Approximation of controllability and inverse problems for PDE

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PART 2

Minimization of the conjugate functional

$$
\left\{
\begin{array}{l}
\text{Min } \; J^\star(\varphi_0, \varphi_1) = \dfrac{1}{2} \displaystyle\int_0^T \!\! \int_{\Gamma_0} \left| \dfrac{\partial \varphi}{\partial \nu} \right|^2 d\sigma \, dt + < y_0, \, \varphi_t(\cdot, 0) >_{L^2} - \langle y_1, \varphi(\cdot\,, 0) \rangle_{H^{-1}, H_0^1} \\
\text{Subject to } \; (\varphi_0, \varphi_1) \in \boldsymbol{V} = H_0^1(\Omega) \times L^2(\Omega) \quad \text{where} \quad L^\star \varphi = 0
\end{array}
\right.
\tag{1}
$$

Second method to bypass the fact that $L^\star\varphi_h \neq 0$

Since we can not achieve $L^\star\varphi_h = 0$, the idea is to relax the constraint $L^\star\varphi_h = 0$!!!?!!

The idea is to replace the observability inequality

$$\begin{cases} \|\varphi_0,\varphi_1\|_{\boldsymbol{V}}^2 \leq C_{obs}\left\|\dfrac{\partial\varphi}{\partial\nu}\right\|_{L^2(\Gamma_T)}^2, & \forall(\varphi_0,\varphi_1), \\ L^\star\varphi = 0, \quad \varphi_{|\Sigma_T} = 0 \end{cases} \tag{2}$$

by a "generalized observability inequality" :

$$\|\varphi(\cdot,0),\varphi_t(\cdot,0)\|_{\boldsymbol{V}}^2 \leq C_{obs}\left(\left\|\frac{\partial\varphi}{\partial\nu}\right\|_{L^2(\Gamma_T)}^2 + \|L^\star\varphi\|_{L^2(Q_T)}^2\right), \qquad \forall\varphi\in\boldsymbol{\Phi} \tag{3}$$

Why ? Because, if $\varphi_h \in \Phi_h$ a finite dimensional subspace of Φ, then

$$\|\varphi_h(\cdot,0),\varphi_{h,t}(\cdot,0)\|_{\boldsymbol{V}}^2 \leq C_{obs}\left(\left\|\frac{\partial\varphi_h}{\partial\nu}\right\|_{L^2(\Gamma_T)}^2 + \|L^\star\varphi_h\|_{L^2(Q_T)}^2\right), \qquad \forall\varphi_h\in\boldsymbol{\Phi_h} \tag{4}$$

and the constant is still C_{obs} (independant of h) !!!

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From observability to generalized observability

Let $f \in L^2(Q_T)$. We decompose the solution φ of

$$L^\star \varphi = f \quad Q_T \qquad \varphi_{|\Sigma_T} = 0, \qquad (\varphi(\cdot,0), \varphi_t(\cdot,0)) = (\varphi_0, \varphi_1)$$

as $\varphi = \varphi_1 + \varphi_2$ with

$$\begin{cases} L^\star \varphi_1 = 0 \quad Q_T, & \varphi_{1|\Sigma_T} = 0, \qquad (\varphi_1(\cdot,0), \varphi_{1,t}(\cdot,0)) = (\varphi_0, \varphi_1), \\ L^\star \varphi_2 = f \quad Q_T, & \varphi_{2|\Sigma_T} = 0, \qquad (\varphi_2(\cdot,0), \varphi_{2,t}(\cdot,0)) = (0, 0) \end{cases}$$

We have

$$\begin{aligned} \|(\varphi(\cdot,0), \varphi_t(\cdot,0))\|^2_{\boldsymbol{V}} &= \|(\varphi_1(\cdot,0), \varphi_{1,t}(\cdot,0))\|^2_{\boldsymbol{V}} \\ &\leq C_{obs} \left\| \frac{\partial \varphi_1}{\partial \nu} \right\|^2_{L^2(\Gamma_T)} \\ &\leq 2C_{obs} \left\| \frac{\partial \varphi}{\partial \nu} \right\|^2_{L^2(\Gamma_T)} + 2C_{obs} \left\| \frac{\partial \varphi_2}{\partial \nu} \right\|^2_{L^2(\Gamma_T)} \\ &\leq 2C_{obs} \left\| \frac{\partial \varphi}{\partial \nu} \right\|^2_{L^2(\Gamma_T)} + 2C_{obs} C(\Omega, T) \|L^\star \varphi\|^2_{L^2(Q_T)} \end{aligned} \tag{5}$$

Minimization of $J^\star$

We now replace the problem

$$\begin{cases} \text{Min} \;\; J^\star(\varphi_0,\varphi_1) = \dfrac{1}{2}\displaystyle\int_0^T\!\!\int_{\Gamma_0} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt + < y_0,\, \varphi_t(\cdot,0) >_{L^2} - \langle y_1, \varphi(\cdot\,,0)\rangle_{H^{-1},H_0^1} \\ \text{Subject to} \;\; (\varphi_0,\varphi_1) \in \boldsymbol{V} = H_0^1(\Omega)\times L^2(\Omega) \quad \text{where} \quad L^\star\varphi = 0 \end{cases} \tag{6}$$

by the equivalent problem

$$\begin{cases} min\, J^\star(\varphi) = \dfrac{1}{2}\displaystyle\int_0^T\!\!\int_{\Gamma_0} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt + < y_0,\, \varphi_t(\cdot,0) >_{L^2} - \langle y_1, \varphi(\cdot\,,0)\rangle_{H^{-1},H_0^1} \\ \text{Subject to } \varphi \in \boldsymbol{W} := \left\{\varphi : \varphi \in C^0(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega)),\; L^\star\varphi = 0 \in L^2(Q_T)\right\} \end{cases} \tag{7}$$

Remark- If $\varphi \in \boldsymbol{W}$ then $\frac{\partial\varphi}{\partial\nu} \in L^2(\Gamma_T)$

Remark- $\boldsymbol{W}$ endowed with the norm $\|\varphi\|_{\boldsymbol{W}} := \|\frac{\partial\varphi}{\partial\nu}\|_{L^2(\Gamma_T)}$ is an Hilbert space.

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Minimization of $J^\star$

We assume T and Γ_0 large enough. We now replace the problem

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$$\begin{cases} min\, J_r{}^\star(\varphi) = \dfrac{1}{2}\displaystyle\int_0^T\!\!\int_{\Gamma_0}\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt + \frac{r}{2}\|L^\star\varphi\|^2_{L^2(Q_T)} + < y_0,\, \varphi_t(\cdot,0) >_{L^2} - \langle y_1, \varphi(\cdot\,,0)\rangle_{H^{-1},H_0^1} \\ \text{Subject to } \varphi \in \boldsymbol{W} := \left\{\varphi : \varphi \in C^0(0,T; H_0^1(\Omega)) \cap C^1(0,T; L^2(\Omega)),\; L^\star\varphi = 0 \in L^2(Q_T)\right\} \end{cases} \tag{9}$$

for all $r \geq 0$.

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Relaxation of $L^\star\varphi = 0$

In order to address the $L^2(Q_T)$ constraint $L^\star\varphi = 0$, we introduce a Lagrange multiplier $\lambda \in L^2(Q_T)$; we consider the saddle point problem :

$$\begin{cases} \sup_{\lambda\in L^2(Q_T)} \inf_{\varphi\in\mathbf{\Phi}} \mathcal{L}_r(\varphi,\lambda), \\ \mathcal{L}_r(\varphi,\lambda) := J_r(\varphi) + < L^\star\varphi, \lambda >_{L^2(Q_T)} \\ \mathbf{\Phi} := \left\{ \varphi : \varphi \in C^0(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega)), L^\star\varphi \in L^2(Q_T) \right\} \supset \boldsymbol{W} \end{cases} \tag{10}$$

Remark- For all $\eta > 0$, $\mathbf{\Phi}$ is endowed with the scalar product,

$$< \varphi, \overline{\varphi} >_{\mathbf{\Phi}} := < \frac{\partial\varphi}{\partial\nu}, \frac{\partial\overline{\varphi}}{\partial\nu} >_{L^2(\Gamma_T)} + \eta < L^\star\varphi, L^\star\overline{\varphi} >_{L^2(Q_T)}, \quad \forall \varphi, \overline{\varphi} \in \mathbf{\Phi}.$$

$\|\varphi\|_{\mathbf{\Phi}} := \sqrt{< \varphi, \varphi >_{\mathbf{\Phi}}}$ is a norm and $(\mathbf{\Phi}, \|\cdot\|_{\mathbf{\Phi}})$ is an Hilbert space.

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Mixed formulation

Find $(\varphi, \lambda) \in \mathbf{\Phi} \times L^2(Q_T)$ solution of

$$\left\{ \begin{array}{rcll} a_r(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) & = & l(\overline{\varphi}), & \forall \overline{\varphi} \in \mathbf{\Phi} \\ b(\varphi, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{array} \right. \quad (11)$$

where

$$a_r : \mathbf{\Phi} \times \mathbf{\Phi} \to \mathbb{R}, \quad a_r(\varphi, \overline{\varphi}) = < \frac{\partial \varphi}{\partial \nu}, \frac{\partial \overline{\varphi}}{\partial \nu} >_{L^2(\Gamma_T)} + r < L^\star \varphi, L^\star \overline{\varphi} >_{L^2(Q_T)} \quad (12)$$

$$b : \mathbf{\Phi} \times L^2(Q_T) \to \mathbb{R}, \quad b(\varphi, \lambda) = < L^\star \varphi, \lambda >_{L^2(Q_T)} \quad (13)$$

$$l : \mathbf{\Phi} \to \mathbb{R}, \quad l(\varphi) = - < y_0, \varphi_t(\cdot, 0) >_{L^2} + \langle y_1, \varphi(\cdot, 0) \rangle_{H^{-1}, H_0^1} \quad (14)$$

Theorem

For all $r \geq 0$,

1. *The mixed formulation is well-posed.*
2. *The unique solution $(\varphi, \lambda) \in \boldsymbol{\Phi} \times L^2(Q_T)$ is the unique saddle-point of the Lagrangian $\mathcal{L}_r : \boldsymbol{\Phi} \times L^2(Q_T) \to \mathbb{R}$ defined by*

$$\mathcal{L}_r(\varphi, \lambda) = \frac{1}{2} a_r(\varphi, \varphi) + b(\varphi, \lambda) - l(\varphi). \tag{15}$$

3. *The optimal function φ given by 2. satisfies $\varphi \in \boldsymbol{W}$ and is the minimizer of $J_r^\star$ over $\boldsymbol{W}$ while the optimal function $\lambda \in L^2(Q_T)$ is the state of the controlled wave equation in the weak sense.*
4. *We have the following estimates*

$$\|\varphi\|_{\boldsymbol{\Phi}} \leq \|y_0, y_1\|_{\boldsymbol{H}},$$

$$\|\lambda\|_{L^2} \leq \frac{1}{\delta}\left(1 + \max(1, \frac{r}{\eta})\right)\|y_0, y_1\|_{\boldsymbol{H}}, \quad \delta = (C_{\Omega,T} + \eta)^{-1/2}$$

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Well-posedness 2

The kernel $\mathcal{N}(b) = \{\varphi \in \mathbf{\Phi}; b(\varphi, \lambda) = 0 \quad \forall \lambda \in L^2(Q_T)\}$ coincides with $\boldsymbol{W}$: we get

$$a_r(\varphi, \varphi) = \|\varphi\|^2_{\mathbf{\Phi}}, \quad \forall \varphi \in \mathcal{N}(b) = \boldsymbol{W}.$$

It remains to check the inf-sup constant property : $\exists \delta > 0$ such that

$$\inf_{\lambda \in L^2} \sup_{\varphi \in \mathbf{\Phi}} \frac{b(\varphi, \lambda)}{\|\varphi\|_{\mathbf{\Phi}} \|\lambda\|_{L^2}} \geq \delta. \tag{16}$$

For any fixed $\lambda \in L^2(Q_T)$, we define $\varphi^0 \in \mathbf{\Phi}$ as the unique solution of

$$L^\star \varphi^0 = \lambda \text{ in } Q_T, \quad (\varphi^0(\cdot, 0), \varphi^0_t(\cdot, 0)) = (0, 0) \text{ on } \Omega, \quad \varphi^0 = 0 \text{ on } \Sigma_T.$$

We get $b(\varphi^0, \lambda) = \|\lambda\|^2_{L^2}$ and $\|\varphi^0\|^2_{\Phi} = \left\| \frac{\partial \varphi^0}{\partial \nu} \right\|^2_{L^2(\Gamma_T)} + \eta \|\lambda\|^2_{L^2}$.

The estimate $\left\| \frac{\partial \varphi^0}{\partial \nu} \right\|_{L^2(\Gamma_T)} \leq \sqrt{C_{\Omega,T}} \|\lambda\|_{L^2(Q_T)}$ implies that

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It remains to check the inf-sup constant property : $\exists \delta > 0$ such that

$$\inf_{\lambda \in L^2} \sup_{\varphi \in \mathbf{\Phi}} \frac{b(\varphi, \lambda)}{\|\varphi\|_{\mathbf{\Phi}} \|\lambda\|_{L^2}} \geq \delta. \tag{16}$$

For any fixed $\lambda \in L^2(Q_T)$, we define $\varphi^0 \in \mathbf{\Phi}$ as the unique solution of

$$L^\star \varphi^0 = \lambda \text{ in } Q_T, \quad (\varphi^0(\cdot, 0), \varphi^0_t(\cdot, 0)) = (0, 0) \text{ on } \Omega, \quad \varphi^0 = 0 \text{ on } \Sigma_T.$$

We get $b(\varphi^0, \lambda) = \|\lambda\|^2_{L^2}$ and $\|\varphi^0\|^2_{\Phi} = \left\| \frac{\partial \varphi^0}{\partial \nu} \right\|^2_{L^2(\Gamma_T)} + \eta \|\lambda\|^2_{L^2}$.

The estimate $\left\| \frac{\partial \varphi^0}{\partial \nu} \right\|_{L^2(\Gamma_T)} \leq \sqrt{C_{\Omega,T}} \|\lambda\|_{L^2(Q_T)}$ implies that

$$\sup_{\varphi \in \mathbf{\Phi}} \frac{b(\varphi, \lambda)}{\|\varphi\|_{\mathbf{\Phi}} \|\lambda\|_{L^2}} \geq \frac{b(\varphi^0, \lambda)}{\|\varphi^0\|_{\mathbf{\Phi}} \|\lambda\|_{L^2}} \geq \frac{1}{\sqrt{C_{\Omega,T} + \eta}} > 0$$

leading to the inf-sup property with $\delta = (C_{\Omega,T} + \eta)^{-1/2}$.

The multiplier λ

Taking $r = 0$, the first equation reads

$$a_{r=0}(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) = l(\overline{\varphi}), \qquad \forall \overline{\varphi} \in \mathbf{\Phi} \tag{17}$$

i.e.

$$\iint_{\Gamma_T} \frac{\partial \varphi}{\partial \nu} \frac{\partial \overline{\varphi}}{\partial \nu} + \iint_{Q_T} \lambda\, L^{\star} \overline{\varphi} = - < y_0, \overline{\varphi}_t(\cdot, 0) >_{L^2} + \langle y_1, \overline{\varphi}(\cdot, 0) \rangle_{H^{-1}, H_0^1}, \ \forall \overline{\varphi} \in \mathbf{\Phi} \tag{18}$$

which means $\lambda \in L^2(Q_T)$ is solution in the sense of transposition of

$$\begin{cases} L\lambda = 0, \quad \text{in} \quad Q_T \\ (\lambda(\cdot, 0), \lambda_t(\cdot, 0)) = (y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega) \\ (\lambda(\cdot, T), \lambda_t(\cdot, T)) = (0, 0), \\ \lambda = \dfrac{\partial \varphi}{\partial \nu} \quad \text{on} \quad \Gamma_T \end{cases} \tag{19}$$

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$$\lambda \in C^0([0, T], L^2(\Omega)) \cap C^1([0, T], H^{-1}(\Omega))$$

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Dual of the dual - Problem w.r.t. λ

Lemma

Let $\mathcal{P}_r$ be the linear operator from L^2 into L^2 defined by

$$\mathcal{P}_r\lambda := L^\star\varphi, \quad \forall\lambda\in L^2 \quad where \quad \varphi\in\mathbf{\Phi} \quad solves \quad a_r(\varphi,\overline{\varphi}) = b(\overline{\varphi},\lambda), \quad \forall\overline{\varphi}\in\mathbf{\Phi}.$$

For any $r>0$, the operator $\mathcal{P}_r$ is a strongly elliptic, symmetric isomorphism from L^2 into L^2.

$$\exists C>0, \quad \iint_{Q_T}(\mathcal{P}_r\lambda)\lambda\,dx\,dt \geq C\|\lambda\|^2_{L^2(Q_T)}, \quad \forall\lambda\in L^2(Q_T) \quad ?? \tag{20}$$

PROOF- By contradiction, there exists then a sequence $\{\lambda_n\}_{n\geq 0}$ of $L^2(Q_T)$ such that

$$\|\lambda_n\|_{L^2(Q_T)} = 1, \quad \forall n\geq 0, \qquad \lim_{n\to\infty}\iint_{Q_T}(\mathcal{P}_r\lambda_n)\lambda_n\,dx\,dt = 0.$$

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Dual of the dual - Problem w.r.t. λ - 2

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Writing $a_r(\varphi_n, \overline{\varphi}) = b(\overline{\varphi}, \lambda_n), \quad \forall \overline{\varphi} \in \mathbf{\Phi}$, we get

$$\iint_{Q_T} (rL^\star \varphi_n - \lambda_n) L^\star \overline{\varphi}\, dx\, dt + \iint_{\Gamma_T} \varphi_{n,\nu}\, \overline{\varphi}_\nu\, d\sigma\, dt = 0, \quad \forall \overline{\varphi} \in \mathbf{\Phi}. \tag{23}$$

We define the sequence $\{\overline{\varphi}_n\}_{n\geq 0}$ as follows :

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so that $\|\overline{\varphi}_{n,\nu}\|_{L^2(\Gamma_T)} \leq C_{\Omega,T} \|rL^\star \varphi_n - \lambda_n\|_{L^2(Q_T)}$, so that $\overline{\varphi}_n \in \mathbf{\Phi}$. Then, using (23), we get

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Dual of the dual - Problem w.r.t. λ - 2

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Dual of the dual - Problem in λ - 3

Theorem

$$\sup_{\lambda\in L^2}\inf_{\varphi\in\mathbf{\Phi}}\mathcal{L}_r(\varphi,\lambda) = -\inf_{\lambda\in L^2} J_r^{\star\star}(\lambda) \quad + \mathcal{L}_r(\varphi_0, 0)$$

where $\varphi_0\in\mathbf{\Phi}$ *solves* $a_r(\varphi_0,\overline{\varphi}) = l(\overline{\varphi}), \forall\overline{\varphi}\in\mathbf{\Phi}$ *and* $J_r^{\star\star}: L^2\to\mathbb{R}$ *defined by*

$$J_r^{\star\star}(\lambda) := \frac{1}{2} < \mathcal{P}_r\lambda,\lambda >_{L^2(Q_T)} - b(\varphi_0,\lambda)$$

PROOF- For any $\lambda\in L^2(Q_T)$, let us denote by $\varphi_\lambda\in\mathbf{\Phi}$ the minimizer of $\varphi\to\mathcal{L}_r(\varphi,\lambda)$; φ_λ satisfies the equation

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We then have

$$\begin{aligned}\inf_{\varphi\in\Phi}\mathcal{L}_r(\varphi,\lambda) &= \mathcal{L}_r(\varphi_\lambda,\lambda) = \mathcal{L}_r(\psi_\lambda+\varphi_0,\lambda)\\ &= \frac{1}{2}a_r(\psi_\lambda+\varphi_0,\psi_\lambda+\varphi_0) + b(\psi_\lambda+\varphi_0,\lambda) - l(\psi_\lambda+\varphi_0)\\ &:= X_1 + X_2 + X_3\end{aligned}$$

Dual of the dual - Problem in λ - 3

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Dual of the dual - Problem in λ - 4

$$\begin{aligned}\inf_{\varphi\in\Phi}\mathcal{L}_r(\varphi,\lambda) &= \mathcal{L}_r(\varphi_\lambda,\lambda)=\mathcal{L}_r(\psi_\lambda+\varphi_0,\lambda)\\ &= \frac{1}{2}a_r(\psi_\lambda+\varphi_0,\psi_\lambda+\varphi_0)+b(\psi_\lambda+\varphi_0,\lambda)-l(\psi_\lambda+\varphi_0)\\ &:= X_1+X_2+X_3\end{aligned}$$

with

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From the definition of φ_0, $X_2=0$ while $X_3=\mathcal{L}_r(\varphi_0,0)$. Eventually, from the definition of ψ_λ,

$$X_1=-\frac{1}{2}a_r(\psi_\lambda,\psi_\lambda)+b(\varphi_0,\lambda)=-\frac{1}{2}\iint_{Q_T}(\mathcal{P}_r\lambda)\,\lambda\,dx\,dt+b(\varphi_0,\lambda)$$

The control problem is reduced to the minimization of an unconstrained functional with respect to the control state !!!

Dual of the dual - Problem in λ - 4

$$\begin{aligned}\inf_{\varphi\in\Phi} \mathcal{L}_r(\varphi,\lambda) &= \mathcal{L}_r(\varphi_\lambda,\lambda) = \mathcal{L}_r(\psi_\lambda+\varphi_0,\lambda)\\ &= \frac{1}{2}a_r(\psi_\lambda+\varphi_0,\psi_\lambda+\varphi_0) + b(\psi_\lambda+\varphi_0,\lambda) - l(\psi_\lambda+\varphi_0)\\ &:= X_1 + X_2 + X_3\end{aligned}$$

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Conformal Approximation

Let then Φ_h and Λ_h be two finite dimensional spaces parametrized by the variable h such that

$$\Phi_h \subset \mathbf{\Phi}, \quad \Lambda_h \subset L^2(Q_T), \qquad \forall h > 0.$$

Then, we can introduce the following approximated problems : find $(\varphi_h, \lambda_h) \in \Phi_h \times \Lambda_h$ solution of

$$\left\{ \begin{aligned} a_r(\varphi_h, \overline{\varphi}_h) + b(\overline{\varphi}_h, \lambda_h) &= l(\overline{\varphi}_h), \qquad && \forall \overline{\varphi}_h \in \Phi_h \\ b(\varphi_h, \overline{\lambda}_h) &= 0, && \forall \overline{\lambda}_h \in \Lambda_h. \end{aligned} \right. \tag{24}$$

For any $h > 0$, the well-posedness is again a consequence of two properties

- the coercivity of the bilinear form a_r on the subset $\mathcal{N}_h(b) = \{\varphi_h \in \Phi_h ; b(\varphi_h, \lambda_h) = 0 \quad \forall \lambda_h \in \Lambda_h\}$. From the relation

$$a_r(\varphi, \varphi) \geq \frac{r}{\eta} \|\varphi\|_{\Phi}^2, \quad \forall \varphi \in \mathbf{\Phi}$$

the form a_r is coercive on the full space Φ, and so *a fortiori* on $\mathcal{N}_h(b) \subset \Phi_h \subset \Phi$.

- The second property is a discrete inf-sup condition : there exists $\delta > 0$ such that

$$\delta_h := \inf_{\lambda_h \in \Lambda_h} \sup_{\varphi_h \in \Phi_h} \frac{b(\varphi_h, \lambda_h)}{\|\varphi_h\|_{\Phi_h} \|\lambda_h\|_{\Lambda_h}} \geq \delta. \tag{25}$$

A necessary condition is: $\dim(\Phi_h) > \dim(\Lambda_h)$

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Finite dimensional linear system

Let $n_h = \dim \Phi_h$, $m_h = \dim \Lambda_h$ and let the real matrices $A_{r,h} \in \mathbb{R}^{n_h,n_h}$, $B_h \in \mathbb{R}^{m_h,n_h}$, $J_h \in \mathbb{R}^{m_h,m_h}$ and $L_h \in \mathbb{R}^{n_h}$ be defined by

$$\begin{cases} a_r(\varphi_h, \overline{\varphi_h}) = < A_{r,h}\{\varphi_h\}, \{\overline{\varphi_h}\} >_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}, & \forall \varphi_h, \overline{\varphi_h} \in \Phi_h, \\ b(\varphi_h, \lambda_h) = < B_h\{\varphi_h\}, \{\lambda_h\} >_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}}, & \forall \varphi_h \in \Phi_h, \forall \lambda_h \in \Lambda_h, \\ l(\varphi_h) = < L_h, \{\varphi_h\} >, & \forall \varphi_h \in \Phi_h \end{cases}$$

where $\{\varphi_h\} \in \mathbb{R}^{n_h}$ denotes the vector associated to φ_h and $< \cdot, \cdot >_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}$ the usual scalar product over $\mathbb{R}^{n_h}$. Problem (24) reads as follows :

find $\{\varphi_h\} \in \mathbb{R}^{n_h}$ and $\{\lambda_h\} \in \mathbb{R}^{m_h}$ such that

$$\begin{pmatrix} A_{r,h} & B_h^T \\ B_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix} \{\varphi_h\} \\ \{\lambda_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} L_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}. \qquad (26)$$

$A_{r,h}$ is symmetric and positive definite for any $h > 0$ and any $r > 0$.
The full matrix of order $m_h + n_h$ in (26) is symmetric but not positive definite.

Choice of the conformal spaces Φ_h and Λ_h

We introduce a triangulation $\mathcal{T}_h$ such that $\overline{Q_T} = \cup_{K\in\mathcal{T}_h} K$ and we assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family. We note $h := \max\{\mathrm{diam}(K), K \in \mathcal{T}_h\}$.

We define the finite dimensional space

$$\Lambda_h = \{\lambda_h \in C^0(\overline{Q_T}), \lambda_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h,\ \lambda_h = 0 \text{ on } \Sigma_T\} \subset L^2(Q_T)$$

The space Φ_h must be chosen such that $L^\star\varphi_h \in L^2(Q_T)$ for any $\varphi_h \in \Phi_h$. This is guaranteed as soon as φ_h possesses second-order derivatives in $L^2(Q_T)$. A conformal approximation based on standard triangulation of Q_T is obtained with spaces of functions continuously differentiable with respect to both x and t.

We introduce the space Φ_h as follows:

$$\Phi_h = \{\varphi_h \in \Phi_h \in C^1(\overline{Q_T}) : \varphi_h|_K \in \mathbb{P}(K) \quad \forall K \in \mathcal{T}_h,\ \varphi_h = 0 \text{ on } \Sigma_T\} \subset \Phi$$

where $\mathbb{P}(K)$ denotes an appropriate space of polynomial functions in x and t.

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C^1 finite element over Q_T

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We may consider the following choices for $\mathbb{P}(K)$:

1. The *Bogner-Fox-Schmit* (BFS for short) C^1 element defined for rectangles. It involves 16 degrees of freedom, namely the values of $\varphi_h, \varphi_{h,x}, \varphi_{h,t}, \varphi_{h,xt}$ on the four vertices of each rectangle K.

2. The reduced *Hsieh-Clough-Tocher* (HCT for short) C^1 element defined for triangles. This is a so-called composite finite element and involves 9 degrees of freedom, namely the values of $\varphi_h, \varphi_{h,x}, \varphi_{h,t}$ on the three vertices of each triangle K.

[1] P.G. Ciarlet, The finite element for elliptic problems, North-Holland, 1979

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First error estimate

Proposition

Let $h > 0$. Let (φ, λ) and (φ_h, λ_h) be the solution of (11) and of (24) respectively. Let δ_h the discrete inf-sup constant defined by (25). Then,

$$\|\varphi - \varphi_h\|_{\Phi} \leq 2\left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right) d(\varphi, \Phi_h) + \frac{1}{\sqrt{\eta}} d(\lambda, \Lambda_h),$$

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq \left(2 + \frac{1}{\sqrt{\eta}\delta_h}\right) \frac{1}{\delta_h} d(\varphi, \Phi_h) + \frac{3}{\sqrt{\eta}\delta_h} d(\lambda, \Lambda_h)$$

with

$$\begin{cases} d(\lambda, \Lambda_h) := \inf\limits_{\lambda_h \in \Lambda_h} \|\lambda - \lambda_h\|_{L^2(Q_T)}, \\ d(\varphi, \Phi_h) := \inf\limits_{\varphi_h \in \Phi_h} \left(\|\partial_\nu \varphi - \partial_\nu \varphi_h\|^2_{L^2(\Gamma_T)} + \eta \|L^\star(\varphi - \varphi_h)\|^2_{L^2(Q_T)} \right)^{1/2}. \end{cases}$$

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Convergence rate in Φ and in $L^2(Q_T)$

Proposition (BFS element for $N = 1$ - Convergence in Φ)

Let $h > 0$, let $k \leq 2$. If $(\varphi, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$, $\exists K > 0$

$$\|\varphi - \varphi_h\|_{\Phi} \leq K\left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right)h^k,$$

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K\left(\left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right)\frac{1}{\delta_h} + \frac{1}{\sqrt{\eta}\delta_h}\right)h^k.$$

Writing the ineq. obs. for $\varphi - \varphi_h \in \Phi$ and using that $L^\star(\varphi - \varphi_h) = -L^\star\varphi_h$, we get

$$\|\varphi - \varphi_h\|^2_{L^2(Q_T)} \leq C_{\Omega,T}(C_{obs} + 1)(\|\partial_\nu(\varphi - \varphi_h)\|^2_{L^2(\Gamma_T)} + \|L^\star\varphi_h\|^2_{L^2(Q_T)})$$
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Theorem (BFS element for $N = 1$ - Convergence in $L^2(Q_T)$)

Let $h > 0$, let $k \leq 2$. If $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$,

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$$\leq C_{\Omega,T}(C_{obs} + 1)\max(1, \frac{2}{\sqrt{\eta}})\|\varphi - \varphi_h\|_{\Phi}$$

Theorem (BFS element for $N = 1$ - Convergence in $L^2(Q_T)$)

Let $h > 0$, let $k \leq 2$. If $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$,

$$\|\varphi - \varphi_h\|_{L^2(Q_T)} \leq K\max(1, \frac{2}{\sqrt{\eta}})\left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right)h^k.$$

The discrete inf-sup test - Evaluation of δ_h

$$\delta_h := \inf_{\lambda_h \in \Lambda_h} \sup_{\varphi_h \in \Phi_h} \frac{b(\varphi_h, \lambda_h)}{\|\varphi_h\|_{\Phi_h} \|\lambda_h\|_{\Lambda_h}} \geq \delta. \qquad (27)$$

Taking $\eta = r > 0$ so that $a_r(\varphi, \overline{\varphi}) = (\varphi, \overline{\varphi})_\Phi$, we have [2]

$$\delta_h = \inf\left\{ \sqrt{\delta} : B_h A_{r,h}^{-1} B_h^T \{\lambda_h\} = \delta\, J_h \{\lambda_h\}, \quad \forall\, \{\lambda_h\} \in \mathbb{R}^{m_h} \setminus \{0\} \right\}. \qquad (28)$$

The matrix $B_h A_{r,h}^{-1} B_h^T$ is symmetric and positive definite so that $\delta_h > 0$.

Power iteration algorithm: for any $\{v_h^0\} \in \mathbb{R}^{n_h}$ such that $\|\{v_h^0\}\|_2 = 1$, compute for any $n \geq 0$, $\{\varphi_h^n\} \in \mathbb{R}^{n_h}$, $\{\lambda_h^n\} \in \mathbb{R}^{m_h}$ and $\{v_h^{n+1}\} \in \mathbb{R}^{m_h}$ iteratively as follows :

$$\begin{cases} A_{r,h}\{\varphi_h^n\} + B_h^T\{\lambda_h^n\} = 0 \\ B_h\{\varphi_h^n\} = -J_h\{v_h^n\} \end{cases}, \quad \{v_h^{n+1}\} = \frac{\{\lambda_h^n\}}{\|\{\lambda_h^n\}\|_2}.$$

Then $\delta_h = \lim_{n\to\infty} (\|\{\lambda_h^n\}\|_2)^{-1/2}$.

[2] K. Bathe, D. Chapelle, The discrete inf-sup test, (2003)

The discrete inf-sup test - Evaluation of δ_h

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The discrete inf-sup test - Evaluation of δ_h

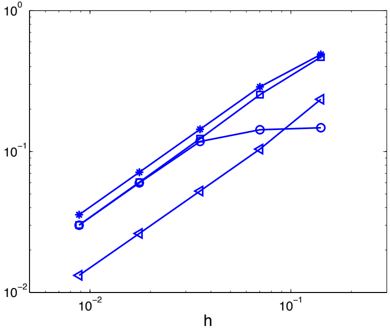

Figure: BFS finite element - Evolution of $\sqrt{r}\delta_{h,r}$ with respect to h for $r = 1$ ($\square$), $r = 10^{-2}$ ($\circ$), $r = h$ ($\star$) and $r = h^2$ ($<$).

$$\delta_h \approx C_r \frac{h}{\sqrt{r}} \quad \text{as} \quad h \to 0^+ \tag{29}$$

Choice of r versus δ_h

With $\eta = r$, we get $\delta_h \approx \frac{h}{\sqrt{r}}$ (as $h \to 0^+$)

$$\|\varphi - \varphi_h\|_{L^2(Q_T)} \leq K \max(1, \frac{2}{\sqrt{r}})\Big(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}\Big) h^k.$$

Optimal parameter: $r \approx 1$ leading to $\|\varphi - \varphi_h\|_{L^2(Q_T)} \leq K h^{k-1}$.

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 \frac{\sqrt{r}}{h}(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}) h^k.$$

Optimal parameter: $r \approx h^2$ leading to $\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 h^{k-1}$.

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Example 1 - $N = 1$ - Numerical experiments

$$\textbf{(EX3)} \qquad y_0(x) = 4x\ 1_{(0,1/2)}(x), \quad y_1(x) = 0, \quad T = 2.4.$$

$$v(t) = 2(1-t)\ 1_{(1/2,3/2)}(t), \quad t \in (0,T), \quad \|v\|_{L^2(0,T)} = 1/\sqrt{3} \approx 0.5773. \qquad (30)$$

Example 1 - $N = 1$ - Numerical experiments

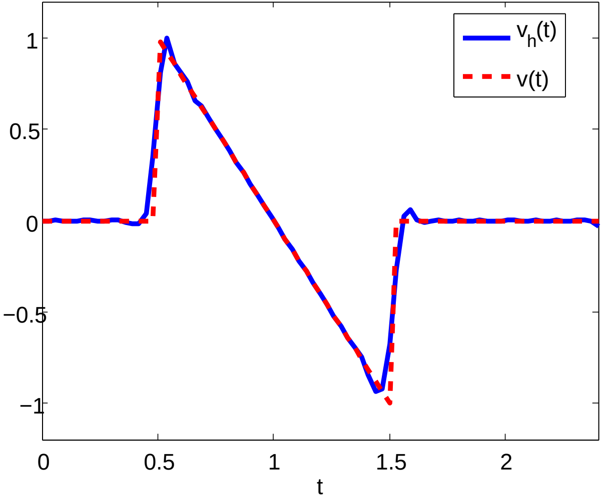

Figure: Control of minimal L^2-norm v and its approximation v_h on $(0, T)$; $r = 10^{-2}$; $h = 2.46 \times 10^{-2}$

Example 1 - $N = 1$ - Numerical experiments

h	1.41×10^{-1}	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\Vert v_h\Vert_{L^2(0,T)}$	0.6003	0.5850	0.5776	0.5752	0.5747
$\Vert v - v_h\Vert_{L^2(0,T)}$	2.87×10^{-1}	2.05×10^{-1}	1.47×10^{-1}	1.08×10^{-1}	8.18×10^{-2}
$\Vert \lambda_h\Vert_{L^2(Q_T)}$	0.62	0.598	0.586	0.581	0.578
$\Vert L^\star \varphi_h\Vert_{L^2(Q_T)}$	1.02×10^{-1}	7.53×10^{-2}	5.8×10^{-2}	4.55×10^{-2}	3.6×10^{-2}
$\Vert L^\star \varphi_h\Vert_{H^{-1}(Q_T)}$	1.92×10^{-16}	3.83×10^{-16}	7.46×10^{-16}	1.51×10^{-15}	2.81×10^{-15}

Table: BFS element - $r = 1$.

$$r = 1: \quad \Vert v - v_h\Vert_{L^2(0,T)} \approx 1.12 \cdot h^{0.52}, \quad \Vert L^\star \varphi_h\Vert_{L^2(Q_T)} \approx 15.67 \cdot h^{0.72},$$

$$r = 10^{-2}: \quad \Vert v - v_h\Vert_{L^2(0,T)} \approx 0.83 \cdot h^{0.45}, \quad \Vert L^\star \varphi_h\Vert_{L^2(Q_T)} \approx 0.24 \cdot h^{0.37}.$$

A curiosity : $\Vert v_h\Vert_{L^2(0,T)}$ is close to $\Vert y_h\Vert_{L^2(Q_T)}$!?!!

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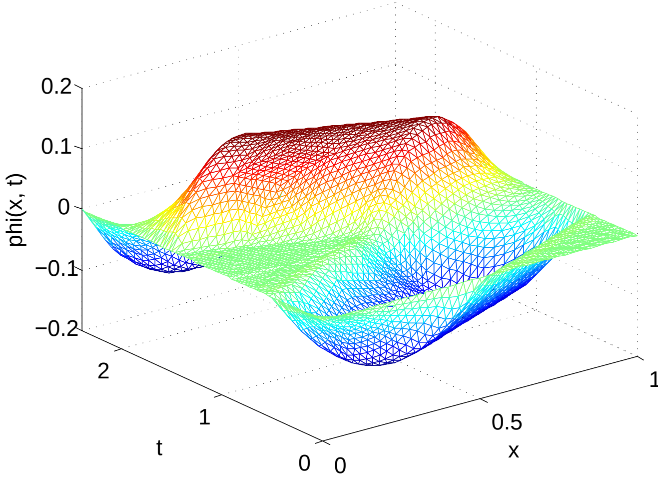

Figure: The dual variable φ_h in Q_T; $h = 2.46 \times 10^{-2}$; $r = 10^{-2}$.

Example 1 - $N = 1$ - Numerical experiments

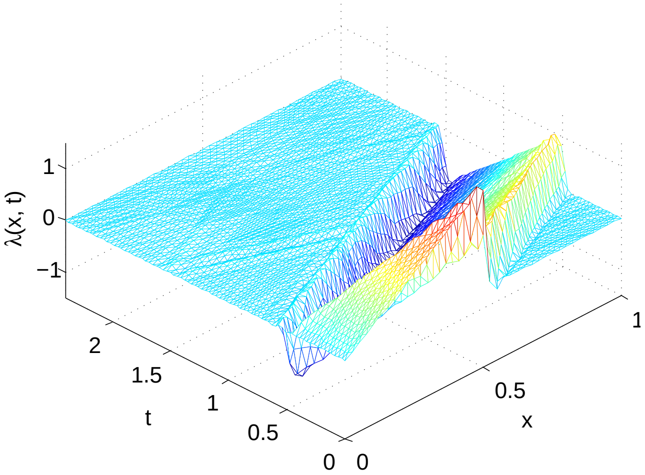

Figure: The primal variable λ_h in Q_T; $h = 2.46 \times 10^{-2}$; $r = 10^{-2}$.

Mesh adaptation

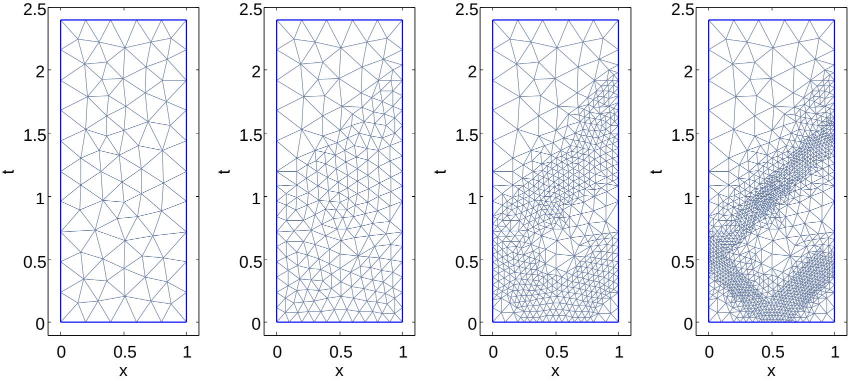

Figure: Iterative refinement of the triangular mesh over Q_T with respect to the variable λ_h: $142, 412, 1\,154, 2\,556$; $r = 2 \times 10^{-3}$.

Example 1 - $N = 1$ - Numerical experiments

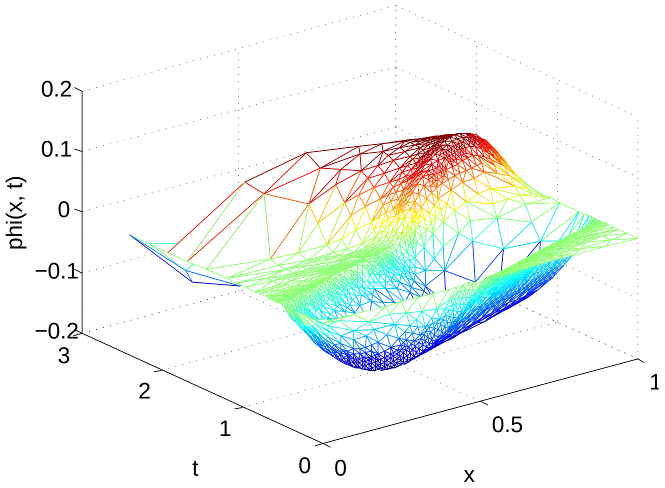

Figure: The dual variable φ_h in Q_T corresponding to the finer mesh; $r = 2 \times 10^{-3}$.

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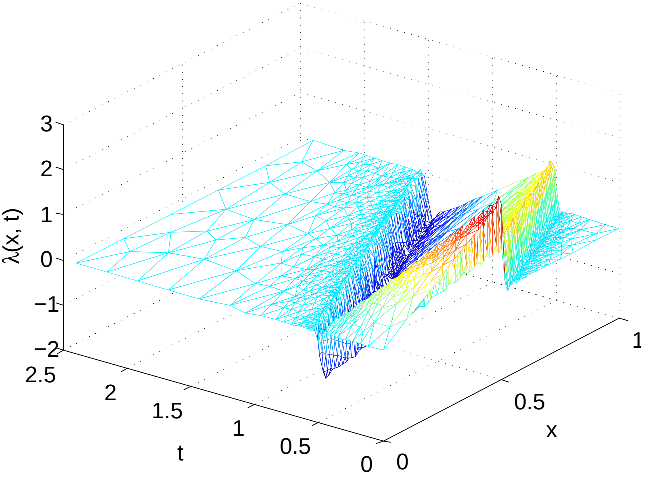

Figure: The primal variable λ_h in Q_T corresponding to the finer mesh.

Minimization of $J_r^{\star\star}$ with respect to λ

$$J_r^{\star\star}(\lambda) := \frac{1}{2} < \mathcal{P}_r\lambda, \lambda >_{L^2(Q_T)} - b(\varphi_0, \lambda)$$

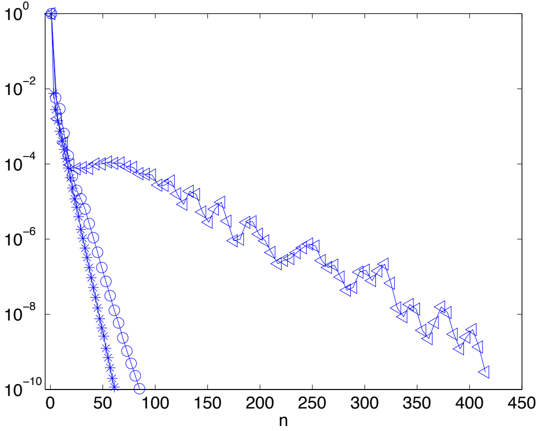

Figure: Evolution of $\|g^n\|_{L^2(Q_T)}/\|g^0\|_{L^2(Q_T)}$ w.r.t. the iterate n for $r = 10^2$ ($\star$), $r = 1$ ($\Box$), $r = 10^{-2}$ ($\circ$) and $r = h^2$ ($<$) ; $h = 9.99 \times 10^{-3}$.

Minimization of $J_r^{\star\star}$ with respect to λ

$$J_r^{\star\star}(\lambda) := \frac{1}{2} < \mathcal{P}_r\lambda, \lambda >_{L^2(Q_T)} - b(\varphi_0, \lambda)$$

h	1.56×10^{-1}	7.92×10^{-2}	3.99×10^{-2}	1.99×10^{-2}	9.99×10^{-3}
$\sharp$ iterates	20	26	31	44	61
$m_h = card(\{\lambda_h\})$	231	840	3 198	12 555	49 749
$\\|\lambda_h(1,\cdot)\\|_{L^2(0,T)}$	0.6089	0.5867	0.5775	0.5746	0.5742
$\\|v - \lambda_h(1,\cdot)\\|_{L^2(0,T)}$	2.40×10^{-1}	1.68×10^{-1}	1.28×10^{-1}	9.69×10^{-2}	7.62×10^{-2}
$\\|\lambda_h\\|_{L^2(Q_T)}$	0.6178	0.5963	0.5857	0.5806	0.5784

Table: BFS element - Conjugate gradient algorithm - $r = 1$.

Remind: $\|v\|_{L^2(0,T)} \approx 0.5773$

Comparison with the bi-harmonic regularization

$$\begin{cases} \min_{(\varphi_0,\varphi_1)\in \tilde{V}} J^\star_\epsilon(\varphi_0,\varphi_1) := J^\star(\varphi_0,\varphi_1) + \frac{\epsilon}{2}\|\varphi_0,\varphi_1\|^2_{\tilde{V}}, \quad \epsilon > 0, \\ \tilde{V} := H^2(0,1)\cap H^1_0(0,1) \times H^1_0(0,1) \end{cases} \tag{31}$$

h	1.56×10^{-1}	7.92×10^{-2}	3.99×10^{-2}	1.99×10^{-2}	9.99×10^{-3}
$\sharp$ iterates	62	> 5000	78	58	39
$card(\{\varphi_{0h},\varphi_{1h}\})$	44	84	164	324	644
$\|v_h\|_{L^2(0,T)}$	0.5484	0.5603	0.5671	0.5712	0.5736
$\|v-v_h\|_{L^2(0,T)}$	2.72×10^{-1}	2.23×10^{-1}	1.81×10^{-1}	1.47×10^{-1}	1.24×10^{-1}
$\|y_h\|_{L^2(Q_T)}$	0.5386	0.5557	0.5649	0.5701	0.5731

Table: Biharmonic Tychonoff regularization; $\epsilon = h^{1.8}$.

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Remark : If $\epsilon = h^2$, the algorithm diverges.

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Stabilized mixed formulation "à la Barbosa-Hughes"

[3]

$\alpha > 0$

$$\begin{cases} \sup_{\lambda\in\Lambda} \inf_{\varphi\in\Phi} \mathcal{L}_{r,\alpha}(\varphi,\lambda), \\ \mathcal{L}_{r,\alpha}(\varphi,\lambda) := \mathcal{L}_r(\varphi,\lambda) - \frac{\alpha}{2}\|L\lambda\|^2_{L^2(H^{-1}(\Omega))} - \frac{\alpha}{2}\|\lambda - \partial_\nu\varphi\|^2_{L^2(\Gamma_T)}. \end{cases} \tag{32}$$

$$\Lambda := \Big\{ \lambda : \lambda \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega)),$$
$$L\lambda \in L^2([0,T];H^{-1}(\Omega)), \lambda(\cdot,0) = \lambda_t(\cdot,0) = 0, \lambda_{|\Gamma_T} \in L^2(\Gamma_T) \Big\}.$$

Λ is a Hilbert space endowed with the following inner product

$$\langle \lambda, \overline{\lambda}\rangle_\Lambda := \int_0^T \langle L\lambda(t), L\overline{\lambda}(t)\rangle_{H^{-1}(\Omega)} dt + \iint_{\Gamma_T} \lambda\overline{\lambda} d\sigma dt, \qquad \forall\, \lambda, \overline{\lambda} \in \Lambda$$

using notably that

$$\|\lambda\|_{L^2(Q_T)} \leq C_{\Omega,T}\sqrt{<\lambda,\lambda>_\Lambda}, \quad \forall \lambda \in \Lambda \tag{33}$$

for some positive constant $C_{\Omega,T}$. We denote $\|\lambda\|_\Lambda := \sqrt{<\lambda,\lambda>_\Lambda}$.

[3] H. Barbosa, T. Hugues : The finite element method with Lagrange multipliers on the boundary: circumventing the Babusÿka-Brezzi condition, 1991

Stabilized mixed formulation "à la Barbosa-Hughes"

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Stabilized mixed formulation "à la Barbosa-Hughes" - 2

Then, $\forall \alpha \in (0,1)$, we consider the following mixed formulation:

$$\begin{cases} a_{r,\alpha}(\varphi,\overline{\varphi}) + b_\alpha(\overline{\varphi},\lambda) = l_1(\overline{\varphi}), & \forall\, \overline{\varphi} \in \Phi \\ b_\alpha(\varphi,\overline{\lambda}) - c_\alpha(\lambda,\overline{\lambda}) = 0, & \forall\, \overline{\lambda} \in \Lambda, \end{cases} \tag{34}$$

where

$$a_{r,\alpha} : \Phi \times \Phi \to \mathbb{R}, \quad a_{r,\alpha}(\varphi,\overline{\varphi}) = (1-\alpha)\iint_{\Gamma_T} \partial_\nu\varphi\, \partial_\nu\overline{\varphi}\, d\sigma dt + r\iint_{Q_T} L^\star\varphi\, L^\star\overline{\varphi} dxdt \tag{35}$$

$$b_\alpha : \Phi \times \Lambda \to \mathbb{R}, \quad b_\alpha(\varphi,\lambda) = \iint_{Q_T} L^\star\varphi\lambda dxdt - \alpha\iint_{\Gamma_T} \partial_\nu\varphi\lambda d\sigma dt \tag{36}$$

$$c_\alpha : \Lambda \times \Lambda \to \mathbb{R}, \quad c_\alpha(\lambda,\overline{\lambda}) = \alpha\int_0^T \langle L\lambda(t), L\overline{\lambda}(t)\rangle_{H^{-1}(\Omega)} dt + \alpha\iint_{\Gamma_T} \lambda\overline{\lambda} d\sigma dt \tag{37}$$

Stabilized mixed formulation "à la Barbosa-Hughes" - 3

Proposition

$\forall\alpha \in (0,1)$, the stabilized mixed formulation (34) is well-posed. Moreover, the unique pair $(\varphi,\lambda) \in \Phi \times \Lambda$ satisfies

$$\theta\|\varphi\|^2_{\Phi} + \alpha\|\lambda\|^2_{\Lambda} \leq \frac{(1-\alpha)^2 + \alpha\theta}{\theta}\|y_0, y_1\|^2_{L^2\times H^{-1}} \tag{38}$$

with $\theta := \min(1-\alpha, r/\eta)$.

Proposition

If $\alpha \in (0,1)$, the solution $(\varphi,\lambda) \in \Phi \times L^2(\Omega)$ coincides with the stabilized solution $(\varphi_\alpha,\lambda_\alpha) \in \Phi \times \Lambda$

Stabilized mixed formulation "à la Barbosa-Hughes" - 3

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Stabilized mixed formulation "à la Barbosa-Hughes" - Numerical approximation

$\alpha \in (0,1)$, $r > 0$.

$$\Phi_h \subset \Phi, \quad \widetilde{\Lambda}_h \subset \Lambda, \qquad \forall h > 0.$$

Find $(\varphi_h, \lambda_h) \in \Phi_h \times \widetilde{\Lambda}_h$ solution of

$$\left\{ \begin{array}{rcll} a_{r,\alpha}(\varphi_h, \overline{\varphi}_h) + b_\alpha(\lambda_h, \overline{\varphi}_h) & = & l_1(\overline{\varphi}_h), & \forall \overline{\varphi}_h \in \Phi_h \\ b_\alpha(\overline{\lambda}_h, \varphi_h) - c_\alpha(\lambda_h, \overline{\lambda}_h) & = & 0, & \forall \overline{\lambda}_h \in \widetilde{\Lambda}_h. \end{array} \right. \tag{39}$$

In view of the properties of $a_{r,\alpha}$, c_α, l_1, this formulation is well-posed.

Lemma

Let $(\varphi, \lambda) \in \Phi \times \Lambda$ be the solution of (34) and $(\varphi_h, \lambda_h) \in \Phi_h \times \widetilde{\Lambda}_h$ be the solution of (39). Then we have,

$$\frac{1}{4}\theta\|\varphi - \varphi_h\|_\Phi^2 + \frac{1}{4}\alpha\|\lambda - \lambda_h\|_\Lambda^2 \le \left(\frac{\|a_{r,\alpha}\|^2_{(\Phi\times\Phi)'}}{\theta} + \frac{\|b_\alpha\|^2_{(\Phi\times\Lambda)'}}{\alpha} + \frac{\theta}{2}\right) \inf_{\overline{\varphi}_h \in \Phi_h} \|\overline{\varphi}_h - \varphi\|_\Phi^2$$
$$+ \left(\frac{\|b_\alpha\|^2_{(\Phi\times\Lambda)'}}{\theta} + \alpha + \frac{\alpha}{2}\right) \inf_{\overline{\lambda}_h \in \widetilde{\Lambda}_h} \|\overline{\lambda}_h - \lambda\|_\Lambda^2$$

with $\|a_{r,\alpha}\|_{(\Phi\times\Phi)'} \le \max(1-\alpha, \eta^{-1} r)$, $\|b_\alpha\|_{(\Phi\times\Lambda)'} \le (C_{\Omega,T} + \alpha)\eta^{-1}$.

Stabilized mixed formulation "à la Barbosa-Hughes" - Numerical approximation

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Find $(\varphi_h, \lambda_h) \in \Phi_h \times \widetilde{\Lambda}_h$ solution of

$$\left\{ \begin{array}{rcll} a_{r,\alpha}(\varphi_h, \overline{\varphi}_h) + b_\alpha(\lambda_h, \overline{\varphi}_h) & = & l_1(\overline{\varphi}_h), & \forall \overline{\varphi}_h \in \Phi_h \\ b_\alpha(\overline{\lambda}_h, \varphi_h) - c_\alpha(\lambda_h, \overline{\lambda}_h) & = & 0, & \forall \overline{\lambda}_h \in \widetilde{\Lambda}_h. \end{array} \right. \tag{39}$$

In view of the properties of $a_{r,\alpha}$, c_α, l_1, this formulation is well-posed.

Lemma

Let $(\varphi, \lambda) \in \Phi \times \Lambda$ be the solution of (34) and $(\varphi_h, \lambda_h) \in \Phi_h \times \widetilde{\Lambda_h}$ be the solution of (39). Then we have,

$$\frac{1}{4}\theta\|\varphi - \varphi_h\|^2_\Phi + \frac{1}{4}\alpha\|\lambda - \lambda_h\|^2_{\widetilde{\Lambda}} \leq \left(\frac{\|a_{r,\alpha}\|^2_{(\Phi\times\Phi)'}}{\theta} + \frac{\|b_\alpha\|^2_{(\Phi\times\Lambda)'}}{\alpha} + \frac{\theta}{2} \right) \inf_{\overline{\varphi}_h \in \Phi_h} \|\overline{\varphi}_h - \varphi\|^2_\Phi$$
$$+ \left(\frac{\|b_\alpha\|^2_{(\Phi\times\Lambda)'}}{\theta} + \alpha + \frac{\alpha}{2} \right) \inf_{\overline{\lambda}_h \in \widetilde{\Lambda_h}} \|\overline{\lambda}_h - \lambda\|^2_\Lambda$$

with $\|a_{r,\alpha}\|_{(\Phi\times\Phi)'} \leq \max(1-\alpha, \eta^{-1} r)$, $\|b_\alpha\|_{(\Phi\times\Lambda)'} \leq (C_{\Omega,T} + \alpha)\,\eta^{-1}$.

Error estimate

Concerning the space $\widetilde{\Lambda}_h$, since $L\lambda_h$ should belong to $L^2(0,T,H^{-1}(\Omega))$, a natural choice is

$$\widetilde{\Lambda}_h = \{\lambda \in \Phi_h; \lambda(\cdot,0) = \lambda_t(\cdot,0) = 0\}. \tag{40}$$

Proposition (BFS element for $N=1$ - Rate of convergence for the norm $\Phi \times \Lambda$)

Let $h>0$, let $k \leq 2$ be a positive integer and $\alpha \in (0,1)$. Let (y,λ) and (y_h,λ_h) be the solution of (34) and (39) respectively. If (y,λ) belongs to $H^{k+2}(Q_T)\times H^{k+2}(Q_T)$, then there exists a positive constant $K = K(\|\varphi\|_{H^{k+2}(Q_T)},\alpha,r,\eta)$ independent of h, such that

$$\|\varphi-\varphi_h\|_\Phi + \|\lambda-\lambda_h\|_\Lambda \leq Kh^k. \tag{41}$$

Remark - no δ_h here !!!! r is arbitrary

Error estimate in $L^2(Q_T)$

Theorem ($N = 1$- Rate of convergence in $L^2(Q_T)$)

Let $h > 0$, let an integer $k \leq 2$. Let (φ, λ) and (φ_h, λ_h) be the solution of (34) and (39) respectively. If the solution (φ, λ) belongs to $H^{k+2}(Q_T) \times H^{k+2}(Q_T)$, then there exist two positives constant $K_i = K_i(\|\varphi\|_{H^{k+2}(Q_T)}, \|\lambda\|_{H^{k+2}(Q_T)}, \alpha, r, \eta)$, $i = 1, 2$ independent of h such that

$$\|\varphi - \varphi_h\|_{L^2(Q_T)} \leq K_1 \frac{h^k}{\sqrt{\eta}}, \quad \|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 h^k. \tag{42}$$

Remark - no δ_h here !!!! r is arbitrary

Remark: The situation is simpler with a different cost !?

$$\begin{cases} \text{Minimize } \; J(y,v) = \frac{1}{2}\iint_{Q_T} |y|^2 \, dx\, dt + \frac{1}{2}\int_0^T \int_{\Gamma_0} |v|^2 \, d\sigma \, dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0, y_1; T) \end{cases} \tag{43}$$

$$v = \frac{\partial \varphi}{\partial \nu} \;\text{ in }\; (0,T)\times\Gamma_0 \;\text{ and }\; y = \mu \;\text{ in }\; Q_T.$$

$$\begin{cases} \text{Minimize } \; J^\star(\mu, \varphi_0, \varphi_1) = \frac{1}{2}\iint_{Q_T} |\mu|^2 \, dx\, dt + \frac{1}{2}\int_0^T \int_{\Gamma_0} \left|\frac{\partial \varphi}{\partial \nu}\right|^2 d\sigma\, dt \\ \qquad\qquad + < (\varphi_0, \varphi_1), (y_0, y_1) > \\ \text{Subject to } \; (\mu, \varphi_0, \varphi_1) \in L^2(Q_T) \times \boldsymbol{V}, \end{cases} \tag{44}$$

where φ solves the nonhomogeneous backward problem

$$L^\star \varphi = \mu \quad \text{in} \quad Q_T, \qquad \varphi = 0 \quad \text{on} \quad \Sigma_T, \qquad (\varphi(\cdot,0), \varphi'(\cdot,0)) = (\varphi_0, \varphi_1) \tag{45}$$

Remark: The situation is much simpler with a different cost !!?! [4]

$$\begin{cases} \text{Minimize } J^\star(\mu, \varphi_0, \varphi_1) = \dfrac{1}{2} \displaystyle\iint_{Q_T} |\mu|^2 \, dx\, dt + \dfrac{1}{2} \int_0^T \int_{\Gamma_0} \left| \dfrac{\partial \varphi}{\partial \nu} \right|^2 d\sigma\, dt \\ \qquad\qquad + < (\varphi_0, \varphi_1), (y_0, y_1) > \\ \text{Subject to } (\mu, \varphi_0, \varphi_1) \in L^2(Q_T) \times \boldsymbol{V}, \end{cases} \tag{46}$$

where φ solves the nonhomogeneous backward problem

$$L^\star \varphi = \mu \quad \text{in} \quad Q_T, \qquad \varphi = 0 \quad \text{on} \quad \Sigma_T, \qquad (\varphi(\cdot, 0), \varphi'(\cdot, 0)) = (\varphi_0, \varphi_1) \tag{47}$$

equivalent to

$$\begin{cases} \text{Minimize } J_1^\star(\varphi) = \dfrac{1}{2} \displaystyle\iint_{Q_T} |L^\star \varphi|^2 \, dx\, dt + \dfrac{1}{2} \int_0^T \int_{\Gamma_0} \left| \dfrac{\partial \varphi}{\partial \nu} \right|^2 d\sigma\, dt \\ \qquad\qquad + < (\varphi_0, \varphi_1), (y_0, y_1) > \\ \text{Subject to } \varphi \in \boldsymbol{\Phi} \end{cases} \tag{48}$$

[4] N. Cindea, E. Fernandez-Cara, AM, Numerical controllability of the wave equation through primal methods and Carleman estimates (2012)

Non constant coefficient: $Ly := y_{tt} - (c(x)y_x)_x + d(x,t)y$

$c \in C^1([0,1])$

$$c(x) = \begin{cases} 1 & x \in [0, 0.45] \\ \in [1., 5.] \quad (c'(x) > 0), & x \in (0.45, 0.55) \\ 5 & x \in [0.55, 1] \end{cases} \tag{49}$$

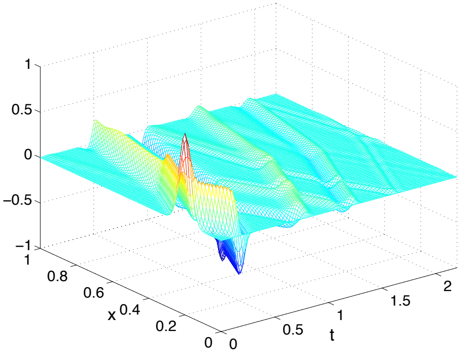


Figure: $y_0(x) \equiv e^{-500(x-0.2)^2}$ and c given by (49) -The solution $\hat{y}_h$ over Q_T - $h = (1/80, 1/80)$.

PART III - CONTROLLABILITY OF THE WAVE EQUATION : DISTRIBUTED CASE

Continuous and discrete case

$$q_T := \omega \times (0, T) \subset \Omega \times (0, T)$$

$$\begin{cases} y_{tt} - \Delta y = v\, 1_{q_T}, & Q_T \\ y = 0, & \Sigma_T \\ (y(\cdot, 0), y_t(\cdot, 0)) = (y_0, y_1) \in \boldsymbol{V}, & \Omega. \end{cases} \quad (50)$$

We assume T and ω "large" enough.

The distributed case

$$\begin{cases} min\, J^\star(\varphi) = \frac{1}{2}\int_0^T\int_\omega |\varphi|^2\, dx\, dt + < y_0, \varphi_t(\cdot,0) >_{H^1,H^{-1}} - \langle y_1, \varphi(\cdot\,,0)\rangle_{L^2} \\ \text{Subject to } \varphi \in \boldsymbol{W} := \left\{\varphi : \varphi \in L^2(q_T),\ \varphi_{|\Sigma_T} = 0, L^\star\varphi = 0 \in L^2(0,T,H^{-1}(\Omega))\right\} \end{cases} \tag{51}$$

Optimal control : $v = \varphi 1_{q_T}$

Generalized observability inequality :

$$\|\varphi_0, \varphi_1\|^2_{\boldsymbol{H}} \leq C_{obs}\left(\|\varphi\|^2_{L^2(q_T)} + \|L^\star\varphi\|^2_{L^2(0,T;H^{-1})}\right), \quad \forall \varphi \in \boldsymbol{\Phi}$$

Multiplier :

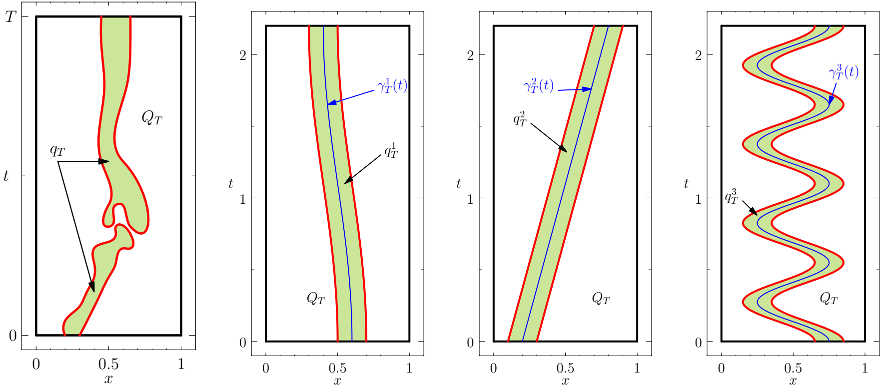
$$b(\varphi,\lambda) = \int_0^T < \lambda(\cdot,t), L^\star\varphi(\cdot,t) >_{H_0^1(\Omega),H^{-1}(\Omega)} dt, \qquad \lambda \in L^2(0,T;H_0^1(\Omega))$$

Non cylindrical situation in 1D with constant coefficient

[5]

[6]

The variational approach is well-adapted to the non cylindrical situation.



Time dependent domains $q_T \subset Q_T = \Omega \times (0, T)$

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[5] C. Castro, N. Cindea, A. Münch, Controllability of the 1D wave equation with inner moving force, SICON (2014)]

[6] G. Lebeau, J. Le Rousseau, P. Terpolilli, E. Trélat, Geometric control condition for the wave equation with a time-dependent domain, (2016)

PART IV - INVERSE PROBLEMS FOR THE WAVE EQUATIONS

Continuous case

Hyperbolic equation - Problem statement

$\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ - $T > 0$, $c \in C^1(\overline{\Omega}, \mathbb{R})$, $d \in L^\infty(Q_T)$, $(y_0, y_1) \in \boldsymbol{H}$, $f \in X$.

$$\begin{cases} Ly := y_{tt} - \nabla \cdot (c \nabla y) + dy = f, & Q_T := \Omega \times (0, T) \\ y = 0, & \Sigma_T := \partial\Omega \times (0, T) \\ (y(\cdot, 0), y_t(\cdot, 0)) = (y_0, y_1), & \Omega. \end{cases} \tag{52}$$

- Inverse Problem 1: Distributed observation on $q_T = \omega \times (0, T), \omega \subset \Omega$

$$\begin{cases} \boldsymbol{H} = L^2 \times H^{-1}, X = L^2(H^{-1}), \\ \text{Given } (y_{obs}, f) \in L^2(q_T) \times X, \text{ find } y \text{ s.t. } \{(52) \text{ and } y - y_{obs} = 0 \quad \text{on} \quad q_T\} \end{cases}$$

- Inverse Problem 2: Boundary observation on $\Gamma_T \subset \partial\Omega \times (0, T)$

$$\begin{cases} \boldsymbol{H} = H_0^1 \times L^2, X = L^2(L^2) \\ \text{Given } y_{obs,\nu} \in L^2(\Gamma_T), \text{ find } (y, f) \text{ s.t. } \{(52) \text{ and } \partial_\nu y - y_{obs,\nu} = 0 \quad \text{on} \quad \Gamma_T\} \end{cases}$$

Hyperbolic equation - Problem statement

$\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ - $T > 0$, $c \in C^1(\overline{\Omega}, \mathbb{R})$, $d \in L^\infty(Q_T)$, $(y_0, y_1) \in \boldsymbol{H}$, $f \in X$.

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- Inverse Problem 2: Boundary observation on $\Gamma_T \subset \partial\Omega \times (0, T)$

$$\begin{cases} \boldsymbol{H} = H_0^1 \times L^2, X = L^2(L^2) \\ \text{Given } y_{obs,\nu} \in L^2(\Gamma_T), \text{find } (y, f) \; s.t. \{(52) \text{ and } \partial_\nu y - y_{obs,\nu} = 0 \quad \text{on} \quad \Gamma_T\} \end{cases}$$

Hyperbolic equation - Problem statement

$\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ - $T > 0$, $c \in C^1(\overline{\Omega}, \mathbb{R})$, $d \in L^\infty(Q_T)$, $(y_0, y_1) \in \boldsymbol{H}$, $f \in X$.

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Inverse problem 1

$$Z := \left\{ y : y \in C([0,T], L^2(\Omega)) \cap C^1([0,T], H^{-1}(\Omega)), Ly \in X, y_{|\Sigma_T} = 0 \right\}.$$

Introducing the operator $P : Z \to X \times L^2(q_T)$

$$P\, y := (Ly, y_{|q_T}),$$

Inverse Problem 1 is reformulated as :

$$\textit{find } y \in Z \textit{ solution of } P\, y = (f, y_{obs}). \qquad (IP)$$

If unique continuation property holds for (52) and if y_{obs} is a restriction to q_T of a solution of (52), then (IP) is well-posed: the state y corresponding to the pair (y_{obs}, f) is unique.

Most natural approach: Relaxation via Least-squares method

The most natural (and widely used in practice) approach consists in introducing a least-squares type technic, i.e. consider the extremal problem

$$(LS) \quad \begin{cases} \text{minimize} \quad J(y_0, y_1) := \dfrac{1}{2}\|y - y_{obs}\|^2_{L^2(q_T)} \\ \text{subject to} \quad (y_0, y_1) \in \boldsymbol{H} \\ \text{where} \quad y \quad \text{solves} \quad (52) \end{cases}$$

The "Discretization then Inverse problem" procedure is discussed in [L. Baudouin, M. De Buhan, S. Ervedoza, 2013]

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Keeping y as the main variable ...

Without loss of generality, $f \equiv 0$.

$$Z := \{y : y \in C([0,T], L^2(\Omega)) \cap C^1([0,T], H^{-1}(\Omega)), Ly \in X, y_{|\Sigma_T} = 0\}.$$

Hypothesis (Generalized Observability Inequality)

Assume that $\exists C_{obs} = C(\omega, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)})$ *s.t. :*

$$(\mathcal{H}) \qquad \|y(\cdot,0), y_t(\cdot,0)\|_{\boldsymbol{H}}^2 \leq C_{obs}\left(\|y\|^2_{L^2(q_T)} + \|Ly\|^2_X\right), \quad \forall y \in Z. \qquad (53)$$

- in 1-D, (53) if $T \geq T^\star(c,d)$ [Fernandez-Cara, Cindea,Münch, COCV 2013],
- in N-D, for $c = 1$ and $d = 0$, (53) if (Ω, ω, T) satisfies geometric optic condition [Bardos, Lebeau, Rauch, 1992]

$$\|y\|^2_{L^2(Q_T)} \leq C_{\Omega,T}\left(C_{obs}\|y\|^2_{L^2(q_T)} + (1 + C_{obs})\|Ly\|^2_X\right) \quad \forall y \in Z. \qquad (54)$$

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Equivalent formulation of IP

Within this hypothesis, for any $\eta > 0$, we define on Z the bilinear form

$$\langle y, \overline{y}\rangle_Z := \iint_{q_T} y\,\overline{y}\,dxdt + \eta \int_0^T \langle Ly,\, L\overline{y}\rangle_{H^{-1}(\Omega)}\,dt \quad \forall y, \overline{y} \in Z. \tag{55}$$

$(Z, \|\cdot\|)$ is a Hilbert space.
Then, we consider the extremal problem :

$$(\mathcal{P}) \quad \begin{cases} \inf J(y) := \dfrac{1}{2}\|y - y_{obs}\|^2_{L^2(q_T)} + \dfrac{r}{2}\|Ly\|^2_X, \quad r \geq 0 \\ \text{subject to} \quad y \in W := \{y \in Z;\ Ly = 0 \text{ in } X\} \end{cases}$$

$(\mathcal{P})$ is well posed : J is continuous over W, strictly convex and $J(y) \to +\infty$ as $\|y\|_W \to \infty$.

The solution of $(\mathcal{P})$ in W does not depend on η.

From (53), the solution y in Z of $(\mathcal{P})$ satisfies $(y(\cdot,0), y_t(\cdot,0)) \in \boldsymbol{H}$, so that problem $(\mathcal{P})$ is equivalent to the minimization of J w.r.t $(y_0, y_1) \in \boldsymbol{H}$.

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Optimality of $(\mathcal{P})$

In order to solve $(\mathcal{P})$, we have to deal with the constraint eq. which appears in W. We introduce a Lagrange multiplier $\lambda \in X'$ and the following mixed formulation: find $(y, \lambda) \in Z \times X'$ solution of

$$\begin{cases} a_r(y, \overline{y}) + b(\overline{y}, \lambda) & = & l(\overline{y}), & \forall \overline{y} \in Z \\ b(y, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in \Lambda, \end{cases} \tag{56}$$

where

$$a_r : Z \times Z \to \mathbb{R}, \quad a_r(y, \overline{y}) := \iint_{q_T} y\,\overline{y}\,dxdt + r \int_0^T \langle Ly,\, L\overline{y} \rangle_{H^{-1}(\Omega)}\,dt,$$

$$b : Z \times X' \to \mathbb{R}, \quad b(y, \lambda) := \int_0^T \langle \lambda,\, Ly \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt,$$

$$l : Z \to \mathbb{R}, \quad l(y) := \iint_{q_T} y_{obs}\, y\, dxdt.$$

System (56) is the optimality system corresponding to the extremal problem $(\mathcal{P})$.

[7]

[7] N. Cindea, AM, Inverse problem for linear hyperbolic equations using mixed formulations, Inverse Problems (2015)

Inverse problem 2: Simultaneous reconstruction of y and the source from $\partial_\nu y$

$f(x,t) = \sigma(t)\mu(x)$

$c := 1$, $d(x,t) = d(x) \in L^p(\Omega)$, $\sigma \in C^1([0,T]), \sigma(0) \neq 0$, $\mu \in H^{-1}(\Omega)$

Theorem (Yamamoto-Zhang 2001)

Assume that (Γ_T, T, Q_T) satisfies the geometric optic condition. Let $y = y(\mu) \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ be the weak solution of (52) with $c := 1$ and $(y_0, y_1) = (0,0)$. $\exists C > 0$ s.t.

$$C^{-1}\|\mu\|_{H^{-1}(\Omega)} \leq \|\partial_\nu y\|_{L^2(\Gamma_T)} \leq C\|\mu\|_{H^{-1}(\Omega)}, \quad \forall \mu \in H^{-1}(\Omega).$$

This leads to the extremal problem :

$$\begin{cases} \inf J(y,\mu) := \frac{1}{2}\|c(x)(\partial_\nu y - y_{\nu,obs})\|^2_{L^2(\Gamma_T)} + \frac{r}{2}\iint_{Q_T}(Ly - \sigma\mu)^2\,dxdt, \\ \text{subject to } (y,\mu) \in W := \Big\{(y,\mu);\ y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \\ \qquad \mu \in H^{-1}(\Omega), Ly - \sigma\mu = 0 \text{ in } Q_T, y(\cdot,0) = y_t(\cdot,0) = 0\Big\}. \end{cases} \quad (\mathcal{P}_{y,\mu})$$

Attached to $\|(y,\mu)\|_W := \|c(x)\partial_\nu y\|_{L^2(\Gamma_T)}$, W is a Hilbert space.

Inverse problem 2: Simultaneous reconstruction of y and the source from $\partial_\nu y$

$f(x,t) = \sigma(t)\mu(x)$

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Assume that (Γ_T, T, Q_T) *satisfies the geometric optic condition. Let* $y = y(\mu) \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ *be the weak solution of (52) with* $c := 1$ *and* $(y_0, y_1) = (0,0)$*.* $\exists C > 0$ *s.t.*

$$C^{-1}\|\mu\|_{H^{-1}(\Omega)} \leq \|\partial_\nu y\|_{L^2(\Gamma_T)} \leq C\|\mu\|_{H^{-1}(\Omega)}, \quad \forall \mu \in H^{-1}(\Omega).$$

This leads to the extremal problem :

$$\begin{cases} \inf J(y,\mu) := \frac{1}{2}\|c(x)(\partial_\nu y - y_{\nu,obs})\|^2_{L^2(\Gamma_T)} + \frac{r}{2}\iint_{Q_T}(Ly - \sigma\mu)^2\,dxdt, \\ \text{subject to } (y,\mu) \in W := \Big\{(y,\mu);\ y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \\ \qquad \mu \in H^{-1}(\Omega), Ly - \sigma\mu = 0 \text{ in } Q_T, y(\cdot,0) = y_t(\cdot,0) = 0\Big\}. \end{cases} \quad (\mathcal{P}_{y,\mu})$$

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$$C^{-1}\|\mu\|_{H^{-1}(\Omega)} \leq \|\partial_\nu y\|_{L^2(\Gamma_T)} \leq C\|\mu\|_{H^{-1}(\Omega)}, \quad \forall \mu \in H^{-1}(\Omega).$$

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$$\begin{cases} \inf J(y,\mu) := \dfrac{1}{2}\|c(x)(\partial_\nu y - y_{\nu,obs})\|^2_{L^2(\Gamma_T)} + \dfrac{r}{2}\displaystyle\iint_{Q_T}(Ly - \sigma\mu)^2\,dxdt, \\ \text{subject to } (y,\mu) \in W := \Big\{(y,\mu);\ y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \\ \qquad \mu \in H^{-1}(\Omega), Ly - \sigma\mu = 0 \text{ in } Q_T, y(\cdot,0) = y_t(\cdot,0) = 0\Big\}. \end{cases} \quad (\mathcal{P}_{y,\mu})$$

Attached to $\|(y,\mu)\|_W := \|c(x)\partial_\nu y\|_{L^2(\Gamma_T)}$, W is a Hilbert space.

Recovering the solution and the source f when the pair (y,f) is unique

$$Y := \Big\{(y,\mu); y \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)), \mu \in H^{-1}(\Omega),$$
$$Ly - \sigma\mu \in L^2(Q_T), y(\cdot,0) = y_t(\cdot,0) = 0\Big\}. \tag{57}$$

Hypothesis

$\exists C_{obs} = C(\Gamma_T, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)}) > 0$ *s.t. :*

$$\|\mu\|^2_{H^{-1}(\Omega)} \le C_{obs}\Big(\|c(x)\partial_\nu y\|^2_{L^2(\Gamma_T)} + \|Ly - \sigma\mu\|^2_{L^2(Q_T)}\Big), \quad \forall (y,\mu) \in Y. \tag{$\mathcal{H}_2$}$$

Then, $\forall \eta > 0$, we define on Y the bilinear form

$$\langle (y,\mu),(\overline{y},\overline{\mu})\rangle_Y := \iint_{\Gamma_T} (c(x))^2\, \partial_\nu y\, \partial_\nu \overline{y}\, d\sigma dt + \eta \iint_{Q_T} (Ly - \sigma\mu)\,(L\overline{y} - \sigma\overline{\mu})\, dxdt \quad \forall y, \overline{y} \in Z. \tag{58}$$

$\|(y,z)\|_Y := \sqrt{<(y,\mu),(y,\mu)>_Y}$.

Lemma

Under the hypotheses $(\mathcal{H}_2)$, the space $(Y, \|\cdot\|_Y)$ is a Hilbert space.

Recovering the solution and the source f when the pair (y, f) is unique

$$Y := \Big\{ (y,\mu); y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \mu \in H^{-1}(\Omega), \quad Ly - \sigma\mu \in L^2(Q_T), y(\cdot,0) = y_t(\cdot,0) = 0 \Big\}. \tag{57}$$

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$$\|\mu\|^2_{H^{-1}(\Omega)} \le C_{obs}\Big(\|c(x)\partial_\nu y\|^2_{L^2(\Gamma_T)} + \|Ly - \sigma\mu\|^2_{L^2(Q_T)} \Big), \quad \forall (y,\mu) \in Y. \tag{$\mathcal{H}_2$}$$

Then, $\forall \eta > 0$, we define on Y the bilinear form

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Under the hypotheses $(\mathcal{H}_2)$, the space $(Y, \|\cdot\|_Y)$ is a Hilbert space.

Recovering the solution and the source f when the pair (y, f) is unique

$$Y := \Big\{(y,\mu); y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \mu \in H^{-1}(\Omega),$$
$$Ly - \sigma\mu \in L^2(Q_T), y(\cdot,0) = y_t(\cdot,0) = 0\Big\}. \tag{57}$$

Hypothesis

$\exists C_{obs} = C(\Gamma_T, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)}) > 0$ *s.t.* :

$$\|\mu\|^2_{H^{-1}(\Omega)} \le C_{obs}\Big(\|c(x)\partial_\nu y\|^2_{L^2(\Gamma_T)} + \|Ly - \sigma\mu\|^2_{L^2(Q_T)}\Big), \quad \forall (y,\mu) \in Y. \tag{$\mathcal{H}_2$}$$

Then, $\forall \eta > 0$, we define on Y the bilinear form

$$\langle (y,\mu), (\overline{y},\overline{\mu})\rangle_Y := \iint_{\Gamma_T} (c(x))^2\, \partial_\nu y\, \partial_\nu \overline{y}\, d\sigma dt + \eta \iint_{Q_T} (Ly - \sigma\mu)\,(L\overline{y} - \sigma\overline{\mu})\, dxdt \quad \forall y, \overline{y} \in Z.$$
$$\|(y,z)\|_Y := \sqrt{<(y,\mu),(y,\mu)>_Y}. \tag{58}$$

Lemma

Under the hypotheses $(\mathcal{H}_2)$, the space $(Y, \|\cdot\|_Y)$ is a Hilbert space.

Recovering the solution and the source f: mixed formulation

Find $((y,\mu),\lambda) \in Y \times L^2(Q_T)$ solution of

$$\left\{ \begin{array}{rcll} a_r((y,\mu),(\overline{y},\overline{\mu})) + b((\overline{y},\overline{\mu}),\lambda) & = & l(\overline{y},\overline{\mu}), & \forall (\overline{y},\overline{\mu}) \in Y \\ b((y,\mu),\overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{array} \right. \tag{59}$$

where

$$\begin{aligned} a_r : Y \times Y \to \mathbb{R}, \quad a_r((y,\mu),(\overline{y},\overline{\mu})) &:= \iint_{\Gamma_T} c^2(x)\partial_\nu y \partial_\nu \overline{y}\, d\sigma dt \\ &\quad + r \iint_{Q_T} (Ly - \sigma\mu)(L\overline{y} - \sigma\overline{\mu})\, dxdt, \; r \geq 0 \\ b : Y \times L^2(Q_T) \to \mathbb{R}, \quad b((y,\mu),\lambda) &:= \iint_{Q_T} \lambda(Ly - \sigma\mu) dx\, dt, \\ l : Y \to \mathbb{R}, \quad l(y,\mu) &:= \iint_{\Gamma_T} c^2(x)\, \partial_\nu y\, y_{\nu,obs}\, d\sigma dt. \end{aligned}$$

[8]

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[8] N. Cindea, AM, Simultaneous reconstruction of the solution and the source of hyperbolic equations from boundary measurements: a robust numerical approach, Inverse Problems (2016)

PART IV - INVERSE PROBLEMS FOR THE WAVE EQUATIONS

Discrete case - Experiments

Numerical illustration - $N = 1$

$$\textbf{(EX1)} \quad y_0(x) = 1 - |2x - 1|, \quad y_1(x) = 1_{(1/3,2/3)}(x), \qquad x \in (0,1)$$

in $H_0^1 \times L^2$ for which the Fourier coefficients are

$$a_k = \frac{4\sqrt{2}}{\pi^2 k^2} \sin(\pi k/2), \quad b_k = \frac{1}{\pi k}(\cos(\pi k/3) - \cos(2\pi k/3)), \quad k > 0$$

$f = 0$. $T = 2$ - The corresponding solution of (52) with $c \equiv 1, d \equiv 0$ is given by

$$y(x,t) = \sum_{k>0} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sqrt{2} \sin(k\pi x)$$

Example 1 - $N = 1$ - Observation on q_T

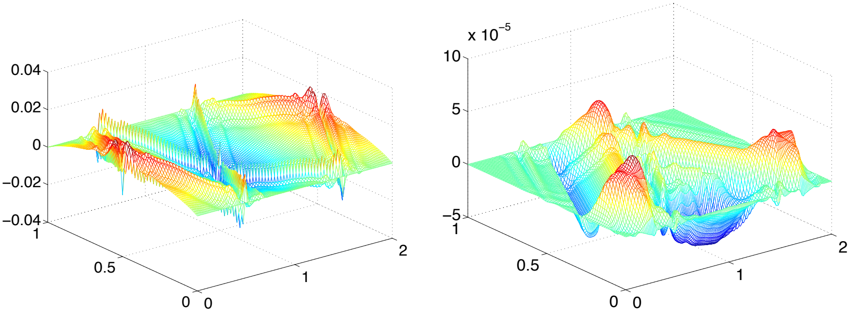
$q_T = (0.1, 0.3) \times (0, T)$

h	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\frac{\|y-y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}}$	1.01×10^{-1}	4.81×10^{-2}	2.34×10^{-2}	1.15×10^{-2}	5.68×10^{-3}
$\frac{\|y-y_h\|_{L^2(q_T)}}{\|y\|_{L^2(q_T)}}$	1.34×10^{-1}	5.05×10^{-2}	2.37×10^{-2}	1.16×10^{-2}	5.80×10^{-3}
$\|Ly_h\|_{L^2(Q_T)}$	7.18×10^{-2}	6.59×10^{-2}	6.11×10^{-2}	5.55×10^{-2}	5.10×10^{-2}
$\|\lambda_h\|_{L^2(Q_T)}$	1.07×10^{-4}	4.70×10^{-5}	2.32×10^{-5}	1.15×10^{-5}	5.76×10^{-6}
♯ CG iterates	29	46	83	133	201

$$\frac{\|y-y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} = \mathcal{O}(h^{0.574}), \quad \frac{\|y-y_h\|_{L^2(q_T)}}{\|y\|_{L^2(q_T)}} = \mathcal{O}(h^{0.94}). \tag{60}$$

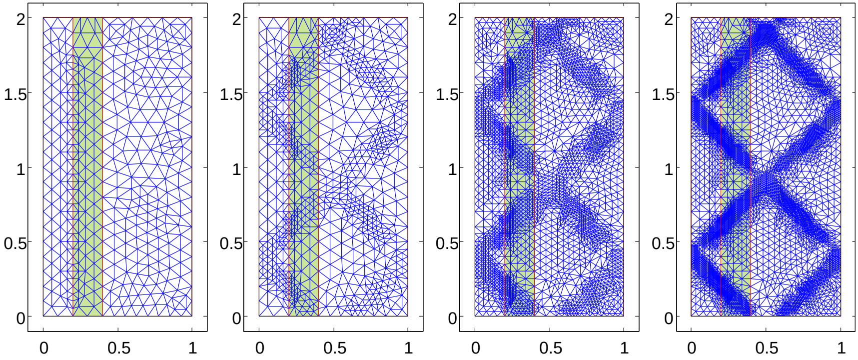
$$\|Ly_h\|_{L^2(Q_T)} = \mathcal{O}(h^{0.123}). \tag{61}$$

Example 2 - $N = 1$ - Observation on q_T



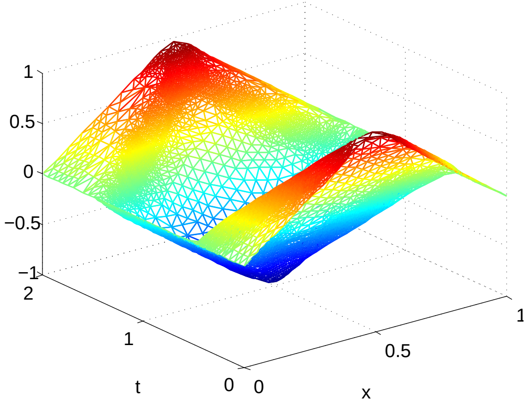
$y - y_h$ and λ_h in Q_T

Example 1 - $N = 1$ - Mesh adaptation



Iterative local refinement of the mesh according to the gradient of y_h

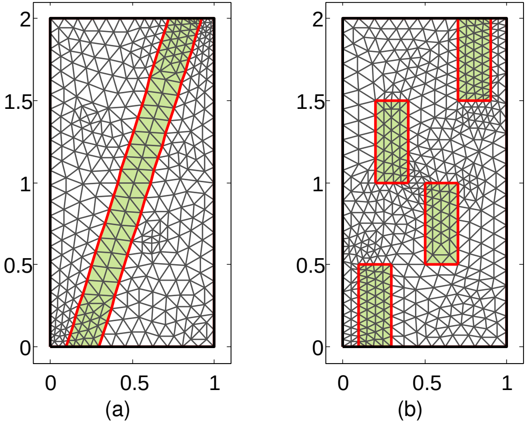
Example 1 - $N = 1$ - Mesh adaptation



Reconstructed state y_h on the adapted mesh

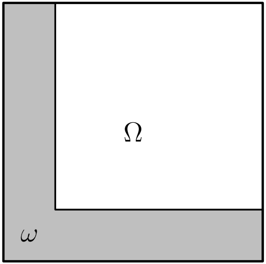
Exemple 2 : $N = 1$ - Non cylindrical domain q_T

Triangular meshes - reduced HCT elements

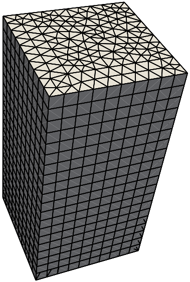


Domain q_T^1 (a) and domain q_T^2 (b) triangulated using some coarse meshes.

$2D$ example: $\Omega = (0,1)^2$ - Observation on q_T



(a)



(b)

Mesh Number	0	1	2	3
Number of elements	5 320	15 320	31 740	120 160
Number of nodes	3 234	8 799	17 670	64 411

Characteristics of the three meshes associated with Q_T.

$2D$ example: $\Omega = (0,1)^2$ - Observation on q_T

$(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$:

$$\textbf{(EX2–2D)} \quad \begin{cases} y_0(x_1,x_2) = (1-|2x_1-1|)(1-|2x_2-1|) \\ y_1(x_1,x_2) = \mathbf{1}_{(\frac{1}{3},\frac{2}{3})^2}(x_1,x_2) \end{cases} \quad (x_1,x_2) \in \Omega. \qquad (62)$$

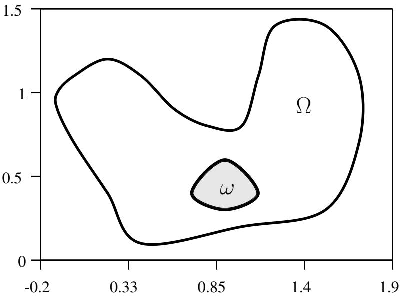
The Fourier coefficients of the corresponding solution are

$$a_{kl} = \frac{2^5}{\pi^4 k^2 l^2} \sin\frac{\pi k}{2} \sin\frac{\pi l}{2}$$
$$b_{kl} = \frac{1}{\pi^2 kl}\left(\cos\frac{\pi k}{3} - \cos\frac{2\pi k}{3}\right)\left(\cos\frac{\pi l}{3} - \cos\frac{2\pi l}{3}\right).$$

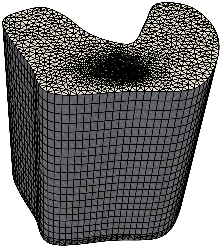
Mesh number	0	1	2	3
$\frac{\Vert y-y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}}$	4.74×10^{-2}	3.72×10^{-2}	2.4×10^{-2}	1.35×10^{-2}
$\Vert Ly_h\Vert_{L^2(Q_T)}$	1.18	0.89	0.99	0.99
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	3.21×10^{-5}	1.46×10^{-5}	1.02×10^{-5}	3.56×10^{-6}

Table: Example **EX2–2D** – $r = h^2$

$2D$ example - Observation on q_T



(a)



(b)

Mesh number	0	1	2
Number of elements	5 730	44 900	196 040
Number of nodes	3 432	24 633	103 566

Characteristics of the three meshes associated with Q_T.

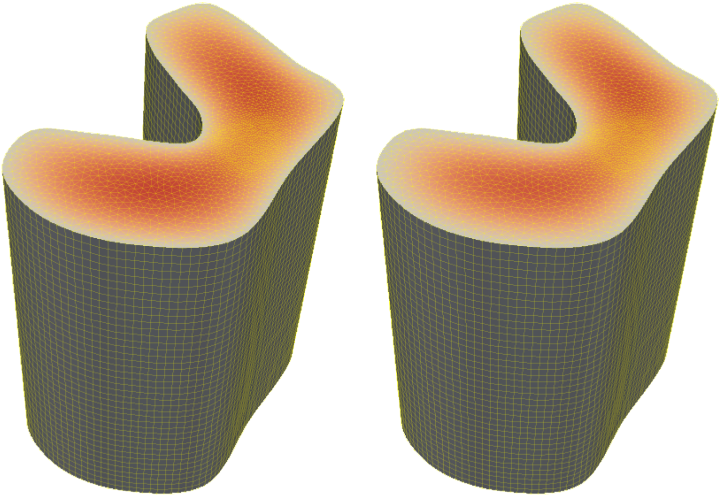
$2D$ example - Observation on q_T

$$\begin{cases} -\Delta y_0 = 10, & \text{in } \Omega \\ y_0 = 0, & \text{on } \partial\Omega, \end{cases} \qquad y_1 = 0. \tag{63}$$

Mesh number	0	1	2
$\frac{\Vert\overline{y}_h - y_h\Vert_{L^2(Q_T)}}{\Vert\overline{y}_h\Vert_{L^2(Q_T)}}$	1.88×10^{-1}	8.04×10^{-2}	5.41×10^{-2}
$\Vert Ly_h\Vert_{L^2(Q_T)}$	3.21	2.01	1.17
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	8.26×10^{-5}	3.62×10^{-5}	2.24×10^{-5}

$$r = h^2 - T = 2$$

$2D$ example - Observation on q_T



(a) (b)

y and y_h in Q_T

Numerical illustration - $N = 1$ - Observation on Γ_T

$f = 0$ - $T = 2$

$$\textbf{(EX2)} \quad y_0(x) = 1 - |2x - 1|, \quad y_1(x) = 1_{(1/3,2/3)}(x), \qquad x \in (0,1)$$

in $H_0^1 \times L^2$ for which the Fourier coefficients are

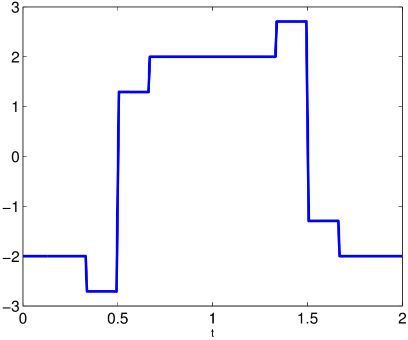


Figure: The observation $y_{\nu,obs}$ on $\{1\} \times (0, T)$ associated to initial data **EX1**.

Numerical illustration - $N = 1$ - Observation on Γ_T

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\frac{\Vert y-y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}}$	1.63×10^{-2}	6.63×10^{-3}	2.78×10^{-3}	1.29×10^{-3}	5.72×10^{-4}
$\frac{\Vert\partial_\nu(y-y_h)\Vert_{L^2(\Gamma_T)}}{\Vert\partial_\nu y\Vert_{L^2(\Gamma_T)}}$	7.67×10^{-3}	4.95×10^{-3}	3.24×10^{-3}	2.16×10^{-3}	1.48×10^{-3}
$\Vert Ly_h\Vert_{L^2(Q_T)}$	0.937	1.204	1.496	1.798	2.135
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	7.74×10^{-3}	3.74×10^{-3}	1.72×10^{-3}	7.90×10^{-4}	3.60×10^{-4}
card($\{\lambda_h\}$)	861	3 321	13 041	51 681	205 761
♯ CG iterates	57	103	172	337	591

$$r = h^2 : \quad \frac{\Vert y-y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}} = \mathcal{O}(h^{1.20}), \quad \frac{\Vert\partial_\nu(y-y_h)\Vert_{L^2(\Gamma_T)}}{\Vert\partial_\nu y\Vert_{L^2(\Gamma_T)}} = \mathcal{O}(h^{0.59}),$$
$$\Vert\lambda_h\Vert_{L^2(Q_T)} = \mathcal{O}(h^{1.11}), \quad \Vert Ly_h\Vert_{L^2(Q_T)} = \mathcal{O}(h^{-0.29}). \qquad (64)$$

Example 2 - $N = 2$ - The stadium

$T = 3$

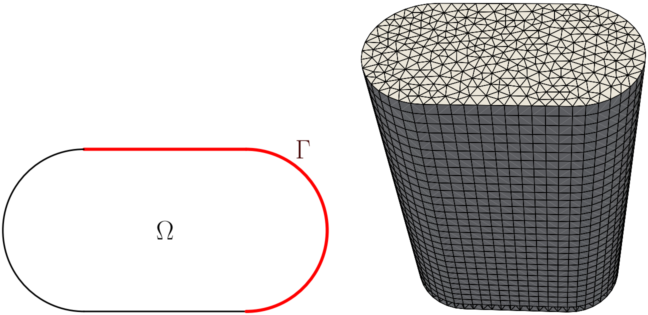


Figure: Bunimovich's stadium and the subset Γ of $\partial\Omega$ on which the observations are available. Example of mesh of the domain Q_T.

Example 2 - $N = 2$ - Recovering of the initial data

$T = 3$

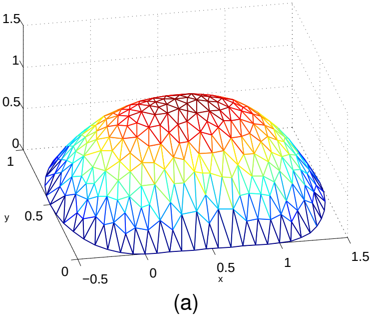


Figure: (a) Initial data y_0 given by (63). (b) Reconstructed initial data $y_h(\cdot, 0)$.

$N = 1$ - Reconstruction of y and μ from the boundary

$\sigma(t) = 1 + t$, $T = 2$

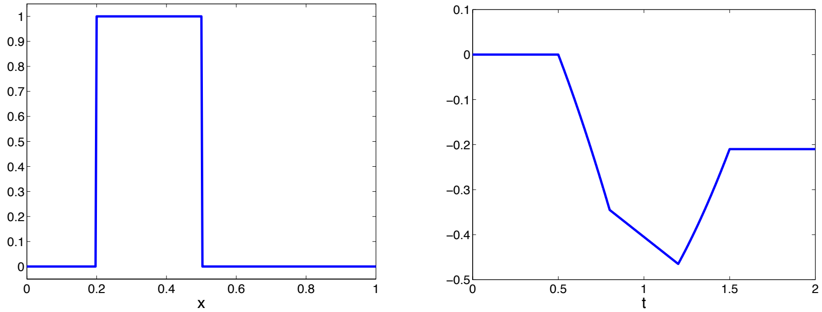


Figure: $\mu(x)$ and corresponding $\partial_\nu y|_{q_T} = y_x(1, t)$ on $(0, T)$.

$N = 1$ - Reconstruction of y and μ from the boundary

$\Delta x = \Delta t = 1/160$

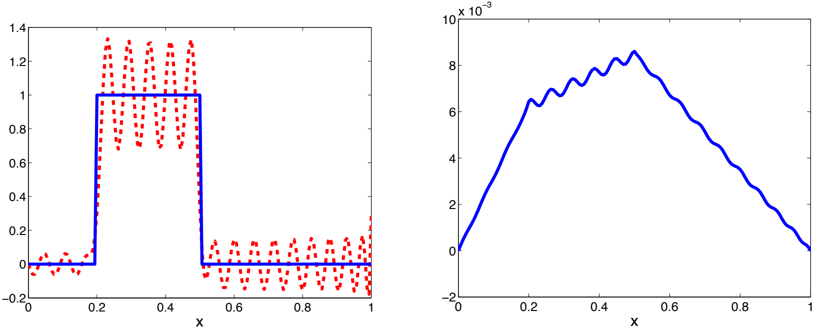


Figure: μ_h, μ and $\frac{-\Delta^{-1}(\mu-\mu_h)}{\|-\Delta^{-1}(\mu)\|_{H_0^1}}$.

$$\frac{\|\mu-\mu_h\|_{H^{-1}(\Omega)}}{\|\mu\|_{H^{-1}(\Omega)}} \approx 7.18 \times 10^{-2}, \qquad \|y - y_h\|_{L^2(Q_T)} \approx 8.68 \times 10^{-4}$$

$N = 1$ - Reconstruction of y and μ from the boundary

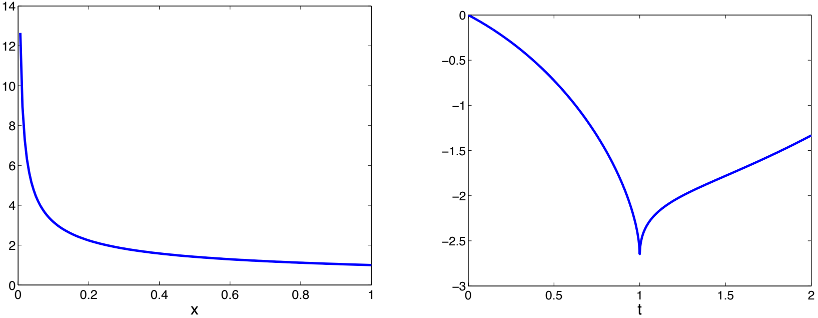


Figure: $\mu(x) = \frac{1}{\sqrt{x}}$ and corresponding $\partial_\nu y|_{q_T} = y_x(1,t)$ on $(0, T)$.

$N = 1$ - Reconstruction of y and μ from the boundary

$\Delta x = \Delta t = \frac{1}{160}$

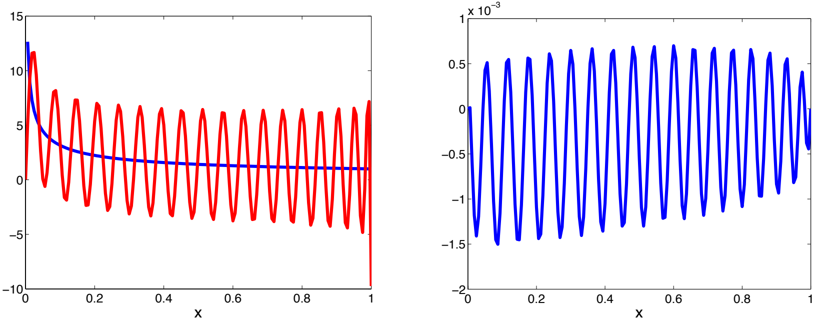


Figure: μ_h, μ and $\frac{-\Delta^{-1}(\mu-\mu_h)}{\|-\Delta^{-1}(\mu)\|_{H_0^1}}$.

$$\frac{\|\mu-\mu_h\|_{H^{-1}(\Omega)}}{\|\mu\|_{H^{-1}(\Omega)}} \approx 2.21 \times 10^{-2}, \qquad \|y - y_h\|_{L^2(Q_T)} \approx 3.56 \times 10^{-5}$$

$N = 1$ - Reconstruction of y and μ from the boundary

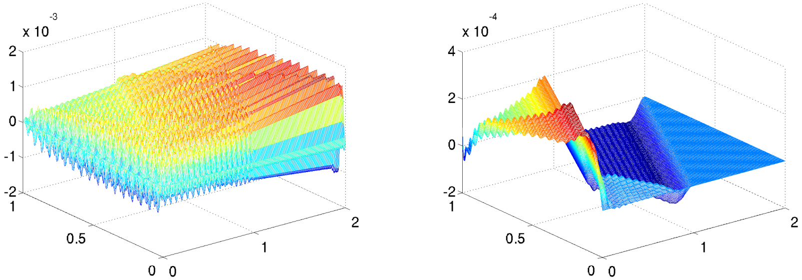


Figure: $y - y_h$ and λ_h