NUMERICAL APPROXIMATION OF THE BOUNDARY CONTROL OF THE 2-D WAVE EQUATION WITH MIXED FINITE ELEMENTS

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Abstract

This paper studies the numerical approximation of the HUM boundary control for the 2-D wave equation. It is known that the discrete and semi-discrete models obtained by discretizing the wave equation with the classical finite difference or finite element method do not provide convergent sequences of approximations to the boundary control of the continuous wave equation, as the mesh size goes to zero (see [9] and [22]). Here we introduce a new semi-discrete model based on the discretization of the wave equation using a mixed finite element method with two different basis functions for the position and velocity. This allows us to construct a convergent sequence of approximations to the HUM control, as the space discretization parameter h tends to zero.

We also introduce a fully-discrete system, obtained from our semi-discrete scheme, that is likely to provide a convergent sequence of discrete approximations as both h and Δt, the time discretization parameter, go to zero. We illustrate this fact with several numerical experiments.

Contents

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1 Introduction

Let us consider $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$ with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ divided as follows

$$\begin{cases} \Gamma_0 = \{(x,0) \,:\, 0 \le x \le 1\} \cup \{(0,y) \,:\, 0 \le y \le 1\}, \\ \Gamma_1 = \{(x,1) \,:\, 0 < x < 1\} \cup \{(1,y) \,:\, 0 < y < 1\}. \end{cases} \tag{1.1}$$

We are concerned with the following exact boundary controllability property for the wave equation in Ω: given $T > 2\sqrt{2}$ and $(u^0,u^1) \in L^2(\Omega)\times H^{-1}(\Omega)$ there exists a control function $(v(t,y),z(t,x)) \in [L^2((0,T)\times(0,1))]^2$ such that the solution of the equation

$$\begin{cases} u'' - \Delta u = 0 & \text{for } (x,y) \in \Omega, \;\; t>0, \\ u(t,x,y) = 0 & \text{for } (x,y) \in \Gamma_0, \;\; t>0, \\ u(t,1,y) = v(t,y) & \text{for } y \in (0,1), \;\; t>0, \\ u(t,x,1) = z(t,x) & \text{for } x \in (0,1), \;\; t>0, \\ u(0,x,y) = u^0(x,y) & \text{for } (x,y) \in \Omega, \\ u'(0,x,y) = u^1(x,y) & \text{for } (x,y) \in \Omega, \end{cases} \tag{1.2}$$

satisfies

$$u(T,\cdot) = u'(T,\cdot) = 0. \tag{1.3}$$

By $'$ we denote the time derivative.

The Hilbert Uniqueness Method (HUM) introduced by J.-L. Lions offered a way to solve this and other multi-dimensional similar problems (see [12]).

In the last years many works have dealt with the numerical approximations of the control problem (1.2)-(1.3) using the HUM approach. For instance, in [6], [8] and [9], a numerical algorithm based on the finite difference approximation of (1.2) was described. However, in these articles a bad behavior of the approximative controls was observed.

Let us briefly explain this fact. When we are dealing with the exact controllability problem, a uniform time $T>0$ for the control of *all solutions* is required. This time T depends, roughly, on the size of the domain and the velocity of propagation of waves. Note that, for the continuous wave equation (1.2), the velocity of propagation of all waves is one and the bound of the minimal controllability time, $T > 2\sqrt{2}$, is exactly the minimum time that requires a wave, starting at any $x \in \Omega$ in any direction, to arrive to the controllability zone.

On the other hand, in general, any semi-discrete dynamics generates spurious high-frequency oscillations that do not exist at the continuous level. Moreover, a numerical dispersion phenomenon appears and the velocity of propagation of some high frequency numerical waves may possibly converge to zero when the mesh size h does. In this case, the controllability property for the semidiscrete system will not be uniform, as $h \to 0$, for a fixed time T and, consequently, there will be initial data (even very regular ones) for which the corresponding controls of the semi-discrete model will diverge, in the L^2-norm, as h tends to zero. This is the case when the semi-discrete model is obtained by discretizing the wave equation with the classical finite differences or finite element method (see [10] for a detailed analysis of the 1-D case and [22] for the 2-D case, in the context of the dual observability problem).

From the numerical point of view, several techniques have been proposed as possible cures of the high frequency spurious oscillations. For example, in [9] a Tychonoff regularization procedure was successfully implemented in several experiments. Roughly speaking, this method introduces an additional control, tending to zero with the mesh size, but acting on the interior of the domain. Other proposed numerical techniques are multi-grid or mixed finite element methods (see [7]). To our knowledge, no proof of convergence has been given for any of these methods, as $h \to 0$, so far.

In this paper we construct, for any fixed $T > 2\sqrt{3}$, a convergent sequence (as $h \to 0$) of semi-discrete approximations of the HUM control (v, z) of (1.2), for any initial data. The main idea is to introduce a new space discretization scheme for the wave equation (1.2), based on a *mixed finite element method*, in which different base functions for the position u and the velocity u' are considered. More precisely, while the classical first order splines are used for the former, discontinuous elements approximate the latter. This new scheme still has spurious high-frequency oscillations but, in this case, the numerical dispersion makes them to have larger velocity of propagation as $h \to 0$. We prove that this fast numerical waves are not an obstacle to the uniform controllability of the semi-dicrete scheme.

The semi-discrete approximations $(v_h, z_h)_{h>0}$ of the HUM control (v, z) of (1.2) are obtained, for any initial data, by minimizing the HUM functional of the associated semi-discrete adjoint system. The main result of the paper is Theorem 6.2 which says, roughly, that if a weakly convergent sequence of approximations of the continuous initial data to be controlled is considered, then $(v_h, z_h)_{h>0}$ converges weakly to (v, z). This result is based on a uniform (in h) observability inequality for the corresponding adjoint system (see Theorem 4.1 below).

The scheme introduced in this paper is different to the mixed element method used in [7] where u and ∇u are approximated in different finite dimensional spaces.

We also introduce a fully-discrete approximation of the wave equation for which the velocity of propagation of all numerical waves does not vanish as both h and Δt, the time discretization parameter, tend to zero. Based on this fact, we conjecture that this scheme also provides convergent approximations of the control as $h, \Delta t \to 0$, but we do not have a proof of this. However we show some numerical examples that somehow exhibits this convergence.

To our knowledge, this mixed finite element approach was used by the first time in the context of the wave equation in [2], in order to obtain a uniform decay rate of the energy associated to the semi-discrete wave equation by a boundary dissipation. It is well known that this boundary stabilization problem is closely related to the boundary controllability problem stated in this paper. However, in [2] the study of the 2-D case is not complete and no rigorous proof is given for the uniform decay property.

We also mention that the convergence and error estimates of the mixed finite element method described in this paper, for the (uncontrolled) wave equation, is given in [11].

In this paper, we concentrate on the simplest 2-D domain consisting of a unit square. However, the method is easily adapted to general 2-D domains where corresponding boundary controllability properties should hold.

Finally, we refer to [4] for the analysis and numerical implementation of the 1-D version of the numerical method described in this article.

The rest of the paper is organized in the following way. The second section briefly recalls some controllability results for the wave equation (1.2). In the third section the semi-discrete model under consideration is deduced. In the fourth section the main properties of this system are discussed and, in particular, a uniform observability inequality which will be fundamental for our study (Theorem 4.1). Its very technical proof is given in an Appendix at the end of the paper. In the fifth section an approximation sequence is constructed and in the sixth section its convergence to the HUM control of the continuous equation (1.2) is proved. The final section is devoted to present the fully-discrete scheme and the numerical results.

2 The continuous problem: results and notations

In this section we recall some of the controllability properties of the wave equation (1.2) and introduce some notations that will be used in the article. The following classical result may be found, for instance, in [12].

Theorem 2.1 *For any $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control function $(v, z) \in [L^2((0,T) \times (0,1))]^2$ such that the solution (u, u') of (1.2) verifies (1.3).*

In fact, given $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, a control of minimal L^2-norm may be obtained. To do that, let us introduce the map $\mathcal{J} : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{J}(w^0, w^1) = \frac{1}{2} \int_0^T \int_0^1 (w_x)^2(t,1,y)dydt + \frac{1}{2} \int_0^T \int_0^1 (w_y)^2(t,x,1)dxdt \\ + \int_\Omega u^0(x,y) w'(0,x,y)dxdy - \left\langle u^1, w(0,\,\cdot\,)\right\rangle_{-1,1}, \end{aligned} \tag{2.1}$$

where (w, w') is the solution of the backward homogeneous equation

$$\begin{cases} w'' - \Delta w = 0, & \text{for } (x,y) \in \Omega,\ \ t > 0, \\ w(t,0,y) = w(t,x,0) = w(t,x,1) = w(t,1,y) = 0, & \text{for } x, y \in (0,1),\ \ t > 0, \\ w(T,x,y) = w^0(x,y),\ \ w'(T,x,y) = w^1(x,y), & \text{for } (x,y) \in \Omega. \end{cases} \tag{2.2}$$

In (2.1), $< \cdot, \cdot >_{-1,1}$ denotes the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Theorem 2.2 *For any $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, $\mathcal{J}$ has an unique minimizer $(\widehat{w}^0, \widehat{w}^1) \in H_0^1(\Omega) \times L^2(\Omega)$. If $(\widehat{w}, \widehat{w}')$ is the corresponding solution of (2.2) with initial data $(\widehat{w}^0, \widehat{w}^1)$, then*

$$(v(t,y), z(t,x)) = (\widehat{w}_x(t,1,y), \widehat{w}_y(t,x,1)), \tag{2.3}$$

is the control of (1.2) with minimal L^2-norm.

We recall that the main ingredient of the proof of the Theorem 2.2 is the following observability inequality for (2.2): given $T > 2\sqrt{2}$ there exists a constant $C > 0$ such that, for any solution of (2.2),

$$E(t) \leq C \left(\int_0^T \int_0^1 |w_x(t,1,y)|^2 dydt + \int_0^T \int_0^1 |w_y(t,x,1)|^2 dxdt \right), \tag{2.4}$$

where

$$E(t) = \frac{1}{2} \int_\Omega \left(|\nabla w|^2 + |w_t|^2 \right) dxdy, \tag{2.5}$$

is the energy corresponding to (2.2).

Remark 2.1 *The control (v, z) from Theorem 2.2 is usually called the HUM control. It may be characterized by the following two properties:*

1. *(v, z) is a control for (1.2), or equivalently,*

$$\begin{aligned} \int_0^T \int_0^1 v(t,y) w_x(t,1,y) dydt + \int_0^T \int_0^1 z(t,x) w_y(t,x,1) dxdt \\ =< u^1, w(0) >_{-1,1} - \int_\Omega u^0(x,y) w'(0,x,y) dxdy, \end{aligned} \tag{2.6}$$

   for any $(w^0, w^1) \in H_0^1(\Omega) \times L^2(\Omega)$, being w the solution of the adjoint equation (2.2).

2. *There exists $(\widehat{w}^0, \widehat{w}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ such that $v(t,y) = \widehat{w}_x(t,1,y)$ and $z(t,x) = \widehat{w}_y(t,x,1)$, where $(\widehat{w}, \widehat{w}')$ is the solution of the adjoint system (2.2) with initial data $(\widehat{w}^0, \widehat{w}^1)$.*

Much of our analysis will be based on Fourier expansion of solutions. Therefore, let us now introduce the eigenvalues of the wave equation (2.2)

$$\lambda^{nm} = \text{sgn}(n) \sqrt{n^2 + m^2} \pi, \tag{2.7}$$

and the corresponding eigenfunctions

$$\Psi^{nm}(x,y) = \sqrt{2} \begin{pmatrix} (\mathbf{i}\lambda^{nm})^{-1} \\ -1 \end{pmatrix} \sin(n\pi x) \sin(m\pi y), \quad (n,m) \in \mathbb{Z}^* \times \mathbb{N}^*, \quad \mathbf{i} = \sqrt{-1}. \tag{2.8}$$

The sequence $(\Psi^{nm})_{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*}$ forms an orthonormal basis in $H_0^1(\Omega) \times L^2(\Omega)$. Moreover,

$$||\Psi^{nm}||_{L^2(\Omega) \times H^{-1}(\Omega)} = \frac{1}{\lambda^{nm}}.$$

Remark 2.2 *Note that it is sufficient to show that (2.6) is verified by $(w^0, w^1) = \Psi^{nm}$ for all $(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*$. Indeed, from the continuity of the linear form $\Lambda : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{C}$, defined by*

$$\begin{aligned} \Lambda(w^0, w^1) = \int_0^T \int_0^1 v(t,y) w_x(t,1,y) dydt + \int_0^T \int_0^1 z(t,x) w_y(t,x,1) dxdt \\ - < u^1, w(0) >_{H^{-1}, H_0^1} + \int_\Omega u^0(x,y) w'(0,x,y) dxdy, \end{aligned} \tag{2.9}$$

it follows that (2.6) holds for any $(w^0, w^1) \in H_0^1(\Omega) \times L^2(\Omega)$ *if and only if it is verified on a basis of the space* $H_0^1(\Omega) \times L^2(\Omega)$*. But* $(\Psi^{n,m})_{(n,m)\in\mathbb{Z}^*\times\mathbb{N}^*}$ *is exactly such a basis.*

Thus, by considering $(w^0, w^1) = \Psi^{nm}$*, we obtain that the control* (v, z) *drives to zero the initial data*

$$(u^0, u^1) = \sum_{(n,m)\in\mathbb{Z}^*\times\mathbb{N}^*} \alpha^0_{nm}\Phi^{nm},$$

of (1.2) if and only if

$$\int_0^T e^{i\lambda^{nm}t}\left((-1)^n n\int_0^1 v(t,y)\sin(m\pi y)dy + (-1)^m m\int_0^1 z(t,x)\sin(n\pi x)dx\right)dt = \frac{\alpha^0_{nm}}{\sqrt{2}\pi}, \quad (2.10)$$

for all $(n, m) \in \mathbb{Z}^* \times \mathbb{N}^*$*.*

3 The semi-discrete problem

In this section we introduce a suitable semi-discretization of the adjoint homogeneous equation (2.2), which characterizes the HUM control in the continuous case. Then, by minimizing the HUM functional corresponding to this semi-discrete system, a convergent sequence of discrete approximations $(v_h, z_h)_{h>0}$ of the HUM control (v, z) of (1.2) is obtained.

We introduce $N \in \mathbb{N}^*$, $h = 1/(N+1)$ and we consider the following partition of the square $(x, y) \in \Omega$

$$(x_i, y_j) = (ih, jh), \quad 0 \le i, j \le N+1, \quad (3.1)$$

and we denote $w_{ij} = w(x_i, y_j)$.

Let us also introduce the new variable $\zeta(t, x, y) = w'(t, x, y)$. Equation (2.2) may be written in the following variational form:

$$\begin{cases} \text{Find} \quad (w,\zeta) = (w,\zeta)(t,x,y) \quad \text{with} \quad (w(t),\zeta(t)) \in (H_0^1(\Omega)\times L^2(\Omega)), \forall t \in (0,T), \text{ such that} \\ \dfrac{d}{dt}\displaystyle\int_0^1\int_0^1 w(t,x,y)\psi(x,y)dxdy = \int_0^1\int_0^1 \zeta(t,x,y)\psi(x,y)dxdy, \qquad \forall \psi \in L^2(\Omega), \\ \dfrac{d}{dt} < \zeta(t,\cdot), \varphi >_{-1,1} = \displaystyle\int_0^1\int_0^1 \nabla w(t,x,y)\nabla\varphi(x,y)dxdy, \qquad \forall\varphi \in H_0^1(\Omega), \\ w(T,x,y) = w^0(x,y),\ \zeta(T,x,y) = w^1(x,y), \qquad \forall (x,y) \in \Omega. \end{cases} \quad (3.2)$$

We now discretize (3.2) by using a mixed finite elements method (see, for instance, [3] or [19]). We approximate the position w in the space $\mathbb{Q}_1$ of piecewise-polynomials of degree one and the velocity ζ in the space $\mathbb{Q}_0$ of discontinuous piecewise constant functions. More precisely, for each $1 \le i, j \le N$, let $Q^h_{ij} = (x_i, x_{i+1}) \times (y_j, y_{j+1})$ be such that $\cup_{0\le i,j\le N} Q^h_{i,j} = \Omega = (0,1)^2$ and define the functions

$$\begin{cases} \psi_{ij} = \begin{cases} \frac{1}{2} & \text{if } (x,y) \in Q^h_{ij} \cup Q^h_{i-1j} \cup Q^h_{ij-1} \cup Q^h_{i-1j-1}, \\ 0 & \text{otherwise,} \end{cases} \\ \varphi_{ij}|_{Q^h_{kl}} \in \mathbb{Q}_1, \quad \varphi_{ij}(x_k, y_l) = \delta^{kl}_{ij}. \end{cases} \quad (3.3)$$

The variational formulation (3.2) is then reduced to find

$$w_h(t,x,y) = \sum_{i,j=1}^N w_{ij}(t)\varphi_{ij}(x,y), \quad \text{and} \quad \zeta_h(t,x,y) = \sum_{i,j=1}^N \zeta_{ij}(t)\psi_{ij}(x,y), \quad (3.4)$$

that satisfy of the semi-discrete system

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_0^1\int_0^1 w_h(t,x,y)\psi_{ij}(x,y)dxdy = \int_0^1\int_0^1 \zeta_h(t,x)\psi_{ij}(x,y)dxdy, & \forall\, 1\le i,j\le N,\\ \dfrac{d}{dt} < \zeta_h(t,\cdot),\varphi_{ij} >_{-1,1} = \displaystyle\int_0^1\int_0^1 \nabla w_h(t,x,y)\nabla\varphi_{ij}(x,y)dxdy, & \forall\, 1\le i,j\le N,\\ w_h(T,x,y)=w_h^0(x,y),\ \zeta_h(T,x,y)=w_h^1(x,y), & \forall (x,y)\in\Omega. \end{cases} \tag{3.5}$$

A straightforward computation shows that the variables ζ_{ij} may be eliminated in (3.4)-(3.5) leading to the following semi-discrete system for $w_{ij}(t)$, in $t\in(0,T)$:

$$\begin{cases} \frac{h^2}{16}\left(4w''_{ij}+2w''_{i+1j}+2w''_{i-1j}+2w''_{ij+1}+2w''_{ij-1}+w''_{i+1j+1}+w''_{i+1j-1}+w''_{i-1j+1}+w''_{i-1j-1}\right)\\ +\frac{1}{3}\left(8w_{ij}-w_{i+1j}-w_{i-1j}-w_{ij+1}-w_{ij-1}-w_{i+1j+1}-w_{i+1j-1}-w_{i-1j+1}-w_{i-1j-1}\right)=0,\\ \text{for } 1\le i,j\le N,\\ w_{i0}=w_{iN+1}=0, \quad \text{for } 0\le i\le N+1,\\ w_{0j}=w_{N+1j}=0, \quad \text{for } 0\le j\le N+1,\\ w_{ij}(T)=w^0_{ij}, \quad w'_{ij}(T)=w^1_{ij}, \ \text{for } 0\le i,j\le N+1. \end{cases} \tag{3.6}$$

It is easy to see that the semi-discrete system (3.6) is consistent of order two in space with the continuous wave equation (2.2). We shall consider that the initial data are zero on the boundary of Ω, which in the discrete equation corresponds to

$$\begin{cases} w^0_{0,j}=w^1_{0,j}=0, \quad w^0_{N+1,j}=w^1_{N+1,j}=0, \ \text{for } 0\le j\le N+1,\\ w^0_{i,0}=w^1_{i,0}=0, \quad w^0_{i,N+1}=w^1_{i,N+1}=0, \ \text{for } 0\le i\le N+1. \end{cases} \tag{3.7}$$

The same property will be also satisfied by the corresponding solutions of (3.6).

Now we write (3.6) in an equivalent vectorial form. In order to do this we first define the following tri-diagonal matrices from $\mathcal{M}_{N\times N}(\mathbb{R})$

$$A_h=\frac{1}{3}\begin{pmatrix} 8 & -1 & & & \\ -1 & 8 & -1 & & (0)\\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1\\ & (0) & & -1 & 8 \end{pmatrix}_{N\times N}, B_h=\frac{1}{3}\begin{pmatrix} -1 & -1 & & & \\ -1 & -1 & -1 & & (0)\\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1\\ & (0) & & -1 & -1 \end{pmatrix}_{N\times N}, \tag{3.8}$$

$$C_h=\frac{h^2}{16}\begin{pmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & (0)\\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1\\ & (0) & & 1 & 2 \end{pmatrix}_{N\times N}. \tag{3.9}$$

Moreover, we define the matrices K_h and M_h from $\mathcal{M}_{N^2\times N^2}(\mathbb{R})$,

$$K_h = \begin{pmatrix} A_h & B_h & & & \\ B_h & A_h & B_h & & (0) \\ & B_h & \ddots & \ddots & \\ & & \ddots & \ddots & B_h \\ & (0) & & B_h & A_h \end{pmatrix}_{N^2\times N^2}, M_h = \begin{pmatrix} 2C_h & C_h & & & \\ C_h & 2C_h & C_h & & (0) \\ & C_h & \ddots & \ddots & \\ & & \ddots & \ddots & C_h \\ & (0) & & C_h & 2C_h \end{pmatrix}_{N^2\times N^2}, \tag{3.10}$$

If we denote the unknown

$$W_h(t) = (w_{11}(t), w_{21}(t), ..., w_{N1}, ..., w_{1N}(t), w_{2N}(t), ..., w_{NN}(t))^T,$$

then equation (3.6) may be written in vectorial form as follows

$$\begin{cases} M_h W_h''(t) + K_h W_h(t) = 0, & \text{for } t > 0, \\ W_h(T) = W_h^0, \ W_h'(T) = W_h^1, \end{cases} \tag{3.11}$$

where $(W_h^0, W_h^1) = (w_{ij}^0, w_{ij}^1)_{1\le i,j\le N} \in \mathbb{R}^{2N^2}$ are the initial data and the corresponding solution of (3.6) is given by $(W_h, W_h') = (w_{ij}, w_{ij}')_{1\le i,j\le N}$.

4 Properties of the semi-discrete system

In this section we study some of the properties of the semi-discrete adjoint system (3.6), related to the controllability problem. In general, the controllability property of a system may be reduced to study a suitable inverse inequality for the uncontrolled adjoint system. The aim of this section consists precisely in giving a uniform (in h) observability inequality for (3.6). But before that, let us briefly explain why the semi-discretization introduced in this paper is likely to provide a uniform observability property rather than others, like the usual finite difference semi-discretization implemented in [8].

It is well known that in order to have a uniform observability property for the wave equation (2.2) of the type (2.4) it is necessary to consider T sufficiently large. This is due to the fact that the velocity of waves is one and then any perturbation of the initial data will last some time to arrive to the observation zone. For the semi-discrete scheme (3.6) we can also define semi-discrete waves as solutions of the form

$$w_{ij} = e^{\mathbf{i}(\xi\cdot(x_i,y_j)-\omega t)}, \qquad \xi = (\xi_1, \xi_2), \qquad (x_i, y_j) = (ih, jh), \qquad \mathbf{i} = \sqrt{-1}.$$

When substituting in (3.6) the following relation between modes ξ and frequencies ω holds

$$\omega_s(\xi) = \frac{2}{h}\sqrt{\tan^2\left(\frac{\xi_1 h}{2}\right) + \tan^2\left(\frac{\xi_2 h}{2}\right) + \frac{2}{3}\tan^2\left(\frac{\xi_1 h}{2}\right)\tan^2\left(\frac{\xi_2 h}{2}\right)},$$

and $\xi \in (-\pi/h, \pi/h)^2$.

The group velocity associated to a mode ξ in a direction $v = (v_1, v_2)$ is given by $\nabla_\xi \omega_s \cdot v$. Clearly, a necessary condition in order to have a uniform (in h) observability property in finite time $T > 0$ is that the group velocity associated to any mode ξ is strictly bounded from below with a certain constant $\mu > 0$ (independent of ξ and h) for at least one direction v. Otherwise some solutions of the semi-discrete system would propagate so slowly in any direction that the observability will require larger time T as $h \to 0$.

To guarantee that we have a group velocity uniformly bounded from below for at least one direction v it is sufficient to have a uniform bound from below (in ξ and h) for $|\nabla_\xi \omega_s| = \sqrt{|\partial_{\xi_1}\omega_s|^2 + \partial_{\xi_2}\omega_s|^2}$. Note that for the continuous wave equation (2.2), $\omega(\xi) = |\xi|$ and $|\nabla_\xi \omega| = 1$. For the semi-discrete scheme (3.6), a straightforward computation shows that the minimum value of $|\nabla_\xi \omega_s|$ is obtained for $\xi = (0,0)$ and that $|\nabla_\xi \omega_s(0,0)| = 1$ (see Figure 1).

On the other hand, the usual semi-discrete scheme based on classical finite differences in space has the following relation

$$\omega_{fd}(\xi) = \frac{2}{h}\sqrt{\sin^2\left(\frac{\xi_1 h}{2}\right) + \sin^2\left(\frac{\xi_2 h}{2}\right)},$$

for which $|\nabla_\xi \omega_{fd}| \to 0$ as $\xi \to (\pi/h, 0)$, for example. In other words, the group velocity of high frequencies associated to this finite differences scheme becomes very small (see Figure 1). This may explain the lack of convergence of the gradient conjugate algorithm implemented in [8], which is based in this semi-discrete scheme.

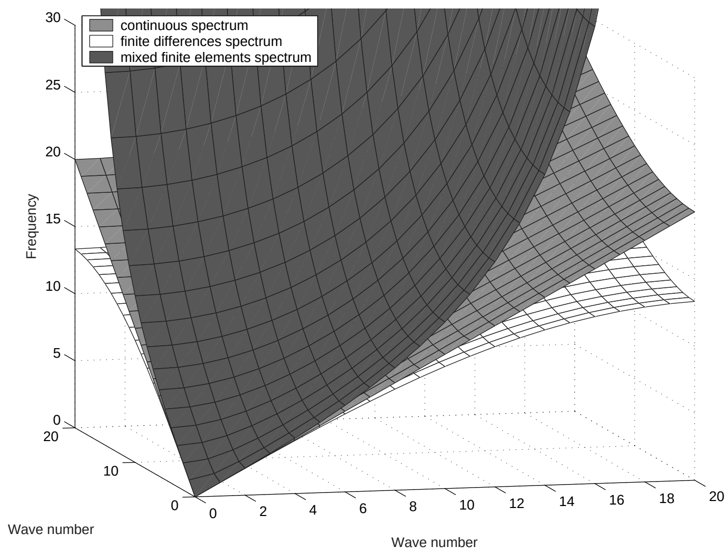

Figure 1: $\omega(\xi)$ with $\xi \in [0, \pi/h)^2$ and $h = 1/21$ for the mixed finite element semi-discretization (upper surface), continuous wave equation (medium surface) and the usual finite differences semi-discretization (lower surface). We observe that the norm of the gradient $|\nabla_\xi \omega(\xi)|$ is always one in the continuous case, it is greater than one for the mixed finite element scheme and it becomes zero for the usual finite differences scheme as ξ approaches $(\pi/h, 0)$

.

We do not know if the above spectral condition on $\nabla_\xi \omega_s$ is sufficient to guarantee a uniform (in h) observability inequality for the semi-discrete system (3.6). In the rest of this section we prove, using a different approach, that indeed this property holds for system (3.6).

We introduce the following discrete version of the continuous energy (2.5)

$$E_h(t) = \frac{h^2}{2}\sum_{i,j=0}^{N}\left\{\left(\frac{w'_{ij}+w'_{ij+1}+w'_{i+1j+1}+w'_{i+1j}}{4}\right)^2\right.$$
$$+\frac{1}{3}\left[\left(\frac{w_{i+1j}-w_{ij}}{h}\right)^2+\left(\frac{w_{ij+1}-w_{ij}}{h}\right)^2\right]$$
$$\left.+\frac{2}{3}\left[\left(\frac{w_{i+1j+1}-w_{ij}}{\sqrt{2}h}\right)^2+\left(\frac{w_{i+1j}-w_{ij+1}}{\sqrt{2}h}\right)^2\right]\right\}. \quad (4.1)$$

Note that in the expression of E_h two different types of finite differences are considered for the discretization of the gradient in (2.5).

The matrices M_h and K_h are definite positives. Let us now define the inner product

$$< (f_1, f_2), (g_1, g_2) >_0 = < K_h f_1, g_1 > + < M_h f_2, g_2 >, \quad (4.2)$$

for any $(f_1, f_2), (g_1, g_2) \in \mathbb{R}^{2N^2}$, where $< \cdot, \cdot >$ denotes the canonical inner product. The corresponding norm will be denoted $|| \cdot ||_0$.

Remark that

$$E_h(t) = \frac{1}{2}\left|\left|(W_h, W'_h)(t)\right|\right|_0^2. \quad (4.3)$$

The following proposition shows that, as in the corresponding continuous case, the energy E_h defined by (4.1) is conserved along trajectories.

Proposition 4.1 *For any $h > 0$ and any solution of the discrete system (3.6) the following holds*

$$E_h(t) = E_h(0), \quad \forall t > 0. \quad (4.4)$$

Proof: Multiplying (3.11) by W'_h, we obtain that

$$0 = < M_h W''_h, W'_h > + < K_h W_h, W'_h > = \frac{1}{2}\left[< M_h W'_h, W'_h > + < K_h W_h, W_h >\right]' = \frac{d}{dt}E_h(t),$$

and the proof finishes. ∎

The following result shows that a discrete version of the observability inequality (2.4) is valid for the solutions of system (3.6).

Theorem 4.1 *Given $T > 2\sqrt{3}$, there exists a constant $C(T) > 0$ independent of the discretization step h such that the following inequality holds*

$$E_h(0) \le C(T)\left\{\frac{h^3}{8}\int_0^T\left[\sum_{i=1}^{N}\left(\frac{w'_{iN}+w'_{i+1N}}{2h}\right)^2+\sum_{j=1}^{N}\left(\frac{w'_{Nj}+w'_{Nj+1}}{2h}\right)^2\right]dt+\right.$$
$$\left.+\frac{h}{2}\int_0^T\left[\sum_{j=1}^{N}\frac{w_{Nj-1}+w_{Nj}+w_{Nj+1}}{3h}\frac{w_{Nj}}{h}+\sum_{i=1}^{N}\frac{w_{i-1N}+w_{iN}+w_{i+1N}}{3h}\frac{w_{iN}}{h}\right]dt\right\}. \quad (4.5)$$

Remark 4.1 *We were able to prove the uniform observability inequality (4.5) only for $T > 2\sqrt{3}$. However, probably the same is true for $T > 2\sqrt{2}$, as in the continuous case.*

The proof of Theorem 4.1 is very technical and it will be given in the Appendix. Remark that (4.5) may be written in the following equivalent form

$$\begin{aligned} E_h(0) \leq C(T)\frac{h}{2}\left\{\int_0^T \left[\frac{1}{h^2} < C_h W'_{N.}, W'_{N.} > + \frac{1}{h^2} < C_h W'_{.N}, W'_{.N} >\right] dt - \right. \\ \left. - \int_0^T \left[\frac{1}{h^2} < B_h W_{N.}, W_{N.} > + \frac{1}{h^2} < B_h W_{.N}, W_{.N} >\right] dt \right\}, \end{aligned} \tag{4.6}$$

where $W_{N.} = (w_{Nj})_{1\leq j\leq N} \in \mathbb{R}^N$ and $W_{.N} = (w_{iN})_{1\leq i\leq N} \in \mathbb{R}^N$. Similarly, the following direct inequality also holds.

Proposition 4.2 *Given $T > 0$, there exists a constant $C(T) > 0$ such that the following holds for any $h > 0$*

$$\begin{aligned} &\frac{h^3}{8}\int_0^T \left[\sum_{i=1}^N \left(\frac{w'_{iN} + w'_{i+1N}}{2h}\right)^2 + \sum_{j=1}^N \left(\frac{w'_{Nj} + w'_{Nj+1}}{2h}\right)^2\right] dt + \\ &+ \frac{h}{2}\int_0^T \left[\sum_{j=1}^N \frac{w_{Nj-1} + w_{Nj} + w_{Nj+1}}{3h}\frac{w_{Nj}}{h} + \sum_{i=1}^N \frac{w_{i-1N} + w_{iN} + w_{i+1N}}{3h}\frac{w_{iN}}{h}\right] dt \leq C(T)E_h(0). \end{aligned} \tag{4.7}$$

The proof of Proposition 4.2 is given at the end of the Appendix, after the proof of Theorem 4.1.

5 Construction of the discrete approximations

In this section we explicitly construct a sequence of approximations $(v_h, z_h)_{h>0}$ of the HUM control (v, z) of (1.2). This will be done by minimizing the HUM functional of the semi-discrete adjoint system (3.6).

Suppose that $(U_h^0, U_h^1) = (u_j^0, u_j^1)_{1\leq j\leq N} \in \mathbb{R}^{2N^2}$ is a discretization of the continuous initial data of (1.2) to be controlled. We define the functional $\mathcal{J} : \mathbb{R}^{2N^2} \to \mathbb{R}$,

$$\begin{aligned} \mathcal{J}((W_h^0, W_h^1)) =& - < (-K_h^{-1}M_h U_h^1, U_h^0), (W_h(0), W_h'(0)) >_0 \\ &+ \frac{1}{2h}\int_0^T [< C_h W'_{N.}, W'_{N.} > + < C_h W'_{.N}, W'_{.N} >] dt \\ &+ \frac{1}{2h}\int_0^T [< B_h W_{N.}, W_{N.} > + < B_h W_{.N}, W_{.N} >] dt, \end{aligned} \tag{5.1}$$

where (W_h, W_h') is the solution of (3.11) with initial data $(W_h^0, W_h^1) \in \mathbb{R}^{2N^2}$, and we have noted $W_{N.} = (w_{Nj})_{1\leq j\leq N} \in \mathbb{R}^N$ and $W_{.N} = (w_{iN})_{1\leq i\leq N} \in \mathbb{R}^N$.

We show now that $\mathcal{J}$ has a minimizer $(\widehat{W}_h^0, \widehat{W}_h^1)$. The main tool in the proof of this result is the observability inequality stated in Theorem 4.1 above.

Lemma 5.1 *Assume that $T > 2\sqrt{3}$. The functional $\mathcal{J}$ defined by (5.1) has an unique minimizer $(\widehat{W}_h^0, \widehat{W}_h^1)$.*

Proof: Since $\mathcal{J}$ is continuous, convex and defined in a finite dimensional space, the lemma is proved if we show that $\mathcal{J}$ is coercive. This is a consequence of (4.5). More precisely,

$$\mathcal{J}(W_h^0, W_h^1) \geq \frac{h}{32}\int_0^T \left(\sum_{j=0}^{N} |w'_{Nj+1}(t) + w'_{Nj}(t)|^2 + \sum_{i=0}^{N} |w'_{i+1N}(t) + w'_{iN}(t)|^2 \right) dt$$
$$+\frac{1}{6h}\int_0^T \left(\sum_{j=0}^{N} |w_{Nj+1}(t) + w_{Nj}(t)|^2 + \sum_{i=0}^{N} |w_{i+1N}(t) + w_{iN}(t)|^2 \right) dt$$
$$-\frac{1}{6h}\int_0^T \left(\sum_{j=0}^{N} |w_{Nj}(t)|^2 + \sum_{i=0}^{N} |w_{iN}(t)|^2 \right) dt - ||(-K_h^{-1}M_hU_h^1, U_h^0)||_0\, ||(W_h(0), W_h'(0))||_0$$
$$\geq C(T)||(W_h^0, W_h^1)||_0^2 - ||(-K_h^{-1}M_hU_h^1, U_h^0)||_0\, ||(W_h^0, W_h^1)||_0,$$

and therefore

$$\lim_{||(W_h^0, W_h^1)||_0 \to \infty} \mathcal{J}(W_h^0, W_h^1) = \infty.$$

■

Definition 5.1 *Let* $(\widehat{W}_h^0, \widehat{W}_h^1)$ *be the minimizer of the functional* $\mathcal{J}$ *given by Lemma 5.1. We define* $v_h = (v_{h,j})_{1\leq j\leq N} \in L^2(0,T;\mathbb{R}^N)$ *and* $z_h = (z_{h,i})_{1\leq i\leq N} \in L^2(0,T;\mathbb{R}^N)$ *by*

$$v_{h,j}(t) = -\frac{\widehat{w}_{Nj}}{h}, \quad z_{h,i}(t) = -\frac{\widehat{w}_{iN}}{h}, \quad \forall\, 1\leq i,j\leq N, \tag{5.2}$$

where $(\widehat{W}_h, \widehat{W}_h')$ *is the solution of (3.11) with initial data* $(\widehat{W}_h^0, \widehat{W}_h^1)$.

Remark 5.1 *The optimality condition for the minimizer of* $\mathcal{J}$ *provides the following characterization of* v_h *and* z_h

$$< (-K_h^{-1}M_hU_h^1, U_h^0), (W_h(0), W_h'(0)) >_0 =$$
$$\frac{h^2}{16}\int_0^T \left(\sum_{j=1}^{N} (2v'_{h,j} + v'_{h,j+1} + v'_{h,j-1})w'_{Nj} + \sum_{i=1}^{N} (2z'_{h,i} + z'_{h,i+1} + z'_{h,i-1})w'_{iN} \right) dt +$$
$$+\frac{1}{3}\int_0^T \left(\sum_{j=1}^{N} (v_{h,j} + v_{h,j+1} + v_{h,j-1})w_{Nj} + \sum_{i=1}^{N} (z_{h,i} + z_{h,i+1} + z_{h,i-1})w_{iN} \right) dt = 0, \tag{5.3}$$

for any $(W_h^0, W_h^1) \in \mathbb{R}^{2N^2}$, *where* (W_h, W_h') *is the corresponding solution of (3.11).*

Remark that v_h *and* z_h **are not controls** *for the semi-discrete system corresponding to (1.2), unless* $v_h'(0) = v_h'(T) = 0$ *and* $z_h'(0) = z_h'(T) = 0$. *However, we shall show that the sequence* $(v_h, z_h)_{h>0}$ *converges to a control of the continuous equation. Therefore, in the sequel we refer to* (v_h, z_h) *as* **discrete controls**.

Our aim is to show that the sequence $(v_h, z_h)_{h>0}$ converges to a control (v, z) of the continuous equation (1.2). Since v_h and z_h belong to $L^2(0,T;\mathbb{R}^N)$ whereas v and z are in $L^2(0,T;L^2(0,1))$ the convergence is stated in terms of the Fourier coefficients. This is done in the next section.

In the rest of this section we introduce the eigenfunctions and the eigenvalues of the semi-discrete problem (3.11) and some notation.

Definition 5.2

$$\mathcal{I}_N = \left\{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^* \,:\, 1 \le |n| \le N, 1 \le m \le N\right\}. \tag{5.4}$$

Lemma 5.2 *The eigenvalues λ_h^{nm}, $(n,m) \in \mathcal{I}_N$, of the semi-discrete problem (3.11) are given by*

$$\lambda_h^{nm} = sgn(n)\frac{2}{h}\sqrt{\tan^2\left(\frac{m\pi h}{2}\right) + \tan^2\left(\frac{n\pi h}{2}\right) + \frac{2}{3}\tan^2\left(\frac{m\pi h}{2}\right)\tan^2\left(\frac{n\pi h}{2}\right)}. \tag{5.5}$$

The corresponding eigenfunctions are

$$\Psi_h^{nm} = \frac{\sqrt{2}}{\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\begin{pmatrix} (i\lambda_h^{nm})^{-1}\,\Phi_h^{nm} \\ -\Phi_h^{nm} \end{pmatrix}, \quad \forall (n,m) \in \mathcal{I}_N, \tag{5.6}$$

where $\Phi_h^{nm} = (\phi_h^n \sin(pm\pi h))_{1\le p\le N} \in \mathbb{R}^{N^2}$ and $\phi_h^n = (\sin(jn\pi h))_{1\le j\le N} \in \mathbb{R}^N$.

A straightforward computation shows that $(\Psi_h^{nm})_{(n,m)\in\mathcal{I}_N}$ constitutes an orthonormal basis in R^{2N^2} with respect to the inner product $< \cdot, \cdot >_0$.

For any $(f^1,f^2),(g^1,g^2) \in \mathbb{R}^{2N^2}$ we introduce the notations

$$\begin{aligned} &< (f^1,f^2),(g^1,g^2) >_{-1} = < (-K_h^{-1}M_h f^2, f^1), (-K_h^{-1}M_h g^2, g^1) >_0, \\ &||(f^1,f^2)||_{-1} = ||(-K_h^{-1}M_h f^2, f^1)||_0. \end{aligned} \tag{5.7}$$

Remark that $< \cdot, \cdot >_{-1}$ is an inner product and $|| \cdot ||_{-1}$ is a norm on $\mathbb{R}^{2N^2}$.

6 Convergence of the discrete approximations

In this section we prove the weak convergence of the sequence $(v_h, z_h)_{h>0}$ to the HUM control of the continuous equation (1.2). Let us first show the following property of the initial data that allows us to construct (v_h, z_h).

Theorem 6.1 *Assume that $T > 2\sqrt{3}$. The sequence of minimizers of $\mathcal{J}$ given by Lemma 5.1, $(\widehat{W}_h^0, \widehat{W}_h^1)_{h>0}$, verify*

$$||(\widehat{W}_h^0, \widehat{W}_h^1)||_0 \le \frac{1}{C}||(-K_h^{-1}M_h U_h^1, U_h^0)||_0, \tag{6.1}$$

where C is the observability constant of (4.5) which is independent of h.

If the sequence of discretizations $(U_h^0, U_h^1)_{h>0}$ is uniformly bounded in the $|| \cdot ||_{-1}-$norm then the sequence $(\widehat{W}_h^0, \widehat{W}_h^1)_{h>0}$ is bounded in the $|| \cdot ||_0-$norm.

Proof: From the observability inequality we have that

$$\begin{aligned} C||(\widehat{W}_h^0, \widehat{W}_h^1)||_0^2 \le &\frac{h}{2}\int_0^T \left[< C_h v_h', v_h' > + < C_h z_h', z_h' >\right] dt \\ &- \frac{h}{2}\int_0^T \left[< B_h v_h, v_h > + < B_h z_h, z_h >\right] dt \\ = &\,\mathcal{J}(\widehat{W}_h^0, \widehat{W}_h^1) + < (-K_h^{-1}M_h U_h^1, U_h^0), (\widehat{W}_h(0), \widehat{W}_h'(0)) >_0 . \end{aligned} \tag{6.2}$$

Now, since $\mathcal{J}(\widehat{W}_h^0, \widehat{W}_h^1) \le \mathcal{J}(0,0) = 0$, it follows that

$$\begin{aligned} C\|(\widehat{W}_h^0, \widehat{W}_h^1)\|_0^2 \le & < (-K_h^{-1} M_h U_h^1, U_h^0), (\widehat{W}_h(0), \widehat{W}_h'(0)) >_0 \\ \le & \|(-K_h^{-1} M_h U_h^1, U_h^0)\|_0 \|(W_h(0), W_h'(0))\|_0 \\ = & \|(-K_h^{-1} M_h U_h^1, U_h^0)\|_0 \|(W_h^0, W_h^1)\|_0, \end{aligned} \tag{6.3}$$

which is equivalent to (6.1). ■

Remark 6.1 *Theorem 6.1 shows that the sequence of initial data* $(\widehat{W}_h^0, \widehat{W}_h^1)_{h>0}$ *which give* (v_h, z_h) *is uniformly bounded in h in the* $\| \, \|_0-$*norm if the sequence of discretizations* $(U_h^0, U_h^1)_{h>0}$ *is bounded in the* $\| \cdot \|_{-1}-$*norm. The sequences* $(v_h, z_h)_{h>0}$ *verifies the following inequality*

$$\begin{aligned} \frac{h}{2} \int_0^T & \left[< C_h v_h', v_h' > + < C_h z_h', z_h' > - < B_h v_h, v_h > - < B_h z_h, z_h > \right] dt \\ & \le \frac{1}{C} \|(-K_h^{-1} M_h U_h^1, U_h^0)\|_0^2 = \frac{1}{C} \|(U_h^0, U_h^1)\|_{-1}^2. \end{aligned} \tag{6.4}$$

6.1 Weak convergence of the approximations

Assume that the sequence of discretizations of the continuous initial data on (1.2), $(U_h^0, U_h^1)_{h>0}$, converges weakly to (u^0, u^1) in $L^2(\Omega) \times H^{-1}(\Omega)$. This should be understood in the sense of the convergence of the Fourier coefficients. More precisely, if

$$(U_h^0, U_h^1) = \sum_{(n,m) \in \mathcal{I}_N} \alpha_{nm}^h \Phi_h^{nm}, \quad (u^0, u^1) = \sum_{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*} \alpha_{nm} \Phi^{nm},$$

then the following weak convergence holds in ℓ^2

$$\left(\frac{\alpha_{nm}^h}{\lambda_h^{nm}} \right)_{(n,m) \in \mathcal{I}_N} \rightharpoonup \left(\frac{\alpha_{nm}}{\lambda^{nm}} \right)_{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*}, \quad \text{when} \quad h \to 0. \tag{6.5}$$

Now, assume that the minimizer $(\widehat{W}_h^0, \widehat{W}_h^1)$ has the following expansion

$$(\widehat{W}_h^0, \widehat{W}_h^1) = \sum_{(n,m) \in \mathcal{I}_N} a_{nm}^h \Psi_h^{nm}. \tag{6.6}$$

Inequality (6.1) is equivalent to

$$\sum_{(n,m) \in \mathcal{I}_N} |a_{nm}^h|^2 = \|(\widehat{W}_h^0, \widehat{W}_h^1)\|_0^2 \le \frac{1}{C^2} \|(-K_h^{-1} M_h U_h^1, U_h^0)\|_0^2 = \frac{1}{C^2} \sum_{(n,m) \in I_N} \left| \frac{\alpha_{nm}^h}{\lambda_h^{nm}} \right|^2.$$

Here, the right hand side is bounded due to the weak convergence stated in (6.5). Hence, the sequence of Fourier coefficients $(a_{nm}^h)_{(n,m) \in \mathcal{I}_N}$ is bounded in ℓ^2 and there exists a subsequence, denoted in the same way, and $(a_{nm})_{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*} \in \ell^2$ such that

$$(a_{nm}^h)_{(n,m) \in \mathcal{I}_N} \rightharpoonup (a_{nm})_{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*} \text{ in } \ell^2 \quad \text{when} \quad h \to 0. \tag{6.7}$$

Let us now introduce the continuous initial data

$$(\widehat{w}^0, \widehat{w}^1) = \sum_{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*} a_{nm} \Psi^{nm} \in H_0^1(\Omega) \times L^2(\Omega), \tag{6.8}$$

and the corresponding solution $(w,w')\in C([0,T];H_0^1(\Omega)\times L^2(\Omega))$. We have that

$$\begin{aligned} w_x(t,1,y) &= \sum_{m\in\mathbb{N}^*}\left(\sum_{n\in\mathbb{Z}^*} i\,a_{nm}(-1)^{n+1}\tfrac{\sqrt{2}n\pi}{\lambda^{nm}}e^{i\lambda^{nm}t}\right)\sin(m\pi y),\\ w_y(t,x,1) &= \sum_{n\in\mathbb{Z}^*}\left(\sum_{m\in\mathbb{N}^*} i\,a_{nm}(-1)^{m+1}\tfrac{\sqrt{2}m\pi}{\lambda^{nm}}e^{i\lambda^{nm}t}\right)\sin(n\pi x).\end{aligned} \tag{6.9}$$

If $(\widehat{W}_h,\widehat{W}_h')$ is the corresponding solution of (3.11) with initial data $(\widehat{W}_h^0,\widehat{W}_h^1)$, it follows that

$$\begin{aligned} v_h &= \sum_{1\le m\le N}\left(\sum_{1\le|n|\le N} i\,a_{nm}^h(-1)^{n+1}\frac{\sqrt{2}}{\lambda_h^{nm}\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\sin(n\pi h)e^{i\lambda_h^{nm}t}\right)\phi_h^m,\\ z_h &= \sum_{1\le|n|\le N}\left(\sum_{1\le m\le N} i\,a_{nm}^h(-1)^{m+1}\frac{\sqrt{2}}{\lambda_h^{nm}\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\sin(m\pi h)e^{i\lambda_h^{nm}t}\right)\phi_h^n.\end{aligned} \tag{6.10}$$

We denote

$$\begin{aligned} b_m^h &= \begin{cases} \sum_{1\le|n|\le N} i\,a_{nm}^h(-1)^{n+1}\frac{\sqrt{2}}{\lambda_h^{nm}\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\sin(n\pi h)e^{i\lambda_h^{nm}t}, & \text{if } 1\le m\le N,\\ 0, & \text{if } m>N,\end{cases}\\ b_m &= \sum_{n\in\mathbb{Z}^*} i\,a_{nm}(-1)^{n+1}\frac{\sqrt{2}n\pi}{\lambda^{nm}}e^{i\lambda^{nm}t},\\ d_n^h &= \begin{cases} \sum_{1\le m\le N} i\,a_{nm}^h(-1)^{m+1}\frac{\sqrt{2}}{\lambda_h^{nm}\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\sin(m\pi h)e^{i\lambda_h^{nm}t}, & \text{if } 1\le|n|\le N,\\ 0, & \text{if } |n|>N,\end{cases},\\ d_n &= \sum_{m\in\mathbb{N}^*} i\,a_{nm}(-1)^{m+1}\frac{\sqrt{2}m\pi}{\lambda^{nm}}e^{i\lambda^{nm}t}.\end{aligned}$$

Theorem 6.2 *Assume that the sequence of discretizations $(U_h^0,U_h^1)_{h>0}$ converges weakly to (u^0,u^1) in the sense of (6.5). The following convergencies hold weakly in $L^2(0,T;\ell^2)$ when h tends to zero*

$$\begin{aligned} (b_m^h)_{m\in\mathbb{N}^*}\rightharpoonup(b_m)_{m\in\mathbb{N}^*},\quad (d_n^h)_{n\in\mathbb{Z}^*}\rightharpoonup(d_n)_{n\in\mathbb{Z}^*},\\ (h(b_m^h)')_{m\in\mathbb{N}^*}\rightharpoonup 0,\quad (h(d_n^h)')_{n\in\mathbb{Z}^*}\rightharpoonup 0.\end{aligned} \tag{6.11}$$

Remark 6.2 *Theorem 6.2 establishes the convergence of the Fourier coefficients of the controls $(v_h,z_h)_{h>0}$ to those of the control (v,z). This means in particular that $(v_h,z_h)_{h>0}$ converges weakly to (v,z) in $[L^2((0,T)\times(0,1))]^2$.*

Proof: We show the first convergence, the other ones being similar. If $(\varphi_m)_{m\ge1}\in\mathcal{D}(0,T;\ell^2)$ we prove that

$$\int_0^T\sum_{m\ge1} b_m^h(t)\varphi_m(t)dt\longrightarrow\int_0^T\sum_{m\ge1} b_m(t)\varphi_m(t)dt\quad\text{when}\quad h\to0, \tag{6.12}$$

which is equivalent to

$$\int_0^T\sum_{m\ge1}\tilde{b}_m^h(t)\varphi_m''(t)dt\longrightarrow\int_0^T\sum_{m\ge1}\tilde{b}_m(t)\varphi_m''(t)dt\quad\text{when}\quad h\to0, \tag{6.13}$$

where

$$\begin{aligned}\tilde{b}^h_m(t) &= \sum_{1\le|n|\le N} i\, a^h_{nm}(-1)^{n+1}\frac{\sqrt{2}\sin(n\pi h)}{\lambda^{nm}_h\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{1}{(\lambda^{nm}_h)^2}e^{i\lambda^{nm}_h t},\\ \tilde{b}_m(t) &= \sum_{n\in\mathbb{Z}^*} i\, a_{nm}(-1)^{n+1}\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{1}{(\lambda^{nm})^2}e^{i\lambda^{nm}t}.\end{aligned}$$

Note that (6.13) follows if the following holds

$$\int_0^T\sum_{m\ge1}|\tilde{b}^h_m(t)-\tilde{b}_m(t)|^2dt\longrightarrow 0 \quad \text{when} \quad h\to 0. \tag{6.14}$$

In order to prove (6.14) we consider an arbitrary $\varepsilon > 0$ and show that there exists N sufficiently large (or, equivalently, h sufficiently small) such that

$$\int_0^T\sum_{m>N}|\tilde{b}_m(t)|^2dt\le\frac{\varepsilon}{2}, \tag{6.15}$$

and

$$\int_0^T\sum_{1\le m\le N}|\tilde{b}^h_m(t)-\tilde{b}_m(t)|^2dt\le\frac{\varepsilon}{2}. \tag{6.16}$$

Remark that (6.15) and (6.16) imply (6.14) immediately.

To prove (6.15) note that, since $(a_{nm})\in\ell^2$, there exists $N_1>0$ independent of h such that, for any $N>N_1$, we have

$$\int_0^T\sum_{m>N}|\tilde{b}_m(t)|^2dt\le\int_0^T\sum_{m>N}\left(\sum_{n\in\mathbb{Z}^*}\frac{1}{|\lambda^{nm}|^4}\right)\left(\sum_{n\in\mathbb{Z}^*}\left|i\, a_{nm}(-1)^{n+1}\frac{\sqrt{2}n\pi}{\lambda^{nm}}e^{i\lambda^{nm}t}\right|^2\right)dt$$

$$\le\sqrt{2}\left(\sum_{m>N}\sum_{n\in\mathbb{Z}^*}\frac{1}{|\lambda^{nm}|^4}\right)\int_0^T\left(\sum_{m>N}\sum_{n\in\mathbb{Z}^*}|a_{nm}|^2\,dt\right)\le C(T)\sum_{m>N}\sum_{n\in\mathbb{Z}^*}|a_{nm}|^2\le\frac{\varepsilon}{2}.$$

Let us now show that, for h sufficiently small (or, equivalently, for N sufficiently large), (6.16) also holds. We have that

$$\frac{1}{2}\sum_{1\le m\le N}\left|\tilde{b}^h_m-\tilde{b}_m\right|^2\le$$

$$\sum_{1\le m\le N}\left|\sum_{1\le|n|\le N}(-1)^{n+1}ia^h_{nm}\left(\frac{\sqrt{2}\sin(n\pi h)}{\lambda^{nm}_h\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{1}{(\lambda^{nm}_h)^2}e^{i\lambda^{nm}_h t}-\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{1}{(\lambda^{nm})^2}e^{i\lambda^{nm}t}\right)\right|^2$$

$$+\sum_{1\le m\le N}\left|\sum_{1\le|n|\le N}i\,(-1)^{n+1}(a^h_{nm}-a_{nm})\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{1}{(\lambda^{nm})^2}e^{i\lambda^{nm}t}\right|^2.$$

According to the weak convergence of the sequence $(a^h_{nm})_{nm}$ to $(a_{nm})_{nm}$ and the presence of the weights $1/(\lambda^{nm})^2$, for h sufficiently small,

$$\sum_{1\le m\le N}\left|\sum_{1\le|n|\le N} i\,(-1)^{n+1}(a^h_{nm}-a_{nm})\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{1}{(\lambda^{nm})^2}e^{i\lambda^{nm}t}\right|^2 \le \sum_{1\le m\le N}\left(\sum_{1\le|n|\le N}|a^h_{nm}-a_{nm}|\frac{1}{(\lambda^{nm})^2}\right)^2 \le \frac{\varepsilon}{4}. \tag{6.17}$$

On the other hand,

$$\sum_{1\le m\le N}\left|\sum_{1\le|n|\le N}(-1)^{n+1}ia^h_{nm}\left(\frac{\sqrt{2}\sin(n\pi h)}{\lambda^{nm}_h\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{e^{i\lambda^{nm}_h t}}{(\lambda^{nm}_h)^2}-\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{e^{i\lambda^{nm}t}}{(\lambda^{nm})^2}\right)\right|^2$$

$$\le \sum_{1\le m\le N}\left(\sum_{1\le|n|\le N}|a^h_{nm}|^2\sum_{1\le|n|\le N}\left|\frac{\sqrt{2}\sin(n\pi h)}{\lambda^{nm}_h\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{e^{i\lambda^{nm}_h t}}{(\lambda^{nm}_h)^2}-\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{e^{i\lambda^{nm}t}}{(\lambda^{nm})^2}\right|^2\right).$$

Since $(a^h_{nm})_{nm}$ is bounded in ℓ^2 there exists $c>0$ such that

$$\sum_{1\le|n|\le N}|a^h_{nm}|^2 \le \sum_{1\le m\le N}\sum_{1\le|n|\le N}|a^h_{nm}|^2\le c,$$

and (6.16) follows if we prove that

$$\sum_{1\le m\le N}\sum_{1\le|n|\le N}\left|\frac{\sqrt{2}\sin(n\pi h)}{\lambda^{nm}_h\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{1}{(\lambda^{nm}_h)^2}e^{i\lambda^{nm}_h t}-\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{1}{(\lambda^{nm})^2}e^{i\lambda^{nm}t}\right|^2\le\frac{\varepsilon}{4c}. \tag{6.18}$$

Note that

$$\max\left\{\left|\frac{\sqrt{2}\sin(n\pi h)}{\lambda^{nm}_h\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\right|,\ \frac{\sqrt{2}n\pi}{\lambda^{nm}}\right\}\le\sqrt{3}.$$

It follows that there exists $n_\varepsilon>0$ independent of h such that

$$\sum_{1\le m\le N}\sum_{n_\varepsilon+1\le|n|\le N}\left|\frac{\sqrt{2}\sin(n\pi h)}{\lambda^{nm}_h\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{1}{(\lambda^{nm}_h)^2}e^{i\lambda^{nm}_h t}-\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{1}{(\lambda^{nm})^2}e^{i\lambda^{nm}t}\right|^2$$

$$+\sum_{n_\varepsilon+1\le m\le N}\sum_{1\le|n|\le n_\varepsilon}\left|\frac{\sqrt{2}\sin(n\pi h)}{\lambda^{nm}_h\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{1}{(\lambda^{nm}_h)^2}e^{i\lambda^{nm}_h t}-\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{1}{(\lambda^{nm})^2}e^{i\lambda^{nm}t}\right|^2$$

$$\le 2\sqrt{3}\sum_{1\le m\le N}\sum_{n_\varepsilon+1\le|n|\le N}\frac{1}{(\lambda^{nm})^2}+2\sqrt{3}\sum_{n_\varepsilon+1\le m\le N}\sum_{1\le|n|\le n_\varepsilon}\frac{1}{(\lambda^{nm})^2}\le\frac{\varepsilon}{8c}.$$

Let us now analyze the case $1\le m,|n|\le n_\varepsilon$. Since $\lambda^{nm}_h\to\lambda^{nm}$ when h tends to zero, it follows that, for h sufficiently small,

$$\left|\frac{\sqrt{2}\sin(n\pi h)}{\lambda^{nm}_h\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{1}{(\lambda^{nm}_h)^2}e^{i\lambda^{nm}_h t}-\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{1}{(\lambda^{nm})^2}e^{i\lambda^{nm}t}\right|^2$$

$$\leq \frac{\sqrt{2}}{(\lambda^{nm})^4}\left|\frac{\frac{\sin(n\pi h)}{n\pi}\lambda^{nm}}{\lambda_h^{nm}\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{(\lambda^{nm})^2}{(\lambda_h^{nm})^2}e^{i(\lambda_h^{nm}-\lambda^{nm})t}-1\right|^2 \leq \frac{\varepsilon}{8cn_\varepsilon^2}.$$

Consequently

$$\sum_{1\leq m\leq n_\varepsilon}\sum_{1\leq |n|\leq n_\varepsilon}\left|\frac{\sqrt{2}\sin(n\pi h)}{\lambda_h^{nm}\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{1}{(\lambda_h^{nm})^2}e^{i\lambda_h^{nm}t}-\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{1}{(\lambda^{nm})^2}e^{i\lambda^{nm}t}\right|^2 \leq \frac{\varepsilon}{8c}.$$

Thus,

$$\sum_{1\leq m\leq N}\sum_{1\leq |n|\leq N}\left|\frac{\sqrt{2}\sin(n\pi h)}{\lambda_h^{nm}\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{1}{(\lambda_h^{nm})^2}e^{i\lambda_h^{nm}t}-\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{1}{(\lambda^{nm})^2}e^{i\lambda^{nm}t}\right|^2$$

$$=\sum_{1\leq m\leq n_\varepsilon}\sum_{1\leq |n|\leq n_\varepsilon}\left|\frac{\sqrt{2}\sin(n\pi h)}{\lambda_h^{nm}\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{1}{(\lambda_h^{nm})^2}e^{i\lambda_h^{nm}t}-\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{1}{(\lambda^{nm})^2}e^{i\lambda^{nm}t}\right|^2$$

$$+\sum_{1\leq m\leq N}\sum_{n_\varepsilon+1\leq |n|\leq N}\left|\frac{\sqrt{2}\sin(n\pi h)}{\lambda_h^{nm}\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{1}{(\lambda_h^{nm})^2}e^{i\lambda_h^{nm}t}-\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{1}{(\lambda^{nm})^2}e^{i\lambda^{nm}t}\right|^2$$

$$+\sum_{n_\varepsilon+1\leq m\leq N}\sum_{1\leq |n|\leq n_\varepsilon}\left|\frac{\sqrt{2}\sin(n\pi h)}{\lambda_h^{nm}\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}\frac{1}{(\lambda_h^{nm})^2}e^{i\lambda_h^{nm}t}-\frac{\sqrt{2}n\pi}{\lambda^{nm}}\frac{1}{(\lambda^{nm})^2}e^{i\lambda^{nm}t}\right|^2$$

$$\leq \frac{\varepsilon}{8c}+\frac{\varepsilon}{8c}=\frac{\varepsilon}{4c},$$

and the proof ends. ■

6.2 Identification of the limit control

In this section we show that the limit of $(v_h, z_h)_{h>0}$ is the HUM control for the continuous equation (1.2).

Theorem 6.3 *We have that $(v,z)=(\widehat{w}_x(t,1,y),\widehat{w}_y(t,x,1))$ is the HUM control for (1.2), where $(\widehat{w},\widehat{w}')$ is the solution of (2.2) with initial data $(\widehat{w}^0,\widehat{w}^1)$ given by (6.8).*

Proof: By taking into account remarks 2.1 and 2.2, the proof consists of verifying (2.10). In order to do that we consider (5.3) and evaluate it for $(W_h^0, W_h^1)=\Psi_h^{nm}$. We obtain that, for any $(n,m)\in\mathcal{I}_N$,

$$\frac{\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2})}{\sqrt{2}} < (-K_h^{-1}M_hU_h^1, U_h^0), \Psi_h^{mn}e^{i\lambda_h^{mn}T} >_0$$

$$=\int_0^T e^{i\lambda_h^{nm}(t-T)}\left[(-1)^{n+1}\sin(n\pi h) < C_h v_h', \phi_h^m > +(-1)^{m+1}\sin(m\pi h) < C_h z_h', \phi_h^n >\right]dt$$

$$+\int_0^T \frac{e^{i\lambda_h^{nm}(t-T)}}{i\lambda_h^{nm}}\left[(-1)^{n+1}\sin(n\pi h) < B_h v_h, \phi_h^m > +(-1)^{m+1}\sin(m\pi h) < B_h z_h, \phi_h^n >\right]dt,$$

which is equivalent to

$$
\begin{aligned}
i\,\cos(\frac{n\pi h}{2})\cos(\frac{m\pi h}{2}) &< (-K_h^{-1}M_hU_h^1, U_h^0), \Psi_h^{mn} >_0 \\
&= \frac{\sqrt{2}h^2 i}{4}\int_0^T e^{i\lambda_h^{nm}t}\Big[(-1)^{n+1}\sin(n\pi h)\cos^2(\frac{m\pi h}{2})(v_h', \phi_h^m) \\
&+ (-1)^{m+1}\sin(m\pi h)\cos^2(\frac{n\pi h}{2})(z_h', \phi_h^n)\Big]\,dt \\
&- \frac{\sqrt{2}}{3\lambda_h^{nm}}\int_0^T e^{i\lambda_h^{nm}t}\left[(-1)^{n+1}\sin(n\pi h)(1+2\cos(m\pi h))(v_h, \phi_h^m)\right. \\
&\left. + (-1)^{m+1}\sin(m\pi h)(1+2\cos(n\pi h))(z_h, \phi_h^n)\right]dt.
\end{aligned}
\tag{6.19}
$$

We have that

$$
< (-K_h^{-1}M_hU_h^1, U_h^0), \Psi_h^{mn} >_0 = \frac{1}{i\lambda_h^{nm}}\alpha_{nm}^h,
$$

and

$$
< v_h, \phi_h^m > = \frac{1}{2h}b_h^m(t), \quad < z_h, \phi_h^n > = \frac{1}{2h}d_h^n(t).
$$

By taking into account that, for every fixed $(n,m) \in \mathcal{I}_N$, when h tends to zero we have that

$$
\begin{gathered}
\alpha_{nm}^h \to \alpha_{nm}, \quad \lambda_h^{nm} \to \lambda^{nm}, \\
b_m^h(t) \to b_m(t), \quad d_m^h(t) \to d_m(t) \text{ in } L^2(0,T), \\
h(b_m^h)'(t) \to 0, \quad h(d_m^h)'(t) \to 0 \text{ in } L^2(0,T),
\end{gathered}
$$

and by passing to the limit in (6.19) we obtain (2.10). ■

7 Numerical experiments

The aim of this section is to present simple numerical experiments in order to confirm the theoretical results that indicate the efficiency of the introduced scheme to restore the uniform controllability. This is done over a fully-discrete approximation.

A rigorous analysis of the results in this paper to the fully discrete case remains to be done. However, the recent results in [4], [16], [17] and its applications to fully-discrete approximation of the 1-D wave equation suggest that, very likely, the scheme permits to restore uniform properties in this case too. We first present and discuss briefly the full discrete scheme used (derived from the semi-discrete one studied in the previous sections) and then consider three different examples with different kind of regularity and location of control. In good agreement with the theoretical results, this scheme will appear numerically robust displaying good results.

7.1 Description of a fully discrete finite-difference scheme

We first introduce a fully discrete - in space and time - scheme associated to system (2.2). The scheme is precisely the time discretization of the semi-discrete scheme (3.6). Let us denote by w_{ij}^k the approximation of the solution w of (2.2) at the point of coordinates (x_i, y_j) and at time $t^k = k\Delta t$: $w_{ij}^k \approx w(k\Delta t, x_i, y_j)$. Δt designates the time-step and k a nonnegative integer in the set $\{0, M\}$. M and Δt are defined such that $T = M\Delta t$. The scheme is then obtained by replacing the time derivative $w''_{ij}(t^k)$ by the finite difference $(w_{ij}^{k+1} - 2w_{ij}^k + w_{ij}^{k-1})/(\Delta t^2)$. Then, noting $W^k = (w_{ij}^k)_{1\le i,j\le N} \in \mathbb{R}^{N^2}$, for $0 \le k \le M$, the vectorial form (3.6) becomes

$$\begin{cases} M_h \frac{W^{k+1}-2W^k+W^{k-1}}{\Delta t^2} + K_h W^k = 0, \quad \forall 0 \le k \le M, \\ W^M = w^0, \frac{W^{M+1}-W^{M-1}}{2\Delta t} = w^1. \end{cases} \tag{7.1}$$

The scheme (7.1) is consistent of order 2 in time and space with the continuous system (2.2). Furthermore, we recall that this latter scheme is stable under the so-called *Courant-Friedrichs-Lewy* (CFL) condition (see [5])

$$\frac{\Delta t^2}{4} \sup_{W\in\mathbb{R}^{N^2}, W\neq 0} \frac{(K_h W, W)}{(M_h W, W)} < 1, \quad \forall h, \Delta t > 0. \tag{7.2}$$

The discrete spectrum $(\lambda_{h,\Delta t}^{mn})_{1\le m,n\le N}$ associated to the scheme (7.1) is

$$\lambda_{h,\Delta t}^{mn} = \frac{2}{\Delta t} \arcsin\left(\frac{\Delta t}{2}\lambda_h^{mn}\right), \quad 1 \le m, n \le N \tag{7.3}$$

with λ_h^{mn} defined by (5.5). Therefore (7.2) implies the following condition:

$$\Delta t \le C h^3, \quad \forall C > 0. \tag{7.4}$$

This condition is too much restrictive from a numerical point of view because it implies a very small time step Δt and therefore a costly scheme. In order to relaxe the stability condition we use a Newmark method ([4], [5], [16]) replacing the term $K_h W^k$ in (7.1) by $1/4 K_h(W^{k+1} + 2W^k + W^{k-1})$. This leads to the following scheme

$$\begin{cases} (M_h + \frac{\Delta t^2}{4} K_h) \frac{W^{k+1}-2W^k+W^{k-1}}{\Delta t^2} + K_h W^k = 0, \quad \forall 0 \le k \le M, \\ W^M = w^0, \frac{W^{M+1}-W^{M-1}}{2\Delta t} = w^1. \end{cases} \tag{7.5}$$

This new scheme remains consistent with the continuous system (2.2). In addition, it is unconditionally stable whatever the value of Δt, thanks to the inequality:

$$\frac{\Delta t^2}{4} \sup_{W\in\mathbb{R}^{N^2}, W\neq 0} \frac{(K_h W, W)}{((M_h + \Delta t^2/4 K_h)W, W)} < 1, \quad \forall h, \Delta t > 0. \tag{7.6}$$

Consequently, $\Delta t = C_1 h$, for all $C_1 > 0$ is an admissible value.

Let us now analyze if this fully-discrete system conserves the observability properties of the semi-discrete scheme. Following the analysis in Section 4 above we study the group velocity of discrete plane waves of the form

$$w_{ij}^k = e^{i(\xi\cdot(x_i,x_j)-\omega t^k)}, \qquad \xi = (\xi_1, \xi_2).$$

For the discrete system (7.5) the following relation between modes ξ and frequencies ω holds

$$\omega(\xi) = \frac{2}{\Delta t} \arcsin\left(\frac{\Delta t}{2}\sqrt{\frac{\omega_s(\xi)^2}{1+\frac{\Delta t^2}{4}\omega_s(\xi)^2}}\right),$$

where

$$\omega_s(\xi) = \frac{2}{h}\sqrt{\tan^2(\xi_1 h/2) + \tan^2(\xi_2 h/2) + \frac{2}{3}\tan^2(\xi_1 h/2)\tan^2(\xi_2 h/2)},$$

and $\xi \in [-\pi/h, \pi/h]^2$.

The group velocity associated to a mode ξ in a direction $v = (v_1, v_2)$ is given by $\nabla_\xi \omega \cdot v$ and a necessary condition in order to have a uniform (in h and Δt) observability property in finite time $T > 0$ is to have a uniform bound from below (in ξ, h and Δt) for $|\nabla_\xi \omega| = \sqrt{|\partial_{\xi_1}\omega|^2 + \partial_{\xi_2}\omega|^2}$. A straightforward computation shows that the minimum value of $|\nabla_\xi \omega|$ is obtained for $\xi = (\pi/h, \pi/h)$ and that

$$|\nabla_\xi \omega(\pi/h, \pi/h)| \sim h^{3/2}\Delta t^{-1}$$

Therefore, this is uniformly bounded from below if

$$\Delta t = Ch^{3/2}, \quad \forall C > 0. \tag{7.7}$$

Thus, we have seen that, even if the scheme (7.5) is stable for any discretization step Δt, this is not sufficient to guarantee a uniform (in h and Δt) controllable scheme. In fact, a necessary condition is given by (7.7). In other words, if we do not have condition (7.7) then the time discretization lose the uniform controllability property of the semi-discrete scheme.

If we compare with the initial scheme (7.1) for which the stability is ensured provided that (7.4) holds, the Newmark strategy permits to gain a factor $h^{3/2}$ in the ratio $\Delta t/h$.

Remark 7.1 *The situation is thus different from the 1-D case where a Newmark scheme leads to a uniform controllable system with Δt of the order of h (see [16]). By optimizing the Newmark parameters, one might design uniform controllable schemes associated to less restrictive conditions on the ratio $\Delta t/h$ (we refer to [16] for the 1-D case).*

7.2 Numerical examples

In this section, we present some numerical experiments for three different initial conditions. The first one concerns the simplest regular initial condition involving only one frequency mode (see eq. (7.8) below). The second example is the well-known pathological test proposed by Glowinski-Li-Lions in [8](see eq. (7.9) below). Finally, the third one is a very singular one involving a discontinuous initial solution u^0 at time $t = 0$ (see eq. (7.14) below) . Each one of these examples are defined on the unit square. For these three examples, we compare the results obtain from the usual finite difference scheme (FDS), the mixed finite element scheme we have introduced (MFES) and the bi-grid method (BI-GRID) (see [1],[8]).

To compute the control, we have used the HUM method which reduces the control problem to the determination of suitable initial conditions $(\hat{w}^0, \hat{w}^1)$ of a forward wave equation. Following [8], the iterative gradient conjugate algorithm is used with the initialization $(\hat{w}^0, \hat{w}^1) = (0, 0)$. We assume that the convergence is obtained when the relative residual is lower than a given *eps* > 0. Finally, the computation are performed using the MATLAB Toolbox and the double precision.

7.2.1 Example 1: Regular initial conditions

We consider the following initial condition

$$u^0(x, y) = 10\sin(\pi x)\sin(\pi y) \quad ; \quad u^1(x, y) = 0; \quad (x, y) \in (0, 1)^2. \tag{7.8}$$

We assume that the control is active on Γ_1 defined by (1.1). The time at which we want the solution to be controlled is taken equal to $T = 3 > 2\sqrt{3}$. Finally, we take *eps* $= 1.e - 06$.

According to the previous remarks on stability and uniform observability, we used the MFES scheme with $\Delta t = h^{3/2}$. Furthermore, we remind that the FDS and BI-GRID schemes are stable under the condition $\Delta t \leq h/\sqrt{2}$. Table 1 compares the results obtained with the three methods for a mesh size $h = 1/31$. At a first glance, the results are quite similar, particularly the L^2-norm of the control $||\boldsymbol{v_h}||_{L^2(0,1)} + ||\boldsymbol{z_h}||_{L^2(0,1)}$ also depicted on Figure 2. The BI-GRID and MFES methods provide good results with few iterations. Table 2 gives the results obtained with the MFES method for different values of h and illustrates in particular the convergence of $||\boldsymbol{v_h}||_{L^2(0,1)} + ||\boldsymbol{z_h}||_{L^2(0,1)}$.

On the other hand, note that the number of iterations necessary to obtain convergence with the FDS is larger. The situation get worse when the ratio $\Delta t/h = \sqrt{h}$ decreases (see Table 1) or h is smaller. Actually, for this example, if we consider h lower than 1/31 or *eps* lower than $1.e-06$ then the FDS does not converge. In addition, the initial conditions $(\hat{w}^0, \hat{w}^1)$ of the forward wave equation are very badly approximated (see Figure 3).

	FDS	FDS	MFES	BI-GRID	BI-GRID
h	1/31	1/31	1/31	1/31	1/101
Δt	$1/\sqrt{2}h$	$h^{3/2}$	$h^{3/2}$	$1/\sqrt{2}h$	$1/\sqrt{2}h$
Number of iterations	45	$\boxed{199}$	17	4	4
$\|\|\boldsymbol{\hat{w}_h^0}\|\|_{L^2(\Omega)}$	0.02277	0.03062	0.02092	0.01976	0.02044
$\|\|\boldsymbol{\hat{w}_h^0}\|\|_{H^1(\Omega)}$	$\boxed{0.26788}$	$\boxed{0.55847}$	0.18261	0.16079	0.18501
$\|\|\boldsymbol{\hat{w}_h^1}\|\|_{L^2(\Omega)}$	$\boxed{1.85936}$	$\boxed{2.2638}$	1.7944	1.79034	1.7874
$\frac{\|\|\boldsymbol{u^0}-\boldsymbol{u_h^0}\|\|_{L^2(\Omega)}}{\|\|u_0\|\|_{L^2(\Omega)}}$	0.01267	0.00834	0.00843	0.01240	0.00772
$\|\|\boldsymbol{u_h^1}\|\|_{H^{-1}(\Omega)}$	0.00239	0.00204	0.00118	0.07132	0.03011
$\frac{\|\|\boldsymbol{u_h(T)}\|\|_{L^2(\Omega)}}{\|\|\boldsymbol{u_h(0)}\|\|_{L^2(\Omega)}}$	0.01147	0.00159	0.00450	0.02527	0.01147
$\|\|\boldsymbol{v_h}\|\|_{L^2(0,1)} + \|\|\boldsymbol{z_h}\|\|_{L^2(0,1)}$	2.98066	2.9869	2.98164	2.97492	2.98106

Table 1: Comparative results between FDS, MFES and BI-GRID methods (Example 1).

7.2.2 Example 2: Irregular initial conditions - Discontinuity of the initial velocity

Let us now consider the following initial conditions:

$$u^0(x,y) = \phi_0(0,x,y) + \phi_1(0,x,y) \quad ; \quad u^1(x,y) = \frac{\partial \phi_0}{\partial t}(0,x,y) + \frac{\partial \phi_1}{\partial t}(0,x,y) \tag{7.9}$$

with

$$\phi_0(t,x,y) = -\pi\sqrt{2}\cos(\pi\sqrt{2})\left(t - \frac{1}{4\sqrt{2}}\right)\Big(\sin(\pi x)\cos(2\pi y) + \cos(2\pi x)\sin(\pi y)\Big) \tag{7.10}$$

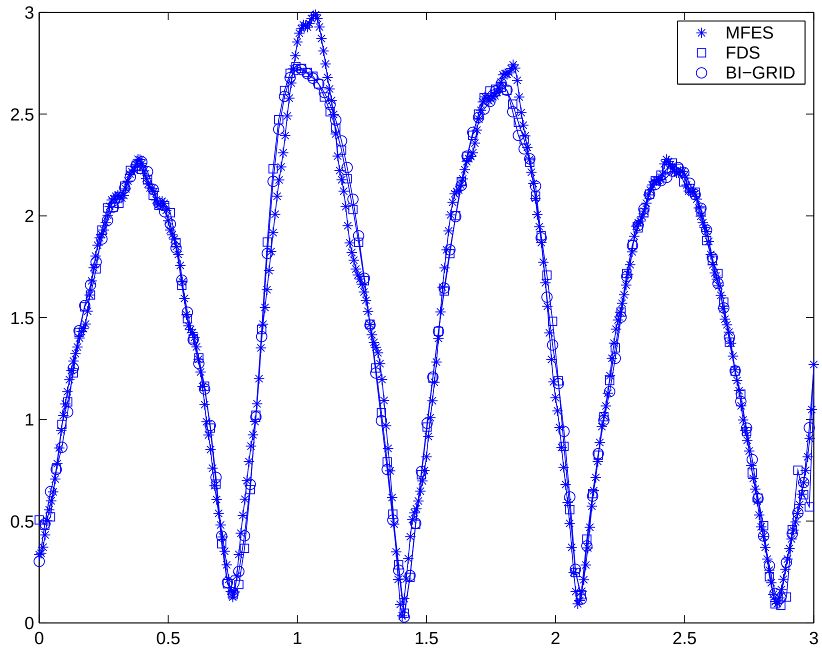

Figure 2: $||\boldsymbol{v_h}||_{L^2(0,1)} + ||\boldsymbol{z_h}||_{L^2(0,1)}$ vs. $t \in [0, T]$ obtained with the FDS ($\Delta t = 1/\sqrt{2}h$), MFES ($\Delta t = h^{3/2}$) and BI-GRID ($\Delta t = 1/\sqrt{2}h$); $h = 1/31$ (Example 1).

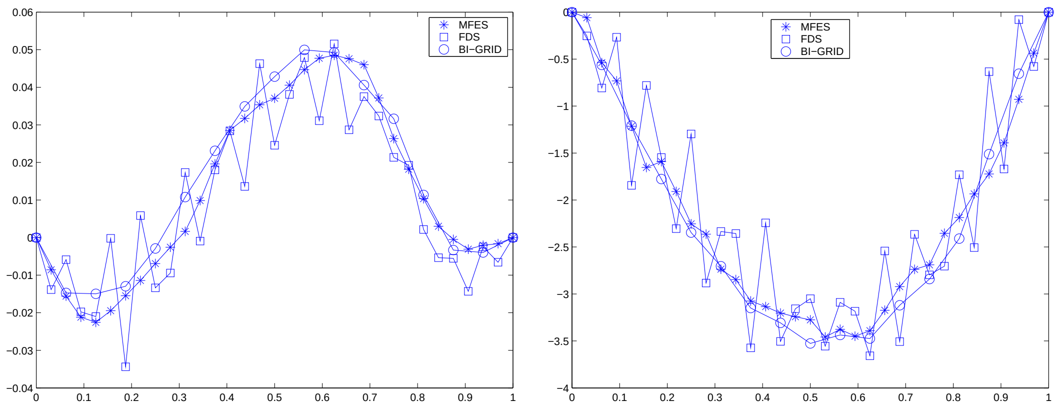

Figure 3: $\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{0}}(x, y = 1/2)$ (left) and $\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{1}}(x, y = 1/2)$ (right) vs. $x \in [0, 1]$ obtained with the FDS ($\Delta t = 1/\sqrt{2}h$),MFES ($\Delta t = h^{3/2}$) and BI-GRID ($\Delta t = 1/\sqrt{2}h$); $h = 1/31$ (Example 1).

	h=1/15	h=1/21	h=1/25	h=1/31	h=1/35
Nb. of iterations	9	12	12	17	19
$\lvert\lvert\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{0}}\rvert\rvert_{L^2(\Omega)}$	0.023014	0.021306	0.021007	0.020920	0.020761
$\lvert\lvert\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{0}}\rvert\rvert_{H^1(\Omega)}$	0.175284	0.176373	0.174714	0.182618	0.182817
$\lvert\lvert\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{1}}\rvert\rvert_{L^2(\Omega)}$	1.81206	1.80115	1.79666	1.7944	1.79184
$\frac{\lvert\lvert\boldsymbol{u^0}-\boldsymbol{u_h^0}\rvert\rvert_{L^2(\Omega)}}{\lvert\lvert\boldsymbol{u^0}\rvert\rvert_{L^2(\Omega)}}$	0.0082343	0.0125101	0.0078352	0.0084316	0.0092339
$\lvert\lvert\boldsymbol{u_h^1}\rvert\rvert_{H^{-1}(\Omega)}$	0.0013481	0.0015728	0.0011591	0.0011838	0.0010488
$\frac{\lvert\lvert\boldsymbol{u_h(T)}\rvert\rvert_{L^2(\Omega)}}{\lvert\lvert\boldsymbol{u_h(0)}\rvert\rvert_{L^2(\Omega)}}$	0.006617	0.005767	0.0051284	0.0045045	0.004359
$\lvert\lvert\boldsymbol{v_h}\rvert\rvert_{L^2(0,1)}+\lvert\lvert\boldsymbol{z_h}\rvert\rvert_{L^2(0,1)}$	2.98449	2.98265	2.98138	2.98164	2.98127

Table 2: Results obtained with the MFES for different values of h, $\Delta t = h^{3/2}$ (Example 1).

and

$$
\begin{aligned}
\phi_1(t,x,y) = {} & 4\pi(T-t)\sin(\pi\sqrt{2})\left(t-\frac{1}{4\sqrt{2}}\right) - \frac{28}{3\sqrt{2}}\sin(\pi\sqrt{2}(t-T))\sin(\pi x)\sin(\pi y) \\
& + 4\sin(\pi x)\sum_{p\geq 3, p \text{ odd}} \frac{p}{p^2-1}\left[\frac{2}{\sqrt{1+p^2}}\sin(\pi\sqrt{1+p^2}(t-T)) \right. \\
& \qquad \left. + \frac{3\sqrt{2}}{p^2-4}\cos\left(\pi\sqrt{2}\left(t-\frac{1}{4\sqrt{2}}\right)\right)\right]\sin(p\pi y) \\
& + 4\sin(\pi y)\sum_{p\geq 3, p \text{ odd}} \frac{p}{p^2-1}\left[\frac{2}{\sqrt{1+p^2}}\sin(\pi\sqrt{1+p^2}(t-T)) \right. \\
& \qquad \left. + \frac{3\sqrt{2}}{p^2-4}\cos\left(\pi\sqrt{2}\left(t-\frac{1}{4\sqrt{2}}\right)\right)\right]\sin(p\pi x)
\end{aligned}
\tag{7.11}
$$

with $T = 15/4\sqrt{2}$. These irregular initial conditions introduced by Glowinski, Li and Lions [8] are well-known to produce spurious oscillations and pathological numerical effects. We remind that u^0 is a Lipschitz continuous function not belonging to $C^1(\overline{\Omega})$ whereas u^1 belongs to $L^\infty(\Omega)$ but not to $C^0(\overline{\Omega})$. The functions u^0 and u^1 are depicted on Figures 4.

The main advantage is that the analytical solution is known. More precisely, the initial conditions $(\hat{w}^0, \hat{w}^1)$ of the forward wave system are given by:

$$
\hat{w}_0(x,y) = \sin(\pi x)\sin(\pi y), \quad \hat{w}_1(x,y) = \pi\sqrt{2}\sin(\pi x)\sin(\pi y) \tag{7.12}
$$

leading to the solution

$$
\phi(t,x,y) = \sqrt{2}\cos\left(\pi\sqrt{2}\left(t-\frac{1}{4\sqrt{2}}\right)\right)\sin(\pi x)\sin(\pi y) \tag{7.13}
$$

and then to the analytical expression of the control $V = \frac{\partial\phi}{\partial t}_{|\partial\Omega}$ acting on the whole boundary $\partial\Omega$. The L^2-norm of the control with respect to time is represented in Figure 7.

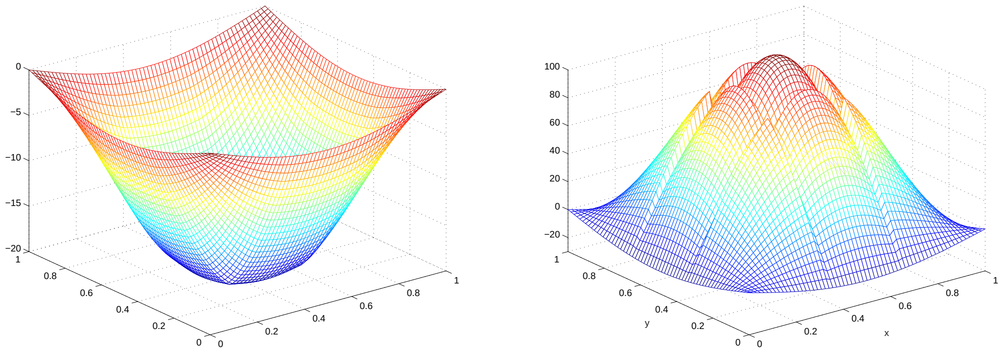

Figure 4: Initial condition u^0 (left) and u^1 (right) (Example 2).

Let us consider $eps = 1.e{-}07$. Figures 5 and 6 depicts the numerical approximation $(\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{0}}, \hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{1}})$ obtained for $(\hat{w}^0, \hat{w}^1)$. The MFES reduces significantly the oscillations observed with the FDS. As already noticed in [8], we point out that these oscillations remains when considering, for the FDS, a smaller ratio $\Delta t/h$. For instance, for the ratio $\Delta t/h = \sqrt{h}$ used for MFES, the FDS algorithm diverges.

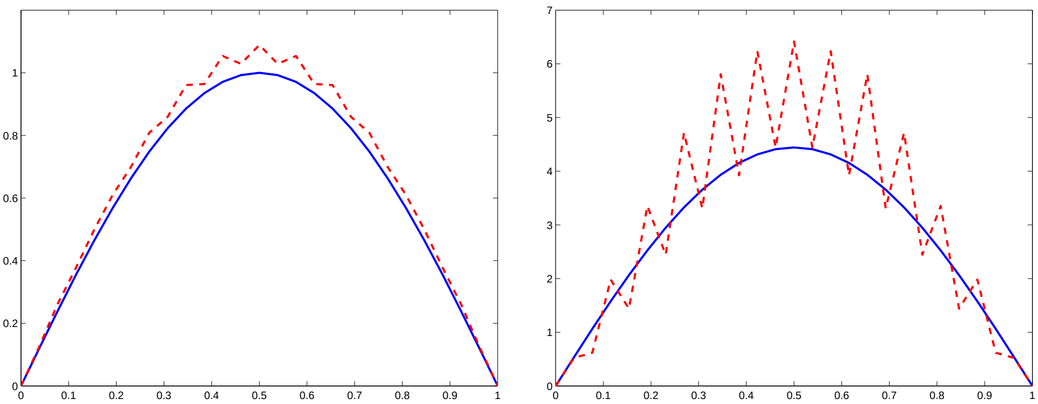

Figure 5: $\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{0}}(x, y = 1/2)$ and $\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{1}}(x, y = 1/2)$ vs. $x \in [0, 1]$ obtained with the FDS: exact (—) and numerical (-.-) solution ; $h = 1/25$, $\Delta t = 1/\sqrt{2}h$ (Example 2).

Figure 7 depicts the evolution of the numerical control in $L^2(\partial\Omega)$-norm with respect to time in $[0, T]$ obtained for the FDS and the MFES. The deterioration of the numerical results is less significant on this quantity. Figure 8 gives the evolution of the residual with respect to the iterations for $h = 1/25$ and $h = 1/61$. The BI-GRID produced an error lower than $1.e-07$ after 5 iterations. For $h = 1/25$, the MFES algorithm needs 16 iterations whereas the FDS algorithm needs 29 iterations to reach the same order of error. For smaller value of h - in particular $h = 1/61$ - the FDS algorithm does not converge.

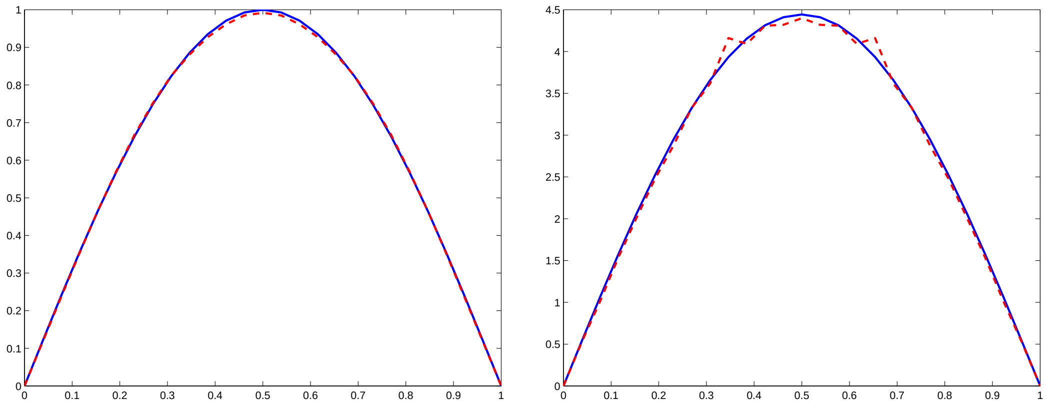

Figure 6: $\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\mathbf{0}}(x, y = 1/2)$ and $\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\mathbf{1}}(x, y = 1/2)$ vs. $x \in [0, 1]$ obtained with the MFES: exact (—) and numerical (-.-) solution ; $h = 1/25$, $\Delta t = h^{3/2}$ (Example 2)

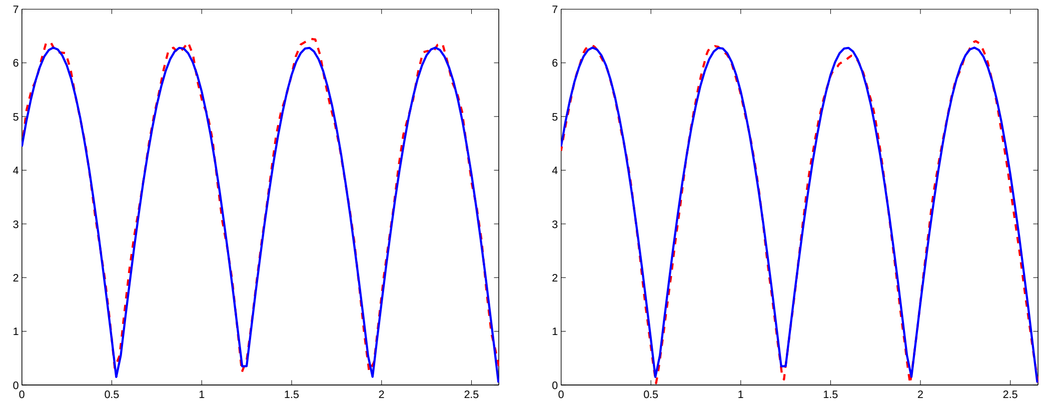

Figure 7: $||\boldsymbol{V}_{\boldsymbol{h}}(t)||_{L^2(\partial\Omega)}$ vs. $t \in [0, T]$ obtained with the FDS (left) and the MFES (right): exact (—) and numerical (-.-) solution ; $h = 1/25$ (Example 2).

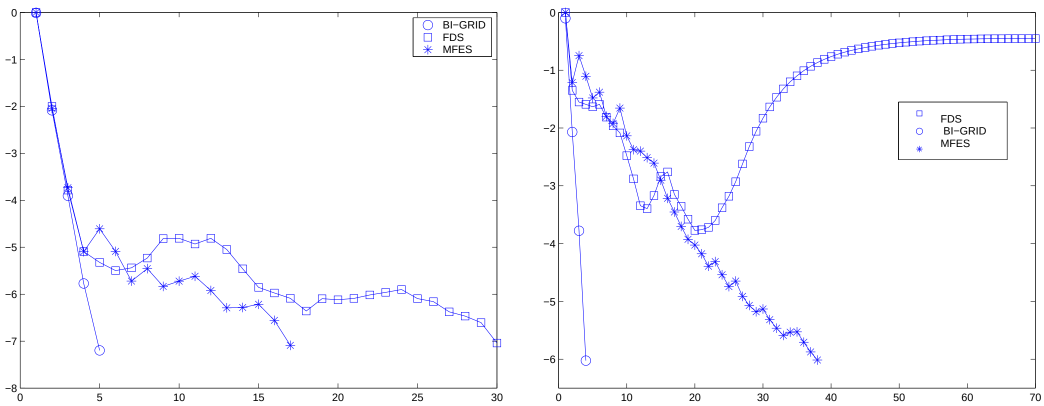

Figure 8: Log10 (Relative error on the residual) vs. iteration of the gradient conjugate algorithm obtained for the FDS, MFES and BI-GRID ; Left: $h = 1/25$ - Right: $h = 1/61$ (Example 2).

	FDS	MFES	BI-GRID
Nb. of iterations	29	16	5
Δt	$1/\sqrt{2}h$	$h^{3/2}$	$1/\sqrt{2}h$
$\frac{\|\|\hat{\boldsymbol{w}}_{\boldsymbol{0}}-\hat{\boldsymbol{w}}_{\boldsymbol{0h}}\|\|_{L^2(\Omega)}}{\|\|\hat{\boldsymbol{w}}_{\boldsymbol{0}}\|\|_{L^2(\Omega)}}$	0.0480827	0.00779022	0.0481913
$\frac{\|\|\hat{\boldsymbol{w}}_{\boldsymbol{0}}-\hat{\boldsymbol{w}}_{\boldsymbol{0h}}\|\|_{H^1(\Omega)}}{\|\|\hat{\boldsymbol{w}}_{\boldsymbol{0}}\|\|_{H^1(\Omega)}}$	0.0610629	0.0252612	0.0106558
$\frac{\|\|\hat{\boldsymbol{w}}_{\boldsymbol{1}}-\hat{\boldsymbol{w}}_{\boldsymbol{1h}}\|\|_{L^2(\Omega)}}{\|\|\hat{\boldsymbol{w}}_{\boldsymbol{1}}\|\|_{L^2(\Omega)}}$	0.3227	0.0336811	0.0172094
$\frac{\|\|\boldsymbol{u}^{\boldsymbol{0}}-\boldsymbol{u}_{\boldsymbol{h}}^{\boldsymbol{0}}\|\|_{L^2(\Omega)}}{\|\|\boldsymbol{u}^{\boldsymbol{0}}\|\|_{L^2(\Omega)}}$	0.00251389	0.00147872	0.00738625
$\frac{\|\|\boldsymbol{u}^{\boldsymbol{1}}-\boldsymbol{u}_{\boldsymbol{h}}^{\boldsymbol{1}}\|\|_{H^{-1}(\Omega)}}{\|\|\boldsymbol{u}^{\boldsymbol{1}}\|\|_{H^{-1}(\Omega)}}$	0.000338676	0.000163876	0.0282561
$\|\|\boldsymbol{V_h}\|\|_{L^2(\partial\Omega)}$	7.47136	7.37811	7.36179

Table 3: Comparative results for $h = 1/25$ (Example 2).

	h=1/15	h=1/21	h=1/25	h=1/31	h=1/35	h=1/41
Nb. of iterations	8	15	16	22	18	19
$\frac{\|\|\hat{\boldsymbol{w}}^{\boldsymbol{0}}-\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{0}}\|\|_{L^2(\Omega)}}{\|\|\hat{\boldsymbol{w}}_{\boldsymbol{0}}\|\|_{L^2(\Omega)}}$	0.0176528	0.0087720	0.0077902	0.0045873	0.0044670	0.0034090
$\frac{\|\|\hat{\boldsymbol{w}}^{\boldsymbol{0}}-\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{0}}\|\|_{H^1(\Omega)}}{\|\|\hat{\boldsymbol{w}}_{\boldsymbol{0}}\|\|_{H^1(\Omega)}}$	0.0365394	0.0240179	0.0252612	0.0155487	0.0166127	0.0146926
$\frac{\|\|\hat{\boldsymbol{w}}^{\boldsymbol{1}}-\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{0}}\|\|_{L^2(\Omega)}}{\|\|\hat{\boldsymbol{w}}_{\boldsymbol{1}}\|\|_{L^2(\Omega)}}$	0.0477631	0.0314514	0.0336811	0.0287866	0.0188667	0.0153937
$\frac{\|\|\boldsymbol{u}^{\boldsymbol{0}}-\boldsymbol{u}_{\boldsymbol{h}}^{\boldsymbol{0}}\|\|_{L^2(\Omega)}}{\|\|\boldsymbol{u}^{\boldsymbol{0}}\|\|_{L^2(\Omega)}}$	0.0035010	0.0017632	0.0014787	0.0011760	0.0006623	0.0005863
$\frac{\|\|\boldsymbol{u}^{\boldsymbol{1}}-\boldsymbol{u}_{\boldsymbol{h}}^{\boldsymbol{1}}\|\|_{H^{-1}(\Omega)}}{\|\|\boldsymbol{u}^{\boldsymbol{1}}\|\|_{H^{-1}(\Omega)}}$	0.0002294	0.0001223	0.0001638	0.0001006	9.5612e-05	0.0001241
$\|\|\boldsymbol{V_h}\|\|_{L^2(\partial\Omega)}$	7.39264	7.36749	7.37811	7.37243	7.38466	7.38576

Table 4: Results obtained with the MFES for several values of h(Example 2).

7.2.3 Example 3: Irregular initial conditions - Discontinuity of the initial position

In this third example, we consider the most singular situation with a discontinuous initial condition u^0. More precisely, we consider, still on the unit square $(0,1)^2$, the following functions

$$u^0(x,y) = \begin{cases} 40 & (x,y) \in (\frac{1}{3}, \frac{2}{3})^2 \\ 0 & \text{elsewhere} \end{cases} \quad ; \quad u^1(x,y) = 0. \tag{7.14}$$

We assume that the control is active on Γ_1 and we take $T = 3$ and $eps = 1.e-06$. With this discontinuous initial position, the usual finite difference scheme (FDS) completely fails, even for relative large values of h - for instance $h = 1/21$. On the contrary, the relative error associated to the MFES decreases (see Figure 9). The L^2-norm of the control converges as indicated by the results of Table 5, as h goes to zero, and the solution of the discrete wave system is driven at rest at time T (see figure 10).

	h=1/15	h=1/21	h=1/25	h=1/31	h=1/35
Nb. of iterations	25	33	38	43	46
$\|\|\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\mathbf{0}}\|\|_{L^2(\Omega)}$	0.165546	0.116949	0.097492	0.0878891	0.0817421
$\|\|\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\mathbf{0}}\|\|_{H^1(\Omega)}$	1.4625	1.34044	1.13954	1.1905	1.26147
$\|\|\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\mathbf{1}}\|\|_{L^2(\Omega)}$	13.612	12.9784	11.3383	10.8551	10.4097
$\frac{\|\|\boldsymbol{u}^{\mathbf{0}}-\boldsymbol{u}_{\boldsymbol{h}}^{\mathbf{0}}\|\|_{L^2(\Omega)}}{\|\|\boldsymbol{u}^{\mathbf{0}}\|\|_{L^2(\Omega)}}$	0.0173194	0.00915911	0.0113164	0.00723832	0.00613776
$\|\|\boldsymbol{u}_{\boldsymbol{h}}^{\mathbf{1}}\|\|_{H^{-1}(\Omega)}$	0.0038072	0.00544011	0.004714	0.00458297	0.0033676
$\frac{\|\|\boldsymbol{u}_{\boldsymbol{h}}(T)\|\|_{L^2(\Omega)}}{\|\|\boldsymbol{u}_{\boldsymbol{h}}(0)\|\|_{L^2(\Omega)}}$	0.0118741	0.0091952	0.0147911	0.0110379	0.00855918
$\|\|\boldsymbol{v}_{\boldsymbol{h}}\|\|_{L^2(0,1)} + \|\|\boldsymbol{z}_{\boldsymbol{h}}\|\|_{L^2(0,1)}$	10.6906	10.2347	9.65102	9.477	9.3831

Table 5: Results obtained with the MFES for several values of h (Example 3).

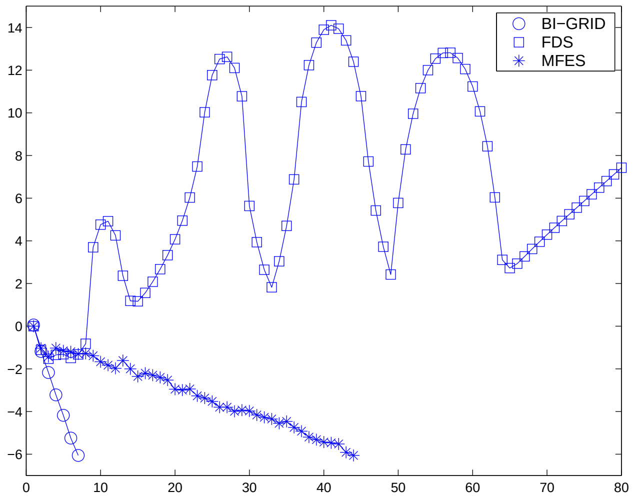

Figure 9: Log10(Relative error on the residual) vs. iteration of the gradient conjugate algorithm obtained for the FDS, MFES and BI-GRID, $h = 1/31$ (Example 3).

Let us finally discuss the results obtained with the BI-GRID method. Similarly with the previous examples, the residual becomes lower than eps after less than 10 iterations (see Figure 9). However, the results summarized in Table 6 indicates that the control obtained do not drive the solution at rest at time T. In particular, the quantities $||\boldsymbol{u}^{\mathbf{0}} - \boldsymbol{u}_{\boldsymbol{h}}^{\mathbf{0}}||_{L^2(\Omega)}/||\boldsymbol{u}^{\mathbf{0}}||_{L^2(\Omega)}$, $||\boldsymbol{u}_{\boldsymbol{h}}^{\mathbf{1}}||_{H^{-1}(\Omega)}$, and $||\boldsymbol{u}_{\boldsymbol{h}}(T)||_{L^2(\Omega)}/||\boldsymbol{u}_{\boldsymbol{h}}(0)||_{L^2(\Omega)}$ do not converge toward zero with h, like it should be. This result is not a contradiction. Firstly, the theoretical proof of the efficiency of the bi-grid algorithm remains to be done in 2-D. It was recently proved, as a semi-discrete level and in 1-D, that the control obtained by this method controls only the projection of the discrete wave system on the coarse mesh (see [18]). In our case, the bi-grid procedure has no regularization effect on the initial irregular position u^0 (the projection of u^0 on a coarser mesh is still discontinuous

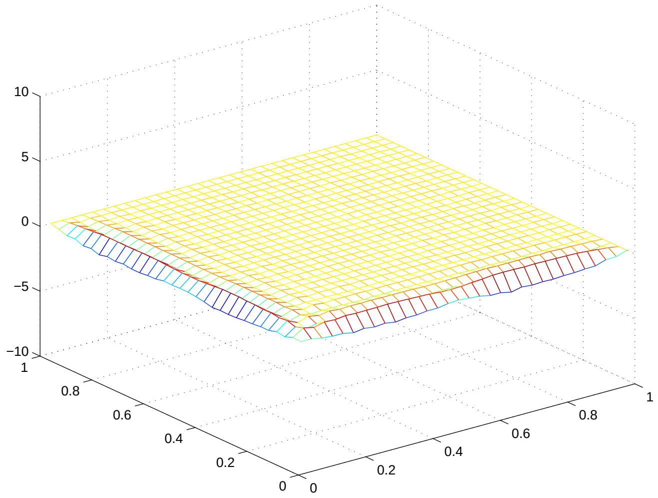

Figure 10: Solution at time $T = 3$ obtained with the MFES, $h = 1/31$ (Example 3).

with a jump independent of h), and therefore the control obtained is not comparable to the one obtained with the MFES. In the previous example, the situation was different, the initial condition u^0 being continuous. This example shows that the test on the relative residual commonly used in the literature should be replaced or at least confirmed by a test on the quantity $||\boldsymbol{u_h}(T)||_{L^2(\Omega)}/||\boldsymbol{u_h}(0)||_{L^2(\Omega)}$ or $E_h(T)/E_h(0)$ (if we designate by E_h the energy associated to the discrete system.). Figure 11 depicts the L^2-norm of the control obtained with MFES and BI-GRID. As expected, the curve corresponding to the BI-GRID is smoother. However, this control do not drive the solution at rest at time T while the control obtained with the MFES does.

	h=1/21	h=1/41	h=1/61	h=1/81	h=1/101
Nb. of iterations	6	6	7	7	8
$\|\|\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{0}}\|\|_{L^2(\Omega)}$	0.0738504	0.0692983	0.0630167	0.0593113	0.059985
$\|\|\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{0}}\|\|_{H^1(\Omega)}$	0.676801	0.854368	0.824286	0.819461	0.846549
$\|\|\hat{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{1}}\|\|_{L^2(\Omega)}$	6.73463	6.93969	6.30109	6.04344	6.19305
$\frac{\|\|\boldsymbol{u}^{\boldsymbol{0}}-\boldsymbol{u}_h^{\boldsymbol{0}}\|\|_{L^2(\Omega)}}{\|\|\boldsymbol{u}^{\boldsymbol{0}}\|\|_{L^2(\Omega)}}$	0.332749	0.265029	0.219566	0.189398	0.174985
$\|\|\boldsymbol{u}_{\boldsymbol{h}}^{\boldsymbol{1}}\|\|_{H^{-1}(\Omega)}$	0.661024	0.544694	0.394718	0.384597	0.357067
$\frac{\|\|\boldsymbol{u_h}(T)\|\|_{L^2(\Omega)}}{\|\|\boldsymbol{u_h}(0)\|\|_{L^2(\Omega)}}$	0.211348	0.160687	0.152281	0.127367	0.127306
$\|\|\boldsymbol{v_h}\|\|_{L^2(0,1)} + \|\|\boldsymbol{z_h}\|\|_{L^2(0,1)}$	9.27014	9.86701	9.16334	8.85218	9.15013

Table 6: Results obtained with BI-GRID for different values of h (Example 3).

Regarding to those results, we may say that the scheme MFES "pass" this very singular test.

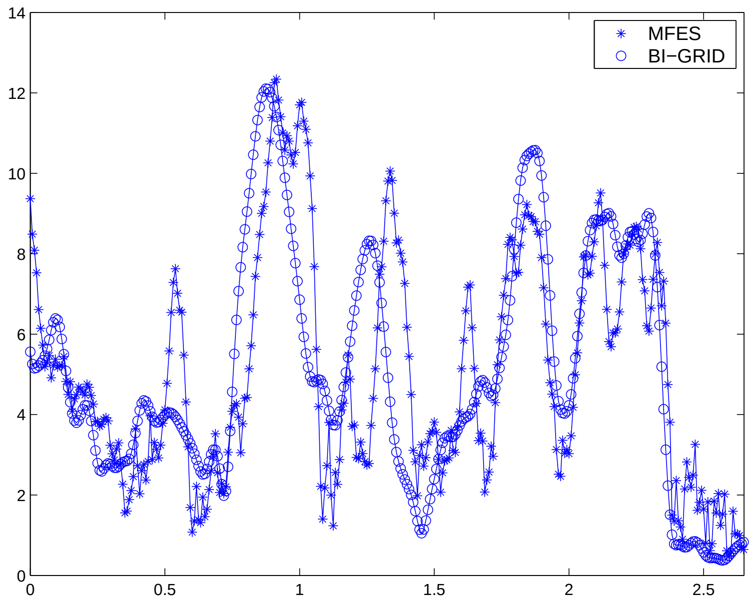

Figure 11: $||\boldsymbol{v_h}||_{L^2(0,1)} + ||\boldsymbol{z_h}||_{L^2(0,1)}$ vs. $t \in [0, T]$, $h = 1/31$ (Example 3).

A Appendix

The aim of this appendix is to prove Theorem 4.1 and Proposition 4.2. To simplify the notation we write

$$\begin{aligned}
a_{ij}^{kl} &= w_{ik} + w_{il} + w_{jk} + w_{jl}, \qquad b_{ij}^{kl} = w'_{ik} + w'_{il} + w'_{jk} + w'_{jl}, \\
c_{ij}^{kl} &= w''_{ik} + w''_{il} + w''_{jk} + w''_{jl}, \\
\Delta_{(1,0)} w_{ij} &= 2w_{ij} - w_{i+1j} - w_{i-1j}, \qquad \Delta_{(0,1)} w_{ij} = 2w_{ij} - w_{ij+1} - w_{ij-1}, \\
\Delta_{(1,1)} w_{ij} &= 2w_{ij} - w_{i+1j+1} - w_{i-1j-1}, \qquad \Delta_{(1,-1)} w_{ij} = 2w_{ij} - w_{i+1j-1} - w_{i-1j+1}.
\end{aligned}$$

A.1 Proof of Theorem 4.1

Multiplying the discrete system by the discrete version of the usual continuous multiplier $(x, y) \cdot \nabla u$, i.e.

$$(ih, jh).(\frac{w_{i+1j} - w_{i-1j}}{2h}, \frac{w_{ij+1} - w_{i-1j}}{2h}) = i\frac{w_{i+1j} - w_{i-1j}}{2} + j\frac{w_{ij+1} - w_{ij-1}}{2} \equiv \frac{m_{ij}}{2}, \tag{A.1}$$

and summing in i and j we obtain

$$\begin{aligned}
0 \;=\; & \underbrace{\frac{h^2}{32} \int_0^T \sum_{i,j=1}^{N} \left(c_{ii+1}^{jj+1} + c_{ii+1}^{j-1j} + c_{i-1i}^{jj+1} + c_{i-1i}^{j-1j} \right) m_{ij} dt}_{\equiv C} \\
& + \frac{1}{6} \int_0^T \underbrace{\sum_{i,j=1}^{N} \left(\Delta_{(1,0)} w_{ij} + \Delta_{(0,1)} w_{ij} + \Delta_{(1,1)} w_{ij} + \Delta_{(1,-1)} w_{ij} \right) m_{ij} \, dt}_{\equiv D}
\end{aligned} \tag{A.2}$$

We study separately C and D. Integrating by parts in C we easily obtain,

$$C = \int_0^T C_1 dt + [C_2]_0^T \,, \tag{A.3}$$

where

$$C_1 = -\sum_{i,j=1}^{N} \left(b_{ii+1}^{jj+1} + b_{ii+1}^{j-1j} + b_{i-1i}^{jj+1} + b_{i-1i}^{j-1j} \right) m'_{i,j}, \tag{A.4}$$

$$C_2 = \sum_{i,j=1}^{N} \left(b_{ii+1}^{jj+1} + b_{ii+1}^{j-1j} + b_{i-1i}^{jj+1} + b_{i-1i}^{j-1j} \right) m_{ij}. \tag{A.5}$$

We first consider the term C_1 above. Changing the indexes in the last three terms of C_1 above (in order to have the common factor b_{ii+1}^{jj+1}) and taking into account that $w_{i,0} = w_{i,N+1} = w_{0,j} = w_{j,N+1} = 0$, we obtain

$$C_1 = 2\sum_{i,j=0}^{N} \left(b_{ii+1}^{jj+1} \right)^2 - (N+1) \left[\sum_{i=1}^{N} \left(w'_{iN} + w'_{i+1N} \right)^2 + \sum_{j=1}^{N} \left(w'_{Nj} + w'_{Nj+1} \right)^2 \right]. \tag{A.6}$$

We now analyze the term D in (A.2). We only make the details for the first term in D since the others can be simplified similarly. The first term in D reads

$$\sum_{i,j=1}^{N} \Delta_{(1,0)} w_{ij} m_{ij} = \sum_{i,j=1}^{N} \Delta_{(1,0)} w_{ij} \left[i \left(w_{i+1j} - w_{i-1j} \right) + j \left(w_{ij+1} - w_{ij-1} \right) \right]. \tag{A.7}$$

We consider separately these two terms. For the second one we have

$$\begin{aligned} &\sum_{i,j=1}^{N} \left(2w_{ij} - w_{i+1j} - w_{i-1j} \right) j \left(w_{ij+1} - w_{ij-1} \right) \\ &\quad = \sum_{i,j=1}^{N} j \left(w_{ij} - w_{i-1j} \right) w_{ij+1} - \sum_{i,j=1}^{N} j \left(w_{i+1j} - w_{ij} \right) w_{ij+1} \\ &\qquad - \left[\sum_{i,j=1}^{N} j \left(w_{ij} - w_{i-1j} \right) w_{ij-1} - \sum_{i,j=1}^{N} j \left(w_{i+1j} - w_{ij} \right) w_{ij-1} \right]. \end{aligned}$$

Changing the indexes to obtain the common factor $(w_{i+1j} - w_{ij})$ in all the terms and taking into account that $w_{i,0} = w_{i,N+1} = w_{0,j} = w_{j,N+1} = 0$, we obtain

$$\begin{aligned} &\sum_{i,j=0}^{N} \left[j \left(w_{i+1j} - w_{ij} \right) \left(w_{i+1j+1} - w_{ij+1} \right) - j \left(w_{i+1j} - w_{ij} \right) \left(w_{i+1j-1} - w_{ij-1} \right) \right] \\ &= \sum_{i,j=0}^{N} j \left(w_{i+1j} - w_{ij} \right) \left(w_{i+1j+1} - w_{ij+1} \right) - \sum_{i,j=0}^{N,N-1} (j+1) \left(w_{i+1j+1} - w_{ij+1} \right) \left(w_{i+1j} - w_{ij} \right) \\ &= -\sum_{i,j=0}^{N} \left(w_{i+1j+1} - w_{ij+1} \right) \left(w_{i+1j} - w_{ij} \right). \end{aligned}$$

An analogous argument allows to simplify the first term in (A.7),

$$\sum_{i,j=1}^{N} \Delta_{(1,0)} w_{ij} m_{ij} = \sum_{i,j=0}^{N} \Big[(w_{i+1j} - w_{ij})^2 - (N+1)(w_{Nj})^2 - (w_{i+1j+1} - w_{ij+1}) (w_{i+1j} - w_{ij}) \Big] .$$

Simplifying the other three terms in D we finally have

$$\begin{aligned} D = &- \sum_{i,j=0}^{N} \left[(w_{i+1j+1} - w_{i+1j}) (w_{ij+1} - w_{ij}) + (w_{i+1j+1} - w_{ij+1}) (w_{i+1j} - w_{ij}) \right] \\ &+ \sum_{i,j=0}^{N} \Big[(w_{i+1j} - w_{ij})^2 + (w_{ij+1} - w_{ij})^2 \Big] - (N+1) \sum_{j=0}^{N} \left[(w_{Nj})^2 + 2 w_{Nj} w_{Nj+1} \right] \\ &- (N+1) \sum_{i=0}^{N} \left[(w_{iN})^2 + 2 w_{iN} w_{i+1N} \right] . \end{aligned} \tag{A.8}$$

By Young's inequality we can estimate the first term in this formula,

$$\begin{aligned} &\sum_{i,j=0}^{N} \left[(w_{i+1j+1} - w_{i+1j}) (w_{ij+1} - w_{ij}) + (w_{i+1j+1} - w_{ij+1}) (w_{i+1j} - w_{ij}) \right] \\ &\leq \frac{1}{2} \sum_{i,j=0}^{N} \Big[(w_{i+1j+1} - w_{i+1j})^2 + (w_{ij+1} - w_{ij})^2 + (w_{i+1j+1} - w_{ij+1})^2 + (w_{i+1j} - w_{ij})^2 \Big] \\ &= \sum_{i,j=0}^{N} \Big[(w_{ij+1} - w_{ij})^2 + (w_{i+1j} - w_{ij})^2 \Big] . \end{aligned}$$

Therefore,

$$\begin{aligned} D &\geq -(N+1) \left[\sum_{j=0}^{N} w_{Nj}^2 + 2 \sum_{j=0}^{N} w_{Nj} w_{Nj+1} + \sum_{i=0}^{N} w_{iN}^2 + 2 \sum_{i=0}^{N} w_{iN} w_{i+1N} \right] \\ &= -(N+1) \Bigg[\sum_{j=1}^{N} w_{Nj}^2 + \sum_{j=1}^{N+1} w_{Nj-1} w_{Nj} + \sum_{j=0}^{N} w_{Nj} w_{Nj+1} \\ &\quad + \sum_{i=1}^{N+1} w_{iN}^2 + \sum_{i=1}^{N+1} w_{i-1N} w_{iN} + \sum_{i=0}^{N} w_{iN} w_{i+1N} \Bigg] \\ &= -(N+1) \left[\sum_{j=1}^{N} (w_{Nj-1} + w_{Nj} + w_{Nj+1}) w_{Nj} + \sum_{i=1}^{N} (w_{i-1N} + w_{iN} + w_{i+1N}) w_{iN} \right] . \end{aligned} \tag{A.9}$$

Substituting (A.3), (A.6) and (A.9) into (A.2) we obtain

$$h^2\int_0^T \sum_{i,j=0}^N \left(\frac{b_{ii+1}^{jj+1}}{4}\right)^2 dt \le \frac{h}{8}\int_0^T \left[\sum_{i=1}^N \left(\frac{w'_{iN}+w'_{i+1N}}{2}\right)^2 + \sum_{j=1}^N \left(\frac{w'_{Nj}+w'_{Nj+1}}{2}\right)^2\right] dt$$
$$+\frac{1}{2}\int_0^T \left[\sum_{j=1}^N \frac{w_{Nj-1}+w_{Nj}+w_{Nj+1}}{3}\frac{w_{Nj}}{h} + \sum_{i=1}^N \frac{w_{i-1N}+w_{iN}+w_{i+1N}}{3}\frac{w_{iN}}{h}\right] dt$$
$$-\frac{h^2}{32}[C_2]_0^T. \tag{A.10}$$

We observe that the term in the left hand side contains only one part of the energy. In order to obtain the full energy we make an equipartition of the energy. The following lemma is a discrete version of the well-known equipartition of energy for the continuous wave equation, which reads

$$0 = -\int_0^T\int_\Omega (|w_t|^2 + |\nabla u|^2) dx dt + \int_\Omega |w_t u|^2 dx\Big]_0^T. \tag{A.11}$$

Lemma A.1 *The following holds:*

$$\begin{aligned} 0 \;=\; & -h^2\int_0^T \left[\sum_{i,j=0}^N \left(\frac{b_{ii+1}^{jj+1}}{4}\right)^2\right] dt + h^2 \left[\sum_{i,j=0}^N \left(\frac{a_{ii+1}^{jj+1}}{4}\right)\left(\frac{b_{ii+1}^{jj+1}}{4}\right)\right]_0^T \\ & + h^2 \sum_{i,j=0}^N \int_0^T \left[\frac{1}{3}\left(\frac{w_{i+1j}-w_{ij}}{h}\right)^2 + \frac{1}{3}\left(\frac{w_{ij+1}-w_{ij}}{h}\right)^2\right. \\ & \left. + \frac{2}{3}\left(\frac{w_{i+1j+1}-w_{ij}}{h\sqrt{2}}\right)^2 + \frac{2}{3}\left(\frac{w_{i+1j}-w_{ij+1}}{h\sqrt{2}}\right)^2\right] dt. \end{aligned}$$

The proof of this lemma is straightforward following the idea of the continuous system where (A.11) is obtained multiplying system (2.2) by u and integrating by parts.

When applying Lemma A.1 to the identity (A.10) we obtain

$$\int_0^T E_h(t)dt + \frac{h^2}{32}\left[\sum_{i,j=0}^N a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2\right]_0^T$$
$$\le \frac{h}{8}\int_0^T \left[\sum_{i=1}^N \left(\frac{w'_{iN}+w'_{i+1N}}{2}\right)^2 + \sum_{j=1}^N \left(\frac{w'_{Nj}+w'_{Nj+1}}{2}\right)^2\right] dt$$
$$+\frac{1}{2}\int_0^T \left[\sum_{j=1}^N \frac{w_{Nj-1}+w_{Nj}+w_{Nj+1}}{3}\frac{w_{Nj}}{h} + \sum_{i=1}^N \frac{w_{i-1N}+w_{iN}+w_{i+1N}}{3}\frac{w_{iN}}{h}\right] dt. \tag{A.12}$$

The following lemma allows us to estimate the the second term in the left hand side of this formula.

Lemma A.2 *The following holds*

$$h^2\left[\sum_{i,j=0}^N a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2\right]_0^T \le 64\sqrt{3}E_h(0). \tag{A.13}$$

Before proving this lemma we finish the proof of Theorem 4.1.

Taking into account the conservation of the energy stated in Proposition 4.1 we have

$$\int_0^T E_h(t)dt + \frac{h^2}{32}\left[\sum_{i,j=0}^{N} a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2\right]_0^T \geq T E_h(0) - 2\sqrt{3}E_h(0) = (T - 2\sqrt{3})E_h(0),$$

which combined with (A.12) provides the following

$$(T - 2\sqrt{3})E_h(0) \leq \frac{h}{8}\int_0^T \left[\sum_{i=1}^{N}\left(\frac{w'_{iN} + w'_{i+1N}}{2}\right)^2 + \sum_{j=1}^{N}\left(\frac{w'_{Nj} + w'_{Nj+1}}{2}\right)^2\right] dt$$
$$+\frac{1}{2}\int_0^T \left[\sum_{j=1}^{N}\frac{w_{Nj-1} + w_{Nj} + w_{Nj+1}}{3}\frac{w_{Nj}}{h} + \sum_{i=1}^{N}\frac{w_{i-1N} + w_{iN} + w_{i+1N}}{3}\frac{w_{iN}}{h}\right] dt.$$

This concludes the proof of Theorem 4.1. ∎

Proof of Lemma A.2. From (A.5) we have

$$\sum_{i,j=0}^{N} a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2 = \sum_{i,j=0}^{N} a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + \sum_{i,j=1}^{N}\left[b_{ii+1}^{jj+1} + b_{ii+1}^{j-1j} + b_{i-1i}^{jj+1} + b_{i-1i}^{j-1j}\right] m_{ij} \qquad \text{(A.14)}$$

To simplify the notation we assume that

$$w_{N+2j} = w_{Nj}, \qquad w_{iN+2} = w_{iN}, \qquad w_{-1j} = w_{i,-1} = 0, \qquad \forall i,j = 0,...,N+1. \qquad \text{(A.15)}$$

We change the indexes in each one of the terms of the right hand side of (A.14) in order to have the common factor b_{ii+1}^{jj+1}. Then we obtain

$$\sum_{i,j=0}^{N} a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2 = \sum_{i,j=0}^{N}\left(a_{ii+1}^{jj+1} + R_{ij}\right) b_{ii+1}^{jj+1}, \qquad \text{(A.16)}$$

where

$$R_{ij} = i\left[(w_{i+1j} - w_{i-1j}) + (w_{i+1j+1} - w_{i-1j+1})\right] + (i+1)\left[(w_{i+2j} - w_{ij}) + (w_{i+2j+1} - w_{ij+1})\right]$$
$$+j\left[(w_{ij+1} - w_{ij-1}) + (w_{i+1j+1} - w_{i+1j-1})\right] + (j+1)\left[(w_{ij+2} - w_{ij}) + (w_{i+1j+2} - w_{i+1j})\right].$$

We estimate the right hand side in (A.16) using the Schwartz inequality. Thus,

$$\sum_{i,j=0}^{N} a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2 \leq \left[\sum_{i,j=0}^{N}\left(a_{ii+1}^{jj+1} + R_{ij}\right)^2\right]^{1/2}\left[\sum_{i,j=0}^{N}(b_{ii+1}^{jj+1})^2\right]^{1/2}. \qquad \text{(A.17)}$$

Now we prove that

$$\sum_{i,j=0}^{N}\left(a_{ii+1}^{jj+1} + R_{ij}\right)^2 \leq \sum_{i,j=0}^{N} R_{ij}^2 + 8\sum_{i=1}^{N}(N+1)(w_{iN})^2 + 8\sum_{j=1}^{N}(N+1)\left(w_{Nj}\right)^2. \qquad \text{(A.18)}$$

Indeed, we have

$$\sum_{i,j=0}^{N}\left[\left(a_{ii+1}^{jj+1}+R_{ij}\right)^2-R_{ij}^2\right]=\sum_{i,j=0}^{N}\left[(a_{ii+1}^{jj+1})^2+2a_{ii+1}^{jj+1}R_{ij}\right], \tag{A.19}$$

and it is not difficult to see that

$$\sum_{i,j=0}^{N}a_{ii+1}^{jj+1}R_{ij}=-2\sum_{i,j=0}^{N}(a_{ii+1}^{jj+1})^2+\sum_{i=1}^{N}(N+1)\left(w_{iN}+w_{i+1N}\right)^2+\sum_{j=1}^{N}(N+1)\left(w_{Nj}+w_{Nj+1}\right)^2.$$

Therefore, using Young's inequality, the right hand side in (A.19) reads

$$\begin{aligned}\sum_{i,j=0}^{N}\left[(a_{ii+1}^{jj+1})^2+2a_{ii+1}^{jj+1}R_{ij}\right]&=-3\sum_{i,j=0}^{N}(a_{ii+1}^{jj+1})^2+2\sum_{i=1}^{N}(N+1)\left(w_{iN}+w_{i+1N}\right)^2\\&+2\sum_{j=1}^{N}(N+1)\left(w_{Nj}+w_{Nj+1}\right)^2\le 8\sum_{i=1}^{N}(N+1)(w_{iN})^2+8\sum_{j=1}^{N}(N+1)\left(w_{Nj}\right)^2.\end{aligned} \tag{A.20}$$

From (A.19)-(A.20) we easily deduce (A.18). Now we estimate the right hand side in (A.18). Concerning the first term we have

$$\begin{aligned}\sum_{i,j=0}^{N}R_{ij}^2=&\sum_{i,j=0}^{N}[i(w_{i+1j}-w_{i-1j})+i(w_{i+1j+1}-w_{i-1j+1})\\&+(i+1)(w_{i+2j}-w_{ij})+(i+1)(w_{i+2j+1}-w_{ij+1})+j(w_{ij+1}-w_{ij-1})\\&+j(w_{i+1j+1}-w_{i+1j-1})+(j+1)(w_{ij+2}-w_{ij})+(j+1)(w_{i+1j+2}-w_{i+1j})]^2\\\le&\frac{32}{h^2}\sum_{i,j=0}^{N}\left[(w_{i+1j}-w_{i-1j})^2+(w_{ij+1}-w_{ij-1})^2\right],\end{aligned} \tag{A.21}$$

where we have used Young's inequality and the fact that $i,j\le h^{-1}$. In (A.21), the first term is estimated as follows

$$\begin{aligned}&\sum_{i,j=0}^{N}(w_{i+1j}-w_{i-1j})^2\\&=\frac{1}{2}\sum_{i,j=0}^{N}\left[(w_{i+1j}-w_{ij}+w_{ij}-w_{i-1j})^2+(w_{i+1j}-w_{ij+1}+w_{ij+1}-w_{i-1j})^2\right]\\&\le\sum_{i,j=0}^{N}\left[(w_{i+1j}-w_{ij})^2+(w_{ij}-w_{i-1j})^2\right]+\sum_{i,j=1}^{N}\left[(w_{i+1j}-w_{ij+1})^2+(w_{ij+1}-w_{i-1j})^2\right]\\&=\sum_{i,j=0}^{N}\left[2(w_{i+1j}-w_{ij})^2+(w_{i+1j}-w_{ij+1})^2+(w_{i+1j+1}-w_{ij})^2\right]-2\sum_{j=0}^{N}(w_{Nj})^2,\end{aligned} \tag{A.22}$$

and an analogous formula holds for the second term in (A.21). Therefore, we have

$$\sum_{i,j=0}^{N} \left[(w_{i+1j}-w_{i-1j})^2+(w_{ij+1}-w_{ij-1})^2\right]$$
$$\leq 2\sum_{i,j=0}^{N} \left[(w_{i+1j}-w_{ij})^2+(w_{ij+1}-w_{ij})^2+(w_{i+1j}-w_{ij+1})^2+(w_{i+1j+1}-w_{ij})^2\right]$$
$$-2\sum_{j=0}^{N}(w_{Nj})^2-2\sum_{i=0}^{N}(w_{iN})^2. \tag{A.23}$$

Substituting (A.21) into (A.18) and taking into account (A.23) we easily obtain

$$\sum_{i,j=0}^{N}\left(a_{ii+1}^{jj+1}+R_{ij}\right)^2$$
$$\leq \frac{64}{h^2}\sum_{i,j=0}^{N}\left[(w_{i+1j}-w_{ij})^2+(w_{ij+1}-w_{ij})^2+(w_{i+1j}-w_{ij+1})^2+(w_{i+1j+1}-w_{ij})^2\right], \tag{A.24}$$

which allows us estimate (A.17). In fact, using Young's inequality we obtain

$$\sum_{i,j=0}^{N} a_{ii+1}^{jj+1}b_{ii+1}^{jj+1}+C_2 \leq \frac{8}{h}\left(\sum_{i,j=0}^{N}(b_{ii+1}^{jj+1})^2\right)^{1/2}$$
$$\times\left(\sum_{i,j=0}^{N}\left[(w_{i+1j}-w_{ij})^2+(w_{ij+1}-w_{ij})^2+(w_{i+1j}-w_{ij+1})^2+(w_{i+1j+1}-w_{ij})^2\right]\right)^{1/2}$$
$$\leq 16\sqrt{3}\left[\sum_{i,j=0}^{N}\left(\frac{b_{ii+1}^{jj+1}}{4}\right)^2+\frac{1}{3}\sum_{i,j=0}^{N}\left[(w_{i+1j}-w_{ij})^2+(w_{ij+1}-w_{ij})^2\right.\right.$$
$$\left.\left.+(w_{i+1j}-w_{ij+1})^2+(w_{i+1j+1}-w_{ij})^2\right]\right]=32\sqrt{3}E_h(t).$$

Therefore,

$$\left[\sum_{i,j=0}^{N} a_{ii+1}^{jj+1}b_{ii+1}^{jj+1}+C_2\right]_0^T \leq 32\sqrt{3}\left(E_h(T)+E_h(0)\right)\leq 64\sqrt{3}E_h(0). \tag{A.25}$$

This concludes the proof of Lemma A.2. ■

A.2 Proof of Proposition 4.2

Coming back to the proof of Theorem 4.1 we consider D defined in (A.2). By (A.8) and Young's inequality we have

$$D\leq 2\sum_{i,j=0}^{N}(w_{i+1j}-w_{ij})^2+2\sum_{i,j=0}^{N}(w_{ij+1}-w_{ij})^2-\sum_{j=1}^{N}(N+1)(w_{Nj-1}+w_{Nj}+w_{Nj+1})w_{Nj}$$
$$-\sum_{i=1}^{N}(N+1)\left(w_{i-1N}+w_{iN}+w_{i+1N}\right)w_{iN}. \tag{A.26}$$

Thus, we can estimate (A.2) as follows

$$
\begin{aligned}
0 \quad = \quad & \frac{h^2}{32}C + \frac{1}{6}\int_0^T D dt \le \frac{h^2}{32}\int_0^T \Bigg[2\sum_{i,j=0}^{N} \left(b_{ii+1}^{jj+1}\right)^2 \\
& -\sum_{i=1}^{N}(N+1)\left(w'_{iN}+w'_{i+1N}\right)^2 - \sum_{j=1}^{N}(N+1)\left(w'_{Nj}+w'_{Nj+1}\right)^2 \Bigg] dt \\
& +\frac{1}{3}\int_0^T \left(\sum_{i,j=0}^{N} (w_{i+1j}-w_{ij})^2 + \sum_{i,j=0}^{N}(w_{ij+1}-w_{ij})^2 \right) dt \\
& -\frac{1}{6}\int_0^T \Bigg[\sum_{j=1}^{N}(N+1)(w_{Nj-1}+w_{Nj}+w_{Nj+1})w_{Nj} \\
& \quad + \sum_{i=1}^{N}(N+1)\left(w_{i-1N}+w_{iN}+w_{i+1N}\right)w_{iN} \Bigg] dt + \frac{h^2}{32}[C_2]_0^T,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
E_h(t) \ge \frac{h}{8}\int_0^T & \left[\sum_{i=1}^{N}\left(\frac{w'_{iN}+w'_{i+1N}}{2}\right)^2 + \sum_{j=1}^{N}\left(\frac{w'_{Nj}+w'_{Nj+1}}{2}\right)^2 \right] dt \qquad \text{(A.27)} \\
+\frac{1}{2}\int_0^T & \left[\sum_{j=1}^{N} \frac{w_{Nj-1}+w_{Nj}+w_{Nj+1}}{3}\frac{w_{Nj}}{h} + \sum_{i=1}^{N}\frac{w_{i-1N}+w_{iN}+w_{i+1N}}{3}\frac{w_{iN}}{h}\right] dt - \frac{h^2}{32}[C_2]_0^T.
\end{aligned}
$$

Therefore, it is enough to find an estimate of C_2 in terms of the energy. Note that

$$
\begin{aligned}
C_2 &= \sum_{i,j=1}^{N} m_{ij}\left[b_{ii+1}^{jj+1}+b_{ii+1}^{j-1j}+b_{i-1i}^{jj+1}+b_{i-1i}^{j-1j}\right] \\
&\le \frac{1}{2}\sum_{i,j=1}^{N}\left[i^2(w_{i+1j}-w_{i-1j})^2 + \left(b_{ii+1}^{jj+1}+b_{ii+1}^{j-1j}+b_{i-1i}^{jj+1}+b_{i-1i}^{j-1j}\right)^2\right] \\
&\quad +\frac{1}{2}\sum_{i,j=1}^{N}\left[j^2(w_{ij+1}-w_{ij-1})^2 + \left(b_{ii+1}^{jj+1}+b_{ii+1}^{j-1j}+b_{i-1i}^{jj+1}+b_{i-1i}^{j-1j}\right)^2\right] \\
&\le \frac{1}{2h^2}\sum_{i,j=1}^{N}\left[(w_{i+1j}-w_{ij}+w_{ij}-w_{i-1j})^2+(w_{ij+1}-w_{ij}+w_{ij}-w_{ij-1})^2\right] + 4\sum_{i,j=1}^{N}\left(b_{ii+1}^{jj+1}\right)^2 \\
&\quad \le \frac{2}{h^2}\sum_{i,j=1}^{N}\left[(w_{i+1j}-w_{ij})^2+(w_{ij+1}-w_{ij})^2\right] + 4\sum_{i,j=1}^{N}\left(b_{ii+1}^{jj+1}\right)^2 \le \frac{16}{h^2}E_h(t).
\end{aligned}
$$

Therefore,

$$
\frac{h^2}{32}[C_2]_0^T \le E_h(0).
$$

This estimate combined with (A.27) provide inequality (4.7). ■

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