On the control of crack growth in elastic media

Patrick Hild [a], Arnaud Münch [a], Yves Ousset [b]

[a] *Laboratoire de Mathématiques, Université de Franche-Comté, UMR CNRS 6623, 25030 Besançon, France*
[b] *ONERA, DMSE, 29 avenue de la division Leclerc, BP 72, 92322, Chatillon, Cedex France*

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Abstract

In the framework of linear fracture theory, the Griffith criterion postulates the growth of any crack if the corresponding so-called energy release rate, defined as the variation of the mechanical energy, reaches a critical value. We consider in this note the optimal location problem which consists in minimizing this rate by applying to the structure an additional boundary load having a support which is disjoint from the support of the initial load possibly responsible of the growth. We give a sufficient well-posedness condition, introduce a relaxed problem in the general case, and then present a numerical experiment which suggests that the original nonlinear problem is actually well-posed. *To cite this article: P. Hild, A. Münch, Y. Ousset, C. R. Mécanique (2007).*

Résumé

Sur le contrôle de la propagation de fissure en milieu élastique. Dans le cadre de la mécanique linéaire de la rupture, le critère de Griffith postule la croissance d'une fissure si le taux de restitution de l'énergie associé excède une valeur critique. On considère dans cette note le problème d'optimisation de position qui consiste à minimiser ce taux en appliquant à la structure un chargement de frontière additionnel de support disjoint du chargement initial. On donne une condition suffisante d'existence de solution, introduit une relaxation du problème dans le cas général, puis présente une simulation numérique suggérant que ce problème non linéaire est en fait bien posé. *Pour citer cet article : P. Hild, A. Münch, Y. Ousset, C. R. Mécanique (2007).*

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Soit une structure élastique occupant au repos le domaine borné Ω de $\mathbb{R}^2$ (muni du repère orthornormé $(O, \boldsymbol{e_1}, \boldsymbol{e_2})$), encastrée sur $\Gamma_0 \subset \partial\Omega$ et soumise à un chargement $\boldsymbol{G} \in (L^2(\partial\Omega\backslash\Gamma_0))^2$ défini par

$$\boldsymbol{G} = \boldsymbol{f}\mathcal{X}_{\Gamma_f} + \boldsymbol{h}\mathcal{X}_{\Gamma_h}, \quad \boldsymbol{f} \in (L^2(\Gamma_f))^2, \boldsymbol{h} \in (L^2(\Gamma_h))^2, \Gamma_f, \Gamma_h \subset \partial\Omega\backslash\Gamma_0, \Gamma_f \cap \Gamma_h = \emptyset, \tag{1}$$

Email addresses: `patrick.hild@univ-fcomte.fr` (Patrick Hild), `arnaud.munch@univ-fcomte.fr` (Arnaud Münch), `yves.ousset@onera.fr` (Yves Ousset).

où $\mathcal{X}_{\Gamma_f} \in L^\infty(\Gamma_f, \{0,1\})$ désigne la fonction caractéristique de Γ_f. On suppose que Ω contient une fissure γ d'extrémité $\boldsymbol{F}$, non chargée (i.e., $\Gamma_f \cap \gamma = \emptyset$, $\Gamma_h \cap \gamma = \emptyset$) et libre (i.e., $\Gamma_0 \cap \gamma = \emptyset$). Le déplacement correspondant noté $\boldsymbol{u} = (u_1, u_2) \in (H^1_{\Gamma_0}(\Omega))^2$ où $H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega), v = 0 \text{ sur } \Gamma_0\}$ minimise à l'équilibre l'énergie élastique (2) et vérifie le système aux limites (3). On suppose pour simplifier que la force $\boldsymbol{G}$ est telle que les lèvres de la fissure ne s'interpénètrent pas.

Dans le cadre de la mécanique linéaire de la rupture [9], le critère de Griffith [6] postule la croissance de la fissure - précisément du point $\boldsymbol{F}$ - lorsque le taux de restitution associé, noté $G_{\boldsymbol{\psi}}$ défini par (6) (fonction de la géométrie, du matériau et du chargement) atteint une valeur critique. Afin de réduire ce taux et ainsi la propagation du défaut, une première approche consiste à optimiser les caractéristiques du matériau, mentionnant dans ce sens, l'essor des matériaux composites. Une seconde approche, abordée dans cette note, suivant les développements plus récents en contrôle actif (voir [8]), consiste à modifier le chargement $\boldsymbol{G}$. Supposant fixes le chargement principal $\boldsymbol{f}$ et son support Γ_f ainsi que $\boldsymbol{h}$, on introduit le problème $(\mathcal{P}_{\Gamma_h})$ défini en (4) qui consiste à minimiser le taux $G_{\boldsymbol{\psi}}$ par rapport au support Γ_h de $\boldsymbol{h}$ considérée ainsi comme une contre-force de $\boldsymbol{f}$. Ce point de vue ne semble avoir été abordé que dans deux notes [2,3] par P. Destuynder dans le contexte simplifié de l'opérateur de Laplace (voir [4] pour des aspects numériques). La référence [2] considère l'équation des ondes et propose une loi de commande sur les facteurs d'intensité de contraintes. Signalons plus récemment [12] qui se propose d'annuler les singularités en fond de fissure par l'ajout de chargements (singuliers !) sur la frontière.

Le problème non linéaire $(\mathcal{P}_{\Gamma_h})$ est potentiellement un problème mal posé dans la mesure où l'infimum peut ne pas être atteint dans la classe des fonctions caractéristiques. Dans un tel cas, le support optimal Γ_h est composé d'un nombre arbitrairement grand de composantes disjointes. L'existence d'au moins un minimum est garanti si le nombre de composantes disjointes de Γ_h est supposé fini (voir Théorème 2.1) : la démonstration repose sur la distance de Hausdorff [1]. Dans le cas général, l'introduction d'une relaxation, c'est-à-dire d'un problème bien posé et dont le minimum égale l'infimum de $(\mathcal{P}_{\Gamma_h})$ est nécéssaire : cette relaxation, notée $(\mathcal{RP}_{\Gamma_h})$, s'obtient ici simplement en remplaçant la classe des fonctions caractéristiques $\mathcal{X}_L$ par son enveloppe convexe pour la topologie faible L^∞-$\star$, c'est à dire l'espace des densités $S_L = \{s \in L^\infty(\Gamma, [0,1]), \|s\|_{L^1(\Gamma)} = L|\Gamma|\}$ (voir Théorème 2.2). Ce résultat, attendu dans la mesure où la variable de position Γ_h apparaît seulement dans le terme d'ordre zéro de l'équation d'état elliptique (par opposition aux problèmes plus difficiles où la relaxation implique l'opérateur différentiel principal - ici l'opérateur de divergence) se démontre par exemple en utilisant l'approche variationnelle non convexe et la mesure de Young (on renvoit à [11] pour la preuve dans un cas similaire), ou plus simplement ici en adaptant la preuve du Théorème 2.1. La résolution numérique du problème $(\mathcal{RP}_{\Gamma_h})$ à l'aide d'une méthode de gradient (reposant sur le théorème 3.1) suggère en fait que la densité optimale est une fonction caractéristique, et que de fait, le problème initial $(\mathcal{P}_{\Gamma_h})$ coïncide avec $(\mathcal{RP}_{\Gamma_h})$ et est bien posé. On renvoit à [7] pour d'autres applications confirmant cette propriété.

1. Problem statement

Let S be an elastic structure occupying a bounded domain Ω of $\mathbb{R}^2$ (referred to the orthonormal frame $(O, \boldsymbol{e_1}, \boldsymbol{e_2})$), fixed on a part $\Gamma_0 \subset \partial\Omega$ and submitted to a normal load $\boldsymbol{G} \in (L^2(\partial\Omega \backslash \Gamma_0))^2$ defined in (1) where $\mathcal{X}_{\Gamma_f} \in L^\infty(\Gamma_f, \{0,1\})$ denotes the characteristic function of Γ_f. The domain Ω is assumed to contain a crack γ of extremity $\boldsymbol{F}$, unloaded i.e. $\Gamma_f \cap \gamma = \emptyset$, $\Gamma_h \cap \gamma = \emptyset$ and free i.e. $\Gamma_0 \cap \gamma = \emptyset$. The corresponding displacement field $\boldsymbol{u} = (u_1, u_2) \in (H^1_{\Gamma_0}(\Omega))^2$ where $H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma_0\}$ minimizes at equilibrium the energy

$$J(\boldsymbol{u}, \gamma) = \frac{1}{2}\int_\Omega \mathrm{Tr}(\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \nabla \boldsymbol{u}) dx - \int_{\Gamma_f} \boldsymbol{f} \cdot \boldsymbol{u} d\sigma - \int_{\Gamma_h} \boldsymbol{h} \cdot \boldsymbol{u} d\sigma \tag{2}$$

(Tr designates the trace operator) and satisfies the following linear partial differential system

$$\begin{cases} -\boldsymbol{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) = 0 \quad \text{in} \quad \Omega, \qquad \boldsymbol{\sigma}(\boldsymbol{u}) = \mathbb{A} : \boldsymbol{\varepsilon}(\boldsymbol{u}), \quad \boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2, \\ \boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_0 \subset \partial\Omega, \qquad \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{\nu} = \boldsymbol{f}\mathcal{X}_{\Gamma_f} + \boldsymbol{h}\mathcal{X}_{\Gamma_h} \quad \text{on} \quad \partial\Omega \backslash \Gamma_0 \end{cases} \tag{3}$$

where $\mathbb{A}$ designates the 2-D elasticity tensor and $\boldsymbol{\nu}$ is the outward normal vector on $\partial\Omega$. We assume here for simplicity that the load $\boldsymbol{G}$ is such that there is not interpenetration of the crack lips.

In the framework of the linear fracture mechanics [9], the well-known Griffith's criterion [6] postulates the static-growth of the crack γ if the corresponding so-called energy release $G_{\boldsymbol{\psi}}$ reaches a critical value. In order to prevent the growth and reduce this rate, one may act on the material characteristics of the structure. We mention in this way the progress achieved with composite materials. On the other hand, following developments in the field of active control (see [8]), one may also act on the boundary load $\boldsymbol{G}$. In this respect, assuming fixed the main load $\boldsymbol{f}$ and its support Γ_f, we consider in this note, for any $L \in (0,1)$, the following nonlinear problem :

$$(\mathcal{P}_{\Gamma_h}) : \quad \inf_{\mathcal{X}_{\Gamma_h} \in \mathcal{X}_L} G_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h}) \tag{4}$$

with

$$\mathcal{X}_L = \{\mathcal{X}_{\Gamma_h} \in L^\infty(\Gamma; \{0,1\}), \|\mathcal{X}_{\Gamma_h}\|_{L^1(\Gamma)} = L|\Gamma|\}, \quad \Gamma \subset \partial\Omega \backslash (\Gamma_0 \cup \Gamma_f \cup \gamma) \tag{5}$$

which consists to find the optimal distribution of the support $\Gamma_h \subset \Gamma$ of the additional load $\boldsymbol{h}$ in order to reduce the rate $G_{\boldsymbol{\psi}}$, and therefore prevent the crack growth. The equality $\|\mathcal{X}_{\Gamma_h}\|_{L^1(\Gamma)} = L|\Gamma|$ imposes a size restriction on the support Γ_h. Remark that this support may be *a priori* composed of several disjoint components. As it is well-known (see [5,10]), the energy release rate defined as minus the variation of the elastic energy with respect to the point $\boldsymbol{F}$ may be expressed in terms of a surface integral (more appropriate for theoretical and numerical analysis). Denoting by $\boldsymbol{\nu_F}$ the orientation of the crack at point $\boldsymbol{F}$ and introducing any velocity field $\boldsymbol{\psi} \in \boldsymbol{W} = \{\boldsymbol{\psi} \in (W^{1,\infty}(\Omega))^2, \boldsymbol{\psi} = 0 \text{ on } \partial\Omega/\gamma, \boldsymbol{\psi}(\boldsymbol{F}) \cdot \boldsymbol{\nu_F} = 1\}$, the expression of the rate is

$$G_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h}) = -\frac{1}{2}\int_\Omega \text{Tr}(\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \nabla \boldsymbol{u}) \text{div}\, \boldsymbol{\psi} dx + \int_\Omega \text{Tr}(\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\psi}) dx \tag{6}$$

where $\boldsymbol{u} = \boldsymbol{u}(\mathcal{X}_{\Gamma_h})$ is the solution of (3). The value of $G_{\boldsymbol{\psi}}$ is independent of $\boldsymbol{\psi} \in \boldsymbol{W}$. In practice, $\boldsymbol{\psi}$ is chosen equal to a radial function constant on the neighborhood of $\boldsymbol{F}$.

2. Well-posedness and Relaxation for the problem $(\mathcal{P}_{\Gamma_h})$

The nonlinear optimal location problem $(\mathcal{P}_{\Gamma_h})$ is a proto-type of ill-posed problem in the sense that the infimum may be not reached in the class $\mathcal{X}_L$ of characteristic functions: the optimal domain Γ_h may then be composed of an infinite number of disjoint components. The well-posed property is ensured under for instance some geometrical assumptions.

Theorem 2.1 *Let $\boldsymbol{h}$ be fixed in $(L^2(\Gamma))^2$. If Γ_h is composed of a finite number of disjoint components, then problem $(\mathcal{P}_{\Gamma_h})$ admits a least a solution.* ■

Proof- Without loss of generality, let us assume that Γ_h is composed of only one part. The energy release rate G is non-negative. This is direct consequence of the fact that $J(\boldsymbol{u}(\gamma_1), \gamma_1) \leq J(\boldsymbol{u}(\gamma_2), \gamma_2)$ when the crack γ_1 contains the crack γ_2, i.e. $\gamma_2 \subset \gamma_1$. The existence of a minimizer is then related to the continuity of G with respect to the variation of Γ_h on $\partial\Omega$ for a given metric of $\mathbb{R}$. Let us consider the Hausdorff distance (see [1]): $d^{\mathcal{H}}(\gamma_1, \gamma_2) = \sup(\text{dist}_{\boldsymbol{x_1} \in \gamma_1}(x_1, \gamma_2), \text{dist}_{\boldsymbol{x_2} \in \gamma_2}(x_2, \gamma_1))$, for all $\gamma_1, \gamma_2 \subset \Gamma$ and a minimizing sequence $(\Gamma_h^n)_{n \geq 0} \subset \Gamma$ for $G_{\boldsymbol{\psi}}$ such that $d^{\mathcal{H}}(\Gamma_h^n, \Gamma_h) \to 0$ as n goes to infinity. The solution $\boldsymbol{u}^n \in (H^1_{\Gamma_0}(\Omega))^2$ associated with Γ_h^n satisfies the formulation

$$\int_\Omega \text{Tr}(\boldsymbol{\sigma}(\boldsymbol{u^n}) \cdot \nabla \boldsymbol{v}) dx = \int_{\Gamma_f} \boldsymbol{f} \cdot \boldsymbol{v} d\sigma + \int_{\Gamma_h^n} \boldsymbol{h} \cdot \boldsymbol{v} d\sigma, \quad \forall \boldsymbol{v} \in (H^1_{\Gamma_0}(\Omega))^2. \tag{7}$$

Assuming $\boldsymbol{h} \in (L^2(\Gamma_h^n))^2$ for all n, and putting $\boldsymbol{v} = \boldsymbol{u^n}$ in (7), we obtain from the Korn inequality that there exists a positive constant C such that $\|\boldsymbol{u^n}\|_{(H^1_{\Gamma_0}(\Omega))^2} \leq C(\|\boldsymbol{h}\|_{(L^2(\Gamma_h^n))^2} + \|\boldsymbol{f}\|_{(L^2(\Gamma_f))^2})$. Moreover, the convergence of Γ_h^n towards Γ_h for the Hausdorff distance implies that $\|\boldsymbol{h}\|_{(L^2(\Gamma_h^n))^2} - ||\boldsymbol{h}||_{(L^2(\Gamma_h))^2} \to 0$ as $n \to \infty$ (see [1]). Consequently, the sequence $(\boldsymbol{u^n})_n$ is uniformly bounded in the reflexive space $(H^1_{\Gamma_0}(\Omega))^2$ and one may extract a subsequence (still denoted by $\boldsymbol{u^n}$) such that $\boldsymbol{u^n}$ weakly converges to $\boldsymbol{u^\star}$ in $(H^1_{\Gamma_0}(\Omega))^2$. By passing to the limit in (7), $\boldsymbol{u^\star}$ verifies the formulation:

$$\int_\Omega \mathrm{Tr}(\boldsymbol{\sigma}(\boldsymbol{u^\star}) \cdot \nabla \boldsymbol{v}) dx = \int_{\Gamma_f} \boldsymbol{f} \cdot \boldsymbol{v} d\sigma + \int_{\Gamma_h} \boldsymbol{h} \cdot \boldsymbol{v} d\sigma, \quad \forall \boldsymbol{v} \in (H^1_{\Gamma_0}(\Omega))^2.$$

Now, observe that the compact embedding of the trace operator $tr : H^{1/2}(\partial\Omega) \to L^2(\partial\Omega)$ implies that the trace $tr(\boldsymbol{u^n})_{|\Gamma_h}$ converges to $tr(\boldsymbol{u^\star})_{|\Gamma_h}$ in $(L^2(\Gamma_h))^2$. Therefore,

$$\int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u^n}) \cdot \nabla \boldsymbol{u^n}) dx = \int_{\Gamma_f} \boldsymbol{f} \cdot \boldsymbol{u^n} d\sigma + \int_{\Gamma_h^n} \boldsymbol{h} \cdot \boldsymbol{u^n} d\sigma = \int_{\Gamma_f} \boldsymbol{f} \cdot \boldsymbol{u^n} d\sigma + \int_{\Gamma_h^n} \boldsymbol{h} \cdot (\boldsymbol{u^n} - \boldsymbol{u^\star}) d\sigma + \int_{\Gamma_h^n} \boldsymbol{h} \cdot \boldsymbol{u^\star} d\sigma \quad (8)$$

converges towards $\int_{\Gamma_f} \boldsymbol{f} \cdot \boldsymbol{u^\star} d\sigma + \int_{\Gamma_h} \boldsymbol{h} \cdot \boldsymbol{u^\star} d\sigma$. Then, using that $\int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u^n}) \cdot \nabla \boldsymbol{u^\star}) dx \to \int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u^\star}) \cdot \nabla \boldsymbol{u^\star}) dx$ and $Tr(\boldsymbol{\sigma}(\boldsymbol{u^\star}) \cdot \nabla \boldsymbol{u^n}) = Tr(\boldsymbol{\sigma}(\boldsymbol{u^n}) \cdot \nabla \boldsymbol{u^\star})$, we obtain from the equality

$$\int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u^n} - \boldsymbol{u^\star}) \cdot \nabla(\boldsymbol{u^n} - \boldsymbol{u^\star})) dx = \int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u^\star}) \cdot \nabla \boldsymbol{u^\star}) dx - 2 \int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u^\star}) \cdot \nabla \boldsymbol{u^n}) dx + \int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u^n}) \cdot \nabla \boldsymbol{u^n}) dx$$

that $\int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u^n} - \boldsymbol{u^\star}) \cdot \nabla(\boldsymbol{u^n} - \boldsymbol{u^\star})) dx \to 0$ as $n \to 0$. Consequently, the sequence $\boldsymbol{u^n}$ converges strongly towards $\boldsymbol{u^\star}$ in $(H^1_{\Gamma_0}(\Omega))^2$. In view of (6), we conclude to the convergence of the decreasing sequence $G_{\boldsymbol{\psi}}(\boldsymbol{u^n}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h^n})$ towards $G_{\boldsymbol{\psi}}(\boldsymbol{u^\star}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h})$. We refer to [7] for the more details. □

Without any geometrical condition on Γ_h, a relaxation of $(\mathcal{P}_{\Gamma_h})$ is *a priori* needed. Let us introduce the following problem

$$(\mathcal{RP}_{\Gamma_h}) : \quad \inf_{s \in S_L} G(\boldsymbol{u}, \boldsymbol{h}, s); \quad S_L = \{s \in L^\infty(\Gamma, [0,1]), ||s||_{L^1(\Gamma)} = L|\Gamma|\}$$

where $L \in (0,1)$ is the real parameter which appears in $(\mathcal{P}_{\Gamma_h})$, and $\boldsymbol{u}$ is the solution of

$$\begin{cases} -\boldsymbol{div}\, \boldsymbol{\sigma}(\boldsymbol{u}) = 0 \quad \text{in} \quad \Omega, \qquad \boldsymbol{\sigma}(\boldsymbol{u}) \equiv \mathbb{A} : \boldsymbol{\varepsilon}(\boldsymbol{u}), \quad \boldsymbol{\varepsilon}(\boldsymbol{u}) \equiv (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2, \\ \boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_0 \subset \partial\Omega, \qquad \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{\nu} = \boldsymbol{f} \mathcal{X}_{\Gamma_f} + s(\boldsymbol{x}) \boldsymbol{h} \mathcal{X}_\Gamma \quad \text{on} \quad \partial\Omega \backslash \Gamma_0. \end{cases} \quad (9)$$

The invariance of $G_\psi(u, \boldsymbol{h}, s)$ with respect to $\boldsymbol{\psi} \in \boldsymbol{W}$ remains since $\boldsymbol{\psi} = 0$ on Γ. Observe that $(\mathcal{RP}_{\Gamma_h})$ is obtained from the original one $(\mathcal{P}_{\Gamma_h})$ simply by replacing the set of characteristic functions $\{\mathcal{X}_{\Gamma_h} \in L^\infty(\Gamma, \{0,1\})\}$ by its convex hull for the L^∞ weak-$\star$ topology, i.e., the set of densities $\{s \in L^\infty(\Gamma, [0,1])\}$.

Theorem 2.2 *The problem $(\mathcal{RP}_{\Gamma_h})$ is a full well-posed relaxation of $(\mathcal{P}_{\Gamma_h})$ in the sense that problem $(\mathcal{RP}_{\Gamma_h})$ is well-posed and the minimum of $(\mathcal{RP}_{\Gamma_h})$ equals the infimum of $(\mathcal{P}_{\Gamma_h})$. Moreover, to the optimal density s^{opt} solution of $(\mathcal{RP}_{\Gamma_h})$, one may associate [through a Young measure process] a minimizing sequence $(\Gamma_h^{(k)})_{(k>0)}$ for the problem $(\mathcal{P}_{\Gamma_h})$, i.e., such that $||\mathcal{X}_{\Gamma_h^{(k)}}||_{L^1(\Gamma)} = ||s^{opt}||_{L^1(\Gamma)} = L|\Gamma|$, for all $k > 0$ and $\lim_{k\to\infty} G_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h^{(k)}}) = G_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, s^{opt})$.* ■

Sketch of the proof - Observe that this result is very natural since the location variable Γ_h appears only in the lower order part of the elliptic state equation (in contrast to standard optimal problems where the relaxation involves the differential operator - here the divergence one - itself). The result may be obtained using the non convex variational approach based on the computation of quasi-convexified function through Young measure (taking advantage of the divergence free from of the equation). We refer for instance to [11] for a proof in a similar context. In our simple situation, the result is obtained directly by replacing in the proof of Theorem 2.1, the Haussdorff (compact) convergence for Γ_h^n by the convergence induced by the topology of L^∞-$\star$ for any minimizing sequence s^n (and using the weak -$\star$ compactness of S_L). □

Theorem 2.2 is very valuable both from theoretical and numerical point of view. First, because this result replaces the minimization over domains by a simpler minimization over a set of density functions. This in particular avoids the computation of shape derivatives and the use of a level set approach. Secondly, the property of the optimal density may give valuable information on the nature of the original problem $(\mathcal{P}_{\Gamma_h})$.

3. A numerical experiment

The numerical resolution of $(\mathcal{RP}_{\Gamma_h})$ may be done efficiently using a gradient algorithm based on the

Theorem 3.1 *The first variation of $G_{\boldsymbol{\psi}}$ with respect to s in the direction $s_1 \in L^\infty(\Gamma, [0,1])$ takes the following expression*

$$\frac{\partial G_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, s)}{\partial s} \cdot s_1 = -\int_\Gamma s_1(\boldsymbol{x}) \boldsymbol{h} \cdot \boldsymbol{p} \, d\sigma, \quad \forall s_1 \in L^\infty(\Gamma, [0,1]) \tag{10}$$

where $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{u}, s) \in (H^1_{\Gamma_0}(\Omega))^2$ is the solution of the following (weak) adjoint problem:

$$\int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{p}) \cdot \nabla \boldsymbol{\phi}) dx - \int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \nabla \boldsymbol{\phi}) div\, \boldsymbol{\psi} dx + \int_\Omega (Tr(\boldsymbol{\sigma}(\boldsymbol{\phi}) \cdot \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\psi}) + Tr(\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \nabla \boldsymbol{\phi} \cdot \nabla \boldsymbol{\psi})) dx = 0 \tag{11}$$

for all $\boldsymbol{\phi} \in (H^1_{\Gamma_0}(\Omega))^2$ and $\boldsymbol{u} = \boldsymbol{u}(s)$ is the solution of (9). ■

The descent algorithm then takes the form: $s^{(k+1)}(\boldsymbol{x}) = s^{(k)}(\boldsymbol{x}) + \eta_s(\boldsymbol{x}) s_1^{(k)}(\boldsymbol{x})$ for all $k \geq 0$ and any s^0 (initial density) given in S_L. The descent direction is $s_1^{(k)} = \boldsymbol{h} \cdot \boldsymbol{p}^{(k)} - \lambda^{(k)}$ on Γ where $\lambda^{(k)} = (\|s^{(k)}\|_{L^1(\Gamma)} - L|\Gamma| + \int_\Gamma \eta_s \boldsymbol{h} \cdot \boldsymbol{p}^{(k)} d\sigma)/\|\eta_s\|_{L^1(\Gamma)}$ is a multiplier in order to enforce the size restriction and η_s is a positive function small enough and chosen in order to enforce the condition $s^{(k)}(x) \in [0,1]$ in Γ: $\eta_s = \epsilon s^{(k)}(1 - s^{(k)})$ with $\epsilon << 1$ is a candidate. As example, we consider the structure S occupying the domain $\Omega = (0,1)^2$ of area 1 square meter, fixed on $\Gamma_0 = \{1\} \times [0,1]$ with a crack $\gamma = [0, 1/2] \times \{1/2\}$, and submitted to the load $\boldsymbol{f} = (0, 10^6 N/m)$ on $\Gamma_f = [0.3, 0.6] \times \{1\}$. We assume that S is isotropic with a Young modulus $E = 2 \times 10^{11}$ and a Poisson ratio $\nu = 0.3$. The value of the rate without extra force (i.e., $\Gamma_h = \emptyset$), is $g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, \emptyset) \approx 1.147 N/m$ (see Figure 1-top left for the corresponding deformation of S). As expected, the opening mode (mode I) is predominant. We now take $\psi_1(\boldsymbol{x}) = \zeta(dist(\boldsymbol{x}, \boldsymbol{F}))$ with $\zeta(r) = \mathcal{X}_{0 \leq r < r_1} + (r_1 - r_2)^{-3}(r - r_2)^2(3r_1 - r_2 - 2r)\mathcal{X}_{r_1 \leq r \leq r_2}$, $r_1 = 0.1$ and $r_2 = 0.4$ and solve problem $(\mathcal{RP}_{\Gamma_h})$ assuming that $\Gamma = [0,1] \times \{0\}$ (i.e., the lower edge of the structure), $\boldsymbol{h} = (0, h_2)$ with $h_2 = 10^6 N/m$ and $L = 0.3$ so that $\int_\Gamma s(\boldsymbol{x}) h_2 d\sigma = \int_{\Gamma_f} f_2 d\sigma$. The algorithm is initialized with the constant density function $s^{(0)} \equiv L$ in Γ which does not privilege any location for Γ_h: the rate corresponding to this uniform load on Γ is $G_{\boldsymbol{\psi}}(\boldsymbol{u}, (0, 0.3 f_2), \Gamma_h) \approx 0.783 N/m$ (see Figure 1-top right); in this case, the mode II (in-plane shear mode) is predominant. At the convergence of the algorithm, we observe that the optimal density depicted on Figure 1-bottom left is a characteristic function $s^{opt} \approx \mathcal{X}_{[0.42, 0.72]}$ leading to $G_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, s^{opt}) \approx 0.464 N/m$. Mathematically, this suggests, at least for these data, that the well-posed relaxed problem $(\mathcal{RP}_{\Gamma_h})$ coincides with the original one $(\mathcal{P}_{\Gamma_h})$, and therefore, that this latter is well-posed. This load provides a mixed mode I-II (see Figure 1-bottom right for the corresponding deformation of S). We were not able to exhibit a case leading to a non characteristic optimal density. We therefore conjecture that the problem $(\mathcal{P}_{\Gamma_h})$ is always well-posed. Finally, we refer to [7] where the similar (and simpler) problem which consists to minimize $G_{\boldsymbol{\psi}}$ with respect to $\boldsymbol{h}$, the support Γ_h being fixed, is studied.

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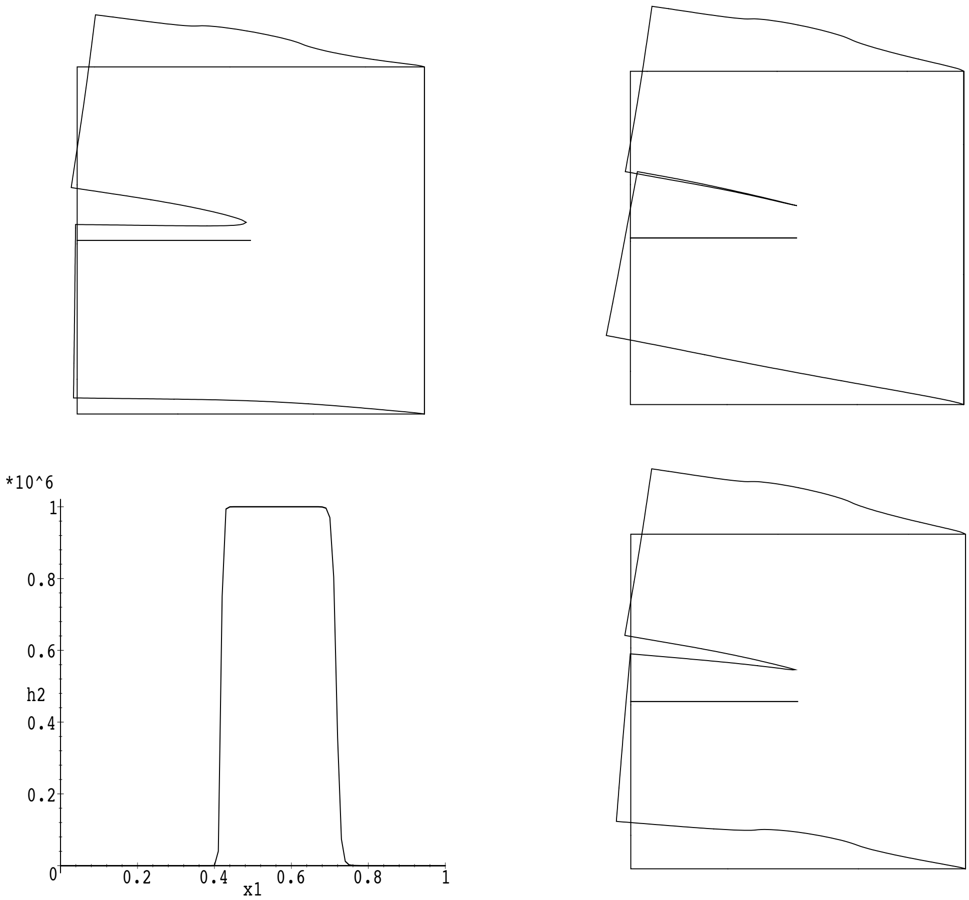

Figure 1. Three deformations of the cracked unit square for different extra-loads on the bottom (the deformations are amplified by a factor 20000) and one optimal density.

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