p,q} - a + \alpha^2\right)}{\lambda^{\pm}_{p,q}\sqrt{\left(\lambda^{\pm}_{p,q} - a\right)^2 + \alpha^2p^2\pi^2}} v_{2,q} + \frac{q\pi\left(\lambda^{\pm}_{p,q} - a\right)}{\lambda^{\pm}_{p,q}\sqrt{\left(\lambda^{\pm}_{p,q} - a\right)^2 + \alpha^2p^2\pi^2}} v_{1,p}. \tag{3.4}$$

By the same arguments, we obtain

$$\left\langle \theta, e_q \right\rangle_H = \frac{\alpha}{a\sqrt{2}} v_{2,q}. \tag{3.5}$$

This proves that if (3.1) has a solution θ, then this solution is unique and it writes

$$\theta = \sum_{p,q} \left\langle \theta, e^{+}_{p,q} \right\rangle_H e^{+}_{p,q} + \sum_{p,q} \left\langle \theta, e^{-}_{p,q} \right\rangle_H e^{-}_{p,q} + \sum_{q} \left\langle \theta, e_q \right\rangle_H e_q,$$

with $\left\langle \theta, e^{\pm}_{p,q} \right\rangle_H$ and $\left\langle \theta, e_q \right\rangle_H$ given by (3.4) and (3.5). Now, we have to check that such a θ is an element of H. Formula (3.4) gives

$$\left|\left\langle \theta, e^{+}_{p,q} \right\rangle_H\right| \leq \frac{p^{\epsilon+1}\pi\left(\lambda^{+}_{p,q} - a + \alpha^2\right)}{\lambda^{+}_{p,q}\sqrt{\left(\lambda^{+}_{p,q} - a\right)^2 + \alpha^2p^2\pi^2}} \frac{|v_{2,q}|}{p^\epsilon} + \frac{q^{\epsilon+1}\pi\left(\lambda^{+}_{p,q} - a\right)}{\lambda^{+}_{p,q}\sqrt{\left(\lambda^{+}_{p,q} - a\right)^2 + \alpha^2p^2\pi^2}} \frac{|v_{1,p}|}{q^\epsilon}.$$

From the asymptotic property $\lambda^{+}_{p,q} \underset{\|(p,q)\| \to +\infty}{\sim} \mu_{pq}$, we have

$$\frac{\pi\left(\lambda^{+}_{p,q} - a + \alpha^2\right)}{\lambda^{+}_{p,q}\sqrt{\left(\lambda^{+}_{p,q} - a\right)^2 + \alpha^2p^2\pi^2}} \underset{\|(p,q)\| \to +\infty}{\sim} \frac{\pi\left(\lambda^{+}_{p,q} - a\right)}{\lambda^{+}_{p,q}\sqrt{\left(\lambda^{+}_{p,q} - a\right)^2 + \alpha^2p^2\pi^2}} \underset{\|(p,q)\| \to +\infty}{\sim} \frac{\pi}{\mu_{pq}}.$$

Consequently, there exists a positive constant c_1, independant of v, such that for every $(p,q)\in\mathbb{N}^*\times\mathbb{N}^*$,

$$\frac{\pi\left(\lambda^+_{p,q}-a+\alpha^2\right)}{\lambda^+_{p,q}\sqrt{\left(\lambda^+_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\le\frac{c_1}{\mu_{pq}}\quad and\quad\frac{\pi\left(\lambda^+_{p,q}-a\right)}{\lambda^+_{p,q}\sqrt{\left(\lambda^+_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\le\frac{c_1}{\mu_{pq}}.$$

Hence

$$\frac{p^{\epsilon+1}\pi\left(\lambda^+_{p,q}-a+\alpha^2\right)}{\lambda^+_{p,q}\sqrt{\left(\lambda^+_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\le c_1\frac{p^{\epsilon+1}}{\mu_{pq}}\le c_1\frac{p^2}{\mu_{pq}}\le c_1\ and\ \frac{q^{\epsilon+1}\pi\left(\lambda^+_{p,q}-a\right)}{\lambda^+_{p,q}\sqrt{\left(\lambda^+_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\le c_1.$$

Finally

$$\left|\left\langle\theta,e^+_{p,q}\right\rangle_H\right|\le c_1\left(\frac{|v_{2,q}|}{p^\epsilon}+\frac{|v_{1,p}|}{q^\epsilon}\right).$$

This gives

$$\sum_{p,q\ge1}\left(\left\langle\theta,e^+_{p,q}\right\rangle_H\right)^2\le 2{c_1}^2\sum_{r\ge1}\frac{1}{r^{2\epsilon}}\left(\sum_{q\in\mathbb{N}^*}(v_{2,q})^2+\sum_{p\in\mathbb{N}^*}(v_{1,p})^2\right)=4{c_1}^2\sum_{r\ge1}\frac{1}{r^{2\epsilon}}\|v\|^2_{L^2(\Gamma)}.\tag{3.6}$$

From (3.4), we also have

$$\left|\left\langle\theta,e^-_{p,q}\right\rangle_H\right|\le\left|\frac{p^{\epsilon+1}\pi\left(\lambda^-_{p,q}-a+\alpha^2\right)}{q^\epsilon\lambda^-_{p,q}\sqrt{\left(\lambda^-_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\right|\frac{q^\epsilon|v_{2,q}|}{p^\epsilon}+\left|\frac{q^{\epsilon+1}\pi\left(\lambda^-_{p,q}-a\right)}{p^\epsilon\lambda^-_{p,q}\sqrt{\left(\lambda^-_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\right|\frac{p^\epsilon|v_{1,p}|}{q^\epsilon}.$$

Using the definition of $\lambda^-_{p,q}$, we can prove that as $\|(p,q)\|\to+\infty$

$$\frac{\pi\left(\lambda^-_{p,q}-a+\alpha^2\right)}{\lambda^-_{p,q}\sqrt{\left(\lambda^-_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\sim\frac{|\alpha|}{\lambda^-_{p,q}}\frac{q^2}{p\left(p^2+q^2\right)},\quad\frac{\pi\left(\lambda^-_{p,q}-a\right)}{\lambda^-_{p,q}\sqrt{\left(\lambda^-_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\sim-\frac{|\alpha|}{\lambda^-_{p,q}}\frac{p}{p^2+q^2}.$$

Since $\left(\lambda^-_{p,q}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$ is bounded away from 0, there exists a constant $c_2>0$ such that for every $(p,q)\in\mathbb{N}^*\times\mathbb{N}^*$:

$$\left|\frac{\pi\left(\lambda^-_{p,q}-a+\alpha^2\right)}{\lambda^-_{p,q}\sqrt{\left(\lambda^-_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\right|\le c_2\frac{q^2}{p\left(p^2+q^2\right)},\quad\left|\frac{\pi\left(\lambda^-_{p,q}-a\right)}{\lambda^-_{p,q}\sqrt{\left(\lambda^-_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\right|\le c_2\frac{p}{p^2+q^2}.$$

This implies that:

$$\left|\frac{p^{\epsilon+1}\pi\left(\lambda^-_{p,q}-a+\alpha^2\right)}{q^\epsilon\lambda^-_{p,q}\sqrt{\left(\lambda^-_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\right|\le c_2\frac{p^\epsilon q^{2-\epsilon}}{p^2+q^2},\quad\left|\frac{q^{\epsilon+1}\pi\left(\lambda^-_{p,q}-a\right)}{p^\epsilon\lambda^-_{p,q}\sqrt{\left(\lambda^-_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\right|\le c_2\frac{p^{1-\epsilon}q^{\epsilon+1}}{p^2+q^2}.$$

We have $\frac{p^\epsilon q^{2-\epsilon}}{p^2+q^2}\le1$ and $\frac{p^{1-\epsilon}q^{\epsilon+1}}{p^2+q^2}\le1$ since $0\le\epsilon\le1$. We conclude that

$$\left|\left\langle\theta,e^-_{p,q}\right\rangle_H\right|\le c_2\left(\frac{q^\epsilon|v_{2,q}|}{p^\epsilon}+\frac{p^\epsilon|v_{1,p}|}{q^\epsilon}\right).$$

Since $v \in H^{\epsilon}(\Gamma)$, we can write from this inequality

$$\sum_{p,q\geq 1}\left(\left\langle \theta, e^{-}_{p,q}\right\rangle_H\right)^2 \leq 2{c_2}^2\sum_{r\geq 1}\frac{1}{r^{2\epsilon}}\left(\sum_{q\in\mathbb{N}^*}\left(q^{\epsilon}v_{2,q}\right)^2+\sum_{p\in\mathbb{N}^*}\left(p^{\epsilon}v_{1,p}\right)^2\right) \leq 4{c_2}^2\sum_{r\geq 1}\frac{1}{r^{2\epsilon}}\left\|v\right\|^2_{H^{\epsilon}(\Gamma)}. \tag{3.7}$$

Besides, formula (3.5) clearly gives $\left(\left\langle \theta, e_q\right\rangle_H\right)_{q\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$, with

$$\sum_{q\in\mathbb{N}^*}\left(\left\langle \theta, e_q\right\rangle_H\right)^2 = \frac{\alpha^2}{2a^2}\sum_{q\in\mathbb{N}^*}\left(v_{2,q}\right)^2 \leq \frac{\alpha^2}{a^2}\left\|v\right\|^2_{L^2(\Gamma)}. \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we obtain $\theta \in H$, with $\left\|\theta\right\|_H \leq c_{\epsilon}\left\|v\right\|_{H^{\epsilon}(\Gamma)}$ and $c_{\epsilon} = \max\left(\frac{2c_1}{\pi}\sqrt{\sum_{r\geq 1}\frac{1}{r^{2\epsilon}}}, 2c_2\sqrt{\sum_{r\geq 1}\frac{1}{r^{2\epsilon}}}, \frac{|\alpha|}{a}\right)$. □

Proposition 3.1 gives that $H^{\epsilon}(\Gamma) \subset D(\mathcal{D})$ for every $\epsilon \in (1/2, 1]$. This implies that $\mathcal{D} : D(\mathcal{D}) \subset L^2(\Gamma) \to H$ is an unbounded operator with dense domain in $L^2(\Gamma)$. Consequently, the adjoint operator $\mathcal{D}^*$ of $\mathcal{D}$ is well-defined as an unbounded operator $\mathcal{D}^* : D(\mathcal{D}^*) \subset H \to L^2(\Gamma)$.

PROPOSITION 3.2. *$\mathcal{D}^*$ is given by*

$$\begin{aligned} &D(\mathcal{D}^*) = \left\{ g \in H, \int_{\Gamma}\left(\frac{\partial\left(A^{-1}g\right)_1}{\partial\nu} + \alpha\left(A^{-1}g\right)_2\nu_1\right)^2 d\sigma < +\infty\right\} \\ &\mathcal{D}^* g = \left(\frac{\partial\left(A^{-1}g\right)_1}{\partial\nu} + \alpha\left(A^{-1}g\right)_2\nu_1\right)_{|\Gamma} \ \textit{where}\ A^{-1}g = \begin{pmatrix} \left(A^{-1}g\right)_1 \\ \left(A^{-1}g\right)_2 \end{pmatrix}, \forall g \in D(\mathcal{D}^*). \end{aligned}$$

Proof. For $v \in D(\mathcal{D})$ and $g \in H$, we have

$$\left\langle \mathcal{D}v,g\right\rangle_H = \sum_{p,q\geq 1}\left(\left\langle \mathcal{D}v,e^{+}_{p,q}\right\rangle_H\left\langle g,e^{+}_{p,q}\right\rangle_H + \left\langle \mathcal{D}v,e^{-}_{p,q}\right\rangle_H\left\langle g,e^{-}_{p,q}\right\rangle_H\right) + \sum_{q\geq 1}\left\langle \mathcal{D}v,e_q\right\rangle_H\left\langle g,e_q\right\rangle_H.$$

By (3.2) and (3.3), we have

$$\begin{aligned} \left\langle \mathcal{D}v,g\right\rangle_H = &\sum_{p\geq 1}v_{1,p}\sum_{q\geq 1}\left(\frac{q\pi\left(\lambda^{+}_{p,q}-a\right)\left\langle g,e^{+}_{p,q}\right\rangle_H}{\lambda^{+}_{p,q}\sqrt{\left(\lambda^{+}_{p,q}-a\right)^2+\alpha^2p^2\pi^2}} + \frac{q\pi\left(\lambda^{-}_{p,q}-a\right)\left\langle g,e^{-}_{p,q}\right\rangle_H}{\lambda^{-}_{p,q}\sqrt{\left(\lambda^{-}_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\right) \\ &+\sum_{q\geq 1}v_{2,q}\left(\sum_{p\geq 1}\left(\frac{p\pi\left(\lambda^{+}_{p,q}-a+\alpha^2\right)\left\langle g,e^{+}_{p,q}\right\rangle_H}{\lambda^{+}_{p,q}\sqrt{\left(\lambda^{+}_{p,q}-a\right)^2+\alpha^2p^2\pi^2}} + \frac{p\pi\left(\lambda^{-}_{p,q}-a+\alpha^2\right)\left\langle g,e^{-}_{p,q}\right\rangle_H}{\lambda^{-}_{p,q}\sqrt{\left(\lambda^{-}_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\right)\right. \\ &\qquad\left. + \frac{\alpha}{a\sqrt{2}}\left\langle g, e_q\right\rangle_H\right). \end{aligned} \tag{3.9}$$

By definition of the adjoint of an unbounded operator, $D(\mathcal{D}^*)$ is the set of the elements $g \in H$ such that $v \mapsto \left\langle \mathcal{D}v, g\right\rangle_H$ is a continuous linear form on $L^2(\Gamma)$. Using (3.9), we see that $D(\mathcal{D}^*)$ is the set of the elements $g \in H$ such that the two following sequences are in $l^2(\mathbb{N}^*)$: $\left(\sum_{q\geq 1}\left(\frac{q\pi\left(\lambda^{+}_{p,q}-a\right)\left\langle g,e^{+}_{p,q}\right\rangle_H}{\lambda^{+}_{p,q}\sqrt{\left(\lambda^{+}_{p,q}-a\right)^2+\alpha^2p^2\pi^2}} + \frac{q\pi\left(\lambda^{-}_{p,q}-a\right)\left\langle g,e^{-}_{p,q}\right\rangle_H}{\lambda^{-}_{p,q}\sqrt{\left(\lambda^{-}_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\right)\right)_{p\geq 1}$,

$\left(\sum_{p\geq 1}\left(\frac{p\pi\left(\lambda_{p,q}^{+}-a+\alpha^{2}\right)\left\langle g,e_{p,q}^{+}\right\rangle_{H}}{\lambda_{p,q}^{+}\sqrt{\left(\lambda_{p,q}^{+}-a\right)^{2}+\alpha^{2}p^{2}\pi^{2}}}+\frac{p\pi\left(\lambda_{p,q}^{-}-a+\alpha^{2}\right)\left\langle g,e_{p,q}^{-}\right\rangle_{H}}{\lambda_{p,q}^{-}\sqrt{\left(\lambda_{p,q}^{-}-a\right)^{2}+\alpha^{2}p^{2}\pi^{2}}}\right)+\frac{\alpha}{a\sqrt{2}}\left\langle g,e_{q}\right\rangle_{H}\right)_{q\geq 1}$. It is easily seen that

$$\left\|\frac{\partial\left(A^{-1}g\right)_{1}}{\partial\nu}+\alpha\left(A^{-1}g\right)_{2}\nu_{1}\right\|_{L^{2}(\Gamma)}^{2}$$
$$=\sum_{q\geq 1}\left(\sum_{p\geq 1}\left(\frac{p\pi\left(\lambda_{p,q}^{+}-a+\alpha^{2}\right)\left\langle g,e_{p,q}^{+}\right\rangle_{H}}{\lambda_{p,q}^{+}\sqrt{\left(\lambda_{p,q}^{+}-a\right)^{2}+\alpha^{2}p^{2}\pi^{2}}}+\frac{p\pi\left(\lambda_{p,q}^{-}-a+\alpha^{2}\right)\left\langle g,e_{p,q}^{-}\right\rangle_{H}}{\lambda_{p,q}^{-}\sqrt{\left(\lambda_{p,q}^{-}-a\right)^{2}+\alpha^{2}p^{2}\pi^{2}}}\right)+\frac{\alpha}{a\sqrt{2}}\left\langle g,e_{q}\right\rangle_{H}\right)^{2}$$
$$+\sum_{p\geq 1}\left(\sum_{q\geq 1}\left(\frac{q\pi\left(\lambda_{p,q}^{+}-a\right)\left\langle g,e_{p,q}^{+}\right\rangle_{H}}{\lambda_{p,q}^{+}\sqrt{\left(\lambda_{p,q}^{+}-a\right)^{2}+\alpha^{2}p^{2}\pi^{2}}}+\frac{q\pi\left(\lambda_{p,q}^{-}-a\right)\left\langle g,e_{p,q}^{-}\right\rangle_{H}}{\lambda_{p,q}^{-}\sqrt{\left(\lambda_{p,q}^{-}-a\right)^{2}+\alpha^{2}p^{2}\pi^{2}}}\right)\right)^{2}.$$

This allows to conclude that $D\left(\mathcal{D}^{*}\right)=\left\{g\in H,\left\|\frac{\partial\left(A^{-1}g\right)_{1}}{\partial\nu}+\alpha\left(A^{-1}g\right)_{2}\nu_{1}\right\|_{L^{2}(\Gamma)}<\infty\right\}$. Now we suppose that $g\in D\left(\mathcal{D}^{*}\right)$. Then we can easily check that (3.9) leads to $\left\langle\mathcal{D}v,g\right\rangle_{H}=\int_{\Gamma}v\left(\frac{\partial\left(A^{-1}g\right)_{1}}{\partial\nu}+\alpha\left(A^{-1}g\right)_{2}\nu_{1}\right)d\sigma$ and then $\mathcal{D}^{*}g=\left(\frac{\partial\left(A^{-1}g\right)_{1}}{\partial\nu}+\alpha\left(A^{-1}g\right)_{2}\nu_{1}\right)_{|\Gamma}$.
□

Proposition 3.3. *$\mathcal{D}:D\left(\mathcal{D}\right)\subset L^{2}\left(\Gamma\right)\to H$ is a closed operator on $L^{2}\left(\Gamma\right)$.*

Proof. Let $\left(v_{n}\right)_{n\in\mathbb{N}}$ be a sequence in $D\left(\mathcal{D}\right)$ which converges to a certain v in $L^{2}\left(\Gamma\right)$ and such that $\mathcal{D}v_{n}$ converges in H to an element θ. Let $v_{1,p}^{n}$ and $v_{2,q}^{n}$ denote

$$v_{1,p}^{n}=2\int_{0}^{1}v_{n}(x,0)\sin\left(p\pi x\right)dx,\quad v_{2,q}^{n}=2\int_{0}^{1}v_{n}(0,y)\sin\left(q\pi y\right)dy.$$

Similarly, let $v_{1,p}=2\int_{0}^{1}v(x,0)\sin\left(p\pi x\right)dx$ and $v_{2,q}=2\int_{0}^{1}v(0,y)\sin\left(q\pi y\right)dy$. Since

$$\left\|v_{n}-v\right\|_{L^{2}(\Gamma)}^{2}=\frac{1}{2}\left(\sum_{p\geq 1}\left|v_{1,p}^{n}-v_{1,p}\right|^{2}+\sum_{q\geq 1}\left|v_{2,q}^{n}-v_{2,q}\right|^{2}\right),$$

we clearly have: $v_{1,p}^{n}\underset{n\to+\infty}{\to}v_{1,p}$ for any $p\in\mathbb{N}^{*}$ and $v_{2,q}^{n}\underset{n\to+\infty}{\to}v_{2,q}$ for any $q\in\mathbb{N}^{*}$. From (3.2) and (3.3) we deduce that for every $(p,q)\in\mathbb{N}^{*}\times\mathbb{N}^{*}$:

$$\left\langle\mathcal{D}v_{n},e_{p,q}^{\pm}\right\rangle_{H}\underset{n\to+\infty}{\to}\frac{p\pi\left(\lambda_{p,q}^{\pm}-a+\alpha^{2}\right)}{\lambda_{p,q}^{\pm}\sqrt{\left(\lambda_{p,q}^{\pm}-a\right)^{2}+\alpha^{2}p^{2}\pi^{2}}}v_{2,q}+\frac{q\pi\left(\lambda_{p,q}^{\pm}-a\right)}{\lambda_{p,q}^{\pm}\sqrt{\left(\lambda_{p,q}^{\pm}-a\right)^{2}+\alpha^{2}p^{2}\pi^{2}}}v_{1,p}$$
$$\left\langle\mathcal{D}v_{n},e_{q}\right\rangle_{H}\underset{n\to+\infty}{\to}\frac{\alpha}{a\sqrt{2}}v_{2,q}.$$

Besides, the convergence of $\mathcal{D}v_{n}$ to θ in H implies the convergence of $\left\langle\mathcal{D}v_{n},e_{p,q}^{\pm}\right\rangle_{H}$ (resp. $\left\langle\mathcal{D}v_{n},e_{q}\right\rangle_{H}$) to $\left\langle\theta,e_{p,q}^{\pm}\right\rangle_{H}$ (resp. $\left\langle\theta,e_{q}\right\rangle_{H}$) for every $(p,q)\in\mathbb{N}^{*}\times\mathbb{N}^{*}$. We deduce that, for all $(p,q)\in\mathbb{N}^{*}\times\mathbb{N}^{*}$

$$\left\langle\theta,e_{p,q}^{\pm}\right\rangle_{H}=\frac{p\pi\left(\lambda_{p,q}^{\pm}-a+\alpha^{2}\right)}{\lambda_{p,q}^{\pm}\sqrt{\left(\lambda_{p,q}^{\pm}-a\right)^{2}+\alpha^{2}p^{2}\pi^{2}}}v_{2,q}+\frac{q\pi\left(\lambda_{p,q}^{\pm}-a\right)}{\lambda_{p,q}^{\pm}\sqrt{\left(\lambda_{p,q}^{\pm}-a\right)^{2}+\alpha^{2}p^{2}\pi^{2}}}v_{1,p}$$
$$\left\langle\theta,e_{q}\right\rangle_{H}=\frac{\alpha}{a\sqrt{2}}v_{2,q}.$$

From (3.2) and (3.3), it means that $\theta=\mathcal{D}v$. Since $\theta\in H$, this implies that $v\in D\left(\mathcal{D}\right)$. We have thus proved that $\mathcal{D}$ is closed. □

3.2. Toward an internal control problem. We introduce the following notations:

$$X = H \times H_{-1/2},\ X_{-1} = H_{-1/2} \times H_{-1},\ X_1 = H_{1/2} \times H,$$

and

$$L : D\left(L\right) \subset X_{-1} \to X_{-1},\ L = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix},\ D\left(L\right) = X.$$

Note that the operator occuring in the definition of L is the extension of A from H to H_{-1}.

In this part we transform the boundary control problem (1.1) into the familiar form of an internal control problem:

$$\begin{cases} Z' = LZ + Bv & \text{in } Q_T \\ Z(0) = \left(u^0, u^1\right)^T & \text{in } \Omega. \end{cases} \tag{3.10}$$

The originality of this problem is that B is an unbounded control operator from $L^2\left(\Gamma\right)$ to X_{-1}.

Assume that v is an element of $H^1\left([0,T], D\left(\mathcal{D}\right)\right)$. For the moment we denote by Z the vector $Z = \left(u, u'\right)^T - \left(\mathcal{D}v, 0\right)^T$ where u is solution of (1.1). Then Z is solution of

$$Z' = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix} Z - \begin{pmatrix} \mathcal{D}v' \\ 0 \end{pmatrix}.$$

Since $v \in H^1\left([0,T], D\left(\mathcal{D}\right)\right)$, we have that $\mathcal{D}v' \in L^2\left([0,T], H\right)$ so that $\left(\mathcal{D}v', 0\right)^T$ is an element of $L^2\left([0,T], D\left(L\right)\right)$. Therefore Z is solution of

$$Z' = LZ - \begin{pmatrix} \mathcal{D}v' \\ 0 \end{pmatrix} \quad \text{in} \quad X_{-1}$$

and the semigroup theory gives

$$Z(t) = S(t)Z(0) - \int_0^t S(t-s) \begin{pmatrix} \mathcal{D}v'(s) \\ 0 \end{pmatrix} ds \tag{3.11}$$

where $\left(S(t)\right)_{t \geq 0}$ is the C^0-semigroup associated with the maximal and dissipative operator L. Integrating by parts in (3.11) and using $\begin{pmatrix} \mathcal{D}v(s) \\ 0 \end{pmatrix} \in D\left(L\right)$, we obtain

$$Z(t) = S(t)Z(0) - \begin{pmatrix} \mathcal{D}v(t) \\ 0 \end{pmatrix} + S(t) \begin{pmatrix} \mathcal{D}v(0) \\ 0 \end{pmatrix} - \int_0^t S(t-s)L \begin{pmatrix} \mathcal{D}v(s) \\ 0 \end{pmatrix} ds.$$

Replacing $Z(t)$ by its definition, we obtain the following expression for $\left(u(t), u'(t)\right)^T$:

$$\begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} = S(t) \begin{pmatrix} u^0 \\ u^1 \end{pmatrix} + \int_0^t S(t-s)Bv(s)ds \tag{3.12}$$

where

$$B = \begin{pmatrix} 0 \\ A\mathcal{D} \end{pmatrix} : D(B) \subset L^2\left(\Gamma\right) \to X_{-1} \tag{3.13}$$

is an unbounded operator with dense domain $D(B) = D(\mathcal{D})$. Formula (3.12) means that $(u, u')^T$ is the mild solution Z of the internal control system (3.10).

THEOREM 3.4. *For every* $(u^0, u^1) \in X$ *and every* $v \in H^1([0,T], D(\mathcal{D}))$*, system (1.1) has a unique solution* u *in* X_{-1}*. Moreover,* $u \in C([0,T], H) \cap C^1([0,T], H_{-1/2}) \cap C^2([0,T], H_{-1})$.

Proof. Let $(u^0, u^1) \in X$ and $v \in H^1([0,T], D(\mathcal{D}))$. Set $Z^0 = (u^0, u^1)^T$. We have to solve system (3.10) in X_{-1}. Let us prove that $Bv \in H^1([0,T], X_{-1})$:

$$\begin{aligned}
\|Bv\|^2_{H^1([0,T],X_{-1})} &= \|Bv\|^2_{L^2([0,T],X_{-1})} + \|Bv'\|^2_{L^2([0,T],X_{-1})} \\
&= \|A\mathcal{D}v\|^2_{L^2([0,T],H_{-1})} + \|A\mathcal{D}v'\|^2_{L^2([0,T],H_{-1})} \\
&= \|\mathcal{D}v\|^2_{L^2([0,T],H)} + \|\mathcal{D}v'\|^2_{L^2([0,T],H)} \\
&= \|Dv\|^2_{H^1([0,T],H)} \\
&\leq \|v\|^2_{H^1([0,T],D(\mathcal{D}))} < +\infty.
\end{aligned}$$

Since $Bv \in H^1([0,T], X_{-1})$ and $Z^0 \in X$, the semigroup theory (see [20, Theorem 4.1.6 page 113]) ensures that system (3.10) has a unique solution Z in X_{-1} and that $Z \in C([0,T], X) \cap C^1([0,T], X_{-1})$. The existence and uniqueness of the solution u of (1.1) with the regularity $u \in C([0,T], H) \cap C^1([0,T], H_{-1/2}) \cap C^2([0,T], H_{-1})$ follow. □

4. Controllability and observability.

4.1. Formulation of the observability inequality. We introduce the control operator $L_T : D(L_T) \subset L^2(\Gamma \times (0,T)) \to X$ defined by

$$L_T v = \int_0^T S(T-t)Bv(t)dt, \tag{4.1}$$

with domain $D(L_T) = \{v \in L^2(\Gamma \times (0,T)), L_T v \in X\}$. The exact boundary controllability problem for system (1.1) is the following: given $T > 0$ large enough, initial data $(u^0, u^1) \in X$ and final data $(u^0_T, u^1_T) \in X$, to find a control function v in $D(L_T)$, such that the solution u of system (1.1) satisfies $u(.,T) = u^0_T$, $u'(.,T) = u^1_T$ in Ω. Therefore, system (1.1) is exactly controllable at time T if and only if the operator L_T is onto.

The following proposition may be proved using similar arguments as in Proposition 3.3.

PROPOSITION 4.1. $L_T : D(L_T) \subset L^2(\Gamma \times (0,T)) \to X$ *is a closed operator.*

By Theorem 3.4, $D(L_T)$ contains $H^1([0,T], D(\mathcal{D}))$, which is dense in $L^2(\Gamma \times (0,T))$ since $D(\mathcal{D})$ is dense in $L^2(\Gamma)$. Hence $D(L_T)$ is dense in $L^2(\Gamma \times (0,T))$. This allows to compute the adjoint L_T^* of L_T. Since L_T is closed, the surjectivity of L_T is equivalent (see for instance [4, Theorem II.19 page 29]) to the existence of a positive constant c such that

$$\left\| L_T^* \begin{pmatrix} \Psi^0 \\ \Psi^1 \end{pmatrix} \right\|^2_{L^2(\Sigma_T)} \geq c \left\| \begin{pmatrix} \Psi^0 \\ \Psi^1 \end{pmatrix} \right\|^2_X, \ \forall \begin{pmatrix} \Psi^0 \\ \Psi^1 \end{pmatrix} \in D(L_T^*).$$

This last inequality is also equivalent to:

$$\left\| L_T^* L \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} \right\|^2_{L^2(\Sigma_T)} \geq c \left\| \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} \right\|^2_{X_1}, \ \forall \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} \in D(L_T^* L). \tag{4.2}$$

Consequently, the exact controllability problem for system (1.1) relies on the "observability inequality" (4.2). In what follows, we compute $L_T^* L$ to translate (4.2) in terms of the adjoint system (1.3) of system (1.1).

By the definition of L_T in (4.1) we have $L_T^* = B^* S(T-\cdot)^*$. So we have to compute the adjoint of B.

LEMMA 4.2. *The adjoint $B^* : D(B^*) \subset X_1 \to L^2(\Gamma)$ of the operator B defined in (3.13) is given by:*

$$\begin{aligned} &D(B^*) = \left\{ \left(\Phi^0, \Phi^1\right)^T \in X_1 \ / \ \Phi^1 \in D(\mathcal{D}^*) \right\} \\ &B^* \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} = \mathcal{D}^* \Phi^1, \ \forall \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} \in D(B^*). \end{aligned} \tag{4.3}$$

Moreover for every $\begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} \in D(B^* L)$ *we have*

$$B^* L \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} = -\left(\frac{\partial \Phi_1^0}{\partial \nu} + \alpha \Phi_2^0 \nu_1 \right)_{|\Gamma}, \ \textit{where } \Phi^0 = \left(\Phi_1^0, \Phi_2^0\right)^T. \tag{4.4}$$

Proof. Let $v \in D(B) = D(\mathcal{D})$ and $\begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} \in X_1$. We recall that since A is self-adjoint in H, L is skew-adjoint in X_{-1} (i.e. $L^* = -L$). This allows to write

$$\begin{aligned} \left\langle \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix}, Bv \right\rangle_{X_1, X_{-1}} &= \left\langle \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix}, LL^{-1}Bv \right\rangle_{X_1, X_{-1}} = \left\langle L^* \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix}, L^{-1}Bv \right\rangle_X \\ &= -\left\langle L \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix}, L^{-1}Bv \right\rangle_X = -\left\langle \begin{pmatrix} \Phi^1 \\ -A\Phi^0 \end{pmatrix}, L^{-1}Bv \right\rangle_X. \end{aligned}$$

It is easily seen that $L^{-1} = \begin{pmatrix} 0 & -A^{-1} \\ I & 0 \end{pmatrix}$ so that $L^{-1}Bv = \begin{pmatrix} -\mathcal{D}v \\ 0 \end{pmatrix}$. It follows that

$$\left\langle \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix}, Bv \right\rangle_{X_1, X_{-1}} = \left\langle \Phi^1, \mathcal{D}v \right\rangle_H.$$

This gives (4.3). Now suppose that $\begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} \in D(B^* L)$. Then

$$B^* L \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} = B^* \begin{pmatrix} \Phi^1 \\ -A\Phi^0 \end{pmatrix} = -\mathcal{D}^* \left(A\Phi^0\right).$$

Proposition 3.2 easily gives (4.4). ☐

PROPOSITION 4.3. *The operator $L_T^* L : D(L_T^* L) \subset X_1 \to L^2(\Gamma \times (0,T))$ is given by:*

$$\begin{aligned} &D(L_T^* L) = \left\{ \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} \in X_1 \ / \ \frac{\partial \varphi}{\partial \nu} + \alpha \psi \nu_1 \in L^2(\Gamma \times (0,T)) \right\} \\ &L_T^* L \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} = -\left(\frac{\partial \varphi}{\partial \nu} + \alpha \psi \nu_1 \right)_{|\Gamma} \end{aligned} \tag{4.5}$$

where $\Phi = (\varphi, \psi)^T$ is the solution of the backward adjoint system of (1.1):

$$\begin{cases} \Phi'' + A\Phi = 0 & \text{in } Q_T \\ \Phi(.,T) = \Phi^0, \Phi'(.,T) = \Phi^1 & \text{in } \Omega. \end{cases} \tag{4.6}$$

Proof. Let $\begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} \in X_1$ such that $\begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} \in D(L_T^* L)$. Then

$$L_T^* L \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} = B^* S(T-\cdot)^* L \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} = B^* L S(T-\cdot)^* \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} = B^* L \begin{pmatrix} \Phi \\ \widetilde{\Phi} \end{pmatrix}$$

where $\begin{pmatrix} \Phi \\ \widetilde{\Phi} \end{pmatrix}$ is the solution of

$$\begin{cases} \begin{pmatrix} \Phi \\ \widetilde{\Phi} \end{pmatrix}' = -L^* \begin{pmatrix} \Phi \\ \widetilde{\Phi} \end{pmatrix} = L \begin{pmatrix} \Phi \\ \widetilde{\Phi} \end{pmatrix} & \text{in } [0,T] \\ \begin{pmatrix} \Phi \\ \widetilde{\Phi} \end{pmatrix}(T) = \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix}. \end{cases}$$

By (4.4) we have $B^* L \begin{pmatrix} \Phi \\ \widetilde{\Phi} \end{pmatrix} = -\left(\frac{\partial \varphi}{\partial \nu} + \alpha \psi \nu_1\right)_{|\Gamma}$, where $\Phi = (\varphi, \psi)^T$. This proves the proposition. □

Thanks to Proposition 4.3, the observability inequality (4.2) consists in the following inequality:

$$\forall \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} \in D(L_T^* L) \quad \int_0^T \int_\Gamma \left(\frac{\partial \varphi}{\partial \nu} + \alpha \psi \nu_1 \right)^2 d\sigma dt \geq c \left\| \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix} \right\|_{X_1}^2$$

where $\Phi = (\varphi, \psi)^T$ is the solution of the backward adjoint system (4.6). Remark that this is also equivalent to the same inequality when $\Phi = (\varphi, \psi)^T$ denotes the solution of the forward adjoint system (1.3). Consequently we have the following characterization of the controllability:

COROLLARY 4.4. *System (1.1) is exactly controllable at time T if and only if there exists a constant $C(T) > 0$ such that for all initial data $(\Phi^0, \Phi^1)^T \in D(L_T^* L)$, the solution $\Phi = (\varphi, \psi)^T$ of the adjoint system (1.3) with initial data (Φ^0, Φ^1) satisfies the following observability inequality*

$$\left\| (\Phi^0, \Phi^1) \right\|_{X_1}^2 \leq C(T) \int_0^T \int_\Gamma \left(\frac{\partial \varphi}{\partial \nu} + \alpha \psi \nu_1 \right)^2 d\sigma dt. \tag{4.7}$$

4.2. Observability inequality in $H^+_{1/2} \times H^+$. From now on, we assume that $a \leq 2\pi^2$. The main result of this paper is the following uniform observability of (1.3) for initial data in $D(L_T^* L) \cap \left(H^+_{1/2} \times H^+\right)$. Our proof adapts an Ingham type theorem due to Mehrenberger (see [17]) which allows to prove, by a spectral method, the observability inequality for the wave equation. The crucial point here is that the discrete part $\{\lambda^+_{p,q}\}_{p,q \geq 1}$ of $\sigma(A)$ has the same asymptotic behavior as $\sigma(-\Delta)$.

THEOREM 4.5. *Let* $\gamma = \frac{\pi\sqrt{\pi}}{4\sqrt{2\pi+|\alpha|}}$ *and* $T_0 = \frac{2\pi}{\gamma}\sqrt{1+2\frac{\left(\lambda^+_{1,1}-a+\alpha^2\right)^2}{\left(\lambda^+_{1,1}-a\right)^2}}$. *If* $a \le 2\pi^2$, *then for any* $T > T_0$ *there exists a positive constant* $C^+(T)$ *such that for all initial data* $\left(\Phi^0,\Phi^1\right)^T$ *in* $D\left(L_T^* L\right) \cap \left(H^+_{1/2} \times H^+\right)$ *the solution of (1.3) satisfies the following observability inequality:*

$$\left\|\left(\Phi^0,\Phi^1\right)\right\|^2_{X_1} \le C^+(T)\int_0^T\int_\Gamma\left(\frac{\partial\varphi}{\partial\nu}+\alpha\psi\nu_1\right)^2 d\sigma dt. \tag{4.8}$$

4.2.1. The observability inequality in terms of Fourier series. Fix $\left(\Phi^0,\Phi^1\right)^T \in D\left(L_T^* L\right) \cap \left(H^+_{1/2} \times H^+\right)$. By definition of $H^+_{1/2}$ and H^+, Φ^0 and Φ^1 may be written as

$$\Phi^0 = \sum_{p,q\ge1}\Phi^0_{p,q}e^+_{p,q}, \quad \Phi^1 = \sum_{p,q\ge1}\Phi^1_{p,q}e^+_{p,q}$$

with $\sum_{p,q\ge1}\lambda^+_{p,q}\left(\Phi^0_{p,q}\right)^2 < +\infty$ and $\sum_{p,q\ge1}\left(\Phi^1_{p,q}\right)^2 < +\infty$. The solution $\Phi = (\varphi,\psi)^T$ of system (1.3) with initial data $\left(\Phi^0,\Phi^1\right)$ is given by

$$\Phi(t) = \frac{1}{2}\sum_{p,q\ge1}\left[\left(\Phi^0_{p,q} - i\frac{\Phi^1_{p,q}}{\sqrt{\lambda^+_{p,q}}}\right)e^{i\sqrt{\lambda^+_{p,q}}t} + \left(\Phi^0_{p,q} + i\frac{\Phi^1_{p,q}}{\sqrt{\lambda^+_{p,q}}}\right)e^{-i\sqrt{\lambda^+_{p,q}}t}\right]e^+_{p,q}.$$

Using the definition of $e^+_{p,q}$ and the Parseval equality, we can easily prove that

$$\begin{aligned}
&\int_0^T\int_\Gamma\left(\frac{\partial\varphi}{\partial\nu}+\alpha\psi\nu_1\right)^2 d\sigma dt \\
&= \frac{1}{2}\sum_{q\ge1}\int_0^T\left|\sum_{p\ge1}p\pi\left[\left(\Phi^0_{p,q}-i\frac{\Phi^1_{p,q}}{\sqrt{\lambda^+_{p,q}}}\right)e^{i\sqrt{\lambda^+_{p,q}}t}+\left(\Phi^0_{p,q}+i\frac{\Phi^1_{p,q}}{\sqrt{\lambda^+_{p,q}}}\right)e^{-i\sqrt{\lambda^+_{p,q}}t}\right]\frac{\lambda^+_{p,q}-a+\alpha^2}{\sqrt{\left(\lambda^+_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\right|^2 dt \\
&+ \frac{1}{2}\sum_{p\ge1}\int_0^T\left|\sum_{q\ge1}q\pi\left[\left(\Phi^0_{p,q}-i\frac{\Phi^1_{p,q}}{\sqrt{\lambda^+_{p,q}}}\right)e^{i\sqrt{\lambda^+_{p,q}}t}+\left(\Phi^0_{p,q}+i\frac{\Phi^1_{p,q}}{\sqrt{\lambda^+_{p,q}}}\right)e^{-i\sqrt{\lambda^+_{p,q}}t}\right]\frac{\lambda^+_{p,q}-a}{\sqrt{\left(\lambda^+_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}\right|^2 dt.
\end{aligned} \tag{4.9}$$

NOTATION 4.6.

1. *For every* $(p,q) \in \mathbb{N}^* \times \mathbb{N}^*$ *we set* $a_{p,q} = \Phi^0_{p,q} - i\frac{\Phi^1_{p,q}}{\sqrt{\lambda^+_{p,q}}}$.
2. *For* $p \in \mathbb{Z}^*$ *and* $q \in \mathbb{N}^*$ *we define*

$$x_{p,q} = \begin{cases} \pi a_{p,q}\frac{\lambda^+_{p,q}-a+\alpha^2}{\sqrt{\left(\lambda^+_{p,q}-a\right)^2+\alpha^2p^2\pi^2}} & \text{if } p \ge 1 \\ -\overline{x_{-p,q}} & \text{if } p \le -1. \end{cases}$$

3. *For* $p \in \mathbb{N}^*$ *and* $q \in \mathbb{Z}^*$ *we define*

$$y_{p,q} = \begin{cases} \pi a_{p,q}\frac{\lambda^+_{p,q}-a}{\sqrt{\left(\lambda^+_{p,q}-a\right)^2+\alpha^2p^2\pi^2}} & \text{if } q \ge 1 \\ -\overline{y_{p,-q}} & \text{if } q \le -1. \end{cases}$$

The left-hand side of (4.8) is given by

$$\left\|\left(\Phi^0,\Phi^1\right)\right\|_{X_1}^2=\sum_{p,q\geq 1}\left(\lambda_{p,q}^+\left(\Phi_{p,q}^0\right)^2+\left(\Phi_{p,q}^1\right)^2\right)=\sum_{p,q\geq 1}\lambda_{p,q}^+\left|a_{p,q}\right|^2.$$

Furthermore, using Notation 4.6, we can write (4.9) in the form

$$2\int_0^T\int_\Gamma\left(\frac{\partial\varphi}{\partial\nu}+\alpha\psi\nu_1\right)^2d\sigma dt=\sum_{q\in\mathbb{N}^*}\int_0^T\left|\sum_{p\in\mathbb{N}^*}p\left(x_{p,q}e^{i\sqrt{\lambda_{p,q}^+}t}+\overline{x_{p,q}}e^{-i\sqrt{\lambda_{p,q}^+}t}\right)\right|^2dt$$
$$+\sum_{p\in\mathbb{N}^*}\int_0^T\left|\sum_{q\in\mathbb{N}^*}q\left(y_{p,q}e^{i\sqrt{\lambda_{p,q}^+}t}+\overline{y_{p,q}}e^{-i\sqrt{\lambda_{p,q}^+}t}\right)\right|^2dt. \quad (4.10)$$

Consequently, the observability inequality (4.8) is equivalent to the following inequality

$$C(T)\sum_{p,q\geq 1}\lambda_{p,q}^+\left|a_{p,q}\right|^2\leq\sum_{q\in\mathbb{N}^*}\int_0^T\left|\sum_{p\in\mathbb{N}^*}p\left(x_{p,q}e^{i\sqrt{\lambda_{p,q}^+}t}+\overline{x_{p,q}}e^{-i\sqrt{\lambda_{p,q}^+}t}\right)\right|^2dt$$
$$+\sum_{p\in\mathbb{N}^*}\int_0^T\left|\sum_{q\in\mathbb{N}^*}q\left(y_{p,q}e^{i\sqrt{\lambda_{p,q}^+}t}+\overline{y_{p,q}}e^{-i\sqrt{\lambda_{p,q}^+}t}\right)\right|^2dt \quad (4.11)$$

where $C(T)$ is a positive constant which does not depend on $\left(a_{p,q}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$. We recall below the main theorem of [17] for the observability of the wave equation in two space dimension:

THEOREM 4.7 (Mehrenberger [17]). *We assume the existence of $\gamma_1>0$ and $\gamma_2>0$ such that for every p, p', q and q' in $\mathbb{N}^*$*

$$\begin{array}{l} p\leq\max\left(q,q'\right)\Rightarrow\left|\sqrt{\mu_{pq}}\pm\sqrt{\mu_{pq'}}\right|\geq\gamma_1\left|q\pm q'\right| \\ q\leq\max\left(p,p'\right)\Rightarrow\left|\sqrt{\mu_{pq}}\pm\sqrt{\mu_{p'q}}\right|\geq\gamma_2\left|p\pm p'\right|. \end{array} \quad (4.12)$$

Then for any $T>2\pi\sqrt{\frac{1}{\gamma_1^2}+\frac{1}{\gamma_2^2}}$, there exists a constant $C(T)>0$ such that

$$C(T)\sum_{p,q\geq 1}\left(p^2+q^2\right)\left|x_{p,q}\right|^2\leq\sum_{p\in\mathbb{N}^*}\int_0^T\left|\sum_{q\in\mathbb{N}^*}q\left(x_{p,q}e^{i\sqrt{\mu_{pq}}t}+\overline{x_{p,q}}e^{-i\sqrt{\mu_{pq}}t}\right)\right|^2dt$$
$$+\sum_{q\in\mathbb{N}^*}\int_0^T\left|\sum_{p\in\mathbb{N}^*}p\left(x_{p,q}e^{i\sqrt{\mu_{pq}}t}+\overline{x_{p,q}}e^{-i\sqrt{\mu_{pq}}t}\right)\right|^2dt \quad (4.13)$$

for every complex sequence $\left(x_{p,q}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$ such that the sums involved are finite.

The use of Ingham type methods (see [11, 12]) is based on some gap properties (here given by (4.12)). Our observability inequality (4.11) is similar to (4.13) since $\lambda_{p,q}^+\sim\mu_{pq}$ as $\|(p,q)\|\to+\infty$. The difference is the presence of an other sequence $\left(y_{p,q}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$ which is different from the sequence $\left(x_{p,q}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$. Consequently, we cannot apply directly Theorem 4.7 to obtain the observability inequality (4.11). However, from the definition of $x_{p,q}$ and $y_{p,q}$ we can prove that both $x_{p,q}$ and $y_{p,q}$

are equivalent to the same term $a_{p,q} = \Phi^0_{p,q} - i\Phi^1_{p,q}/\sqrt{\lambda^+_{p,q}}$. This allows us to adapt the proof of Theorem 4.7. The keypoint is to prove that we have some good gap properties. To do this, we consider the sequence $(\Lambda_{p,q})_{(p,q)\in\mathbb{Z}^*\times\mathbb{Z}^*}$ defined below.

NOTATION 4.8. *For $p \in \mathbb{Z}^*$ and $q \in \mathbb{Z}^*$ let $\Lambda_{p,q}$ denote*

$$\Lambda_{p,q} = \begin{cases} \sqrt{\lambda^+_{p,q}} & \text{if } p \geq 1 \text{ and } q \geq 1 \\ -\sqrt{\lambda^+_{p,-q}} & \text{if } p \geq 1 \text{ and } q \leq -1 \\ -\sqrt{\lambda^+_{-p,q}} & \text{if } p \leq -1 \text{ and } q \geq 1. \end{cases}$$

With Notation 4.6 we deduce from (4.10) that

$$2\int_0^T\int_\Gamma\left(\frac{\partial\varphi}{\partial\nu} + \alpha\psi\nu_1\right)^2 d\sigma dt = \sum_{q\geq1}\int_0^T\left|\sum_{p\in\mathbb{Z}^*} px_{p,q}e^{i\Lambda_{p,q}t}\right|^2 dt + \sum_{p\geq1}\int_0^T\left|\sum_{q\in\mathbb{Z}^*} qy_{p,q}e^{i\Lambda_{p,q}t}\right|^2 dt.$$

Thus the observability inequality (4.8) that we have to prove is the following inequality

$$C(T)\sum_{p,q\geq1}\lambda^+_{p,q}|a_{p,q}|^2 \leq \sum_{q\geq1}\int_0^T\left|\sum_{p\in\mathbb{Z}^*} px_{p,q}e^{i\Lambda_{p,q}t}\right|^2 dt + \sum_{p\geq1}\int_0^T\left|\sum_{q\in\mathbb{Z}^*} qy_{p,q}e^{i\Lambda_{p,q}t}\right|^2 dt \quad (4.14)$$

where $C(T)$ is a positive constant which does not depend on the sequence $(a_{p,q})_{(p,q)}$.

4.2.2. Some gap properties. To adapt the proof of Mehrenberger (see [17]) we prove the following gap properties for the sequence $(\Lambda_{p,q})_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$.

PROPOSITION 4.9. *Let γ be as in Theorem 4.5.*

1. *For all $p \in \mathbb{N}^*$ and all $(q, q') \in \mathbb{Z}^* \times \mathbb{Z}^*$ such that $p \leq \max(q, q')$,*

$$|\Lambda_{p,q} - \Lambda_{p,q'}| \geq \gamma\,|q - q'|\,.$$

2. *For all $q \in \mathbb{N}^*$ and all $(p, p') \in \mathbb{Z}^* \times \mathbb{Z}^*$ such that $q \leq \max(p, p')$,*

$$|\Lambda_{p,q} - \Lambda_{p',q}| \geq \gamma\,|p - p'|\,.$$

Proof. 1. According to the definition of $\Lambda_{p,q}$, it is sufficient to show that

- for all $p \in \mathbb{N}^*$ and all $(q, q') \in \mathbb{N}^* \times \mathbb{N}^*$ such that $p \leq \max(q, q')$, we have $\left|\sqrt{\lambda^+_{p,q}} - \sqrt{\lambda^+_{p,q'}}\right| \geq \gamma\,|q - q'|$.
- for all p, q and q' in $\mathbb{N}^*$, we have $\sqrt{\lambda^+_{p,q}} + \sqrt{\lambda^+_{p,q'}} \geq \gamma\,(q + q')$.

Let us first consider $p \in \mathbb{N}^*$ and $(q, q') \in \mathbb{N}^* \times \mathbb{N}^*$ such that $p \leq \max(q, q')$. We can easily check that

$$\left|\lambda^+_{p,q} - \lambda^+_{p,q'}\right| = \frac{1}{2}\left|q^2 - q'^2\right|\pi^2\left(1 + \frac{\left(p^2+q^2\right)\pi^2 - a + \left(p^2+q'^2\right)\pi^2 - a}{\sqrt{\delta_{p,q}} + \sqrt{\delta_{p,q'}}}\right)$$

with

$$\delta_{p,q} = \left(\left(p^2+q^2\right)\pi^2 - a\right)^2 + 4\alpha^2p^2\pi^2. \quad (4.15)$$

From $a \leq 2\pi^2$, we have $\left(p^2+q^2\right)\pi^2 - a \geq 2\pi^2 - a \geq 0$. Therefore,

$$\left|\lambda_{p,q}^+ - \lambda_{p,q'}^+\right| \geq \frac{1}{2}\left|q^2 - q'^2\right|\pi^2.$$

Writing $\left|\sqrt{\lambda_{p,q}^+} - \sqrt{\lambda_{p,q'}^+}\right| = \frac{\left|\lambda_{p,q}^+ - \lambda_{p,q'}^+\right|}{\sqrt{\lambda_{p,q}^+}+\sqrt{\lambda_{p,q'}^+}}$, we obtain

$$\left|\sqrt{\lambda_{p,q}^+} - \sqrt{\lambda_{p,q'}^+}\right| \geq \frac{\pi^2}{2}\frac{q+q'}{\sqrt{\lambda_{p,q}^+}+\sqrt{\lambda_{p,q'}^+}}\left|q-q'\right|. \tag{4.16}$$

Consequently, we are reduced to bound from below the quantity $\frac{q+q'}{\sqrt{\lambda_{p,q}^+}+\sqrt{\lambda_{p,q'}^+}}$. We can remark that

$$\begin{aligned}
\sqrt{\lambda_{p,q}^+} &= \frac{1}{\sqrt{2}}\sqrt{\left(p^2+q^2\right)\pi^2 + a + \sqrt{\left(\left(p^2+q^2\right)\pi^2 - a\right)^2 + 4\alpha^2p^2\pi^2}} \\
&\leq \frac{1}{\sqrt{2}}\sqrt{\left(p^2+q^2\right)\pi^2 + a + \left(p^2+q^2\right)\pi^2 - a + 2\left|\alpha\right|p\pi} \\
&= \sqrt{\pi}\sqrt{\left(p^2+q^2\right)\pi + \left|\alpha\right|p}.
\end{aligned}$$

Thus

$$\frac{q+q'}{\sqrt{\lambda_{p,q}^+}+\sqrt{\lambda_{p,q'}^+}} \geq \frac{1}{\sqrt{\pi}}\frac{q+q'}{\sqrt{\left(p^2+q^2\right)\pi + \left|\alpha\right|p} + \sqrt{\left(p^2+q'^2\right)\pi + \left|\alpha\right|p}}. \tag{4.17}$$

By assumption, $p \leq \max(q,q')$. Without loss of generality we can assume that $q \leq q'$. From (4.17) it follows that

$$\frac{q+q'}{\sqrt{\lambda_{p,q}^+}+\sqrt{\lambda_{p,q'}^+}} \geq \frac{1}{\sqrt{\pi}}\frac{1+\frac{q}{q'}}{\sqrt{\left(\frac{p^2}{q'^2}+\frac{q^2}{q'^2}\right)\pi + \left|\alpha\right|\frac{p}{q'^2}} + \sqrt{\left(\frac{p^2}{q'^2}+1\right)\pi + \left|\alpha\right|\frac{p}{q'^2}}} \geq \frac{1}{\sqrt{\pi}}\frac{1}{2\sqrt{2\pi+\left|\alpha\right|}}.$$

Combining this inequality with (4.16) we conclude that

$$\left|\sqrt{\lambda_{p,q}^+} - \sqrt{\lambda_{p,q'}^+}\right| \geq \frac{\pi\sqrt{\pi}}{4\sqrt{2\pi+\left|\alpha\right|}}\left|q-q'\right| = \gamma\left|q-q'\right|.$$

Now, consider any p, q and q' in $\mathbb{N}^*$. Writing

$$\sqrt{\lambda_{p,q}^+} = \frac{1}{\sqrt{2}}\sqrt{\left(p^2+q^2\right)\pi^2 + a + \sqrt{\delta_{p,q}}} \geq \frac{\pi}{\sqrt{2}}q$$

with $\delta_{p,q}$ defined by (4.15), we obviously obtain

$$\sqrt{\lambda_{p,q}^+} + \sqrt{\lambda_{p,q'}^+} \geq \frac{\pi}{\sqrt{2}}\left(q+q'\right) \geq \gamma\left(q+q'\right).$$

This completes the proof of item 1.

2. The second assertion of Proposition 4.9 can be proved in the same way by interchanging (p,p') and (q,q'). □

4.2.3. Proof of the observability inequality.

NOTATION 4.10. *Given $T > T_0$, we denote by k the function*

$$k(t) := \begin{cases} \sin\left(\frac{\pi t}{T}\right) & \text{if } 0 \le t \le T \\ 0 & \text{else} \end{cases}$$

and we define the following quantities:

$$I_1 := \sum_{p \ge 1} \int_0^T k(t) \left| \sum_{q \ge p} q y_{p,q} e^{i\Lambda_{p,q} t} \right|^2 dt, \quad I_2 := \sum_{q \ge 1} \int_0^T k(t) \left| \sum_{p \ge q} p x_{p,q} e^{i\Lambda_{p,q} t} \right|^2 dt$$

$$I_3 := \left| \sum_{p \ge 1} \int_0^T k(t) \left(\sum_{\substack{q \in \mathbb{Z}^\star \\ q < p}} q y_{p,q} e^{i\Lambda_{p,q} t} \right) \left(\sum_{\substack{q \in \mathbb{Z}^\star \\ q \ge p}} q \overline{y_{p,q}} e^{-i\Lambda_{p,q} t} \right) dt \right|$$

$$I_4 := \left| \sum_{q \ge 1} \int_0^T k(t) \left(\sum_{\substack{p \in \mathbb{Z}^\star \\ p < q}} p x_{p,q} e^{i\Lambda_{p,q} t} \right) \left(\sum_{\substack{p \in \mathbb{Z}^\star \\ p \ge q}} p \overline{x_{p,q}} e^{-i\Lambda_{p,q} t} \right) dt \right|.$$

It is easily seen that the Fourier transform of k is given by

$$\widehat{k}(\xi) = e^{-i\frac{\xi T}{2}} T\sqrt{2\pi} \frac{\cos\left(\frac{\xi T}{2}\right)}{\pi^2 - T^2\xi^2}, \quad \forall \xi \in \mathbb{R}. \tag{4.18}$$

In particular, $|\widehat{k}|$ is an even function and $\widehat{k}(0) = \frac{T}{\pi}\sqrt{\frac{2}{\pi}}$ is real. In the following lemma we bound from below the right-hand side of the observability inequality (4.14), using the quantities I_j for $j = 1, \ldots, 4$.

LEMMA 4.11.

$$\sum_{q \ge 1} \int_0^T \left| \sum_{p \in \mathbb{Z}^*} p x_{p,q} e^{i\Lambda_{p,q} t} \right|^2 dt + \sum_{p \ge 1} \int_0^T \left| \sum_{q \in \mathbb{Z}^*} q y_{p,q} e^{i\Lambda_{p,q} t} \right|^2 dt \ge I_1 + I_2 - 2\left(I_3 + I_4\right).$$

Proof. If we prove that $\sum_{p \ge 1} \int_0^T \left| \sum_{q \in \mathbb{Z}^\star} q y_{p,q} e^{i\Lambda_{p,q} t} \right|^2 dt \ge I_1 - 2I_3$, then, replacing p by q and $y_{p,q}$ by $x_{p,q}$, we deduce that $\sum_{q \ge 1} \int_0^T \left| \sum_{p \in \mathbb{Z}^\star} p x_{p,q} e^{i\Lambda_{p,q} t} \right|^2 dt \ge I_2 - 2I_4$.

Since $0 \leq k(t) \leq 1$ for all $t \in [0,T]$, it follows that

$$
\begin{aligned}
\sum_{p\geq 1}\int_0^T \left|\sum_{q\in\mathbb{Z}^\star} q y_{p,q} e^{i\Lambda_{p,q}t}\right|^2 dt \geq & \sum_{p\geq 1}\int_0^T k(t)\left|\sum_{q\in\mathbb{Z}^\star} q y_{p,q} e^{i\Lambda_{p,q}t}\right|^2 dt \\
= & \sum_{p\geq 1}\int_0^T k(t)\left|\sum_{\substack{q\in\mathbb{Z}^\star\\ q\geq p}} q y_{p,q} e^{i\Lambda_{p,q}t} + \sum_{\substack{q\in\mathbb{Z}^\star\\ q< p}} q y_{p,q} e^{i\Lambda_{p,q}t}\right|^2 dt \\
\geq & \sum_{p\geq 1}\int_0^T k(t)\left|\sum_{\substack{q\in\mathbb{Z}^\star\\ q\geq p}} q y_{p,q} e^{i\Lambda_{p,q}t}\right|^2 dt + \sum_{p\geq 1}\int_0^T k(t)\left|\sum_{\substack{q\in\mathbb{Z}^\star\\ q< p}} q y_{p,q} e^{i\Lambda_{p,q}t}\right|^2 dt \\
& -2\left|\sum_{p\geq 1}\int_0^T k(t)\left(\sum_{\substack{q\in\mathbb{Z}^\star\\ q\geq p}} q \overline{y_{p,q}} e^{-i\Lambda_{p,q}t}\right)\left(\sum_{\substack{q\in\mathbb{Z}^\star\\ q< p}} q y_{p,q} e^{i\Lambda_{p,q}t}\right) dt\right| \\
= & I_1 + \sum_{p\geq 1}\int_0^T k(t)\left|\sum_{\substack{q\in\mathbb{Z}^\star\\ q< p}} q y_{p,q} e^{i\Lambda_{p,q}t}\right|^2 dt - 2I_3 \ \geq\ I_1 - 2I_3.
\end{aligned}
$$

□

As a consequence of Lemma 4.11, the observability inequality will be established if we bound from below I_1 and I_2 and bound from above I_3 and I_4.

LEMMA 4.12. *[Lower bound of I_1]*

$$
I_1 \geq \frac{2T}{\pi}\left(1-\left(\frac{2\pi}{\gamma T}\right)^2\right)\sum_{p\geq 1}\sum_{q\geq p} q^2 \left|y_{p,q}\right|^2.
$$

Proof. First, we remark that:

$$
\begin{aligned}
I_1 = & \sum_{p\geq 1}\int_0^T k(t)\left|\sum_{q\geq p} q y_{p,q} e^{i\Lambda_{p,q}t}\right|^2 dt \\
= & \sum_{p\geq 1}\int_0^T k(t)\left(\sum_{q\geq p} q y_{p,q} e^{i\Lambda_{p,q}t}\right)\left(\sum_{q'\geq p} q' \overline{y_{p,q'}} e^{-i\Lambda_{p,q'}t}\right) dt \\
= & \sqrt{2\pi}\sum_{p\geq 1}\sum_{q\geq p}\sum_{q'\geq p} q y_{p,q} q' \overline{y_{p,q'}}\,\widehat{k}\left(\Lambda_{p,q'}-\Lambda_{p,q}\right) \\
\geq & \sqrt{2\pi}\sum_{p\geq 1}\left(\widehat{k}(0)\sum_{q\geq p} q^2 |y_{p,q}|^2 - \sum_{q\geq p}\sum_{\substack{q'\geq p\\ q'\neq q}} q\left|y_{p,q}\right| q'\left|y_{p,q'}\right|\left|\widehat{k}\left(\Lambda_{p,q'}-\Lambda_{p,q}\right)\right|\right) \\
\geq & \sqrt{2\pi}\sum_{p\geq 1}\left(\widehat{k}(0)\sum_{q\geq p} q^2 |y_{p,q}|^2 - \sum_{q\geq p}\sum_{\substack{q'\geq p\\ q'\neq q}} \frac{q^2|y_{p,q}|^2+q'^2|y_{p,q'}|^2}{2}\left|\widehat{k}\left(\Lambda_{p,q'}-\Lambda_{p,q}\right)\right|\right). \qquad (4.19)
\end{aligned}
$$

Fix $p \in \mathbb{N}^*$. From parity of $\left|\widehat{k}\right|$ it follows

$$\sum_{q\geq p}\sum_{\substack{q'\geq p\\ q'\neq q}}\frac{q^2|y_{p,q}|^2+q'^2|y_{p,q'}|^2}{2}\left|\widehat{k}\left(\Lambda_{p,q'}-\Lambda_{p,q}\right)\right|=\sum_{q\geq p}q^2|y_{p,q}|^2\left(\sum_{\substack{q'\geq p\\ q'\neq q}}\left|\widehat{k}\left(\Lambda_{p,q'}-\Lambda_{p,q}\right)\right|\right). \tag{4.20}$$

Now, consider $q \in \mathbb{Z}^*$ and $q' \in \mathbb{Z}^*$ such that $q \geq p$, $q' \geq p$ and $q \neq q'$. Proposition 4.9 yields

$$|\Lambda_{p,q'} - \Lambda_{p,q}| \geq \gamma\,|q - q'| > \gamma. \tag{4.21}$$

Since $T > T_0 = \frac{2\pi}{\gamma}\sqrt{1 + 2\frac{\left(\lambda_{1,1}^{+}-a+\alpha^2\right)^2}{\left(\lambda_{1,1}^{+}-a\right)^2}} \geq \frac{2\pi}{\gamma}$, we obtain

$$|\Lambda_{p,q'} - \Lambda_{p,q}| > \gamma > \frac{2\pi}{T}. \tag{4.22}$$

Using (4.18), (4.22) and then (4.21), we get

$$\begin{aligned}\left|\widehat{k}\left(\Lambda_{p,q'} - \Lambda_{p,q}\right)\right| &\leq \frac{T\sqrt{2\pi}}{T^2\left(\Lambda_{p,q'} - \Lambda_{p,q}\right)^2 - \pi^2} = \widehat{k}(0)\left(\frac{2\pi}{\gamma T}\right)^2 \frac{1}{4\left(\frac{\Lambda_{p,q'}-\Lambda_{p,q}}{\gamma}\right)^2 - \left(\frac{2\pi}{\gamma T}\right)^2}\\ &\leq \widehat{k}(0)\left(\frac{2\pi}{\gamma T}\right)^2 \frac{1}{4\left(q-q'\right)^2 - 1}.\end{aligned}$$

Suming over $q' \geq p$, we obtain

$$\begin{aligned}\sum_{\substack{q'\geq p\\ q'\neq q}}\left|\widehat{k}\left(\Lambda_{p,q'} - \Lambda_{p,q}\right)\right| &\leq \widehat{k}(0)\left(\frac{2\pi}{\gamma T}\right)^2\sum_{\substack{q'\geq p\\ q'\neq q}}\frac{1}{4\left(q-q'\right)^2 - 1}\\ &\leq \widehat{k}(0)\left(\frac{2\pi}{\gamma T}\right)^2\sum_{r\in\mathbb{Z}^*}\frac{1}{4r^2 - 1} = \widehat{k}(0)\left(\frac{2\pi}{\gamma T}\right)^2.\end{aligned}$$

By (4.20), we can assert that

$$\sum_{q\geq p}\sum_{\substack{q'\geq p\\ q'\neq q}} q\,|y_{p,q}|\,q'\,|y_{p,q'}|\left|\widehat{k}\left(\Lambda_{p,q'} - \Lambda_{p,q}\right)\right| \leq \widehat{k}(0)\left(\frac{2\pi}{\gamma T}\right)^2\sum_{q\geq p}q^2\,|y_{p,q}|^2. \tag{4.23}$$

Combining this inequality with (4.19), we conclude that

$$I_1 \geq \sqrt{2\pi}\widehat{k}(0)\left(1 - \left(\frac{2\pi}{\gamma T}\right)^2\right)\sum_{\substack{q\geq p\\ p\geq 1}}q^2\,|y_{p,q}|^2 = \frac{2T}{\pi}\left(1 - \left(\frac{2\pi}{\gamma T}\right)^2\right)\sum_{\substack{q\geq p\\ p\geq 1}}q^2\,|y_{p,q}|^2,$$

which proves the lemma. □

Interchanging p and q and replacing $y_{p,q}$ by $x_{p,q}$, we also get

$$I_2 \geq \frac{2T}{\pi}\left(1 - \left(\frac{2\pi}{\gamma T}\right)^2\right)\sum_{q\geq 1}\sum_{p\geq q}p^2\,|x_{p,q}|^2.$$

Since $\lambda^+_{p,q} > a$, it is easily seen that $|x_{p,q}| \geq |y_{p,q}|$ (see Notation 4.6). It follows that

$$I_2 \geq \frac{2T}{\pi}\left(1-\left(\frac{2\pi}{\gamma T}\right)^2\right)\sum_{q\geq 1}\sum_{p\geq q} p^2\left|y_{p,q}\right|^2. \tag{4.24}$$

Combining Lemma 4.12 with (4.24), we obtain

$$I_1 + I_2 \geq \frac{2T}{\pi}\left(1-\left(\frac{2\pi}{\gamma T}\right)^2\right)\left(\sum_{p\geq 1}\sum_{q\geq p} q^2\left|y_{p,q}\right|^2 + \sum_{q\geq 1}\sum_{p\geq q} p^2\left|y_{p,q}\right|^2\right).$$

Remarking that

$$\sum_{p\geq 1}\sum_{q\geq 1}\left(p^2+q^2\right)\left|y_{p,q}\right|^2 \leq 2\left(\sum_{p\geq 1}\sum_{q\geq p} q^2\left|y_{p,q}\right|^2 + \sum_{q\geq 1}\sum_{p\geq q} p^2\left|y_{p,q}\right|^2\right)$$

we get

$$I_1 + I_2 \geq \frac{T}{\pi}\left(1-\left(\frac{2\pi}{\gamma T}\right)^2\right)\sum_{p,q\geq 1}(p^2+q^2)|y_{p,q}|^2.$$

Replacing $y_{p,q}$ by its definition gives

$$I_1 + I_2 \geq \pi T\left(1-\left(\frac{2\pi}{\gamma_1 T}\right)^2\right)\sum_{p,q\geq 1}\left(p^2+q^2\right)\left|a_{p,q}\right|^2 \frac{\left(\lambda^+_{p,q}-a\right)^2}{\left(\lambda^+_{p,q}-a\right)^2+\alpha^2p^2\pi^2}. \tag{4.25}$$

Now, let us bound from above I_3.

LEMMA 4.13. *[Upper bound of I_3]*

$$I_3 \leq \pi T\left(\frac{2\pi}{\gamma T}\right)^2\sum_{p\geq 1}\sum_{q\geq 1} q^2\left|a_{p,q}\right|^2 \frac{\left(\lambda^+_{p,q}-a\right)^2}{\left(\lambda^+_{p,q}-a\right)^2+\alpha^2p^2\pi^2}. \tag{4.26}$$

Proof. By definition of I_3 (see Notation 4.10), we have

$$\begin{aligned} I_3 &= \left|\sum_{p\geq 1}\int_0^T k(t)\left(\sum_{\substack{q\in\mathbb{Z}^\star\\ q<p}} q y_{p,q} e^{i\Lambda_{p,q}t}\right)\left(\sum_{\substack{q'\in\mathbb{Z}^\star\\ q'\geq p}} q'\overline{y_{p,q'}}e^{-i\Lambda_{p,q'}t}\right)dt\right| \\ &= \sqrt{2\pi}\left|\sum_{p\geq 1}\sum_{\substack{q\in\mathbb{Z}^\star\\ q<p}}\sum_{q'\geq p} q y_{p,q} q'\overline{y_{p,q'}}\widehat{k}\left(\Lambda_{p,q'}-\Lambda_{p,q}\right)\right| \\ &\leq \frac{\sqrt{2\pi}}{2}\sum_{p\geq 1}\sum_{\substack{q\in\mathbb{Z}^\star\\ q<p}}\sum_{q'\geq p}\left(q^2\left|y_{p,q}\right|^2 + q'^2\left|y_{p,q'}\right|^2\right)\left|\widehat{k}\left(\Lambda_{p,q'}-\Lambda_{p,q}\right)\right|. \end{aligned}$$

Analysis similar to that in the proof of Lemma 4.12 shows that for all $p\in\mathbb{N}^*$ and all $q\in\mathbb{Z}^*$ such that $q<p$ we have

$$\begin{aligned} \sum_{q'\geq p}\left|\widehat{k}\left(\Lambda_{p,q'}-\Lambda_{p,q}\right)\right| &\leq \widehat{k}(0)\left(\frac{2\pi}{\gamma T}\right)^2\sum_{q'\geq p}\frac{1}{4\left(q-q'\right)^2-1} \\ &\leq \widehat{k}(0)\left(\frac{2\pi}{\gamma T}\right)^2\sum_{r\geq 1}\frac{1}{4r^2-1} = \frac{\widehat{k}(0)}{2}\left(\frac{2\pi}{\gamma T}\right)^2. \end{aligned}$$

Therefore

$$\sum_{\substack{q\in\mathbb{Z}^\star\\ q<p}}\sum_{q'\geq p} q^2\left|y_{p,q}\right|^2\left|\widehat{k}\left(\Lambda_{p,q'}-\Lambda_{p,q}\right)\right| \leq \frac{\widehat{k}(0)}{2}\left(\frac{2\pi}{\gamma T}\right)^2\sum_{\substack{q\in\mathbb{Z}^\star\\ q<p}} q^2\left|y_{p,q}\right|^2.$$

Similarly we can prove that

$$\sum_{q'\geq p}\sum_{\substack{q\in\mathbb{Z}^\star\\ q<p}} q'^2\left|y_{p,q'}\right|^2\left|\widehat{k}\left(\Lambda_{p,q'}-\Lambda_{p,q}\right)\right| \leq \frac{\widehat{k}(0)}{2}\left(\frac{2\pi}{\gamma T}\right)^2\sum_{q'\geq p} q'^2\left|y_{p,q'}\right|^2.$$

Adding these last two inequalities we obtain: $I_3 \leq \frac{\sqrt{2\pi}}{4}\widehat{k}(0)\left(\frac{2\pi}{\gamma T}\right)^2\sum_{p\geq 1}\sum_{q\in\mathbb{Z}^\star} q^2\left|y_{p,q}\right|^2$. Replacing $y_{p,q}$ by its definition gives the desired inequality. □

Similarly,

$$I_4 \leq \pi T\left(\frac{2\pi}{\gamma T}\right)^2\sum_{p,q\geq 1} p^2\left|a_{p,q}\right|^2\frac{\left(\lambda_{p,q}^+-a+\alpha^2\right)^2}{\left(\lambda_{p,q}^+-a\right)^2+\alpha^2p^2\pi^2}. \tag{4.27}$$

Adding (4.26) and (4.27) gives

$$I_3+I_4 \leq \pi T\left(\frac{2\pi}{\gamma T}\right)^2\sum_{p,q\geq 1}\left(p^2+q^2\right)\left|a_{p,q}\right|^2\frac{\left(\lambda_{p,q}^+-a+\alpha^2\right)^2}{\left(\lambda_{p,q}^+-a\right)^2+\alpha^2p^2\pi^2}. \tag{4.28}$$

From (4.25) and (4.28) it follows that

$$I_1+I_2-2\left(I_3+I_4\right) \geq \pi T\sum_{p,q\geq 1}\left(p^2+q^2\right)\left|a_{p,q}\right|^2 b_{p,q}, \tag{4.29}$$

where $\left(b_{p,q}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$ is defined by

$$b_{p,q} = \left(1-\left(\frac{2\pi}{\gamma T}\right)^2\right)\frac{\left(\lambda_{p,q}^+-a\right)^2}{\left(\lambda_{p,q}^+-a\right)^2+\alpha^2p^2\pi^2} - 2\left(\frac{2\pi}{\gamma T}\right)^2\frac{\left(\lambda_{p,q}^+-a+\alpha^2\right)^2}{\left(\lambda_{p,q}^+-a\right)^2+\alpha^2p^2\pi^2}.$$

To bound from below $I_1+I_2-2\left(I_3+I_4\right)$ by $\sum_{p,q\geq 1}\lambda_{p,q}^+\left|a_{p,q}\right|^2$, it suffices to prove that the sequence $\left(\frac{\left(p^2+q^2\right)b_{p,q}}{\lambda_{p,q}^+}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$ is bounded from below. Actually

$$b_{p,q} = \frac{1}{T^2}\frac{\left(\lambda_{p,q}^+-a\right)^2}{\left(\lambda_{p,q}^+-a\right)^2+\alpha^2p^2\pi^2}\left(T^2-\left(\frac{2\pi}{\gamma}\right)^2\left(1+2\frac{\left(\lambda_{p,q}^+-a+\alpha^2\right)^2}{\left(\lambda_{p,q}^+-a\right)^2}\right)\right)$$

and it is easy to check that

$$\sup_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}\frac{\left(\lambda_{p,q}^+-a+\alpha^2\right)^2}{\left(\lambda_{p,q}^+-a\right)^2} = \frac{\left(\lambda_{1,1}^+-a+\alpha^2\right)^2}{\left(\lambda_{1,1}^+-a\right)^2}.$$

Since $T > T_0 = \frac{2\pi}{\gamma}\sqrt{1+2\frac{\left(\lambda_{1,1}^+-a+\alpha^2\right)^2}{\left(\lambda_{1,1}^+-a\right)^2}}$, this implies that $b_{p,q} > 0$ for any (p,q) in $\mathbb{N}^*\times\mathbb{N}^*$. Besides, from the asymptotic property $\lambda_{p,q}^+ \underset{\|(p,q)\|\to+\infty}{\sim} \mu_{pq} = \left(p^2+q^2\right)\pi^2$

we deduce that

$$\frac{\left(p^2+q^2\right)b_{p,q}}{\lambda_{p,q}^+} \underset{\|(p,q)\|\to+\infty}{\sim} \frac{1}{\pi^2}b_{p,q}.$$

It is easily seen that $\lim\limits_{\|(p,q)\|\to+\infty} b_{p,q} = 1-3\left(\frac{2\pi}{\gamma T}\right)^2$. Since $T > T_0 > \sqrt{3}\frac{2\pi}{\gamma}$, it follows that $\lim\limits_{\|(p,q)\|\to+\infty} b_{p,q} > 0$. Consequently, the sequence $\left(\frac{\left(p^2+q^2\right)b_{p,q}}{\lambda_{p,q}^+}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$ which is positive with a positive limit is uniformly bounded from below by a positive constant, denoted by c. It follows from (4.29) that

$$I_1+I_2-2\left(I_3+I_4\right)\geq \pi c T\sum_{p,q\geq 1}\lambda_{p,q}^+\left|a_{p,q}\right|^2,$$

which gives inequality (4.14) (according to Lemma 4.11) and then the observability inequality (4.8).

4.2.4. Uniform controllability in $H^+\times H^+_{-1/2}$. Theorem 4.5 implies, by usual duality arguments (see [14, 20] and the references therein) the following controllability result.

THEOREM 4.14. *Let $\gamma = \frac{\pi\sqrt{\pi}}{4\sqrt{2\pi+|\alpha|}}$ and $T_0 = \frac{2\pi}{\gamma}\sqrt{1+2\frac{\left(\lambda_{1,1}^+-a+\alpha^2\right)^2}{\left(\lambda_{1,1}^+-a\right)^2}}$. Assume that $a\leq 2\pi^2$. For any $T>T_0$, any initial data $\left(u^0,u^1\right)\in H^+\times H^+_{-1/2}$ and any target $\left(u_T^0,u_T^1\right)\in H^+\times H^+_{-1/2}$, there exist control functions $v\in D\left(L_T\right)$ such that the unique solution u of system (1.1) satisfies $u(\cdot,T)=u_T^0$ and $u'(\cdot,T)=u_T^1$ in Ω.*

5. Lack of controllability and dependance with respect to α.

5.1. Controllability in $H^a\times H^a_{-1/2}$. We also obtain the observability of the adjoint system in $D\left(L_T^*L\right)\cap(H^a_{1/2}\times H^a)$, the eigenspace associated with the eigenvalue a of infinite multiplicity. In that case, the observability inequality is simply

$$\left\|\left((0,\psi^0),(0,\psi^1)\right)\right\|_{X_1}^2 \leq C(T)\int_0^T\int_0^1 \alpha^2\psi^2(y,t)dydt \tag{5.1}$$

with $\psi(y,t)=\sum_{q\geq 1}\left(\Phi_q^0\cos(\sqrt{a}t)+\Phi_q^1\frac{\sin(\sqrt{a}t)}{\sqrt{a}}\right)e_q(y)$. Simple computations permit to show that (5.1) is true for any $T>\frac{\pi}{2\sqrt{a}}$. Remark that, in that specific case, the observation is simply from $\Gamma_2=\{0\}\times[0,1]$. The corresponding controllability result is the following:

PROPOSITION 5.1. *If $T>\frac{\pi}{2\sqrt{a}}$, then for every $\left(u^0,u^1\right)\in H^a\times H^a_{-1/2}$ and any $\left(u_T^0,u_T^1\right)\in H^a\times H^a_{-1/2}$, there exists $v\in D\left(L_T\right)$ such that the solution u of (1.1) satisfies $u(\cdot,T)=u_T^0$, $u'(\cdot,T)=u_T^1$ in Ω.*

Proof. Let $\left(\Phi^0,\Phi^1\right)^T\in D\left(L_T^*L\right)\cap\left(H^a_{1/2}\times H^a\right)$. We write $\Phi^0=\left(0,\psi^0\right)$ and $\Phi^1=\left(0,\psi^1\right)$ in the form $\Phi^0=\sum\limits_{q\geq 1}\Phi_q^0 e_q$ and $\Phi^1=\sum\limits_{q\geq 1}\Phi_q^1 e_q$, with $\left\|\left(\Phi^0,\Phi^1\right)\right\|_{X_1}^2 = \sum\limits_{q\geq 1}\left(a\left(\Phi_q^0\right)^2+\left(\Phi_q^1\right)^2\right)$. The solution $\Phi=(\varphi,\psi)^T$ of system (1.3) with initial data $\left(\Phi^0,\Phi^1\right)$ is clearly given by

$$\Phi(t)=\frac{1}{2}\sum_{q\geq 1}\left(\left(\Phi_q^0-i\frac{\Phi_q^1}{\sqrt{a}}\right)e^{i\sqrt{a}t}+\left(\Phi_q^0+i\frac{\Phi_q^1}{\sqrt{a}}\right)e^{-i\sqrt{a}t}\right)e_q.$$

Set $c_q = \Phi_q^0 - i\frac{\Phi_q^1}{\sqrt{a}}$. This gives for every $((x,y),t) \in \Omega \times [0,T]$, $\varphi((x,y),t) = 0$ and $\psi((x,y),t) = \frac{1}{2}\sum_{q\geq 1}\left(c_q e^{i\sqrt{a}t} + \overline{c_q}e^{-i\sqrt{a}t}\right)\sqrt{2}\sin(q\pi y)$. Consequently,

$$\int_0^T\int_\Gamma\left(\frac{\partial\varphi}{\partial\nu} + \alpha\psi\nu_1\right)^2 d\sigma dt = \frac{\alpha^2}{4}\int_0^T\sum_{q\geq 1}\left(c_q e^{i\sqrt{a}t} + \overline{c_q}e^{-i\sqrt{a}t}\right)^2 dt.$$

Since $0 \leq k(t) \leq 1$ for all $t \in [0,T]$, we obtain

$$\begin{aligned}\int_0^T\int_\Gamma\left(\frac{\partial\varphi}{\partial\nu} + \alpha\psi\nu_1\right)^2 d\sigma dt &\geq \frac{\alpha^2}{4}\int_0^T k(t)\sum_{q\geq 1}\left(c_q e^{i\sqrt{a}t} + \overline{c_q}e^{-i\sqrt{a}t}\right)^2 dt \\ &= \frac{\alpha^2}{4}\sum_{q\geq 1}\int_0^T k(t)\left(c_q e^{i\sqrt{a}t} + \overline{c_q}e^{-i\sqrt{a}t}\right)^2 dt. \qquad (5.2)\end{aligned}$$

Let us study the integral $\int_0^T k(t)\left(c_q e^{i\sqrt{a}t} + \overline{c_q}e^{-i\sqrt{a}t}\right)^2 dt$. Using the parity of $\left|\widehat{k}\right|$ we can easily prove that $\int_0^T k(t)\left(c_q e^{i\sqrt{a}t} + \overline{c_q}e^{-i\sqrt{a}t}\right)^2 dt = 2|c_q|^2\widehat{k}(0) + \left((c_q)^2 + (\overline{c_q})^2\right)\widehat{k}(2\sqrt{a})$. Thus

$$\begin{aligned}\int_0^T k(t)\left(c_q e^{i\sqrt{a}t} + \overline{c_q}e^{-i\sqrt{a}t}\right)^2 dt &= 2|c_q|^2\widehat{k}(0) + 2\Re\left((c_q)^2\right)\widehat{k}\left(2\sqrt{a}\right) \\ &\geq 2|c_q|^2\left(\widehat{k}(0) - \left|\widehat{k}\left(2\sqrt{a}\right)\right|\right). \qquad (5.3)\end{aligned}$$

If $T > \frac{\pi}{2\sqrt{a}}$, then, using (4.18), we obtain $\left|\widehat{k}(2\sqrt{a})\right| \leq \frac{\sqrt{2\pi}T}{|\pi^2 - 4aT^2|} < \frac{\sqrt{2\pi}T}{\pi^2} = \widehat{k}(0)$. From (5.2), (5.3) and $\sum_{q\geq 1}|c_q|^2 = \left\|\left(\Phi^0, \Phi^1\right)\right\|_{X_1}^2$, we finally obtain

$$\int_0^T\int_\Gamma\left(\frac{\partial\varphi}{\partial\nu} + \alpha\psi\nu_1\right)^2 d\sigma dt \geq \frac{\alpha^2}{2}\left(\widehat{k}(0) - \left|\widehat{k}\left(2\sqrt{a}\right)\right|\right)\left\|\left(\Phi^0, \Phi^1\right)\right\|_{X_1}^2$$

with $\widehat{k}(0) - \left|\widehat{k}(2\sqrt{a})\right| > 0$. This gives (5.1). ☐

5.2. Lack of controllability in $H^- \times H^-_{-1/2}$. In agreement with the general result [10] we pointed out in the introduction, the lack of observability is related to essential spectrum.

PROPOSITION 5.2. *For any $T > 0$ and any $\epsilon > 0$, there exist initial data $\left(\Phi^0, \Phi^1\right) \in H^-_{1/2} \times H^-$ for which the solution $\Phi = (\varphi, \psi)^T$ of (1.3) satisfies*

$$\left\|\left(\Phi^0, \Phi^1\right)\right\|_{X_1}^{-2}\int_0^T\int_\Gamma\left(\frac{\partial\varphi}{\partial\nu} + \alpha\psi\nu_1\right)^2 d\sigma dt < \epsilon. \qquad (5.4)$$

Proof. We consider the two sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ of positive integers given by $p_n = n(n+1)/2$ and $q_n = n$. Let $\Phi_n = (\varphi_n, \psi_n)^T$ be the solution of the adjoint system (1.3) with initial data $\left(\Phi_n^0, \Phi_n^1\right) = \left(e^-_{p_n,q_n}, 0\right) \in H^-_{1/2} \times H^-$. The norm of the initial data is given by $\left\|\left(\Phi_n^0, \Phi_n^1\right)\right\|_{X_1}^2 = \left\|e^-_{p_n,q_n}\right\|_{H_{1/2}}^2 = \lambda^-_{p_n,q_n}$. We can write (p_n, q_n) in the polar coordinates $(p_n, q_n) = r_n(\cos(\theta_n), \sin(\theta_n))$. It is easily seen that $\lim_{n\to+\infty} r_n = +\infty$ and $\lim_{n\to+\infty} \theta_n = 0$. This readily implies $\lim_{n\to+\infty} \lambda^-_{p_n,q_n} = a - \alpha^2$. Since $a > \alpha^2$, we obtain $\lim_{n\to+\infty} \left\|\left(\Phi^0_n, \Phi^1_n\right)\right\|^2_{X_1} > 0$. On the other hand, $\Phi_n(t) = \cos\left(\sqrt{\lambda^-_{p_n,q_n}}t\right) e^-_{p_n,q_n}$ implies that

$$\int_0^T\int_\Gamma\left(\frac{\partial\varphi_n}{\partial\nu} + \alpha\psi_n\nu_1\right)^2 d\sigma dt$$
$$= \pi^2\left(T + \frac{\sin\left(2\sqrt{\lambda^-_{p_n,q_n}}T\right)}{2\sqrt{\lambda^-_{p_n,q_n}}}\right)\frac{\cos^2\theta_n\left(\lambda^-_{p_n,q_n} - a + \alpha^2\right)^2 + \sin^2\theta_n\left(\lambda^-_{p_n,q_n} - a\right)^2}{\frac{\left(\lambda^-_{p_n,q_n}-a\right)^2}{r_n^2} + \alpha^2\cos^2\theta_n\pi^2}.$$

Letting $n \to +\infty$, we obtain $\lim_{n\to+\infty} \int_0^T\int_\Gamma\left(\frac{\partial\varphi_n}{\partial\nu} + \alpha\psi_n\nu_1\right)^2 d\sigma dt = 0$. This contradicts the uniform observability since $\lim_{n\to+\infty} \left\|\left(\Phi^0_n, \Phi^1_n\right)\right\|^2_{X_1} > 0$. □

Remark that the counterexample is obtained for (Φ^0, Φ^1) composed of only one eigenfunction for which the limit of the associated eigenvalue is $\lambda = a-\alpha^2$. This value is very particular because any other datum (Φ^0, Φ^1) composed of one eigenfunction associated with $\lambda \in (a - \alpha^2, a]$ does not contradict the uniform observability (for instance, we refer to previous section for $\lambda = a$). The loss of observability may be exhibited by considering a (non trivial) combination of such modes (as done in [10] using Weil sequence), in order to enhance the lack of spectral gap.

Simpler, this phenomenon may be observed numerically as follows. Let $\mathbb{H}^\pm_N$ be the space of the initial data $\left(\Phi^0, \Phi^1\right)$ in $H^\pm_{1/2} \times H^\pm$ spanned by $\{e^\pm_{p,q}\}_{1\le p,q\le N}$. If we denote by $\boldsymbol{\Phi}^\pm \in \mathbb{R}^{2N^2}$ the components of $\left(\Phi^0, \Phi^1\right)^T$ in the basis $\{e^\pm_{p,q}\}_{1\le p,q\le N}$, then we can write

$$\left\|\left(\Phi^0, \Phi^1\right)\right\|^2_{X_1} = (\boldsymbol{A}^\pm\boldsymbol{\Phi}^\pm, \boldsymbol{\Phi}^\pm)_{\mathbb{R}^{2N^2}}, \quad \int_0^T\int_\Gamma\left(\frac{\partial\varphi}{\partial\nu} + \alpha\psi\nu_1\right)^2 d\sigma dt = (\boldsymbol{B}^\pm\boldsymbol{\Phi}^\pm, \boldsymbol{\Phi}^\pm)_{\mathbb{R}^{2N^2}}$$

for all $\left(\Phi^0, \Phi^1\right)$ in $\mathbb{H}^\pm_N$, where $\boldsymbol{A}^\pm, \boldsymbol{B}^\pm \in \mathbb{R}^{2N^2\times 2N^2}$ denote real symmetric matrices and $(\cdot,\cdot)_{\mathbb{R}^{2N^2}}$ denotes the scalar product in $\mathbb{R}^{2N^2}$. $\boldsymbol{A}^\pm$ is diagonal. On $\mathbb{H}^\pm_N$, the observability inequality formally writes

$$(\boldsymbol{A}^\pm\boldsymbol{\Phi}^\pm, \boldsymbol{\Phi}^\pm)_{\mathbb{R}^{2N^2}} \le C^\pm_N(T)(\boldsymbol{B}^\pm\boldsymbol{\Phi}^\pm, \boldsymbol{\Phi}^\pm)_{\mathbb{R}^{2N^2}}.$$

The observability constant $C^\pm_N(T)$ whose behavior allows to detect the lack of observability, is then solution of the generalized eigenvalue problem

$$C^\pm_N(T) = \max\{\lambda > 0,\ \boldsymbol{A}^\pm\boldsymbol{\Phi} = \lambda\boldsymbol{B}^\pm\boldsymbol{\Phi},\ \boldsymbol{\Phi} \in \mathbb{R}^{2N^2} \setminus \{0\}\}. \tag{5.5}$$

In practice, since $\boldsymbol{A}^\pm$ is diagonal, it is easier to evaluate $(C^\pm_N)^{-1}$ equal to the lowest eigenvalue of $\boldsymbol{B}^\pm(\boldsymbol{A}^\pm)^{-1}$. Table 5.1 gives the value of $C^-_N(T)$ for various values of N and clearly exhibits the non-uniform boundedness with respect to N, in constrast to $C^+_N(T)$. This is in agreement with Theorem 4.14.

	$N = 5$	$N = 10$	$N = 20$	$N = 40$	$N = 80$
$C_N^+(T)$	5.01×10^{-1}	5.43×10^{-1}	5.71×10^{-1}	5.95×10^{-1}	6.02×10^{-1}
$C_N^-(T)$	2.42×10^{1}	4.41×10^{2}	3.24×10^{3}	8.6×10^{4}	1.01×10^{6}

TABLE 5.1
Evolution of the observability constant $C_N^{\pm}(T)$ vs. N for $(a, \alpha, T) = (4, 1, 3)$.

5.3. Controllability with respect to α and T. If $\alpha = 0$, then system (1.1) degenerates into an uncoupled system and the control only acts on the variable u_1. However, we observe from the proof of Theorem 4.5, that the minimal time T_0 as well as the observability constant $C^+(T)$ are uniformly bounded with respect to α. Therefore, the controllability holds uniformly w.r.t. α in $H^+ \times H^+_{-1/2}$ and by classical arguments, the corresponding controls converge toward controls for u_1, the solution of the wave equation with initial condition (u_1^0, u_1^1). This property is related to the fact that the second component of $\{e^+_{p,q}\}_{p,q \geq 1}$ degenerates as α goes to zero. Numerically, we observe that the variation of $C_N^+(T)$ with respect to α is very low. For $T = 3$, $N = 50$ and $a = 4$, we obtain $C_N^+(T) = 6.02217 \times 10^{-1}$ for $\alpha = 2$ and $C_N^+(T) = 6.02224 \times 10^{-1}$ for $\alpha = 2/100$. Remark that the uniform controllability with respect to α does not hold in $H^- \times H^-_{-1/2}$, nor in $H^a \times H^a_{-1/2}$ (the right term of (5.1) going to zero with α). The eigenvalue problem (5.5) allows also to estimate numerically the minimal controllability time for a, α, N fixed. Figure 5.1 depicts the evolution of $C_N^+(T)$ with respect to T for $(a, \alpha, N) = (4, 1, 50)$ and suggests that the minimal controllability time is about 2.5. The lower bound time T_0 in Theorem 4.14 leading for $(a, \alpha, N) = (4, 1, 50)$ to $T_0 \approx 21.96$ is thus not sharp.

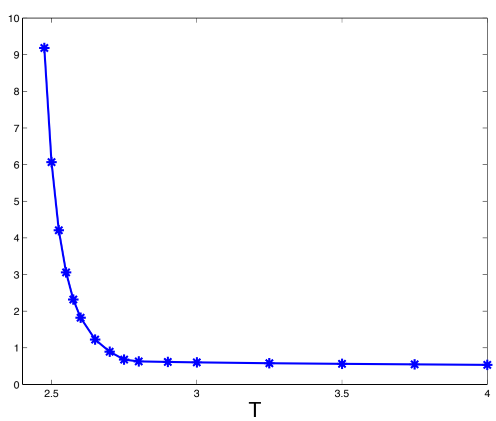

FIG. 5.1. *Evolution of $C_N^+(T)$ with respect to T for $(a, \alpha, N) = (4, 1, 50)$.*

6. Concluding remarks and comments.

1) **Characterization of the controllable data -** We can sum up the different controllability results we have obtained in the following theorem:

THEOREM 6.1. *For every $N \in \mathbb{N}^*$, let us denote by H^{N^-} (resp. $H^{N^-}_{-1/2}$) the Hilbert subspace of H (resp. $H_{-1/2}$) spanned by the $e^-_{p,q}$ for $1 \leq p, q \leq N$. If $a \leq 2\pi^2$, then for any $T > \max\left(T_0, \frac{\pi}{2\sqrt{a}}\right)$, any $N \in \mathbb{N}^*$, any initial data (u^0, u^1) and final data (u_T^0, u_T^1) in $\left(H^a \oplus H^+ \oplus H^{N^-}\right) \times \left(H^a_{-\frac{1}{2}} \oplus H^+_{-\frac{1}{2}} \oplus H^{N^-}_{-1/2}\right)$ there exists a control* *function v in $D(L_T)$ such that the solution u of (1.1) satisfies*

$$u(\cdot,T)=u_T^0,\ u'(\cdot,T)=u_T^1 \quad \text{in } \Omega.$$

2) **Partial controllability -** Following [13], one may analyze the uniform partial controllability which consists in controlling to rest only the first component u_1. In that weaker situation, the controllability is uniform with respect to the data $\left(u^0,u^1\right)$. From the second equation of (1.1), we express the component u_2 in terms of u_1 as follows $u_2(\cdot,t)=-\alpha\int_0^t \partial_x u_1(\cdot,s)\sin(\sqrt{a}(t-s))ds$ in Q_T, assuming for simplicity that $\left(u_2^0,u_2^1\right)=(0,0)$. The variable u_1 to be controlled is then solution of

$$\begin{cases} u_1''=\Delta u_1-\alpha^2\int_0^t \partial_{xx}u_1(\cdot,s)\sin(\sqrt{a}(t-s))ds & \text{in} \quad Q_T,\\ u_1=v\,1_\Gamma & \text{on} \quad \Sigma_T,\\ (u_1(\cdot,0),u_1'(\cdot,0))=\left(u_1^0,u_1^1\right) & \text{in} \quad \Omega. \end{cases}$$

The corresponding spectrum is $\{\lambda_{p,q}^-\}_{p,q\geq 1}\cup\{\lambda_{p,q}^+\}_{p,q\geq 1}$ with corresponding eigenfunctions $\{e_{p,q}^+\}_{p,q\geq 1}$ and $\{e_{p,q}^-\}_{p,q\geq 1}$. The difference with respect to the full controllability problem, is that the Fourier coefficients in φ_1, the adjoint solution of u_1, are all connected to each other. This allows a compensation of the modes $\{e_{p,q}^-\}_{p,q}$ by the modes $\{e_{p,q}^+\}_{p,q}$ (we refer to [13] for the analysis on a similar system). The analysis remains to be fully written.

3) **Null boundary controllability of a cylindrical membrane shell -** The operator which describes a membrane cylindrical elastic shell is as follows (see [19])

$$A=\begin{pmatrix} -a\partial_{xx}^2-c\partial_{yy}^2 & -(b+c)\partial_{xy}^2 & -ar^{-1}\partial_x\\ -(b+c)\partial_{xy}^2 & -c\partial_{xx}^2-a\partial_{yy}^2 & -br^{-1}\partial_y\\ r^{-1}a\partial_x & r^{-1}b\partial_y & r^{-2}a \end{pmatrix}$$

with $a=8\mu(\lambda+\mu)/(\lambda+2\mu)$, $b=4\lambda\mu/(\lambda+2\mu)$ and $c=2\mu$. $\lambda,\mu>0$ denote the Lamé coefficients. $r^{-1}>0$ denotes the curvature of the cylinder and is the coupling parameter between the tangential displacement (u_1,u_2) and the normal displacement u_3 of the shell. Ω is still equal to $(0,1)^2$. This mixed order and self-adjoint operator enters in the framework of [6] so that we can compute $\sigma_{ess}(A)$ using [7]. We obtain $\sigma_{ess}(A)=\left[0,2r^{-2}(3\lambda+2\mu)/(\lambda+\mu)\right]$ .The spectrum of A is therefore composed of two distinct parts, the essential spectrum plus a discrete spectrum with asymptotic behavior equal, up to some constant, to $\sigma(-\Delta)$. The difficulty here is that the discrete spectrum is not known explicitly.

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