Constructive exact controls for semilinear PDEs

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Seminário de EDP e Matemáticada Aplicadada - December 2021

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General objective

GIVEN some semilinear uniformly exactly controllable PDEs

$$\boxed{\begin{cases} PDE(y,v)=0, \\ y=y(x,t) - \text{state}, \quad v=v(x,t) - \text{control function}, \\ +\text{ initial conditions and boundary conditions} \end{cases}} \tag{1}$$

FIND a sequence $(y_k, v_k)_{k\in\mathbb{N}}$ such that $(y_k, v_k) \to (y, v)$ as $k \to \infty$, with (y, v) a controlled pair for (1) ?

• Largely open issue because in many situations, proofs are based on non constructive fixed point arguments.

• We focus on the case of the wave and heat eq. with distributed controls.

• We design constructive strongly convergent sequence $(y_k, v_k)_{k\in\mathbb{N}}$ using least-squares approaches and extend ideas used for the NS-direct problem in [1], [2]

[1] Lemoine, Münch, *Resolution of the Implicit Euler scheme for the Navier-Stokes equation through a least-squares method*, Numerische Mathematik 2021

[2] Lemoine, Münch, *A fully space-time least-squares method for the unsteady Navier-Stokes system*, J. Mathematical Fluids Mechanics 2021

• Let $\Omega := (0,1)$, $\omega := (\ell_1, \ell_2)$ with $0 \leq \ell_1 < \ell_2 \leq 1$, $T > 0$. We set $Q_T := \Omega \times (0,T)$, $q_T := \omega \times (0,T)$ and $\Sigma_T := \partial\Omega \times (0,T)$.

$$\boxed{\begin{cases} \partial_{tt} y - \partial_{xx} y + f(y) = v 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases}} \qquad (2)$$

• $(u_0, u_1) \in \boldsymbol{V} := H_0^1(\Omega) \times L^2(\Omega)$, $v \in L^2(q_T)$. $f \in C^1(\mathbb{R};\mathbb{R})$
• $|f(r)| \leq C(1+|r|)\ln^2(2+|r|)$ $\forall r \in \mathbb{R}$
• $y \in \mathcal{C}^0([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ is unique.

• (2) is exactly controllable in time T IFF for any $(u_0,u_1), (z_0,z_1) \in \boldsymbol{V}$, $\exists$ a control function $v \in L^2(q_T)$ such that $(y(\cdot,T), \partial_t y(\cdot,T)) = (z_0, z_1)$.

Theorem (Zuazua'93)

Assume that $T > 2\max(\ell_1, 1-\ell_2)$. There exists $\beta > 0$ (only depending on Ω and T) such that, if

$$(\mathbf{H_1}) \quad \limsup_{|r|\to\infty} \frac{|f(r)|}{|r|\ln^2|r|} < \beta$$

then (2) is exactly controllable in time T.

[3] E. Zuazua, Exact controllability for semilinear wave equations in one space dimension, Ann. Inst. H. Poincaré Anal. Non Linéaire

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The proof given in Zuazua'93 is based on a Leray Schauder fixed point argument: Let $K : L^\infty(Q_T) \to L^\infty(Q_T)$, where $y := \Lambda(\xi)$ is a controlled solution with the control function v_ξ of the linear boundary value problem

$$\begin{cases} \partial_{tt} y - \partial_{xx} y + \widehat{f}(\xi)\, y = -f(0) + v 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \qquad \widehat{f}(r) := \begin{cases} \dfrac{f(r) - f(0)}{r} & \text{if } r \neq 0 \\ f'(0) & \text{if } r = 0 \end{cases} \tag{3}$$

satisfying $(y(\cdot,T), \partial_t y(\cdot,T)) = (z_0, z_1)$. The control f is the one of minimal $L^2(q_T)$ norm.

Then, Λ has a fixed point. In particular,

$$\boxed{\|K(\xi)\|_\infty \le C\Big(\|u_0, u_1\|_{\boldsymbol{V}} + \|f(0)\|_2\Big)(1 + \|\xi\|_\infty)^{(1+C)\sqrt{\beta}}, \quad \forall \xi \in L^\infty(Q_T).}$$

Stability result: If $(1 + C)\sqrt{\beta} < 1$, then $\exists M > 0$ s.t. $\|\xi\|_\infty \le M \Longrightarrow \|\Lambda(\xi)\|_\infty \le M$.

A key point in the analysis is

Lemma (A priori estimates for the linearized eq.)

$A \in L^\infty(Q_T)$, $B \in L^2(Q_T)$, $(z_0, z_1) \in \boldsymbol{V}$. $T > 2\max(\ell_1, 1-\ell_2)$. There exists a unique control of minimal $L^2(q_T)$ norm that the solution of

$$\begin{cases} \partial_{tt} z - \partial_{xx} z + Az = v 1_\omega + B & \textit{in } Q_T, \\ z = 0 & \textit{on } \Sigma_T, \\ (z(\cdot,0), \partial_t z(\cdot,0)) = (z_0, z_1) & \textit{in } \Omega, \end{cases} \tag{4}$$

satisfies $(z(\cdot,T), \partial_t z(\cdot,T)) = (0,0)$ in Ω. Moreover, for $C = C(\Omega, T)$,

$$\|v\|_{2,q_T} + \|(z, \partial_t z)\|_{L^\infty(0,T;\boldsymbol{V})} \leq C\Big(\|B\|_2\, e^{(1+C)\sqrt{\|A\|_\infty}} + \|z_0, z_1\|_{\boldsymbol{V}}\Big) e^{C\sqrt{\|A\|_\infty}} \tag{5}$$

A first idea is to consider the Picard iterations $(y_k)_{k\in\mathbb{N}}$ associated with the operator Λ:

$$\boxed{\begin{cases} y_0 \in L^\infty(Q_T) \quad \text{given} \\ y_{k+1} = \Lambda(y_k),\ k \geq 0 \end{cases}} \tag{6}$$

[i.e. for any y_k, find an exact control v_{k+1} for y_{k+1} solution of

$$\begin{cases} \partial_{tt} y_{k+1} - \partial_{xx} y_{k+1} + y_{k+1}\,\widehat{f}(y_k) = -f(0) + v_{k+1} 1_\omega & \text{in} \quad Q_T, \\ y_{k+1} = 0 \text{ on } \Sigma_T, \quad (y_{k+1}(\cdot,0), \partial_t y_{k+1}(\cdot,0)) = (u_0, u_1) & \text{in } \Omega.] \end{cases} \tag{7}$$

Such a strategy usually fails since the operator Λ is in general not contracting, even if f is globally Lipschitz.

Contracting property is ensured notably under smallness assumptions :

Lemma (Contraction property under smallness assumption on f)

Let $M = M(\|u_0, u_1\|_{\boldsymbol{V}}, \beta)$ be such that Λ maps $B_\infty(0, M)$ into itself and assume that $\hat{f}' \in L^\infty(0, M)$. For any $\xi^i \in B_\infty(0, M)$, $i = 1, 2$, there exists $c(M) > 0$ such that

$$\|\Lambda(\xi^2) - \Lambda(\xi^1)\|_\infty \leq c(M)\|\hat{f}'\|_{L^\infty(0,M)}\|\xi^2 - \xi^1\|_\infty.$$

A least-squares approach

We consider the Hilbert space

$$\mathcal{H} := \Big\{(y,v) \in L^2(Q_T) \times L^2(q_T) \;\mid\; \partial_{tt}y - \partial_{xx}y \in L^2(Q_T),\; y = 0 \text{ on } \Sigma_T,$$
$$(y(\cdot,0), \partial_t y(\cdot,0)) \in \boldsymbol{V}\Big\}$$

and the subspaces of $\mathcal{H}$ defined by

$$\mathcal{A} := \Big\{(y,v) \in \mathcal{H} \;\mid\; (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0,u_1),\; (y(\cdot,T), \partial_t y(\cdot,T)) = (z_0,z_1) \text{ in } \Omega\Big\},$$
$$\mathcal{A}_0 := \Big\{(y,v) \in \mathcal{H} \;\mid\; (y(\cdot,0), \partial_t y(\cdot,0)) = (0,0),\; (y(\cdot,T), \partial_t y(\cdot,T)) = (0,0) \text{ in } \Omega\Big\},$$

Note that $\mathcal{A} = (\overline{y}, \overline{f}) + \mathcal{A}_0$ for any $(\overline{y}, \overline{f}) \in \mathcal{A}$.
We define the least-squares functional $E : \mathcal{A} \to \mathbb{R}$ by

$$\boxed{E(y,v) := \frac{1}{2}\big\|\partial_{tt}y - \partial_{xx}y + f(y) - v1_\omega\big\|^2_{L^2(Q_T)}}$$

and consider the nonconvex minimization problem

$$\boxed{\inf_{(y,v)\in\mathcal{A}} E(y,v)} \qquad (8)$$

Proposition

$\forall (y,f) \in \mathcal{A}$, $\exists C = C(\Omega, T)$

$$\frac{1}{\sqrt{2}\max\left(1, \|f'(y)\|_\infty\right)}\|E'(y,v)\|_{\mathcal{A}_0'} \le \sqrt{E(y,v)} \le Ce^{C\sqrt{\|f'(y)\|_\infty}}\|E'(y,v)\|_{\mathcal{A}_0'}. \quad (9)$$

Consequence:
Any *critical* point $(y,v) \in \mathcal{A}$ of E (i.e., $E'(y,v) = 0$) is a zero of E, and thus is a pair solution of the controllability problem. Moreover:

given any sequence $(y_k, v_k)_{k\in\mathbb{N}}$ *in* $\mathcal{A}$ *such that* $\|E'(y_k,v_k)\|_{\mathcal{A}_0'} \underset{k\to+\infty}{\longrightarrow} 0$ *and such that* $\|f'(y_k)\|_\infty$ *is uniformly bounded, we have* $E(y_k,v_k) \underset{k\to+\infty}{\longrightarrow} 0$.

Thanks to this instrumental property, a minimizing sequence for E cannot be stuck in a local minimum, even though E fails to be convex (it has multiple zeros).

Second property of the least-squares functional E

For any $(y, v) \in \mathcal{A}$, let $(Y^1, V^1) \in \mathcal{A}_0$ be the solution of

$$\begin{cases} \partial_{tt} Y^1 - \partial_{xx} Y^1 + f'(y) Y^1 = V^1 1_\omega + \big(\partial_{tt} y - \partial_{xx} y + f(y) - v 1_\omega\big) & \text{in } Q_T, \\ Y^1 = 0 & \text{on } \Sigma_T, \\ (Y^1(\cdot, 0), \partial_t Y^1(\cdot, 0)) = (0, 0) & \text{in } \Omega, \end{cases} \tag{10}$$

Proposition

For all $(y, v) \in \mathcal{A}$, *for some* $C = C(\Omega, T)$

•$E'(y, v) \cdot (Y^1, V^1) = 2E(y, v)$

•$\|(Y^1, \partial_t Y^1)\|_{L^\infty(0,T;\boldsymbol{V})} + \|V^1\|_{2,q_T} \leq C e^{C\sqrt{\|f'(y)\|_\infty}} \sqrt{E(y, f)}$

• *Assume that some* $p \in [0, 1]$, $[f']_p := \sup_{\substack{a,b \in \mathbb{R} \\ a \neq b}} \frac{|f'(a) - f'(b)|}{|a-b|^p} < +\infty$. *Then,*

$$E\big((y, v) - \lambda(Y^1, V^1)\big) \leq \Big(|1 - \lambda| + \lambda^{1+p} C(y) E(y, v)^{\frac{p}{2}}\Big)^2 E(y, v) \qquad \forall \lambda \in \mathbb{R} \tag{11}$$

where

$$C(y) := C\,[f']_p \Big(C e^{C\sqrt{\|f'(y)\|_\infty}}\Big)^{1+p}. \tag{12}$$

Assume that $T > 2\max(\ell_1, 1-\ell_2)$. The vector $-(Y^1, V^1)$, solution of minimal control norm of (10), is a descent direction for E. This leads us to define, for any fixed $m \geq 1$, the sequence $(y_k, v_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ defined by

$$\boxed{\begin{cases} (y_0, v_0) \in \mathcal{A} \\ (y_{k+1}, v_{k+1}) = (y_k, v_k) - \lambda_k (Y_k^1, V_k^1) \qquad \forall k \in \mathbb{N} \\ \lambda_k = \underset{\lambda\in[0,m]}{\mathrm{argmin}}\ E\big((y_k, v_k) - \lambda(Y_k^1, V_k^1)\big) \end{cases}} \tag{13}$$

where $(Y_k^1, V_k^1) \in \mathcal{A}_0$ is the solution of minimal control norm of

$$\boxed{\begin{cases} \partial_{tt} Y_k^1 - \partial_{xx} Y_k^1 + f'(y_k) Y_k^1 = V_k^1 1_\omega + (\partial_{tt} y_k - \partial_{xx} y_k + f(y_k) - v_k 1_\omega) & \text{in } Q_T, \\ Y_k^1 = 0 & \text{on } \Sigma_T, \\ (Y_k^1(\cdot,0), \partial_t Y_k^1(\cdot,0)) = (0,0) & \text{in } \Omega. \end{cases}} \tag{14}$$

The real number $m \geq 1$ is arbitrarily fixed. It is used in the proof of convergence to bound the sequence of optimal descent steps λ_k.

Given any $p \in [0,1]$, we set

$$\beta^0(p) := \frac{p^2}{C^2(2p+1)^2} \tag{15}$$

Theorem (Trélat, Münch, 2021)

Assume that $T > 2\max(\ell_1, 1-\ell_2)$, that $[f']_p < +\infty$ for some $p \in [0,1]$, and that there exist $\alpha \geq 0$ and $\beta \in [0, \beta^0(p))$ (with the agreement that $\beta = 0$ if $p = 0$), such that

$$\boxed{|f'(r)| \leq \alpha + \beta \ln^2(1+|r|) \qquad \forall r \in \mathbb{R}.} \tag{16}$$

For $p = 0$ (i.e., $f' \in L^\infty(\mathbb{R})$), we assume moreover that $2\|f'\|_\infty C^2 e^{C\sqrt{\|f'\|_\infty}} < 1$.
Then, as $k \to \infty$

- *For any $(y_0, v_0) \in \mathcal{A}$, $(y_k, v_k) \to (y, v)$ a controlled pair for the nonlinear wave eq.*
- *$\lambda_k \to 1$.*

Moreover, the convergence of these sequences is at least linear, and is at least of order $1+p$ after a finite number of iterations.

4

[4] Münch,Trélat: Constructive exact control of semilinear 1D wave equationd by a least-squares approach. To appear in SICON

Step 1: We prove by induction that, if β is small enough, then $\|y_k\|_{L^\infty(Q_T)}{}_{n\in\mathbb{N}}$ is bounded. Assume $\exists M > 0$ such that $\|y_k\|_{L^\infty(Q_T)} \leq M$, $\forall k \leq n$. Then,

$$\begin{aligned}\|y_{n+1}\|_\infty &\leq \|y_0\|_\infty + \sum_{k=1}^{n} |\lambda_k| \|Y_k^1\|_\infty \\ &\leq \|y_0\|_\infty + m \sum_{k=1}^{n} Ce^{C\sqrt{\|f'(y_k)\|_\infty}} \sqrt{E(y_k, v_k)}\end{aligned}$$

But, $e^{C\sqrt{\|f'(y_k)\|_\infty}} \leq e^{C\sqrt{\alpha}}(1 + \|y_k\|_\infty)^{C\sqrt{\beta}} \leq e^{C\sqrt{\alpha}}(1 + M)^{C\sqrt{\beta}}$, $k \leq n$ so that

$$\boxed{\|y_{n+1}\|_\infty \leq \|y_0\|_\infty + me^{C\sqrt{\alpha}}(1 + M)^{C\sqrt{\beta}} \sum_{k=1}^{n} \sqrt{E(y_k, v_k)}}$$

Moreover, from

$$\sqrt{E((y_k, v_k) - \lambda(Y_k^1, V_k^1))} \leq \left(|1-\lambda| + \lambda^{1+p}\, C(y_k) \sqrt{E(y_k, v_k)}^{\,p}\right) \sqrt{E(y_k, v_k)} \qquad \forall \lambda \in \mathbb{R}$$

where

$$C(y_k) := C\,[f']_p \left(Ce^{C\sqrt{\|f'(y_k)\|_\infty}}\right)^{1+p} \leq C(M) := C\,[f']_p \left(e^{C\sqrt{\alpha}}(1+M)^{C\sqrt{\beta}}\right)^{1+p} \qquad (17)$$

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$$\|y_{n+1}\|_\infty \leq \|y_0\|_\infty + \sum_{k=1}^{n} |\lambda_k| \|Y_k^1\|_\infty$$

$$\leq \|y_0\|_\infty + m \sum_{k=1}^{n} Ce^{C\sqrt{\|f'(y_k)\|_\infty}} \sqrt{E(y_k, v_k)}$$

But, $e^{C\sqrt{\|f'(y_k)\|_\infty}} \leq e^{C\sqrt{\alpha}}(1 + \|y_k\|_\infty)^{C\sqrt{\beta}} \leq e^{C\sqrt{\alpha}}(1 + M)^{C\sqrt{\beta}}$, $k \leq n$ so that

$$\boxed{\|y_{n+1}\|_\infty \leq \|y_0\|_\infty + me^{C\sqrt{\alpha}}(1 + M)^{C\sqrt{\beta}} \sum_{k=1}^{n} \sqrt{E(y_k, v_k)}}$$

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we get that

$$\frac{\sqrt{E(y_{k+1}, v_{k+1})}}{\sqrt{E(y_k, v_k)}} \leq min_{\lambda \in [0,m]} \Big(|1-\lambda| + \lambda^{1+p}\, C(M) E(y_k, v_k)^{\frac{p}{2}} \Big)$$

and (after computations ....) to

$$\sum_{k=0}^{n} \sqrt{E(y_k, v_k)} \leq \bigg(C\, [f']_p \Big(e^{C\sqrt{\alpha}} (1+M)^{C\sqrt{\beta}} \Big)^{1+p} \bigg)^{1/p} \sqrt{E(y_0, v_0)}$$

and then to

$$\boxed{\|y_{n+1}\|_\infty \leq \|y_0\|_\infty + [f']_p^{1/p} \Big(C e^{C\sqrt{\alpha}} (1+M)^{C\sqrt{\beta}} \Big)^{\frac{1+2p}{p}} \sqrt{E(y_0, v_0)}}$$

from which we deduce that, if $C\sqrt{\beta}\,\frac{1+2p}{p} < 1$, then $\|y_{n+1}\|_{L^\infty(Q_T)} \leq M$ (for some M large enough).

Step 2: Once we know that $\{\|y_k\|_{L^\infty}\}_{k\in\mathbb{N}}$ is bounded, we get using again

$$\frac{\sqrt{E(y_{k+1},v_{k+1})}}{\sqrt{E(y_k,v_k)}} \leq min_{\lambda\in[0,m]}\Big(|1-\lambda| + \lambda^{1+p}\, C(y_k)E(y_k,v_k)^{\frac{p}{2}}\Big)^2$$

that the series $\sum_{k=1}^n \|Y_k^1, V_k^1\| \leq C(M)\sum_{k=1}^n \sqrt{E(y_k,f_k)}$ converges and

$$(y_{k+1},v_{k+1}) = (y_0,v_0) - \sum_{k=1}^{n}\lambda_k(Y_k^1,V_k^1) \to (y_0,v_0) - \sum_{k=1}^{\infty}\lambda_k(Y_k^1,V_k^1) := (y,v) \quad \text{in} \quad \mathcal{A}$$

with in particular,

$$\boxed{\|(y,v)-(y_k,v_k)\|_{\mathcal{H}} \leq C\sqrt{E(y_k,v_k)}, \quad \forall k\in\mathbb{N}.} \tag{18}$$

Step 3 We pass to the limit w.r.t. k in (using that $\|Y_k^1, V_k^1\|_{\mathcal{A}_0} \to 0$)

$$\begin{cases} \partial_{tt}Y_k^1 - \partial_{xx}Y_k^1 + f'(y_k)Y_k^1 = V_k^1 1_\omega + (\partial_{tt}y_k - \partial_{xx}y_k + f(y_k) - v_k 1_\omega) & \text{in } Q_T,\\ Y_k^1 = 0 & \text{on } \Sigma_T,\\ (Y_k^1(\cdot,0), \partial_t Y_k^1(\cdot,0)) = (0,0) & \text{in } \Omega. \end{cases} \tag{19}$$

Our assumptions on f :

• For some $p \in [0,1]$

$$(\overline{\mathbf{H}}_{\mathbf{p}}) \quad [f']_p := \sup_{\substack{a,b\in\mathbb{R}\\ a\neq b}} \frac{|f'(a)-f'(b)|}{|a-b|^p} < +\infty$$

and

• For $p \in (0,1]$

$$(\mathbf{H_2}) \quad \exists \alpha \geq 0 \text{ and } \beta \in [0, \beta^\star(p)) \text{ s.t. } |f'(r)| \leq \alpha + \beta \ln^2(1+|r|)\, \forall r \in \mathbb{R}$$

• For $p = 0$,

$$(\mathbf{H_3}) \quad \sqrt{2} C \|f'\|_\infty e^{C\sqrt{\|f'\|_\infty}} < 1$$

are a bit stronger than in Zuazua'93:

$$(\mathbf{H_1}) \quad \limsup_{|r|\to\infty} \frac{|f(r)|}{|r| \ln^2 |r|} < \beta$$

However the function $f(r) = a + br + \beta r \ln^2(1+|r|)\ \forall a, b \in \mathbb{R}$ (which is somehow the limit case in $(\mathbf{H_1})$) satisfies $(\overline{\mathbf{H}}_{\mathbf{2}})$ and $(\mathbf{H_2})$.

Remark 2: Link with a Damped Newton method

Defining $F : \mathcal{A} \to L^2(Q_T)$ by $F(y,v) := (\partial_{tt} y - \partial_{xx} y + f(y) - v 1_\omega)$,

$$E(y,v) = \frac{1}{2} \|F(y,v)\|^2_{L^2(Q_T)}$$

For $\lambda_k = 1$, the least-squares algorithm coincides with the Newton algorithm applied to F (explaining the super-linear convergence property).

Optimizing the parameter λ_k gives a global convergence property of the algorithm and leads to the so-called damped Newton method applied to F.

[Let $f : \mathbb{R} \to \mathbb{R}$ smooth. We want to approximate $x \in \mathbb{R}$ such that $f(x) = 0$. For that, we consider the extremal problem

$$\inf_{x \in \mathbb{R}} g(x), \qquad g(x) = |f(x)|^2.$$

We observe that $g'(x) \cdot \overline{x} = 2f(x)f'(x) \cdot \overline{x}$ so that $\overline{x} = -\frac{f(x)}{f'(x)}$ is a descent direction for g ($g'(x) \cdot \overline{x} = -2g(x) \leq 0$) leading to the descent algorithm with optimal step:

$$\begin{cases} x_0 \in \mathbb{R} \quad \text{given}, \quad x_{k+1} = x_k - \lambda_k \dfrac{f(x_k)}{f'(x_k)}, \quad k \geq 0, \\ \lambda_k \quad \text{minimizes} \quad \lambda \to g\left(x_k - \lambda \dfrac{f(x_k)}{f'(x_k)}\right) \end{cases}$$

Consequently, a descent algorithm with optimal step (using a specific descent direction) for g COINCIDES with the (globally convergent) damped Newton method for f]

If $E(y_0, v_0)$ is small, assumption $|f'(r)| \leq \alpha + \beta \ln^2(1 + |r|) \quad \forall r \in \mathbb{R}$ may be relaxed

Proposition

Assume that $[g']_p < \infty$ for some $p \in (0, 1]$.
$\exists C([f']_p) > 0$ such that, if $E(y_0, v_0) \leq C([f']_p)$, then $(y_k, v_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ converges.
The convergence is at least linear, and is at least of order $1 + p$ after a finite number of iterations.

$E(y_0, v_0)$ is notably small if

- $|f(0)|$ is small;
- The initial guess (y_0, v_0) solves the linear controllability problem (i.e. $f \equiv 0$);
- The initial condition (u_0, u_1) and target (z_0, z_1) are small for the norm $\boldsymbol{V}$;

since then

$$E(y_0, v_0) = \frac{1}{2}\|f(y_0)\|_2^2 \leq |f(0)|^2|Q_T| + C(\alpha, \beta, \epsilon)(\|u_0, u_1\|_{\boldsymbol{V}}^{2+\epsilon} + \|z_0, z_1\|_{\boldsymbol{V}}^{2+\epsilon}), \quad \epsilon > 0$$

The proposition involves notably the (well-known) local controllability of the wave equation.

We may weaken the assumption : $\exists p \in (0,1]$ s.t.

$$(\overline{\mathbf{H}}_{\mathbf{p}}) \quad [f']_p := \sup_{\substack{a,b\in\mathbb{R}\\ a\neq b}} \frac{|f'(a)-f'(b)|}{|a-b|^p} < +\infty$$

by the following one : $\exists p \in (0,1]$ s.t.

$$(\overline{\mathbf{H}}'_{\mathbf{p}}) \quad \text{There exist } \overline{\alpha}, \overline{\beta}, \gamma \in \mathbb{R}^+ \text{ such that}$$
$$|f'(a) - f'(b)| \leq |a-b|^p\big(\overline{\alpha} + \overline{\beta}(|a|^\gamma + |b|^\gamma)\big), \quad \forall a, b \in \mathbb{R}$$

assuming $\frac{\gamma + C\sqrt{\beta}(1+2p)}{p} < 1$.

Multidimensional case: Ω bounded domain of $\mathbb{R}^d$, $d \leq 3$ with $C^{1,1}$ boundary

Using the observability estimate in [Fu, Yong, Zhang 2007] [5] with the potential
$A \in L^\infty(0,T;L^d(\Omega))$ $\left[\|\varphi_0,\varphi_1\|_{L^2\times H^{-1}} \leq Ce^{C\|A\|^2_{L^\infty(L^d)}}\|\varphi\|_{L^2(\omega\times(0,T))}\right]$

Theorem (Bottois, Lemoine, M 21)

Let $(u_0,u_1),(z_0,z_1) \in \boldsymbol{V}$. For any $x_0 \in \mathbb{R}^d\backslash\overline{\Omega}$, let $\Gamma_0 = \{x \in \partial\Omega, (x-x_0)\cdot\nu(x) > 0\}$ and, for any $\epsilon > 0$, $\mathcal{O}_\epsilon(\Gamma_0) = \{y \in \mathbb{R}^d \mid |y-x| \leq \epsilon \, for \, x \in \Gamma_0\}$. Assume

$(\mathbf{H_0})$ $\quad T > 2\max_{x\in\overline{\Omega}}|x-x_0|$ *and* $\omega \supseteq \mathcal{O}_\epsilon(\Gamma_0)\cap\Omega$ *for some* $\epsilon > 0$.

Assume that f' satisfies $(\overline{\mathbf{H}}_\mathbf{p})$ for some $p \in [0,1]$ and

$(\mathbf{H_2})$ $\quad \exists\alpha \geq 0, \beta \in [0,\beta^\star(p))$ *s.t.* $|f'(r)| \leq \alpha + \beta\ln^{1/2}(1+|r|)\forall r \in \mathbb{R}$ $(p>0)$

$(\mathbf{H_3})$ $\quad \sqrt{2}C\|f'\|_\infty e^{C\|f'\|^2_\infty|\Omega|^{2/d}} < 1 (p=0)$

For any $(y_0,f_0) \in \mathcal{A}$, the sequence $(y_k,v_k)_{k\in\mathbb{N}}$ strongly converges to a solution of

$$\begin{cases} \partial_{tt}y - \Delta y + f(y) = v1_\omega, & in \ Q_T, \\ y = 0, & on \ \Sigma_T, \\ (y(\cdot,0),y_t(\cdot,0)) = (u_0,u_1), (y(\cdot,T),y_t(\cdot,T)) = (z_0,z_1) & in \ \Omega. \end{cases} \tag{20}$$

The convergence is at least linear and is at least of order $1+p$ after a finite number of iterations.

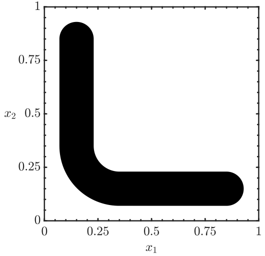
[5] X. Fu, J. Yong, X. Zhang *Exact controllability for multidimensional semilinear hyperbolic equations*, SICON

We consider a two-dimensional case for which $\Omega = (0,1)^2$ and $T = 3$. Moreover, for any real constant c_f, we consider the nonlinear function g defined as follows :

$$f(r) = c_f\, r \ln^{1/2}(2 + |r|), \quad \forall r \in \mathbb{R}.$$

f satisfies $(\overline{\mathbf{H}}_\mathbf{p})$ for $p = 1$ and $(\mathbf{H_2})$ for $|c_f|$ small enough. The unfavorable situation for which the norm of the uncontrolled solution grows corresponds to $c_f < 0$. We consider

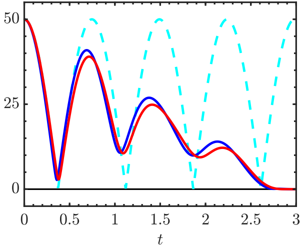
$$(u_0, u_1) = (100 \sin(\pi x_1)\sin(\pi x_2), 0), \quad (z_0, z_1) = (0, 0)$$



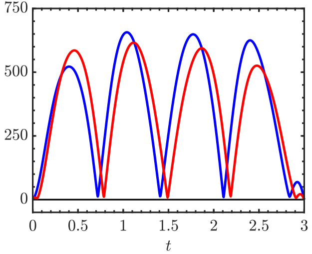
Control domain $\omega \subset \Omega = (0,1)^2$ (black part).

Initialization of the sequence (y_k, v_k) with the controlled solution of the linear equation.

$\sharp$iterate k	$\sqrt{2E(y_k, v_k)}$	$\frac{\|y_k - y_{k-1}\|_{L^2(Q_T)}}{\|y_{k-1}\|_{L^2(Q_T)}}$	$\frac{\|v_k - v_{k-1}\|_{L^2_\chi(q_T)}}{\|v_{k-1}\|_{L^2_\chi(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|v_k\|_{L^2_\chi(q_T)}$	λ_k
0	7.44×10^1	–	–	38.116	732.22	1
1	8.83×10^{-1}	1.65×10^{-1}	3.37×10^{-1}	37.2	697.423	1
2	7.14×10^{-6}	2.66×10^{-4}	9.90×10^{-4}	37.201	697.615	–

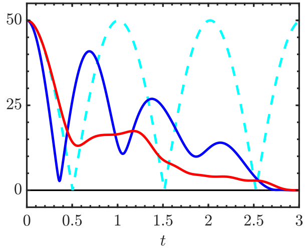


(—) $\|y_2(\cdot, t)\|_{L^2(\Omega)}$; (—) $\|y_0(\cdot, t)\|_{L^2(\Omega)}$; (– –) $\|y(\cdot, t; 0)\|_{L^2(\Omega)}$.

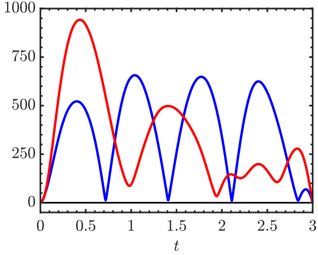


(—) $\|v_2(\cdot, t)\|_{L^2_\chi(\omega)}$; (—) $\|v_0(\cdot, t)\|_{L^2_\chi(\omega)}$.

$\sharp$iterate k	$\sqrt{2E(y_k, v_k)}$	$\frac{\Vert y_k - y_{k-1}\Vert_{L^2(Q_T)}}{\Vert y_{k-1}\Vert_{L^2(Q_T)}}$	$\frac{\Vert v_k - v_{k-1}\Vert_{L^2_\chi(q_T)}}{\Vert v_{k-1}\Vert_{L^2_\chi(q_T)}}$	$\Vert y_k\Vert_{L^2(Q_T)}$	$\Vert v_k\Vert_{L^2_\chi(q_T)}$	λ_k
0	3.72×10^{2}	–	–	38.116	732.22	1
1	4.58×10^{1}	9.01×10^{-1}	1.07×10^{0}	30.219	665.222	1
2	9.12×10^{-1}	6.36×10^{-2}	1.57×10^{-1}	30.563	734.688	1
3	1.69×10^{-4}	6.34×10^{-4}	1.43×10^{-3}	30.567	734.56	1
4	9.31×10^{-11}	1.15×10^{-7}	1.78×10^{-7}	30.567	734.559	–



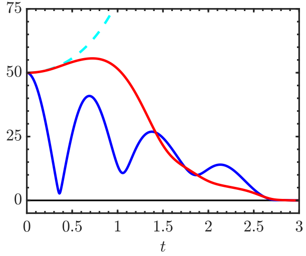
(—) $\Vert y_4(\cdot,t)\Vert_{L^2(\Omega)}$; (—) $\Vert y_0(\cdot,t)\Vert_{L^2(\Omega)}$; (- -) $\Vert y(\cdot,t;0)\Vert_{L^2(\Omega)}$.



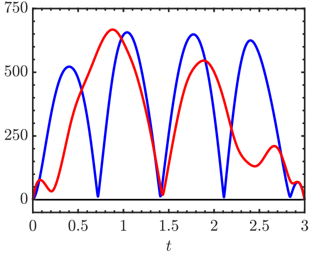
(—) $\Vert v_4(\cdot,t)\Vert_{L^2_\chi(\omega)}$; (—) $\Vert v_0(\cdot,t)\Vert_{L^2_\chi(\omega)}$.

Numerical experiments in the 2d case; $c_f = -10$

$\sharp$iterate k	$\sqrt{2E(y_k, f_k)}$	$\frac{\Vert y_k - y_{k-1}\Vert_{L^2(Q_T)}}{\Vert y_{k-1}\Vert_{L^2(Q_T)}}$	$\frac{\Vert v_k - v_{k-1}\Vert_{L^2_\chi(q_T)}}{\Vert v_{k-1}\Vert_{L^2_\chi(q_T)}}$	$\Vert y_k\Vert_{L^2(Q_T)}$	$\Vert v_k\Vert_{L^2_\chi(q_T)}$	λ_k
0	7.44×10^2	–	–	38.116	732.22	1
1	1.63×10^2	1.79×10^0	9.30×10^{-1}	58.691	667.602	1
2	1.62×10^0	8.42×10^{-2}	1.41×10^{-1}	60.781	642.643	1
3	1.97×10^{-3}	1.21×10^{-3}	4.66×10^{-3}	60.745	643.784	1
4	5.11×10^{-10}	6.43×10^{-7}	2.63×10^{-6}	60.745	643.785	–



(red) $\Vert y_4(\cdot,t)\Vert_{L^2(\Omega)}$; (blue) $\Vert y_0(\cdot,t)\Vert_{L^2(\Omega)}$; (dashed) $\Vert y(\cdot,t;0)\Vert_{L^2(\Omega)}$.



(red) $\Vert v_4(\cdot,t)\Vert_{L^2_\chi(\omega)}$; (blue) $\Vert v_0(\cdot,t)\Vert_{L^2_\chi(\omega)}$.

Zuazua fixed point operator; $c_f = -5$

$$\begin{cases} \partial_{tt} y_{k+1} - \Delta y_{k+1} + y_{k+1}\, \widehat{f}(y_k) = -f(0) + v_{k+1} 1_\omega, & \text{in} \quad Q_T, \\ y_{k+1} = 0, & \text{on } \Sigma_T, \\ (y_{k+1}(\cdot,0), \partial_t y_{k+1}(\cdot,0)) = (y_0, y_1), & \text{in } \Omega. \end{cases} \tag{21}$$

$\sharp$iterate k	$\sqrt{2E(y_k, v_k)}$	$\frac{\Vert y_{k+1}-y_k\Vert_{L^2(Q_T)}}{\Vert y_k\Vert_{L^2(Q_T)}}$	$\frac{\Vert v_{k+1}-v_k\Vert_{L^2_\chi(q_T)}}{\Vert v_k\Vert_{L^2_\chi(q_T)}}$	$\Vert y_k\Vert_{L^2(Q_T)}$	$\Vert v_k\Vert_{L^2_\chi(q_T)}$
0	3.72×10^{2}	1.02×10^{0}	1.33×10^{0}	38.116	732.22
1	4.79×10^{1}	5.85×10^{-2}	1.73×10^{-1}	37.945	562.213
2	2.65×10^{0}	3.35×10^{-3}	1.55×10^{-2}	36.798	530.787
3	1.54×10^{-1}	3.05×10^{-4}	9.84×10^{-4}	36.812	526.864
4	1.39×10^{-2}	4.70×10^{-5}	8.77×10^{-5}	36.807	527.209
5	2.13×10^{-3}	9.24×10^{-6}	1.81×10^{-5}	36.806	527.221
6	4.20×10^{-4}	1.88×10^{-6}	3.93×10^{-6}	36.806	527.225
7	8.55×10^{-5}	4.07×10^{-7}	8.81×10^{-7}	36.806	527.226
8	1.85×10^{-5}	8.97×10^{-8}	1.99×10^{-7}	36.806	527.226
9	4.08×10^{-6}	–	–	36.806	527.226

Lack of convergence for $|c_f| > 15$

$$\begin{cases} \partial_{tt} y_{k+1} - \Delta y_{k+1} = v_{k+1} 1_\omega - f(y_k), & \text{in} \quad Q_T, \\ y_{k+1} = 0, & \text{on } \Sigma_T, \\ (y_{k+1}(\cdot,0), \partial_t y_{k+1}(\cdot,0)) = (y_0, y_1), & \text{in } \Omega. \end{cases} \tag{22}$$

$\sharp$iterate k	$\sqrt{2E(y_k, v_k)}$	$\frac{\Vert y_{k+1}-y_k\Vert_{L^2(Q_T)}}{\Vert y_k\Vert_{L^2(Q_T)}}$	$\frac{\Vert v_{k+1}-v_k\Vert_{L^2_\chi(q_T)}}{\Vert v_k\Vert_{L^2_\chi(q_T)}}$	$\Vert y_k\Vert_{L^2(Q_T)}$	$\Vert v_k\Vert_{L^2_\chi(q_T)}$
0	3.72×10^2	8.26×10^{-1}	1.71×10^0	38.116	732.22
1	3.11×10^2	3.77×10^{-1}	7.11×10^{-1}	48.341	1330.18
2	1.80×10^2	1.49×10^{-1}	3.32×10^{-1}	46.01	1264.46
3	6.89×10^1	6.16×10^{-2}	1.31×10^{-1}	44.116	965.409
4	2.70×10^1	2.94×10^{-2}	4.71×10^{-2}	43.696	879.298
5	1.27×10^1	1.64×10^{-2}	2.22×10^{-2}	43.66	859.09
6	7.08×10^0	9.78×10^{-3}	1.24×10^{-2}	43.702	849.733
7	4.24×10^0	6.07×10^{-3}	7.53×10^{-3}	43.757	844.619
8	2.64×10^0	3.85×10^{-3}	4.71×10^{-3}	43.804	841.847
9	1.68×10^0	2.48×10^{-3}	3.02×10^{-3}	43.841	840.273
–	–	–	–	–	–
39	8.71×10^{-6}	–	–	43.929	838.099

Lack of convergence for $|c_f| > 7$

Let $\omega \subset \Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$.

$$\begin{cases} \partial_t y - \Delta y + f(y) = v 1_\omega \quad \text{in} \quad Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = u_0 \text{ in } \Omega, \end{cases} \tag{23}$$

where $u_0 \in L^2(\Omega)$ is the initial state of y and $v \in L^2(q_T)$ is a *control* function such that $y(T, \cdot) = 0$.

Theorem (Fernández-Cara, Zuazua, 2000)

Let $T > 0$ be given. Assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is locally Lipschitz continuous, satisfies $f(0) = 0$ and

$$(\mathbf{H_0}) \quad |f'(r)| \leq C(1 + |r|^{4+d}) \text{ a.e. in } \mathbb{R}.$$

There exists a $\beta^\star > 0$ such that if

$$(\mathbf{H_1}) \quad \limsup_{|r| \to \infty} \frac{|f(r)|}{|r| \ln_+^{3/2} |r|} \leq \beta^\star$$

then system (23) is globally exactly controllable to 0 at time T with controls in $L^\infty(q_T)$.

[6]

[6] E. Fernández-Cara and E. Zuazua, *Null and approximate controllability for weakly blowing up semilinear,* Ann. Inst. H. Poincaré Anal. Non Linéaire 2000

Theorem (Lemoine, Münch 21)

Let $T > 0$ be given. Let $d = 1$. Assume that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies $f(0) = 0$ and the growth condition

$$(\mathbf{H_1'}) \quad \exists \alpha > 0, \ s.t. \ |f'(r)| \leq (\alpha + \beta^\star \ln_+ |r|)^{3/2}, \quad \forall r \in \mathbb{R}$$

for some $\beta^\star > 0$ small enough and

$$(\overline{\mathbf{H}}_\mathbf{p}) \quad \exists p \in [0,1] \ \textit{such that} \ \sup_{\substack{a,b \in \mathbb{R} \\ a \neq b}} \frac{|f'(a) - f'(b)|}{|a-b|^p} < +\infty.$$

Then, for any $u_0 \in H_0^1(\Omega)$, one can construct a sequence $(y_k, v_k)_{k \in \mathbb{N}}$ converging strongly to a controlled pair for (23) satisfying $y(T) = 0$. Moreover, after a finite number of iterations, the convergence is of order at least $1 + p$.

[7]

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[7] Lemoine, Münch, *Constructive exact control of semilinear 1D heat equations*. To appear in MCRF.

Set up of the least-squares approach

We introduce, for all $s \geq 0$, the vectorial space $\mathcal{A}_0(s)$

$$\mathcal{A}_0(s) := \Big\{(y,v) : \rho(s)\, y \in L^2(Q_T),\ \rho_0(s) v \in L^2(q_T),$$
$$\rho_0(s)(\partial_t y - \partial_{xx} y) \in L^2(Q_T),\ y(\cdot,0) = 0 \text{ in } \Omega,\ y = 0 \text{ on } \Sigma_T\Big\} \quad (24)$$

where $\rho_i(s)$ Carleman weights of the form $\boxed{\rho_i(s) \approx e^{\frac{s\varphi(x)}{T-t}}}$ and the convex space

$$\mathcal{A}(s) := \Big\{(y,v) : \rho(s)\, y \in L^2(Q_T),\ \rho_0(s) v \in L^2(q_T),$$
$$\rho_0(s)(\partial_t y - \partial_{xx} y) \in L^2(Q_T),\ y(\cdot,0) = u_0 \text{ in } \Omega,\ y = 0 \text{ on } \Sigma_T\Big\} \quad (25)$$

We define the least-squares functional $E : \mathcal{A}(s) \to \mathbb{R}$ by

$$\boxed{E_s(y,v) := \frac{1}{2}\|\rho_0(s)(\partial_t y - \partial_{xx} y + f(y) - v 1_\omega)\|^2_{L^2(Q_T)}}$$

and consider the nonconvex minimization problem

$$\boxed{\inf_{(y,v)\in\mathcal{A}(s)} E_s(y,v)}$$

For any fixed $m \geq 1$, a minimizing sequence $(y_k, v_k)_{k\in\mathbb{N}} \in \mathcal{A}(s)$ as follows:

$$\begin{cases} (y_0, f_0) \in \mathcal{A}(s), \\ (y_{k+1}, v_{k+1}) = (y_k, v_k) - \lambda_k(Y_k^1, V_k^1), \quad k \geq 0, \\ \lambda_k = \text{argmin}_{\lambda\in[0,m]} E_s((y_k, v_k) - \lambda(Y_k^1, V_k^1)) \end{cases} \tag{26}$$

where $(Y_k^1, V_k^1) \in \mathcal{A}_0(s)$ is the minimal controlled pair solution of

$$\begin{cases} \partial_t Y_k^1 - \partial_{xx} Y_k^1 + f'(y_k)Y_k^1 = V_k^1 1_\omega + \partial_t y_k - \partial_{xx} y_k + f(y_k) - v_k 1_\omega \quad \text{in} \quad Q_T, \\ Y_k^1 = 0 \ \text{ on } \ \Sigma_T, \quad Y_k^1(\cdot, 0) = 0 \ \text{ in } \ \Omega \end{cases} \tag{27}$$

associated with $(y_k, v_k) \in \mathcal{A}(s)$. In particular, the pair (Y_k^1, V_k^1) vanishes when $E_s(y_k, v_k)$ vanishes.

Rk. In order to give a meaning to (26), we need to prove that we can choose the parameter s independent of k, that is $s \geq \max\left(\|f'(y_k)\|^{2/3}_{L^\infty(Q_T)}, s_0\right)$ for all $k \in \mathbb{N}$. In this respect, it suffices to prove that there exists $M > 0$ such that $\|y_k\|_{L^\infty(Q_T)} \leq M$ for every $k \in \mathbb{N}$.

The proof requires fine estimates of the controlled pair for a linearized eq.

$$\begin{cases} \partial_t z - \partial_{xx} z + Az = v 1_\omega + B & \text{in} \quad Q_T, \\ z = 0 \text{ on } \Sigma_T, \quad z(\cdot, 0) = z_0 \text{ in } \Omega \end{cases} \tag{28}$$

Theorem

Assume $A \in L^\infty(Q_T)$, $s \geq \max(\|A\|^{2/3}_{L^\infty(Q_T)}, s_0)$, $B \in L^2(\rho_0(s), Q_T)$ and $z_0 \in H^1_0(\Omega)$. The unique solution z which minimizes together with the corresponding control v the functional $J : L^2(\rho(s); Q_T) \times L^2(\rho_0(s); q_T) \to \mathbb{R}^+$ defined by

$$J(z, v) := \frac{1}{2} \|\rho(s)\, z\|^2_{L^2(Q_T)} + \frac{s^{-3}}{2} \|\rho_0(s)\, v\|^2_{L^2(q_T)}$$

satisfies the following

$$\|z\|_{L^\infty(Q_T)} \leq C e^{-\frac{3}{2}s} (1 + \|A\|_{L^\infty(Q_T)}) \big(\|\rho_0(s) B\|_{L^2(Q_T)} + e^{cs} \|z_0\|_{H^1_0(\Omega)} \big). \tag{29}$$

A simpler linearization (leading to linear convergence)

Theorem (Ervedoza, Lemoine, Münch)

Let $T > 0$ be given. Let $d \leq 5$ and $s > 0$ large enough. If $f \in \mathcal{C}^1(\mathbb{R})$ satisfies

$$(\mathbf{H}_1') \quad \exists \alpha > 0, \ s.t. \ |f'(r)| \leq (\alpha + \beta^\star \ln_+ |r|)^{3/2}, \quad \forall r \in \mathbb{R}$$

for some $\beta^\star > 0$ small enough and $f(0) = 0$ then for any $u_0 \in L^\infty(\Omega)$, one can construct a sequence $(y_k, v_k)_{k\in\mathbb{N}}$ converging strongly in $L^2(Q_T) \times L^2(Q_T)$ to a controlled pair for (23). Besides, the convergence of $(y_k, v_k)_{k\in\mathbb{N}}$ holds at least with a linear rate for the norm $L^2(\rho_0(s), Q_T) \times L^2(\rho_0(s), q_T)$, where s is chosen suitably large depending on $\|u_0\|_{L^\infty(\Omega)}$.

Sketch: For any $\widehat{y}$ in a suitable class $\mathcal{C}(s)$ depending on a free parameter $s \geq 1$, let v be a null for y solution of

$$\begin{cases} \partial_t y - \Delta y = v 1_\omega - f(\widehat{y}) & \text{in} \quad Q_T \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = u_0 & \text{in} \quad \Omega \end{cases}, \tag{30}$$

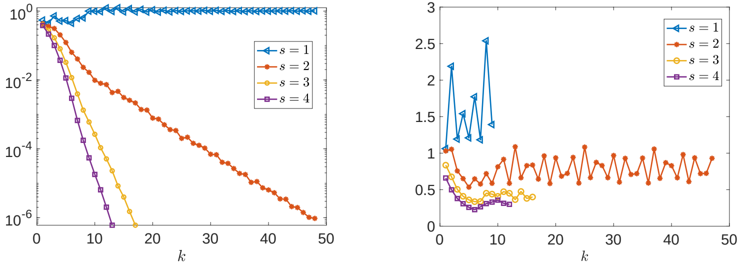
satisfies $y(\cdot, T) = 0$ in Ω. For any s sufficiently large depending on $\|u_0\|_{L^\infty(\Omega)}$, the operator $\Lambda_s : \widehat{y} \mapsto y$ from some suitable class $\mathcal{C}(s)$ into itself, is contracting. Moreover, the constant of contraction increases with s.[8]

[8] Ervedoza, Lemoine, Munch, Exact controllability of semilinear heat equations through a constructive approach

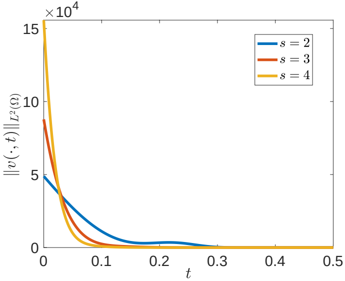
$\Omega = (0,1)$, $\omega = (0.2, 0.8)$, $T = \frac{1}{2}$;

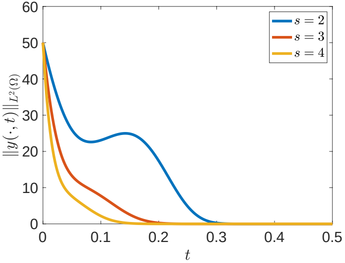
$$f(r) = c_f \big(1 + \ln(1 + |r|)\big)^{3/2} r, \quad c_f < 0$$

f satisfies $(\mathbf{H_1})$, $(\mathbf{H'_1})$ and also $(\overline{\mathbf{H}}_\mathbf{p})$ for every $p \in [0,1]$. In particular, $f'' \in L^\infty(Q_T)$.
$u_0(x) = c_{u_0} \sin(\pi x)$ parametrized by $c_{u_0} > 0$.



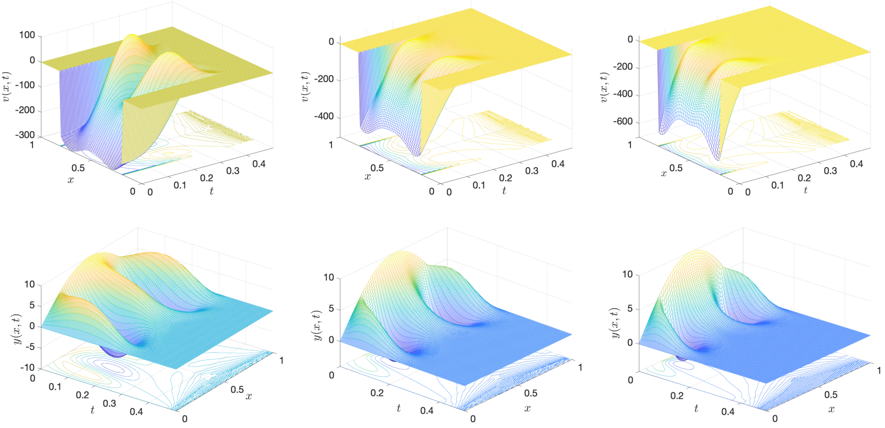
Relative error $\frac{\|\rho_0(s)(y_{k+1}-y_k)\|_{L^2(Q_T)}}{\|\rho_0(s)y_k\|_{L^2(Q_T)}}$ (**Left**) and $\frac{\|\rho_0(s)(y_{k+1}-y_k)\|_{L^2(Q_T)}}{\|\rho_0(s)(y_k-y_{k-1})\|_{L^2(Q_T)}}$ (**Right**) w.r.t. k for $s \in \{1,2,3,4\}$.$c_f = -5$, $c_{u_0} = 10$;





Evolution of $\|v_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ and $\|y_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ w.r.t. $t \in [0,T]$ for $c_{u_0} = 10$, $c_f = -5$ and $s \in \{2,3,4\}$.

$c_{u_0} = 10$, $c_f = -5$



Control $v_{k^\star}$ (**top**) and controlled solution $y_{k^\star}$ (**bottom**) in Q_T for $s \in \{2, 3, 4\}$.

$$u_0(x) = c_{u_0} \sin(\pi x); \quad f(r) = -2\big(1 + \ln(1 + |r|)\big)^{3/2} r; \quad s = 3 \tag{31}$$

c_{u_0}	$\lVert y_{k^\star} \rVert_{L^2(Q_T)}$	$\lVert \rho(s) y_{k^\star} \rVert_{L^2(Q_T)}$	$\lVert v_{k^\star} \rVert_{L^2(q_T)}$	$\lVert \rho_0(s) v_{k^\star} \rVert_{L^2(q_T)}$	$\lVert v_{k^\star} \rVert_{L^\infty(q_T)}$	$k^\star$
10	1.14031	41.0689	34.0273	292.739	271.86	8
100	12.4721	576.084	420.613	3722.92	3381.68	12
500	69.8357	4170.43	2443.36	22324.9	20055.3	15
1000	149.215	10045.6	5213.5	48679.6	43195	17
2000	322.25	24509.6	11144.3	107260	93000.8	20
3000	507.998	41520.3	17405	171395	145648	22
4000	703.063	60489.4	23901.8	239863	200195	23
5000	905.632	81095	30586.9	311987	256227	24
6000	1114.55	103128	37430.5	387325	313419	25
7000	1329.02	126440	44412	465555	371629	26
8000	1548.43	150917	51516.2	546433	430756	28
9000	1772.31	176471	58731.6	629764	490642	28
10000	2000.29	203029	66048.7	715388	551209	30
20000	4455.89	513872	143566	1.67×10^6	1.18×10^6	39

$$k^\star = \min\left\{ k \in \mathbb{N}, \frac{\lVert \rho_0(s)(y_{k+1} - y_k) \rVert_{L^2(Q_T)}}{\lVert \rho_0(s) y_k \rVert_{L^2(Q_T)}} \leq 10^{-6} \right\} \tag{32}$$

Some conclusions

Constructive proof of controllability (without fixed point arguments)

Introducing weighted functional involving both the control and the state, one may design, for any initial guess, convergent algorithms to a controlled pair solution of the semilinear equation (wave and heat equation)

The various methods require a growth condition at infinity of the derivative of the nonlinearity : $|f'(r)| \leq \beta \ln_+^{3/2}(r)$ for large r and β small.

The Carleman parameter s is taken large enough in order to absorb the lower order terms and get contraction properties.

The method may be extended to the systems for which a precise Carleman estimate (for the linearized eq.) is available.

The approach allows to get a numerical approximation v_k^h of a nonlinear control v, for k large enough and h (numerical space-time discretization parameter) small enough :

$$\|v - v_k^h\| \leq \|v - v_k\| + \|v_k - v_k^h\|$$

May be very likely adapted to (nonlinear) inverse/assimilation problem

On going work: extension to Burgers equation and Navier-Stokes system !

THANK YOU VERY MUCH FOR YOUR ATTENTION

• Bottois, Lemoine, Munch. *Constructive proof of the exact controllability for semi-linear multi-dimensional wave equations*, arxiv.org/abs/2101.06446.

• Munch, Trélat. *Constructive exact control of semilinear 1D wave equations by a least-squares approach*, To appear in SICON

• Ervedoza, Lemoine, Münch. *Exact controllability of semilinear heat equations through a constructive approach*, hal-03350534

• Lemoine, Munch. *A fully space-time least-squares method for the unsteady Navier-Stokes system*, J. Mathematical Fluids Mechanics 2021

• Lemoine, Munch. *Resolution of the Implicit Euler scheme for the Navier-Stokes equation through a least-squares method*, Numerische Mathematik 2021.

• Lemoine, Munch, Pedregal, *Analysis of continuous H^{-1} least-squares methods for the steady Navier-Stokes system*, Applied. Math. Optimization 2021.

• Lemoine, Marin-Gayte, Munch, *Approximation of null controls for semilinear heat equations using a least-squares approach*. ESAIM:COCV 2021

• Zuazua Exact controllability for semilinear wave equations in one space dimension, Ann. Inst. H. Poincaré Anal. Non Linéaire 1993

• Fernández-Cara, Zuazua, *Null and approximate controllability for weakly blowing up semilinear,* Ann. Inst. H. Poincaré Anal. Non Linéaire 2000

• X. Fu, J. Yong, X. Zhang, *Exact controllability for multidimensional semilinear hyperbolic equations*, SICON 2007