Inverse problems for linear PDEs via variational formulations : robust numerical approximations

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General context and purpose

Given a suitable observation $y_{obs} (= B(y))$ of y, unique solution of a linear well-posed PDE

$$\begin{cases} PDE(y, \nabla y, ...) = f, & \Omega \times (0, T), \\ + \text{boundary and initial conditions} & \end{cases},$$

find a convergent (numerical) approximation of the following linear inverse problem :

reconstruct the solution y and the source f such that $B(y) = y_{obs}$.

The main aim is to highlight that space-time variational approach of first and second order leads to robust approximation.

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Hyperbolic situation

Hyperbolic equation - Problem statement

$\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ - $T > 0$, $c \in C^1(\overline{\Omega}, \mathbb{R})$, $d \in L^\infty(Q_T)$, $(y_0, y_1) \in \boldsymbol{H}$, $f \in X$.

$$\begin{cases} Ly := y_{tt} - \nabla \cdot (c \nabla y) + dy = f, & Q_T := \Omega \times (0, T) \\ y = 0, & \Sigma_T := \partial\Omega \times (0, T) \\ (y(\cdot, 0), y_t(\cdot, 0)) = (y_0, y_1), & \Omega. \end{cases} \tag{1}$$

- Inverse Problem 1: Distributed observation on $q_T = \omega \times (0, T), \omega \subset \Omega$

$$\begin{cases} \boldsymbol{H} = L^2 \times H^{-1}, X = L^2(H^{-1}), \\ \text{Given } (y_{obs}, f) \in L^2(q_T) \times X, \text{ find } y \; s.t. \; \{(1) \text{ and } y - y_{obs} = 0 \quad \text{on} \quad q_T\} \end{cases}$$

- Inverse Problem 2: Boundary observation on $\Gamma_T \subset \partial\Omega \times (0, T)$

$$\begin{cases} \boldsymbol{H} = H_0^1 \times L^2, X = L^2(L^2) \\ \text{Given } y_{obs,\nu} \in L^2(\Gamma_T), \text{ find } (y, f) \; s.t. \; \{(1) \text{ and } \partial_\nu y - y_{obs,\nu} = 0 \quad \text{on} \quad \Gamma_T\} \end{cases}$$

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Inverse problem 1

$$Z := \Big\{ y : y \in C([0,T], L^2(\Omega)) \cap C^1([0,T], H^{-1}(\Omega)), Ly \in X, y_{|\Sigma_T} = 0 \Big\}.$$

Introducing the operator $P : Z \to X \times L^2(q_T)$

$$P\,y := (Ly, y_{|q_T}),$$

Inverse Problem 1 is reformulated as :

$$\textit{find } y \in Z \textit{ solution of } P\,y = (f, y_{obs}). \tag{IP}$$

If unique continuation property holds for (1) and if y_{obs} is a restriction to q_T of a solution of (1), then (IP) is well-posed: the state y corresponding to the pair (y_{obs}, f) is unique.

Most natural approach: Relaxation via Least-squares method

The most natural (and widely used in practice) approach consists in introducing a least-squares type technic, i.e. consider the extremal problem

$$(LS) \quad \begin{cases} \text{minimize} \quad J(y_0, y_1) := \frac{1}{2}\|y - y_{obs}\|^2_{L^2(q_T)} \\ \text{subject to} \quad (y_0, y_1) \in \boldsymbol{H} \\ \text{where} \quad y \quad \text{solves} \quad (1) \end{cases}$$

A minimizing sequence $(y_{0k}, y_{1k})_{(k>0)} \in \boldsymbol{H}$ is defined in term of an adjoint problem.

Drawback : it is difficult to minimize over a finite dimensional subspace of the set of constraints

The minimization procedure first requires the discretization of J and of the system (1);

This raises the issue of uniform coercivity property of the discrete functional w.r.t. the approximation parameter.

The "Discretization then Inverse problem" procedure is discussed in [L. Baudouin, M. De Buhan, S. Ervedoza, 2013]

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A not so different approach : Luenberger observers type approach

[Auroux-Blum 2005],[Chapelle,Cindea,Moireau,2012], [Ramdani-Haine 2011], etc...

Define a dynamic

$$\begin{aligned} L\overline{y} &= G(y_{obs}, q_T) \\ \overline{y}(\cdot, 0) &\quad \text{fixed} \end{aligned}$$

such that

$$\|\overline{y}(\cdot, t) - y(\cdot, t)\|_{N(\Omega)} \to 0 \quad \text{as} \quad t \to \infty$$

$N(\Omega)$ - appropriate norm
The reversibility of the eq. then allows to recover y for any time.

But again, on a numerically point of view, this method requires to prove uniform discrete observability properties.

Klibanov and co-workers approach: Quasi-reversibility for ill-posed problem

[Klibanov, Beilina 20xx], [Bourgeois, Darde 2010]

QR_ε method (Quasi-Reversibility): for any $\varepsilon > 0$, find $y_\varepsilon \in Z$ such that

$$\langle Py_\varepsilon, P\overline{y}\rangle_{X\times L^2(q_T)} + \varepsilon\langle y_\varepsilon, \overline{y}\rangle_{\mathcal{Z}} = \langle (f, y_{obs}), P\overline{y}\rangle_{X\times L^2(q_T), X\times L^2(q_T)}, \qquad (QR)$$

for all $\overline{y} \in Z$,

equivalent to the minimization over Z of

$$\begin{aligned} y \to & \|Py - (f, y_{obs})\|^2_{X\times L^2(q_T)} + \varepsilon\|y\|^2_Z \\ & = \|Ly - f\|^2_X + \|y - y_{obs}\|^2_{L^2(q_T)} + \varepsilon\|y\|^2_Z \end{aligned} \qquad (2)$$

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Keeping y as the main variable ...

Without loss of generality, $f \equiv 0$.

$$Z := \{y : y \in C([0,T], L^2(\Omega)) \cap C^1([0,T], H^{-1}(\Omega)), Ly \in X, y_{|\Sigma_T} = 0\}.$$

Hypothesis (Generalized Observability Inequality)

Assume that $\exists C_{obs} = C(\omega, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)})$ *s.t. :*

$$(\mathcal{H}) \qquad \|y(\cdot,0), y_t(\cdot,0)\|_H^2 \leq C_{obs}\left(\|y\|^2_{L^2(q_T)} + \|Ly\|^2_X\right), \quad \forall y \in Z. \qquad (3)$$

- in 1-D, (3) if $T \geq T^\star(c,d)$ [Fernandez-Cara, Cindea,Münch, COCV 2013],
- in N-D, for $c = 1$ and $d = 0$, (3) if (Ω, ω, T) satisfies geometric optic condition [Bardos, Lebeau, Rauch, 1992]

$$\|y\|^2_{L^2(Q_T)} \leq C_{\Omega,T}\left(C_{obs}\|y\|^2_{L^2(q_T)} + (1 + C_{obs})\|Ly\|^2_X\right) \quad \forall y \in Z. \qquad (4)$$

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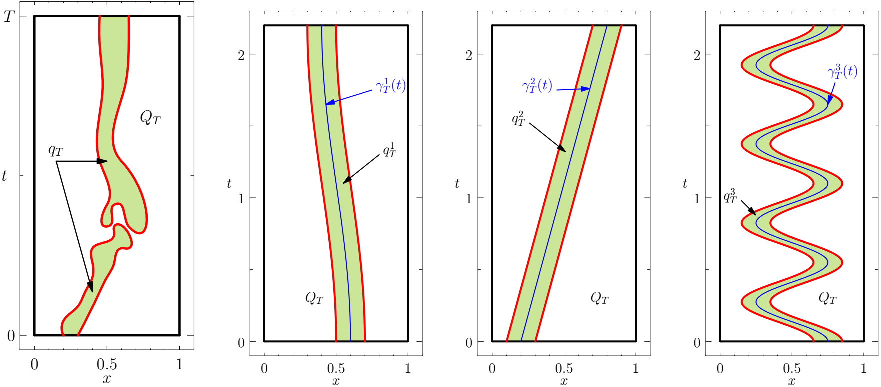
Non cylindrical situation in 1D

[Castro-Cindea-Münch, SICON 2014] [1], [Lebeau, 2017][2]

In 1D with $c \equiv 1$ and $d \equiv 0$, the observability ineq. also holds for non cylindrical domains.



Time dependent domains $q_T \subset Q_T = \Omega \times (0, T)$

[1] C. Castro, N. Cindea, A. Münch, Controllability of the 1D wave equation with inner moving force, SICON (2014)]

[2] G. Lebeau, J. Le Rousseau, P. Terpolilli, E. Trélat, Geometric control condition for the wave equation with a time-dependent domain, (2017)

Equivalent formulation of IP

Within this hypothesis, for any $\eta > 0$, we define on Z the bilinear form

$$\langle y, \overline{y} \rangle_Z := \iint_{q_T} y\, \overline{y}\, dxdt + \eta \int_0^T \langle Ly,\, L\overline{y} \rangle_{H^{-1}(\Omega)}\, dt \quad \forall y, \overline{y} \in Z. \tag{5}$$

$(Z, \|\cdot\|)$ is a Hilbert space.

Then, we consider the extremal problem :

$$(\mathcal{P}) \quad \begin{cases} \inf J(y) := \frac{1}{2}\|y - y_{obs}\|^2_{L^2(q_T)} + \frac{r}{2}\|Ly\|^2_X, \quad r \geq 0 \\ \text{subject to} \quad y \in W := \{y \in Z;\ Ly = 0 \text{ in } X\} \end{cases}$$

$(\mathcal{P})$ is well posed : J is continuous over W, strictly convex and $J(y) \to +\infty$ as $\|y\|_W \to \infty$.

The solution of $(\mathcal{P})$ in W does not depend on η.

From (3), the solution y in Z of $(\mathcal{P})$ satisfies $(y(\cdot,0), y_t(\cdot,0)) \in \boldsymbol{H}$, so that problem $(\mathcal{P})$ is equivalent to the minimization of J w.r.t $(y_0, y_1) \in \boldsymbol{H}$.

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Optimality of $(\mathcal{P})$

In order to solve $(\mathcal{P})$, we have to deal with the constraint eq. which appears in W. We introduce a Lagrange multiplier $\lambda \in X'$ and the following mixed formulation: find $(y, \lambda) \in Z \times X'$ solution of

$$\begin{cases} a_r(y, \overline{y}) + b(\overline{y}, \lambda) & = & l(\overline{y}), & \forall \overline{y} \in Z \\ b(y, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in \Lambda, \end{cases} \tag{6}$$

where

$$a_r : Z \times Z \to \mathbb{R}, \quad a_r(y, \overline{y}) := \iint_{q_T} y\, \overline{y}\, dxdt + r \int_0^T \langle Ly,\ L\overline{y} \rangle_{H^{-1}(\Omega)}\, dt,$$

$$b : Z \times X' \to \mathbb{R}, \quad b(y, \lambda) := \int_0^T \langle \lambda,\ Ly \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt,$$

$$l : Z \to \mathbb{R}, \quad l(y) := \iint_{q_T} y_{obs}\, y\, dxdt.$$

System (22) is the optimality system corresponding to the extremal problem $(\mathcal{P})$.

3

[3] N. Cindea, AM, Inverse problem for linear hyperbolic equations using mixed formulations, Inverse Problems (2015)

Well-posedness of the mixed formulation

Theorem

For all $r \geq 0$,

1. *The mixed formulation (22) is well-posed.*
2. *The unique solution* $(y, \lambda) \in Z \times X'$ *is the unique saddle-point of the Lagrangian* $\mathcal{L} : Z \times X' \to \mathbb{R}$ *defined by*

$$\mathcal{L}(y, \lambda) := \frac{1}{2} a_r(y, y) + b(y, \lambda) - l(y).$$

3. *We have the estimate*

$$\|y\|_Y = \|y\|_{L^2(q_T)} \leq \|y_{obs}\|_{L^2(q_T)}, \quad \|\lambda\|_{X'} \leq 2\sqrt{C_{\Omega,T} + \eta}\|y_{obs}\|_{L^2(q_T)}. \qquad (7)$$

Well-posedness

The kernel $\mathcal{N}(b) = \{y \in Z; b(y,\lambda) = 0 \quad \forall\lambda \in X'\}$ coincides with W: we easily get

$$a_r(y,y) = \|y\|_Z^2, \quad \forall y \in \mathcal{N}(b) = W.$$

It remains to check the inf-sup constant property : $\exists\delta > 0$ such that

$$\inf_{\lambda\in X'} \sup_{y\in Z} \frac{b(y,\lambda)}{\|y\|_Z\|\lambda\|_{X'}} \geq \delta. \tag{8}$$

For any fixed $\lambda \in X'$, we define $y^0 \in Z$ as the unique solution of

$$Ly^0 = -\Delta\lambda \ \text{ in } \ Q_T, \quad (y^0(\cdot,0), y_t^0(\cdot,0)) = (0,0) \ \text{ on } \ \Omega, \quad y^0 = 0 \ \text{ on } \ \Sigma_T.$$

We get $b(y^0,\lambda) = \|\lambda\|_{X'}^2$ and $\|y^0\|_Z^2 = \|y^0\|_{L^2(q_T)}^2 + \eta\|\lambda\|_{X'}^2$.

The estimate $\|y^0\|_{L^2(q_T)} \leq \sqrt{C_{\Omega,T}}\|\lambda\|_{X'}$ implies that

$$\sup_{y\in Z} \frac{b(y,\lambda)}{\|y\|_Y\|\lambda\|_{X'}} \geq \frac{b(y^0,\lambda)}{\|y^0\|_Y\|\lambda\|_{X'}} \geq \frac{1}{\sqrt{C_{\Omega,T}+\eta}} > 0$$

leading to (8) with $\delta = (C_{\Omega,T}+\eta)^{-1/2}$.

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We get $b(y^0,\lambda) = \|\lambda\|_{X'}^2$ and $\|y^0\|_Z^2 = \|y^0\|_{L^2(q_T)}^2 + \eta\|\lambda\|_{X'}^2$.

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$$\sup_{y \in Z} \frac{b(y,\lambda)}{\|y\|_Y \|\lambda\|_{X'}} \geq \frac{b(y^0,\lambda)}{\|y^0\|_Y \|\lambda\|_{X'}} \geq \frac{1}{\sqrt{C_{\Omega,T} + \eta}} > 0$$

leading to (8) with $\delta = (C_{\Omega,T} + \eta)^{-1/2}$.

Well-posedness

The kernel $\mathcal{N}(b) = \{y \in Z; b(y,\lambda) = 0 \quad \forall \lambda \in X'\}$ coincides with W: we easily get

$$a_r(y,y) = \|y\|_Z^2, \quad \forall y \in \mathcal{N}(b) = W.$$

It remains to check the inf-sup constant property : $\exists \delta > 0$ such that

$$\inf_{\lambda \in X'} \sup_{y \in Z} \frac{b(y,\lambda)}{\|y\|_Z \|\lambda\|_{X'}} \geq \delta. \tag{8}$$

For any fixed $\lambda \in X'$, we define $y^0 \in Z$ as the unique solution of

$$Ly^0 = -\Delta\lambda \text{ in } Q_T, \quad (y^0(\cdot,0), y^0_t(\cdot,0)) = (0,0) \text{ on } \Omega, \quad y^0 = 0 \text{ on } \Sigma_T.$$

We get $b(y^0,\lambda) = \|\lambda\|^2_{X'}$ and $\|y^0\|^2_Z = \|y^0\|^2_{L^2(q_T)} + \eta\|\lambda\|^2_{X'}$.

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leading to (8) with $\delta = (C_{\Omega,T} + \eta)^{-1/2}$.

Remark 1

Taking $r = 0$, the first equation of the mixed formulation reads

$$\iint_{q_T} (y - y_{obs})\overline{y} dtdx + \int_0^T < \lambda, L\overline{y} >_{H_0^1, H^{-1}} dt = 0, \quad \forall \overline{y} \in Z$$

which means that the multiplier $\lambda \in X'$ solves in the sense of transposition

$$\begin{cases} L\lambda = -(y - y_{obs}) 1_{q_T}, \quad \lambda = 0 \quad \text{in} \quad \Sigma_T, \\ \lambda(\cdot, T) = \lambda_t(\cdot, T) = 0 \quad \text{in}\, \Omega \end{cases} \tag{9}$$

Therefore, λ coincides with the weak solution of the wave equation controlled by v.

$$\lambda \in C^0([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$$

If y_{obs} is the restriction to q_T of a solution of (1), then λ vanishes everywhere.

In that case, $\sup_{\lambda \in \Lambda} \inf_{y \in Y} \mathcal{L}_r(y, \lambda) = \inf_{y \in Y} \mathcal{L}_r(y, 0)$ and the mixed formulation is reduced to : find $y \in Z$ such that

$$a_r(y, \overline{y}) = \iint_{q_T} y\, \overline{y}\, dxdt + r \int_0^T \langle Ly, L\overline{y} \rangle_{H^{-1}, H^{-1}(\Omega)} dt = l(\overline{y}), \quad \forall \overline{y} \in Z.$$

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Remark 2

In the general case, the mixed formulation can be rewritten as follows: find $(z, \lambda) \in Z \times X'$ solution of

$$\begin{cases} \langle P_r y, P_r \overline{y} \rangle_{X \times L^2(q_T)} + \langle L\overline{y}, \lambda \rangle_{X,X'} = \langle (0, y_{obs}), P_r \overline{y} \rangle_{X \times L^2(q_T)}, & \forall \overline{y} \in Z, \\ \langle Ly, \overline{\lambda} \rangle_{X,X'} = 0, & \forall \overline{\lambda} \in X' \end{cases}$$

with $P_r y := (\sqrt{r} L\, y, y_{|q_T})$.

Analogy with the quasi-reversibility method [Klibanov-Beilina 08, Bourgeois-Darde 10]: for any $\varepsilon > 0$, find $y_\varepsilon \in Z$ such that

$$\langle P y_\varepsilon, P\overline{y} \rangle_{X \times L^2(q_T)} + \varepsilon \langle y_\varepsilon, \overline{y} \rangle_Z = \langle (f, y_{obs}), P\overline{y} \rangle_{X \times L^2(q_T), X \times L^2(q_T)}, \quad \forall \overline{y} \in Z, \quad (QR)$$

equivalent to the minimization over Z of

$$\begin{aligned} y \rightarrow & \| Py - (f, y_{obs}) \|^2_{X \times L^2(q_T)} + \varepsilon \|y\|^2_Z \\ & = \|Ly - f\|^2_X + \|y - y_{obs}\|^2_{L^2(q_T)} + \varepsilon \|y\|^2_Z \end{aligned} \quad (10)$$

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Remark 3: Stabilized mixed formulation "à la Barbosa-Hughes" [4]

$$\Lambda := \left\{\lambda \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), L\lambda \in L^2(Q_T), \lambda(\cdot,T) = \lambda_t(\cdot,T) = 0\right\}.$$

$$\begin{cases} \sup_{\lambda\in\Lambda} \inf_{y\in Z} \mathcal{L}_{r,\alpha}(y,\lambda) \\ \mathcal{L}_{r,\alpha}(y,\lambda) := \mathcal{L}_r(y,\lambda) - \frac{\alpha}{2}\|L\lambda + (y - y_{obs})1_\omega\|^2_{L^2(Q_T)}, \qquad \alpha > 0. \end{cases}$$

For $\alpha \geq 0$, find $(y,\lambda) \in Z \times \Lambda$ such that

$$\begin{cases} a_{r,\alpha}(y,\overline{y}) + b_\alpha(\overline{y},\lambda) = l_{1,\alpha}(\overline{y}), & \forall \overline{y} \in Y \\ b_\alpha(y,\overline{\lambda}) - c_\alpha(\lambda,\overline{\lambda}) = l_{2,\alpha}(\overline{\lambda}), & \forall \overline{\lambda} \in \widetilde{\Lambda}, \end{cases} \tag{11}$$

$$a_{r,\alpha} : Z \times Z \to \mathbb{R}, \quad a_{r,\alpha}(y,\overline{y}) := (1-\alpha)\iint_{q_T} y\overline{y}\,dxdt + r\int_0^T (Ly, L\overline{y})_{H^{-1}(\Omega)}dt,$$

$$b_\alpha : Z \times \Lambda \to \mathbb{R}, \quad b_\alpha(y,\lambda) := \int_0^T \langle \lambda, Ly\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}dt - \alpha\iint_{q_T} y\,L\lambda\,dxdt,$$

$$c_\alpha : \Lambda \times \Lambda \to \mathbb{R}, \quad c_\alpha(\lambda,\overline{\lambda}) := \alpha\iint_{Q_T} L\lambda\,L\overline{\lambda},\ dxdt$$

$$l_{1,\alpha} : Z \to \mathbb{R}, \quad l_{1,\alpha}(y) := (1-\alpha)\iint_{q_T} y_{obs}\,y\,dxdt,$$

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Remark 3: Stabilized mixed formulation "à la Barbosa-Hughes"

Proposition

Under the hypothesis $(\mathcal{H})$, *for any* $\alpha \in (0,1)$, *the corresponding mixed formulation is well-posed. The unique pair* (y,λ) *in* $Z \times \Lambda$ *satisfies*

$$\theta_1 \|y\|_Z^2 + \theta_2 \|\lambda\|_\Lambda^2 \leq \left(\frac{(1-\alpha)^2}{\theta_1} + \frac{\alpha^2}{\theta_2} \right) \|y_{obs}\|_{L^2(q_T)}^2. \quad (12)$$

with $\theta_1 := \min\left(1-\alpha, r\,\eta^{-1}\right), \theta_2 := \frac{1}{2}\min\left(\alpha, C_{\Omega,T}^{-1}\right)$.

Proposition

If $\alpha \in (0,1)$, *the solution* $(y,\lambda) \in Z \times X'$ *of (22) coincides with the stabilized* α *solution in* $Z \times \Lambda$.

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Remark 4 - Link with controllability

The mixed formulation has a structure very closed to the one we get when we address - using the same approach - the null controllability of (1): the control of minimal $L^2(q_T)$-norm which drives to rest $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ is given by $v = \varphi\, 1_{q_T}$ where $(\varphi, \lambda) \in \Phi \times L^2(0, T; H_0^1(\Omega))$ solves

$$\begin{cases} a(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) = l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi \\ b(\varphi, \overline{\lambda}) = 0, & \forall \overline{\lambda} \in L^2(0, T; H_0^1(\Omega)), \end{cases} \tag{13}$$

where

$$a : \Phi \times \Phi \to \mathbb{R}, \quad a(\varphi, \overline{\varphi}) = \iint_{q_T} \varphi(x,t)\overline{\varphi}(x,t)\, dx\, dt$$

$$b : \Phi \times L^2(0, T; H_0^1(0,1)) \to \mathbb{R}, \quad b(\varphi, \lambda) = \int_0^T < L\varphi, \lambda >_{H^{-1}, H_0^1} dt$$

$$l : \Phi \to \mathbb{R}, \quad l(\varphi) = - < \varphi_t(\cdot, 0), y_0 >_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_0^1 \varphi(\cdot, 0)\, y_1\, dx.$$

with $\Phi = \{\varphi \in L^2(q_T),\ \varphi = 0 \text{ on } \Sigma_T \text{ such that } L\varphi \in L^2(0, T; H^{-1}(0,1))\}$.
[Cîndea- Münch, Calcolo 2015]

Remark 5

"Reversing the order of priority" between the constraint $y - y_{obs} = 0$ in $L^2(q_T)$ and $Ly - f = 0$ in X, a possibility could be to minimize the functional

$$\begin{cases} \text{minimize} \quad J(y) := \|Ly - f\|_X^2 + \varepsilon\|y\|_{\mathcal{A}}^2 \\ \text{subject to } y \in Z \quad \text{and to} \quad y - y_{obs} = 0 \quad \text{in} \quad L^2(q_T) \end{cases} \tag{14}$$

via the introduction of a Lagrange multiplier in $L^2(q_T)$.

The proof of the inf-sup property : there exists $\delta > 0$ such that

$$\inf_{\lambda \in L^2(q_T)} \sup_{y \in Z} \frac{\iint_{q_T} \lambda y \, dxdt}{\|\lambda\|_{L^2(q_T)} \|y\|_Y} \geq \delta$$

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Remark 6 : Dual of the mixed problem - Minimization over λ

Theorem

$$\sup_{\lambda \in X'} \inf_{y \in Z} \mathcal{L}_r(y, \lambda) = - \inf_{\lambda \in X'} J_r^{\star\star}(\lambda) \quad + \mathcal{L}_r(y_0, 0)$$

where $y_0 \in Z$ solves $a_r(y_0, \overline{y}) = l(\overline{y}), \forall \overline{y} \in Y$ and

$$J_r^{\star\star} : X' \to \mathbb{R}, \qquad J_r^{\star\star}(\lambda) = \frac{1}{2} \int_0^T \langle \mathcal{P}_r \lambda, \lambda \rangle_{X'} dt - b(y_0, \lambda).$$

Lemma

Let $\mathcal{P}_r$ be the linear operator from X' into X' defined by

$$\mathcal{P}_r \lambda := -\Delta^{-1}(Ly), \quad \forall \lambda \in X' \quad \textit{where} \quad y \in Z \quad \textit{solves} \quad a_r(y, \overline{y}) = b(\overline{y}, \lambda), \forall \overline{y} \in Z.$$

i.e.

$$\iint_{q_T} y \overline{y} dxdt + r \int_0^T < Ly, L\overline{y} >_{H^{-1}} dt = \int_0^T < Ly, \lambda >_{X, X'} dt, \forall \overline{y} \in Z. \quad (15)$$

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Remark 7 - Boundary observation

$(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ - Ω of class C^2

The results apply if the distributed observation on q_T is replaced by a Neumann boundary observation on a sufficiently large subset Σ_T of $\partial\Omega \times (0, T)$ (i.e. assuming $\frac{\partial y}{\partial \nu} = y_{\nu,obs} \in L^2(\Sigma_T)$ is known on Σ_T).

If (Q_T, Σ_T, T) satisfy some geometric condition, then there exists a positive constant $C_{obs} = C(\omega, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)})$ such that

$$\|y(\cdot,0), y_t(\cdot,0)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq C_{obs}\left(\left\|\frac{\partial y}{\partial \nu}\right\|^2_{L^2(\Sigma_T)} + \|Ly\|^2_{L^2(Q_T)}\right), \quad \forall y \in Z \qquad (16)$$

It suffices to re-define the form a in by $a(y, y) := \iint_{\Sigma_T} \frac{\partial y}{\partial \nu}\frac{\partial \overline{y}}{\partial \nu}\, d\sigma dx$ and the form l by $l(y) := \iint_{\Sigma_T} \frac{\partial y}{\partial \nu} y_{obs}\, d\sigma dx$ for all $y, \overline{y} \in Z$.

Inverse problem 2: Simultaneous reconstruction of y and the source from $\partial_\nu y$

$f(x,t) = \sigma(t)\mu(x)$

$c := 1$, $d(x,t) = d(x) \in L^p(\Omega)$, $\sigma \in C^1([0,T]), \sigma(0) \neq 0$, $\mu \in H^{-1}(\Omega)$

Theorem (Yamamoto-Zhang 2001)

Assume that (Γ_T, T, Q_T) satisfies the geometric optic condition. Let $y = y(\mu) \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ be the weak solution of (1) with $c := 1$ and $(y_0, y_1) = (0,0)$. $\exists C > 0$ s.t.

$$C^{-1}\|\mu\|_{H^{-1}(\Omega)} \leq \|\partial_\nu y\|_{L^2(\Gamma_T)} \leq C\|\mu\|_{H^{-1}(\Omega)}, \quad \forall \mu \in H^{-1}(\Omega).$$

This leads to the extremal problem :

$$\begin{cases} \inf J(y,\mu) := \dfrac{1}{2}\|c(x)(\partial_\nu y - y_{\nu,obs})\|^2_{L^2(\Gamma_T)} + \dfrac{r}{2}\displaystyle\iint_{Q_T}(Ly - \sigma\mu)^2\,dxdt, \\ \text{subject to } (y,\mu) \in W := \Big\{(y,\mu);\ y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \\ \qquad \mu \in H^{-1}(\Omega), Ly - \sigma\mu = 0 \text{ in } Q_T, y(\cdot,0) = y_t(\cdot,0) = 0\Big\}. \end{cases} \quad (\mathcal{P}_{y,\mu})$$

Attached to $\|(y,\mu)\|_W := \|c(x)\partial_\nu y\|_{L^2(\Gamma_T)}$, W is a Hilbert space.

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Inverse problem 2: Simultaneous reconstruction of y and the source from $\partial_\nu y$

$f(x,t) = \sigma(t)\mu(x)$
$c := 1$, $d(x,t) = d(x) \in L^p(\Omega)$, $\sigma \in C^1([0,T]), \sigma(0) \neq 0$, $\mu \in H^{-1}(\Omega)$

Theorem (Yamamoto-Zhang 2001)

Assume that (Γ_T, T, Q_T) satisfies the geometric optic condition. Let $y = y(\mu) \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ be the weak solution of (1) with $c := 1$ and $(y_0, y_1) = (0,0)$. $\exists C > 0$ s.t.

$$C^{-1}\|\mu\|_{H^{-1}(\Omega)} \leq \|\partial_\nu y\|_{L^2(\Gamma_T)} \leq C\|\mu\|_{H^{-1}(\Omega)}, \quad \forall \mu \in H^{-1}(\Omega).$$

This leads to the extremal problem :

$$\begin{cases} \inf J(y,\mu) := \frac{1}{2}\|c(x)(\partial_\nu y - y_{\nu,obs})\|^2_{L^2(\Gamma_T)} + \frac{r}{2}\iint_{Q_T}(Ly - \sigma\mu)^2\,dxdt, \\ \text{subject to } (y,\mu) \in W := \Big\{(y,\mu);\ y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \\ \qquad \mu \in H^{-1}(\Omega), Ly - \sigma\mu = 0 \text{ in } Q_T, y(\cdot,0) = y_t(\cdot,0) = 0\Big\}. \end{cases} \quad (\mathcal{P}_{y,\mu})$$

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Attached to $\|(y,\mu)\|_W := \|c(x)\partial_\nu y\|_{L^2(\Gamma_T)}$, W is a Hilbert space.

Recovering the solution and the source f when the pair (y, f) is unique

$$Y := \Big\{(y,\mu); y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \mu \in H^{-1}(\Omega),$$
$$Ly - \sigma\mu \in L^2(Q_T), y(\cdot,0) = y_t(\cdot,0) = 0\Big\}. \tag{17}$$

Hypothesis

$\exists C_{obs} = C(\Gamma_T, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)}) > 0$ *s.t. :*

$$\|\mu\|^2_{H^{-1}(\Omega)} \leq C_{obs}\Big(\|c(x)\partial_\nu y\|^2_{L^2(\Gamma_T)} + \|Ly - \sigma\mu\|^2_{L^2(Q_T)}\Big), \quad \forall (y,\mu) \in Y. \qquad (\mathcal{H}_2)$$

Then, $\forall \eta > 0$, we define on Y the bilinear form

$$\langle (y,\mu),(\overline{y},\overline{\mu})\rangle_Y := \iint_{\Gamma_T} (c(x))^2\,\partial_\nu y\,\partial_\nu \overline{y}\,d\sigma dt + \eta\iint_{Q_T}(Ly - \sigma\mu)\,(L\overline{y} - \sigma\overline{\mu})\,dxdt \quad \forall y, \overline{y} \in Z. \tag{18}$$

$$\|(y,z)\|_Y := \sqrt{<(y,\mu),(y,\mu)>_Y}.$$

Lemma

Under the hypotheses $(\mathcal{H}_2)$, the space $(Y, \|\cdot\|_Y)$ is a Hilbert space.

Recovering the solution and the source f when the pair (y, f) is unique

$$Y := \Big\{(y,\mu); y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \mu \in H^{-1}(\Omega),$$
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$$\|(y,z)\|_Y := \sqrt{< (y,\mu), (y,\mu) >_Y}.$$

Lemma

Under the hypotheses $(\mathcal{H}_2)$*, the space* $(Y, \|\cdot\|_Y)$ *is a Hilbert space.*

Recovering the solution and the source f: mixed formulation

Find $((y,\mu),\lambda) \in Y \times L^2(Q_T)$ solution of

$$\left\{ \begin{aligned} a_r((y,\mu),(\overline{y},\overline{\mu})) + b((\overline{y},\overline{\mu}),\lambda) &= l(\overline{y},\overline{\mu}), & \forall(\overline{y},\overline{\mu}) \in Y \\ b((y,\mu),\overline{\lambda}) &= 0, & \forall\overline{\lambda} \in L^2(Q_T), \end{aligned} \right. \tag{19}$$

where

$$a_r : Y \times Y \to \mathbb{R}, \quad a_r((y,\mu),(\overline{y},\overline{\mu})) := \iint_{\Gamma_T} c^2(x)\partial_\nu y \partial_\nu \overline{y}\, d\sigma dt$$
$$+ r \iint_{Q_T} (Ly - \sigma\mu)(L\overline{y} - \sigma\overline{\mu})\, dxdt,\ r \geq 0$$
$$b : Y \times L^2(Q_T) \to \mathbb{R}, \quad b((y,\mu),\lambda) := \iint_{Q_T} \lambda(Ly - \sigma\mu) dx\, dt,$$
$$l : Y \to \mathbb{R}, \quad l(y,\mu) := \iint_{\Gamma_T} c^2(x)\, \partial_\nu y\, y_{\nu,obs}\, d\sigma dt.$$

[5]

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[5] N. Cindea, AM, Simultaneous reconstruction of the solution and the source of hyperbolic equations from boundary measurements: a robust numerical approach, Inverse Problems (2016)

Parabolic situation

$\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ - $T > 0$, $c \in C^1(\overline{\Omega}, \mathbb{R}))$, $d \in L^\infty(Q_T)$, $y_0 \in \boldsymbol{H}$

$$\begin{cases} Ly := y_t - \nabla \cdot (c\nabla y) + dy = f, & Q_T := \Omega \times (0, T) \\ y = 0, & \Sigma_T := \partial\Omega \times (0, T) \\ y(\cdot, 0) = y_0, & \Omega. \end{cases} \tag{20}$$

- Inverse Problem : Distributed observation on $q_T = \omega \times (0, T), \omega \subset \Omega$

$$\begin{cases} X = L^2(q_T), \\ \text{Given } (y_{obs}, f) \in (L^2(q_T), X), \text{find } y \; s.t. \{(20) \text{ and } y - y_{obs} = 0 \quad \text{on} \quad q_T\} \end{cases}$$

Well-known difficulty:

$$\left(Ly \in L^2(Q_T), y \in L^2(q_T), y_{|\Sigma_T} = 0 \right) \Longrightarrow y \in C([\delta, T], H_0^1(\Omega)), \quad \forall \delta > 0$$

Observability inequality for the heat eq. (Carleman inequality)

Let $\rho_c, \rho_{c,0}$ be some Carleman weights of the form

$$\rho_c(t) = t^{\alpha} \exp(1/t), \quad \rho_{c,0}(t) = t^{\beta} \exp(1/t)$$

$$\left\{ \begin{array}{l} \displaystyle\iint_{Q_T} \rho_{c,0}^{-2} |y|^2 \, dx \, dt \\ \displaystyle\qquad\qquad \leq C \left(\iint_{Q_T} \rho_c^{-2} |Ly|^2 \, dx \, dt + \iint_{q_T} \rho_{c,0}^{-2} |y|^2 \, dx \, dt \right), \forall y \in Y. \end{array} \right. \tag{21}$$

Second order mixed formulation .... as for the wave equation

We then define the following extremal problem :

$$\begin{cases} \text{Minimize } \; J(y) := \dfrac{1}{2} \displaystyle\iint_{q_T} \rho_0^{-2} |y(x,t) - y_{obs}(x,t)|^2 \, dx\, dt + r \iint_{Q_T} (\rho^{-1} L\, y)^2 \, dx\, dt \\ \text{Subject to } \; y \in \mathcal{W} := \left\{ y \in \mathcal{Y} : \rho^{-1} L y = 0 \text{ in } L^2(Q_T) \right\} \end{cases} \tag{P}$$

with $\rho_0, \rho \in \mathcal{R}$ where ($\rho_\star \in \mathbb{R}_\star^+$)

$$\mathcal{R} := \{ w : w \in C(Q_T); w \geq \rho_\star > 0 \text{ in } Q_T; w \in L^\infty(\Omega \times (\delta, T)) \; \forall \delta > 0 \}$$

Let $\mathcal{Y}_0 := \left\{ y \in C^2(\overline{Q}_T) : y = 0 \text{ on } \Sigma_T \right\}$ and for $\eta > 0$, $\rho \in \mathcal{R}$, the bilinear form by

$$(y, \overline{y})_{\mathcal{Y}_0} := \iint_{q_T} \rho_0^{-2} y\, \overline{y} \, dx\, dt + \eta \iint_{Q_T} \rho^{-2} L\, y L\, \overline{y} \, dx\, dt, \quad \forall y, \overline{y} \in \mathcal{Y}_0.$$

Let $\mathcal{Y}$ be the completion of $\mathcal{Y}_0$ for this scalar product endowed with the norm

$$\|y\|_{\mathcal{Y}}^2 := \|\rho_0^{-1} y\|_{L^2(q_T)}^2 + \eta \|\rho^{-1} L\, y\|_{L^2(Q_T)}^2, \quad \forall y \in \mathcal{Y}.$$

Second order mixed formulation .... as for the wave equation

We then define the following extremal problem :

$$\begin{cases} \text{Minimize } \; J(y) := \frac{1}{2}\iint_{q_T} \rho_0^{-2}|y(x,t)-y_{obs}(x,t)|^2\,dx\,dt + r\iint_{Q_T}(\rho^{-1}L\,y)^2\,dx\,dt \\ \text{Subject to } \; y \in \mathcal{W} := \left\{ y \in \mathcal{Y} : \rho^{-1}Ly = 0 \text{ in } L^2(Q_T)\right\} \end{cases} \qquad (P)$$

with $\rho_0, \rho \in \mathcal{R}$ where $(\rho_\star \in \mathbb{R}_\star^+)$

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Mixed formulation

Find $(y, \lambda) \in \mathcal{Y} \times L^2(Q_T)$ solution of

$$\begin{cases} a_r(y, \overline{y}) + b(\overline{y}, \lambda) & = & l(\overline{y}) & \forall \overline{y} \in \mathcal{Y}, \\ b(y, \overline{\lambda}) & = & 0 & \forall \overline{\lambda} \in L^2(Q_T), \end{cases} \tag{22}$$

where

$$a_r : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}, \quad a(y, \overline{y}) := \iint_{q_T} \rho_0^{-2} y\, \overline{y}\, dx\, dt + r \iint_{Q_T} \rho^{-2} Ly\, L\overline{y}\, dx\, dt$$

$$b : \mathcal{Y} \times L^2(Q_T) \to \mathbb{R}, \quad b(y, \lambda) := \iint_{Q_T} \rho^{-1} Ly\, \lambda\, dx\, dt$$

$$l : \mathcal{Y} \to \mathbb{R}, \quad l(y) := \iint_{q_T} \rho_0^{-2} y\, y_{obs}\, dx\, dt.$$

Theorem

Let $\rho_0 \in \mathcal{R}$ and $\rho \in \mathcal{R} \cap L^\infty(Q_T)$.

1. *The mixed formulation (22) is well-posed.*
2. *The unique solution $(y, \lambda) \in \mathcal{Y} \times L^2(Q_T)$ is the unique saddle-point of the Lagrangian $\mathcal{L}_r : \mathcal{Y} \times L^2(Q_T) \to \mathbb{R}$ defined by*

$$\mathcal{L}_r(y,\lambda) := \frac{1}{2} a_r(y,y) + b(y,\lambda) - l(y).$$

3. *The solution (y, λ) satisfies the estimates*

$$\|y\|_{\mathcal{Y}} \le \|\rho_0^{-1} y_{obs}\|_{L^2(q_T)}, \quad \|\lambda\|_{L^2(Q_T)} \le 2\sqrt{\rho_\star^{-2}\|\rho\|^2_{L^\infty(Q_T)} + \eta}\, \|\rho_0^{-1} y_{obs}\|_{L^2(q_T)}.$$

Corollary

Let $\rho_0 \in \mathcal{R}, \rho \in \mathcal{R} \cap L^\infty(Q_T)$ and assume $\exists K$ s.t.

$$\rho_0 \le K\rho_{c,0}, \quad \rho \le K\rho_c \quad in \quad Q_T.$$

If (y,λ) is the solution of the mixed formulation (22), then $\exists C > 0$ such that

$$\|\rho_{c,0}^{-1} y\|_{L^2(Q_T)} \le C\|y\|_{\mathcal{Y}}.$$

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If (y, λ) is the solution of the mixed formulation (22), then $\exists C > 0$ such that

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Stabilization

The first equation of the mixed formulation (22) reads as follows:

$$\iint_{q_T} \rho_0^{-2}\, y\, \overline{y}\, dx\, dt + \iint_{Q_T} \rho^{-1} L\overline{y}\, \lambda\, dx\, dt = \iint_{q_T} \rho_0^{-2}\, y_{obs}\, \overline{y}\, dx\, dt \quad \forall \overline{y} \in \mathcal{Y}.$$

$\rho^{-1}\lambda \in L^2(Q_T)$ solves the parabolic equation in the transposition sense, i.e. $\rho^{-1}\lambda$ solves the problem :

$$\left\{ \begin{array}{lll} L^\star\,(\rho^{-1}\lambda) = -\rho_0^{-2}\,(y - y_{obs})1_{q_T} & \text{in} & Q_T, \\ \rho^{-1}\lambda = 0 & \text{on} & \Sigma_T, \\ (\rho^{-1}\lambda)(\cdot, T) = 0 & \text{in} & \Omega. \end{array} \right. \tag{23}$$

Therefore, $\rho^{-1}\lambda$ belongs to $C^0([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$.

$$\begin{aligned} \Lambda := \{\lambda : \rho^{-1}\lambda \in C^0([0,T]; L^2(\Omega)), \rho_0\, L^\star(\rho^{-1}\lambda) \in L^2(Q_T), \\ \rho^{-1}\lambda = 0 \text{ on } \Sigma_T,\ (\rho^{-1}\lambda)(\cdot, T) = 0\}. \end{aligned} \tag{24}$$

$$\left\{ \begin{array}{l} \sup\limits_{\lambda \in \Lambda} \inf\limits_{y \in \mathcal{Y}} \mathcal{L}_{r,\alpha}(y, \lambda), \\ \\ \mathcal{L}_{r,\alpha}(y, \lambda) := \mathcal{L}_r(y, \lambda) - \dfrac{\alpha}{2} \left\| \rho_0 \left(L^\star(\rho^{-1}\lambda) + \rho_0^{-2}(y - y_{obs})1_\omega \right) \right\|^2_{L^2(Q_T)}. \end{array} \right.$$

Dual formulation

For any $r > 0$, let us define the linear operator $\mathcal{T}_r$ from $L^2(Q_T)$ into $L^2(Q_T)$ by

$$\mathcal{T}_r\lambda := \rho^{-1}Ly, \quad \forall\lambda \in L^2(Q_T)$$

where $y \in \mathcal{Y}$ is the unique solution to

$$a_r(y, \overline{y}) = b(\overline{y}, \lambda), \quad \forall\overline{y} \in \mathcal{Y}. \tag{25}$$

Lemma

For any $r > 0$, the operator $\mathcal{T}_r$ is a strongly elliptic, symmetric isomorphism from $L^2(Q_T)$ into $L^2(Q_T)$.

Proposition

For any $r > 0$, let $y_0 \in \mathcal{Y}$ be the unique solution of

$$a_r(y_0, \overline{y}) = l(\overline{y}), \quad \forall\overline{y} \in \mathcal{Y}$$

and let $J_r^{\star\star} : L^2(Q_T) \to L^2(Q_T)$ be the functional defined by

$$J_r^{\star\star}(\lambda) := \frac{1}{2}\iint_{Q_T} (\mathcal{T}_r\lambda)\lambda \, dx\, dt - b(y_0, \lambda).$$

The following equality holds :

$$\sup_{\lambda\in L^2(Q_T)} \inf_{y\in\mathcal{Y}} \mathcal{L}_r(y, \lambda) = -\inf_{\lambda\in L^2(Q_T)} J_r^{\star\star}(\lambda) \quad + \mathcal{L}_r(y_0, 0).$$

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$H_0^1 - L^2$ first order formulation

First order formulation involving y and the flux $\mathbf{p} = c(x)\nabla y$.

$$\begin{cases} \mathcal{I}(y,\mathbf{p}) := y_t - \nabla\cdot\mathbf{p} + d\,y = f, \quad \mathcal{J}(y,\mathbf{p}) := c(x)\nabla y - \mathbf{p} = \mathbf{0} & \text{in} \quad Q_T, \\ y = 0 & \text{on} \quad \Sigma_T, \\ y(x,0) = y_0(x) & \text{in} \quad \Omega. \end{cases} \tag{26}$$

$$(y_0, f) \in L^2(\Omega)\times L^2(Q_T) \Longrightarrow p \in \mathbf{L}^2(Q_T), y \in L^2(0,T,H_0^1(\Omega)), y_t \in L^2(0,T,H^{-1}(\Omega))$$

- Inverse Problem : Distributed observation on $q_T = \omega \times (0,T), \omega \subset \Omega$

$$\begin{cases} X = L^2(q_T), \\ \text{Given } (y_{obs}, f) \in (L^2(q_T), X), \text{ find } (y,\mathbf{p}) \; s.t. \{(26) \text{ and } y - y_{obs} = 0 \quad \text{on} \quad q_T\} \end{cases}$$

[6] D. de Souza, AM, Inverse problems for linear parabolic equations using mixed formulations - Part 1 : Theoretical analysis, Journal of Inverse and Ill posed problems (2017)

Parabolic case: $H_0^1 - L^2$ first order formulation of the parabolic

The extremal problem is then :

$$\begin{cases} \text{Minimize } \; J(y,\mathbf{p}) := \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^{-2}|y(x,t) - y_{obs}(x,t)|^2\,dx\,dt + \mathbf{r}.... \\ (y,\mathbf{p}) \in \mathcal{V} := \Big\{ (y,\mathbf{p}) \in \mathcal{U} \,:\, \rho_1^{-1}\mathcal{J}(y,\mathbf{p}) = 0 \text{ in } \mathbf{L}^2(Q_T), \quad \rho^{-1}\mathcal{I}(y,\mathbf{p}) = 0 \text{ in } L^2(Q_T) \Big\} \end{cases}$$

$\mathcal{U}$ - completion of $\mathcal{U}_0 := \Big\{ (y,\mathbf{p}) \in C^1(\overline{Q}_T) \times \mathbf{C}^1(\overline{Q}_T) : y = 0 \text{ on } \Sigma_T \Big\}$ for

$$\begin{aligned} ((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}))_{\mathcal{U}_0} = & \iint_{q_T} \rho_0^{-2} y\,\overline{y}\,dx\,dt + \eta_1 \iint_{Q_T} \rho_1^{-2}\mathcal{J}(y,\mathbf{p})\cdot\mathcal{J}(\overline{y},\overline{\mathbf{p}})\,dx\,dt \\ & + \eta_2 \iint_{Q_T} \rho^{-2}\mathcal{I}(y,\mathbf{p})\mathcal{I}(\overline{y},\overline{\mathbf{p}})\,dx\,dt \quad \forall (y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}) \in \mathcal{U}_0. \end{aligned}$$

for any $\eta_1, \eta_2 > 0$ and any $\rho,\ \rho_0,\ \rho_1 \in \mathcal{R}$

$$\|(y,\mathbf{p})\|_{\mathcal{U}}^2 := \|\rho_0^{-1}y\|_{L^2(q_T)}^2 + \eta_1\|\rho_1^{-1}\mathcal{J}(y,\mathbf{p})\|_{\mathbf{L}^2(Q_T)}^2 + \eta_2\|\rho^{-1}\mathcal{I}(y,\mathbf{p})\|_{L^2(Q_T)}^2.$$

Parabolic case: $H_0^1 - L^2$ first order formulation of the parabolic

The extremal problem is then :

$$\begin{cases} \text{Minimize } J(y,\mathbf{p}) := \frac{1}{2}\iint_{q_T} \rho_0^{-2}|y(x,t) - y_{obs}(x,t)|^2\,dx\,dt + \mathbf{r}.... \\ (y,\mathbf{p}) \in \mathcal{V} := \Big\{(y,\mathbf{p}) \in \mathcal{U} \,:\, \rho_1^{-1}\mathcal{J}(y,\mathbf{p}) = 0 \text{ in } \mathbf{L}^2(Q_T), \quad \rho^{-1}\mathcal{I}(y,\mathbf{p}) = 0 \text{ in } L^2(Q_T)\Big\} \end{cases}$$

$\mathcal{U}$ - completion of $\mathcal{U}_0 := \Big\{(y,\mathbf{p}) \in C^1(\overline{Q}_T) \times \mathbf{C}^1(\overline{Q}_T) : y = 0 \text{ on } \Sigma_T\Big\}$ for

$$\begin{aligned} ((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}))_{\mathcal{U}_0} = &\iint_{q_T} \rho_0^{-2} y\,\overline{y}\,dx\,dt + \eta_1 \iint_{Q_T} \rho_1^{-2}\mathcal{J}(y,\mathbf{p})\cdot\mathcal{J}(\overline{y},\overline{\mathbf{p}})\,dx\,dt \\ &+ \eta_2 \iint_{Q_T} \rho^{-2}\mathcal{I}(y,\mathbf{p})\mathcal{I}(\overline{y},\overline{\mathbf{p}})\,dx\,dt \quad \forall (y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}) \in \mathcal{U}_0. \end{aligned}$$

for any $\eta_1, \eta_2 > 0$ and any $\rho,\ \rho_0,\ \rho_1 \in \mathcal{R}$

$$\|(y,\mathbf{p})\|_{\mathcal{U}}^2 := \|\rho_0^{-1}y\|_{L^2(q_T)}^2 + \eta_1\|\rho_1^{-1}\mathcal{J}(y,\mathbf{p})\|_{\mathbf{L}^2(Q_T)}^2 + \eta_2\|\rho^{-1}\mathcal{I}(y,\mathbf{p})\|_{L^2(Q_T)}^2.$$

Parabolic case: $H^1_0 - L^2$ first order formulation - Mixed formulation

Precisely, we set $\mathcal{X} := L^2(Q_T) \times \mathbf{L}^2(Q_T)$ and then we consider the following mixed formulation : find $((y,\mathbf{p}),(\lambda,\boldsymbol{\mu})) \in \mathcal{U} \times \mathcal{X}$ solution of

$$\begin{cases} a_{\mathbf{r}}((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}})) + b((\overline{y},\overline{\mathbf{p}}),(\lambda,\boldsymbol{\mu})) = l(\overline{y},\overline{\mathbf{p}}) & \forall(\overline{y},\overline{\mathbf{p}}) \in \mathcal{U}, \\ b((y,\mathbf{p}),(\overline{\lambda},\overline{\boldsymbol{\mu}})) = 0 & \forall(\overline{\lambda},\overline{\boldsymbol{\mu}}) \in \mathcal{X}, \end{cases} \tag{27}$$

where

$$a_{\mathbf{r}} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}, \quad a_{\mathbf{r}}((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}})) := \iint_{q_T} \rho_0^{-2} y\, \overline{y}\, dx\, dt$$

$$+ r_1 \iint_{Q_T} \rho_1^{-2} \mathcal{J}(y,\mathbf{p}) \cdot \mathcal{J}(\overline{y},\overline{\mathbf{p}})\, dx\, dt + r_2 \iint_{Q_T} \rho^{-2} \mathcal{I}(y,\mathbf{p}) \mathcal{I}(\overline{y},\overline{\mathbf{p}})\, dx\, dt$$

$$b : \mathcal{U} \times \mathcal{X} \to \mathbb{R}, \quad b((y,\mathbf{p}),(\lambda,\boldsymbol{\mu})) := \iint_{Q_T} \rho_1^{-1} \mathcal{J}(y,\mathbf{p}) \cdot \boldsymbol{\mu}\, dx\, dt + \iint_{Q_T} \rho^{-1} \mathcal{I}(y,\mathbf{p}) \lambda\, dx\, dt$$

$$l : \mathcal{U} \to \mathbb{R}, \quad l(y,\mathbf{p}) := \iint_{q_T} \rho_0^{-2} y\, y_{obs}\, dx\, dt.$$

$\forall \mathbf{r} = (r_1, r_2) \in (\mathbb{R}^+)^2$

Proposition (Imanuvilov-Puel-Yamamoto, 2010)

$$\rho_p(x,t) := \exp\left(\frac{\beta(x)}{t^2}\right), \quad \beta(x) := K_1\left(e^{K_2} - e^{\beta_0(x)}\right),$$

$$\rho_{p,0}(x,t) := t\rho_p(x,t), \quad \rho_{p,1}(x,t) := t^{-1}\rho_p(x,t), \quad \rho_{p,2}(x,t) := t^{-2}\rho_p(x,t)$$

$\exists C = C(\omega, T) > 0$ *s.t.*

$$\|\rho_{p,0}^{-1}y\|^2_{L^2(Q_T)} + \|\rho_{p,1}^{-1}\nabla y\|^2_{L^2(Q_T)} \leq C\left(\|\rho_p^{-1}\mathbf{G}\|^2_{\mathbf{L}^2(Q_T)} + \|\rho_{p,2}^{-1}g\|^2_{L^2(Q_T)} + \|\rho_{p,0}^{-1}y\|^2_{L^2(q_T)}\right),$$

for any

$$\begin{cases} y \in \mathcal{K} := \left\{y \in L^2(0,T;H_0^1(\Omega)) : y_t \in L^2(0,T;H^{-1}(\Omega))\right\}, \\ Ly = g + \nabla\cdot\mathbf{G} \text{ in } Q_T, \quad (g,\mathbf{G}) \in L^2(Q_T)\times\mathbf{L}^2(Q_T). \end{cases}$$

$$\begin{cases} Ly = \mathcal{I}(y,\mathbf{p}) - \nabla\cdot\mathcal{J}(y,\mathbf{p}), \\ \mathcal{J}(y,\mathbf{p}) := c(x)\nabla y - \mathbf{p}, \quad \mathcal{I}(y,\mathbf{p}) := y_t - \nabla\cdot\mathbf{p} + dy \end{cases}$$

Finite dimensional approximation

Conformal Approximation of the mixed formulation (boundary observation case, to fix idea)

Let then Z_h and Λ_h be two finite dimensional spaces parametrized by the variable h such that

$$Z_h \subset Z, \quad \Lambda_h \subset L^2(Q_T), \qquad \forall h > 0.$$

Then, we can introduce the following approximated problems : find $(z_h, \lambda_h) \in Z_h \times \Lambda_h$ solution of

$$\left\{ \begin{array}{rcll} a_r(y_h, \overline{y}_h) + b(\overline{y}_h, \lambda_h) & = & l(\overline{y}_h), & \forall \overline{y}_h \in Z_h \\ b(y_h, \overline{\lambda}_h) & = & 0, & \forall \overline{\lambda}_h \in \Lambda_h. \end{array} \right. \tag{28}$$

For any $h > 0$, the well-posedness is again a consequence of two properties

- the coercivity of the bilinear form a_r on the subset $\mathcal{N}_h(b) = \{y_h \in Z_h;\, b(y_h, \lambda_h) = 0 \quad \forall \lambda_h \in \Lambda_h\}$. From the relation

$$a_r(y, y) \geq \frac{r}{\eta} \|y\|_Z^2, \quad \forall y \in Z$$

the form a_r is coercive on the full space Z, and so *a fortiori* on $\mathcal{N}_h(b) \subset Z_h \subset Z$.
- The second property is a discrete inf-sup condition : there exists $\delta > 0$ such that

$$\delta_h := \inf_{\lambda_h \in \Lambda_h} \sup_{y_h \in Z_h} \frac{b(y_h, \lambda_h)}{\|y_h\|_{Z_h} \|\lambda_h\|_{\Lambda_h}} \geq \delta. \tag{29}$$

A necessary condition is: $\dim(Z_h) > \dim(\Lambda_h)$

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Linear system

Let $n_h = \dim Z_h$, $m_h = \dim \Lambda_h$ and let the real matrices $A_{r,h} \in \mathbb{R}^{n_h,n_h}$, $B_h \in \mathbb{R}^{m_h,n_h}$, $J_h \in \mathbb{R}^{m_h,m_h}$ and $L_h \in \mathbb{R}^{n_h}$ be defined by

$$\begin{cases} a_r(y_h, \overline{y_h}) = \langle A_{r,h}\{y_h\}, \{\overline{y_h}\}\rangle_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}} & \forall y_h, \overline{y_h} \in Z_h, \\ b(y_h, \lambda_h) = \langle B_h\{y_h\}, \{\lambda_h\}\rangle_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}} & \forall y_h \in Z_h, \lambda_h \in \Lambda_h, \\ \iint_{Q_T} \lambda_h \overline{\lambda_h}\, dx\, dt = \langle J_h\{\lambda_h\}, \{\overline{\lambda_h}\}\rangle_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}} & \forall \lambda_h, \overline{\lambda_h} \in \Lambda_h, \\ l(y_h) = \langle L_h, \{y_h\}\rangle_{\mathbb{R}^{n_h}} & \forall y_h \in Z_h, \end{cases} \quad (30)$$

where $\{y_h\} \in \mathbb{R}^{n_h}$ denotes the vector associated to y_h and $\langle \cdot, \cdot \rangle_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}$ the usual scalar product over $\mathbb{R}^{n_h}$. With these notations, the problem (28) reads as follows: find $\{y_h\} \in \mathbb{R}^{n_h}$ and $\{\lambda_h\} \in \mathbb{R}^{m_h}$ such that

$$\begin{pmatrix} A_{r,h} & B_h^T \\ B_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix} \{y_h\} \\ \{\lambda_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} L_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}. \quad (31)$$

The matrix of order $m_h + n_h$ is symmetric but not positive definite.

First estimate

Proposition

Let $h > 0$. Let (y, λ) and (y_h, λ_h) be the solution of (22) and of (28) respectively. Let δ_h the discrete inf-sup constant defined by (29). Then,

$$\|y - y_h\|_Z \leq 2\left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right) d(y, Z_h) + \frac{1}{\sqrt{\eta}} d(\lambda, \Lambda_h),$$

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq \left(2 + \frac{1}{\sqrt{\eta}\delta_h}\right) \frac{1}{\delta_h} d(y, Z_h) + \frac{3}{\sqrt{\eta}\delta_h} d(\lambda, \Lambda_h)$$

$d(\lambda, \Lambda_h) := \inf_{\lambda_h \in \Lambda_h} \|\lambda - \lambda_h\|_{L^2(Q_T)}$

Choice of the conformal spaces Z_h and Λ_h

We introduce a triangulation $\mathcal{T}_h$ such that $\overline{Q_T} = \cup_{K \in \mathcal{T}_h} K$ and we assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family. We note $h := \max\{\mathrm{diam}(K), K \in \mathcal{T}_h\}$.

We define the finite dimensional space

$$\Lambda_h = \{\lambda_h \in C^0(\overline{Q_T}), \lambda_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h,\ \lambda_h = 0 \text{ on } \Sigma_T\} \subset L^2(Q_T)$$

The space Z_h must be chosen such that $Ly_h \in L^2(Q_T)$ for any $y_h \in Z_h$. This is guaranteed as soon as y_h possesses second-order derivatives in $L^2(Q_T)$. A conformal approximation based on standard triangulation of Q_T is obtained with spaces of functions continuously differentiable with respect to both x and t.

We introduce the space Z_h as follows:

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C^1 finite element over Q_T

[7]

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where $\mathbb{P}(K)$ denotes an appropriate space of polynomial functions in x and t.

We may consider the following choices for $\mathbb{P}(K)$:

1. The *Bogner-Fox-Schmit* (BFS for short) C^1 element defined for rectangles. It involves 16 degrees of freedom, namely the values of $y_h, y_{h,x}, y_{h,t}, y_{h,xt}$ on the four vertices of each rectangle K.

2. The reduced *Hsieh-Clough-Tocher* (HCT for short) C^1 element defined for triangles. This is a so-called composite finite element and involves 9 degrees of freedom, namely the values of $y_h, y_{h,x}, y_{h,t}$ on the three vertices of each triangle K.

[7] P.G. Ciarlet, The finite element for elliptic problems, North-Holland, 1979

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Convergence rate in Z and in $L^2(Q_T)$

Proposition (BFS element for $N=1$ - Convergence in Z)

Let $h>0$, let $k\leq 2$. If $(y,\lambda)\in H^{k+2}(Q_T)\times H^k(Q_T)$, $\exists K>0$

$$\|y-y_h\|_Z\leq K\Big(1+\frac{1}{\sqrt{\eta}\delta_h}+\frac{1}{\sqrt{\eta}}\Big)h^k,\quad \|\lambda-\lambda_h\|_{L^2(Q_T)}\leq K\Big(\Big(1+\frac{1}{\sqrt{\eta}\delta_h}\Big)\frac{1}{\delta_h}+\frac{1}{\sqrt{\eta}\delta_h}\Big)h^k.$$

Writing the ineq. obs. for $y-y_h\in Z$ and using that $L(y-y_h)=-Ly_h$, we get

$$\begin{aligned}\|y-y_h\|^2_{L^2(Q_T)}&\leq C_{\Omega,T}(C_{obs}+1)(\|\partial_\nu(y-y_h)\|^2_{L^2(\Gamma_T)}+\|Ly_h\|^2_{L^2(Q_T)})\\&\leq C_{\Omega,T}(C_{obs}+1)\max(1,\frac{2}{\sqrt{\eta}})\|y-y_h\|_Z\end{aligned}$$

Theorem (BFS element for $N=1$ - Convergence in $L^2(Q_T)$)

Let $h>0$, let $k\leq 2$. If $(y,\lambda)\in H^{k+2}(Q_T)\times H^k(Q_T)$,

$$\|y-y_h\|_{L^2(Q_T)}\leq K\max(1,\frac{2}{\sqrt{\eta}})\Big(1+\frac{1}{\sqrt{\eta}\delta_h}+\frac{1}{\sqrt{\eta}}\Big)h^k.$$

Convergence rate in Z and in $L^2(Q_T)$

Proposition (BFS element for $N = 1$ - Convergence in Z)

Let $h > 0$, let $k \leq 2$. If $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$, $\exists K > 0$

$$\|y - y_h\|_Z \leq K\left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right)h^k, \quad \|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K\left(\left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right)\frac{1}{\delta_h} + \frac{1}{\sqrt{\eta}\delta_h}\right)h^k.$$

Writing the ineq. obs. for $y - y_h \in Z$ and using that $L(y - y_h) = -Ly_h$, we get

$$\begin{aligned}\|y - y_h\|^2_{L^2(Q_T)} &\leq C_{\Omega,T}(C_{obs} + 1)(\|\partial_\nu(y - y_h)\|^2_{L^2(\Gamma_T)} + \|Ly_h\|^2_{L^2(Q_T)})\\ &\leq C_{\Omega,T}(C_{obs} + 1)\max(1, \frac{2}{\sqrt{\eta}})\|y - y_h\|_Z\end{aligned}$$

Theorem (BFS element for $N = 1$ - Convergence in $L^2(Q_T)$)

Let $h > 0$, let $k \leq 2$. If $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$,

$$\|y - y_h\|_{L^2(Q_T)} \leq K\max(1, \frac{2}{\sqrt{\eta}})\left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right)h^k.$$

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Proposition (BFS element for $N = 1$ - Convergence in Z)

Let $h > 0$, let $k \leq 2$. If $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$, $\exists K > 0$

$$\|y - y_h\|_Z \leq K\left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right)h^k, \quad \|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K\left(\left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right)\frac{1}{\delta_h} + \frac{1}{\sqrt{\eta}\delta_h}\right)h^k.$$

Writing the ineq. obs. for $y - y_h \in Z$ and using that $L(y - y_h) = -Ly_h$, we get

$$\|y - y_h\|^2_{L^2(Q_T)} \leq C_{\Omega,T}(C_{obs} + 1)(\|\partial_\nu(y - y_h)\|^2_{L^2(\Gamma_T)} + \|Ly_h\|^2_{L^2(Q_T)})$$
$$\leq C_{\Omega,T}(C_{obs} + 1)\max(1, \frac{2}{\sqrt{\eta}})\|y - y_h\|_Z$$

Theorem (BFS element for $N = 1$ - Convergence in $L^2(Q_T)$)

Let $h > 0$, let $k \leq 2$. If $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$,

$$\|y - y_h\|_{L^2(Q_T)} \leq K\max(1, \frac{2}{\sqrt{\eta}})\left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right)h^k.$$

The discrete inf-sup test - Evaluation of δ_h

Taking $\eta = r > 0$ so that $a_r(\varphi, \overline{\varphi}) = (\varphi, \overline{\varphi})_\Phi$, we have [8]

$$\delta_h = \inf\left\{ \sqrt{\delta} : B_h A_{r,h}^{-1} B_h^T \{\lambda_h\} = \delta\, J_h\{\lambda_h\}, \quad \forall \{\lambda_h\} \in \mathbb{R}^{m_h} \setminus \{0\} \right\}$$

$$\delta_{r,h} \approx C_r \frac{h}{\sqrt{r}} \quad \text{as} \quad h \to 0^+, \qquad C_r > 0 \tag{32}$$

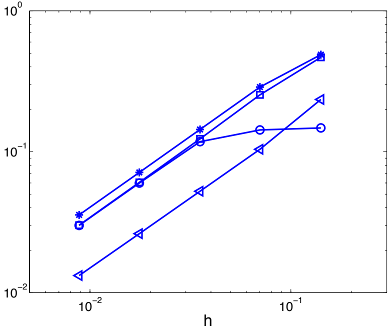


Figure: BFS finite element - Evolution of $\sqrt{r}\delta_{h,r}$ with respect to h for $r = 1$ ($\square$), $r = 10^{-2}$ ($\circ$), $r = h$ ($\star$) and $r = h^2$ ($<$).

[8] K. Bathe, D. Chapelle, The discrete inf-sup test, (2003)

Choice of r versus δ_h

$$\|y - y_h\|_{L^2(Q_T)} \leq K \max(1, \frac{2}{\sqrt{r}})\left(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}\right) h^k.$$

Optimal parameter: $r \approx 1$ leading to $\|y - y_h\|_{L^2(Q_T)} \leq K h^{k-1}$.

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 \frac{\sqrt{r}}{h}(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}) h^k.$$

Optimal parameter: $r \approx h^2$ leading to $\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 h^{k-1}$.

Choice of r versus δ_h

$$\|y - y_h\|_{L^2(Q_T)} \leq K \max(1, \frac{2}{\sqrt{r}})\left(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}\right) h^k.$$

Optimal parameter: $r \approx 1$ leading to $\|y - y_h\|_{L^2(Q_T)} \leq K h^{k-1}$.

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 \frac{\sqrt{r}}{h}(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}) h^k.$$

Optimal parameter: $r \approx h^2$ leading to $\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 h^{k-1}$.

$\alpha \in (0, 1)$ - Stabilized mixed formulation

The problem (11) becomes : find $(y_h, \lambda_h) \in Z_h \times \Lambda_h$ solution of

$$\begin{cases} a_{r,\alpha}(y_h, \overline{y}_h) + b_\alpha(\lambda_h, \overline{y}_h) &= l_{1,\alpha}(\overline{y}_h), \qquad \forall \overline{y}_h \in Z_h \\ b_\alpha(\overline{\lambda}_h, y_h) - c_\alpha(\lambda_h, \overline{\lambda}_h) &= l_{2,\alpha}(\overline{\lambda}_h), \qquad \forall \overline{\lambda}_h \in \widetilde{\Lambda}_h, \end{cases} \tag{33}$$

$$\Lambda_h = \{\lambda \in Z_h; \lambda(\cdot, T) = \lambda_t(\cdot, T) = 0\}.$$

Proposition (BFS element for $N = 1$ - Rates of convergence)

Let $h > 0$, let $k \in \{0, 2\}$. If the solution $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$, $\exists K > 0$

$$\|y - y_h\|_Z + \|\lambda - \lambda_h\|_\Lambda \leq K h^k.$$

Recovering the solution and the source $\mu \in H^{-1}(\Omega)$

$$\begin{cases} a_r((y_h,\mu_h),(\overline{y}_h,\overline{\mu}_h)) + b(\overline{y}_h,\lambda_h) = l(\overline{y}_h), & \forall(\overline{y}_h,\overline{\mu}_h) \in Y_h \\ b((y_h,\mu_h),\overline{\lambda}_h) = 0, & \forall\overline{\lambda}_h \in \Lambda_h. \end{cases} \quad (34)$$

Theorem (BFS element for $N = 1$ - Rate of convergence $L^2(Q_T)$)

Let $h > 0$, let $k, q \in \{0, 2\}$ be two nonnegative integers. If $((y,\mu),\lambda) \in H^{k+2}(Q_T) \times H^q(\Omega) \times H^k(Q_T)$, $\exists$

$$K = K(\|y\|_{H^{k+2}(Q_T)}, \|\mu\|_{H^k(\Omega)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)}),$$

independent of h, such that

$$\|y - y_h\|_{L^2(Q_T)} \leq K C_{\Omega,T}(1 + \|\sigma\|_{L^2(0,T)}\sqrt{C_{obs}}) \max(1, \frac{1}{\sqrt{\eta}}) \left[\left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right)h^k + \left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right)(\Delta x)^q\right].$$

First order versus second order - Heat case

Heat eq. ; Observation on q_T

- Second order formulation in (y, λ) ;

  $\mathbb{Q}_3 \times \mathbb{Q}_1$ approximation $\Longrightarrow \delta_h \approx \frac{C_r}{\sqrt{r}}$

$$(y, \lambda) \in H^3(Q_T) \times H^1(Q_T) \Longrightarrow \|\rho_{1,c}^{-1}(y - y_h)\|_{L^2(Q_T)} \leq K \frac{h}{\sqrt{r}}$$

- First order formulation in $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu}))$;

  $(\mathbb{Q}_1 \times \mathbb{Q}_1) \times (\mathbb{Q}_1 \times \mathbb{Q}_1)$ approximation; $\Longrightarrow \delta_h = 0 \; \forall r, h > 0 \; (Ker(B_h^\star) \neq \{0\})$

- First order stabilized formulation in $((y, \mathbf{p}), (\lambda, \mu))$ is needed

  If $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu})) \in (H^2(Q_T) \times \boldsymbol{H}^2(Q_T)) \times (H^1(Q_T) \times \boldsymbol{H}^1(Q_T))$

$$\|\rho_{1,p}^{-1}(y - y_h)|_{L^2(Q_T)} \leq Kh$$

First order versus second order - Heat case

Heat eq. ; Observation on q_T

- Second order formulation in (y, λ) ;

  $\mathbb{Q}_3 \times \mathbb{Q}_1$ approximation $\Longrightarrow \delta_h \approx \frac{C_r}{\sqrt{r}}$

  $$(y, \lambda) \in H^3(Q_T) \times H^1(Q_T) \Longrightarrow \|\rho_{1,c}^{-1}(y - y_h)\|_{L^2(Q_T)} \leq K \frac{h}{\sqrt{r}}$$

- First order formulation in $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu}))$;

  $(\mathbb{Q}_1 \times \mathbb{Q}_1) \times (\mathbb{Q}_1 \times \mathbb{Q}_1)$ approximation; $\Longrightarrow \delta_h = 0 \; \forall r, h > 0 \; (Ker(B_h^\star) \neq \{0\})$

- First order stabilized formulation in $((y, \mathbf{p}), (\lambda, \mu))$ is needed

  If $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu})) \in (H^2(Q_T) \times \boldsymbol{H}^2(Q_T)) \times (H^1(Q_T) \times \boldsymbol{H}^1(Q_T))$

  $$\|\rho_{1,p}^{-1}(y - y_h)\|_{L^2(Q_T)} \leq Kh$$

First order versus second order - Heat case

Heat eq. ; Observation on q_T

- Second order formulation in (y, λ) ;

  $\mathbb{Q}_3 \times \mathbb{Q}_1$ approximation $\Longrightarrow \delta_h \approx \frac{C_r}{\sqrt{r}}$

  $$(y, \lambda) \in H^3(Q_T) \times H^1(Q_T) \Longrightarrow \|\rho_{1,c}^{-1}(y - y_h)\|_{L^2(Q_T)} \leq K\frac{h}{\sqrt{r}}$$

- First order formulation in $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu}))$;

  $(\mathbb{Q}_1 \times \mathbb{Q}_1) \times (\mathbb{Q}_1 \times \mathbb{Q}_1)$ approximation; $\Longrightarrow \delta_h = 0 \ \forall r, h > 0 \ (Ker(B_h^\star) \neq \{0\})$

- First order stabilized formulation in $((y, \mathbf{p}), (\lambda, \mu))$ is needed

  If $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu})) \in (H^2(Q_T) \times \boldsymbol{H}^2(Q_T)) \times (H^1(Q_T) \times \boldsymbol{H}^1(Q_T))$

  $$\|\rho_{1,p}^{-1}(y - y_h)|_{L^2(Q_T)} \leq Kh$$

EXPERIMENTS

Numerical illustration - $N = 1$

$$(\textbf{EX1}) \quad y_0(x) = 1 - |2x - 1|, \quad y_1(x) = 1_{(1/3,2/3)}(x), \qquad x \in (0,1)$$

in $H_0^1 \times L^2$ for which the Fourier coefficients are

$$a_k = \frac{4\sqrt{2}}{\pi^2 k^2}\sin(\pi k/2), \quad b_k = \frac{1}{\pi k}(\cos(\pi k/3) - \cos(2\pi k/3)), \quad k > 0$$

$f = 0$. $T = 2$ - The corresponding solution of (1) with $c \equiv 1, d \equiv 0$ is given by

$$y(x,t) = \sum_{k>0}\left(a_k \cos(k\pi t) + \frac{b_k}{k\pi}\sin(k\pi t)\right)\sqrt{2}\sin(k\pi x)$$

Example 1 - $N = 1$ - Observation on q_T

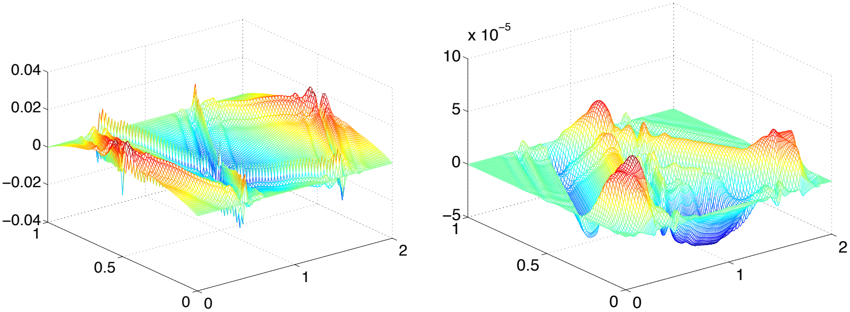
$q_T = (0.1, 0.3) \times (0, T)$

h	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\frac{\|y-y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}}$	1.01×10^{-1}	4.81×10^{-2}	2.34×10^{-2}	1.15×10^{-2}	5.68×10^{-3}
$\frac{\|y-y_h\|_{L^2(q_T)}}{\|y\|_{L^2(q_T)}}$	1.34×10^{-1}	5.05×10^{-2}	2.37×10^{-2}	1.16×10^{-2}	5.80×10^{-3}
$\|Ly_h\|_{L^2(Q_T)}$	7.18×10^{-2}	6.59×10^{-2}	6.11×10^{-2}	5.55×10^{-2}	5.10×10^{-2}
$\|\lambda_h\|_{L^2(Q_T)}$	1.07×10^{-4}	4.70×10^{-5}	2.32×10^{-5}	1.15×10^{-5}	5.76×10^{-6}
$\sharp$ CG iterates	29	46	83	133	201

$$\frac{\|y-y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} = \mathcal{O}(h^{0.574}), \quad \frac{\|y-y_h\|_{L^2(q_T)}}{\|y\|_{L^2(q_T)}} = \mathcal{O}(h^{0.94}). \tag{35}$$

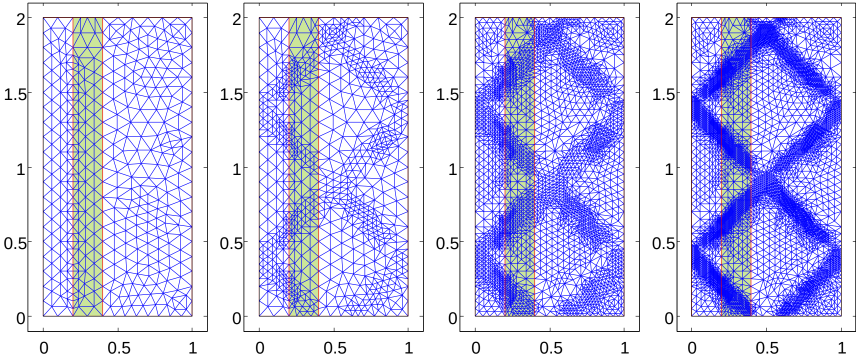
$$\|Ly_h\|_{L^2(Q_T)} = \mathcal{O}(h^{0.123}). \tag{36}$$

Example 2 - $N = 1$ - Observation on q_T



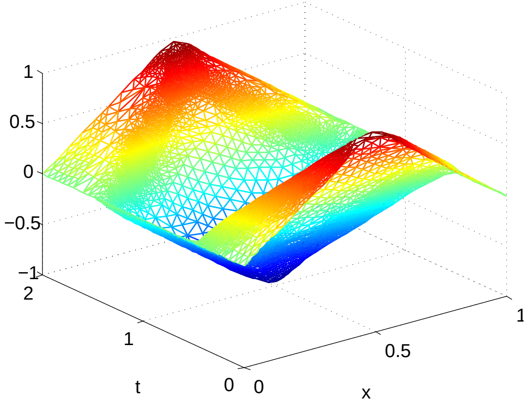
$y - y_h$ and λ_h in Q_T

Example 1 - $N = 1$ - Mesh adaptation



Iterative local refinement of the mesh according to the gradient of y_h

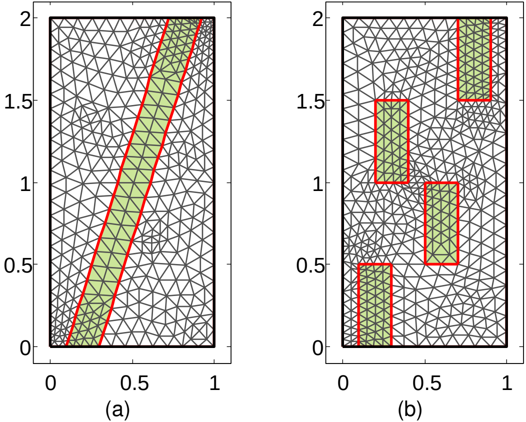
Example 1 - $N = 1$ - Mesh adaptation



Reconstructed state y_h on the adapted mesh

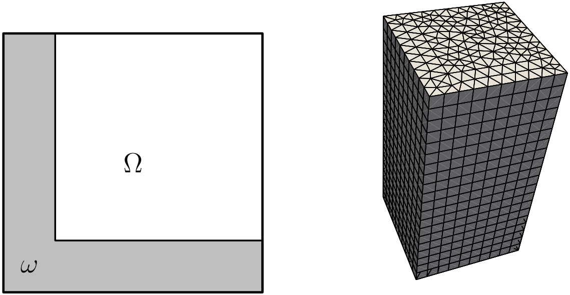
Exemple 2 : $N = 1$ - Non cylindrical domain q_T

Triangular meshes - reduced HCT elements



Domain q_T^1 (a) and domain q_T^2 (b) triangulated using some coarse meshes.

$2D$ example: $\Omega = (0,1)^2$ - Observation on q_T



(a) (b)

Mesh Number	0	1	2	3
Number of elements	5 320	15 320	31 740	120 160
Number of nodes	3 234	8 799	17 670	64 411

Characteristics of the three meshes associated with Q_T.

$2D$ example: $\Omega = (0,1)^2$ - Observation on q_T

$(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$:

$$\textbf{(EX2–2D)} \quad \begin{cases} y_0(x_1,x_2) = (1-|2x_1-1|)(1-|2x_2-1|) \\ y_1(x_1,x_2) = \mathbf{1}_{(\frac{1}{3},\frac{2}{3})^2}(x_1,x_2) \end{cases} \quad (x_1,x_2)\in\Omega. \qquad (37)$$

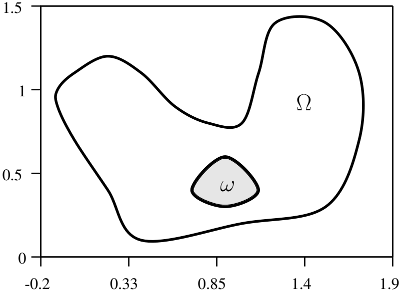
The Fourier coefficients of the corresponding solution are

$$a_{kl} = \frac{2^5}{\pi^4 k^2 l^2} \sin\frac{\pi k}{2} \sin\frac{\pi l}{2}$$
$$b_{kl} = \frac{1}{\pi^2 kl}\left(\cos\frac{\pi k}{3} - \cos\frac{2\pi k}{3}\right)\left(\cos\frac{\pi l}{3} - \cos\frac{2\pi l}{3}\right).$$

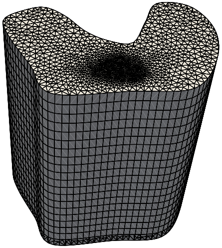
Mesh number	0	1	2	3
$\frac{\lVert y-y_h\rVert_{L^2(Q_T)}}{\lVert y\rVert_{L^2(Q_T)}}$	4.74×10^{-2}	3.72×10^{-2}	2.4×10^{-2}	1.35×10^{-2}
$\lVert Ly_h\rVert_{L^2(Q_T)}$	1.18	0.89	0.99	0.99
$\lVert\lambda_h\rVert_{L^2(Q_T)}$	3.21×10^{-5}	1.46×10^{-5}	1.02×10^{-5}	3.56×10^{-6}

Table: Example **EX2–2D** – $r = h^2$

$2D$ example - Observation on q_T



(a)



(b)

Mesh number	0	1	2
Number of elements	5 730	44 900	196 040
Number of nodes	3 432	24 633	103 566

Characteristics of the three meshes associated with Q_T.

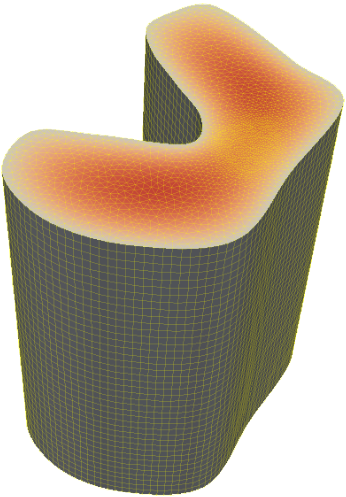
$2D$ example - Observation on q_T

$$\begin{cases} -\Delta y_0 = 10, & \text{in } \Omega \\ y_0 = 0, & \text{on } \partial\Omega, \end{cases} \qquad y_1 = 0. \tag{38}$$

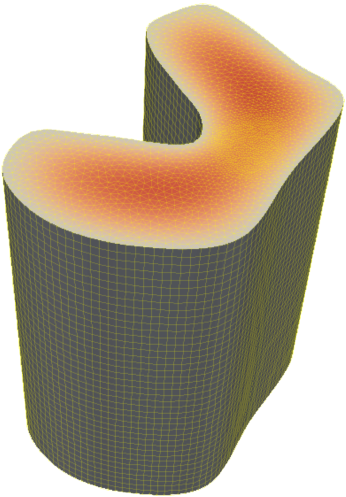
Mesh number	0	1	2
$\frac{\lVert\overline{y}_h - y_h\rVert_{L^2(Q_T)}}{\lVert\overline{y}_h\rVert_{L^2(Q_T)}}$	1.88×10^{-1}	8.04×10^{-2}	5.41×10^{-2}
$\lVert Ly_h\rVert_{L^2(Q_T)}$	3.21	2.01	1.17
$\lVert\lambda_h\rVert_{L^2(Q_T)}$	8.26×10^{-5}	3.62×10^{-5}	2.24×10^{-5}

$$r = h^2 - T = 2$$

$2D$ example - Observation on q_T



(a)



(b)

y and y_h in Q_T

Numerical illustration - $N = 1$ - Observation on Γ_T

$f = 0$ - $T = 2$

$$(\textbf{EX2}) \quad y_0(x) = 1 - |2x - 1|, \quad y_1(x) = 1_{(1/3,2/3)}(x), \qquad x \in (0,1)$$

in $H_0^1 \times L^2$ for which the Fourier coefficients are

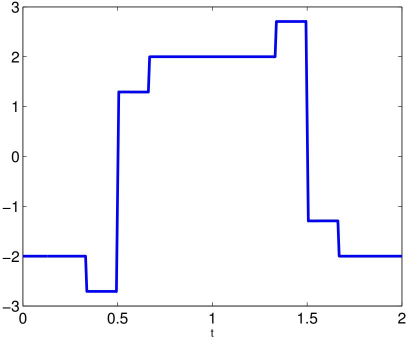


Figure: The observation $y_{\nu,obs}$ on $\{1\} \times (0, T)$ associated to initial data **EX1**.

Numerical illustration - $N = 1$ - Observation on Γ_T

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\frac{\Vert y-y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}}$	1.63×10^{-2}	6.63×10^{-3}	2.78×10^{-3}	1.29×10^{-3}	5.72×10^{-4}
$\frac{\Vert \partial_\nu (y-y_h)\Vert_{L^2(\Gamma_T)}}{\Vert \partial_\nu y\Vert_{L^2(\Gamma_T)}}$	7.67×10^{-3}	4.95×10^{-3}	3.24×10^{-3}	2.16×10^{-3}	1.48×10^{-3}
$\Vert Ly_h\Vert_{L^2(Q_T)}$	0.937	1.204	1.496	1.798	2.135
$\Vert \lambda_h\Vert_{L^2(Q_T)}$	7.74×10^{-3}	3.74×10^{-3}	1.72×10^{-3}	7.90×10^{-4}	3.60×10^{-4}
card($\{\lambda_h\}$)	861	3 321	13 041	51 681	205 761
$\sharp$ CG iterates	57	103	172	337	591

$$r = h^2 : \quad \frac{\Vert y - y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}} = \mathcal{O}(h^{1.20}), \quad \frac{\Vert \partial_\nu (y - y_h)\Vert_{L^2(\Gamma_T)}}{\Vert \partial_\nu y\Vert_{L^2(\Gamma_T)}} = \mathcal{O}(h^{0.59}), \tag{39}$$

$$\Vert \lambda_h\Vert_{L^2(Q_T)} = \mathcal{O}(h^{1.11}), \quad \Vert Ly_h\Vert_{L^2(Q_T)} = \mathcal{O}(h^{-0.29}).$$

Example 2 - $N = 2$ - The stadium

$T = 3$

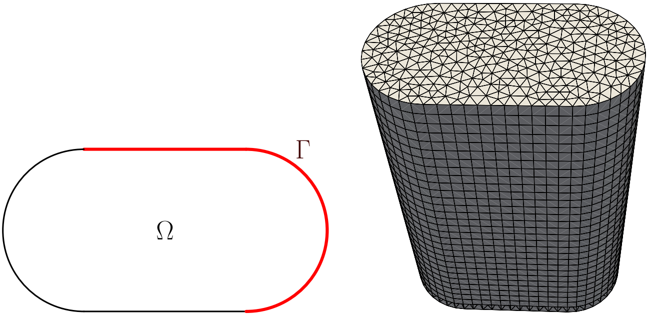


Figure: Bunimovich's stadium and the subset Γ of $\partial\Omega$ on which the observations are available. Example of mesh of the domain Q_T.

Example 2 - $N = 2$ - Recovering of the initial data

$T = 3$

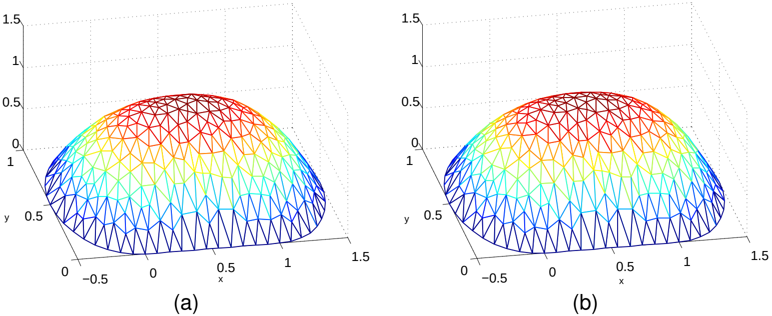


Figure: (a) Initial data y_0 given by (38). (b) Reconstructed initial data $y_h(\cdot, 0)$.

$N = 1$ - Reconstruction of y and μ from the boundary

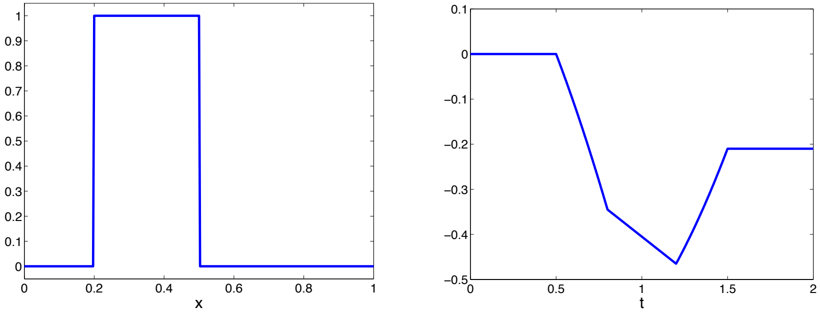


Figure: $\mu(x)$ and corresponding $\partial_\nu y|_{q_T} = y_x(1, t)$ on $(0, T)$.

$N = 1$ - Reconstruction of y and μ from the boundary

$\Delta x = \Delta t = 1/160$

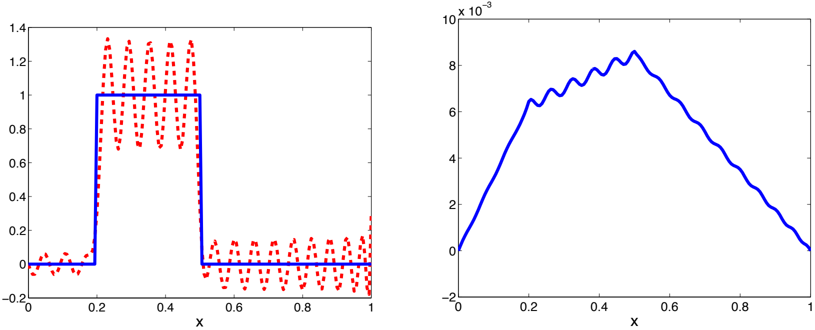


Figure: μ_h, μ and $\frac{-\Delta^{-1}(\mu - \mu_h)}{\|-\Delta^{-1}(\mu)\|_{H_0^1}}$.

$$\frac{\|\mu - \mu_h\|_{H^{-1}(\Omega)}}{\|\mu\|_{H^{-1}(\Omega)}} \approx 7.18 \times 10^{-2}, \qquad \|y - y_h\|_{L^2(Q_T)} \approx 8.68 \times 10^{-4}$$

$N = 1$ - Reconstruction of y and μ from the boundary

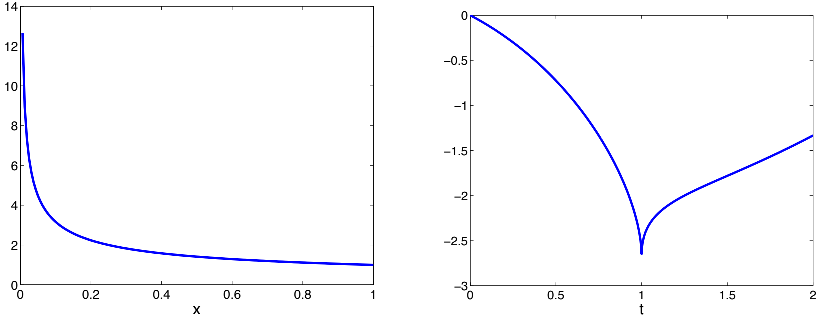
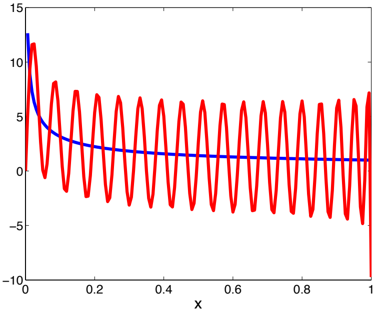


Figure: $\mu(x) = \frac{1}{\sqrt{x}}$ and corresponding $\partial_\nu y|_{q_T} = y_x(1, t)$ on $(0, T)$.

$N = 1$ - Reconstruction of y and μ from the boundary

$\Delta x = \Delta t = \frac{1}{160}$



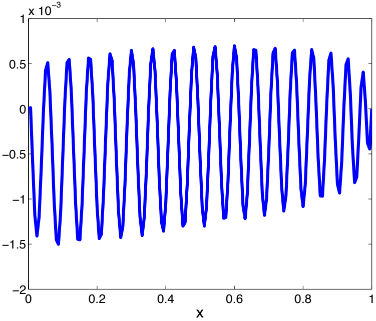


Figure: μ_h, μ and $\frac{-\Delta^{-1}(\mu-\mu_h)}{\|-\Delta^{-1}(\mu)\|_{H_0^1}}$.

$$\frac{\|\mu-\mu_h\|_{H^{-1}(\Omega)}}{\|\mu\|_{H^{-1}(\Omega)}} \approx 2.21 \times 10^{-2}, \qquad \|y - y_h\|_{L^2(Q_T)} \approx 3.56 \times 10^{-5}$$

$N = 1$ - Reconstruction of y and μ from the boundary

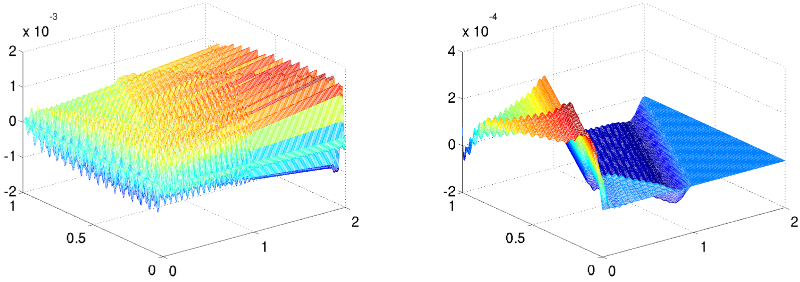
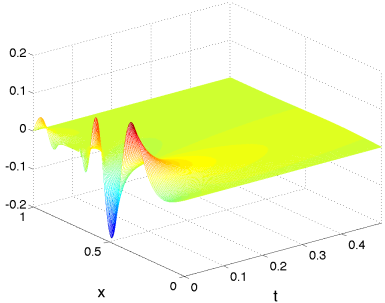


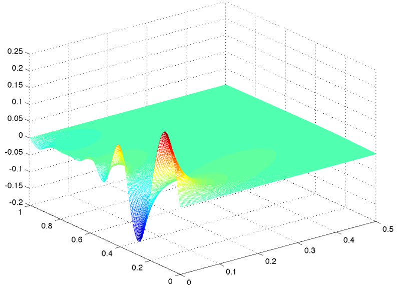
Figure: $y - y_h$ and λ_h

$N = 1$ - Heat eq. Comparison with the standard method

$$y_0(x) = \sin(\pi x)^{20}, \quad Q_T = (0,1)\times(0,T), \quad q_T = (0.7, 0.8)\times(0,T), \quad T = 1/2$$

$$\min_{y_{0h}} \left(J_h(y_{0h}) + \frac{h^2}{2}\|y_{0h}\|^2_{L^2(\Omega)} \right) \quad vs. \quad \min_{\lambda_h} J^{\star\star}(\lambda_h) \quad \text{over} \quad \Lambda_h \qquad (40)$$





$$\frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} \approx 5.86 \times 10^{-2}, \qquad \frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} \approx 7.70 \times 10^{-2}$$

$N = 1$ - Comparison with the standard method

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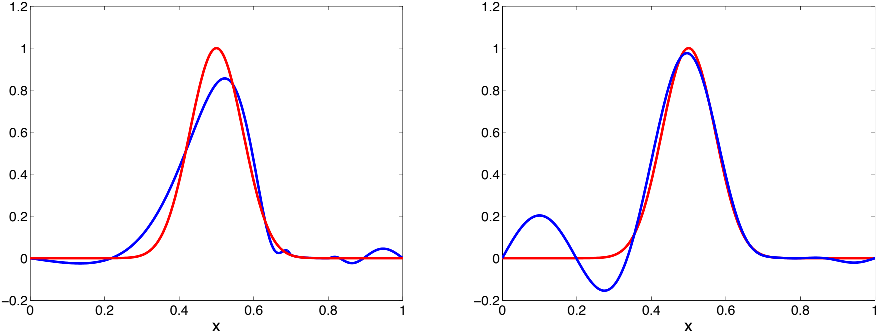

Restriction at $(0,1) \times \{0\}$

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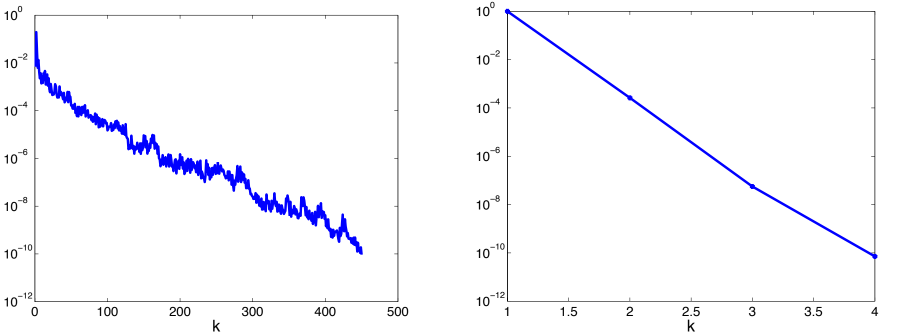

Evolution of the relative residu $\frac{\|g^k\|}{\|g^0\|}$ w.r.t. iterate k

Concluding remarks

MIXED FORMULATION ALLOWS TO RECONSTRUCT SOLUTION AND SOURCE

DIRECT AND ROBUST METHOD - STRONG CONVERGENCE

SPACE-TIME FORMULATIONS ARE VERY SUITABLE FOR MESH ADAPTATION AND MOVING ZONE OF OBSERVATION

NO NEED TO PROVE UNIFORM DISCRETE OBSERVABILITY ESTIMATE

THE MINIMIZATION OF $J_r^{\star\star}(\lambda)$ SEEMS VERY ROBUST AND FAST CONTRARY TO THE MINIMIZATION OF $J(y_0, y_1)$ (INVERSION OF SYMMETRIC DEFINITE POSITIVE AND VERY SPARSE MATRIX WITH DIRECT CHOLESKY SOLVERS)

DIRECT APPROACH CAN BE USED FOR MANY OTHER OBSERVABLE SYSTEMS FOR WHICH A GENERALIZED OBS. ESTIMATE IS AVAILABLE.

PERSPECTIVE: RECONSTRUCTION OF POTENTIAL AND COEFFICIENT

THANK YOU FOR YOUR ATTENTION

Concluding remarks

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Concluding remarks

- Mixed formulation allows to reconstruct solution and source
- Direct and robust Method - Strong convergence
- Space-time formulations are very suitable for mesh adaptation and moving zone of observation
- No need to prove uniform discrete observability estimate
- The minimization of $J_r^{\star\star}(\lambda)$ seems very robust and fast contrary to the minimization of $J(y_0, y_1)$ (inversion of symmetric definite positive and very sparse matrix with direct Cholesky solvers)
- Direct approach can be used for many other observable systems for which a generalized obs. estimate is available.
- Perspective: Reconstruction of potential and coefficient

Thank you for your attention

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