Relaxation of an optimal design problem in Fracture Mechanic

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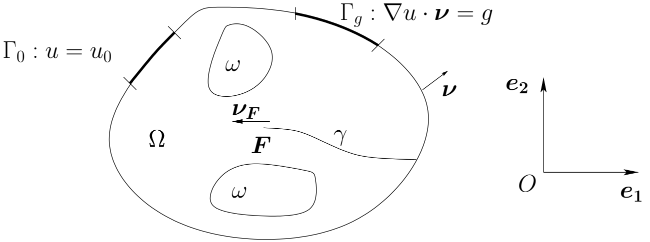

Figure: Crack domain Ω in $\mathbb{R}^2$

$$a_{\mathcal{X}_\omega}(\boldsymbol{x}) = \alpha \mathcal{X}_\omega(\boldsymbol{x}) + \beta(1 - \mathcal{X}_\omega(\boldsymbol{x})), \quad \boldsymbol{x} = (x_1, x_2) \in \Omega \tag{1}$$

$$\begin{cases} -\, div(a_{\mathcal{X}_\omega}(\boldsymbol{x})\nabla u) = 0 & \Omega, \\ u = u_0 & \Gamma_0 \subset \partial\Omega, \\ \beta\ \nabla u \cdot \boldsymbol{\nu} = g & \Gamma_g \subset \partial\Omega \end{cases} \tag{2}$$

Remark

If $g \in H^{1/2}(\partial\Omega)$ *and* $u_0 \in L^2(\partial\Omega) \Longrightarrow u \in H^1(\Omega)$.

1

[1] P. Grisvard, Singularities in boundary value problems, Springer Berlin, 1992

Definition

The energy release rate $G(u, \mathcal{X}_\omega, \boldsymbol{F})$ *is defined as minus the variation of the energy*

$$E(u, \mathcal{X}_\omega, \boldsymbol{F}) = \frac{1}{2}\int_\Omega a_{\mathcal{X}_\omega} |\nabla u|^2 dx - \int_{\Gamma_g} gud\sigma \quad (3)$$

with respect to the variation of $\boldsymbol{F}$ *(in the direction* e_1*).*

Criterion (Griffith, 1920)[2]

If $G(\boldsymbol{u}, \mathcal{X}_\omega, \boldsymbol{F}) \geq G_c$ *then* $\boldsymbol{F}$ *grows.*

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We assume that, in the neighborhood of $\boldsymbol{F}$, the crack γ is rectilinear and (without loss of generalities) oriented along $\boldsymbol{e_1}$. We introduce any velocity field $\boldsymbol{\psi} = (\psi_1, \psi_2) \in W \equiv \{\boldsymbol{\psi} \in (W^{1,\infty}(\Omega, \mathbb{R}))^2, \boldsymbol{\psi} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega/\gamma\}$, where $\boldsymbol{\nu}$ designates the unit outward normal to Ω. Moreover, we assume that the support of the function $\boldsymbol{\psi}$ is disjoint from the support Γ_g of the load. Let $\eta > 0$ and the transformation $\mathcal{F}^\eta : \boldsymbol{x} \rightarrow \boldsymbol{x} + \eta\boldsymbol{\psi}(\boldsymbol{x})$ so that $\mathcal{F}^\eta(\boldsymbol{F}) = \boldsymbol{F^\eta}$ and $\mathcal{F}^\eta(\gamma) = \gamma^\eta$; we first recall the following definition [3].

Definition (Energy release rate)

Let u be the solution of (2). The derivative of the functional $-E(u, \gamma)$ with respect to a variation of γ (precisely $\boldsymbol{F}$) in the direction $\boldsymbol{\psi}$ is defined as the Fréchet derivative in W at 0 of the application $\eta \rightarrow -E(u, (Id + \eta\boldsymbol{\psi})(\gamma))$, i.e.

$$E(u, (Id + \eta\boldsymbol{\psi})(\gamma)) = E(u, \gamma) - \eta \frac{\partial E(u, \gamma)}{\partial \gamma}.\boldsymbol{\psi} + o(\eta^2). \tag{4}$$

In the sequel, we denote this derivative by $g_{\boldsymbol{\psi}}(u, \mathcal{X}_\omega)$. ■

[3] H.D. Bui, Mécanique de la rupture fragile, Masson, 1978

Lemma

The first derivative of $-E$ with respect to γ in the direction $\boldsymbol{\psi} = (\psi_1, \psi_2) \in W$ is given by

$$\begin{aligned} g_{\boldsymbol{\psi}}(u, \mathcal{X}_\omega) &= \int_\Omega a_{\mathcal{X}_\omega}(\boldsymbol{x})\nabla u \cdot (\nabla \boldsymbol{\psi} \cdot \nabla u)dx - \frac{1}{2}\int_\Omega a_{\mathcal{X}_\omega}(\boldsymbol{x})|\nabla u|^2 div(\boldsymbol{\psi})dx \\ &= \int_\Omega a_{\mathcal{X}_\omega}(\boldsymbol{x})(A_{\boldsymbol{\psi}}(\boldsymbol{x})\nabla u, \nabla u)dx \end{aligned} \tag{5}$$

with

$$\begin{aligned} A_{\boldsymbol{\psi}}(\boldsymbol{x}) &= \nabla \boldsymbol{\psi} - \frac{1}{2}div(\boldsymbol{\psi})I_2 = \nabla \boldsymbol{\psi} - \frac{1}{2}Tr(\nabla \boldsymbol{\psi})I_2 \\ &= \frac{1}{2}\begin{pmatrix} \psi_{1,1} - \psi_{2,2} & 2\psi_{1,2} \\ 2\psi_{2,1} & \psi_{2,2} - \psi_{1,1} \end{pmatrix}. \end{aligned} \tag{6}$$

Remark

Since $\boldsymbol{F}$ moves along e_1, we can take $\psi_2 = 0$ so that

$$\begin{aligned} A_{\boldsymbol{\psi}}(\boldsymbol{x}) &= \nabla \boldsymbol{\psi} - \frac{1}{2}div(\boldsymbol{\psi})I_2 = \nabla \boldsymbol{\psi} - \frac{1}{2}Tr(\nabla \boldsymbol{\psi})I_2 \\ &= \frac{1}{2}\begin{pmatrix} \psi_{1,1} & 2\psi_{1,2} \\ 0 & -\psi_{1,1} \end{pmatrix}. \end{aligned} \tag{7}$$

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Lemma ((Local) Energy release rate)

Let us assume the isotropic situation ($\alpha = \beta$). Let $C(\boldsymbol{F}, r)$ be the circle of center $\boldsymbol{F}$ and radius $r > 0$, $\boldsymbol{\nu_c} = (\nu_{c,1}, \nu_{c,2})$ its outward normal and

$$G_r(u, \mathcal{X}_\omega) = \frac{1}{2}\int_{C(\boldsymbol{F},r)} a_{\mathcal{X}_\omega}(\boldsymbol{x}) u_{i,j} u_{j,i} \psi_k \nu_{c,k} d\sigma - \int_{C(\boldsymbol{F},r)} a_{\mathcal{X}_\omega}(\boldsymbol{x}) u_{k,j} u_{j,i} \psi_i \nu_{c,k} d\sigma.$$

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$$g_\psi(u, \mathcal{X}_\omega) = \lim_{r \to 0} G_r(u, \mathcal{X}_\omega)\, (\psi \cdot \boldsymbol{\nu})_{|\boldsymbol{F}} \equiv G(u, \mathcal{X}_\omega)\, \boldsymbol{\psi(F)} \cdot \boldsymbol{\nu_F}, \quad \forall \psi \in W, \tag{8}$$

where $\boldsymbol{\nu_F} = (\nu_{F,1}, \nu_{F,2})$ designates the orientation of the crack γ at the point $\boldsymbol{F}$. ∎

When $\alpha \neq \beta$, this invariance is true if and only if the function ψ is such that $\{\boldsymbol{x} \in \Omega, \psi(\boldsymbol{x}) \neq 0\} \cap \partial\omega = \emptyset$: this simply requires to have a uniform material (α or β) in a neighborhood, say $\mathcal{D} \subset \Omega$, of $\boldsymbol{F}$, so that $\overline{\omega} \cap \mathcal{D} = \emptyset$ and $\{\boldsymbol{x} \in \Omega, \psi(\boldsymbol{x}) \neq 0\} \subset \mathcal{D}$:

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Non diagonal case: $\psi_1 = \psi_1(x_1, x_2)$

$$\psi_1(\boldsymbol{x}) = \zeta(\mathrm{dist}(\boldsymbol{x}, \boldsymbol{F}))\nu_{F,1}, \quad \forall \boldsymbol{x} \in \Omega \tag{11}$$

defining the function $\zeta \in C^1(\mathbb{R}^+; [0,1])$ as follows:

$$\zeta(r) = \begin{cases} 1 & r \leq r_1 \\ \dfrac{(r - r_2)^2(3r_1 - r_2 - 2r)}{(r_1 - r_2)^3} & r_1 \leq r \leq r_2 \\ 0 & r \geq r_2 \end{cases} \tag{12}$$

with $0 < r_1 < r_2 < \mathrm{dist}(\partial\Omega/\gamma, \boldsymbol{F}) = \inf_{\boldsymbol{x} \in \partial\Omega/\gamma} \mathrm{dist}(\boldsymbol{x}, \boldsymbol{F})$.

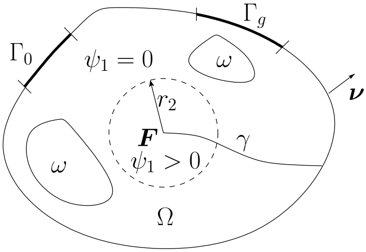


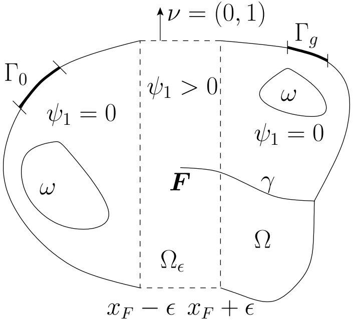
Figure: Choice of a radial function $\psi_1(\boldsymbol{x})$ leading to a non diagonal matrix $A_{\boldsymbol{\psi}}$.

$$A_{\boldsymbol{\psi}}(\boldsymbol{x}) = \frac{1}{2}\begin{pmatrix} \psi_{1,1} & 2\psi_{1,2} \\ 0 & -\psi_{1,1} \end{pmatrix}. \tag{13}$$

$$\zeta(x_1) = \begin{cases} 0 & x_1 \leq r_1 \\ \frac{(x_1 - r_1)^2(2x_1 + r_1 - 3r_2)}{r_1 - r_2} & r_1 \leq x_1 \leq r_2, \\ 1 & r_2 \leq x_1 \leq r_3, \\ \frac{(x_1 - r_4)^2(2x_1 + r_4 - 3r_3)}{r_4 - r_3} & r_3 \leq x_1 \leq r_4, \\ 0 & x_1 \geq r_4 \end{cases} \quad (14)$$

with

$$r_1 = x_F - \frac{2\epsilon}{3}, \quad r_2 = x_F - \frac{\epsilon}{3}, \quad r_3 = x_F + \frac{\epsilon}{3}, \quad r_4 = x_F + \frac{2\epsilon}{3}. \quad (15)$$



$$A_\psi(\boldsymbol{x}) = \frac{1}{2}\begin{pmatrix} \psi_{1,1} & 0 \\ 0 & -\psi_{1,1} \end{pmatrix}. \quad (16)$$

Figure: Choice of a function $\psi_1(\boldsymbol{x}) = \psi_1(x_1)\mathcal{X}_{\Omega_\epsilon}$ leading to a diagonal matrix A_ψ assuming the existence of a domain Ω_ϵ.

We introduce the linear manifolds $\Lambda_\gamma = \{(\lambda,\rho) \in \mathbb{R}^2 \times \mathbb{R}^2 : \rho = \gamma\lambda\}$ and

$$W(\boldsymbol{x},\rho,\lambda) = \begin{cases} \alpha A_{\psi}(\boldsymbol{x}) \otimes \lambda\lambda^T & \text{if} \quad (\rho,\lambda) \in \Lambda_\alpha, \\ \beta A_{\psi}(\boldsymbol{x}) \otimes \lambda\lambda^T & \text{if} \quad (\rho,\lambda) \in \Lambda_\beta, \\ +\infty & \text{else}, \end{cases} \tag{17}$$

and

$$V(\rho,\lambda) = \begin{cases} 1 & \text{if} \quad (\rho,\lambda) \in \Lambda_\alpha, \\ 0 & \text{if} \quad (\rho,\lambda) \in \Lambda_\beta, \\ +\infty & \text{else}. \end{cases} \tag{18}$$

Then we check that (P) is equivalent to the following new problem

$$(VP): \qquad \inf_{G,u} \int_\Omega W(\boldsymbol{x}, G(\boldsymbol{x}), \nabla u(\boldsymbol{x}))dx \tag{19}$$

subject to

$$\begin{cases} G \in L^2(\Omega;\mathbb{R}^2),\ div\ G = 0 \text{ in } H^{-1}(\Omega,\mathbb{R}),\ \boldsymbol{G(x) = \beta\nabla u(x)} \textbf{ in } \boldsymbol{\mathcal{D} \cup \partial\Omega}, \\ u \in H^1(\Omega;\mathbb{R}),\ u = u_0 \text{ on } \Gamma_0,\ \beta\nabla u \cdot \boldsymbol{\nu} = g \text{ on } \Gamma_g \subset \partial\Omega\backslash(\gamma \cup \Gamma_0), \\ \int_\Omega V(G(\boldsymbol{x}), \nabla u(\boldsymbol{x}))dx = L|\Omega|. \end{cases} \tag{20}$$

$$(RP): \qquad \min_{s,G,u} \int_\Omega CQW(\boldsymbol{x}, s(\boldsymbol{x}), G(\boldsymbol{x}), \nabla u(\boldsymbol{x}))dx \tag{21}$$

for s such that

$$s \in L^\infty(\Omega, [0,1]), \ \boldsymbol{s = 0} \text{ in } \boldsymbol{\mathcal{D} \cup \partial\Omega}, \int_\Omega s(\boldsymbol{x})dx = L|\Omega|. \tag{22}$$

The constrained quasi-convex density CQW is computed by solving the problem in measures :

$$\begin{aligned} &CQW(\boldsymbol{x}, s(\boldsymbol{x}), G(\boldsymbol{x}), \nabla u(\boldsymbol{x})) \\ &= \inf_\nu \Big\{ \alpha s(\boldsymbol{x}) A_\psi(\boldsymbol{x}) \otimes \int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_{x,\alpha}(\lambda) + \beta(1 - s(\boldsymbol{x})) A_\psi(\boldsymbol{x}) \otimes \int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_{x,\beta}(\lambda) \Big\} \end{aligned} \tag{23}$$

for any measure ν subject to

$$\begin{cases} \nu = \{\nu_x\}_{x\in\Omega}, \quad \nu_x = s(\boldsymbol{x})\nu_{x,\alpha} + (1 - s(\boldsymbol{x}))\nu_{x,\beta}, \ \text{supp}(\nu_{x,\gamma}) \subset \Lambda_\gamma, \\ \nu \quad \text{is div-curl Young measure}, \\ G(\boldsymbol{x}) = \int_{\mathbb{R}^2} \rho d\nu_x(\lambda, \rho), \quad div\ G = 0 \text{ weakly in } \Omega, \\ \nabla u(\boldsymbol{x}) = \int_{\mathbb{R}^2} \lambda d\nu_x(\lambda, \rho). \end{cases} \tag{24}$$

Concerning the first moment of ν, we may write

$$(\lambda,\rho)=\int_\Lambda (x,y)d\nu(x,y)=s\int_{\mathbb{R}^2}(x,\alpha x)d\nu_\alpha^{(1)}(x)+(1-s)\int_{\mathbb{R}^2}(x,\beta x)d\nu_\beta^{(1)}(x) \quad (25)$$

where $\nu_\gamma^{(1)}$ is the projection of ν_γ onto the first copy of $\mathbb{R}^2$ of the product $\mathbb{R}^2\times\mathbb{R}^2$. By introducing

$$\lambda_\gamma=\int_{\mathbb{R}^2} x d\nu_\gamma^{(1)}(x), \quad (26)$$

we have $\lambda=s\lambda_\alpha+(1-s)\lambda_\beta$, $\rho=s\alpha\lambda_\alpha+(1-s)\beta\lambda_\beta$, and then

$$\lambda_\alpha=\frac{1}{s(\beta-\alpha)}(\beta\lambda-\rho),\quad \lambda_\beta=\frac{1}{(1-s)(\beta-\alpha)}(\rho-\alpha\lambda). \quad (27)$$

Moreover, the commutation with the inner product yields the relation

$$\lambda^T\rho=\int_\Lambda x^Ty d\nu(x,y)=\alpha s\int_{\mathbb{R}^2}x^Txd\nu_\alpha^{(1)}(x)+\beta(1-s)\int_{\mathbb{R}^2}x^Txd\nu_\beta^{(1)}(x). \quad (28)$$

To find a lower bound of *CQW*, we retain just the relevant property expressed in the commutation (28), so that we regard feasible measures ν as Young measures which satisfy this commutation property, but are not necessarily a div-curl Young measure. We introduce

$$\boldsymbol{X_\gamma=\int_{\mathbb{R}^2}xx^Td\nu_\gamma^{(1)}(x),\quad \gamma=\alpha,\beta} \quad (29)$$

a convex combination of symmetric rank-one matrices. It is well-known that

$$\boldsymbol{X_\gamma\geq\lambda_\gamma\lambda_\gamma^T,\quad \gamma=\alpha,\beta} \quad (30)$$

in the usual sense of symmetric matrices, i.e. that $X_\gamma-\lambda_\gamma\lambda_\gamma^T$ is semi-definite positive. The relation (28) becomes

$$\lambda^T\rho=\lambda\cdot\rho=\alpha sTr(X_\alpha)+\beta(1-s)Tr(X_\beta). \quad (31)$$

Similarly, the cost may be written in term of the variable X_γ as follows :

$$s\alpha A_\psi \otimes X_\alpha + (1-s)\beta A_\psi \otimes X_\beta = s\alpha Tr(A_\psi X_\alpha) + (1-s)\beta Tr(A_\psi X_\beta) \tag{32}$$

from the relation $A_\psi \otimes X_\gamma = Tr(A_\psi X_\gamma)$, $\gamma = \alpha, \beta$. Consequently, in seeking a lower bound of the constrained quasiconvexification, we are led to consider the mathematical programming problem

$$\min_{X_\alpha, X_\beta} C(X_\alpha, X_\beta) = \alpha s Tr(A_\psi X_\alpha) + \beta(1-s) Tr(A_\psi X_\beta) \tag{33}$$

subject to the constraints

$$\lambda^T \rho = \lambda \cdot \rho = \alpha s Tr(X_\alpha) + \beta(1-s) Tr(X_\beta), \quad X_\gamma \geq \lambda_\gamma \lambda_\gamma^T. \tag{34}$$

We first realize that the set of vectors for which the constraints yield a non-empty set takes place if

$$\alpha s Tr(\lambda_\alpha \lambda_\alpha^T) + \beta(1-s) Tr(\lambda_\beta \lambda_\beta^T) \leq \lambda \cdot \rho \tag{35}$$

i.e. if

$$\begin{aligned} B(\rho, \lambda) &\equiv \lambda \cdot \rho - \alpha s |\lambda_\alpha|^2 - \beta(1-s)|\lambda_\beta|^2 \geq 0 \\ &= (\lambda_\beta - \lambda_\alpha) \cdot (\beta \lambda_\beta - \alpha \lambda_\alpha) \end{aligned} \tag{36}$$

using that $Tr(\lambda \rho^T) = \lambda \cdot \rho$.

Proposition (Non diagonal case)

For any $s \in L^\infty(\Omega)$ and $(\lambda, \rho) = (\nabla u, G)$ satisfying all the constraints,

$$m(s,\lambda,\rho) = \begin{cases} \frac{1}{2}\Big[-\sqrt{\psi_{1,1}^2+\psi_{1,2}^2}(\rho\cdot\lambda - \alpha s|\lambda_\alpha|^2 - \beta(1-s)|\lambda_\beta|^2) \\ \quad + \psi_{1,1}(\alpha s\lambda_{\alpha,1}^2 + (1-s)\beta\lambda_{\beta,1}^2) - \psi_{1,1}(\alpha s\lambda_{\alpha,2}^2 + (1-s)\beta\lambda_{\beta,2}^2) \\ \quad + 2\psi_{1,2}(\alpha s\lambda_{\alpha,1}\lambda_{\alpha,2} + (1-s)\beta\lambda_{\beta,1}\lambda_{\beta,2})\Big] & \text{if } \mathbf{B}(\rho,\lambda)\geq 0 \\ +\infty & \text{else} \end{cases} \quad (37)$$

is a lower bound for the constrained quasi-convexified CQW of W:

$$m(s,\lambda,\rho) \leq CQW(s,\lambda,\rho). \quad (38)$$

$\lambda_\gamma = \lambda_\gamma(s,\lambda,\rho)$, $\gamma = \alpha,\beta$ are defined by (27). ■

We note

$$A_\psi = \frac{1}{2}\begin{pmatrix} \psi_{1,1} & 2\psi_{1,2} \\ 0 & -\psi_{1,1} \end{pmatrix} \equiv \begin{pmatrix} a & 2b \\ 0 & -a \end{pmatrix} \tag{39}$$

and made the change of variables $\boldsymbol{Y_\gamma = X_\gamma - \lambda_\gamma \lambda_\gamma^T}$ so that the cost and the constraints are transformed into

$$\min_{Y_{\gamma,11},Y_{\gamma,22},Y_{\gamma,12}} \alpha s(a(Y_{\alpha,11}-Y_{\alpha,22})+2bY_{\alpha,12})+\beta(1-s)((a(Y_{\beta,11}-Y_{\beta,22})+2bY_{\beta,12}))+A \tag{40}$$

and

$$\begin{cases} s\alpha(Y_{\alpha,11}+Y_{\alpha,22})+(1-s)\beta(Y_{\beta,11}+Y_{\beta,22}) = \boldsymbol{B}, \\ Y_{\gamma,11}+Y_{\gamma,22}\geq 0, \quad Y_{\gamma,11}Y_{\gamma,22}\geq Y^2_{\gamma,12} \quad \gamma=\alpha,\beta \end{cases} \tag{41}$$

where the constant A is defined by

$$A = \alpha s(a(\lambda^2_{\alpha,1}-\lambda^2_{\alpha,2})+2b\lambda_{\alpha,1}\lambda_{\alpha,2})+\beta(1-s)((a(\lambda^2_{\beta,1}-\lambda^2_{\beta,2})+2b\lambda_{\beta,1}\lambda_{\beta,2})). \tag{42}$$

The minimum of the linear cost is reached on the boundary of the convex sets

$$\Gamma_\gamma = \left\{(Y_{\gamma,11},Y_{\gamma,22},Y_{\gamma,12})\in\mathbb{R}^3,\, Y_{\gamma,11}\geq 0,\, Y_{\gamma,22}\geq 0,\, Y_{\gamma,11}Y_{\gamma,22}\geq Y^2_{\gamma,12}\right\}, \quad \gamma=\alpha,\beta \tag{43}$$

which implies $Y_{\gamma,11}Y_{\gamma,22}=Y^2_{\gamma,12}$. Therefore, we can introduce the new variables $Z_\gamma \equiv (Z_{\gamma,11},Z_{\gamma,22})^T$ so that $Y_{\gamma,11}=Z^2_{\gamma,11}$, $Y_{\gamma,22}=Z^2_{\gamma,22}$ and $\epsilon_\gamma=\pm 1$ and then $Z_{\gamma,11}Z_{\gamma,22}=\epsilon_\gamma Y_{\gamma,12}$ reducing the problem to

$$\begin{aligned}\min_{Z_{\gamma,11},Z_{\gamma,22},\epsilon_\gamma} C(Z_\gamma,\epsilon_\gamma) = \alpha s(a(Z^2_{\alpha,11}-Z^2_{\alpha,22})+2b\epsilon_\alpha Z_{\alpha,11}Z_{\alpha,22}) \\ +\beta(1-s)((a(Z^2_{\beta,11}-Z^2_{\beta,22})+2b\epsilon_\beta Z_{\beta,11}Z_{\beta,22}))+A\end{aligned} \tag{44}$$

under the constraint

$$s\alpha(Z^2_{\alpha,11}+Z^2_{\alpha,22})+(1-s)\beta(Z^2_{\beta,11}+Z^2_{\beta,22}) = B. \tag{45}$$

We note

$$A_\psi = \frac{1}{2}\begin{pmatrix} \psi_{1,1} & 2\psi_{1,2} \\ 0 & -\psi_{1,1} \end{pmatrix} \equiv \begin{pmatrix} a & 2b \\ 0 & -a \end{pmatrix} \tag{39}$$

and made the change of variables $\boldsymbol{Y_\gamma = X_\gamma - \lambda_\gamma \lambda_\gamma^T}$ so that the cost and the constraints are transformed into

$$\min_{Y_{\gamma,11}, Y_{\gamma,22}, Y_{\gamma,12}} \alpha s(a(Y_{\alpha,11} - Y_{\alpha,22}) + 2bY_{\alpha,12}) + \beta(1-s)((a(Y_{\beta,11} - Y_{\beta,22}) + 2bY_{\beta,12})) + A \tag{40}$$

and

$$\begin{cases} s\alpha(Y_{\alpha,11} + Y_{\alpha,22}) + (1-s)\beta(Y_{\beta,11} + Y_{\beta,22}) = \boldsymbol{B}, \\ Y_{\gamma,11} + Y_{\gamma,22} \geq 0, \quad Y_{\gamma,11}Y_{\gamma,22} \geq Y^2_{\gamma,12} \quad \gamma = \alpha, \beta \end{cases} \tag{41}$$

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$$\begin{aligned} \min_{Z_{\gamma,11}, Z_{\gamma,22}, \epsilon_\gamma} C(Z_\gamma, \epsilon_\gamma) = \alpha s(a(Z^2_{\alpha,11} - Z^2_{\alpha,22}) + 2b\epsilon_\alpha Z_{\alpha,11}Z_{\alpha,22}) \\ + \beta(1-s)((a(Z^2_{\beta,11} - Z^2_{\beta,22}) + 2b\epsilon_\beta Z_{\beta,11}Z_{\beta,22})) + A \end{aligned} \tag{44}$$

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under the constraint

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Introducing the Lagrangian L and the multiplier p

$$L(Z_\gamma, p) = C(Z_\gamma, \epsilon_\gamma) - p\Big(s\alpha(Z^2_{\alpha,11} + Z^2_{\alpha,22}) + (1-s)\beta(Z^2_{\beta,11} + Z^2_{\beta,22}) - \mathbf{B}\Big), \tag{46}$$

we arrive at the optimality conditions :

$$A_{\psi,\epsilon_\gamma} Z_\gamma = pZ_\gamma, \qquad A_{\psi,\epsilon_\gamma} = \begin{pmatrix} a & b\epsilon_\gamma \\ b\epsilon_\gamma & -a \end{pmatrix}. \tag{47}$$

The resolution of a spectral problem leads to

$$p = -\sqrt{a^2+b^2}, \quad Z_\gamma = a_\gamma\Big(b\epsilon_\gamma, -(a+\sqrt{a^2+b^2})\Big)^T \tag{48}$$

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for any $a_\gamma \in \mathbb{R}^*$. Now, writing that $a(Z^2_{\gamma,11} - Z^2_{\gamma,22}) + 2b\epsilon_\gamma Z_{\gamma,11}Z_{\gamma,22} = A_{\psi,\epsilon_\gamma} Z_\gamma \cdot Z_\gamma$, we may write from (47) that

$$\begin{aligned} C(Z_\gamma, \epsilon_\gamma) =& \alpha s A_{\psi,\epsilon_\alpha} Z_\alpha \cdot Z_\alpha + \beta(1-s)A_{\psi,\epsilon_\gamma} Z_\beta \cdot Z_\beta + A \\ =& p(\alpha s|Z_\alpha|^2 + \beta(1-s)|Z_\beta|^2) + A \\ =& p\mathbf{B} + A \end{aligned} \tag{50}$$

Therefore, the cost, independent of ϵ_γ is obtained for the lowest eigenvalue (independent here of the sign of a) :

$$\min C(Z_\gamma, \epsilon_\gamma) = -\sqrt{a^2+b^2}\mathbf{B} + A \tag{51}$$

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$$(a^2_\alpha s\alpha + a^2_\beta(1-s)\beta)(b^2 + (a+\sqrt{a^2+b^2})^2) = \mathbf{B}. \tag{52}$$

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$$\begin{aligned} C(Z_\gamma, \epsilon_\gamma) =& \alpha s A_{\psi,\epsilon_\alpha} Z_\alpha \cdot Z_\alpha + \beta(1-s) A_{\psi,\epsilon_\gamma} Z_\beta \cdot Z_\beta + A \\ =& p(\alpha s |Z_\alpha|^2 + \beta(1-s)|Z_\beta|^2) + A \\ =& p\boldsymbol{B} + A \end{aligned} \tag{50}$$

Therefore, the cost, independent of ϵ_γ is obtained for the lowest eigenvalue (independent here of the sign of a) :

$$\min C(Z_\gamma, \epsilon_\gamma) = -\sqrt{a^2+b^2}\boldsymbol{B} + A \tag{51}$$

for $Z_\gamma = a_\gamma(b\epsilon_\gamma, -(a + \sqrt{a^2+b^2}))^T$. The constraint (45) then gives the relation

$$(a^2_\alpha s\alpha + a^2_\beta(1-s)\beta)(b^2 + (a + \sqrt{a^2+b^2})^2) = \boldsymbol{B}. \tag{52}$$

According to the previous computation, the optimal second moment are of the form

$$X_\gamma = \lambda_\gamma \lambda_\gamma^T + a_\gamma^2 \begin{pmatrix} \psi_{1,2}^2 & -\psi_{1,2}(\psi_{1,1} + \sqrt{\psi_{1,1}^2 + \psi_{1,2}^2}) \\ -\psi_{1,2}(\psi_{1,1} + \sqrt{\psi_{1,1}^2 + \psi_{1,2}^2}) & (\psi_{1,1} + \sqrt{\psi_{1,1}^2 + \psi_{1,2}^2})^2 \end{pmatrix} \tag{53}$$

leading to the cost $-\sqrt{\psi_{1,1}^2 + \psi_{1,2}^2}\boldsymbol{B} + A$. But, on $\Omega/\mathcal{D}$, the radial function ψ is zero so that,

$$\boldsymbol{X_\gamma = \lambda_\gamma \lambda_\gamma^T, \quad x \in \Omega/\mathcal{D}} \tag{54}$$

i.e. in particular

$$X_{\gamma,ii} = \int_{\mathbb{R}} x_i^2 d\nu_\gamma^{(1,i)}(x_i) = \left(\int_{\mathbb{R}} x_i d\nu_\gamma^{1,i}(x_i)\right)^2 = (\lambda_{\gamma,i})^2, \quad i = 1,2 \tag{55}$$

where $\nu_\gamma^{(1,i)}$ denotes the projection of $\nu^{(1)}$ onto the i-th copy of $\mathbb{R}^2$. From the strict convexity of the square function, this implies that $\nu_\gamma^{(1,i)} = \delta_{\lambda_{\gamma,i}}$, i.e.

$$\nu_\alpha^{(1,i)} = \delta_{\frac{\beta\lambda_i - \rho_i}{s(\beta-\alpha)}}, \quad \nu_\beta^{(1,i)} = \delta_{\frac{\rho_i - \alpha\lambda_i}{(1-s)(\beta-\alpha)}}. \tag{56}$$

Remark that this is compatible with the third equality $X_{\gamma,12} = \lambda_{\gamma,1}\lambda_{\gamma,2}^T$. This also implies (see for instance (52)) the equality in (35), i.e. that

$$\boldsymbol{B} = \lambda \cdot \rho - \alpha s|\lambda_\alpha|^2 - \beta(1-s)|\lambda_\beta|^2 = 0. \tag{57}$$

Consequently, the optimal value $m(s, \lambda, \rho)$ may be recovered by the following measure

$$\boldsymbol{\nu = s\delta_{(\alpha\lambda_\alpha, \lambda_\alpha)} + (1-s)\delta_{(\beta\lambda_\beta, \lambda_\beta)}} \tag{58}$$

which is a first order (div-curl) laminate, the div-curl condition $(\beta\lambda_\beta - \alpha\lambda_\alpha)\cdot(\lambda_\beta - \lambda_\alpha) = 0$ (analogous to a rank one condition for H^1 −gradient Young measure) being equivalent precisely to $\boldsymbol{B} = 0$.

Theorem

The variational problem

$$(RP): \qquad \min_{s,u,G} \int_\Omega m(s, \nabla u, G) dx \tag{59}$$

subject to

$$\begin{cases} s \in L^\infty(\Omega, [0,1]),\ s = 0 \ \textit{in}\ \mathcal{D} \cup \partial\Omega,\ \int_\Omega s(\boldsymbol{x}) dx = L|\Omega|, \\ u \in H^1(\Omega), \quad u = u_0 \ \textit{on}\ \Gamma_0, \quad \beta \nabla u \cdot \boldsymbol{\nu} = g \ \textit{on}\ \Gamma_g, \\ G \in (L^2(\Omega))^2, \quad \mathit{div}\ G = 0 \ \textit{weakly in}\ \Omega, \end{cases} \tag{60}$$

where m is defined by (37) is a relaxation of (VP) in the sense that the minimum of (RP) exists and equals the minimum of (VP). Moreover, the underlying Young measure associated with (RP) can be found in the form of a first order laminate whose direction of lamination are given explicitly in terms of the optimal solution (u, G): precisely, the normal are orthogonal to $\lambda_\beta - \lambda_\alpha$.

Simplified conclusion

The above formulation may be simplified by taking into account that $\boldsymbol{B} = 0$. Precisely, we use (27) to express $\boldsymbol{B} = (\beta\lambda_\beta - \alpha\lambda_\alpha) \cdot (\lambda_\beta - \lambda_\alpha) = 0$ as follows

$$(\rho - \lambda^-(s)\lambda) \cdot (\rho - \lambda^+(s)\lambda) = 0 \tag{61}$$

in terms of the harmonic and arithmetic mean of α, β with weight s.

Theorem

The variational problem

$$(\overline{RP}) : \qquad \min_{s,u,G} \int_\Omega F(s, \nabla u, G)dx \tag{62}$$

subject to

$$\begin{cases} s \in L^\infty(\Omega, [0,1]), s = 0 \text{ in } \mathcal{D} \cup \partial\Omega, \int_\Omega s(\boldsymbol{x})dx = L|\Omega|, \\ u \in H^1(\Omega), \quad u = u_0 \text{ on } \Gamma_0, \quad \beta\nabla u \cdot \boldsymbol{\nu} = g \text{ on } \Gamma_g, \\ G \in (L^2(\Omega))^2, \quad div\ G = 0 \text{ weakly in } \Omega, \\ (G - \lambda^-(s)\nabla u) \cdot (G - \lambda^+(s)\nabla u) = 0 \text{ in } L^2(\Omega), \end{cases} \tag{63}$$

where F, deduced from m, is defined

$$\begin{aligned} F(s, \lambda, \rho) = \frac{1}{2} \Big[& \psi_{1,1}(\alpha s \lambda_{\alpha,1}^2 + (1-s)\beta\lambda_{\beta,1}^2) - \psi_{1,1}(\alpha s \lambda_{\alpha,2}^2 + (1-s)\beta\lambda_{\beta,2}^2) \\ & + 2\psi_{1,2}(\alpha s \lambda_{\alpha,1}\lambda_{\alpha,2} + (1-s)\beta\lambda_{\beta,1}\lambda_{\beta,2}) \Big] \end{aligned} \tag{64}$$

is a relaxation of (VP) in the sense that the minimum of $(\overline{RP})$ exists and equals the minimum of (VP). ■

Following [5] we remark that $\boldsymbol{B} = 0$ is equivalent to

$$\left|\rho - \frac{\lambda^+(s) + \lambda^-(s)}{2}\lambda\right|^2 = \left(\frac{\lambda^+(s) - \lambda^-(s)}{2}\right)^2 |\lambda|^2. \tag{65}$$

Therefore, by introducing the additional variable $t(\boldsymbol{x}) \in \mathbb{R}^2$ such that $|t| = 1$, we may write $\rho = G(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$ under the form (we use that $\lambda^-(s) \leq \lambda^+(s)$ for all $s \in (0, 1)$)

$$\rho = \underbrace{\frac{\lambda^+(s) + \lambda^-(s)}{2}}_{\equiv A(s)} \lambda + \underbrace{\frac{\lambda^+(s) - \lambda^-(s)}{2}}_{\equiv C(s)} |\lambda| t \equiv \phi(s, t, \lambda). \tag{66}$$

We have

$$A(s) = \frac{2\alpha\beta + s(1-s)(\beta-\alpha)^2}{2(\alpha(1-s)+\beta s)}, \qquad C(s) = \frac{s(1-s)(\beta-\alpha)^2}{2(\alpha(1-s)+\beta s)}. \tag{67}$$

The relation $div\ G = 0$ then permits to recover u as the solution of a *nonlinear* equation under a divergence form (having in mind that $\lambda = \nabla u$):

$$\begin{cases} div(A(s)\nabla u + C(s)|\nabla u| t) = 0, & \text{in } \Omega, \\ u = u_0, & \text{on } \Gamma_0, \\ \beta \nabla u \cdot \boldsymbol{\nu} = g, & \text{on } \Gamma_g. \end{cases} \tag{68}$$

We assume that this problem is well-posed in $H^1(\Omega)$.

[5] P. Pedregal, *Div-Curl Young measures and optimal design in any dimension*, Rev. Mat. Complut., (2007).

A transformation

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Theorem

Let F and ϕ be defined respectively by (64) and (66). The following formulation

$$(\underline{RP}) : \qquad \min_{s,t} I(s,t) = \int_\Omega F(s, \nabla u, \phi(s,t,\nabla u))dx \tag{70}$$

subject to the constraints

$$\begin{cases} s \in L^\infty(\Omega, [0,1]), s = 0 \textit{ in } \mathcal{D} \cup \partial\Omega, \int_\Omega s(\boldsymbol{x})dx = L|\Omega|, \\ t \in L^\infty(\Omega, \mathbb{R}^2), \quad |t| = 1, \\ u \in H^1(\Omega), \quad u = u_0 \textit{ on } \Gamma_0, \quad \beta\nabla u \cdot \boldsymbol{\nu} = g \textit{ on } \Gamma_g, \\ div\ \phi(s,t,\nabla u) = 0 \textit{ weakly in } \Omega \end{cases} \tag{71}$$

is equivalent to the relaxation (RP). In particular, $(\underline{RP})$ is a full well-posed relaxation of (VP). ■

Remark

Since $s = 0$ in $\mathcal{D}$, $\int_\Omega F(s, \nabla u, \phi(s,t,\nabla u))dx = \int_\Omega \beta(A_\psi \nabla u, \nabla u)dx$.

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Theorem

The first variation of I with respect to s and t in the direction δs and δt exist and are given respectively by

$$\frac{dI(s,t,u,p)}{ds}\cdot\delta s = \int_\Omega F_{,s}(s,\nabla u,\phi(s,t,\nabla u))\cdot\delta s\, dx + \int_\Omega \Big(A_{,s}(s)\nabla u\cdot\nabla p + B_{,s}(s)|\nabla u| t\cdot\nabla p\Big)\cdot\delta s\, dx \tag{72}$$

and

$$\frac{dI(s,t,u,p)}{dt}\cdot\delta t = \int_\Omega F_{,t}(s,\nabla u,\phi(s,t,\nabla u))\cdot\delta t\, dx + \int_\Omega B(s)|\nabla u|\delta t\cdot\nabla p\, dx \tag{73}$$

where $p\in H^1_{\Gamma_0}(\Omega)=\{v\in H^1(\Omega), v=0 \text{ on } \Gamma_0\}$ solves the adjoint problem

$$\int_\Omega F_{,u}(s,\nabla u,\phi(s,t,\nabla u))\cdot v\, dx + \int_\Omega \Big(A(s)\nabla v\cdot\nabla p + B(s)\frac{\nabla u\cdot\nabla v}{|\nabla u|}t\cdot\nabla p\Big) dx = 0, \tag{74}$$

for all v in $H^1_{\Gamma_0}(\Omega)$. $A_{,s}$ and $B_{,s}$ denote the partial derivative of A and B with respect to s and $F_{,t}$ the partial derivative of F with respect to t. ■

At each iteration k, the solution u of the variational formulation

$$\int_\Omega \left(A(s^{(k)}) \nabla u \cdot \nabla v + B(s^{(k)}) |\nabla u| t^{(k)} \cdot \nabla v \right) dx = \int_{\Gamma_g} gv \, d\sigma, \quad \forall v \in H^1_{\Gamma_0}(\Omega) \tag{75}$$

(we use that $s = 0$ on $\partial\Omega$ and that $A(0) = \beta$, $B(0) = 0$)

is soved using the full Newton algorithm:

$$\begin{cases} u^0 \in H^1(\Omega), u^0 = u_0 \text{ on } \Gamma_0, \\ \int_\Omega \left(A(s^{(k)}) \nabla u^{n+1} \cdot \nabla v + B(s^{(k)}) \dfrac{\nabla u^{n+1} \cdot \nabla u^n}{|\nabla u^n|} \, t^{(k)} \cdot \nabla v \right) dx = \int_{\Gamma_g} gv \, d\sigma, \forall n > 0, \forall v \in H^1_{\Gamma_0}(\Omega). \end{cases} \tag{76}$$

$$
\begin{aligned}
&\Omega = (0,1)^2, \quad \gamma = [1/2, 1] \times \{a\}(a \in (0,1)), \quad \boldsymbol{F} = (1/2, a),\\
&\Gamma_0 = \Gamma_{0,1} \cup \Gamma_{0,2}, \quad u_0 = 0 \text{ on } \Gamma_{0,1} = \{0\} \times [0,1], \quad u_0 = 1/2 \text{ on } \Gamma_{0,2} = \{1\} \times [0.5, 0.8],\\
&\Gamma_g = \emptyset, \qquad\qquad (77)\\
&\mathcal{D} = \{\boldsymbol{x} \in \Omega, \|\boldsymbol{x} - \boldsymbol{F}\| \le r_3\}, \quad r_3 = 0.05,\\
&r_1 = 0.015, \ r_2 = 0.045 < 0.3, \ \nu_{F,1} = -1
\end{aligned}
$$

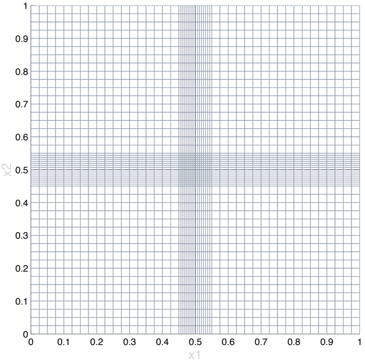

Figure: Example of quadrangulation of the unit square with a refinement on the support of the radial function ψ_1 (52×52 finite elements - 2916 degrees of freedom) around the point $\boldsymbol{F} = (1/2, 1/2)$.

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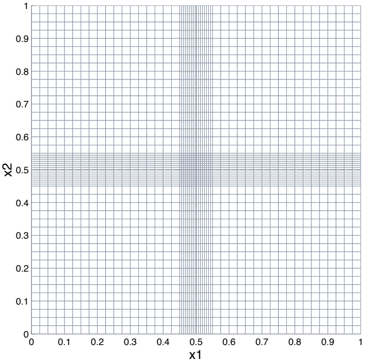

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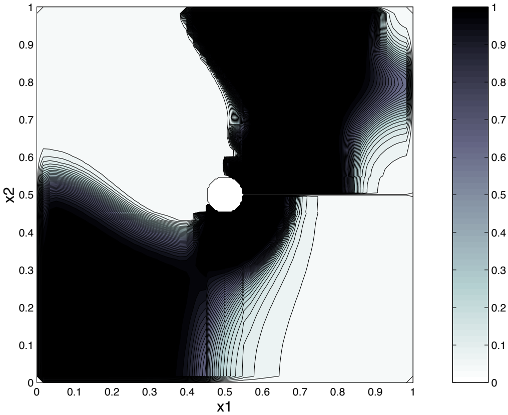
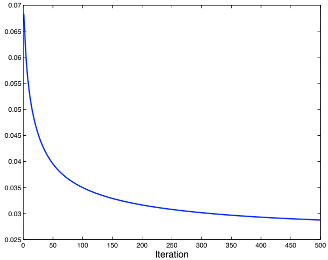


Figure: $(\alpha, \beta) = (1, 2)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - Iso-value of the density s^{opt} on the crack domain Ω with $s^{opt} = 0$ on $\partial\Omega$.

$$\|\boldsymbol{B}(\lambda, \rho)\|_{L^2(\Omega)} = \|(\rho - \lambda^-(s^{opt})\lambda) \cdot (\rho - \lambda^+(s^{opt})\lambda)\|_{L^2(\Omega)} \approx 1.32 \times 10^{-6}. \tag{78}$$

Moreover, we obtain

$$\|\rho - \lambda^+(s^{opt})\lambda\|_{L^2(\Omega)} \approx 3.13 \times 10^{-4}, \quad \|\rho - \lambda^-(s^{opt})\lambda\|_{L^2(\Omega)} \approx 4.21 \times 10^{-3}. \tag{79}$$



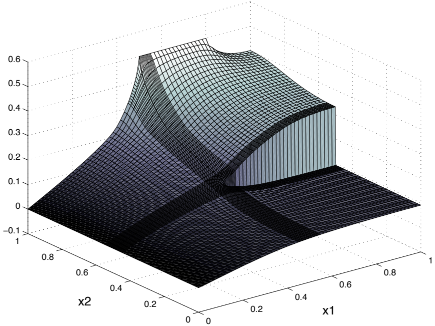


Figure: $(\alpha, \beta) = (1, 2)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - Evolution of the relaxed cost $I(s^{(k)}, t^{(k)})$ w.r.t the iteration (**Left**) and final solution u on Ω (**Right**).

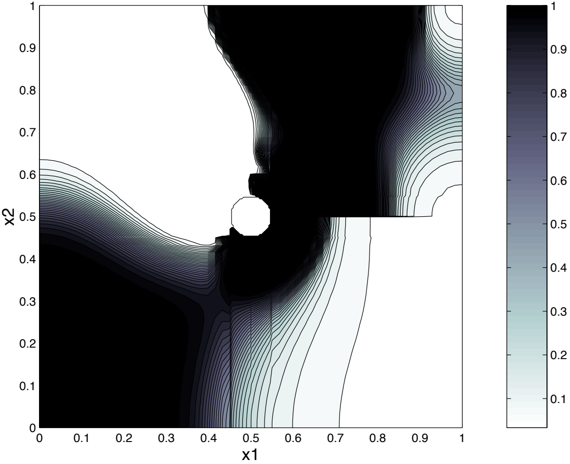


Figure: $(\alpha, \beta) = (1, 2)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - Iso-value of the density s on the crack domain with s free on $\partial\Omega$.

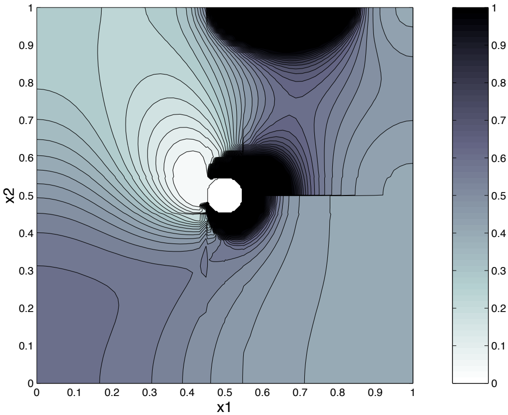
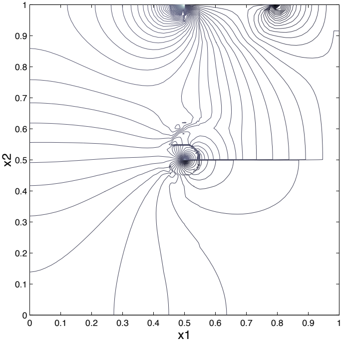


Figure: $(\alpha, \beta) = (1, 10)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - Iso-values of the density s on the crack domain.

$$\|\boldsymbol{B}(\lambda, \rho)\|_{L^2(\Omega)} = \|(\rho - \lambda^-(s^{opt})\lambda) \cdot (\rho - \lambda^+(s^{opt})\lambda)\|_{L^2(\Omega)} \approx 1.32 \times 10^{-5} \tag{80}$$

but

$$\|\rho - \lambda^+(s^{opt})\lambda\|_{L^2(\Omega)} \approx 8.21 \times 10^{-1}, \quad \|\rho - \lambda^-(s^{opt})\lambda\|_{L^2(\Omega)} \approx 4.09 \times 10^{-1}. \tag{81}$$



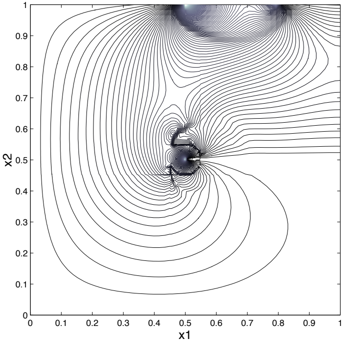


Figure: $(\alpha, \beta) = (1, 10)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - Iso-values of the components of the vector $\lambda_\beta - \lambda_\alpha$.

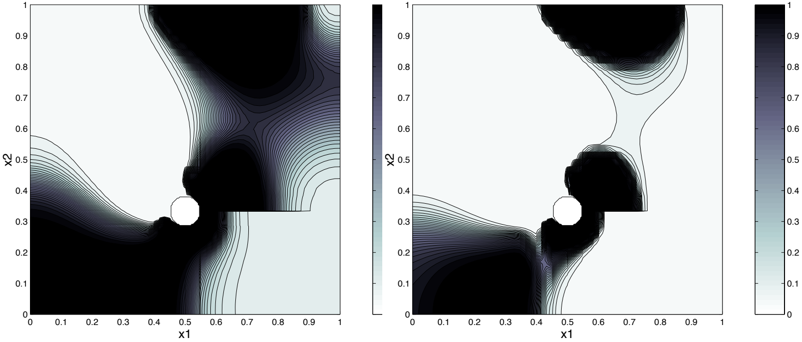


Figure: $(\alpha, \beta) = (1, 2)$- $\boldsymbol{F} = (1/2, 1/3)$ - Iso-values of the density s for $L = 2/5$ (**Left**) and $L = 1/5$ (**Right**).

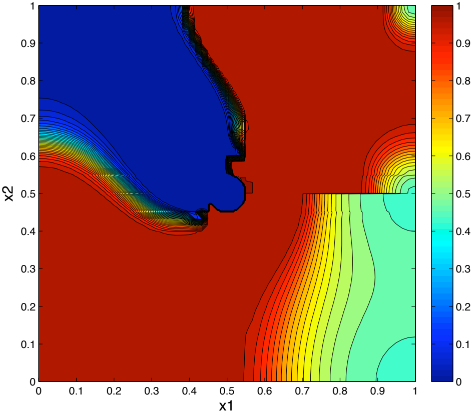


Figure: $(\alpha, \beta) = (1, 2)$- $\boldsymbol{F} = (1/2, 1/2)$ - Iso-value of the density s on the crack domain with s free on $\partial\Omega$

The optimal distribution corresponds to $L \approx 0.65$.

Conclusion

- The optimisation of the rate with respect to material seems original.
- The relaxation makes appear an original nonlinear divergence free form system.
- It would be interesting to optimize with respect to the shape of Omega (i.e. $\alpha \to 0$).

- The main drawback is the β- mechanical assumption around the crack tip F.
- Due to that, the effect of the singularity can not be cancelled !!

One idea is to replace the rate g_ψ by the quotient

$$\frac{E(u^\eta, \mathcal{X}_\omega, \boldsymbol{F}^\eta) - E(u, \mathcal{X}_\omega, \boldsymbol{F})}{\eta} \quad (82)$$

with η small ???????????
A second possibility is to introduce a stress criterion.

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THANK YOU - MUCHAS GRACIAS - MERCI BEAUCOUP