Maintien hors-gel des chaussées: un problème de contrôle optimal non linéaire

ARNAUD MÜNCH
Lab. de mathématiques Blaise Pascal - Clermont-Ferrand - France

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avec Frédéric Bernardin (Cerema Clermont)

Collaboration avec le CEREMA (Centre d'études et d'expertises sur les risques, l'environnement, la mobilité et l'aménagement) dans le cadre du projet européen "Routes de 5ième génération" :

- Réduction du bruit
- Récupération d'énergie / Panneau solaire
- Utilisation de matériaux recyclable
- Incrustation luminescente interactive
- etc .....

Projet : Chaussées chauffantes et récupératrices d'énergie par circulation d'un fluide caloporteur au sein d'une couche poreuse de la chaussée

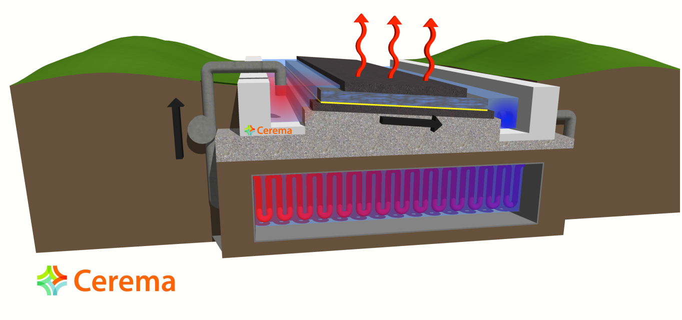


Figure: Schéma du démonstrateur (cas chauffant)

Figure: Le demonstrateur d'Egletons

Figure: Voie de circulation à Egletons

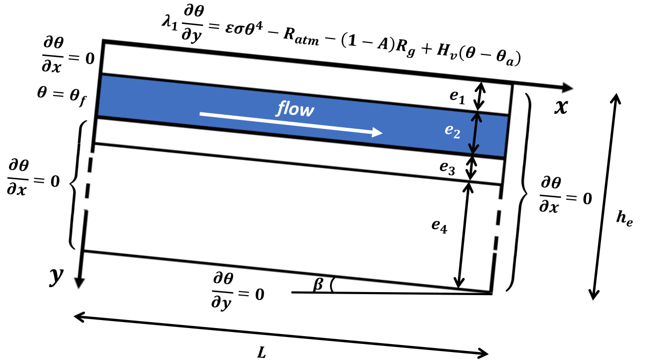


Figure: Schéma transversal de la structure avec condition aux limites: θ_f est la température d'injection du fluide

The road is assumed to have no longitudinal slant and to be infinite in its third dimension. h and L denote the height of the road structure and its length, respectively. The hydraulic regime is assumed stationary with hydraulic parameters independent of temperature T. Denoting by $1 \leq i \leq 4$ the indices of the road layers, the thermo-hydraulic model is as follows. For $0 \leq x \leq L$ and $0 \leq y \leq h$:

$$\begin{cases} C_i \dfrac{\partial \theta}{\partial t}(x,y,t) - \lambda_i \Delta \theta(x,y,t) = 0, \quad i \in \{1,3,4\}, \\ C_2 \dfrac{\partial \theta}{\partial t}(x,y,t) + C_f v \dfrac{\partial \theta}{\partial x}(x,y,t) - (\lambda_2 + \phi_2 \lambda_f)\Delta \theta(x,y,t) = 0, \\ v = -K \dfrac{H_2 - H_1}{L}, \end{cases} \tag{1}$$

where

$(\rho C)_i\,, \lambda_i, \phi_i$	specific heat, thermal conductivity and porosity of layer i
$(\rho C)_f\,, \lambda_f$	specific heat, thermal conductivity of the fluid
v	Darcy fluid velocity along x
K	hydraulic conductivity of the porous asphalt
H_1, H_2	hydraulic heads imposed upstream and downstream of fluid circulating in porous draining asphalt layer

Road surface boundary condition expresses the energy balance between road and atmosphere [1]:

$$\lambda_1 \frac{\partial \theta}{\partial y}(x,0,t) = \sigma\varepsilon(t)\theta^4(x,0,t) + H_v(t)(\theta(x,0,t) - \theta_a(t)) - R_{atm}(t) - (1 - A(t))R_g(t) + L_f I(t) \tag{2}$$

ε, A : emissivity and albedo of the road surface,

σ : Stefan-Boltzmann constant (5.67×10^{-8} W/m^{2}K^4),

R_{atm}, R_g : atmospheric and global radiation (W/m^2),

θ_a : air temperature (K),

H_v : convection heat transfer coefficient (W/m^{2}K),

I : snow rate (mm.s^{-1}),

L_f : latent heat of fusion of the ice per kg (J.kg^{-1}).

The convection coefficient is defined by $H_v = Cp_a \times \rho_a(V_{wind}C_d + C_{d_1})$ where the following notations are used :

Cp_a : thermal capacity (J/kg.K) of the air,
V_{wind} : wind velocity (m/s),
ρ_a : density of the air (kg/m^3),
C_d, C_{d_1} : two convection coefficients (-).

[1] Asfour, Bernardin, 2015 : Experimental validation of 2d hydrothermal modelling of porous pavement

The injection temperature of the fluid is imposed :

$$\forall\, e_1 \leq y \leq e_1 + e_2,\ \forall\, t \geq 0,\ \theta(t,0,y) = \theta_f(y,t) = q(t). \tag{3}$$

The optimal control problem is the following

$$\begin{cases} \inf\limits_{q \in H_0^1(0,T)} J(q), \text{ subject to:} \\ q \geq 0 \ a.e.\ t \in (0,T), \\ \theta \geq \underline{\theta},\ a.e. on\ \Sigma_b \times (0,T), \\ \theta \text{ solves } (1). \end{cases} \tag{4}$$

where J is defined as follows :

$$J(q) = \frac{1}{2} v\, e_2\, C_f \left(\int_0^T \int_{e_1}^{e_1+e_2} \left(q(t) - \theta(L,y,t)\right)^+ dydt \right)^2 + \frac{1}{2}\alpha \int_0^T q_t^2(t) dt. \tag{5}$$

The functional J represents the energy expended by the power q over the period $[0, T]$.

1D - model: $\theta = \theta(y,t)$ denotes the temperature in Kelvin at the point y and at time t solves

$$\begin{cases} c(y)\dfrac{\partial \theta}{\partial t}(y,t) - \dfrac{\partial}{\partial y}\left(k(y)\dfrac{\partial \theta}{\partial y}(y,t)\right) = q(t)\delta(y^0), \quad (y,t) \in Q_T = (0,L)\times(0,T), \\ -k(0)\dfrac{\partial \theta}{\partial y}(0,t) = f_1(t) - f_2(t)\theta(0,t) - \sigma\varepsilon(t)\theta^4(0,t), \quad t \in (0,T), \\ \theta(y,0) = \theta_0(y), \quad y \in (0,L), \quad \dfrac{\partial \theta}{\partial y}(L,t) = 0, \quad t \in (0,T), \\ f_1(t) = (1 - A(t))R_g(t) + R_{atm}(t) + H_v(t)\theta_a(t) - \dfrac{L_f}{3600} I(t), \quad f_2(t) = H_v(t). \end{cases} \tag{6}$$

Optimal control problem:

$$\begin{cases} \inf\limits_{q\in K} J(q) := \displaystyle\int_0^T q(t)dt, \\ \text{subject to } q \in K = \Big\{ q \in L^1(0,T), (t) \geq 0,\ \theta(0,t) \geq \underline{\theta},\ \forall t \in (0,T),\ \theta = \theta(q) \text{ solves } (6) \Big\} \end{cases}$$

q (W/m^2); J (Wh ou J)

Let us then introduce the following regularity assumptions :

$$(\mathcal{H}) \quad \begin{cases} q_s, f_1 \in H^1(0,T), \theta_0 \in V, (k(\theta_0)_y)_y \in V', c, k \in L^\infty(0,h_e), \\ \varepsilon \in H^1(0,T), \varepsilon^{-1}\varepsilon_t \in L^\infty(0,T), \varepsilon(t) \geq 0, \forall t \in (0,T), \\ f_2 \in H^1(0,T), f_{2,t} \in L^\infty(0,T). \end{cases}$$

Theorem

Assume $(\mathcal{H})$. Assume moreover that the control q satisfies $q(0) = 0$ and that the initial condition θ_0 satisfies the compatibility condition

$$-k(0)(\theta_0)_y(0) = f_1(0) - f_2(0)\theta_0(0) - \sigma\varepsilon(0)\theta_0^3(0)|\theta_0(0)|, \quad (\theta_0)_y(h_e) = 0, \qquad (7)$$

at the point $y = 0$ and $y = h_e$ respectively. There exists a unique solution θ with $\theta, \theta_t \in L^2(0,T,V) \cap L^\infty(0,T,H)$ such that

$$\|\sqrt{c}\,\theta_t\|_{L^\infty(0,T,H)} + \|\theta_t\|_{L^2(0,T,V)} \leq C_2(\|k(\theta_0)_y\|_H + \|q\|_{H^1(0,T)} + \|f_1\|_{H^1(0,T)}).$$

for a constant $C_2 = C(C_1, \|f_2\|_{H^1(0,T)}, \|\varepsilon^{-1}\varepsilon_t\|_{L^\infty(0,T)}, \|f_{2,t}\|_{L^\infty(0,T)}) > 0$

Moreover, the solution enjoys the following comparison principle.

Proposition

Assume the hypothesis of Theorem 1. Let θ and $\hat{\theta}$ the solutions of (6) associated with the pair (q, θ_0) and $(\hat{q}, \hat{\theta}_0)$ respectively. If $q \geq \hat{q}$ in $[0, T]$ and $\theta_0 \geq \hat{\theta}_0$ in $[0, L]$, then $\theta \geq \hat{\theta}$ in Q_T.

$$\inf_{q\in\mathcal{C}} J_\alpha(q) := \frac{1}{2}\left(\|q\|^2_{L^1(0,T)} + \alpha\|q\|^2_{H^1(0,T)}\right) \tag{8}$$

where the constraint set is given by

$$\mathcal{C} := \left\{ q \in H^1(0,T), q(0)=0, q(t)\geq 0, \theta(0,t)\geq \underline{\theta}, \forall t\in[0,T], \theta=\theta(q) \text{ solves } (6)\right\}.$$

Lemma

- *Let us assume that $\theta_0 \geq \underline{\theta}$ on $(0, h_e)$. If*

$$f_1(t) - f_2(t)\underline{\theta} - \sigma\varepsilon(t)\underline{\theta}^4 \geq 0, \ \forall t\in(0,T),$$

then $\mathcal{C}$ is not empty. In particular, if $q \geq \underline{\theta}$ then $\theta \geq \underline{\theta}$.

- *$\mathcal{C}$ is a closed convex subset of $H^1(0,T)$.*

Let $H^1_{0,0}(0,T) = \{q \in H^1(0,T), q(0) = 0\}$.

$$(\mathcal{P}_\epsilon): \quad \inf_{q\in\mathcal{D}} J_{\alpha,\epsilon}(q) := \frac{1}{2}\|q\|^2_{L^1(0,T)} + \frac{\alpha}{2}\|q\|^2_{H^1(0,T)} + \frac{\epsilon^{-1}}{2}\left\|(\theta(0,\cdot) - \underline{\theta})^-\right\|^2_{L^2(0,T)}$$

Theorem

For any $\alpha > 0, \epsilon > 0$, the functional $J_{\alpha,\epsilon}$ is Gâteaux differentiable on the set $\mathcal{D}$ and its derivative at $q \in \mathcal{D}$ in the admissible direction $\overline{q}$ (i.e. $\overline{q} \in H^1_{0,0}(0,T)$ such that $q + \eta\overline{q} \in \mathcal{D}$ for all $\eta \neq 0$ small) is given by

$$< J'_{\alpha,\epsilon}(q), \overline{q} >= \int_0^T \Big(\|q\|_{L^1(0,T)} - p(y_0,\cdot)\Big)\overline{q}\,dt + \alpha\int_0^T (q\overline{q} + q_t\overline{q}_t)dt \tag{9}$$

where p solves the adjoint problem

$$\begin{cases} -c(y)p_t(y,t) - \big(k(y)p_y(y,t)\big)_y = 0, \quad (y,t) \in Q_T, \\ -k(0)p_y(0,t) = -f_2(t)p(0,t) - 4\sigma\varepsilon(t)\theta_q(0,t)^3 p(0,t) - \epsilon^{-1}(\theta_q(0,t) - \underline{\theta})^-, \ t \in (0,T), \\ p_y(h_e,t) = 0, \quad t \in (0,T), \\ p(y,T) = 0, \quad y \in (0,h_e), \end{cases} \tag{10}$$

and θ_q solves (6).

Euler implicite en temps + Elément finis $\mathbb{P}_1$ en espace
Let

$$V_h = \{\theta_h \in C^1([0,h_e]), \theta_{h|_{[x_i,x_{i+1}]}} \in \mathbb{P}_1 \quad \forall i = 1,\dots,N_y - 1\}$$

$\pi_h : V \to V_h$ is the projection operator over V_h. Let $(t_n)_{n=1,\cdot,N_t}$ such that $[0,T] = \cup_{n=0}^{N_t-1}[t_n, t_{n+1}]$.

We note by (θ_h^n) an approximation of $\theta_h(\cdot, t_n)$ the solution of the following implicit Euler type scheme:

$$\begin{cases} \theta_h^0 = \pi_h(\theta_0), \\ \left(c\dfrac{\theta_h^{n+1}-\theta_h^n}{\Delta t}, \phi_h\right)_H + a(t_{n+1}, \theta_h^{n+1}, \phi_h) + 4\sigma\varepsilon(t_{n+1})(\theta_h^n(0))^3\theta_h^{n+1}(0)\phi_h(0) \\ \qquad -3\sigma\varepsilon(t_n)(\theta_h^n(0))^4\phi_h(0) = q(t_{n+1})\phi_h(y_0) + f_1(t_{n+1})\phi_h(0), \\ \forall \phi_h \in V_h, n \geq 0. \end{cases}$$

$$-k(0)\frac{\partial\theta}{\partial y}(0,t) = f_1(t) - f_2(t)\theta(0,t) - \sigma\varepsilon(t)\theta^4(0,t), \quad t \in (0,T),$$

$$f_1(t) = (1 - A(t))R_g(t) + R_{atm}(t) + H_v(t)\theta_a(t) - \frac{L_f}{3600} I(t),$$

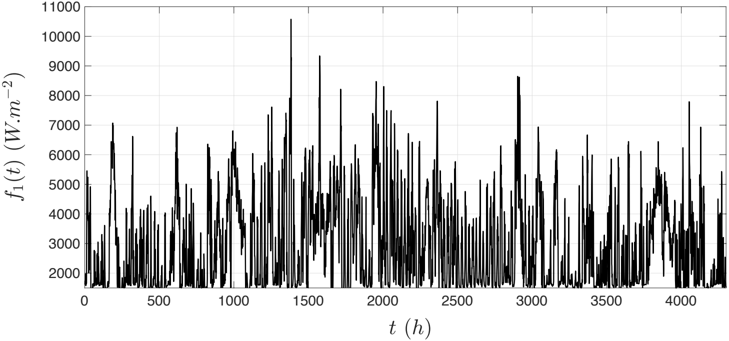


Figure: The function f_1 from data of the french highway A75 in Cantal (1100 m altitude) - October 2009- March 2010

$$J_{\alpha,\epsilon}(q) := \frac{1}{2}\|q\|^2_{L^1(0,T)} + \frac{\alpha}{2}\left(T\|q\|^2_{L^2(0,T)} + \frac{T^3}{4\pi^2}\|q_t\|^2_{L^2(0,T)}\right) + \frac{\epsilon^{-1}}{2}\left\|(\theta_q(0,\cdot) - \underline{\theta})^-\right\|^2_{L^2(0,T)}.$$

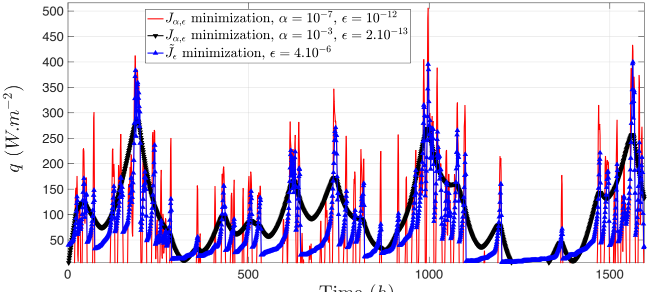


Figure: Optimal controls q on $[0, 1600]$ corresponding to the minimization of $J_{\alpha,\epsilon}$ and $\tilde{J}_\epsilon$.

$$J_{\alpha,\epsilon}(q) := \frac{1}{2}\|q\|^2_{L^1(0,T)} + \frac{\alpha}{2}\left(T\|q\|^2_{L^2(0,T)} + \frac{T^3}{4\pi^2}\|q_t\|^2_{L^2(0,T)}\right) + \frac{\epsilon^{-1}}{2}\left\|(\theta_q(0,\cdot) - \underline{\theta})^-\right\|^2_{L^2(0,T)}. \quad (11)$$

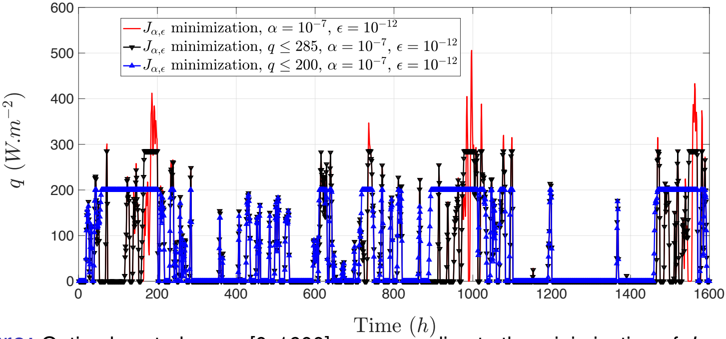


Figure: Optimal controls q on $[0, 1600]$ corresponding to the minimization of $J_{\alpha,\epsilon}$ under the additional constraint $\|q\|_\infty \leq \lambda$ for $\lambda = 200$ and 285.

$$J_{\alpha,\epsilon}(q) := \frac{1}{2}\|q\|^2_{L^1(0,T)} + \frac{\alpha}{2}\left(T\|q\|^2_{L^2(0,T)} + \frac{T^3}{4\pi^2}\|q_t\|^2_{L^2(0,T)}\right) + \frac{\epsilon^{-1}}{2}\left\|(\theta_q(0,\cdot) - \underline{\theta})^-\right\|^2_{L^2(0,T)}.$$

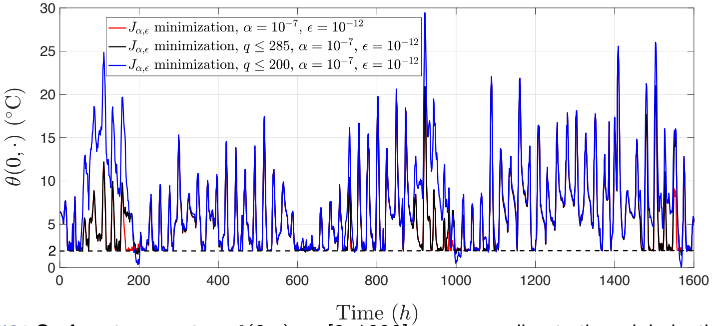


Figure: Surface temperature $\theta(0,\cdot)$ on $[0, 1600]$ corresponding to the minimization of $J_{\alpha,\epsilon}$ for different bounds of $\|q\|_\infty$.

$$J_{\alpha,\epsilon}(q) := \frac{1}{2}\|q\|^2_{L^1(0,T)} + \frac{\alpha}{2}\left(T\|q\|^2_{L^2(0,T)} + \frac{T^3}{4\pi^2}\|q_t\|^2_{L^2(0,T)}\right) + \frac{\epsilon^{-1}}{2}\left\|(\theta_q(0,\cdot) - \underline{\theta})^-\right\|^2_{L^2(0,T)}.$$

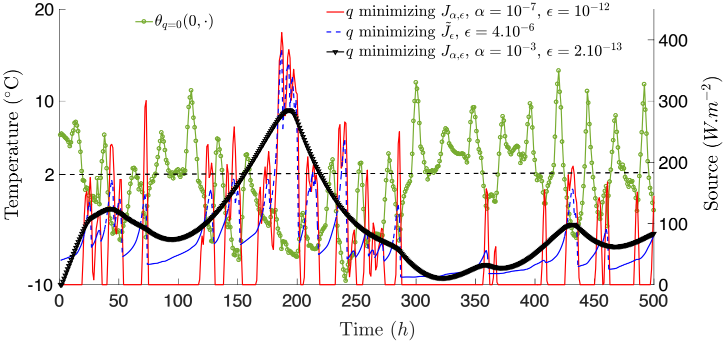


Figure: Surface temperature $\theta_{q=0}(0,\cdot)$, and sources q represented on $[0, 500]$ corresponding to the minimization of $J_{\alpha,\epsilon}$ with $\alpha = 10^{-3}$ and $\epsilon = 2.10^{-13}$, $\alpha = 10^{-7}$ and $\epsilon = 10^{-12}$ and $\tilde{J}_\epsilon$ with $\epsilon = 4.10^{-6}$.

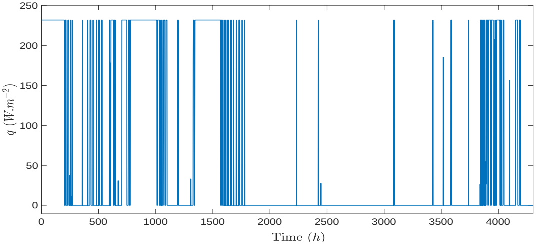


Figure: Optimal bang-bang control q on $[0, T]$ corresponding to $L = 1/4$. $q(t) = \lambda s(t)$ with $\lambda \approx 2.34 \times 10^2$.

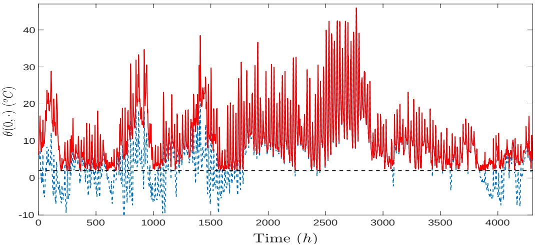


Figure: Temperature $\theta(0, \cdot)$ at the road surface on $[0, T]$ in the controlled (red full line)

Conclusion de l'étude

• The total energy needed to keep the road surface temperature over 2°C during a winter with snow is about 5.10^8 J $\simeq$ 139 kWh per m^2 of road, with minimal and maximal values per m^2 respectively equal to 124 kWh and 213 kWh.

• The L^∞-norm of the optimal power q ranges in 240-500 W/m^2.

• Some experiments for de-icing obtained by the circulation of a coolant in pipes inserted in the road. equals $100 - 170$ kWh/m^2.

According to the mathematical analysis, if the source q acts on the top of the road ($y_0 = 0$) satisfies the condition $q + f_1(t) - f_2(t)\underline{\theta} - \sigma\varepsilon(t)\underline{\theta}^4 \geq 0$, then the corresponding variable θ_q satisfies $\theta_q(0,t) - \underline{\theta} \geq 0$ for all $t \in (0,T)$. This suggests to consider,the following explicit source

$$\boxed{q(t) = \max\Big(0, -(f_1(t) - f_2(t)\underline{\theta} - \sigma\varepsilon(t)\underline{\theta}^4) + \delta\Big)}$$

for some real $\delta \geq 0$ large enough, dependent of y_0. Table 1 gives the L^1-norm of q and the corresponding value of $\min((\theta(0,\cdot) - \underline{\theta})^-)$ for some values of δ. The value $\delta = 55$ is large enough to satisfy the condition $\theta(0,\cdot) \geq 2^oC$ at the road surface. The corresponding L^1-norm $\|q\|_{L^1(0,T)} \approx 7.52 \times 10^8$ is of the same order as in the previous section.

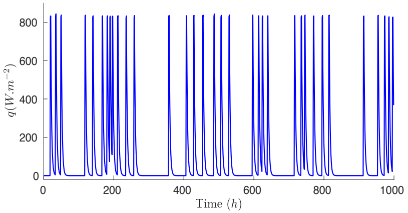
δ	0	50	54	55
$\|q\|_{L^1(0,T)}$	4.01×10^8	7.2×10^8	7.52×10^8	7.60×10^8
$\|q\|_{L^\infty(0,T)}$	2.72×10^2	3.22×10^2	3.26×10^2	3.27×10^2
$\|(\theta(0,\cdot) - \underline{\theta})^-\|_{L^2(0,T)}$	5.49×10^2	1.29×10^1	1.71	0.
$\|(\theta(0,\cdot) - \underline{\theta})^-\|_{L^\infty(0,T)}$	1.91	1.74×10^{-1}	2.92×10^{-2}	0.

Table: Characteristics of the temperature θ.

The source from the previous law is active on some period where the value of $\theta(0,\cdot)$ is (significantly) above $\underline{\theta}$. This is due to the large variations of the functions f_1 and f_2. A third way is therefore to consider source term q which depends explicitly on the variable θ_q, for instance as follows:

$$q(t) = \begin{cases} 0 & \text{if} \quad \theta(0,t-\delta) \geq \theta_m, \\ 0 & \text{if} \quad \underline{\theta} \leq \theta(0,t-\delta) \leq \theta_m \quad \text{and} \quad \theta'(0,t-\delta) > 0, \\ f(t,\theta)\Big(\theta(0,t-\delta) - \theta_m\Big)^{-} & \text{else} \end{cases}$$

for some reals $\theta_m > \underline{\theta}$, $\delta \in (0,T)$ and a negative function f which depends only at time t on the temperature $\theta(s)$, $s \in (0,t)$. Figure below depicts the source associated with $\theta_m = 273.15 + 3$, $\delta = 1$ hour and to the corresponding temperature $\theta(0,\cdot)$.



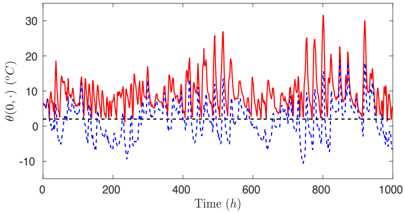


Figure: Source q for $t \in [0,1000]$ and corresponding temperature $\theta(0,\cdot)$.

- Analyse mathématique du cas 1d

F. Bernardin, AM: Modeling and optimizing a road de-icing device by a nonlinear heating ESAIM Math. Model. Numer. Anal, 2019.

- Lien sur arte.tv (émission du 2 avril 2016):

https://sites.arte.tv/futuremag/fr/les-routes-de-demain-futuremag

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