Approximation of control and inverse problems for PDEs using variational methods

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with NICOLAE CÎNDEA (CF), ENRIQUE FERNÁNDEZ-CARA (Sevilla)

Context

We discuss hyperbolic and parabolic equations and try to emphasize the interest of space-time variational methods with respect to time marching methods.

Wave like equation with initial data in $L^2 \times H^{-1}$

$\Omega \subset \mathbb{R}^N$ bounded domain with C^2-boundary; $T > 0$; $Q_T := \Omega \times (0, T)$; $c \in C^1(\overline{\Omega}, \mathbb{R})$; $d \in L^\infty(Q_T)$; $\Gamma_0 \subset \partial\Omega$

$$\begin{cases} Ly := y_{tt} - \nabla \cdot (c(x)\nabla y) + d(x,t)y = 0, & Q_T := \Omega \times (0, T), \\ y = v\, 1_{\Gamma_0}(x), & \Sigma_T := \partial\Omega \times (0, T), \\ (y(\cdot, 0), y_t(\cdot, 0)) = (y_0, y_1) \in \boldsymbol{H} := L^2(\Omega) \times H^{-1}(\Omega), & \Omega. \end{cases} \quad (1)$$

$v = v(t)$ - *control* function in $L^2(\Sigma_T)$.

Existence - Uniqueness (Lions'88)
$\exists! y = y(v) \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ and

$$\|y\|_{L^\infty(0,T;L^2(\Omega))} \leq C_{\Omega,T}\left(\|y_0, y_1\|_{\boldsymbol{H}} + \|v\|_{L^2(\Sigma_T)}\right)$$

Null controllability (Lions'88, Lebeau'92, Lasiecka'93, ...) If (T, Γ_0, Ω) satisfies a geometric optic condition, system (1) is null controllable at time T uniformly with respect to the initial condition (y_0, y_1): there exist control functions $v \in L^2(\Sigma_T)$ such that

$$(y_v(\cdot\,, T), (y_v)_t(\cdot\,, T)) = (0, 0), \quad \text{in } \Omega. \quad (2)$$

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Link with the observability for the adjoint system

The controllability property of the hyperbolic equation is equivalent to the observability for the corresponding adjoint problem :

$$\begin{cases} L^\star \varphi := \varphi_{tt} - \nabla \cdot (c(x)\nabla\varphi) + d\varphi = 0 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ (\varphi(\cdot\,,T), \varphi_t(\cdot\,,T)) = (\varphi_0, \varphi_1) \in \boldsymbol{V} & \text{in } \Omega \end{cases} \tag{3}$$

$\boldsymbol{V} := H_0^1(\Omega) \times L^2(\Omega) = \boldsymbol{H}'$.

OBSERVABILITY INEQUALITY- System (3) is observable in time T if there exists a positive constant $C_{obs} > 0$ such that

$$\|(\varphi_0, \varphi_1)\|_{\boldsymbol{V}}^2 \leq C_{obs} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma\, dt \quad \forall (\varphi_0, \varphi_1) \in \boldsymbol{V}. \tag{4}$$

$C_{obs} = C_{obs}(T, \Gamma_0, \Omega, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(Q_T)})$ - Observability constant

Minimal L^2-norm control

$$\begin{cases} \text{Minimize } \; J(y,v) = \dfrac{1}{2}\displaystyle\int_0^T\int_{\Gamma_0} |v|^2\, dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0,y_1;T) \end{cases} \qquad (5)$$

where $\mathcal{C}(y_0,y_1;T)$ denotes the non-empty linear manifold

$$\mathcal{C}(y_0,y_1;T) = \{\,(y,v) : v \in L^2(\Sigma_T),\; y \text{ solves (1) and satisfies (2)}\,\}.$$

Using the Fenchel-Rockafellar theorem [Ekeland-Temam 74], [Brezis 84] we get that

$$\inf_{(y,v)\in\mathcal{C}(y_0,y_1;T)} J(y,v) = -\min_{(\varphi_0,\varphi_1)\in \boldsymbol{V}} J^\star(\varphi_0,\varphi_1)$$

$$\begin{cases} \text{Minimize } \; J^\star(\varphi_0,\varphi_1) = \dfrac{1}{2}\displaystyle\int_0^T\int_{\Gamma_0} \left|\dfrac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt + < (\varphi_0,\varphi_1),(y_0,y_1) > \\ \text{Subject to } \; (\varphi_0,\varphi_1) \in \boldsymbol{V} \quad \text{where} \quad L^\star\varphi = 0 \quad + IC \end{cases} \qquad (6)$$

$$< (\varphi_0,\varphi_1),(y_0,y_1) > := < y_0,\varphi_1 >_{L^2,L^2} - < y_1,\varphi_0 >_{H^{-1},H_0^1}$$

Optimal control: $v = \frac{\partial\varphi}{\partial\nu} 1_{\Gamma_0}$

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Approximation and minimization of $J^\star$ over $\boldsymbol{V} := H_0^1(\Omega) \times L^2(\Omega)$

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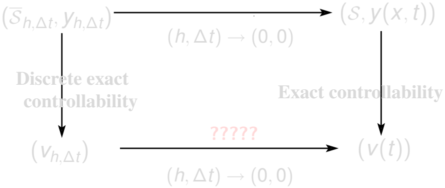
The numerical minimization over a finite dimensional space of $\boldsymbol{V}$ w.r.t. (φ_0, φ_1) may be done using iterative gradient method.
The "difficulty" then is to respect at the finite dimensional level the constraint $L^\star \varphi = 0$!!!!
The usual "trick", developed initially by Glowinski [1] is first to discretize the hyperbolic equation and then to exactly control the corresponding finite dimensional system.



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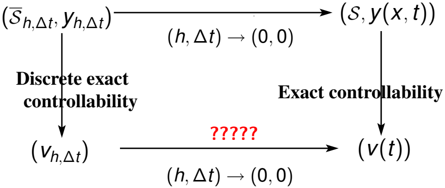

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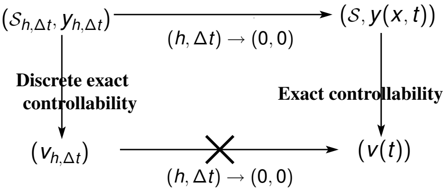
1D - Negative Commutation diagram

Centered finite difference in space and time - Uniform discretization - Constant coefficients $c := 1$, $d := 0$

$$(\overline{\mathcal{S}}_{h,\Delta t}) \begin{cases} \Delta_{\Delta t} y_{h,\Delta t} - \Delta_h y_{h,\Delta t} = 0, \\ + \text{Initial conditions and Boundary terms} \end{cases} \tag{8}$$

produces a non discrete uniformly bounded and convergent control under the (CFL) condition $\Delta t < h$.



For high frequency components of the discrete solution, the discrete observability constant $C_{obs,h}$ blows up as $h \to 0$
[Glowinski-Lions'90] then [Zuazua team later].

Numerical example

$\Omega = (0,1)$ - $\Gamma_0 = \{1\}$ - $T = 2.4$

$$y_0(x) = \begin{cases} 16x & x \in [0, 1/2], \\ 0 & x \in]1/2, 1]. \end{cases} \quad ; \quad y_1(x) = 0. \tag{9}$$

The control v with minimal L^2-norm is discontinuous :

$$v(t) = \begin{cases} 0 & t \in [0, 0.9] \cup [1.9, T], \\ 8(t - 1.4) & t \in]0.9, 1.9[, \end{cases} \tag{10}$$

leading to $||v||_{L^2(0,T)} = 4/\sqrt{3} \approx 2.3094$.

Usual centered finite difference scheme - control

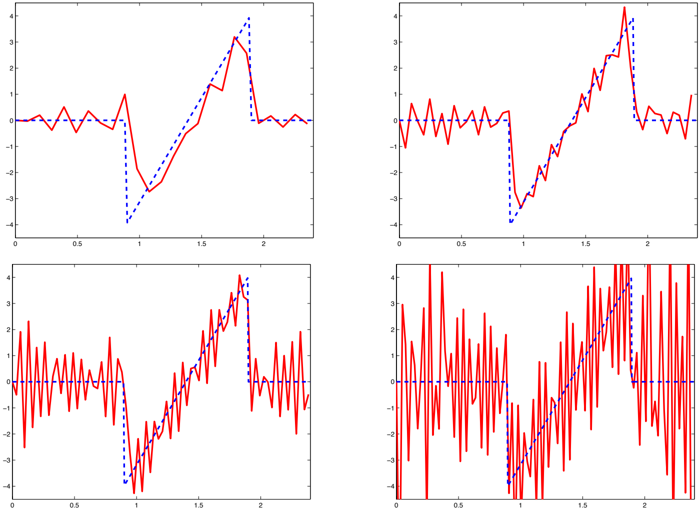


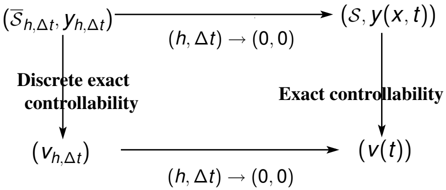
Figure: Control $P(\mathbf{v_h})(t)$ vs. $t \in [0, T]$, $\Delta t/h = 0.98$, $T = 2.4$ and $h = 1/10, 1/20, 1/30$ and $h = 1/40$.

1D - Positive Commutation diagram with a modified scheme

[2]

$$(\overline{\mathcal{S}}_{h,\Delta t})\begin{cases}\Delta_{\Delta t}y_{h,\Delta t}+\frac{1}{4}(h^2-\Delta t^2)\Delta_h\Delta_{\Delta t}y_{h,\Delta t}-\Delta_h y_{h,\Delta t}=0,\\ +\text{Initial conditions and Boundary terms}\end{cases}\tag{11}$$

produces a discrete uniformly bounded and converging control under the condition $\Delta t < h\sqrt{T/2}$.



Within this approach (discretize then control), remedies in the general case (general domain, non constant coefficients) are unknown.

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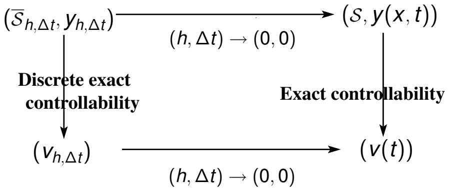

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Modified scheme - control

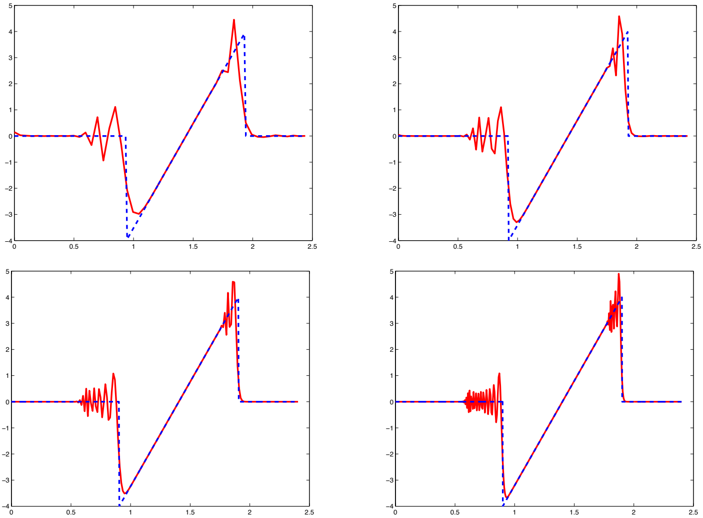

Figure: Modified scheme - Control $P(\boldsymbol{v_h})(t)$ vs. $t \in [0, T]$ - $\Delta t = 1.095445h$, $T = 2.4$ and $h = 1/20, 1/40, 1/80, 1/160$.

Second method to bypass the fact that $L^\star\varphi_h \neq 0$

Since we can not achieve $L^\star\varphi_h = 0$, the idea is to relax the constraint $L^\star\varphi_h = 0$!!!?!!

The idea is to replace the observability inequality

$$\begin{cases} \|\varphi_0,\varphi_1\|_V^2 \leq C_{obs}\left\|\dfrac{\partial\varphi}{\partial\nu}\right\|_{L^2(\Gamma_T)}^2, & \forall(\varphi_0,\varphi_1), \\ L^\star\varphi = 0, \quad \varphi_{|\Sigma_T} = 0 \end{cases} \tag{12}$$

by a "generalized observability inequality" :

$$\|\varphi(\cdot,0),\varphi_t(\cdot,0)\|_V^2 \leq C_{\Omega,T}(1+C_{obs})\left(\left\|\frac{\partial\varphi}{\partial\nu}\right\|_{L^2(\Gamma_T)}^2 + \|L^\star\varphi\|_{L^2(Q_T)}^2\right), \qquad \forall\varphi\in\mathbf{\Phi} \tag{13}$$

Why ? If $\varphi_h \in \mathbf{\Phi_h}$ a finite dimensional subspace of $\mathbf{\Phi}$, then

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with a constant $C_{\Omega,T}(1+C_{obs})$ independant of h !!!

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with a constant $C_{\Omega,T}(1 + C_{obs})$ independant of h !!!

Minimization of $J^\star$

We now replace the problem

$$\begin{cases} \text{Min} \;\; J^\star(\varphi_0,\varphi_1) = \dfrac{1}{2}\displaystyle\int_0^T\!\!\int_{\Gamma_0}\left|\dfrac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt + < y_0,\, \varphi_t(\cdot,0) >_{L^2} - \langle y_1, \varphi(\cdot\,,0)\rangle_{H^{-1},H_0^1} \\ \text{Subject to} \;\; (\varphi_0,\varphi_1) \in \boldsymbol{V} = H_0^1(\Omega)\times L^2(\Omega) \quad \text{where} \quad L^\star\varphi = 0 \end{cases} \tag{15}$$

by the equivalent problem

$$\begin{cases} min\, J^\star(\varphi) = \dfrac{1}{2}\displaystyle\int_0^T\!\!\int_{\Gamma_0}\left|\dfrac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt + < y_0,\, \varphi_t(\cdot,0) >_{L^2} - \langle y_1, \varphi(\cdot\,,0)\rangle_{H^{-1},H_0^1} \\ \text{Subject to } \varphi \in \boldsymbol{W} := \left\{\varphi : \varphi \in C^0(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega)),\; L^\star\varphi = 0 \in L^2(Q_T)\right\} \end{cases} \tag{16}$$

Remark- $\boldsymbol{W}$ endowed with the norm $\|\varphi\|_{\boldsymbol{W}} := \|\frac{\partial\varphi}{\partial\nu}\|_{L^2(\Gamma_T)}$ is an Hilbert space.

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for all $r \geq 0$.

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Relaxation of $L^\star\varphi = 0$

In order to address the $L^2(Q_T)$ constraint $L^\star\varphi = 0$, we introduce a Lagrange multiplier $\lambda \in L^2(Q_T)$; we consider the saddle point problem [3] :

$$\begin{cases} \sup_{\lambda\in L^2(Q_T)} \inf_{\varphi\in\mathbf{\Phi}} \mathcal{L}_r(\varphi,\lambda), \\ \mathcal{L}_r(\varphi,\lambda) := J_r(\varphi) + < L^\star\varphi, \lambda >_{L^2(Q_T)} \\ \mathbf{\Phi} := \left\{\varphi : \varphi \in C^0(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega)), L^\star\varphi \in L^2(Q_T)\right\} \supset \boldsymbol{W} \end{cases} \tag{19}$$

Remark- For all $\eta > 0$, $\mathbf{\Phi}$ is endowed with the scalar product,

$$< \varphi, \overline{\varphi} >_{\mathbf{\Phi}} := < \frac{\partial\varphi}{\partial\nu}, \frac{\partial\overline{\varphi}}{\partial\nu} >_{L^2(\Gamma_T)} + \eta < L^\star\varphi, L^\star\overline{\varphi} >_{L^2(Q_T)}, \quad \forall\varphi,\overline{\varphi} \in \mathbf{\Phi}.$$

$\|\varphi\|_{\mathbf{\Phi}} := \sqrt{< \varphi,\varphi >_{\mathbf{\Phi}}}$ is a norm and $(\mathbf{\Phi}, \|\cdot\|_{\mathbf{\Phi}})$ is an Hilbert space.

[3] N. Cindea, AM, A mixed formulation for the direct approximation of the control of minimal L^2-norm for linear type wave equations, (2015)

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Mixed formulation

Find $(\varphi, \lambda) \in \mathbf{\Phi} \times L^2(Q_T)$ solution of

$$\left\{ \begin{array}{rcll} a_r(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) & = & l(\overline{\varphi}), & \forall \overline{\varphi} \in \mathbf{\Phi} \\ b(\varphi, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{array} \right. \tag{20}$$

where

$$a_r : \mathbf{\Phi} \times \mathbf{\Phi} \to \mathbb{R}, \quad a_r(\varphi, \overline{\varphi}) =< \frac{\partial \varphi}{\partial \nu}, \frac{\partial \overline{\varphi}}{\partial \nu} >_{L^2(\Gamma_T)} + r < L^\star \varphi, L^\star \overline{\varphi} >_{L^2(Q_T)} \tag{21}$$

$$b : \mathbf{\Phi} \times L^2(Q_T) \to \mathbb{R}, \quad b(\varphi, \lambda) =< L^\star \varphi, \lambda >_{L^2(Q_T)} \tag{22}$$

$$l : \mathbf{\Phi} \to \mathbb{R}, \quad l(\varphi) = - < y_0, \varphi_t(\cdot, 0) >_{L^2} + \langle y_1, \varphi(\cdot, 0) \rangle_{H^{-1}, H_0^1} \tag{23}$$

Theorem

For all $r \geq 0$,

1. *The mixed formulation is well-posed.*
2. *The unique solution $(\varphi, \lambda) \in \mathbf{\Phi} \times L^2(Q_T)$ is the unique saddle-point of the Lagrangian $\mathcal{L}_r : \mathbf{\Phi} \times L^2(Q_T) \to \mathbb{R}$ defined by*

$$\mathcal{L}_r(\varphi, \lambda) = \frac{1}{2} a_r(\varphi, \varphi) + b(\varphi, \lambda) - l(\varphi). \tag{24}$$

3. *The optimal function φ given by 2. satisfies $\varphi \in \boldsymbol{W}$ and is the minimizer of $J_r^\star$ over $\boldsymbol{W}$ while the optimal function $\lambda \in L^2(Q_T)$ is the state of the controlled wave equation in the weak sense.*
4. *We have the following estimates*

$$\|\varphi\|_{\mathbf{\Phi}} \leq \|y_0, y_1\|_{\boldsymbol{H}},$$

$$\|\lambda\|_{L^2} \leq \frac{1}{\delta}\left(1 + \max(1, \frac{r}{\eta})\right)\|y_0, y_1\|_{\boldsymbol{H}}, \quad \delta = (C_{\Omega,T} + \eta)^{-1/2}$$

Well-posedness 2

The kernel $\mathcal{N}(b) = \{\varphi \in \mathbf{\Phi}; b(\varphi,\lambda) = 0 \quad \forall \lambda \in L^2(Q_T)\}$ coincides with $\boldsymbol{W}$: we get

$$a_r(\varphi,\varphi) = \|\varphi\|^2_{\mathbf{\Phi}}, \quad \forall \varphi \in \mathcal{N}(b) = \boldsymbol{W}.$$

It remains to check the inf-sup constant property : $\exists \delta > 0$ such that

$$\inf_{\lambda \in L^2} \sup_{\varphi \in \mathbf{\Phi}} \frac{b(\varphi,\lambda)}{\|\varphi\|_{\mathbf{\Phi}} \|\lambda\|_{L^2}} \geq \delta. \tag{25}$$

For any fixed $\lambda \in L^2(Q_T)$, we define $\varphi^0 \in \mathbf{\Phi}$ as the unique solution of

$$L^\star \varphi^0 = \lambda \text{ in } Q_T, \quad (\varphi^0(\cdot,0), \varphi^0_t(\cdot,0)) = (0,0) \text{ on } \Omega, \quad \varphi^0 = 0 \text{ on } \Sigma_T.$$

We get $b(\varphi^0,\lambda) = \|\lambda\|^2_{L^2}$ and $\|\varphi^0\|^2_{\mathbf{\Phi}} = \left\|\frac{\partial \varphi^0}{\partial \nu}\right\|^2_{L^2(\Gamma_T)} + \eta \|\lambda\|^2_{L^2}$.

The estimate $\left\|\frac{\partial \varphi^0}{\partial \nu}\right\|_{L^2(\Gamma_T)} \leq \sqrt{C_{\Omega,T}} \|\lambda\|_{L^2(Q_T)}$ implies that

$$\sup_{\varphi \in \mathbf{\Phi}} \frac{b(\varphi,\lambda)}{\|\varphi\|_{\mathbf{\Phi}} \|\lambda\|_{L^2}} \geq \frac{b(\varphi^0,\lambda)}{\|\varphi^0\|_{\mathbf{\Phi}} \|\lambda\|_{L^2}} \geq \frac{1}{\sqrt{C_{\Omega,T} + \eta}} > 0$$

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The multiplier λ

Taking $r = 0$, the first equation reads

$$a_{r=0}(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) = l(\overline{\varphi}), \qquad \forall \overline{\varphi} \in \mathbf{\Phi} \tag{26}$$

i.e.

$$\iint_{\Gamma_T} \frac{\partial \varphi}{\partial \nu} \frac{\partial \overline{\varphi}}{\partial \nu} + \iint_{Q_T} \lambda L^\star \overline{\varphi} = - < y_0, \overline{\varphi}_t(\cdot, 0) >_{L^2} + \langle y_1, \overline{\varphi}(\cdot, 0) \rangle_{H^{-1}, H_0^1}, \ \forall \overline{\varphi} \in \mathbf{\Phi} \tag{27}$$

which means $\lambda \in L^2(Q_T)$ is solution in the sense of transposition of

$$\begin{cases} L\lambda = 0, \quad \text{in} \quad Q_T \\ (\lambda(\cdot, 0), \lambda_t(\cdot, 0)) = (y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega) \\ (\lambda(\cdot, T), \lambda_t(\cdot, T)) = (0, 0), \\ \lambda = \dfrac{\partial \varphi}{\partial \nu} \quad \text{on} \quad \Gamma_T \end{cases} \tag{28}$$

Therefore, λ coincides with the weak solution of the wave equation controlled by v.

$$\lambda \in C^0([0, T], L^2(\Omega)) \cap C^1([0, T], H^{-1}(\Omega))$$

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Dual of the dual - Minimization w.r.t. λ

Lemma

Let $\mathcal{P}_r$ be the linear operator from L^2 into L^2 defined by

$$\mathcal{P}_r\lambda := L^\star\varphi, \quad \forall\lambda \in L^2 \quad \textit{where} \quad \varphi \in \mathbf{\Phi} \quad \textit{solves} \quad a_r(\varphi, \overline{\varphi}) = b(\overline{\varphi}, \lambda), \quad \forall\overline{\varphi} \in \mathbf{\Phi}.$$

For any $r > 0$, the operator $\mathcal{P}_r$ is a strongly elliptic, symmetric isomorphism from L^2 into L^2.

Theorem

$$\sup_{\lambda\in L^2}\inf_{\varphi\in\mathbf{\Phi}} \mathcal{L}_r(\varphi,\lambda) = -\inf_{\lambda\in L^2} J_r^{\star\star}(\lambda) \quad + \mathcal{L}_r(\varphi_0, 0)$$

where $\varphi_0 \in \mathbf{\Phi}$ solves $a_r(\varphi_0, \overline{\varphi}) = l(\overline{\varphi}), \forall\overline{\varphi} \in \mathbf{\Phi}$ and $J_r^{\star\star} : L^2 \to \mathbb{R}$ defined by

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Conformal Approximation

Let then Φ_h and Λ_h be two finite dimensional spaces parametrized by the variable h such that

$$\Phi_h \subset \mathbf{\Phi}, \quad \Lambda_h \subset L^2(Q_T), \qquad \forall h > 0.$$

Then, we can introduce the following approximated problems : find $(\varphi_h, \lambda_h) \in \Phi_h \times \Lambda_h$ solution of

$$\begin{cases} a_r(\varphi_h, \overline{\varphi}_h) + b(\overline{\varphi}_h, \lambda_h) & = & l(\overline{\varphi}_h), & \forall \overline{\varphi}_h \in \Phi_h \\ b(\varphi_h, \overline{\lambda}_h) & = & 0, & \forall \overline{\lambda}_h \in \Lambda_h. \end{cases} \tag{29}$$

For any $h > 0$, the well-posedness is again a consequence of two properties

- the coercivity of the bilinear form a_r on the subset $\mathcal{N}_h(b) = \{\varphi_h \in \Phi_h; b(\varphi_h, \lambda_h) = 0 \quad \forall \lambda_h \in \Lambda_h\}$. From

$$a_r(\varphi, \varphi) \geq \frac{r}{\eta} \|\varphi\|_{\Phi}^2, \quad \forall \varphi \in \mathbf{\Phi}$$

the form a_r is coercive on the full space Φ, and so *a fortiori* on $\mathcal{N}_h(b) \subset \Phi_h \subset \Phi$.

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Necessary condition: $\dim(\Phi_h) > \dim(\Lambda_h)$

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Finite dimensional linear system

Let $n_h = \dim \Phi_h$, $m_h = \dim \Lambda_h$ and let the real matrices $A_{r,h} \in \mathbb{R}^{n_h,n_h}$, $B_h \in \mathbb{R}^{m_h,n_h}$, $J_h \in \mathbb{R}^{m_h,m_h}$ and $L_h \in \mathbb{R}^{n_h}$ be defined by

$$\begin{cases} a_r(\varphi_h, \overline{\varphi_h}) = < A_{r,h}\{\varphi_h\}, \{\overline{\varphi_h}\} >_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}, & \forall \varphi_h, \overline{\varphi_h} \in \Phi_h, \\ b(\varphi_h, \lambda_h) = < B_h\{\varphi_h\}, \{\lambda_h\} >_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}}, & \forall \varphi_h \in \Phi_h, \forall \lambda_h \in \Lambda_h, \\ l(\varphi_h) = < L_h, \{\varphi_h\} >, & \forall \varphi_h \in \Phi_h \end{cases}$$

where $\{\varphi_h\} \in \mathbb{R}^{n_h}$ denotes the vector associated to φ_h and $< \cdot, \cdot >_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}$ the usual scalar product over $\mathbb{R}^{n_h}$. Problem (29) reads as follows :

find $\{\varphi_h\} \in \mathbb{R}^{n_h}$ and $\{\lambda_h\} \in \mathbb{R}^{m_h}$ such that

$$\begin{pmatrix} A_{r,h} & B_h^T \\ B_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix} \{\varphi_h\} \\ \{\lambda_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} L_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}. \qquad (31)$$

$A_{r,h}$ is symmetric and positive definite for any $h > 0$ and any $r > 0$.
The full matrix of order $m_h + n_h$ in (31) is symmetric but not positive definite.

Choice of the conformal spaces Φ_h and Λ_h

We introduce a triangulation $\mathcal{T}_h$ such that $\overline{Q_T} = \cup_{K\in\mathcal{T}_h} K$ and we assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family. We note $h := \max\{\mathrm{diam}(K), K \in \mathcal{T}_h\}$.

We define the finite dimensional space

$$\Lambda_h = \{\lambda_h \in C^0(\overline{Q_T}), \lambda_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h,\ \lambda_h = 0 \text{ on } \Sigma_T\} \subset L^2(Q_T)$$

The space Φ_h must be chosen such that $L^\star\varphi_h \in L^2(Q_T)$ for any $\varphi_h \in \Phi_h$. This is guaranteed as soon as φ_h possesses second-order derivatives in $L^2(Q_T)$. A conformal approximation based on standard triangulation of Q_T is obtained with spaces of functions continuously differentiable with respect to both x and t.

We introduce the space Φ_h as follows:

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We introduce a triangulation $\mathcal{T}_h$ such that $\overline{Q_T} = \cup_{K\in\mathcal{T}_h} K$ and we assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family. We note $h := \max\{\mathrm{diam}(K), K \in \mathcal{T}_h\}$.

We define the finite dimensional space

$$\Lambda_h = \{\lambda_h \in C^0(\overline{Q_T}), \lambda_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h,\ \lambda_h = 0 \text{ on } \Sigma_T\} \subset L^2(Q_T)$$

The space Φ_h must be chosen such that $L^\star\varphi_h \in L^2(Q_T)$ for any $\varphi_h \in \Phi_h$. This is guaranteed as soon as φ_h possesses second-order derivatives in $L^2(Q_T)$. A conformal approximation based on standard triangulation of Q_T is obtained with spaces of functions continuously differentiable with respect to both x and t.

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C^1 finite element over Q_T

[4]

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1. The *Bogner-Fox-Schmit* (BFS for short) C^1 element defined for rectangles. It involves 16 degrees of freedom, namely the values of $\varphi_h, \varphi_{h,x}, \varphi_{h,t}, \varphi_{h,xt}$ on the four vertices of each rectangle K.

2. The reduced *Hsieh-Clough-Tocher* (HCT for short) C^1 element defined for triangles. This is a so-called composite finite element and involves 9 degrees of freedom, namely the values of $\varphi_h, \varphi_{h,x}, \varphi_{h,t}$ on the three vertices of each triangle K.

[4] P.G. Ciarlet, The finite element for elliptic problems, North-Holland, 1979

C^1 finite element over Q_T

4

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Convergence rate in Φ and in $L^2(Q_T)$

Proposition (BFS element for $N = 1$ - Convergence in Φ)

Let $h > 0$, let $k \leq 2$. If $(\varphi, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$, $\exists K > 0$

$$\|\varphi - \varphi_h\|_{\Phi} \leq K\left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right)h^k,$$

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K\left(\left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right)\frac{1}{\delta_h} + \frac{1}{\sqrt{\eta}\delta_h}\right)h^k.$$

Writing the ineq. obs. for $\varphi - \varphi_h \in \Phi$ and using that $L^\star(\varphi - \varphi_h) = -L^\star\varphi_h$, we get

$$\begin{aligned}\|\varphi - \varphi_h\|^2_{L^2(Q_T)} &\leq C_{\Omega,T}(C_{obs} + 1)(\|\partial_\nu(\varphi - \varphi_h)\|^2_{L^2(\Gamma_T)} + \|L^\star\varphi_h\|^2_{L^2(Q_T)}) \\ &\leq C_{\Omega,T}(C_{obs} + 1)\max(1, \frac{2}{\sqrt{\eta}})\|\varphi - \varphi_h\|_{\Phi}\end{aligned}$$

Theorem (BFS element for $N = 1$ - Convergence in $L^2(Q_T)$)

Let $h > 0$, let $k \leq 2$. If $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$,

$$\|\varphi - \varphi_h\|_{L^2(Q_T)} \leq K\max(1, \frac{2}{\sqrt{\eta}})\left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right)h^k.$$

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$N = 1$ - Numerical experiments

$\Omega = (0,1) - \Gamma_0 = \{1\} - T = 2.4$

$$(\textbf{EX}) \qquad y_0(x) = 4x \, 1_{(0,1/2)}(x), \quad y_1(x) = 0, \qquad x \in \Omega$$

$$v(t) = 2(1-t)\, 1_{(1/2,3/2)}(t), \quad t \in (0,T), \quad \|v\|_{L^2(0,T)} = 1/\sqrt{3} \approx 0.5773. \qquad (32)$$

$N = 1$ - Numerical experiments

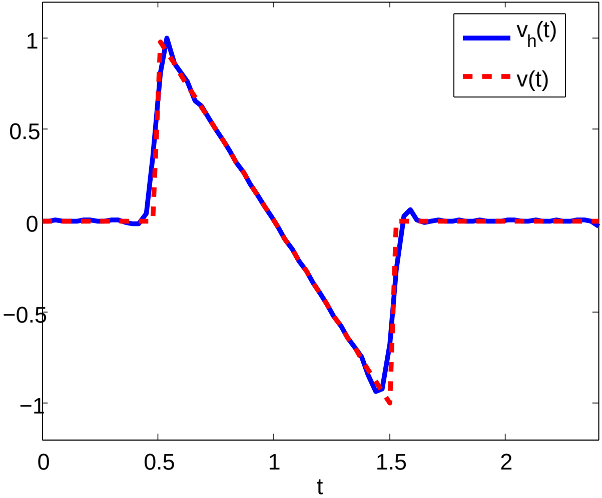

Figure: Control of minimal L^2-norm v and its approximation v_h on $(0, T)$; $r = 10^{-2}$; $h = 2.46 \times 10^{-2}$

Example 1 - $N = 1$ - Numerical experiments

h	1.41×10^{-1}	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\|v_h\|_{L^2(0,T)}$	0.6003	0.5850	0.5776	0.5752	0.5747
$\|v - v_h\|_{L^2(0,T)}$	2.87×10^{-1}	2.05×10^{-1}	1.47×10^{-1}	1.08×10^{-1}	8.18×10^{-2}
$\|\lambda_h\|_{L^2(Q_T)}$	0.62	0.598	0.586	0.581	0.578
$\|L^\star \varphi_h\|_{L^2(Q_T)}$	1.02×10^{-1}	7.53×10^{-2}	5.8×10^{-2}	4.55×10^{-2}	3.6×10^{-2}
$\|L^\star \varphi_h\|_{H^{-1}(Q_T)}$	1.92×10^{-16}	3.83×10^{-16}	7.46×10^{-16}	1.51×10^{-15}	2.81×10^{-15}

Table: BFS element - $r = 1$.

$$r = 1: \qquad \|v - v_h\|_{L^2(0,T)} \approx 1.12 \cdot h^{0.52}, \quad \|L^\star \varphi_h\|_{L^2(Q_T)} \approx 15.67 \cdot h^{0.72},$$

$$r = 10^{-2}: \qquad \|v - v_h\|_{L^2(0,T)} \approx 0.83 \cdot h^{0.45}, \quad \|L^\star \varphi_h\|_{L^2(Q_T)} \approx 0.24 \cdot h^{0.37}.$$

A curiosity : $\|v_h\|_{L^2(0,T)}$ is close to $\|y_h\|_{L^2(Q_T)}$!?!!

Example 1 - $N = 1$ - Numerical experiments

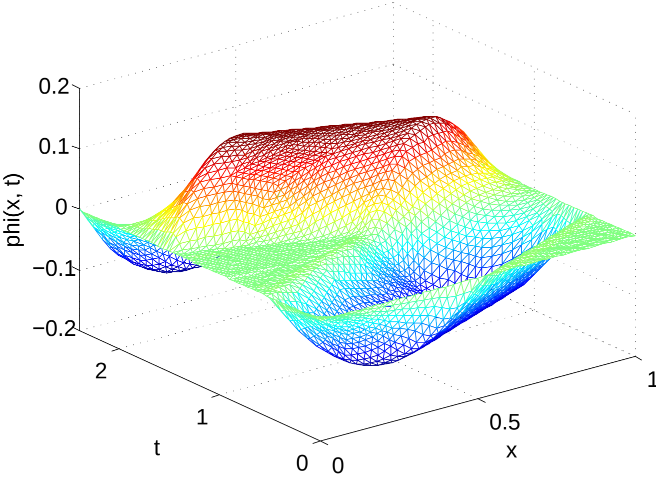

Figure: The dual variable φ_h in Q_T; $h = 2.46 \times 10^{-2}$; $r = 10^{-2}$.

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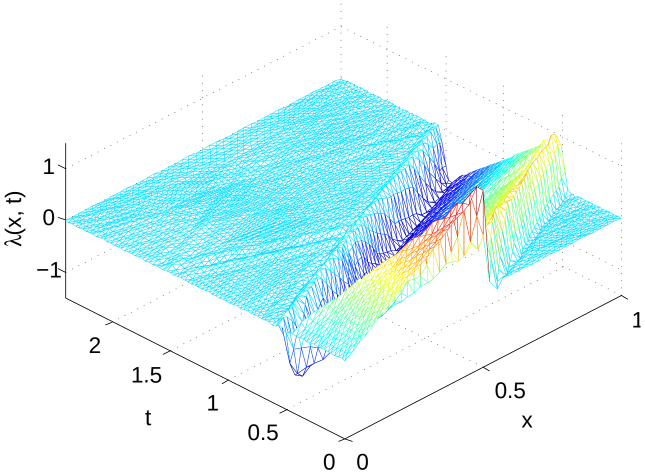

Figure: The primal variable λ_h in Q_T; $h = 2.46 \times 10^{-2}$; $r = 10^{-2}$.

Mesh adaptation

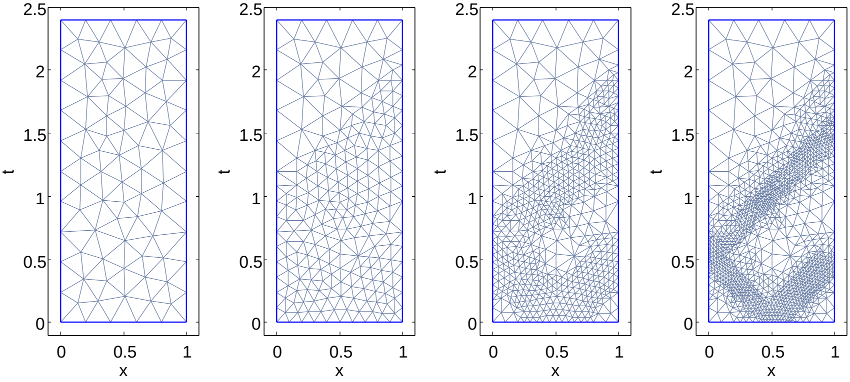

Figure: Iterative refinement of the triangular mesh over Q_T with respect to the variable λ_h: $142, 412, 1\,154, 2\,556$; $r = 2 \times 10^{-3}$.

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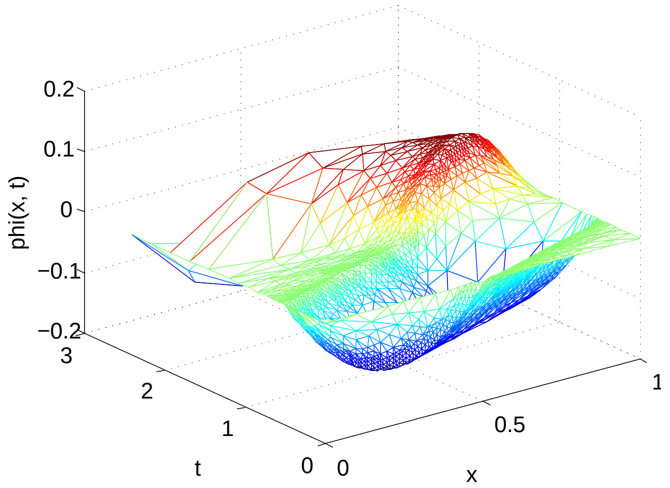

Figure: The dual variable φ_h in Q_T corresponding to the finer mesh; $r = 2 \times 10^{-3}$.

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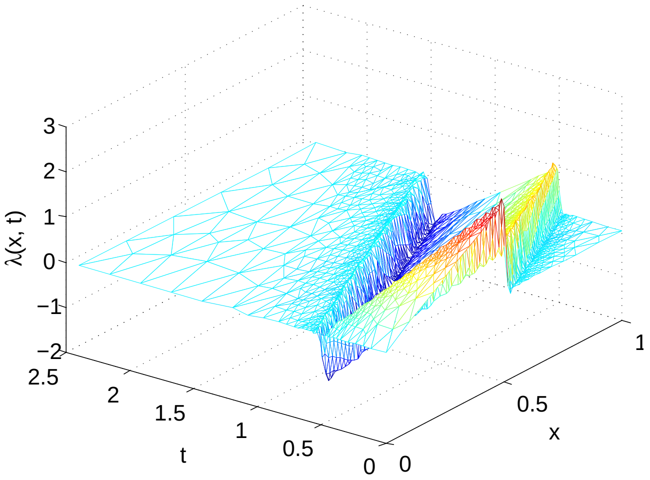

Figure: The primal variable λ_h in Q_T corresponding to the finer mesh.

Minimization of $J_r^{\star\star}$ with respect to λ

$$J_r^{\star\star}(\lambda) := \frac{1}{2} < \mathcal{P}_r\lambda, \lambda >_{L^2(Q_T)} - b(\varphi_0, \lambda)$$

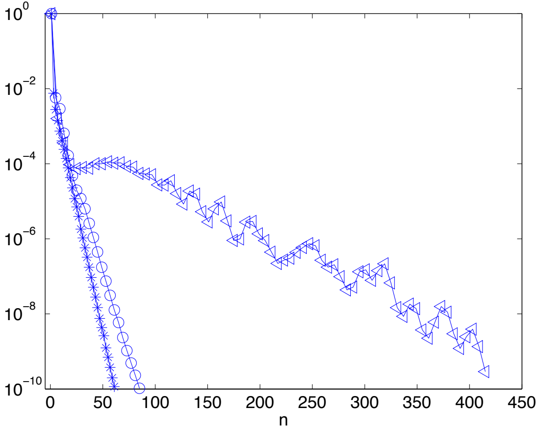

Figure: Evolution of $\|g^n\|_{L^2(Q_T)}/\|g^0\|_{L^2(Q_T)}$ w.r.t. the iterate n for $r = 10^2$ ($\star$), $r = 1$ ($\square$), $r = 10^{-2}$ ($\circ$) and $r = h^2$ ($<$) ; $h = 9.99 \times 10^{-3}$.

Minimization of $J_r^{\star\star}$ with respect to λ

$$J_r^{\star\star}(\lambda) := \frac{1}{2} < \mathcal{P}_r\lambda, \lambda >_{L^2(Q_T)} - b(\varphi_0, \lambda)$$

h	1.56×10^{-1}	7.92×10^{-2}	3.99×10^{-2}	1.99×10^{-2}	9.99×10^{-3}
$\sharp$ iterates	20	26	31	44	61
$m_h = card(\{\lambda_h\})$	231	840	3 198	12 555	49 749
$\|\lambda_h(1,\cdot)\|_{L^2(0,T)}$	0.6089	0.5867	0.5775	0.5746	0.5742
$\|v - \lambda_h(1,\cdot)\|_{L^2(0,T)}$	2.40×10^{-1}	1.68×10^{-1}	1.28×10^{-1}	9.69×10^{-2}	7.62×10^{-2}
$\|\lambda_h\|_{L^2(Q_T)}$	0.6178	0.5963	0.5857	0.5806	0.5784

Table: BFS element - Conjugate gradient algorithm - $r = 1$.

Remind: $\|v\|_{L^2(0,T)} \approx 0.5773$

Comparison with the bi-harmonic regularization

[Glowinski'92]

$$\begin{cases} \min_{(\varphi_0,\varphi_1)\in\tilde{V}} J^\star_\epsilon(\varphi_0,\varphi_1) := J^\star(\varphi_0,\varphi_1) + \frac{\epsilon}{2}\|\varphi_0,\varphi_1\|^2_{\tilde{V}}, \quad \epsilon > 0, \\ \tilde{V} := H^2(\Omega)\cap H^1_0(\Omega)\times H^1_0(\Omega) \end{cases} \tag{33}$$

Time Marching method here ! : $h = \Delta x$; $\Delta t = 0.8\Delta x$

h	1.56×10^{-1}	7.92×10^{-2}	3.99×10^{-2}	1.99×10^{-2}	9.99×10^{-3}
$\sharp$ iterates	62	>5000	78	58	39
$card(\{\varphi_{0h},\varphi_{1h}\})$	44	84	164	324	644
$\|v_h\|_{L^2(0,T)}$	0.5484	0.5603	0.5671	0.5712	0.5736
$\|v-v_h\|_{L^2(0,T)}$	2.72×10^{-1}	2.23×10^{-1}	1.81×10^{-1}	1.47×10^{-1}	1.24×10^{-1}
$\|y_h\|_{L^2(Q_T)}$	0.5386	0.5557	0.5649	0.5701	0.5731

Table: Biharmonic Tychonoff regularization; $\epsilon = h^{1.8}$.

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Remark : If $\epsilon = h^2$, the CG algorithm diverges.

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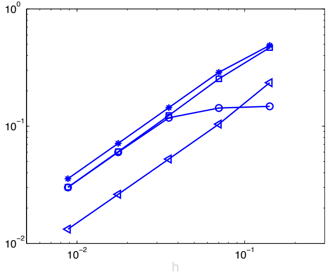
The discrete inf-sup test - Evaluation of δ_h

$$\delta_h := \inf_{\lambda_h \in \Lambda_h} \sup_{\varphi_h \in \Phi_h} \frac{b(\varphi_h, \lambda_h)}{\|\varphi_h\|_{\Phi_h} \|\lambda_h\|_{\Lambda_h}} \geq \delta. \tag{34}$$

Taking $\eta = r > 0$ so that $a_r(\varphi, \overline{\varphi}) = (\varphi, \overline{\varphi})_\Phi$, we have [5]

$$\delta_h = \inf \left\{ \sqrt{\delta} : B_h A_{r,h}^{-1} B_h^T \{\lambda_h\} = \delta \, J_h \{\lambda_h\}, \quad \forall \{\lambda_h\} \in \mathbb{R}^{m_h} \setminus \{0\} \right\}. \tag{35}$$



$\delta_h \approx C_r \frac{h}{\sqrt{r}}$ as $h \to 0^+$

If $r = h^2$, (Φ_h, Λ_h) passes the discrete inf-sup test !

BFS finite element - $h \to \sqrt{r}\delta_{h,r}$ for $r = 1$ ($\Box$), $r = 10^{-2}$ ($\circ$), $r = h$ ($\star$) and $r = h^2$ ($<$)

[5] K. Bathe, D. Chapelle, The discrete inf-sup test, (2003)

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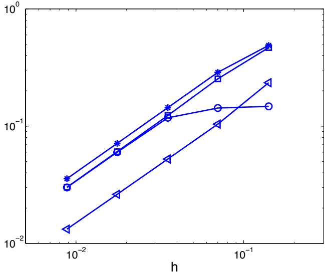

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[5] K. Bathe, D. Chapelle, The discrete inf-sup test, (2003)

Stabilized mixed formulation "à la Barbosa-Hughes"

[6]

$\alpha > 0$

$$\begin{cases} \sup_{\lambda\in\Lambda} \inf_{\varphi\in\Phi} \mathcal{L}_{r,\alpha}(\varphi,\lambda), \\ \mathcal{L}_{r,\alpha}(\varphi,\lambda) := \mathcal{L}_r(\varphi,\lambda) - \frac{\alpha}{2}\|L\lambda\|^2_{L^2(H^{-1}(\Omega))} - \frac{\alpha}{2}\|\lambda - \partial_\nu\varphi\|^2_{L^2(\Gamma_T)}. \end{cases} \tag{36}$$

$$\Lambda := \Big\{\lambda : \lambda \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega)),$$
$$L\lambda \in L^2([0,T];H^{-1}(\Omega)), \lambda(\cdot,0) = \lambda_t(\cdot,0) = 0, \lambda_{|\Gamma_T} \in L^2(\Gamma_T)\Big\}.$$

Λ is a Hilbert space endowed with the following inner product

$$\langle\lambda,\overline{\lambda}\rangle_\Lambda := \int_0^T \langle L\lambda(t), L\overline{\lambda}(t)\rangle_{H^{-1}(\Omega)} dt + \iint_{\Gamma_T} \lambda\overline{\lambda} d\sigma dt, \qquad \forall\, \lambda, \overline{\lambda} \in \Lambda$$

using notably that

$$\|\lambda\|_{L^2(Q_T)} \le C_{\Omega,T}\sqrt{<\lambda,\lambda>_\Lambda}, \quad \forall\lambda \in \Lambda \tag{37}$$

for some positive constant $C_{\Omega,T}$. We denote $\|\lambda\|_\Lambda := \sqrt{<\lambda,\lambda>_\Lambda}$.

[6] H. Barbosa, T. Hugues : The finite element method with Lagrange multipliers on the boundary: circumventing the Babusÿka-Brezzi condition, 1991

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Λ is a Hilbert space endowed with the following inner product

$$\langle\lambda, \overline{\lambda}\rangle_\Lambda := \int_0^T \langle L\lambda(t), L\overline{\lambda}(t)\rangle_{H^{-1}(\Omega)} dt + \iint_{\Gamma_T} \lambda\overline{\lambda} d\sigma dt, \qquad \forall\, \lambda,\ \overline{\lambda} \in \Lambda$$

using notably that

$$\|\lambda\|_{L^2(Q_T)} \le C_{\Omega,T}\sqrt{<\lambda,\lambda>_\Lambda}, \quad \forall\lambda\in\Lambda \tag{37}$$

for some positive constant $C_{\Omega,T}$. We denote $\|\lambda\|_\Lambda := \sqrt{<\lambda,\lambda>_\Lambda}$.

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[6] H. Barbosa, T. Hugues : The finite element method with Lagrange multipliers on the boundary: circumventing the Babusÿka-Brezzi condition, 1991

Stabilized mixed formulation "à la Barbosa-Hughes" - 2

Then, $\forall \alpha \in (0,1)$, we consider the following mixed formulation:

$$\left\{ \begin{array}{rcll} a_{r,\alpha}(\varphi,\overline{\varphi}) + b_\alpha(\overline{\varphi},\lambda) & = & l_1(\overline{\varphi}), & \forall\, \overline{\varphi} \in \Phi \\ b_\alpha(\varphi,\overline{\lambda}) - c_\alpha(\lambda,\overline{\lambda}) & = & 0, & \forall\, \overline{\lambda} \in \Lambda, \end{array} \right. \tag{38}$$

where

$$a_{r,\alpha} : \Phi \times \Phi \to \mathbb{R}, \quad a_{r,\alpha}(\varphi,\overline{\varphi}) = (1-\alpha) \iint_{\Gamma_T} \partial_\nu \varphi\, \partial_\nu \overline{\varphi}\, d\sigma dt + r \iint_{Q_T} L^\star \varphi\, L^\star \overline{\varphi} dxdt \tag{39}$$

$$b_\alpha : \Phi \times \Lambda \to \mathbb{R}, \quad b_\alpha(\varphi,\lambda) = \iint_{Q_T} L^\star \varphi \lambda dxdt - \alpha \iint_{\Gamma_T} \partial_\nu \varphi \lambda d\sigma dt \tag{40}$$

$$c_\alpha : \Lambda \times \Lambda \to \mathbb{R}, \quad c_\alpha(\lambda,\overline{\lambda}) = \alpha \int_0^T \langle L\lambda(t), L\overline{\lambda}(t)\rangle_{H^{-1}(\Omega)} dt + \alpha \iint_{\Gamma_T} \lambda\overline{\lambda} d\sigma dt \tag{41}$$

Stabilized mixed formulation "à la Barbosa-Hughes" - 3

Proposition

$\forall \alpha \in (0,1)$, *the stabilized mixed formulation* (38) *is well-posed. Moreover, the unique pair* $(\varphi, \lambda) \in \Phi \times \Lambda$ *satisfies*

$$\theta \|\varphi\|_{\Phi}^2 + \alpha \|\lambda\|_{\Lambda}^2 \leq \frac{(1-\alpha)^2 + \alpha\theta}{\theta} \|y_0, y_1\|_{L^2 \times H^{-1}}^2 \tag{42}$$

with $\theta := \min(1-\alpha, r/\eta)$.

Proposition

If $\alpha \in (0,1)$, *the solution* $(\varphi, \lambda) \in \Phi \times L^2(\Omega)$ *coincides with the stabilized solution* $(\varphi_\alpha, \lambda_\alpha) \in \Phi \times \Lambda$

Stabilized mixed formulation "à la Barbosa-Hughes" - 3

Proposition

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Proposition

If $\alpha \in (0,1)$, the solution $(\varphi,\lambda) \in \Phi \times L^2(\Omega)$ coincides with the stabilized solution $(\varphi_\alpha, \lambda_\alpha) \in \Phi \times \Lambda$

Stabilized mixed formulation "à la Barbosa-Hughes" - Numerical approximation

$\alpha \in (0,1)$, $r > 0$.

$$\Phi_h \subset \Phi, \quad \widetilde{\Lambda}_h \subset \Lambda, \qquad \forall h > 0.$$

Find $(\varphi_h, \lambda_h) \in \Phi_h \times \widetilde{\Lambda}_h$ solution of

$$\left\{ \begin{array}{rcll} a_{r,\alpha}(\varphi_h, \overline{\varphi}_h) + b_\alpha(\lambda_h, \overline{\varphi}_h) & = & l_1(\overline{\varphi}_h), & \forall \overline{\varphi}_h \in \Phi_h \\ b_\alpha(\overline{\lambda}_h, \varphi_h) - c_\alpha(\lambda_h, \overline{\lambda}_h) & = & 0, & \forall \overline{\lambda}_h \in \widetilde{\Lambda}_h. \end{array} \right. \tag{43}$$

In view of the properties of $a_{r,\alpha}$, c_α, l_1, this formulation is well-posed.

$$\widetilde{\Lambda}_h = \{\lambda \in \Phi_h ; \lambda(\cdot, 0) = \lambda_t(\cdot, 0) = 0\}. \tag{44}$$

Proposition (BFS element for $N = 1$ - Rate of convergence for the norm $\Phi \times \Lambda$)

Let $h > 0$, let $k \leq 2$ be a positive integer and $\alpha \in (0,1)$. Let (y, λ) and (y_h, λ_h) be the solution of (38) and (43) respectively. If (y, λ) belongs to $H^{k+2}(Q_T) \times H^{k+2}(Q_T)$, then there exists a positive constant $K = K(\|\varphi\|_{H^{k+2}(Q_T)}, \alpha, r, \eta)$ independent of h, such that

$$\|\varphi - \varphi_h\|_\Phi + \|\lambda - \lambda_h\|_\Lambda \leq K h^k. \tag{45}$$

Remark - no δ_h here !!!! $r > 0$ is arbitrary

Stabilized mixed formulation "à la Barbosa-Hughes" - Numerical approximation

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Remark 1: The situation may be simpler with a different cost !?

$$\begin{cases} \text{Minimize } \; J(y,v) = \dfrac{1}{2}\displaystyle\iint_{Q_T} |y|^2 \, dx \, dt + \dfrac{1}{2}\int_0^T\!\!\int_{\Gamma_0} |v|^2 \, d\sigma \, dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0, y_1; T) \end{cases} \tag{46}$$

$$v = \frac{\partial \varphi}{\partial \nu} \;\text{ in }\; (0,T)\times\Gamma_0 \;\text{ and }\; y = \mu \;\text{ in }\; Q_T.$$

$$\begin{cases} \text{Minimize } \; J^\star(\mu, \varphi_0, \varphi_1) = \dfrac{1}{2}\displaystyle\iint_{Q_T} |\mu|^2 \, dx \, dt + \dfrac{1}{2}\int_0^T\!\!\int_{\Gamma_0} \left|\dfrac{\partial \varphi}{\partial \nu}\right|^2 d\sigma \, dt \\ \qquad\qquad + < (\varphi_0, \varphi_1), (y_0, y_1) > \\ \text{Subject to } \; (\mu, \varphi_0, \varphi_1) \in L^2(Q_T) \times \boldsymbol{V}, \end{cases} \tag{47}$$

where φ solves the nonhomogeneous backward problem

$$L^\star \varphi = \mu \quad \text{in} \quad Q_T, \qquad \varphi = 0 \quad \text{on} \quad \Sigma_T, \qquad (\varphi(\cdot,0), \varphi'(\cdot,0)) = (\varphi_0, \varphi_1) \tag{48}$$

Remark 1: The situation may be much simpler with a different cost !!?!

[7]

Replacing μ by $L^\star\varphi$ and miniminiz over φ lead to

$$\left\{\begin{array}{l} \text{Minimize } J_1^\star(\varphi) = \frac{1}{2}\iint_{Q_T} |L^\star\varphi|^2\, dx\, dt + \frac{1}{2}\int_0^T\int_{\Gamma_0}\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt \\ \qquad + < (\varphi(\cdot,0),\varphi_t(\cdot,0)),(y_0,y_1) > \\ \text{Subject to } \varphi\in\mathbf{\Phi} \end{array}\right. \tag{49}$$

and to the well-posed variational formulation: find $\varphi \in \mathbf{\Phi}$ such that

$$\iint_{Q_T} L^\star\varphi\, L^\star\overline{\varphi}\, dx\, dt + \int_0^T\int_{\Gamma_0}\frac{\partial\varphi}{\partial\nu}\frac{\partial\overline{\varphi}}{\partial\nu}\, d\sigma\, dt = < (\overline{\varphi}(\cdot,0),\overline{\varphi}_t(\cdot,0)),(y_0,y_1) >, \quad \forall\overline{\varphi}\in\mathbf{\Phi} \tag{50}$$

[7] N. Cindea, E. Fernandez-Cara, AM, Numerical controllability of the wave equation through primal methods and Carleman estimates (2012)

Non constant coefficient: $Ly := y_{tt} - (c(x)y_x)_x + d(x,t)y$

$c \in C^1([0,1])$

$$c(x) = \begin{cases} 1 & x \in [0, 0.45] \\ \in [1., 5.] \quad (c'(x) > 0), & x \in (0.45, 0.55) \\ 5 & x \in [0.55, 1] \end{cases} \tag{51}$$

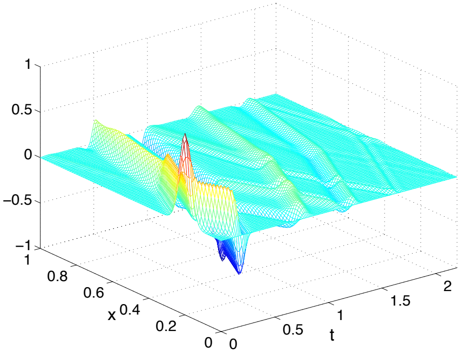


Figure: $y_0(x) \equiv e^{-500(x-0.2)^2}$ and c given by (51) -The solution $\hat{y}_h$ over Q_T - $h = (1/80, 1/80)$.

Remark 2: The distributed case

$$Ly = v\,1_{q_T}, \quad q_T = \omega \times (0,T) \subset \Omega \times (0,T)$$

$$\begin{cases} min\, J^\star(\varphi) = \frac{1}{2}\int_0^T\int_\omega |\varphi|^2\, dx\, dt + < y_0, \varphi_t(\cdot,0) >_{H^1,H^{-1}} - \langle y_1, \varphi(\cdot\,,0)\rangle_{L^2} \\ \text{Subject to } \varphi \in \boldsymbol{W} := \left\{ \varphi : \varphi \in L^2(q_T),\, \varphi_{|\Sigma_T} = 0, L^\star\varphi = 0 \in L^2(0,T,H^{-1}(\Omega)) \right\} \end{cases} \tag{52}$$

Optimal control : $v = \varphi\,1_{q_T}$

Generalized observability inequality : $\exists C_{obs}$ s.t.

$$\|\varphi_0,\varphi_1\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C_{obs}\left(\|\varphi\|^2_{L^2(q_T)} + \|L^\star\varphi\|^2_{L^2(0,T;H^{-1})}\right), \quad \forall\varphi \in \boldsymbol{\Phi}$$

Multiplier :

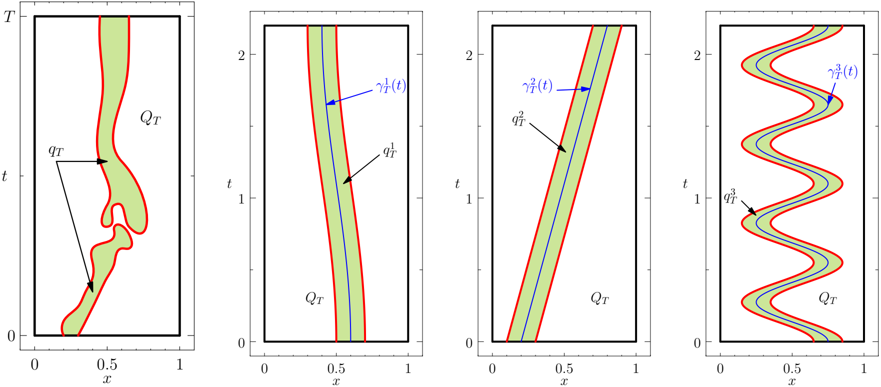
$$b(\varphi,\lambda) = \int_0^T < \lambda(\cdot,t), L^\star\varphi(\cdot,t) >_{H_0^1(\Omega),H^{-1}(\Omega)} dt, \qquad \lambda \in L^2(0,T;H_0^1(\Omega))$$

The distributed case : Non cylindrical situation in 1D with constant coefficient

[8]

[9]

The variational approach is well-adapted to the non cylindrical situation.



Time dependent domains $q_T \subset Q_T = \Omega \times (0, T)$

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[8] C. Castro, N. Cindea, A. Münch, Controllability of the 1D wave equation with inner moving force, SICON (2014)]

[9] G. Lebeau, J. Le Rousseau, P. Terpolilli, E. Trélat, Geometric control condition for the wave equation with a time-dependent domain, (2016)

Remark 3 : Inverse problems -

Given a distributed observation $y_{obs} \in L^2(q_T)$, $f \in X := L^2(H^{-1})$, reconstruct y such that

$$Ly = f \quad \text{in} \quad Q_T, \qquad y = 0 \quad \text{on} \quad \Sigma_T, \qquad y - y_{obs} = 0 \quad \text{on} \quad q_T$$

$$(LS) \quad \begin{cases} \text{minimize} \quad J(y_0, y_1) := \frac{1}{2}\|y - y_{obs}\|^2_{L^2(q_T)} \\ \text{subject to} \quad (y_0, y_1) \in L^2 \times H^{-1} \text{where} \quad Ly - f = 0 \end{cases}$$

The "Discretization then Inverse problem" procedure is discussed in [L. Baudouin, M. De Buhan, S. Ervedoza, 2013]

Keeping y as the main variable [10]....

$$(\mathcal{P}) \quad \begin{cases} \inf J(y) := \frac{1}{2}\|y - y_{obs}\|^2_{L^2(q_T)} + \frac{r}{2}\|Ly - f\|^2_X, \quad r \geq 0 \\ \text{subject to} \quad y \in W := \{y \in Z;\ Ly - f = 0 \text{ in } X\} \end{cases}$$

The multiplier $\lambda \in X'$ is a "measure" of the quality of y_{obs} to reconstruct y.

[10] N. Cindea, AM, Inverse problem for linear hyperbolic equations using mixed formulations, Inverse Problems, (2015).

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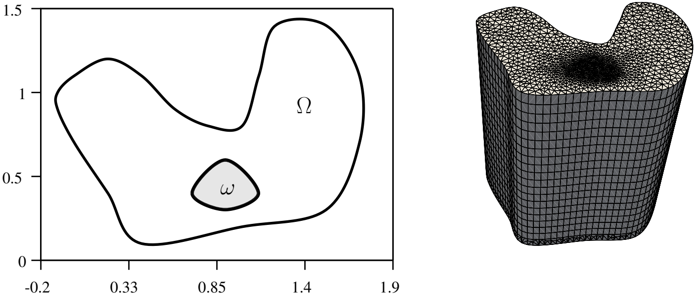
$2D$ example - Observation on q_T



(a) (b)

Mesh number	0	1	2
Number of elements	5 730	44 900	196 040
Number of nodes	3 432	24 633	103 566

Characteristics of the three meshes associated with Q_T.

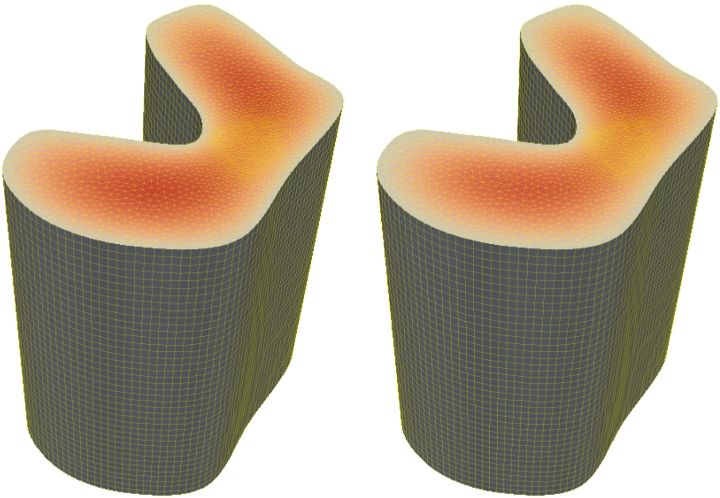
$2D$ example - Observation on q_T

$$\begin{cases} -\Delta y_0 = 10, & \text{in } \Omega \\ y_0 = 0, & \text{on } \partial\Omega, \end{cases} \qquad y_1 = 0. \tag{53}$$

Mesh number	0	1	2
$\frac{\Vert\overline{y}_h - y_h\Vert_{L^2(Q_T)}}{\Vert\overline{y}_h\Vert_{L^2(Q_T)}}$	1.88×10^{-1}	8.04×10^{-2}	5.41×10^{-2}
$\Vert Ly_h\Vert_{L^2(Q_T)}$	3.21	2.01	1.17
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	8.26×10^{-5}	3.62×10^{-5}	2.24×10^{-5}

$$r = h^2 - T = 2$$

$2D$ example - Observation on q_T



(a) (b)

y and y_h in Q_T

Example 2 - $N = 2$ - The stadium

$T = 3$

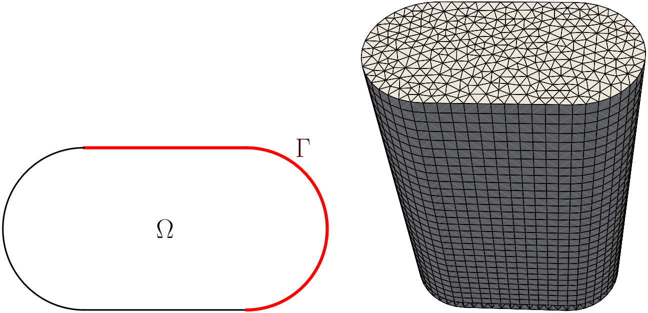


Figure: Bunimovich's stadium and the subset Γ of $\partial\Omega$ on which the observations are available. Example of mesh of the domain Q_T.

Example 2 - $N = 2$ - Recovering of the initial data

$T = 3$

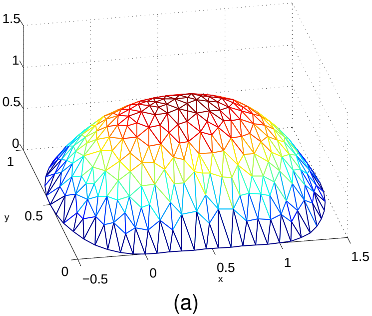


Figure: (a) Initial data y_0 given by (53). (b) Reconstructed initial data $y_h(\cdot, 0)$.

Parabolic case

$\Omega \subset \mathbb{R}^N$; $Q_T = \Omega \times (0, T)$; $q_T = \omega \times (0, T)$

$$\begin{cases} y_t - \nabla \cdot (c(x)\nabla y) + d(x,t)y = v\,1_\omega, & \text{in} \quad Q_T, \\ y = 0, & \text{in} \quad \Sigma_T, \\ y(x,0) = y_0(x), & \text{in} \quad \Omega. \end{cases} \tag{54}$$

$c := (c_{i,j}) \in C^1(\overline{\Omega}; \mathcal{M}_N(\mathbb{R}))$; $(c(x)\xi, \xi) \geq c_0|\xi|^2$ in $\overline{\Omega}$ $(c_0 > 0)$,

$d \in L^\infty(Q_T)$, $y_0 \in L^2(\Omega)$;

$v = v(x,t)$ is the *control* $\quad y = y(x,t)$ is the associated state.

We introduce the linear manifold

$$\mathcal{C}(y_0, T) = \{\, (y, v) : v \in L^2(q_T), \;\; y \text{ solves (54) and satisfies } y(T, \cdot) = 0 \,\}.$$

non empty (see FURSIKOV-IMANUVILOV'96, ROBBIANO-LEBEAU'95)).

NOTATIONS -
$Ly := y_t - \nabla \cdot (c(x)\nabla y) + d(x,t)y; \quad L^\star \varphi := -\varphi_t - \nabla \cdot (c(x)\nabla \varphi) + d(x,t)\varphi$

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$N = 1$ - $L^2(q_T)$-norm of the HUM control with respect to time

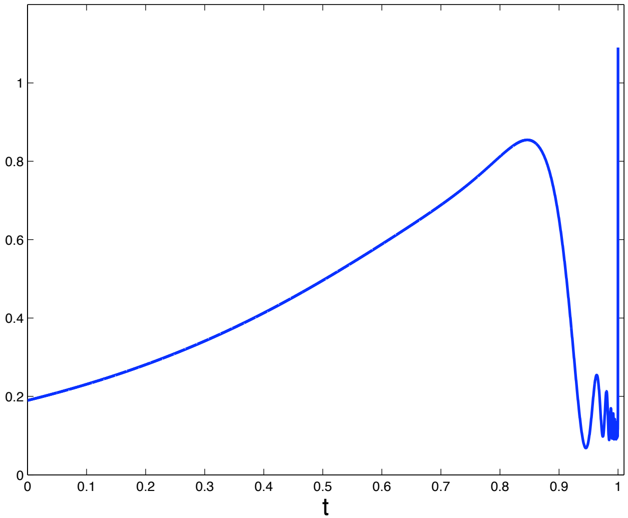

Figure: $y_0(x) = \sin(\pi x)$ - $T = 1$ - $\omega = (0.2, 0.8)$ - $t \longrightarrow \|v(\cdot, t)\|_{L^2(0,1)}$ in $[0, T]$

$N = 1$ - L^2-norm of the HUM control with respect to time: Zoom near T

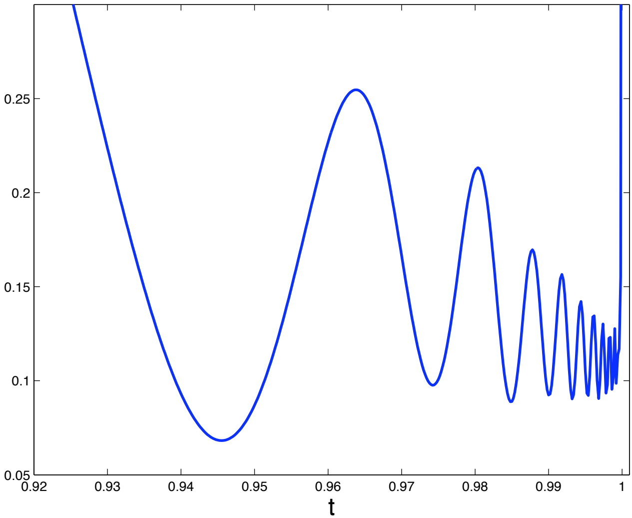

Figure: $y_0(x) = \sin(\pi x)$ - $T = 1$ - $\omega = (0.2, 0.8)$ - $t \to \|v(\cdot, t)\|_{L^2(0,1)}$ in $[0.92T, T]$

Minimal L^2 norm control

Since it is difficult to construct pairs $(v, y) \in \mathcal{C}(y_0, T)$ (*a fortiori* minimizing sequences for J !), it is standard to consider the corresponding dual :

$$\inf_{(y,v)\in\mathcal{C}(y_0,T)} J(y,v) = -\inf_{\phi_T\in H} J^\star(\phi_T), \; J^\star(\phi_T) = \frac{1}{2}\int_{q_T} \phi^2 dxdt + \int_\Omega \phi(0,\cdot)y_0 dx$$

where ϕ solves the backward system

$$\begin{cases} L^\star\phi = 0 & Q_T = (0,T)\times\Omega, \\ \phi = 0 \quad \Sigma_T = (0,T)\times\partial\Omega, & \phi(T,\cdot) = \phi_T \quad \Omega. \end{cases}$$

The Hilbert space H is defined as the completion of $\mathcal{D}(\Omega)$ with respect to the norm

$$\|\phi_T\|_H = \left(\int_{q_T} \phi^2(t,x)dxdt\right)^{1/2}.$$

From the observability inequality

$$\|\phi(0,\cdot)\|^2_{L^2(\Omega)} \leq C_{obs}(\omega,T)\|\phi_T\|^2_H \quad \forall \phi_T \in L^2(\Omega),$$

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Minimal L^2 norm control

Since it is difficult to construct pairs $(v, y) \in \mathcal{C}(y_0, T)$ (*a fortiori* minimizing sequences for J !), it is standard to consider the corresponding dual :

$$\inf_{(y,v)\in\mathcal{C}(y_0,T)} J(y,v) = -\inf_{\phi_T\in H} J^\star(\phi_T), \; J^\star(\phi_T) = \frac{1}{2}\int_{q_T} \phi^2 dxdt + \int_\Omega \phi(0,\cdot)y_0 dx$$

where ϕ solves the backward system

$$\begin{cases} L^\star\phi = 0 & Q_T = (0,T)\times\Omega, \\ \phi = 0 \quad \Sigma_T = (0,T)\times\partial\Omega, & \phi(T,\cdot) = \phi_T \quad \Omega. \end{cases}$$

The Hilbert space H is defined as the completion of $\mathcal{D}(\Omega)$ with respect to the norm

$$\|\phi_T\|_H = \left(\int_{q_T} \phi^2(t,x)dxdt\right)^{1/2}.$$

From the observability inequality

$$\|\phi(0,\cdot)\|^2_{L^2(\Omega)} \leq C_{obs}(\omega,T)\|\phi_T\|^2_H \quad \forall\phi_T \in L^2(\Omega),$$

$J^\star$ is coercive on H. The HUM control is given by $v = \phi\mathcal{X}_\omega$ on Q_T.

Ill-posedness

- The completed space H is huge:

$$H^{-s} \subset H \quad \forall s > 0!$$

(H may also contain elements which are not distribution !!):
Micu [11] proved in 1D that

the set of initial data y_0, for which the corresponding ϕ_T, minimizer of $J^\star$, does not belong to any negative Sobolev spaces, is dense in $L^2(0,1)$!!!

-The dual variable ϕ_T is the Lagrange multiplier for the constraint $y(\cdot,T)=0$ may belong to a "large" dual space, much larger than $L^2(\Omega)$:

$$< y(\cdot,T), \phi_T >= 0$$

-Ill-posedness here is therefore related to the hugeness of H, poorly approximated numerically.

-This phenomenon is unavoidable (unless $\omega=\Omega$!) and is independent of the choice of the norm !

[11] S. Micu, *Regularity issues for the null-controllability of the linear 1-d heat equation, 2011*

Optimal backward solution ϕ on $\partial\omega \times [0, T]$

$$T = 1, \quad y_0(x) = \sin(\pi x), \quad a(x) = a_0 = 1/10, \quad \omega = (0.2, 0.8)$$

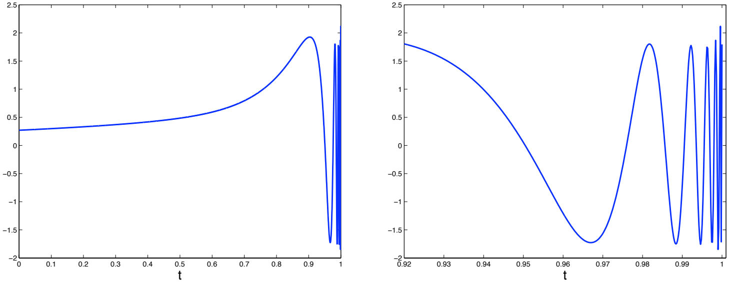

Figure: $T = 1$ - $\omega = (0.2, 0.8)$ - $\phi^N(\cdot, 0.8)$ for $N = 80$ on $[0, T]$ (**Left**) and on $[0.92T, T]$ (**Right**).

[Carthel-Glowinski-Lions'94, JOTA], [AM-Zuazua'11, Inverse Problems]

Remedies : Carleman weights !!

Change of the norm : framework of Fursikov-Imanuvilov'96 [12]

$$\begin{cases} \text{Minimize } \; J(y,v) = \dfrac{1}{2}\displaystyle\iint_{Q_T} \rho^2 |y|^2 \, dx\, dt + \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^2 |v|^2 \, dx\, dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0, T). \end{cases} \tag{55}$$

where ρ, ρ_0 are non-negative continuous weights functions such that $\rho, \rho_0 \in L^\infty(Q_{T-\delta}) \quad \forall \delta > 0$.

[12] A.V. Fursikov and O. Yu. Imanuvilov, *Controllability of Evolution Equations,* Lecture Notes Series, number 34. Seoul National University, Korea, (1996) 1–163.

Primal (direct) approach

Following Fursikov-Imanuvilov'96, we assume Carleman type weights :

$$\left\{\begin{array}{l}\rho(x,t)=\exp\left(\dfrac{\beta(x)}{T-t}\right),\quad \rho_0(x,t)=(T-t)^{3/2}\rho(x,t),\quad \beta(x)=K_1\left(e^{K_2}-e^{\beta_0(x)}\right)\\ \text{where the } K_i \text{ are sufficiently large positive constants (depending on } T,\ c_0 \text{ and } \|c\|_{C^1})\\ \text{and } \beta_0\in C^\infty(\overline{\Omega}),\ \beta_0>0 \text{ in } \Omega,\ (\beta_0)_{|\partial\Omega}=0,\ |\nabla\beta_0|>0 \text{ outside } \omega.\end{array}\right. \tag{56}$$

We introduce

$$P_0=\{\,q\in C^2(\overline{Q}_T):q=0\ \text{ on }\ \Sigma_T\,\}.$$

In this linear space, the bilinear form

$$(p,q)_P:=\iint_{Q_T}\rho^{-2}L^*p\,L^*q\,dx\,dt+\iint_{q_T}\rho_0^{-2}p\,q\,dx\,dt$$

is a scalar product (unique continuation property).

Let P be the completion of P_0 for this scalar product.

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We introduce

$$P_0 = \{\, q \in C^2(\overline{Q}_T) : q = 0 \ \text{ on } \ \Sigma_T \,\}.$$

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$$(p,q)_P := \iint_{Q_T} \rho^{-2} L^* p\, L^* q\, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q\, dx\, dt$$

is a scalar product (unique continuation property).

Let P be the completion of P_0 for this scalar product.

Carleman estimates

Lemma (Fursikov-Imanuvilov'96, Fernández-Cara-Guerrero'06)

Let ρ *and* ρ_0 *be given by* (56). *For any* $\delta > 0$*, one has*

$$P \hookrightarrow C^0([0, T-\delta]; H_0^1(\Omega)),$$

where the embedding is continuous. In particular, there exists $C > 0$*, only depending on* ω*,* T*,* a_0 *and* $\|a\|_{C^1}$*, such that, for all* $q \in P$*,*

$$\|q(\cdot\,,0)\|^2_{H_0^1(\Omega)} \leq C\left(\iint_{Q_T} \rho^{-2}|L^*q|^2\,dx\,dt + \iint_{q_T} \rho_0^{-2}|q|^2\,dx\,dt\right). \qquad (57)$$

Primal (direct) approach

Proposition

Let ρ and ρ_0 be given by (56). *Let* (y, v) *be the corresponding optimal pair for* J. *Then there exists* $p \in P$ *such that*

$$y = \rho^{-2} L^* p, \quad v = -\rho_0^{-2} p|_{q_T} . \tag{58}$$

The function p is the unique solution in P of

$$\iint_{Q_T} \rho^{-2} L^* p \, L^* q \, dx \, dt + \iint_{q_T} \rho_0^{-2} p \, q \, dx \, dt = \int_0^1 y_0(x) \, q(x,0) \, dx, \quad \forall q \in P \tag{59}$$

Remark

p solves, at least in $\mathcal{D}'$, the following differential problem, that is second order in time and fourth order in space:

$$\begin{cases} L(\rho^{-2} L^* p) + \rho_0^{-2} p \, 1_\omega = 0, & (x,t) \in (0,1) \times (0,T) \\ p(x,t) = 0, \ \ (-\rho^{-2} L^* p)(x,t) = 0 & (x,t) \in \{0,1\} \times (0,T) \\ (-\rho^{-2} L^* p)(x,0) = y_0(x), \ \ (-\rho^{-2} L^* p)(x,T) = 0, & x \in (0,1). \end{cases} \tag{60}$$

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Conformal approximation

For large integers N_x and N_t, we set $\Delta x = 1/N_x$, $\Delta t = T/N_t$ and $h = (\Delta x, \Delta t)$. Let us introduce the associated uniform triangulation $\mathcal{T}_h$, with

$$\overline{Q_T} = \bigcup_{K \in \mathcal{T}_h} K.$$

The following (conformal) finite element approximations of the space P are introduced:

$$P_h = \{ \, q_h \in C^{1,0}_{x,t}(\overline{Q}_T) : q_h|_K \in (\mathbb{P}_{3,x} \otimes \mathbb{P}_{1,t})(K) \;\; \forall K \in \mathcal{T}_h, \;\; q_h|_{\Sigma_T} \equiv 0 \, \},$$

where $C^{1,0}_{x,t}(\overline{Q}_T)$ is the space of the functions in $C^0(\overline{Q}_T)$ that are continuously differentiable with respect to x in $\overline{Q}_T$.

The variational equality (59) is approximated as follows:

$$\begin{cases} \displaystyle\iint_{Q_T} \rho^{-2} L^* p_h \, L^* q_h \, dx \, dt + \iint_{q_T} \rho_0^{-2} p_h \, q_h \, dx \, dt = \int_0^1 y_0(x) \, q_h(x,0) \, dx \\ \forall q_h \in P_h; \quad p_h \in P_h. \end{cases} \tag{61}$$

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Conformal approximation

Theorem (Fernandez-Cara, AM)

Let $p_h \in P_h$ be the unique solution to (62). Let us set

$$y_h := \rho^{-2} L^\star p_h, \quad v_h := -\rho_0^{-2} p_h \, 1_{q_T}.$$

Then one has

$$\|y - y_h\|_{L^2(Q_T)} \to 0 \text{ and } \|v - v_h\|_{L^2(q_T)} \to 0, \qquad as \quad h \to 0$$

where (y, v) is the minimizer of J.

In practice, we introduce the variable $m_h := \rho_0^{-1} p_h \in \rho_0^{-1} P_h \subset \rho_0^{-1} P \subset C([0,T], H_0^1(\Omega))$ and we solve

$$\begin{cases} \iint_{Q_T} \rho^{-2} L^*(\rho_0 m_h)\, L^*(\rho_0 \overline{m_h})\, dx\, dt + \iint_{q_T} m_h\, \overline{m_h}\, dx\, dt = \int_0^1 y_0\, \rho_0(\cdot,0) \overline{m_h}(\cdot,0)\, dx \\ \forall m_h \in \rho_0^{-1} P_h; \quad \overline{m_h} \in \rho_0^{-1} P_h. \end{cases} \tag{62}$$

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Experiment with $\omega = (0.2, 0.8)$

$\Delta x = \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$	$1/320$
conditioning	1.33×10^{14}	1.76×10^{22}	7.86×10^{32}	2.17×10^{44}	2.30×10^{54}
$\Vert p_h(\cdot, T)\Vert_{L^2(0,1)}$	2.85×10^{1}	2.04×10^{2}	1.59×10^{3}	4.70×10^{4}	6.12×10^{6}
$\Vert y_h(\cdot, T)\Vert_{L^2(0,1)}$	4.37×10^{-2}	2.18×10^{-2}	1.09×10^{-2}	5.44×10^{-3}	2.71×10^{-3}
$\Vert v_h\Vert_{L^2(q_T)}$	1.228	1.251	1.269	1.281	1.288

Table: $T = 1/2$, $y_0(x) \equiv \sin(\pi x)$, $a(x) \equiv 10^{-1}$. $\Vert y_h(\cdot, T)\Vert_{L^2(0,1)} = \mathcal{O}(h)$.

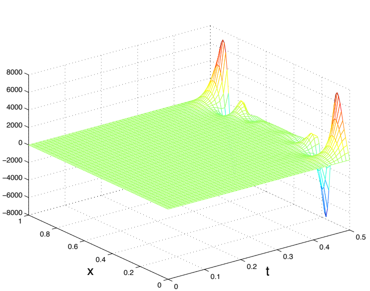


Figure: $\omega = (0.2, 0.8)$. The adjoint state p_h and its restriction to $(0, 1) \times \{T\}$.

Experiments with $\omega = (0.2, 0.8)$

$$T = 1/2,\ y_0(x) \equiv \sin(\pi x),\ a(x) \equiv 10^{-1}.$$

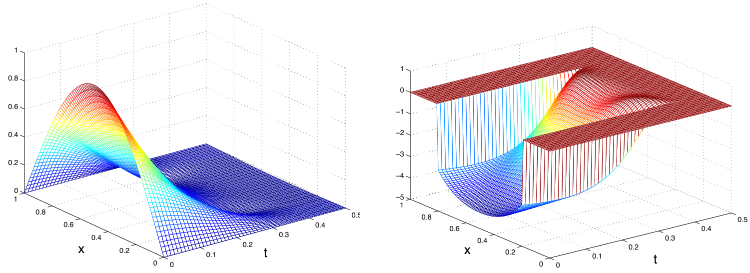


Figure: $\omega = (0.2, 0.8)$. The state y_h (**Left**) and the control v_h (**Right**).

Experiments with $\omega = (0.3, 0.4)$

$\Delta x = \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$	$1/320$
conditioning	3.06×10^{14}	5.24×10^{22}	2.13×10^{33}	5.11×10^{44}	4.03×10^{54}
$\|p_h(\cdot, T)\|_{L^2(0,1)}$	1.37×10^{3}	5.51×10^{3}	5.12×10^{4}	2.16×10^{6}	3.90×10^{6}
$\|y_h(\cdot, T)\|_{L^2(0,1)}$	1.55×10^{-1}	9.46×10^{-2}	6.12×10^{-2}	3.91×10^{-2}	2.41×10^{-2}
$\|v_h\|_{L^2(q_T)}$	5.813	8.203	10.68	13.20	15.81

Table: $T = 1/2$, $y_0(x) \equiv \sin(\pi x)$, $a(x) \equiv 10^{-1}$. $\|y_h(\cdot, T)\|_{L^2(0,1)} = \mathcal{O}(h^{0.66})$.

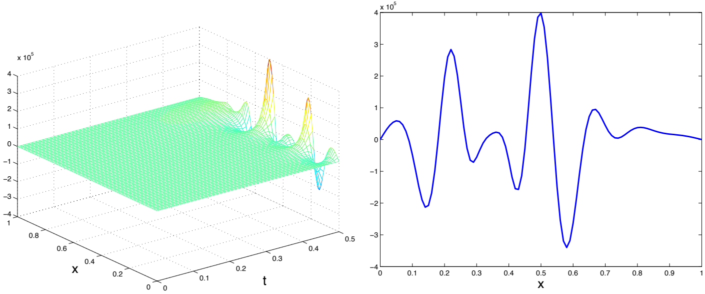
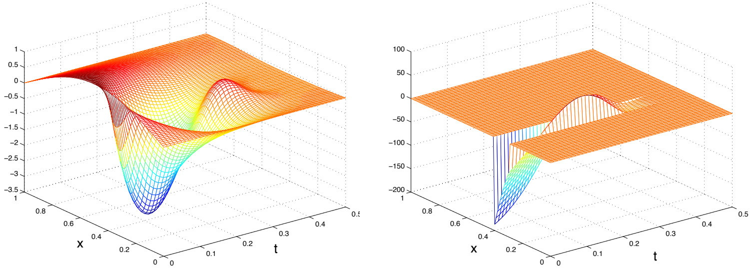


Figure: $\omega = (0.3, 0.4)$. The adjoint state p_h in Q_T (**Left**) and its restriction to $(0, 1) \times \{T\}$ (**Right**).

Experiments with $\omega = (0.3, 0.4)$



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[13] E. Fernández-Cara and AM, *Numerical null controllability of the 1-d heat equation: primal algorithms,* (2013),

[14] E. Fernández-Cara and A. Münch, *Numerical null controllability of the 1-d heat equation: Carleman weights and duality*, JOTA, (2013)

L^2-weighted norm

$$\begin{cases} \text{Minimize } \; J(y,v) = \dfrac{1}{2} \displaystyle\iint_{q_T} \rho_0^2 |v|^2 \, dx \, dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0, T). \end{cases} \tag{63}$$

where ρ_0 are non-negative continuous weights functions such that
$\rho, \rho_0 \in L^\infty(Q_{T-\delta}) \quad \forall \delta > 0.$

$$\min_{\varphi \in \widetilde{W}_{\rho_0,\rho}} \hat{J}^\star(\varphi) = \frac{1}{2} \iint_{q_T} \rho_0^{-2} |\varphi(x,t)|^2 dx \, dt + (y_0, \varphi(\cdot, 0))_{L^2(\Omega)}. \tag{64}$$

$$\widetilde{W}_{\rho_0,\rho} = \{\varphi \in \widetilde{\Phi}_{\rho_0,\rho} : \rho^{-1} L^\star \varphi = 0 \text{ in } L^2(Q_T)\}$$

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[15] D. Araujo de Souza, AM, A mixed formulation for the direct approximation of the control of minimal L^2-weighted norm for the linear heat equation. (2016)

L^2-weighted norm

$$\begin{cases} \text{Minimize } \; J(y,v) = \dfrac{1}{2} \displaystyle\iint_{q_T} \rho_0^2 |v|^2 \, dx \, dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0, T). \end{cases} \tag{63}$$

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15

[15] D. Araujo de Souza, AM, A mixed formulation for the direct approximation of the control of minimal L^2-weighted norm for the linear heat equation. (2016)

Application: Controllability for semi-linear heat equation [16]

$$\begin{cases} y_t - 0.1 y_{xx} - 5ylog^{1.4}(1+|y|) = v 1_{(0.2,0.8)}, & (x,t) \in (0,1)\times(0,1/2) \\ y(x,t) = 0, & (x,t) \in \{0,1\}\times(0,1/2) \\ y(x,0) = 40\sin(\pi x), & x \in (0,1). \end{cases} \quad (65)$$

Without control, blow up at $t \approx 0.318$.

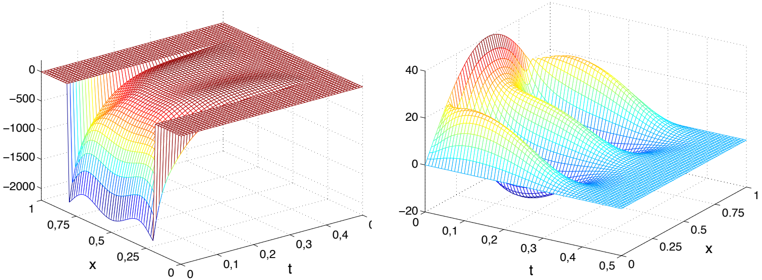


Figure: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) = 40\sin(\pi x)$ - Control v_h (**Left**) and corresponding controlled solution y_h (**Right**) in Q_T.

[16] E. Fernández-Cara and AM, *Numerical null controllability of semi-linear 1D heat equations : fixed point, least squares and Newton methods*, (2012)

We define the non-empty set [17]

$$\mathcal{A} = \Big\{(y, v);\ y \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)); y' \in L^2(0, T, H^{-1}(\Omega)),$$
$$y(\cdot, 0) = y_0, y(\cdot, T) = 0, v \in L^2(q_T)\Big\}$$

and find $(y, v) \in \mathcal{A}$ solution of the heat eq. !

For any $(y, v) \in \mathcal{A}$, we define the "corrector" $c = c(y, v) \in H^1(Q_T)$ solution of the Q_T-elliptic problem

$$\begin{cases} -c_{tt} - \nabla \cdot (a(x)\nabla c) + (Ly - v\,1_\omega) = 0, & (x, t) \in Q_T, \\ c_t = 0, & x \in \Omega, t \in \{0, T\} \\ c = 0, & x \in \Sigma_T. \end{cases} \tag{66}$$

[17] AM, P. Pedregal, Numerical null controllability of the heat equation through a least squares and variational approach. European Journal of Applied Mathematics, (2014).

A space-time Least-squares approach for controllability

We define the non-empty set [17]

$$\mathcal{A} = \Big\{ (y,v);\ y \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega));\ y' \in L^2(0,T,H^{-1}(\Omega)),$$
$$y(\cdot,0) = y_0, y(\cdot,T) = 0, v \in L^2(q_T) \Big\}$$

and find $(y,v) \in \mathcal{A}$ solution of the heat eq. !

For any $(y,v) \in \mathcal{A}$, we define the "corrector" $c = c(y,v) \in H^1(Q_T)$ solution of the Q_T-elliptic problem

$$\begin{cases} -c_{tt} - \nabla \cdot (a(x)\nabla c) + (Ly - v\,1_\omega) = 0, & (x,t) \in Q_T, \\ c_t = 0, & x \in \Omega, t \in \{0,T\} \\ c = 0, & x \in \Sigma_T. \end{cases} \tag{66}$$

[17] AM, P. Pedregal, Numerical null controllability of the heat equation through a least squares and variational approach. European Journal of Applied Mathematics, (2014).

Least-squares approach (2)

Theorem

y is a controlled solution of the heat eq. by the control function $v\,1_\omega \in L^2(q_T)$ if and only if (y, v) is a solution of the extremal problem

$$\inf_{(y,v)\in\mathcal{A}} E(y, v) := \frac{1}{2}\iint_{Q_T} (|c_t|^2 + a(x)|\nabla c|^2)dx\,dt. \qquad (67)$$

Theorem

Any minimizing sequence $\{y_k, v_k\}_{k>0}$ for E converges strongly to a minimizer (which depend on (y_0, v_0)).

The numerical analysis has yet to be done ! You are welcome !

Least-squares approach (2)

Theorem

y is a controlled solution of the heat eq. by the control function $v\,1_\omega \in L^2(q_T)$ if and only if (y, v) is a solution of the extremal problem

$$\inf_{(y,v)\in\mathcal{A}} E(y,v) := \frac{1}{2}\iint_{Q_T} (|c_t|^2 + a(x)|\nabla c|^2)dx\,dt. \tag{67}$$

Theorem

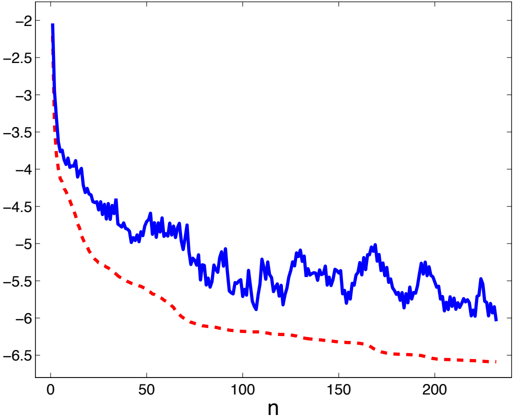
Any minimizing sequence $\{y_k, v_k\}_{k>0}$ for E converges strongly to a minimizer (which depend on (y_0, v_0)).

The numerical analysis has yet to be done ! You are welcome !

A numerical application in 1D (inner controllability)

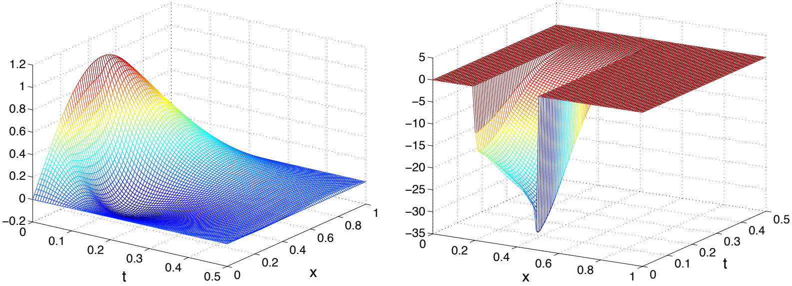
$N = 1$, $\Omega = (0,1)$, $\omega = (0.2, 0.5)$, $y_0(x) = sin(\pi x)$, $a(x) = a_0 = 0.25$, $T = 1/2$, $d := 0$

Starting point of the algorithm: $(y, f) = (y_0(x)(1 - t/T)^2, 0) \in \mathcal{A}$



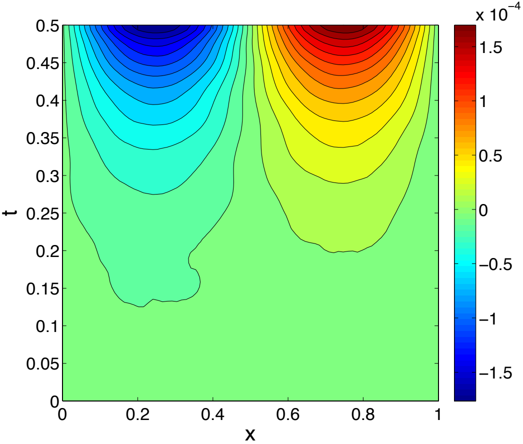
$u_0(x) = \sin(\pi x)$ - Control acting on $\omega = (0.2, 0.5)$ - $\varepsilon = 10^{-6}$ - $\log_{10}(E_h(y_h^n)$ (**dashed line**) and $\log_{10}(\|g_h^n\|_{\mathcal{A}})$ (**full line**) vs. the iteration n of the CG algorithm.

A numerical application in 1D (inner controllability)



$(y, v) \in \mathcal{A}$ along Q_T at convergence

A numerical application in 1D (inner controllability)



Isovalues along Q_T of the corresponding corrector c: $\|c\|_{H^1(Q_T)} \approx 10^{-4}$

INVERSE PROBLEM FOR HEAT - RECONSTRUCTION OF y FROM y_{q_T}

$\Omega \subset \mathbb{R}^N$ ($N \geq 1$) - $T > 0$, $c \in C^1(\overline{\Omega}, \mathbb{R})$), $d \in L^\infty(Q_T)$, $y_0 \in \boldsymbol{H}$

$$\begin{cases} Ly := y_t - \nabla \cdot (c \nabla y) + dy = f, & Q_T := \Omega \times (0, T) \\ y = 0, & \Sigma_T := \partial\Omega \times (0, T) \\ y(\cdot, 0) = y_0, & \Omega. \end{cases} \tag{68}$$

- Inverse Problem : Distributed observation on $q_T = \omega \times (0, T), \omega \subset \Omega$

$$\begin{cases} X = L^2(q_T), \\ \text{Given } (y_{obs}, f) \in (L^2(q_T), X), \text{find } y \; s.t. \{(68) \text{ and } y - y_{obs} = 0 \quad \text{on} \quad q_T\} \end{cases}$$

WELL-KNOWN DIFFICULTY:

$$\left(Ly \in L^2(Q_T), y \in L^2(q_T), y_{|\Sigma_T} = 0\right) \Longrightarrow y \in C([\delta, T], H_0^1(\Omega)), \quad \forall \delta > 0$$

18

[18] D. Araujo de Souza, AM, Inverse problems for linear parabolic equations using mixed formulations - Part 1 : Theoretical analysis. (2016)

Second order mixed formulation .... as in the previous part

We then define the following extremal problem :

$$\begin{cases} \text{Minimize } \; J(y) := \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^{-2}|y(x,t) - y_{obs}(x,t)|^2\,dx\,dt + r\iint_{Q_T}(\rho^{-1}L\,y)^2\,dx\,dt \\ \text{Subject to } \; y \in \mathcal{W} := \left\{ y \in \mathcal{Y} : \rho^{-1}Ly = 0 \text{ in } L^2(Q_T)\right\} \end{cases} \tag{P}$$

with $\rho_0, \rho \in \mathcal{R}$ where ($\rho_\star \in \mathbb{R}_\star^+$)

$$\mathcal{R} := \{w : w \in C(Q_T); w \geq \rho_\star > 0 \text{ in } Q_T; w \in L^\infty(\Omega \times (\delta, T)) \; \forall \delta > 0\}$$

$H_0^1 - L^2$ first order formulation

First order formulation involving y and the flux $\mathbf{p} = c(x)\nabla y$.

$$\begin{cases} \mathcal{I}(y,\mathbf{p}) := y_t - \nabla\cdot\mathbf{p} + d\,y = f, \quad \mathcal{J}(y,\mathbf{p}) := c(x)\nabla y - \mathbf{p} = \mathbf{0} & \text{in} \quad Q_T, \\ y = 0 & \text{on} \quad \Sigma_T, \\ y(x,0) = y_0(x) & \text{in} \quad \Omega. \end{cases} \tag{69}$$

$$(y_0, f) \in L^2(\Omega)\times L^2(Q_T) \Longrightarrow p \in \mathbf{L}^2(Q_T), y \in L^2(0,T,H_0^1(\Omega)), y_t \in L^2(0,T,H^{-1}(\Omega))$$

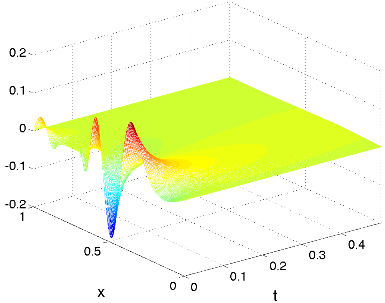
- Inverse Problem : Distributed observation on $q_T = \omega\times(0,T), \omega\subset\Omega$

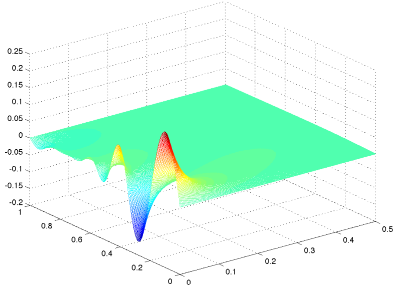
$$\begin{cases} X = L^2(q_T), \\ \text{Given } (y_{obs}, f) \in (L^2(q_T), X), \text{ find } (y,\mathbf{p})\ s.t.\ \{(69) \text{ and } y - y_{obs} = 0 \quad \text{on} \quad q_T\} \end{cases}$$

$N = 1$ - Heat eq. Comparison with the standard method

$$y_0(x) = \sin(\pi x)^{20}, \quad Q_T = (0,1)\times(0,T), \quad q_T = (0.7, 0.8)\times(0,T), \quad T = 1/2$$

$$\min_{y_{0h}} \left(J_h(y_{0h}) + \frac{h^2}{2}\|y_{0h}\|^2_{L^2(\Omega)} \right) \quad \textit{vs.} \quad \min_{\lambda_h} J_r^{\star\star}(\lambda_h) \quad \text{over} \quad \Lambda_h \tag{70}$$

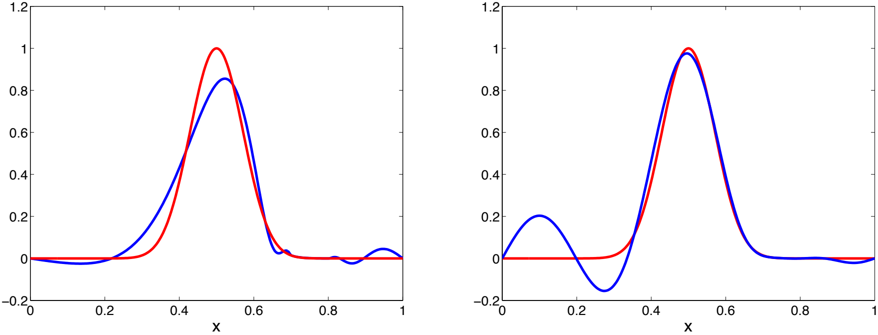




$$\frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} \approx 5.86 \times 10^{-2}, \qquad \frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} \approx 7.70 \times 10^{-2}$$

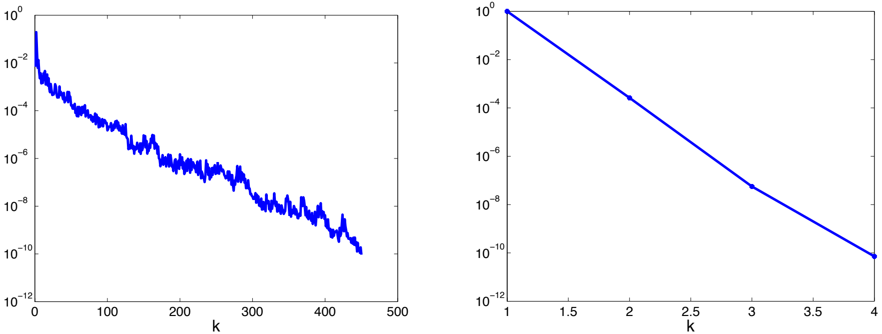
$N = 1$ - Comparison with the standard method

$$y_0(x) = \sin(\pi x)^{20}, \quad Q_T = (0,1) \times (0,T), \quad q_T = (0.7, 0.8) \times (0,T), \quad T = 1/2$$



Restriction at $(0,1) \times \{0\}$

$N = 1$ - Comparison with the standard method



Evolution of the relative residu $\frac{\|g^k\|}{\|g^0\|}$ w.r.t. iterate k

Final comments

THE VARIATIONAL APPROACH CAN BE USED IN THE CONTEXT OF MANY OTHER CONTROLLABLE SYSTEMS FOR WHICH APPROPRIATE CARLEMAN ESTIMATES ARE AVAILABLE.

THE APPROXIMATION IS ROBUST (WE HAVE TO INVERSE SYMMETRIC DEFINITE POSITIVE AND VERY SPARSE MATRICE WITH DIRECT LU AND CHOLESKY SOLVERS)

WITH CONFORMAL TIME-SPACE FINITE ELEMENTS APPROXIMATIONS, WE OBTAIN STRONG CONVERGENCE RESULTS WITH RESPECT TO $h = (\Delta x, \Delta t)$.

THE PRICE TO PAY IS TO USED C^1 FINITE ELEMENTS (AT LEAST IN SPACE) ..... BUT WE MAY INTRODUCE LOWER ORDER SYSTEM.

IN THAT SPACE-TIME APPROACH, THE SUPPORT OF THE CONTROL MAY VARIES IN TIME (WITHOUT ADDITIONAL DIFFICULTIES).

THIS APPROACH MAY BE APPLIED FOR INVERSE PROBLEMS, OBSERVATION PROBLEMS, RECONSTRUCTION OF DATA, ...

MESH ADAPTIVITY MAY BE VERY USEFUL, IN PARTICULAR IN THE PARABOLIC SITUATION

Ongoing works

- Extension to sparse control (L^1 term in the cost)
- Average controllability [19]
- Approximation of observability constant (to infer or not observability property)

$$C_{obs}(T,\omega) = \sup_{\varphi_T \in H_0^1(\Omega)} \frac{\|\varphi(\cdot,0)\|^2_{L^2(\Omega)}}{\|\varphi\|^2_{L^2(\omega\times(0,T))}}, \quad \text{where} \quad L^\star\varphi = 0 \qquad (71)$$

in particular for the VERY SINGULAR case of the transport-diffusion equation

$$\begin{cases} y_t - \epsilon y_{xx} + y_x = 0, & Q_T := (0,1)\times(0,T), \\ y(0,t) = v_\epsilon(t), y(1,t) = 0, \\ y(x,0) = y_0(x) \in L^2(0,1) \end{cases} \qquad (72)$$

as $\epsilon \to 0^+$.

[19] Martinez-Frutos, Kessler, M. Periago, Robust optimal Robin boundary control for the transient heat equation with random input data, (2016).

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