About the controllability of $y_t - \varepsilon y_{xx} + M y_x = 0$ w.r.t. ε: Asymptotic and Numeric

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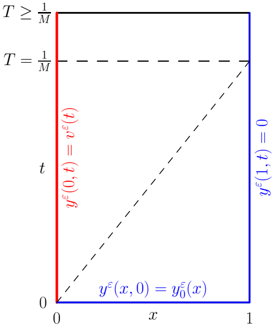
Oberwolfach - April 2018

Introduction - The advection-diffusion equation



Let $T > 0$, $M \in \mathbb{R}$, $\varepsilon > 0$ and $Q_T := (0,1) \times (0,T)$.

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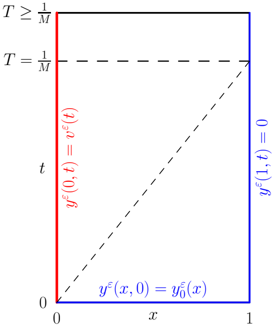
- Well-posednness:

$$\forall y_0^\varepsilon \in H^{-1}(0,1), v^\varepsilon \in L^2(0,T), \quad \exists!\; y^\varepsilon \in L^2(Q_T) \cap \mathcal{C}([0,T]; H^{-1}(0,1))$$

- Null control property: From D.L.Russel'78,

$$\forall T > 0, y_0^\varepsilon \in H^{-1}(0,1), \exists v^\varepsilon \in L^2(0,T) \quad \text{s.t.} \quad y^\varepsilon(\cdot,T) = 0 \quad \text{in } H^{-1}(0,1)$$

- Main concern: Behavior of the controls v^ε as $\varepsilon \to 0$
  - Controllability of conservation law system;
  - Toy model for fluids when Navier-Stokes $\to$ Euler.



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- We note the non empty set of null controls by

$$\mathcal{C}(y_0^\varepsilon, T, \varepsilon, M) := \Big\{ v \in L^2(0,T);\, y = y(v) \text{ solves } (5) \text{ and satisfies } y(\cdot, T) = 0 \Big\}$$

and define, for any $\varepsilon > 0$, the cost of control by the following quantity :

$$K(\varepsilon, T, M) := \sup_{\|y_0^\varepsilon\|_{L^2(0,1)}=1} \left\{ \min_{v \in \mathcal{C}(y_0^\varepsilon, T, \varepsilon, M)} \|v\|_{L^2(0,T)} \right\}.$$

$K(\varepsilon, T, M)$ is the norm of the (linear) operator $y_0^\varepsilon \to v_{HUM}$ where v_{HUM} is the control of minimal L^2-norm.

- We denote by T_M the minimal time for which the cost $K(\varepsilon, T, M)$ is uniformly bounded with respect to ε. In other words, (5) is uniformly controllable with respect to ε if and only if $T \geq T_M$.

- Remark- $K(\varepsilon, T, 0) \sim_{\varepsilon \to 0^+} \varepsilon^{-1/2} e^{\frac{\kappa}{\varepsilon T}}$, $\kappa \in (1/2, 3/4)$ so that $T_0 = \infty$.
We assume $M \neq 0$.

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Main objective : Determine the behavior of the cost $K(\varepsilon, T, M)$ as $\varepsilon \to 0$!!??

Outline :

- Part 1: Facts on the diffusion-advection eq. and literature.
- Part 2: Numerical attempt to estimate $K(\varepsilon, T, M)$.
- Part 3: Asymptotic analysis of the corresponding optimality system

Remark

- *By duality, the controllability property of (5) is related to the existence of a constant* $C > 0$ *such that*

$$\|\varphi(\cdot,0)\|_{L^2(0,1)} \leq C\|\varepsilon\varphi_x(0,\cdot)\|_{L^2(0,T)}, \quad \forall \varphi_T \in H_0^1(0,1) \cap H^2(0,1) \tag{2}$$

where φ *solves the adjoint system*

$$\begin{cases} L_\varepsilon^\star \varphi := \varphi_t + \varepsilon\varphi_{xx} + M\varphi_x = 0 & \text{in} & Q_T, \\ \varphi(0,\cdot) = \varphi(1,\cdot) = 0 & \text{on} & (0,T), \\ \varphi(\cdot,T) = \varphi_T & \text{in} & (0,1). \end{cases}$$

- *The quantity*

$$C_{obs}(\varepsilon,T,M) = \sup_{\varphi_T \in H_0^1(0,1)} \frac{\|\varphi(\cdot,0)\|_{L^2(0,1)}}{\|\varepsilon\varphi_x(0,\cdot)\|_{L^2(0,T)}}.$$

is the smallest constant for which (2) holds true and

$$K(\varepsilon,T,M) = C_{obs}(\varepsilon,T,M).$$

Theorem (Coron- Guerrero, 2005)

Let $T > 0$, $M \in \mathbb{R}^\star$, $y_0 \in L^2(0,1)$ independent of ε. Let $(v^\varepsilon)_{(\varepsilon)}$ be a sequence of functions in $L^2(0,T)$ such that for some $v \in L^2(0,T)$

$$v^\varepsilon \rightharpoonup v \quad in \quad L^2(0,T), \quad as \quad \varepsilon \to 0^+.$$

For $\varepsilon > 0$, let us denote by $y^\varepsilon \in C([0,T];H^{-1}(0,1))$ the weak solution of

$$\begin{cases} y^\varepsilon_t - \varepsilon y^\varepsilon_{xx} + M y^\varepsilon_x = 0 & Q_T, \\ y^\varepsilon(0,\cdot) = v^\varepsilon(t), \; y^\varepsilon(1,\cdot) = 0 & (0,T), \\ y^\varepsilon(\cdot,0) = y_0 & (0,1). \end{cases} \tag{3}$$

Let $y \in C([0,T];L^2(0,1))$ be the weak solution of

$$\begin{cases} y_t + M y_x = 0 & Q_T, \\ y(0,\cdot) = v(t) \quad if \quad M > 0 & (0,T), \\ y(1,\cdot) = 0 \quad if \quad M < 0 & (0,T). \\ y(\cdot,0) = y_0 & (0,1), \end{cases} \tag{4}$$

Then, $y^\varepsilon \rightharpoonup y \quad in \quad L^2(Q_T) \quad as \quad \varepsilon \to 0^+$.

Corollary

If $T < \frac{1}{|M|}$, $\lim_{\varepsilon\to 0} K(\varepsilon, T, M) \to \infty$. *Consequently,* $T_M \geq \frac{1}{|M|}$.

PROOF. Assume that $K(\varepsilon, T, M) \not\to +\infty$. There exists $(\varepsilon_n)_{(n\in\mathbb{N})}$ positive tending to 0 such that $(K(\varepsilon_n, T, M))_{(n\in\mathbb{N})}$ is bounded.

Let v^{ε_n} the optimal control driving y_0 to 0 at time T and y^{ε_n} the corresponding solution. Let $T_0 \in (T, 1/|M|)$. We extend y^{ε_n} and v^{ε_n} by 0 on (T, T_0). From the inequality

$$\|v^{\varepsilon_n}\|_{L^2(0,T_0)} = \|v^{\varepsilon_n}\|_{L^2(0,T)} \leq K(\varepsilon^n, T, M)\|y_0\|_{L^2(0,1)},$$

we deduce that $(v^{\varepsilon_n})_{(n\in\mathbb{N})}$ is bounded in $L^2(0, T_0)$, so we extract a subsequence $(v^{\varepsilon_n})_{(n\in\mathbb{N})}$ such that $v^{\varepsilon_n} \rightharpoonup v$ in $L^2(0, T_0)$. We deduce that $y^{\varepsilon_n} \rightharpoonup y$ in $L^2(Q_{T_0})$ solution of the transport equation. Necessarily, $y \equiv 0$ on $(0, 1) \times (T, T_0)$. Contradiction.

Lower bounds for T_M

We expect $T_M = \frac{1}{|M|}$ and that $\lim_{\varepsilon\to 0} K(\varepsilon, T, M) = 0^+$ because the transport eq. is null controlled at time $T \geq \frac{1}{|M|}$ with $v \equiv 0$!

Theorem (Coron-Guerrero'2005)

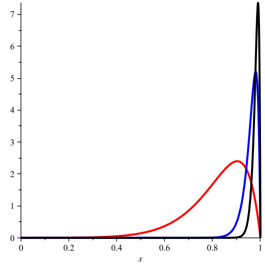
- *If $M > 0$, then $K(\varepsilon, T, M) \geq Ce^{c/\varepsilon}$, $c, C > 0$, when $\varepsilon \to 0$ for $T < \frac{1}{M}$.*

The lower bound are obtained using specific initial condition:

$$y_0(x) = K_\varepsilon e^{\frac{Mx}{2\varepsilon}} \sin(\pi x),$$
$$K_\varepsilon = \mathcal{O}(\varepsilon^{-3/2} e^{\frac{-M}{2\varepsilon}}) \quad \text{s.t.} \quad \|y_0\|_{L^2(0,1)} = 1$$

leading, for $M > 0$, to

$$K(\varepsilon, T, M) \geq C_1 \frac{\varepsilon^{-3/2} T^{-1/2} M^2}{1 + M^3 \varepsilon^{-3}} \exp\left(\frac{M}{2\varepsilon}(1 - TM) - \pi^2 \varepsilon T\right)$$



y_0 for $\varepsilon = 5 \times 10^{-2}$, $\varepsilon = 10^{-2}$, $\varepsilon = 5 \times 10^{-3}$.

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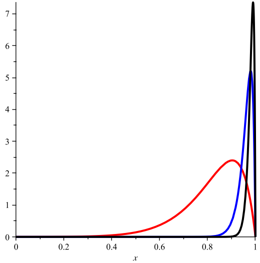

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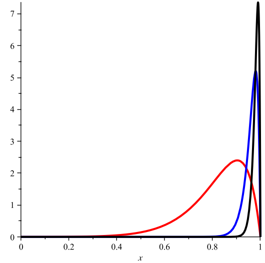

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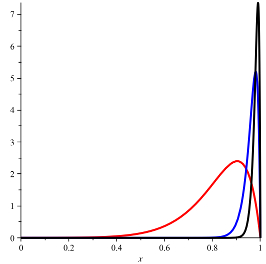

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Lower bounds for T_M, $M < 0$.

Theorem (Coron-Guerrero'2005)

- *If $M < 0$, then $K(\varepsilon, T, M) \geq Ce^{c/\varepsilon}$, $c, C > 0$, when $\varepsilon \to 0$ for $T < \frac{2}{|M|}$.*

With again $y_0(x) = K_\varepsilon e^{\frac{Mx}{2\varepsilon}} \sin(\pi x)$,

$$K(\varepsilon, T, M) \geq C_1 \frac{\varepsilon^{-3/2} T^{-1/2} M^2}{1 + |M|^3 \varepsilon^{-3}} \exp\left(\frac{|M|}{2\varepsilon}(2 - T|M|) - \pi^2 \varepsilon T \right)$$

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Lemma

The free solution (i.e. $v^\varepsilon = 0$) satisfies

$$\|y^\varepsilon(\cdot,t)\|_{L^2(0,1)} \le \|y^\varepsilon(\cdot,0)\|_{L^2(0,1)} e^{-\frac{M^2}{4\varepsilon}\left(t-\frac{1}{|M|}\right)^2}, \quad \forall t > \frac{1}{|M|}.$$

PROOF. Let $\alpha > 0$. We check $z^\varepsilon(x,t) = e^{\frac{-M\alpha x}{2\varepsilon}} y^\varepsilon(x,t)$ solves

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Consequently

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and the result with $\alpha = t - \frac{1}{M} > 0$.

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Theorem (Coron-Guerrero'2005)

- *If* $M > 0$*, then* $K(\varepsilon, T, M) \leq Ce^{-c/\varepsilon}$ *when* $\varepsilon \to 0$ *for* $T \geq \frac{4.3}{M}$.
- *If* $M < 0$*, then* $K(\varepsilon, T, M) \leq Ce^{-c/\varepsilon}$ *when* $\varepsilon \to 0$ *for* $T \geq \frac{57.2}{|M|}$.

Estimates for T_M (Upper and lower bounds)

Theorem (Coron-Guerrero'2005)

$$T_M \in [1, 4.3]\frac{1}{M} \quad \textit{if} \quad M > 0, \qquad [2, 57.2]\frac{1}{|M|} \quad \textit{if} \quad M < 0.$$

Theorem (Glass'2009)

$$T_M \in [1, 4.2]\frac{1}{M} \quad \textit{if} \quad M > 0, \qquad [2, 6.1]\frac{1}{|M|} \quad \textit{if} \quad M < 0.$$

Theorem (Lissy'2015)

$$T_M \in [1, 2\sqrt{3}]\frac{1}{M} \quad \textit{if} \quad M > 0, \qquad [2\sqrt{2}, 2(1+\sqrt{3})]\frac{1}{|M|} \quad \textit{if} \quad M < 0.$$

$(2\sqrt{3} \approx 3.46)$

Theorem (Darde-Ervedoza'2017)

$$T_M \in [1, K]\frac{1}{M} \quad \textit{if} \quad M > 0, K \approx 3.34$$

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Estimates for T_M (Upper and lower bounds)

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Numerical estimate of the cost $K(\varepsilon, T, M)$ w.r.t. ε !??

$$K^2(\varepsilon, T, M) = \sup_{y_0 \in L^2(0,1)} \frac{(\mathcal{A}_\varepsilon y_0, y_0)_{L^2(0,1)}}{(y_0, y_0)_{L^2(0,1)}}$$

where $\mathcal{A}_\varepsilon : L^2(0,1) \to L^2(0,1)$ is the control operator defined by $\mathcal{A}_\varepsilon y_0 := -\hat{\varphi}(0)$ where $\hat{\varphi}$ solves the adjoint system

$$\begin{cases} L^\star_\varepsilon \varphi := \varphi_t + \varepsilon\varphi_{xx} + M\varphi_x = 0 & \text{in} \quad Q_T, \\ \varphi(0,\cdot) = \varphi(1,\cdot) = 0 & \text{on} \quad (0,T), \\ \varphi(\cdot, T) = \varphi_T & \text{in} \quad (0,1), \end{cases} \tag{6}$$

associated to the initial condition $\varphi_T \in H^1_0(0,1)$, solution of the extremal problem

$$\inf_{\varphi_T \in H^1_0(0,1)} J^\star(\varphi_T) := \frac{1}{2}\|\varepsilon\varphi_x(0,\cdot)\|^2_{L^2(0,T)} + (y_0, \varphi(\cdot,0))_{L^2(0,1)}.$$

REFORMULATION - $K(\varepsilon, T, M)$ is solution of the generalized eigenvalue problem :

$$\sup\left\{\sqrt{\lambda} \in \mathbb{R} : \exists\, y_0 \in L^2(0,1), y_0 \neq 0, \text{ s.t. } \mathcal{A}_\varepsilon y_0 = \lambda y_0 \quad \text{in} \quad L^2(0,1)\right\}.$$

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The generalized eigenvalue problem by the power iterated method

In order to get the largest eigenvalue of the operator $\mathcal{A}_\varepsilon$, we may employ the power iterate method (Chatelain'89):

$$\begin{cases} y_0^0 \in L^2(0,1) \quad \text{given such that} \quad \|y_0^0\|_{L^2(0,1)} = 1, \\ \tilde{y}_0^{k+1} = \mathcal{A}_\varepsilon y_0^k, \quad k \geq 0, \\ y_0^{k+1} = \dfrac{\tilde{y}_0^{k+1}}{\|\tilde{y}_0^{k+1}\|_{L^2(0,1)}}, \quad k \geq 0. \end{cases} \tag{7}$$

The real sequence $\{\|\tilde{y}_0^k\|_{L^2(0,1)}\}_{k>0}$ converges to the eigenvalue with largest module of the operator $\mathcal{A}_\varepsilon$:

$$\sqrt{\|\tilde{y}_0^k\|_{L^2(0,1)}} \to K(\varepsilon, T, M) \quad \text{as} \quad k \to \infty. \tag{8}$$

The L^2-sequence $\{y_0^k\}_k$ then converges toward the corresponding eigenvector.

Remark -The first step requires to determine the control of minimal L^2 for (5) with initial condition y_0^k.

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Computation of the control of minimal L^2-norm

For a fixed initial data $y^0 \in L^2(0,1)$ and ε small, the numerical approximation of controls of minimal L^2-norm is a serious challenge :

- the minimization of $J^\star$ is ill-posed : the infimum φ_T lives in a huge dual space !!! this implies that the minimizer φ_T is highly oscillating at time T leading to high oscillations of the control $\varepsilon\varphi_{,x}(0,\cdot)$.
- Tychonoff like regularization

$$\inf_{\varphi_T \in H_0^1(0,1)} J_\beta^\star(\varphi_T) := J^\star(\varphi_T) + \beta\|\varphi_T\|_{H_0^1(0,1)} \longrightarrow \|y^\varepsilon(\cdot,T)\|_{H^{-1}(0,1)} \leq \beta \quad (9)$$

  is meaningless here for $T > 1/|M|$ because the uncontrolled solution $y^\varepsilon(\cdot,T)$ goes to zero with ε.
- Several boundary layers occurs for y^ε and φ^ε and requires fine discretization and adapted meshes.

We use the variational approach developed in [Fernandez-Cara-Münch, 2013], [De Souza-Münch, 2015] leading to convergent approximation with respect to the discretization parameter (ε being fixed).

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Let ρ_0, ρ continuous non negative weights function in $L^\infty([0, T-\delta])$ and $L^\infty((0,1)\times(0,T-\delta))$, $\forall \delta > 0$ and let the optimal problem

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At the finite dimensional (numerical) level, it may not be possible to satisfy the constraint $L^\star_\varepsilon\varphi^\varepsilon = 0$. A classical trick consists in discretizing first the equation then control the discrete equation. This raises the issue of the uniform discrete observability property !

Instead, we consider the minimization with respect to φ :

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The main variable is φ (instead of $\varphi(\cdot, T)$) submitted to the constraint equality $L_\varepsilon^\star \varphi = 0$; a lagrange multiplier $\lambda \in L^2(Q_T)$ is introduced and then the saddle-point problem :

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The main tool to prove the well-posedeness is a generalized observability inequality (or global Carleman inequality): there exists a constant $C > 0$ such that

$$\|\varphi(\cdot,0)\|^2_{L^2(0,1)} \leq C\Big(\|\varepsilon\rho_0^{-1}\varphi_x(0,\cdot)\|^2_{L^2(0,T)} + \|\rho^{-1}L_\varepsilon^\star\varphi\|^2_{L^2(Q_T)}\Big), \forall\varphi\in\Phi \qquad (10)$$

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Overview of a space-time variational method (2)

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and $\Lambda := \{\lambda\in C([0,T],L^2(0,T)), L_\varepsilon\lambda\in L^2(Q_T), \lambda(L,\cdot)=0\}$.

• The adjoint system is preliminary transformed into a first system

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- The adjoint system is preliminary transformed into a first system

$$L^\star_{\varepsilon,1}(\varphi,p) := \varphi_t + p_x + M\varphi_x = 0, \quad L^\star_{\varepsilon,2}(\varphi,p) := p - \varepsilon\varphi_x = 0, \quad Q_T,$$

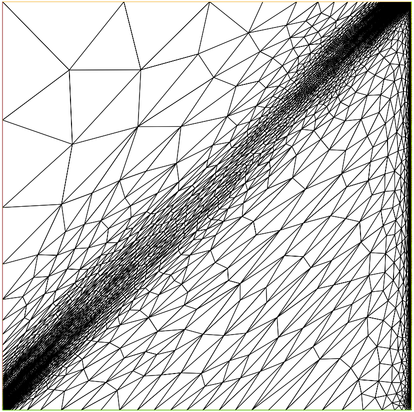
leading to the saddle-point formulation

$$\begin{cases} \sup_{(\lambda_1,\lambda_2)\in\Lambda}\inf_{(\varphi,p)\in\Phi_\beta} \mathcal{L}_{r,\alpha}((\varphi,p),(\lambda_1,\lambda_2)) := \frac{1}{2}\|p(0,\cdot)\|^2_{L^2(0,T)} + (y_0,\varphi(0,\cdot))_{L^2(0,L)} \\ \qquad + < \lambda_1, L^\star_{\varepsilon,1}\varphi >_{L^2(Q_T)} + < \lambda_2, L^\star_{\varepsilon,2}\varphi >_{L^2(Q_T)} \\ \qquad + \frac{r_1}{2}\|L^\star_{\varepsilon,1}(\varphi,p)\|^2_{L^2(Q_T)} + \frac{r_2}{2}\|L^\star_{\varepsilon,2}(\varphi,p)\|^2_{L^2(Q_T)} \\ \qquad - \frac{\alpha_1}{2}\|L_{\varepsilon,1}(\lambda_1,\lambda_2)\|^2_{L^2(Q_T)} - \frac{\alpha_2}{2}\|L_{\varepsilon,2}(\lambda_1,\lambda_2)\|^2_{L^2(Q_T)} \end{cases}$$

with $r_1, r_2 > 0$ (augmentation parameters) and α_1, α_2 (stabilization terms).

```
border bas(s=0,1){x=s; y=0;label=Ntop;};  border droit(s=0,T){x=1;y=s;label=Nright;}
border haut(s=1,0){x=s;y=T;label=Nhaut;} border gauche(s=T,0){x=0;y=s;label=Ngauche;}
mesh Th=buildmesh(bas(50)+droit(50)+haut(50)+gauche(50)) ;

fespace Vh(Th,P3); fespace Ph(Th,P3);
real eps=1.e-3, M=1, r1=1.e-6, r2=1.e-6, alpha1=5.e-2, alpha2=5.e-2;

Vh phi,p,phit,pt; Ph l1,l2,l1t,l2t;  Vh y0 = sin(pi*x)*(1-y);

problem transport([phi,p,l1,l2],[phit,pt,l1t,l2t])=
// Initial conjugate cost
 int1d(Th,Ngauche)  (eps*eps*dx(phi)*dx(phit))+int1d(Th,Nbas)(y0*phit)

 // bilinear adjoint- direct solution terms
 + int2d(Th)((dy(phi)+dx(p)+M*dx(phi))*l1t)
 + int2d(Th)((dy(phit)+dx(pt)+M*dx(phit))*l1)
 + int2d(Th)((p-eps*dx(phi))*l2t)
 + int2d(Th)((pt-eps*dx(phit))*l2)

 // Augmentation terms
 + int2d(Th)(r1*(dy(phi)+dx(p)+M*dx(phi))*  (dy(phit)+dx(pt)+M*dx(phit)))
 + int2d(Th)(r2*  (eps*dx(phi)-p)  *  (eps*dx(phit)-pt))

 // stabilized terms
 -int2d(Th)(alpha1*(dy(l1)+M*dx(l1)-eps*dx(l2))*(dy(l1t)+M*dx(l1t)-eps*dx(l2t)))
 - int2d(Th)(alpha2*(dx(l1)-l2)*(dx(l1t)-l2t))

 // boundary conditions for the adjoint and lagrange multiplier solutions
 + on(Nbas,l1=y0)+on(Ndroit,Ngauche,phi=0.)+on(Ndroit, Nhaut, l1=0.);
```

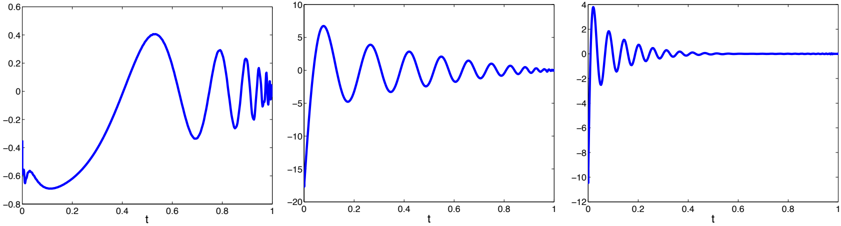
Space-time variational method well suited to mesh adaptivity



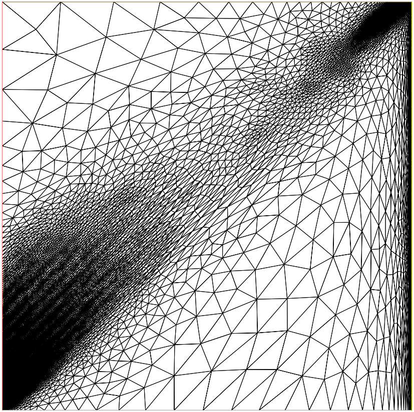
Typical structured space-time meshes used for small values of ε - $M > 0$

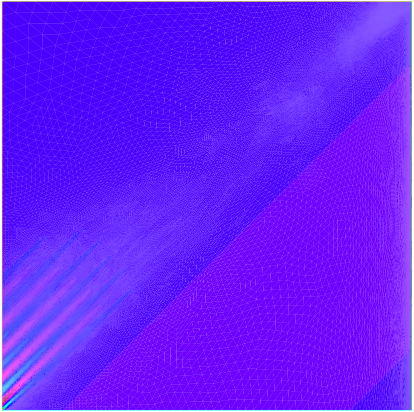
Picture of controls with respect to ε, y_0 fixed

$$y_0(x) = \sin(\pi x); \quad T = 1; \quad M = 1$$



Control of minimal $L^2(0,T)$-norm $v^\varepsilon(t) \in [0,T]$ for $\varepsilon = 10^{-1}, 10^{-2}$ and 10^{-3}.

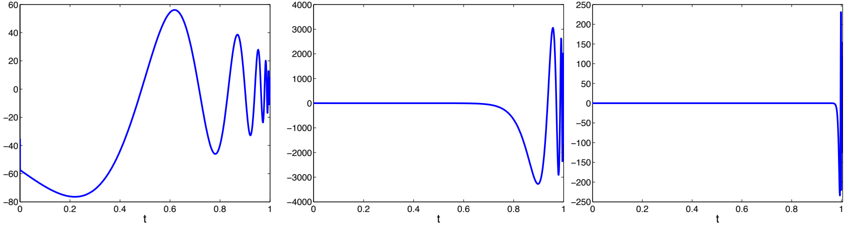




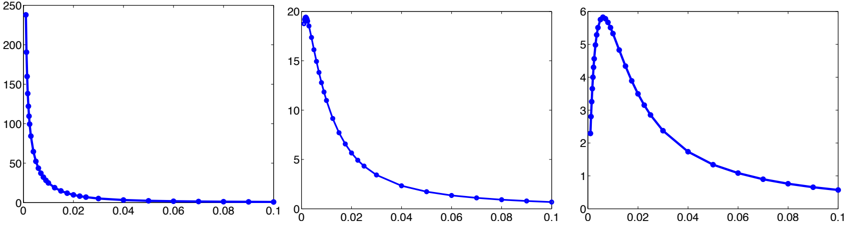
$$y_0(x) = \sin(\pi x) \text{ - } M = 1 \text{ - } \varepsilon = 10^{-3}.$$

Picture of controls with respect to ε, y_0 fixed

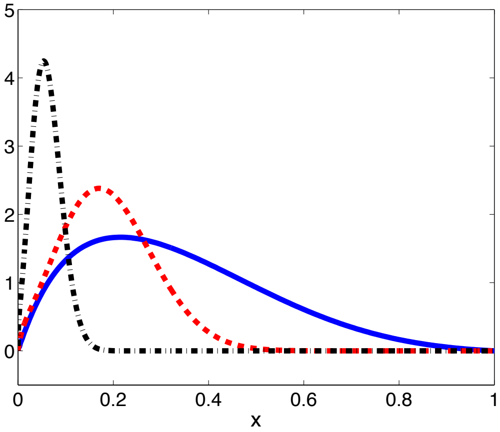
$$y_0(x) = \sin(\pi x); \quad T = 1; \quad M = -1$$



Control of minimal $L^2(0,T)$-norm $v^{\varepsilon}(t) \in [0,T]$ for $\varepsilon = 10^{-1}, 10^{-2}$ and 10^{-3}.

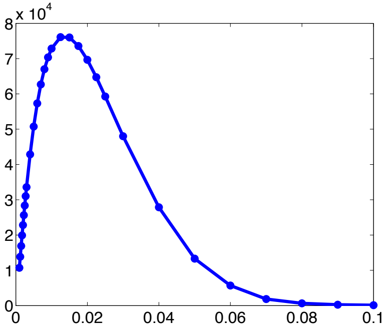


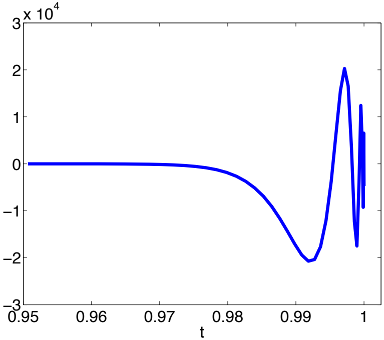
Cost of control w.r.t. $\varepsilon \in [10^{-3}, 10^{-1}]$ for $T = 0.95\frac{1}{M}$, $T = \frac{1}{M}$ and $T = 1.05\frac{1}{M}$



$T = 1$ - $M = 1$ - The optimal initial condition y_0 in $(0,1)$ for $\varepsilon = 10^{-1}$, $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$.

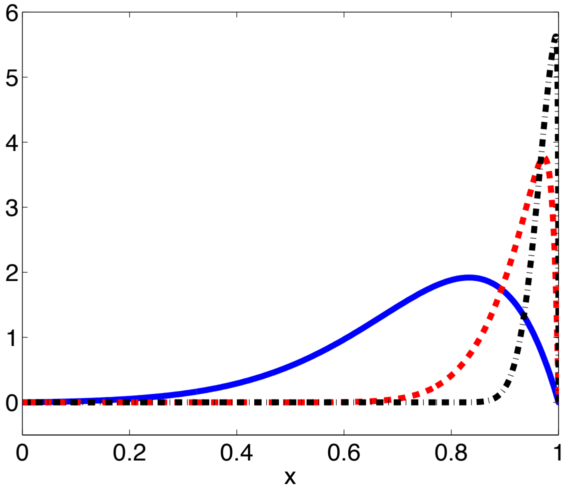
$\Longrightarrow y_0$ is close to $e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x)/\|e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x)\|_{L^2(0,1)}$





Left: Cost of control w.r.t. ε for $T = \frac{1}{|M|}$; **Right**: Corresponding control v^{ε} in the neighborhood of T for $\varepsilon = 10^{-3}$

Corresponding worst initial condition for $M = -1$



$T = 1$ - $M = -1$ - The optimal initial condition y_0 in $(0, 1)$ for $\varepsilon = 10^{-1}$, $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$.

Attempt 2 : Asymptotic analysis w.r.t. ε

We take $M > 0$.

Optimality system :

$$\begin{cases} L_\varepsilon y^\varepsilon = 0, \quad L_\varepsilon^\star \varphi^\varepsilon = 0, & (x,t) \in Q_T, \\ y^\varepsilon(\cdot,0) = y_0^\varepsilon, & x \in (0,1), \\ v^\varepsilon(t) = y^\varepsilon(0,t) = \varepsilon \varphi_x^\varepsilon(0,t), & t \in (0,T), \\ y^\varepsilon(1,t) = 0, . & t \in (0,T), \\ \varphi^\varepsilon(0,t) = \varphi^\varepsilon(1,t) = 0, & t \in (0,T), \\ -\beta(\varepsilon)\varphi_{xx}^\varepsilon(\cdot,T) + y^\varepsilon(\cdot,T) = 0, & x \in (0,1). \end{cases} \tag{11}$$

$\beta(\varepsilon) \geq 0$- Regularization parameter

J.-L. Lions *Perturbations singulières dans les problèmes aux limites et en contrôle optimal.* Lecture Notes in Mathematics. Springer 1973.

Attempt 2 : Asymptotic analysis w.r.t. ε

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$\beta(\varepsilon) \geq 0$- Regularization parameter

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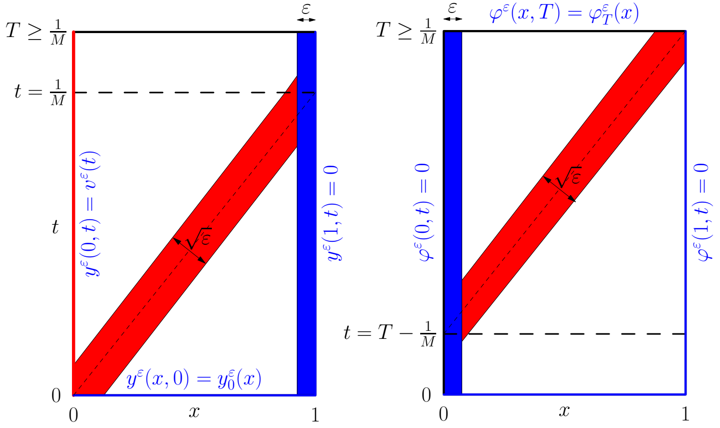
The situation is tricky because (assume $M > 0$)

- y^ε exhibits a boundary layer of size $\mathcal{O}(\varepsilon)$ at $x = 1$ and a boundary layer of size $\mathcal{O}(\sqrt{\varepsilon})$ along the characteristic $\{(x,t) \in Q_T, x - Mt = 0\}$;
- φ^ε exhibits a boundary layer of size $(\mathcal{O}(\varepsilon))$ at $x = 0$ and a boundary layer of size $(\mathcal{O}(\sqrt{\varepsilon}))$ along the characteristic $\{(x,t) \in Q_T, x - M(t-T) - 1 = 0\}$;



Boundary layers zone for y^ε (left) and φ^ε(right) in the case $M > 0$.

$$\begin{cases} y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0, & (x,t) \in (0,1) \times (0,T), \\ y^\varepsilon(0,t) = v^\varepsilon(t) = \sum_{k=0}^{m} \varepsilon^k v^k(t), \quad y^\varepsilon(1,t) = 0, & t \in (0,T), \\ y^\varepsilon(x,0) = y_0(x), & x \in (0,1), \end{cases} \tag{12}$$

$v^0, v^1, \cdots, v^m$ being known.

We construct an asymptotic approximation of the solution y^ε of (12) by using the matched asymptotic expansion method. We consider two formal asymptotic expansions of y^ε:
– the outer expansion

$$\sum_{k=0}^{m} \varepsilon^k y^k(x,t), \quad (x,t) \in (0,T),$$

– the inner expansion. (boundary layer at $x = 1$)

$$\sum_{k=0}^{m} \varepsilon^k Y^k(z,t), \quad z = \frac{1-x}{\varepsilon} \in (0, \varepsilon^{-1}),\ t \in (0,T).$$

Direct problem - Outer expansion - y^k - Case 1

$$y^0(x,t) = \begin{cases} y_0(x - Mt) & x > Mt, \\ v^0\left(t - \frac{x}{M}\right), & x < Mt. \end{cases}$$

Using the method of characteristics we find that, for any $1 \le k \le m$,

$$y^k(x,t) = \begin{cases} \displaystyle\int_0^t y_{xx}^{k-1}(x + (s-t)M, s)ds, & x > Mt, \\ \displaystyle v^k\left(t - \frac{x}{M}\right) + \int_0^{x/M} y_{xx}^{k-1}(sM, t - \frac{x}{M} + s)ds, & x < Mt. \end{cases}$$

For instance,

$$y^1(x,t) = \begin{cases} t\, y_0''(x - Mt), & x > Mt, \\ v^1\left(t - \frac{x}{M}\right) + \frac{x}{M^3}(v^0)''\left(t - \frac{x}{M}\right), & x < Mt, \end{cases}$$

$$y^2(x,t) = \begin{cases} \frac{t^2}{2} y_0^{(4)}(x - Mt), & x > Mt, \\ v^2\left(t - \frac{x}{M}\right) + \frac{x}{M^3}(v^1)''\left(t - \frac{x}{M}\right) \\ \quad - \frac{2x}{M^5}(v^0)^{(3)}\left(t - \frac{x}{M}\right) + \frac{x^2}{2M^6}(v^0)^{(4)}\left(t - \frac{x}{M}\right), & x < Mt. \end{cases}$$

$$y^0(x,t) = \begin{cases} y_0(x-Mt) & x > Mt, \\ v^0\left(t - \frac{x}{M}\right), & x < Mt. \end{cases}$$

Using the method of characteristics we find that, for any $1 \le k \le m$,

$$y^k(x,t) = \begin{cases} \int_0^t y_{xx}^{k-1}(x+(s-t)M, s)ds, & x > Mt, \\ v^k\left(t - \frac{x}{M}\right) + \int_0^{x/M} y_{xx}^{k-1}(sM, t - \frac{x}{M} + s)ds, & x < Mt. \end{cases}$$

For instance,

$$y^1(x,t) = \begin{cases} t\, y_0''(x-Mt), & x > Mt, \\ v^1\left(t - \frac{x}{M}\right) + \frac{x}{M^3}(v^0)''\left(t - \frac{x}{M}\right), & x < Mt, \end{cases}$$

$$y^2(x,t) = \begin{cases} \frac{t^2}{2}\, y_0^{(4)}(x-Mt), & x > Mt, \\ v^2\left(t - \frac{x}{M}\right) + \frac{x}{M^3}(v^1)''\left(t - \frac{x}{M}\right) \\ - \frac{2x}{M^5}(v^0)^{(3)}\left(t - \frac{x}{M}\right) + \frac{x^2}{2M^6}(v^0)^{(4)}\left(t - \frac{x}{M}\right), & x < Mt. \end{cases}$$

Lemma

$$Y^0(z,t) = y^0(1,t)\left(1 - e^{-Mz}\right), \quad (z,t) \in (0,+\infty) \times (0,T).$$

For any $1 \le k \le m$*, the solution reads*

$$Y^k(z,t) = Q^k(z,t) + e^{-Mz} P^k(z,t), \quad (z,t) \in (0,+\infty) \times (0,T), \tag{13}$$

where

$$P^k(z,t) = -\sum_{i=0}^{k} \frac{1}{i!} \frac{\partial^i y^{k-i}}{\partial x^i}(1,t) z^i, \quad Q^k(z,t) = \sum_{i=0}^{k} \frac{(-1)^i}{i!} \frac{\partial^i y^{k-i}}{\partial x^i}(1,t) z^i.$$

Theorem (Amirat, M)

Let y^ε be the solution of problem (12) *and let w_m^ε be the function defined as follows*

$$w_m^\varepsilon(x,t) = \mathcal{X}_\varepsilon(x)\sum_{k=0}^{m}\varepsilon^k y^k(x,t) + (1-\mathcal{X}_\varepsilon(x))\sum_{k=0}^{m}\varepsilon^k Y^k\left(\frac{1-x}{\varepsilon},t\right).$$

Assume that $y_0 \in C^{2m+1}[0,1]$, $v^k \in C^{2(m-k)+1}[0,T]$, $k=0,\cdots,m$ and that the $C^{2(m-k)+1}$- matching conditions are satisfied

$$(v^k)^{(p)}(0) = \sum_{i+j=p-1}(-1)^i M^i \frac{\partial^{p+1}y^{k-1}}{\partial x^{i+2}\partial t^j}(0,0), \quad 0 \le p \le 2(m-k)+1.$$

Then there is a constant c_m independent of ε such that

$$\|y^\varepsilon - w_m^\varepsilon\|_{C([0,T];L^2(0,1))} \le c_m \varepsilon^{\frac{2m+1}{2}\gamma}.$$

Example For $m=0$, y^0 and v^0 should satisfies $y_0 \in C^1[0,1]$, $v^0 \in C^1[0,T]$ and

$$v^0(t=0) = y_0(x=0), \qquad M(v^0)'(t=0) + y_0'(x=0) = 0.$$

Proposition

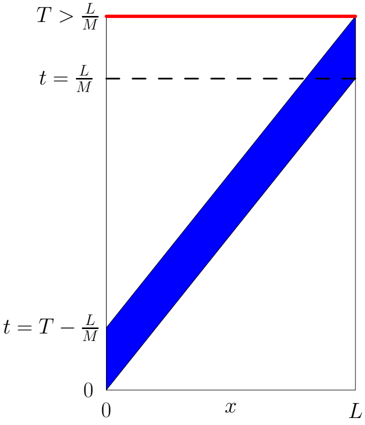
Let $m \in \mathbb{N}$, $T > \frac{1}{M}$ and $a \in]0, T - \frac{1}{M}[$. Assume regularity and matching conditions on the initial condition y_0 and functions v^k, $0 \leq k \leq m$. Assume moreover that

$$v^k(t) = 0, \quad 0 \leq k \leq m, \forall t \in [a, T].$$

Then, the solution y^ε of problem (12) *satisfies the following property*

$$\|y^\varepsilon(\cdot, T)\|_{L^2(0,1)} \leq c_m \, \varepsilon^{\frac{(2m+1)\gamma}{2}}, \quad \forall \gamma \in (0, 1)$$

for some constant $c_m > 0$ independent of ε. The function $v^\varepsilon \in C([0, T])$ defined by $v^\varepsilon := \sum_{k=0}^{m} \varepsilon^k v^k$ is an approximate null control for (5).



Convergence with respect to m under conditions on y_0 and the v^k.

(i) The initial condition y_0 belongs to $C^\infty[0,1]$ and there is $b \in \mathbb{R}$ such that

$$\|y_0^{(k)}\|_{L^2(0,1)} \leq \left\lfloor \frac{k}{2} \right\rfloor!\ b^{\frac{k}{2}}, \quad \forall k \in \mathbb{N}, \tag{14}$$

(ii) $(v^k)_{k\geq 0}$ is a sequence of polynomials of degree $\leq p-1$, $p \geq 1$, uniformly bounded in $C^{p-1}[0,T]$.

(iii) For any $k \in \mathbb{N}$, for any $m \in \mathbb{N}$, the functions v^k and y_0 satisfy the matching conditions.

Theorem

Assume (i)-(ii)-(iii). There exist $\varepsilon_0 > 0$ *and a function* $\tilde{\theta}^\varepsilon \in L^2(0,T;H_0^1(0,1)) \cap C([0,T];L^2(0,1))$ *satisfying an exponential decay, such that, for any fixed* $0 < \varepsilon < \varepsilon_0$*, we have*

$$y_m^\varepsilon - w_m^\varepsilon - \tilde{\theta}^\varepsilon \to 0 \quad \textit{in } C([0,T];L^2(0,1)), \quad \textit{as } m \to +\infty.$$

The function $\tilde{\theta}^\varepsilon$ *satisfies*

$$\|\tilde{\theta}^\varepsilon\|_{C([0,T],L^2(0,1))} \leq c\, e^{-2M\frac{\varepsilon^\gamma}{\varepsilon}},$$

where c is a constant independent of ε*.*

Optimality condition

Using the inner expansion for φ^ε, the equality $v^\varepsilon(t) = \varepsilon \varphi^\varepsilon_x(0,t)$ rewrites as follows

$$v^0(t) + \varepsilon\, v^1(t) + \cdots = \Phi^0_z(0,t) + \varepsilon\, \Phi^1_z(0,t) + \cdots, \quad \forall t \in (0,T).$$

At the zero order, we get therefore the equality $v^0(t) = \Phi^0_z(0,t)$ leading to

$$v^0(t) = M\varphi^0(0,t) = \begin{cases} M\varphi^0_T(M(T-t)), & t \in]T - 1/M, T], \\ 0, & t \in [0, T - 1/M]. \end{cases} \tag{15}$$

leading to $v^0(0) = 0$ and contradicts the matching condition $v^0(0) = y_0(0)$ unless $y_0(0) = 0$! Assuming $y_0(0) = 0$, we determine the optimal function φ^0_T by developing $J^\star_\varepsilon(\varphi^\varepsilon_T) = J^\star_0(\varphi^0_T) + \varepsilon \ldots$ with

$$J^\star_0(\varphi^0_T) := \frac{1}{2}\|v^0\|^2_{L^2(0,T)} - \Big(y_0, \mathcal{X}_\varepsilon \varphi^0(x,0) + (1-\mathcal{X}_\varepsilon)\Phi^0(x,0)\Big)_{L^2(0,1)}$$

leading to $\varphi^0_T = 0$, i.e. $v^0 \equiv 0$. The transport equation in y^0 and φ^0 separates the domain $(0,1)\times(0,T)$ into two distincts part: at the first order, the initial condition y_0 is not seen by the control function v^0.

Corollary

In the class of initial condition $\left\{ y_0 \in C^\infty[0,1], (y_0)^{(m)}(0) = 0, \forall m \in \mathbb{N} \right\}$,

$K(\varepsilon, T, M) \to 0$ *as* $\varepsilon \to 0$ *if* $T \geq \frac{1}{M}$.

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Using the inner expansion for φ^ε, the equality $v^\varepsilon(t) = \varepsilon\varphi^\varepsilon_x(0,t)$ rewrites as follows

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In the class of initial condition $\left\{ y_0 \in C^\infty[0,1], (y_0)^{(m)}(0) = 0, \forall m \in \mathbb{N} \right\}$, $K(\varepsilon, T, M) \to 0$ *as* $\varepsilon \to 0$ *if* $T \geq \frac{1}{M}$.

Using the inner expansion for φ^ε, the equality $v^\varepsilon(t) = \varepsilon\varphi_x^\varepsilon(0,t)$ rewrites as follows

$$v^0(t) + \varepsilon\, v^1(t) + \cdots = \Phi_z^0(0,t) + \varepsilon\,\Phi_z^1(0,t) + \cdots, \quad \forall t \in (0,T).$$

At the zero order, we get therefore the equality $v^0(t) = \Phi_z^0(0,t)$ leading to

$$v^0(t) = M\varphi^0(0,t) = \begin{cases} M\varphi_T^0(M(T-t)), & t \in]T-1/M, T], \\ 0, & t \in [0, T-1/M]. \end{cases} \tag{15}$$

leading to $v^0(0) = 0$ and contradicts the matching condition $v^0(0) = y_0(0)$ unless $y_0(0) = 0$! Assuming $y_0(0) = 0$, we determine the optimal function φ_T^0 by developing $J_\varepsilon^\star(\varphi_T^\varepsilon) = J_0^\star(\varphi_T^0) + \varepsilon \ldots$ with

$$J_0^\star(\varphi_T^0) := \frac{1}{2}\|v^0\|^2_{L^2(0,T)} - \Big(y_0, \mathcal{X}_\varepsilon\varphi^0(x,0) + (1-\mathcal{X}_\varepsilon)\Phi^0(x,0)\Big)_{L^2(0,1)}$$

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Corollary

In the class of initial condition $\left\{ y_0 \in C^\infty[0,1], (y_0)^{(m)}(0) = 0, \forall m \in \mathbb{N} \right\}$, $K(\varepsilon, T, M) \to 0$ *as* $\varepsilon \to 0$ *if* $T \geq \frac{1}{M}$.

We now take into account the boundary layer on the characteristic and consider three formal asymptotic expansions of y^ε:
– the outer expansion

$$\sum_{k=0}^{m} \varepsilon^k y^k(x,t), \quad (x,t) \in Q_T, \quad x - Mt \neq 0$$

– the first inner expansion (on the characteristic $x - Mt = 0$)

$$\sum_{k=0}^{m} \varepsilon^{\frac{k}{2}} W^{k/2}(w,t), \quad w = \frac{x - Mt}{\sqrt{\varepsilon}} \in \left(-\frac{Mt}{\varepsilon^{1/2}}, \frac{1 - Mt}{\sqrt{\varepsilon}}\right), \ t \in (0,T).$$

– the second inner expansion (at $x = 1$)

$$\sum_{k=0}^{m} \varepsilon^{k/2} Y^{k/2}(z,t), \quad z = \frac{1 - x}{\varepsilon} \in (0, \varepsilon^{-1}), \ t \in (0,T).$$

Direct problem - Matched asymptotic expansion method - Case 2 - First order approximation

After computations, the first order approximation of y^{ε} is given by

$$P^0_{\varepsilon}(x,t) = \begin{cases} y_0(x-Mt) + W^0(w,t) - y_0(0) - C^0_{\varepsilon}(t)e^{-Mz}, & x > Mt, \\ \dfrac{y_0(0)+v^0(0)}{2} - C^0_{\varepsilon}(t)e^{-Mz}, & x = Mt, \\ v^0\left(t - \dfrac{x}{M}\right) + W^0(w,t) - v^0(0) - C^0_{\varepsilon}(t)e^{-Mz}, & x < Mt, \end{cases}$$

with $\left(\text{recall that } erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds \right)$

$$\begin{cases} z = \dfrac{1-x}{\varepsilon}, w = \dfrac{x-Mt}{\sqrt{\varepsilon}}, \\ W^0(w,t) = erf\left(\dfrac{w}{2\sqrt{t}}\right)\dfrac{y_0(0)-v^0(0)}{2} + \dfrac{y_0(0)+v^0(0)}{2}, \\ C^0_{\varepsilon}(t) = y^0(1,t) + W^0\left(\dfrac{1-Mt}{\sqrt{\varepsilon}}, t\right) - y_0(0). \end{cases} \quad (16)$$

Direct problem - Matched asymptotic expansion method - Case 2 - First order approximation (1)

After more computations, the next approximation of y^ε is given by $P_\varepsilon^{1/2}(x,t) = P_\varepsilon^0(x,t) + \sqrt{\varepsilon} F_\varepsilon^{1/2}(x,t)$ with

$$F_\varepsilon^{1/2}(x,t) = \begin{cases} W^{1/2}(w,t) - d^+ w - C_\varepsilon^1(t) e^{-Mz}, & x \geq Mt, \\ W^{1/2}(w,t) - d^- w - C_\varepsilon^1(t) e^{-Mz}, & x \leq Mt, \end{cases}$$

and

$$\begin{cases} z = \dfrac{1-x}{\varepsilon}, w_0 = \dfrac{1-Mt}{\sqrt{\varepsilon}}, d^+ = (y_0)^{(1)}(0), \quad d^- = -\dfrac{1}{M}(v^0)^{(1)}(0), \\ C_\varepsilon^1(t) = W^{1/2}\left(\dfrac{1-Mt}{\sqrt{\varepsilon}}, t\right) - d^+ w^0 + z W_z^0(w_0, t), \\ W^{1/2}(w,t) = \dfrac{d^+ - d^-}{2} erf\left(\dfrac{w}{2\sqrt{t}}\right) w + (d^+ - d^-)\dfrac{\sqrt{t}}{\sqrt{\pi}} e^{-\frac{w^2}{4t}} + \dfrac{(d^+ + d^-)}{2} w. \end{cases} \tag{17}$$

Theorem (First order approximation)

Assume $v^0 \in C^1([0,T])$, $y^0 \in C^1([0,1])$. Then $\exists C > 0$ independent of ε s.t.

$$\|y^\varepsilon - P_\varepsilon^{1/2}\|_{C([0,T],L^2(0,1))} \leq C\sqrt{\varepsilon}.$$

After computations and justifications,

Theorem (Amirat, M.)

Assume $y_0 \in C^\infty[0,1]$. $K(\varepsilon, T, M) \to 0$ as $\varepsilon \to 0$ if $T > \frac{1}{M}$.

What is about $y_0^\varepsilon(x) = e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x)$?????

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Assume that the initial condition is of the form $y_0^\varepsilon(x) = c_\varepsilon e^{\frac{M\alpha x}{2\varepsilon}} f(x)$ where f is an arbitrary function independent of ε, $\alpha < 0$ and $c_\varepsilon \in \mathbb{R}^+$. We introduce the following change of variable

$$y^\varepsilon(x,t) = c_\varepsilon e^{l_{\varepsilon,\alpha}(x,t)} z^\varepsilon(x,t), \quad l_{\varepsilon,\alpha}(x,t) := \frac{M\alpha x}{2\varepsilon}\left(x - \frac{(2-\alpha)Mt}{2}\right). \tag{18}$$

We then check that

$$L_\varepsilon y^\varepsilon(x,t) = c_\varepsilon e^{l_{\varepsilon,\alpha}(x,t)}\left(z_t^\varepsilon - \varepsilon z_{xx}^\varepsilon + M_\alpha z_x^\varepsilon\right) := c_\varepsilon e^{l_{\varepsilon,\alpha}(x,t)} L_{\varepsilon,\alpha}(z^\varepsilon)(x,t)$$

with $M_\alpha := M(1-\alpha) > 0$. Consequently, the new variable z^ε solves

$$\begin{cases} L_{\varepsilon,\alpha}(z^\varepsilon) := z_t^\varepsilon - \varepsilon z_{xx}^\varepsilon + M_\alpha z_x^\varepsilon = 0, & (x,t) \in Q_T, \\ z^\varepsilon(0,t) := \overline{v}^\varepsilon(t) = c_\varepsilon^{-1} e^{-l_{\varepsilon,\alpha}(0,t)} v^\varepsilon, \; z^\varepsilon(L,t) = 0, & t \in (0,T), \\ z^\varepsilon(x,0) =: z_0(x) = f(x), & x \in (0,L). \end{cases} \tag{19}$$

The initial data is now independent of ε. On the contrary, the control $\overline{v}^\varepsilon$ depends a priori on ε. We have thus reported the problem on the control part (which is relevant from a controllability viewpoint).

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After computations,

Theorem (Amirat, M.)

Assume $T > 1/M$*. For* $y_0^{\varepsilon}(x) = e^{-\frac{Mx}{2\varepsilon}} \sin(\pi x)$*, the* L^2*-norm of the control of minimal* L^2 *norm is uniformly bounded w.r.t.* ε*.*

The end

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THANK YOU FOR YOUR ATTENTION