Relaxation of an optimal design problem for the heat equation

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August 30, 2007

Abstract

We consider the heat equation in $(0,T) \times \Omega$, $\Omega \subset \mathbb{R}^N, N \geq 1$, and address the nonlinear optimal design problem which consists in finding the distribution in Ω of two given isotropic materials which minimizes a suitable cost functional depending on the heat flux. We obtain well-posed relaxations of the problem by using two well-known approaches: the homogenization method and the classical tools of non-convex, vector, variational problems. We also implement several numerical experiments based on these relaxed formulations in the two-dimensional case which justify the relaxation procedures and support the theoretical results. Finally, we point out some differences and analogies of the two proposed methods.

Résumé

Dans le cadre de l'équation de la chaleur posée sur le cylindre borné $(0,T) \times \Omega$, $\Omega \subset \mathbb{R}^N, N \geq 1$, on adresse le problème non linéaire de la distribution optimale de deux matériaux isotropes minimisant le flux de chaleur dans Ω. En utilisant d'une part la théorie de l'homogénéisation et d'autre part une approche variationnelle basée sur la mesure de Young, on obtient deux relaxations bien posées et équivalentes du problème d'optimisation initial. Enfin, une application numérique dans le cas bi-dimensionnel justifie les procédures de relaxation et permet de confirmer les résultats théoriques puis de comparer les formulations relaxées obtenues.

Key words: Optimal design, Heat equation, Relaxation, Homogenization, Young Measure.

Mathematics Subject Classification (2000): 35K05, 49Q20, 65M60.

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1 Introduction - Problem Formulation

Optimal design problems in which the goal is to know the best way of mixing two different materials in order to optimize some physical quantity associated with the resultant structure have been extensively studied during the last decades, mainly in the case where the underlying state equation is elliptic. We refer the reader to [10, 16]. Among the techniques and tools used to deal with this type of problems, homogenization and variational formulations have played a very important role (see also [1, 2, 5, 18, 21]). More recently, optimal design problems for time-dependent state equations like the wave equation have been also considered ([11, 12, 13]). As far as we know, the case of the heat equation has been treated only from a more applied engineering point of view (see [22] and the references there in).

In this work, we aim to analyze the following nonlinear optimal design problem for the heat equation:

$$\text{(P)} \quad \text{Minimize in } \mathcal{X}: \qquad J(\mathcal{X}) = \frac{1}{2}\int_0^T\int_\Omega K(x)\,\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt$$

where the state variable $u = u(t,x)$ is the solution of the system

$$\begin{cases} \beta(x)\,u'(t,x) - \text{div}\ (K(x)\,\nabla u(t,x)) = f(t,x) & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1}$$

with

$$\begin{cases} \beta(x) = \mathcal{X}(x)\,\beta_1 + (1-\mathcal{X}(x))\,\beta_2, \\ K(x) = \mathcal{X}(x)\,k_1 I_N + (1-\mathcal{X}(x))\,k_2 I_N, \end{cases}$$

and the design variable $\mathcal{X} \in L^\infty(\Omega;\{0,1\})$ satisfies the volume constraint

$$\int_\Omega \mathcal{X}(x)\,dx = L\,|\Omega| \quad \text{for some fixed} \quad 0 < L < 1. \tag{2}$$

We assume that $T > 0$ is a final time and $\Omega \subset \mathbb{R}^N, N \geq 1$ is a bounded domain composed of two homogeneous, isotropic materials with mass densities $\rho_i > 0$, specific heats $c_i > 0$, and thermal conductivities $k_i > 0$, $i = 1,2$ such that $k_1 \neq k_2$. We have put $\beta_i = \rho_i c_i$, $i = 1,2$. I_N denotes the identity matrix of order N, f is an heat source, u_0 the initial temperature, and $u(t,x)$ the temperature at time t and position x. The design variable $\mathcal{X}$ is a characteristic function which indicates the region occupied by the first material (β_1, k_1). As a consequence, the condition (2) constraints the amount of this material that we have at our disposal.

Regarding system (1), it is well-known that for $f \in L^2((0,T)\times\Omega)$ and $u_0 \in L^2(\Omega)$, there exists a unique weak solution

$$u \in L^2\left(0,T;H_0^1(\Omega)\right), \quad \text{with } u' \in L^2\left(0,T;H^{-1}(\Omega)\right)$$

(see for instance [6, Chap. 11] and [7, Chap. 7]).

As for the physical meaning of the cost function $J(\mathcal{X})$, it is a measure of the heat flux during the period of time $(0,T)$. Therefore, the design problem (P) consists in finding the optimal distribution of two different materials in order to minimize the gradient part of the energy for the heat equation. We recall that the energy at time T corresponding to the solution of (1) is defined by

$$E(T) = \frac{1}{2}\int_\Omega \beta(x)\,u^2(T,x)\,dx + \frac{1}{2}\int_0^T\int_\Omega K(x)\,\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt. \tag{3}$$

The same optimal design problem, but with a cost function depending only on the temperature u, was considered in [22] where the numerical simulations suggest the non-existence of optimal designs in the class of characteristic functions. The optimal design is then found in the form of a composite material. For the steady-state case, a counterexample on the non-existence of solutions may be found in [1, p. 206-211]. Relaxation is the appropriate way of dealing mathematically with this type of situations. This basically consists in replacing the original problem by another suitable one which has (at least) a minimizer and, in addition, the optimal cost associated with this new problem coincides with the infimum of the original one. The process is successfully completed whenever we are able to find out the behavior of some minimizing sequences of the original problem from the information codified in the minimizers of the relaxed one.

As indicated above, the homogenization method and the classical tools of non-convex variational problems (in particular, Young measures) are, for the moment, two of the most popular approaches in the mathematical literature to analyze this type of optimal design problems. In this work, we aim to carry out the relaxation procedure in full by using both techniques, homogenization and Young measures. Our study will be not limited to theoretical results. We shall implement several numerical experiments based on both procedures in the two dimensional case.

Finally, we would like to emphasize that this work is only a first step towards a better understanding of optimal design for parabolic equations. Many interesting questions still remain open. We list some of them at the end of the paper.

2 The Homogenization Method

We will obtain a suitable relaxation for the optimal design problem (P). This will be done by using both the homogenization method and a variational approach based on the use of div-curl Young measures. We first focus on the homogenization method, and defer an analysis based on Young measures to the next section.

In order to make this section easier to read we first collect some well-known results on Homogenization theory. Relaxation will follow directly from these results. Throughout this section, we denote by $\mathcal{X}_n \in L^\infty(\Omega; \{0,1\})$, $n = 1, 2, \cdots$, a sequence of characteristic functions and by $K_n \in \mathcal{M}^{N\times N}$ a sequence of tensors of the form

$$K_n = \mathcal{X}_n(x)\, k_1 I_N + (1 - \mathcal{X}_n(x))\, k_2 I_N, \tag{4}$$

with $k_1, k_2 > 0$.

2.1 Previous results on homogenization

The material of this subsection has been taken from [1, Chap. 1 and 2] and [3].

Homogenization is based on the concept of $H-$convergence. Precisely, a sequence of tensors $\{K_n(x)\}_{n\in\mathbb{N}}$ $H-$converges to the tensor $K^* \in L^\infty\left(\Omega; \mathcal{M}^{N\times N}\right)$ if for any $f \in H^{-1}(\Omega)$ the sequence of solutions $u_n \in H_0^1(\Omega)$ of

$$\begin{cases} -\text{div}\,(K_n \nabla u_n) = f & \text{in} \quad \Omega, \\ u_n = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

satisfies

$$\begin{cases} u_n \rightharpoonup u & \text{weak in } H_0^1(\Omega), \\ K_n \nabla u_n \rightharpoonup K^* \nabla u & \text{weak in } \left(L^2(\Omega)\right)^N, \end{cases}$$

where u is the solution of the homogenized system

$$\begin{cases} -\text{div } (K^* \nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We shall write $K_n \xrightarrow{\text{H}} K^*$ to indicate this kind of convergence.

Assume now that there exists $\theta \in L^\infty (\Omega; [0,1])$ and $K^* \in L^\infty \left(\Omega; \mathcal{M}^{N \times N}\right)$ such that

$$\begin{cases} \mathcal{X}_n \rightharpoonup \theta & \text{weak } \star \text{ in } L^\infty (\Omega), \\ K_n \xrightarrow{\text{H}} K^*. \end{cases}$$

The $H-$limit K^* is said to be the homogenized or effective tensor of two isotropic materials obtained by mixing k_1 and k_2 in proportions θ and $1-\theta$, respectively, with a microstructure defined by $\mathcal{X}_n$.

As we will see later on, it is very important to identify all possible homogenized tensors obtained by mixing two given materials with all possible micro-structures. This is the so-called $G-$closure problem. Precisely, we have the following definition.

Definition 2.1 *Given $\theta \in L^\infty (\Omega; [0,1])$, the $G_\theta-$closure of two isotropic materials is defined as the set of tensors $K^* \in L^\infty \left(\Omega; \mathcal{M}^{N \times N}\right)$ such that there exist $\mathcal{X}_n \in L^\infty (\Omega; \{0,1\})$ and K_n of the form (4) satisfying*

$$\begin{cases} \mathcal{X}_n \rightharpoonup \theta & \textit{weak } \star \textit{ in } L^\infty (\Omega), \\ K_n \xrightarrow{H} K^*. \end{cases}$$

Fortunately, for the case of two isotropic materials, the $G_\theta-$closure is well-known.

Theorem 2.2 *Given $\theta \in L^\infty (\Omega; [0,1])$, the $G_\theta-$closure of two isotropic materials $k_i > 0$, $i = 1, 2$, is the set of all symmetric matrices with eigenvalues $\lambda_1, \cdots, \lambda_N$ satisfying*

$$\begin{cases} \lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+, & 1 \leq j \leq N, \\ \sum_{j=1}^N \frac{1}{\lambda_j - k_1} \leq \frac{1}{\lambda_\theta^- - k_1} + \frac{N-1}{\lambda_\theta^+ - k_1}, \\ \sum_{j=1}^N \frac{1}{k_2 - \lambda_j} \leq \frac{1}{k_2 - \lambda_\theta^-} + \frac{N-1}{k_2 - \lambda_\theta^+}, \end{cases}$$

where $\lambda_\theta^- = \left(\frac{\theta}{k_1} + \frac{1-\theta}{k_2}\right)^{-1}$ is the harmonic mean and $\lambda_\theta^+ = \theta k_1 + (1-\theta) k_2$ the arithmetic mean of (k_1, k_2).

We conclude this section with an homogenization result for the heat equation (we refer to [3, Th. 7.1] for the proof).

Theorem 2.3 *Let $\mathcal{X}_n \in L^\infty (\Omega; \{0,1\})$ and let K_n be of the form (4). Assume that*

$$\begin{cases} \mathcal{X}_n \rightharpoonup \theta & \textit{weak } \star \textit{ in } L^\infty (\Omega), \\ K_n \xrightarrow{H} K^*. \end{cases}$$

Consider the system

$$\begin{cases} \beta_n (x) u_n' (t,x) - \textit{div } (K_n (x) \nabla u_n (t,x)) = f (t,x) & \textit{in} \quad (0,T) \times \Omega, \\ u_n = 0 & \textit{on} \quad (0,T) \times \partial\Omega, \\ u_n (0,x) = u_0 (x) & \textit{in} \quad \Omega, \end{cases}$$

where $\beta_n = \mathcal{X}_n \beta_1 + (1 - \mathcal{X}_n)\beta_2$, *with* $\beta_1, \beta_2 > 0$, $f \in L^2((0,T)\times\Omega)$ *and* $u_0 \in L^2(\Omega)$. *Then*

$$\int_0^T \int_\Omega K_n(x)\nabla u_n(t,x)\cdot\nabla u_n(t,x)\,dxdt \to \int_0^T\int_\Omega K^*(x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt, \tag{5}$$

u being the solution of the limit system

$$\begin{cases} \beta(x)u'(t,x) - div\ (K^*(x)\nabla u(t,x)) = f(t,x) & in \quad (0,T)\times\Omega, \\ u = 0 & on \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & in \quad \Omega, \end{cases} \tag{6}$$

with $\beta = \theta\beta_1 + (1-\theta)\beta_2$.

2.2 Relaxation by the homogenization method

As indicated in the introduction, problem (P) is usually ill-posed in the sense that there are no minimizers in the space of *classical designs*

$$\mathbf{CD} = \{\mathcal{X} \in L^\infty(\Omega;\{0,1\}) : \mathcal{X} \text{ satisfies } (2)\}.$$

The idea of relaxation basically consists in considering a larger class of admissible designs with the hope that the optimal design problem to be well-posed in this new class of designs. Having this in mind and based on Theorem 2.2 we introduce the space of *relaxed designs*

$$\mathbf{RD} = \left\{(\theta, K^*) \in L^\infty\left(\Omega;[0,1]\times\mathcal{M}^{N\times N}\right) : K^*(x) \in G_{\theta(x)} \text{ a.e. } x\in\Omega \text{ and } \theta \text{ satisfies } (2)\right\},$$

where $G_{\theta(x)}$ is as in Theorem 2.2.

From Theorem 2.3 is then natural to consider, for $(\theta, K^*) \in \mathbf{RD}$**,** the *relaxed cost*

$$J^*(\theta,K^*) = \frac{1}{2}\int_0^T\int_\Omega K^*(x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt \tag{7}$$

where u is the solution of (6).

Finally, we consider the optimal design problem

$$\text{(RP)} \quad \text{Minimize in } (\theta,K^*)\in\mathbf{RD}: \quad J^*(\theta,K^*)$$

where $J(\theta,K^*)$ is defined by (7). We have the following main result.

Theorem 2.4 *(RP) is a relaxation of (P) in the sense that*

(i) *there exists at least one minimizer for (RP) in the space* **RD***,*

(ii) *up to a subsequence, every minimizing sequence of classical designs* $\mathcal{X}_n$ *converges, weakly* $\star$ *in* $L^\infty(\Omega;[0,1])$, *to a relaxed density* θ, *and its associated sequence of tensors*

$$K_n = \mathcal{X}_n k_1 I_N + (1-\mathcal{X}_n)k_2 I_N$$

$H-$*converges to an effective tensor* K^* *such that* (θ,K^*) *is a minimizer for (RP), and*

(iii) *conversely, every relaxed minimizer* $(\theta,K^*)\in\mathbf{RD}$ *of (RP) is attained by a minimizing sequence* $\mathcal{X}_n$ *of (P) in the sense that*

$$\begin{cases} \mathcal{X}_n \rightharpoonup \theta & weak\ \star\ \ in\ L^\infty(\Omega), \\ K_n \xrightarrow{H} K^*. \end{cases}$$

Proof. The proof of this result follows the same lines as in the static case (see [1, p.p. 213-215]). Anyway, we include it here for completeness.

Let $\mathcal{X}_n$ be a minimizing sequence for (P). Since $\|\mathcal{X}_n\|_{L^\infty(\Omega)} \leq 1$, there exists a subsequence, still denoted by $\mathcal{X}_n$, such that

$$\mathcal{X}_n \rightharpoonup \theta_\infty \quad \text{weak} \star \text{ in } L^\infty(\Omega).$$

Moreover, since $\mathcal{X}_n$ satisfies the volume constraint (2) and $\mathcal{X}_n \rightharpoonup \theta_\infty$ weak $\star$,

$$\int_\Omega \theta_\infty(x)\,dx = L\,|\Omega|.$$

On the other hand, thanks to the compactness of the sequence of tensors K_n with respect to $H-$convergence, up to a subsequence, there exists $K_\infty \in L^\infty\left(\Omega; \mathcal{M}^{N\times N}\right)$ such that $K_n \overset{\text{H}}{\to} K_\infty$. From Theorem 2.3 it follows that

$$\lim_{n\to\infty} J(\mathcal{X}_n) = J^*(\theta_\infty, K_\infty).$$

This proves that

$$m = \inf_{\mathcal{X}} J(\mathcal{X}) = J^*(\theta_\infty, K_\infty). \tag{8}$$

Now let (θ, K^*) be a relaxed design. By the definition of the set G_θ, there exists $\mathcal{X}_n \in L^\infty(\Omega; \{0,1\})$ such that

$$\begin{cases} \mathcal{X}_n \rightharpoonup \theta & \text{weak} \star \text{ in } L^\infty(\Omega), \\ K_n \overset{\text{H}}{\to} K^*. \end{cases}$$

In particular,

$$\lim_{n\to\infty} \int_\Omega \mathcal{X}_n(x)\,dx = \int_\Omega \theta(x)\,dx = L\,|\Omega|,$$

but in principle each individual $\mathcal{X}_n$ does not satisfy the volume constraint (2). Nevertheless, this difficulty may be overcome (see Proposition 2.1). So, assume that $\mathcal{X}_n$ is admissible for (P). By using again Theorem 2.3,

$$J^*(\theta, K^*) = \lim_{n\to\infty} J(\mathcal{X}_n) \geq m.$$

Combining this inequality with (8) we obtain that $(\theta_\infty, K_\infty)$ is a minimizer for (RP). This proves (i) and (ii).

Finally, to prove (iii), let $(\theta, K^*) \in \mathbf{RD}$ be a minimizer for (RP). From the definition of G_θ it follows that there exists $\mathcal{X}_n \in L^\infty(\Omega; \{0,1\})$, which may be assumed to satisfy (2), such that

$$\begin{cases} \mathcal{X}_n \rightharpoonup \theta & \text{weak} \star \text{ in } L^\infty(\Omega) \\ K_n \overset{\text{H}}{\to} K^*, \end{cases}$$

where K_n is the sequence of tensors defined by (4). As before, we also have $J(\theta, K^*) = \lim_{n\to\infty} J(\mathcal{X}_n)$. Obviously, this implies that $\mathcal{X}_n$ is minimizing for (P). ∎

Proposition 2.1 *Let $\mathcal{X}_n \in L^\infty(\Omega; \{0,1\})$ be such that*

$$\begin{cases} (i) & \mathcal{X}_n \rightharpoonup \theta \quad \textit{weak}\star \ \textit{in } L^\infty(\Omega), \\ (ii) & \lim_{n\to\infty} \int_\Omega \mathcal{X}_n(x)\,dx = L\,|\Omega|, \ \textit{and} \\ (iii) & K_n \overset{H}{\to} K, \ \textit{where } K_n \textit{ is as in (4)}. \end{cases}$$

Then there exists $\overline{\mathcal{X}}_n \in L^\infty(\Omega;\{0,1\})$ *such that*

$$\begin{cases} (a) & \overline{\mathcal{X}}_n \rightharpoonup \theta \quad \text{weak}\star \ \text{ in } L^\infty(\Omega), \\ (b) & \int_\Omega \overline{\mathcal{X}}_n(x)\,dx = L|\Omega| \ \text{ for all } n \in \mathbb{N}, \text{ and} \\ (c) & \overline{K}_n \xrightarrow{H} K, \ \text{ where } \overline{K}_n \text{ is as in (4) for } \overline{\mathcal{X}}_n. \end{cases}$$

Proof. From (ii) we may construct a sequence of characteristic functions $\overline{\mathcal{X}}_n$ such that (b) holds and, in addition, the sequence of sets

$$\Omega_n = \left\{x \in \Omega : \mathcal{X}_n(x) \neq \overline{\mathcal{X}}_n(x)\right\}$$

satisfies

$$|\Omega_n| \to 0 \quad \text{as } n \to \infty. \tag{9}$$

From this, it is not difficult to see that, up to a subsequence, not relabelled, we have the convergence stated in (a).

Finally, let us denote by $\overline{K}$ the H-limit of (a subsequence of) $\overline{K}_n$. Again, from (9) and thanks to the locality of H-convergence (see [1, Prop. 1.4.5] or [6, Th. 13.4 (ii)]) it follows that

$$K(x) = \overline{K}(x) \quad \text{a.e } x \in \Omega,$$

which completes the proof. ∎

Theorem 2.4 gives us a relaxation of the original optimal design problem in which we have replaced the original state equation (1) by the relaxed one (6), this last system being written in terms of the homogenized tensor K^* for which we have the information that comes from Theorem 2.2. In the one-dimensional case, we have an explicit expression for the optimal tensor:

Remark 1 *In the 1-D case, the effective coefficient K^* is explicitly known. Indeed, from Theorem 2.2 it follows that K^* equals the harmonic mean, that is,*

$$K^*(x) = \frac{k_1 k_2}{\theta(x)k_2 + (1-\theta(x))k_1}, \quad x \in \Omega.$$

Hence, the relaxed problem (RP) has the simpler form

$$\text{Minimize in } \theta: \quad J^*(\theta) = \frac{1}{2}\int_0^T\int_\Omega \frac{k_1 k_2}{\theta(x)k_2 + (1-\theta(x))k_1}|u_x(t,x)|^2\,dxdt$$

subject to

$$\begin{cases} (\theta\beta_1 + (1-\theta)\beta_2)u' - \left(\frac{k_1k_2}{\theta k_2 + (1-\theta)k_1}u_x\right)_x = f & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{in} \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \\ \theta \in L^\infty(\Omega;[0,1]), \quad \int_\Omega \theta(x)\,dx = L|\Omega|. \end{cases}$$

Once the existence of optimal relaxed designs has been proved in Theorem 2.4, we stop here our study based on the Homogenization method. We will go back to it in the section devoted to the numerical resolution of the relaxed problem (RP).

3 A Young Measure Approach

Next, we will analyze problem (P) from another different perspective. Precisely, we will use the so-called div-curl Young measures as a key tool. We refer the reader to [8, 15, 20] for the main properties of this class of measures and some applications to optimal design in conductivity and stabilization in linear elasticity.

3.1 Div-curl Young measure associated with problem (P)

To begin with, we rewrite the heat equation in system (1) in divergence-free form

$$\operatorname{div}_{(t,x)}\left[\left(-\beta\left(x\right)u\left(t,x\right),K\left(x\right)\nabla u\left(t,x\right)\right)+F\left(t,x\right)\right]=0 \tag{10}$$

where the $\operatorname{div}_{(t,x)}$ operator now includes the time variable t as the first variable and $F(t,x)$ is a vector field such that $\operatorname{div}_{(t,x)}F=f$. Since F will play not an important role, we put $F=0$ for simplicity throughout this section, but it is important to say that all the results that follow hold true for $F\neq 0$.

For $u_0\in H_0^1(\Omega)$, an *integral* solution (or solution in the Young measure sense) of (10) exists (see [9, Section 6]): precisely, we recall that

$$u\in L^{\infty}\left((0,T);H_0^1(\Omega)\right)\quad \text{with } u'\in L^2\left((0,T)\times\Omega\right)$$

is said to be an integral solution of (10) if this equation is satisfied in $H^{-1}\left((0,T)\times\Omega\right)$ and the initial and boundary conditions also hold.

Now let $\mathcal{X}_n$ be an admissible sequence of designs for (P) and let u_n be its corresponding sequence of integral solutions. Consider the two sequences of vector fields

$$\begin{cases} G_n\left(t,x\right)=\left(-\left(\mathcal{X}_n\left(x\right)\beta_1+\left(1-\mathcal{X}_n\left(x\right)\right)\beta_2\right)u_n\left(t,x\right),\ K_n\left(x\right)\nabla u_n\left(t,x\right)\right),\\ H_n\left(t,x\right)=\left(u_n'\left(t,x\right),\ \nabla u_n\left(t,x\right)\right).\end{cases}$$

Since both sequences G_n and H_n are uniformly bounded in $\left(L^2\left((0,T)\times\Omega\right)\right)^{N+1}$, we may associate with (a subsequence of) the pair (G_n,H_n) a family of parameterized measures $\nu=\left\{\nu_{(t,x)}\right\}_{(t,x)\in(0,T)\times\Omega}$. Note also that the pair (G_n,H_n) satisfies

$$\operatorname{div}_{(t,x)}G_n=0\quad\text{and}\quad\operatorname{curl}H_n=0.$$

For this reason, the measure ν is called a div-curl Young measure. In addition, we know that

$$K_n\nabla u_n\cdot\nabla u_n\rightharpoonup K^*\nabla u\cdot\nabla u, \tag{11}$$

essentially due to Theorem 2.3. This condition will translate into a certain commutation property that the underlying measure should verify. We also notice that since u_n is uniformly bounded in $H^1\left((0,T)\times\Omega\right)$, by the Rellich-Kondrachov compactness Theorem,

$$u_n\to u\quad\text{strong in } L^2\left((0,T)\times\Omega\right).$$

Due to the particular form of (G_n,H_n), each individual $\nu_{(t,x)}$ is supported in the union of the two linear manifolds

$$\Lambda_i=\left\{(\rho,\lambda)\in\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}:\rho_1=-\beta_i u,\quad \overline{\rho}=k_i\overline{\lambda}\right\},\quad i=1,2 \tag{12}$$

where

$$\rho=(\rho_1;\overline{\rho})\in\mathbb{R}\times\mathbb{R}^N\quad\text{and}\quad\lambda=\left(\lambda_1;\overline{\lambda}\right)\in\mathbb{R}\times\mathbb{R}^N. \tag{13}$$

Hence, the measure $\nu_{(t,x)}$ may be written as

$$\nu_{(t,x)} = \theta(x)\,\nu_{1,(t,x)} + (1-\theta(x))\,\nu_{2,(t,x)}, \tag{14}$$

with supp $\nu_{i,(t,x)} \subset \Lambda_i$, $i=1,2$, and θ being the weak $\star$ limit in $L^\infty(\Omega)$ of (a subsequence of) $\mathcal{X}_n$.

The importance of having more information on this measure is the following: suppose that $\mathcal{X}_n$ is a minimizing sequence for (P) with the property that its associated $|\nabla u_n|^2$ is equi-integrable. Then, by the fundamental property of Young measures (see [17, Th. 6.2]), we may represent the limit of the costs associated with $\mathcal{X}_n$ through the measure ν. Precisely,

$$\lim_{n\to\infty} J(\mathcal{X}_n) = \frac{1}{2}\int_0^T\int_\Omega \left[k_1\theta(x)\int_{\mathbb{R}^N}\left|\overline{\lambda}\right|^2 d\overline{\nu}^{(2)}_{1,(t,x)} + k_2(1-\theta(x))\int_{\mathbb{R}^N}\left|\overline{\lambda}\right|^2 d\overline{\nu}^{(2)}_{2,(t,x)}\right]dxdt \tag{15}$$

where $\overline{\nu}^{(2)}_{i,(t,x)}$, $i=1,2$ stands for the projection of $\nu_{i,(t,x)}$ onto the last $N-$components of the second copy of $\mathbb{R}^{N+1}$. Therefore, with each minimizing sequence of the original problem (P) we associate an optimal div-curl Young measure. In this sense, optimize with respect to $\mathcal{X}$ is equivalent to optimize with respect to ν. For this reason, from now on, we concentrate on measures rather than on characteristics functions (classical designs).

3.2 Variational reformulation and relaxation

We now proceed to the analysis of problem (P) in a similar fashion as in the stationary case [20]. First step in this process is to put (P) into a variational setting. So, we consider the functions

$$W(\rho,\lambda) = \begin{cases} k_1\left|\overline{\lambda}\right|^2 & \text{if } (\rho,\lambda)\in\Lambda_1, \\ k_2\left|\overline{\lambda}\right|^2 & \text{if } (\rho,\lambda)\in\Lambda_2, \\ +\infty & \text{else,}\end{cases} \tag{16}$$

and

$$V(\rho,\lambda) = \begin{cases} 1 & \text{if } (\rho,\lambda)\in\Lambda_1, \\ 0 & \text{if } (\rho,\lambda)\in\Lambda_2, \\ +\infty & \text{else.}\end{cases} \tag{17}$$

Then we associate with problem (P) the new problem

$$(\widetilde{\mathrm{P}})\quad \text{Minimize in } (G,u):\quad \frac{1}{2}\int_0^T\int_\Omega W\left(G(t,x),\nabla_{(t,x)}u(t,x)\right)dxdt$$

subject to

$$\begin{cases} G\in L^2\left((0,T)\times\Omega;\mathbb{R}^{N+1}\right), & u\in H^1((0,T)\times\Omega;\mathbb{R}), \\ \mathrm{div}_{(t,x)}G = 0 & \text{in } H^{-1}((0,T)\times\Omega), \\ u|_{\partial\Omega}=0 \quad \text{a. e. } t\in[0,T], & u(0)=u_0 \quad \text{in } \Omega, \\ \int_\Omega V(G(t,x),\nabla u(t,x))\,dx = L|\Omega| & \text{a. e. } t\in[0,T].\end{cases}$$

The crucial step in this approach is the computation of the constrained quasi-convexification CQW of the density W because it provides us with a relaxation of $(\widetilde{\mathrm{P}})$. We remind that as is usual in non-convex vector variational problems, a full relaxation of this type of problems is obtained by replacing the original density W by its constrained quasi-convex envelope (see [18, 20] and the references there in). So, we concentrate on the computation of this new relaxed density.

For fixed $(\theta,\rho,\lambda)\in[0,1]\times\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}$ the constrained quasi-convex density $CQW(\theta,\rho,\lambda)$ is computed by solving the problem in measures

$$\text{(MP) Minimize in } \nu: \quad CQW(\theta,\rho,\lambda)=k_1\theta\int_{\mathbb{R}^N}\left|\overline{\lambda}\right|^2 d\overline{\nu}_1^{(2)}+k_2(1-\theta)\int_{\mathbb{R}^N}\left|\overline{\lambda}\right|^2 d\overline{\nu}_2^{(2)}$$

subject to

$$\begin{cases}\nu=\theta\nu_1+(1-\theta)\nu_2, \text{ with supp } \nu_i\subset\Lambda_i,\ i=1,2,\\[2mm] \nu \text{ is a div-curl Young measure verifying the commutation property associated with (11), and}\\[2mm] \rho=\int_{\mathbb{R}^{N+1}}x d\nu^{(1)}(x), \quad \lambda=\int_{\mathbb{R}^{N+1}}x d\nu^{(2)}(x), \text{ with } \nu^{(i)} \text{ the two marginals.}\end{cases}$$

We notice that after solving (MP) we plan to use the localization principle for div-curl Young measures (see [20]) to analyze the optimal cost given by (15). In fact, for almost everywhere $(t,x)\in(0,T)\times\Omega$, we have the identification $\theta=\theta(x)$, $\rho=G(t,x)$ and $\lambda=H(t,x)$, where G and H are the weak limits of G_n and H_n, respectively.

From the expression of the first moment of ν and taking into account (12), (13) and (14), it follows that

$$\begin{cases}\rho_1=-(\theta\beta_1+(1-\theta)\beta_2)u,\\[2mm] \overline{\rho}=k_1\theta\int_{\mathbb{R}^N}\overline{y}d\overline{\nu}_1^{(2)}+k_2(1-\theta)\int_{\mathbb{R}^N}\overline{y}d\overline{\nu}_2^{(2)},\\[2mm] \lambda_1=\theta\int_{\Lambda_1}y_1 d\nu_1+(1-\theta)\int_{\Lambda_2}y_1 d\nu_2,\\[2mm] \overline{\lambda}=\theta\int_{\mathbb{R}^N}\overline{y}d\overline{\nu}_1^{(2)}+(1-\theta)\int_{\mathbb{R}^N}\overline{y}d\overline{\nu}_2^{(2)}.\end{cases}$$

On the other hand, the div-curl condition on ν implies (see [20]) that

$$\int_{\Lambda_1\cup\Lambda_2}x\cdot y\ d\nu(x,y)=\rho\cdot\lambda,$$

where $x\cdot y$ stands for the inner product of x and y. Developing the left-hand side of this expression,

$$\begin{aligned}\int_{\Lambda_1\cup\Lambda_2}x\cdot y\ d\nu(x,y) &= -\theta\beta_1 u\lambda_1^1-(1-\theta)\beta_2 u\lambda_1^2\\ &\quad +k_1\theta\int_{\mathbb{R}^N}|\overline{y}|^2 d\overline{\nu}_1^{(2)}+k_2(1-\theta)\int_{\mathbb{R}^N}|\overline{y}|^2 d\overline{\nu}_2^{(2)}.\end{aligned}$$

Next, we introduce the second moments

$$s_1=\int_{\mathbb{R}^N}|\overline{y}|^2 d\overline{\nu}_1^{(2)} \quad\text{and}\quad s_2=\int_{\mathbb{R}^N}|\overline{y}|^2 d\overline{\nu}_2^{(2)}.$$

Moreover, using this notation we can expressed the commutation property corresponding to (11) in the form

$$\overline{\rho}\cdot\overline{\lambda}=\int_{\Lambda_1\cup\Lambda_2}\overline{x}\cdot\overline{y}\,d\nu(x,y)=\theta k_1 s_1+(1-\theta)k_2 s_2. \tag{18}$$

If we put

$$\lambda_1^1 = \int_{\Lambda_1} y_1 d\nu_1, \quad \lambda_1^2 = \int_{\Lambda_2} y_1 d\nu_2, \quad \overline{\lambda}_1 = \int_{\mathbb{R}^N} \overline{y} d\overline{\nu}_1^{(2)} \quad \text{and} \quad \overline{\lambda}_2 = \int_{\mathbb{R}^N} \overline{y} d\overline{\nu}_2^{(2)},$$

then we can write some of those conditions in the form

$$\begin{cases} \overline{\rho} = k_1 \theta \overline{\lambda}_1 + k_2 (1-\theta) \overline{\lambda}_2, \\ \overline{\lambda} = \theta \overline{\lambda}_1 + (1-\theta) \overline{\lambda}_2, \qquad \lambda_1 = \theta \lambda_1^1 + (1-\theta) \lambda_1^2, \\ k_1 \theta s_1 + k_2 (1-\theta) s_2 - \rho \cdot \lambda = \theta \beta_1 u \lambda_1^1 + (1-\theta) \beta_2 u \lambda_1^2. \end{cases}$$

The first two equations can be used to solve for $\overline{\lambda}_1$ and $\overline{\lambda}_2$, namely,

$$\overline{\lambda}_1 = \frac{1}{\theta (k_1 - k_2)} \left(\overline{\rho} - k_2 \overline{\lambda} \right), \qquad \overline{\lambda}_2 = \frac{1}{(1-\theta)(k_2 - k_1)} \left(\overline{\rho} - k_1 \overline{\lambda} \right).$$

The other two can also be used to solve for λ_1^1 and λ_1^2. If $u \neq 0$, then

$$\begin{cases} \lambda_1^1 = -\frac{1}{u\theta(\beta_1 - \beta_2)} \left(\theta (\beta_2 - \beta_1) u \lambda_1 + \overline{\rho} \cdot \overline{\lambda} - [k_1 \theta s_1 + k_2 (1-\theta) s_2] \right), \\ \lambda_1^2 = -\frac{1}{u(1-\theta)(\beta_1 - \beta_2)} \left((1-\theta) (\beta_2 - \beta_1) u \lambda_1 + \overline{\rho} \cdot \overline{\lambda} - [k_1 \theta s_1 + k_2 (1-\theta) s_2] \right). \end{cases}$$

For $u = 0$ there is an infinity of possibilities for λ_1^1 and λ_1^2, namely

$$\lambda_1^1 = \gamma, \qquad \lambda_1^2 = \frac{1}{(1-\theta)} (\lambda_1 - \theta \gamma)$$

with any $\gamma \in \mathbb{R}$. Recall that we also have the equality (18). With all of these notations, (PM) reads in the simpler form:

$$\text{Minimize in } (s_1, s_2): \quad k_1 \theta s_1 + k_2 (1-\theta) s_2$$

subject to

$$s_1 \geq \frac{\left| \overline{\rho} - k_2 \overline{\lambda} \right|^2}{\theta^2 (k_1 - k_2)^2}, \qquad s_2 \geq \frac{\left| \overline{\rho} - k_1 \overline{\lambda} \right|^2}{(1-\theta)^2 (k_2 - k_1)^2}$$

where the two inequalities appearing in the constraints are a consequence of Jensen's inequality.

It is elementary to realize that the minimum of this problem is attained for

$$s_1 = \frac{\left| \overline{\rho} - k_2 \overline{\lambda} \right|^2}{\theta^2 (k_1 - k_2)^2} \quad \text{and} \quad s_2 = \frac{\left| \overline{\rho} - k_1 \overline{\lambda} \right|^2}{(1-\theta)^2 (k_2 - k_1)^2},$$

and in this case (18) becomes, after some algebra,

$$(\theta k_2 + (1-\theta) k_1) |\overline{\rho}|^2 - \left(\theta (1-\theta) (k_1 - k_2)^2 + 2 k_1 k_2 \right) \overline{\rho} \cdot \overline{\lambda} + (\theta k_1 + (1-\theta) k_2) k_1 k_2 |\overline{\lambda}|^2 = 0. \tag{19}$$

Therefore,

$$CQW(\theta, \rho, \lambda) \geq \begin{cases} k_1 \frac{\left| \overline{\rho} - k_2 \overline{\lambda} \right|^2}{\theta (k_1 - k_2)^2} + k_2 \frac{\left| \overline{\rho} - k_1 \overline{\lambda} \right|^2}{(1-\theta)(k_2 - k_1)^2} & \text{if (19) holds,} \\ +\infty & \text{else.} \end{cases}$$

Here $\rho = (\rho_1, \overline{\rho})$ and $\lambda = (\lambda_1, \overline{\lambda})$.

Our next task is to see if this lower bound can be attained by a first-order div-curl laminate (see [20] for the definition and main properties of this subclass of div-curl Young measures). This would give us more information on the minimizing sequences of $(\widetilde{\mathrm{P}})$.

Note that due to the strict convexity of $|\cdot|^2$, the equality in Jensen's inequality holds if and only if the associated measure is a delta in the corresponding components, that is,

$$\overline{\nu}_1^{(2)} = \delta_{\frac{\overline{\rho}-k_2\overline{\lambda}}{\theta(k_1-k_2)}} \quad \text{and} \quad \overline{\nu}_2^{(2)} = \delta_{\frac{\overline{\rho}-k_1\overline{\lambda}}{(1-\theta)(k_2-k_1)}}.$$

Moreover, since supp $\nu_i \subset \Lambda_i$, $i = 1, 2$, the projection of ν_i onto the last $N-$components of the first copy of $\mathbb{R}^{N+1}$ has the form

$$\overline{\nu}_1^{(1)} = \delta_{k_1 \frac{\overline{\rho}-k_2\overline{\lambda}}{\theta(k_1-k_2)}} \quad \text{and} \quad \overline{\nu}_2^{(1)} = \delta_{k_2 \frac{\overline{\rho}-k_1\overline{\lambda}}{(1-\theta)(k_2-k_1)}}.$$

So, the optimal first-order laminate we are looking for looks like

$$\nu = \theta\delta_{\left(-\beta_1 u, k_1 \frac{\overline{\rho}-k_2\overline{\lambda}}{\theta(k_1-k_2)}; \lambda_1^1, \frac{\overline{\rho}-k_2\overline{\lambda}}{\theta(k_1-k_2)}\right)} + (1-\theta)\,\delta_{\left(-\beta_2 u, k_2 \frac{\overline{\rho}-k_1\overline{\lambda}}{(1-\theta)(k_2-k_1)}; \lambda_1^2, \frac{\overline{\rho}-k_1\overline{\lambda}}{(1-\theta)(k_2-k_1)}\right)}. \tag{20}$$

Remark that according to our previous formulae, we must choose λ_1^1 and λ_1^2 such that

$$\begin{cases} \lambda_1 = \theta\lambda_1^1 + (1-\theta)\,\lambda_1^2, \\ -\theta\beta_1 u\lambda_1^1 - (1-\theta)\,\beta_2 u\lambda_1^2 = \rho\cdot\lambda - \left[k_1\frac{|\overline{\rho}-k_2\overline{\lambda}|^2}{\theta(k_1-k_2)^2} + k_2\frac{|\overline{\rho}-k_1\overline{\lambda}|^2}{(1-\theta)(k_2-k_1)^2}\right], \end{cases}$$

that is, if $u \neq 0$, then

$$\begin{cases} \lambda_1^1 = -\frac{1}{u\theta(\beta_1-\beta_2)}\left(\theta\,(\beta_2-\beta_1)\,u\lambda_1 + \overline{\rho}\cdot\overline{\lambda} - \left[k_1\frac{|\overline{\rho}-k_2\overline{\lambda}|^2}{\theta(k_1-k_2)^2} + k_2\frac{|\overline{\rho}-k_1\overline{\lambda}|^2}{(1-\theta)(k_2-k_1)^2}\right]\right), \\ \lambda_1^2 = -\frac{1}{u(1-\theta)(\beta_1-\beta_2)}\left((1-\theta)\,(\beta_2-\beta_1)\,u\lambda_1 + \overline{\rho}\cdot\overline{\lambda} - \left[k_1\frac{|\overline{\rho}-k_2\overline{\lambda}|^2}{\theta(k_1-k_2)^2} + k_2\frac{|\overline{\rho}-k_1\overline{\lambda}|^2}{(1-\theta)(k_2-k_1)^2}\right]\right), \end{cases}$$

and for $u = 0$ there is an infinity of possibilities for λ_1^1 and λ_1^2, namely

$$\lambda_1^1 = \gamma, \quad \lambda_1^2 = \frac{1}{(1-\theta)}\,(\lambda_1 - \theta\gamma)$$

with $\gamma \in \mathbb{R}$. In this last case, the div-curl compatibility condition reduces to

$$\rho\cdot\lambda = \left[k_1\frac{\left|\overline{\rho}-k_2\overline{\lambda}\right|^2}{\theta\,(k_1-k_2)^2} + k_2\frac{\left|\overline{\rho}-k_1\overline{\lambda}\right|^2}{(1-\theta)\,(k_2-k_1)^2}\right].$$

The above means that optimal measures leading to the exact value for $CQW\,(\theta,\rho,\lambda)$ may be found in the form of first-order laminates in the form (20). As we will see later on, this will enable us to build optimal micro-structures for problem $(\widetilde{\mathrm{P}})$. Note also that thanks to the particular form of this measure, the first component of the vector field G, say G_1, is equal to $-\left(\theta\beta_1 + (1-\theta)\,\beta_2\right) u$. This, together with the divergence-free character of G leads to the equation

$$-\left(\theta\beta_1 + (1-\theta)\,\beta_2\right) u' + \operatorname{div}\overline{G} = 0$$

where we have put $G = \left(G_1, \overline{G}\right)$.

We then find a relaxation of $(\widetilde{\mathrm{P}})$ in the following form:

THEOREM 3.1 *Assume that* Ω *is of class* C^1 *and that* $u_0 \in H_0^1(\Omega)$. *Then the variational problem*

$$(\widetilde{RP}) \quad \text{Minimize in } \left(\theta, \overline{G}, u\right): \quad \overline{J}(\theta, \overline{G}, u) = \frac{1}{2}\int_0^T\int_\Omega CQW\left(\theta(x), G(t,x), \nabla_{(t,x)}u(t,x)\right)dxdt$$

subject to

$$\begin{cases} G \in L^2\left((0,T)\times\Omega;\mathbb{R}^{N+1}\right), & u \in H^1((0,T)\times\Omega;\mathbb{R}), \\ (\theta\beta_1 + (1-\theta)\beta_2)u' - div\,\overline{G} = 0 & in\ H^{-1}((0,T)\times\Omega), \\ u|_{\partial\Omega} = 0 \quad a.\ e.\ t \in [0,T], & u(0) = u_0 \quad in\ \Omega, \\ \theta \in L^\infty(\Omega;[0,1]), \quad \int_\Omega \theta(x)\,dx = L|\Omega|. \end{cases}$$

is a relaxation of $(\widetilde{P})$ *in the sense that*

(i) *there exists at least one minimizer for* $(\widetilde{RP})$,

(ii) *the minimum of* $(\widetilde{P})$ *equals the infimun of* $(\widetilde{RP})$, *and*

(iii) *the underlying Young measure associated with* $(\widetilde{RP})$ *(and therefore the optimal microstructure of* $(\widetilde{P})$*) can be found in the form of a first-order laminate. Moreover, at the points where* $\nabla u \neq 0$, *the normal to this laminate is perpendicular to* ∇u.

Proof. Once the constrained quasi-convex density CQW has been computed, the proof is standard in non-convex vector variational problems (see for instance [15] or [17, Chapter 4]). There is, however, a technical point which deserves an additional explanation. It concerns the equi-integrability property of $|\nabla u_n|^2$ that is needed to represent the limit cost associated with a minimizing sequence of designs through its corresponding Young measure. Note that we also have to face the same problem with the sequence of characteristic functions $\mathcal{X}_n(x)$ as in the homogenization case (see Proposition 2.1).

This problem may be easily overcome if we assume the regularity on the domain Ω and the initial datum u_0 as stated above. Precisely, this implies that the solutions of the heat equation have the regularity

$$u_n \in L^2\left(0,T;H^2(\Omega)\right)$$

with uniform estimates in the norm of this space (see [7, p. 360]). By using the Sobolev embedding theorem it follows that $|\nabla u_n|^2 \in L^{p/2}$ for some $p > 2$ (with uniform bounds) and a. e. $t \in [0,T]$. From this and Hölder inequality one deduces that $|\nabla u_n|^2$ is equi-integrable.

Concerning the direction of lamination, it is not hard to show that if u is a solution of system (1) and $\phi \in H_0^1(\Omega)$, then

$$\int_\Gamma (k_1 - k_2)\nabla u \cdot n\ \phi d\Gamma = 0,$$

with Γ the interface and n the unit normal vector to Γ. This proves our claim. ■

Remark 2 *Note that although the design variable* $\mathcal{X} = \mathcal{X}(x)$ *does not depend on time, however a minimizing sequence of optimal designs* $\mathcal{X}_n(x)$ *are associated with an optimal first-order laminate whose mass points are time dependent. This means that in looking at optimal micro-structures the proportion of the two materials is time independent, but having the same proportion of materials, the way (=direction) in which we should mix the materials depend on time according to* $\nabla u(t,x)$.

Remark 3 *Notice that ($\widetilde{RP}$) may also be written as*

$$\text{Minimize in } \left(\theta, \overline{G}, u\right): \quad \frac{1}{2}\int_0^T\int_\Omega \overline{G}\cdot\nabla u \, dxdt$$

subject to

$$\begin{cases} u \in L^2\left(0,T;H_0^1(\Omega)\right), & \text{with } u' \in L^2\left(0,T;H^{-1}(\Omega)\right) \\ \left|\overline{G}\right|^2 - \left(\lambda_\theta^+ + \lambda_\theta^-\right)\overline{G}\cdot\nabla u + \lambda_\theta^+\lambda_\theta^- \left|\nabla u\right|^2 = 0, & \\ \left(\theta\beta_1 + (1-\theta)\beta_2\right)u' - div\, \overline{G} = 0 & \text{in } (0,T)\times\Omega, \\ u|_{\partial\Omega} = 0 \quad a.\ e.\ t\in[0,T], & u(0) = u_0 \quad \text{in } \Omega, \\ \theta \in L^\infty(\Omega;[0,1]), \quad \int_\Omega \theta(x)\,dx = L|\Omega|. & \end{cases}$$

where λ_θ^+ and λ_θ^- are the arithmetic and harmonic mean, respectively.

3.3 Another relaxation

The constraint expressed in (19) is rather tedious to keep in numerical simulations because it is a point-wise condition. We would like to find another relaxation where we should not keep track of this sort of constraints. In our situation, this is easily achieved. Indeed, it has already been done in Section 3. Notice that all of our computations in the preceding section could have been performed without any reference to (19). In fact, in those computations this constraint does not play a role. This in fact implies that we can forget about this constraint altogether to get a new relaxation which is, we believe, easier to implement numerically as we do not have to bother about that pointwise constraint. Precisely, we have:

THEOREM 3.2 *The problem ($\widehat{RP}$)*

$$\text{Minimize in } \left(\theta, \overline{G}, u\right): \quad \overline{J}(\theta,\overline{G},u) = \frac{1}{2}\int_0^T\int_\Omega \left[k_1\frac{\left|\overline{G}-k_2\nabla u\right|^2}{\theta\left(k_1-k_2\right)^2} + k_2\frac{\left|\overline{G}-k_1\nabla u\right|^2}{(1-\theta)\left(k_2-k_1\right)^2}\right] dxdt$$

subject to

$$\begin{cases} u \in L^2\left(0,T;H_0^1(\Omega)\right), & \text{with } u' \in L^2\left(0,T;H^{-1}(\Omega)\right) \\ \left(\theta\beta_1 + (1-\theta)\beta_2\right)u' - div\, \overline{G} = 0 & \text{in } (0,T)\times\Omega, \\ u|_{\partial\Omega} = 0 \quad a.\ e.\ t\in[0,T], & u(0) = u_0 \quad \text{in } \Omega, \\ \theta \in L^\infty(\Omega;[0,1]), \quad \int_\Omega \theta(x)\,dx = L|\Omega|, & \end{cases}$$

is also a relaxation of the problem ($\widetilde{P}$) with underlying micro-structures which are again first-order laminates.

Let us again stress that one gets this relaxation by simply forgetting about (19). Our claim is that even if we forget this constraint at the outset, at the end the optimal solutions for this relaxation will comply with (19). Even further, we conjecture that the problem

$$(\underline{\text{RP}}) \quad \text{Minimize in } \theta: \quad \underline{J}(\theta) = \frac{1}{2}\int_0^T\int_\Omega \frac{k_1k_2}{\theta k_2 + (1-\theta)k_1}\left|\nabla u\right|^2 dxdt$$

subject to

$$\begin{cases} (\theta\beta_1 + (1-\theta)\,\beta_2)\,u' - \operatorname{div}\,\left(\frac{k_1k_2}{\theta k_2+(1-\theta)k_1}\nabla u\right) = 0 \quad \text{in } (0,T)\times\Omega, \\ u|_{\partial\Omega} = 0 \quad \text{a. e. } t\in[0,T]\,, \qquad\qquad u\,(0) = u_0 \quad \text{in } \Omega, \\ \theta\in L^\infty\,(\Omega;[0,1])\,, \quad \int_\Omega \theta\,(x)\,dx = L\,|\Omega|\,, \end{cases} \tag{21}$$

is also a relaxation for our original problem. Our intuition here is rooted in the fact that if in the expression for CQW, we find the minimum in $\overline{\rho}$ for $\overline{\lambda}$ fixed, with or without constraint (19), then we arrive at a linear relationship given by the harmonic mean between $\overline{\lambda}$ and $\overline{\rho}$

$$\overline{\rho} = \frac{k_1k_2}{(1-\theta)k_1+\theta k_2}\overline{\lambda}.$$

See [19] for more on these ideas for the elliptic case. This computation is elementary. We will try to validate this conjecture in our numerical experiments.

4 Numerical Applications

In this section, we compare numerically in the two dimensional case ($N = 2$) the relaxed formulations (RP) and $(\widehat{RP})$ obtained from the Homogenization and Young measure theory respectively.

4.1 Numerical resolution of the relaxed problems

We first explain the numerical resolution of the relaxed problem (RP) derived from the Homogenization method (see section 2.2).

A convenient way to minimize J^* consists first in using a parametrization of the homogenized tensor $K^*\in G_\theta$ in terms of its Y-transform (we refer to [1, p. 122]): the Y-transform is the map on the set of symmetric matrices defined by

$$Y(K^*) = (\lambda_\theta^+ I_N - K^*)((\lambda_\theta^-)^{-1}K^* - I_N)^{-1}. \tag{22}$$

For $N = 2$, denoting by y_1, y_2 the eigenvalues of $Y(K^*)$, K^* belongs to G_θ if and only if

$$\min(k_1,k_2)^2 \le y_1y_2 \le \max(k_1,k_2)^2, \quad y_1,y_2\ge 0. \tag{23}$$

The advantage is that the set $Y(G_\theta)$ does not depend on θ. Its inverse mapping is

$$K^*(Y) = (\lambda_\theta^+ I_N + Y)((\lambda_\theta^-)^{-1}Y + I_N)^{-1}. \tag{24}$$

We then parameterize a composite design by (θ, Y^*) with $Y^* = Y(A^*)$ for some $A^*\in G_\theta$. The interest is that the constraints on θ and Y are now uncoupled making easier the implementation of gradient algorithm. Consequently, $A^*\in G_\theta$ is parameterized by the density θ, the two eigenvalues y_1 and y_2 and the angle of rotation ϕ such that

$$K^*(\theta,y_1,y_2,\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \frac{\lambda_\theta^+ + y_1}{y_1/\lambda_\theta^- + 1} & 0 \\ 0 & \frac{\lambda_\theta^+ + y_2}{y_2/\lambda_\theta^- + 1} \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{25}$$

Finally, we compute the first derivative of the resulting function (still denoted by J^*) with respect to θ, Y^* and ϕ and apply a gradient algorithm. The first derivative in any direction $(\delta\theta, \delta Y^*, \delta\phi)$ takes the following expression

$$\begin{aligned}\frac{\partial J^*(\theta, Y^*, \phi)}{\partial(\theta, Y^*, \phi)} \cdot (\delta\theta, \delta Y^*, \delta\phi) = &\int_\Omega \int_0^T \left(\frac{1}{2} K^*_\phi \nabla u \cdot \nabla u + K^*_\phi \nabla u \cdot \nabla p\right) dt\ \delta\phi dx \\ &+ \int_\Omega \int_0^T \left(\frac{1}{2} K^*_{Y^*} \nabla u \cdot \nabla u + K^*_{Y^*} \nabla u \cdot \nabla p\right) dt\ \cdot \delta Y^* dx \\ &+ \int_\Omega \int_0^T \left(\frac{1}{2} K^*_\theta \nabla u \cdot \nabla u + K^*_\theta \nabla u \cdot \nabla p + (\beta_1 - \beta_2) u' p\right) dt\ \delta\theta dx \end{aligned} \tag{26}$$

where p designates the adjoint solution of the backward system

$$\begin{cases} -\beta(\theta)p' - \text{div}\ (K^*(\theta, Y^*, \phi)\nabla p) = \text{div}\ (K^*(\theta, Y^*, \phi)\nabla u) & \text{in} \quad (0,T) \times \Omega, \\ p = 0 & \text{on} \quad (0,T) \times \partial\Omega, \\ p\,(T, x) = 0 & \text{in} \quad \Omega \end{cases} \tag{27}$$

and $K^*_\theta, K^*_{Y^*}, K^*_\phi$ the derivatives of K^* with respect to θ, Y^* and ϕ respectively. At last, we use lagrangian multipliers to enforce the constraints $\theta \in L^\infty(\Omega, [0,1])$, $\int_\Omega \theta(x)dx = L|\Omega|$ and (23).

The relaxed problem $(\widehat{RP})$ (see Theorem 3.2) derived from the second approach, although less standard, may be solved in a similar way using a descent algorithm. Precisely, the minimization of $\overline{J}$ is done over θ and $\overline{G}$ while u is determined via the constraint $(\theta\beta_1 + (1-\theta)\beta)u' - div\overline{G} = 0$. The first variation of $\overline{J}$ with respect to $(\theta, \overline{G})$ in any direction $(\delta\theta, \delta\overline{G})$ is given by

$$\begin{aligned}\frac{\partial \overline{J}(\theta, \overline{G}, u)}{\partial(\theta, \overline{G})} \cdot (\delta\theta, \delta\overline{G}) = &\int_\Omega \int_0^T \left[(\beta_1 - \beta_2)u'p - \frac{k_1}{\theta^2} \frac{|\overline{G} - k_2\nabla u|^2}{(k_1 - k_2)^2} + \frac{k_2}{(1-\theta)^2} \frac{|\overline{G} - k_1 \nabla u|^2}{(k_1 - k_2)^2}\right] dt\ \delta\theta dx \\ &+ \int_0^T \int_\Omega \left[\frac{k_1}{\theta} \frac{(\overline{G} - k_2 \nabla u)}{(k_1 - k_2)^2} + \frac{k_2}{1-\theta} \frac{(\overline{G} - k_1\nabla u)}{(k_1 - k_2)^2} + \nabla p\right] \cdot \delta\overline{G} dx dt \end{aligned} \tag{28}$$

where p is solution of the following problem :

$$\begin{cases} (\theta\beta_1 + (1-\theta)\beta_2)p' = \frac{k_1 k_2}{(k_1 - k_2)^2} div \left(\frac{(\overline{G} - k_2 \nabla u)}{\theta} + \frac{(\overline{G} - k_1 \nabla u)}{1-\theta}\right) & \text{in} \quad (0,T) \times \Omega, \\ p = 0 & \text{on} \quad (0,T) \times \partial\Omega, \\ p(T, x) = 0 & \text{in} \quad \Omega. \end{cases} \tag{29}$$

Once again, a multiplier is necessary to deal with the constraints on θ. Finally, the resolution of the problem $(\underline{\text{RP}})$ is standard and we refer to [14] for the details in the context of the wave equation.

For all the variables, we use a continuous finite element approximation of second order with respect to x on a uniform mesh and a finite difference approximation of first order with respect to t. In the resolution of problem $(\widehat{\text{RP}})$, since $\overline{G}$ is a time-space variable, a regularization of the variable p via a viscosity term in (29) is applied (see [13] for a similar phenomenon where the density is time-space dependent).

4.2 Numerical experiments

We consider the following simple initial data on the unit square : $\Omega = (0,1)^2$:

$$u_0(x) = \sin(\pi x_1)\sin(\pi x_2), \quad x = (x_1, x_2) \in \Omega \tag{30}$$

and take $T = 0.5$, $L = 1/2$, $(\beta_1, k_1) = (10, 0.1)$ and $(\beta_2, k_2) = (20, 1)$. At last, the numerical results presented in this section are obtained with the spatial discretisation parameter $h = 1/50$ and with the temporal discretisation parameter $dt = h/4$.

We first give the results obtained for the problem (RP) derived from the Homogenization approach. The algorithm is initialized with constant functions: we take $\theta \equiv L|\Omega|$, $y_i \equiv (k_1+k_2)/2, i = 1, 2$, and $\phi \equiv 0$ on Ω. Figure 1 depicts the functions θ and ϕ, local minima of J^*. Figure 2 depicts the corresponding function y_1 and y_2. We obtain $J^*(\theta, y_1, y_2, \phi) \approx 0.202$ and we observe that θ is a characteristic function in $L^\infty(\Omega, \{0, 1\})$. The corresponding gradient part of the energy with respect to the time is given in Figure 3 highlighting the diffusion of the heat. We also observe - this is the main drawback of gradient method - that the result depends on the initialization. Figure 4 depicts the iso-values of θ and ϕ obtained at convergence of the algorithm initialized still with $\theta = L|\Omega|$, $\phi = 0$ but now with $y_1 = \min(k_1, k_2)$ and $y_2 = \max(k_1, k_2)$. The value of the cost function is however similar highlighting the existence of local minima and a low dependence of J^* with respect to the variables.

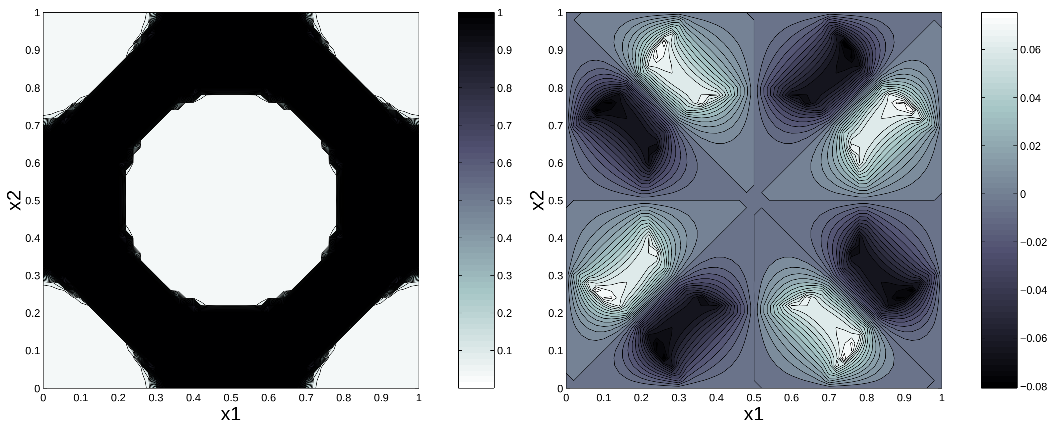

Figure 1: Resolution of (RP) - $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of θ (**Left**) and ϕ (**Right**)- $J^*(\theta, y_1, y_2, \phi) \approx \mathbf{0.202}$.

The results obtained for the relaxed problem ($\widehat{\text{RP}}$) derived from the variational approach are qualitatively different. Once again, the density θ is initialized with $\theta \equiv L|\Omega|$ on Ω which does not privilege any location for the set of the first material (β_1, k_1). On the other hand, the field $\overline{G}$ is initialized by $\overline{G} = \lambda_\theta^- \nabla u$ where u is solution of (21). Figure 5 displays the iso-values of the function θ. The results seem here independent of the initialization of the algorithm: for instance, we get a similar result if we take $\overline{G} = \lambda_\theta^+ \nabla u$. This suggests that the function θ of Figure 5 is the global minimum: we obtain $\overline{J}(\theta, \overline{G}, u) \approx 0.1806$ which is lower than in the previous case. Moreover, we observe that θ is no more a characteristic function which suggests that for these data, the initial design problem (P) is not well-posed, and therefore justifies the whole relaxation procedure. This also indicates that the functions of Figure 1 are only local minima for J^*. Due to the constraints (23), the global minima for J^* seems more difficult to capture. Secondly, we check that the solution $(\theta, \overline{G}, u)$ satisfies

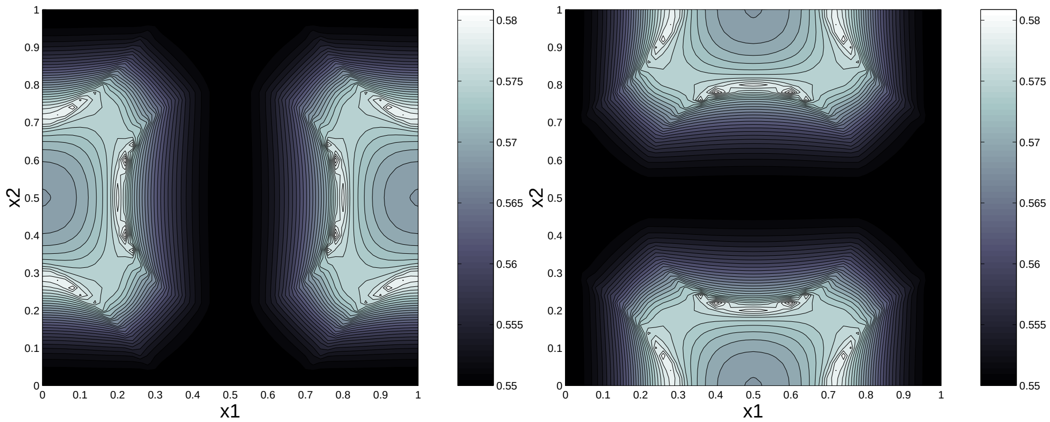

Figure 2: Resolution of (RP) - $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of y_1 (**Left**) and y_2 (**Right**)- $J^*(\theta, y_1, y_2, \phi) \approx$ **0.202**.

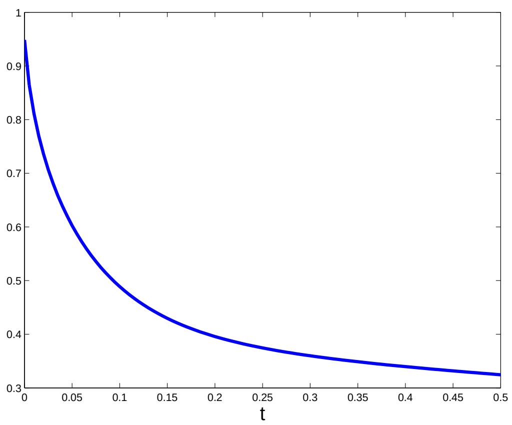

Figure 3: Resolution of (RP) - $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Gradient part of the energy vs. t.

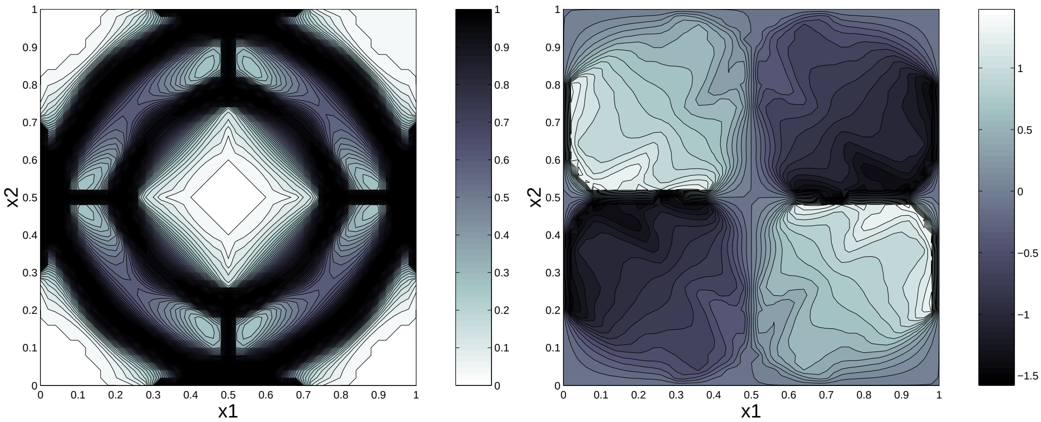

Figure 4: Resolution of (RP) with a different initialization- $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of θ (**Left**) and ϕ (**Right**)- $J^*(\theta, y_1, y_2, \phi) \approx \mathbf{0.224}$.

the relation (19): precisely, we compute that

$$\frac{||\,|\overline{G}|^2 - \left(\lambda_\theta^+ + \lambda_\theta^-\right)\overline{G}\cdot\nabla u + \lambda_\theta^+\lambda_\theta^-\,|\nabla u|^2\,||_{L^2((0,T)\times\Omega)}}{||\,|\overline{G}|^2\,||_{L^2((0,T)\times\Omega)}} \approx 1.34\times 10^{-2} \tag{31}$$

which shows, in agreement with the discussion at the beginning of Section 3.3, that the problem $(\widehat{\mathrm{RP}})$ coincides with the problem $(\widetilde{\mathrm{RP}})$ derived from the variational approach. At last, we compute that

$$\frac{||\overline{G} - \lambda_\theta^-\nabla u||_{L^2((0,T)\times\Omega)^2}}{||\overline{G}||_{L^2((0,T)\times\Omega)^2}} \approx 4.35\times 10^{-3} \tag{32}$$

which provides a numerical evidence that $\overline{G} = \lambda_\theta^-\nabla u$, and that, according to our conjecture of Section 3.3, problems $(\widehat{\mathrm{RP}})$ and $(\underline{\mathrm{RP}})$ coincide. However, if we naively replace this term by the arithmetic mean $\lambda_\theta^+ = k_1\theta + k_2(1-\theta)$, then we obtain the distribution of Figure 6 leading to a greater cost equal to 0.213.

Moreover, similarly to the hyperbolic case (see [13]), we observe that when the gap $k_2 - k_1$ and $\beta_2 - \beta_1$ between the coefficients is small enough (depending on the data of the problem), the density θ is a characteristic function (see Figure 7 obtained for $(\beta_1, k_1) = (10, 0.1)$ and $(\beta_2, k_2) = (10.2, 0.102)$): this suggests that in this case the problem (P) is well-posed.

At last, on a physical point of view, the initial data being fixed, the distribution of the two materials seems to depend mainly on the value of the ratio k_2/k_1 with respect to one. Precisely, the material which have the greater diffusion coefficient (here k_2) is distributed on the center and on the corners of the unit square. The value of the ratio β_2/β_1 and of T seems less preponderant. These observations are related to the exponential diffusion in time of the heat solution u.

5 Concluding remarks

In this work, we have analyzed theoretically and numerically a typical nonlinear optimal design problem for the heat equation. From a theoretical point of view, two relaxations

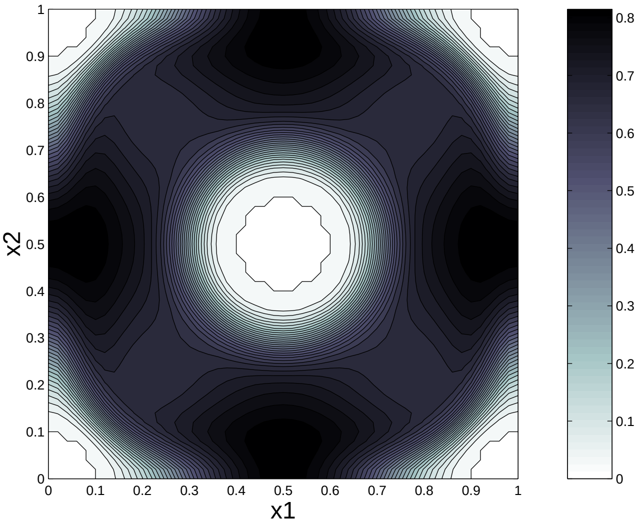

Figure 5: Resolution of $(\widehat{\mathrm{RP}})$ - $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of θ - $\overline{J}(\theta, \overline{G}, u) \approx \mathbf{0.1806}$.

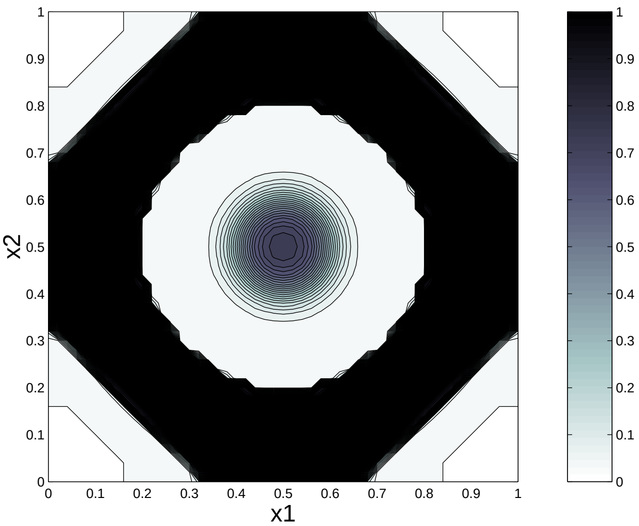

Figure 6: $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of θ when $k_1 \mathcal{X}_\omega + k_2(1 - \mathcal{X}_\omega)$ is directly replaced by the arithmetic mean λ_θ^+- Cost function $\approx \mathbf{0.213}$.

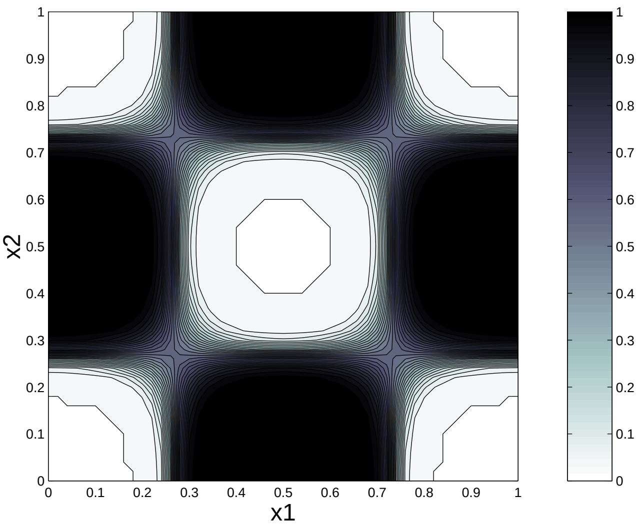

Figure 7: Resolution of $(\widehat{\mathrm{RP}})$ - $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 10.2)$, $(k_1, k_2) = (0.1, 0.102)$ - Iso-values of θ - $\overline{J}(\theta, \overline{G}, u) \approx \mathbf{0.1126}$.

have been obtained by using the Homogenization method and the classical tools of non-convex vector variational problems. The connection between both approaches is given by the identity

$$K^* \nabla u = \overline{G}.$$

The homogenization method permits to obtain quite easily a relaxed formulation and shows that in the one-dimensional case, the optimal tensors K^* is given by the harmonic mean of k_1 and k_2, namely

$$K^* = \left(\frac{\theta}{k_1} + \frac{1-\theta}{k_2} \right)^{-1}.$$

The variational approach, through the Young measure, more involved, leads to a somehow more explicit relaxed problem, although less standard. It is worthwhile to mention that this approach requires an extra-regularity for the initial datum u_0 in order to get the equi-integrability of $|\nabla u_n|^2$. At present, we do not know if this assumption is necessary. Moreover, the numerical simulations, performed in the two-dimensional case, first suggest that the problem (P) may be ill-posed according to the data, and secondly, validate our conjecture on the role of the harmonic mean when the "compliance" cost function is considered.

As indicated in the Introduction, this paper is only a preliminary study on this topic. Many interesting questions remain open. Among them, we could consider more general cost involving anisotropic materials. Very likely, the extension of ([4, 17, 13]) to the parabolic case will provide relaxed formulations. But this remains to be checked. It would also be interesting to consider the case of spatio-temporal design problem where $\mathcal{X} \in L^\infty((0,T) \times \Omega, \{0,1\})$.

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