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A VARIATIONAL APPROACH TO APPROXIMATE CONTROLS FOR SYSTEM WITH ESSENTIAL SPECTRUM : APPLICATION TO MEMBRANAL ARCH

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Abstract. We address the numerical approximation of boundary controls for systems of the form $\boldsymbol{y''} + \boldsymbol{A_M}\boldsymbol{y} = \boldsymbol{0}$ which models dynamical elastic shell structure. The membranal operator $\boldsymbol{A_M}$ is self-adjoint and of mixed order, so that it possesses a non empty and bounded essential spectrum $\sigma_{ess}(\boldsymbol{A_M})$. Consequently, the exact controllability does not hold uniformly with respect to the initial data. Thus the numerical computation of controls by the way of dual approach and gradient method may fail, even if the initial data belongs to the orthogonal of the space spanned by the eigenfunctions associated with $\sigma_{ess}(\boldsymbol{A_M})$. In this work, we adapt a variational approach introduced in [Pablo Pedregal, *Inverse Problems* (26) 015004 (2010)] for the wave equation and obtain a robust method of approximation. This new approach does not require any information on the spectrum of the operator $\boldsymbol{A_M}$. We also show that it allows to extract, from any initial data $(\boldsymbol{y^0}, \boldsymbol{y^1})$, a controllable component for the mixed order system. We illustrate these properties with some numerical experiments. We also consider a relaxed controllability case for which uniform property holds.

1. **Introduction.** This work is a contribution to the boundary controllability of the hyperbolic system

$$\begin{cases} \boldsymbol{y''_\epsilon} + \boldsymbol{A_M}\boldsymbol{y_\epsilon} + \epsilon^2 \boldsymbol{A_F}\boldsymbol{y_\epsilon} = \boldsymbol{0}, & (\boldsymbol{\xi}, t) \in q_T = \omega \times (0, T) \\ (\boldsymbol{y_\epsilon}(\boldsymbol{\xi}, 0), \boldsymbol{y'_\epsilon}(\boldsymbol{\xi}, 0)) = (\boldsymbol{y^0}, \boldsymbol{y^1}), & \boldsymbol{\xi} \in \omega, \end{cases} \tag{1}$$

which models the vibration of an elastic shell-type structure occupying, in local coordinates, the domain $\omega \times] - \epsilon/2, \epsilon/2[$. $\omega \subset \mathbb{R}^2$ denotes the bounded mid-surface of the shell and $\boldsymbol{y_\epsilon} = (\boldsymbol{y_{\epsilon,\alpha}}, y_{\epsilon,3})$ the local displacement field decomposed into the tangential $\boldsymbol{y_{\epsilon,\alpha}} = (y_{\epsilon,1}, y_{\epsilon,2})$ and the normal $y_{\epsilon,3}$ components. ϵ is the constant thickness of the shell and $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \omega$. $\boldsymbol{A_M}$ and $\boldsymbol{A_F}$ denote the membranal and flexural operator respectively, associated with the map defining the mid-surface ω. System (1) is referred to in the literature as the Koiter shell model, derived from the three dimensional elasticity system via kinematical assumptions (see [4, 5, 7, 29]).

The boundary exact controllability problem of (1) consists to drive the initial condition $(\boldsymbol{y^0}, \boldsymbol{y^1})$ - assumed to belong in a suitable space - to a target $(\boldsymbol{y^0_T}, \boldsymbol{y^1_T})$ at a time T large enough through a control $\boldsymbol{v} = (\boldsymbol{v_\alpha}, v_3)$ acting on $\Sigma_T = \Sigma \times (0, T)$,

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$\Sigma \subset \partial\omega$. The controllability of system (1) strongly relies on the spectral property of the operator $\boldsymbol{A^\epsilon} := \boldsymbol{A_M} + \epsilon^2 \boldsymbol{A_F}$. We refer to [17, 19, 30] for some related works.

For any fixed $\epsilon > 0$, the operator $\boldsymbol{A^\epsilon}$ enjoys suitable compactness properties, so that the multiplier method leads to positive controllability results (see [9, 23]). On the other hand, when ϵ goes to zero, the system degenerates, under some assumption on $Ker(\boldsymbol{A_M})$ (see [31]), into the following system

$$\begin{cases} \boldsymbol{y''} + \boldsymbol{A_M y} = \boldsymbol{0}, & (\boldsymbol{\xi}, t) \in q_T \\ (\boldsymbol{y}(\boldsymbol{\xi}, 0), \boldsymbol{y'}(\boldsymbol{\xi}, 0)) = (\boldsymbol{y^0}, \boldsymbol{y^1}), & \boldsymbol{\xi} \in \omega. \end{cases} \tag{2}$$

The controllability of the limit system (2) is much more involved because $\boldsymbol{A_M}$ is a mixed order operator and possesses a non empty essential spectrum [14]. It is shown in [11] that, for such operator, the controllability does not hold uniformly with respect to the data $(\boldsymbol{y^0}, \boldsymbol{y^1})$: the proof relies on a Weyl's characterization of the essential spectrum and on the construction of a singular sequence for which the underlying observability inequality does not hold. For instance, in [2], for any $a, \alpha \in \mathbb{R}$, $a \leq \alpha^2$, $\omega = (0,1)^2$, the mixed order and self-adjoint operator

$$\boldsymbol{A_M} = \begin{pmatrix} -\Delta & -\alpha\partial_{\boldsymbol{\xi}} \\ \alpha\partial_{\boldsymbol{\xi}} & a \end{pmatrix}, \quad \boldsymbol{\xi} \in \omega = (0,1)^2 \tag{3}$$

for which $\sigma_{ess}(\boldsymbol{A_M}) = [a - \alpha^2, a]$ is considered. It is shown that the controllability only holds for initial data spanned by the eigenfunctions associated to the discrete spectrum $\sigma_d(\boldsymbol{A_M})$. The proof is based on the explicit expression of the spectrum $\sigma(\boldsymbol{A_M})$ and the use of 2-D Ingham type theorem [16].

In [25], we provide numerical experiments for the system (1) in the one dimensional case, that is, $\omega = (0,1)$ and the simpler but still very instructive operator $\boldsymbol{A_M}$ defined by

$$\boldsymbol{A_M} = \begin{pmatrix} -\partial^2_{\xi\xi} & -c\partial_\xi \\ c\partial_\xi & c^2 \end{pmatrix}, \quad \xi \in \omega = (0,1) \tag{4}$$

and $D(\boldsymbol{A_M}) = (H^2(\omega) \cap H^1_0(\omega)) \times H^1(\omega)$ for which $\sigma_{ess}(\boldsymbol{A_M}) = \{0\}$. This operator enters in the modelisation of an elastic arch of length one and constant curvature $c > 0$. The experiments exhibit the loss of compactness in the transition shell-membrane, as $\epsilon \to 0$. Precisely, the controls $(v_\epsilon)_{\epsilon>0}$ (obtained by dual arguments and minimization of conjugate functions) are not uniformly bounded with respect to $\epsilon > 0$. In the case $\epsilon = 0$, the practical computation of boundary controls for (2) remains a challenge, unless a precise description of the discrete spectrum is known.

We adapt in this work a different approach based on variational arguments introduced by Pedregal in [27] in the context of the scalar heat and wave equation. We will see that this approach allows to obtain control associated to the controllable part of the initial data $(\boldsymbol{y^0}, \boldsymbol{y^1})$. It is interesting to note that such approach does not require any information on $\sigma(\boldsymbol{A_M})$. It consists first in introducing a class of functions satisfying *a priori* the boundary conditions in space and time - in particular the controllability condition at time T - and then find among this class one element satisfying the state equation of (2).

For the sake of simplicity, we consider the one dimensional case, $\omega = (0,1)$, with the mixed order operator defined by (4). The paper is organized as follow. In Section 2, we use standard dual arguments and analyze some Dirichlet controllability properties of (2) with $\boldsymbol{A_M}$ defined by (4). Two distinct cases are considered: the full controllability case which requires the equality $(\boldsymbol{y}(\cdot, T), \boldsymbol{y'}(\cdot, T)) = (\boldsymbol{y^0_T}, \boldsymbol{y^1_T})$ at the final time and a relaxed situation, where we simply require that $(y_1(\cdot, T), y_1'(\cdot, T)) =$

(y^0_{1T}, y^1_{1T}). In both cases, we characterize the class of initial data which are controllable. In Section 3, we enrich this analysis with some careful numerical experiments, obtained thanks to a complete description of $\sigma(\boldsymbol{A_M})$. Then, in Section 4 we show how to adapt [27] to the operator (4) in these two situations. In Section 5, additional numerical experiments highlight that this variational approach allows to extract from any data $(\boldsymbol{y^0}, \boldsymbol{y^1})$ its controllable part. We conclude with some perspectives in Section 6.

2. **Control problems.** We set $\omega = (0,1)$ and $q_T = \omega \times (0,T)$.

Let us define the spaces $\boldsymbol{H} = L^2(\omega) \times L^2(\omega)$, $\boldsymbol{V} = H^1_0(\omega) \times L^2(\omega)$ and $\boldsymbol{V'} = H^{-1}(\omega) \times L^2(\omega)$ so that one has the usual situation $\boldsymbol{V} \subset \boldsymbol{H} \equiv \boldsymbol{H'} \subset \boldsymbol{V'}$ with $\boldsymbol{H}$ as a pivot space.

The Dirichlet controllability problem for (4) reads as follows: for any $T > 0$ large enough, any $(\boldsymbol{y^0}, \boldsymbol{y^1}), (\boldsymbol{y^0_T}, \boldsymbol{y^1_T}) \in \boldsymbol{H} \times \boldsymbol{V'}$, find a control $v \in L^2(\Sigma_T)$, $\Sigma_T = \{1\} \times (0,T)$ such that the unique solution $\boldsymbol{y} = (y_1, y_3) \in C([0,T], \boldsymbol{H}) \cap C^1([0,T], \boldsymbol{V'})$ of

$$\begin{cases} y_1'' - (y_{1,1} + cy_3)_{,1} = 0, & (\xi,t) \in q_T \\ y_3'' + c(y_{1,1} + cy_3) = 0, & (\xi,t) \in q_T \\ y_1(0,t) = 0, \quad y_1(1,t) = v(t), & t \in (0,T) \\ (\boldsymbol{y}(\xi,0), \boldsymbol{y'}(\xi,0)) = (\boldsymbol{y^0}, \boldsymbol{y^1}), & \xi \in \omega, \end{cases} \tag{5}$$

satisfies $(\boldsymbol{y}(\cdot,T), \boldsymbol{y'}(\cdot,T)) = (\boldsymbol{y^0_T}, \boldsymbol{y^1_T})$ in ω. In (5), $y_{1,1}(\xi,t)$ stands for $\partial y_1(\xi,t)/\partial\xi$.

We also consider a weaker (relaxed) version which consists, under the same hypotheses, to find a control $v \in L^2(\Sigma_T)$ such that simply $(y_1(\cdot,T), y_1'(\cdot,T)) = (y^0_{T,1}, y^1_{T,1})$: this is the so-called *partial controllability* of system (5). In both cases, since y_3 belongs only to $L^2(\omega)$, there is one (boundary) control v for the two components of $\boldsymbol{y}$. (5) models the control of an elastic arch of length one and curvature c. y_1 and y_3 designate the tangential and normal displacement of the arch respectively. The map $\boldsymbol{\varphi} : \overline{\omega} \to \mathbb{R}^2$ which describes the circular line of length one and curvature c is $\boldsymbol{\varphi}(\xi) = (c^{-1}\sin(c\xi), c^{-1}\cos(c\xi))$. $\xi \in \omega$ is the curvilinear abscissae. We refer to [25] for details on the geometrical description.

2.1. **Spectral property of the homogeneous system.** The homogeneous adjoint problem associated with (5) is as follows: for any $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \in \boldsymbol{V} \times \boldsymbol{H}$, find the solution $\boldsymbol{\phi}$ of

$$\begin{cases} \boldsymbol{\phi''} + \boldsymbol{A_M}\boldsymbol{\phi} = \boldsymbol{0}, & (\xi,t) \in q_T \\ \phi_1(0,\cdot) = \phi_1(1,\cdot) = 0, & t \in (0,T) \\ (\boldsymbol{\phi}(\xi,0), \boldsymbol{\phi'}(\xi,0)) = (\boldsymbol{\phi^0}, \boldsymbol{\phi^1}), & \xi \in \omega. \end{cases} \tag{6}$$

Introducing the bilinear and symmetric form $b_M(\boldsymbol{\phi}, \boldsymbol{v}) = (\phi_{1,1} + c\phi_3)(v_{1,1} + cv_3)$, integrations by part then lead to the relation $\int_\omega \boldsymbol{A_M}\boldsymbol{\phi} \cdot \boldsymbol{v}\, d\xi = \int_\omega b_M(\boldsymbol{\phi}, \boldsymbol{v})\, d\xi - [(\phi_{1,1} + c\phi_3)v_1]_0^1$ for all $\boldsymbol{\phi} \in \boldsymbol{V}$ and all $\boldsymbol{v} \in H^1(\omega) \times L^2(\omega)$. Then, from the Lions-Magenes theory [22] we have the following result.

Proposition 1. *For all $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \in \boldsymbol{V} \times \boldsymbol{H}$, there exists a unique weak solution $\boldsymbol{\phi} \in C(0,T;\boldsymbol{V}) \cap C^1(0,T;\boldsymbol{H})$ that satisfies the variational problem*

$$\int_\omega \boldsymbol{\phi''} \cdot \boldsymbol{v}\, d\xi + \int_\omega b_M(\boldsymbol{\phi}, \boldsymbol{v})\, d\xi = 0, \quad \forall \boldsymbol{\phi} \in \boldsymbol{V}.$$

Moreover, the mapping $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \to (\boldsymbol{\phi}(t), \boldsymbol{\phi'}(t))$ is continuous : there exists a constant $C > 0$ such that $|\boldsymbol{\phi}(t)|^2_{\boldsymbol{V}} + |\boldsymbol{\phi'}(t)|^2_{\boldsymbol{H}} \leq C(|\boldsymbol{\phi^0}|^2_{\boldsymbol{V}} + |\boldsymbol{\phi^1}|^2_{\boldsymbol{H}})$, for all $t \in (0,T)$.

The *natural* energy E of the arch, denoted by $E(t,\boldsymbol{\phi})=\frac{1}{2}\int_\omega(|\boldsymbol{\phi}'|^2+b_M(\boldsymbol{\phi},\boldsymbol{\phi}))\,d\xi$, $t\in(0,T)$, is constant along all the trajectories, that is, $E(t,\boldsymbol{\phi})=E_0=\frac{1}{2}\int_\omega(|\boldsymbol{\phi^1}|^2+b_M(\boldsymbol{\phi^0},\boldsymbol{\phi^0}))d\xi$, for all $(\boldsymbol{\phi^0},\boldsymbol{\phi^1})\in\boldsymbol{V}\times\boldsymbol{H}$ and $t>0$.

A simple computation gives that the eigenvalues of

$$\boldsymbol{A_M}\boldsymbol{\psi}=\lambda\boldsymbol{\psi},\ \xi\in\omega,\quad \psi_1=0,\ \xi\in\partial\omega$$

are $\sigma(\boldsymbol{A_M})=\{0,\lambda_k=c^2+(k\pi)^2,k\geq 0\}$. Since

$$Ker\boldsymbol{A_M}=\{v_\zeta=(-c\zeta,\zeta_{,1})\in\boldsymbol{H},\zeta\in H_0^1(\omega)\}$$

and the eigenfunctions associated with λ_0 and λ_k, $k>0$ are respectively :

$$\boldsymbol{v_0}=(0,1),\quad \boldsymbol{v_k}=\left(\sin(k\pi\zeta),\frac{c}{k\pi}\cos(k\pi\zeta)\right),$$

one deduces that the essential spectrum $\sigma_{ess}(\boldsymbol{A_M})=\{0\}$. Let us also remark that an orthogonal basis in $\boldsymbol{H}$ of $Ker\boldsymbol{A_M}$ is

$$\boldsymbol{w_k}=\left(-\frac{c}{k\pi}\sin(k\pi\zeta),\cos(k\pi\zeta)\right),\quad k\geq 1$$

and that $\{\boldsymbol{w_k},\boldsymbol{v_0},\boldsymbol{v_k}\}$ is an orthogonal basis in $\boldsymbol{H}$. This permits to expand the weak solution of the system (6) in term of a Fourier series as follows, setting $\mu_k=\sqrt{\lambda_k}$, $\forall k\geq 0$:

$$\begin{aligned}\boldsymbol{\phi}(\cdot,t)=\sum_{k\geq1}(a_k+b_kt)\boldsymbol{w_k}+(A_0\cos(\mu_0t)+B_0\sin(\mu_0t))\boldsymbol{v_0}\\ +\sum_{k\geq1}(A_k\cos(\mu_kt)+B_k\sin(\mu_kt))\boldsymbol{v_k}\end{aligned}\tag{7}$$

where the coefficients a_k,b_k,A_0,B_0,A_k,B_k are determined from the expansion of the initial data:

$$\boldsymbol{\phi^0}=\sum_{k\geq1}a_k\boldsymbol{w_k}+A_0\boldsymbol{v_0}+\sum_{k\geq1}A_k\boldsymbol{v_k},\quad \boldsymbol{\phi^1}=\sum_{k\geq1}b_k\boldsymbol{w_k}+\mu_0B_0\boldsymbol{v_0}+\sum_{k\geq1}\mu_kB_k\boldsymbol{v_k}.$$

The asumption $(\boldsymbol{\phi^0},\boldsymbol{\phi^1})\in\boldsymbol{V}\times\boldsymbol{H}$ implies that

$$\sum_{k\geq1}(a_k^2+A_k^2)<\infty,\quad \sum_{k\geq1}(ca_k+k\pi A_k)^2<\infty,\quad \sum_{k\geq1}(b_k^2+\lambda B_k^2)<\infty.$$

Observe that if $\boldsymbol{\phi^0},\boldsymbol{\phi^1}\in Ker\boldsymbol{A_M}$ then $\boldsymbol{\phi}(\cdot,t)\in Ker\boldsymbol{A_M}$ for all $t>0$. Similarly, if $\boldsymbol{\phi^0}\in Ker\boldsymbol{A_M}$ and $\boldsymbol{\phi^1}=\boldsymbol{0}$, then $\boldsymbol{\phi}(\cdot,t)=\boldsymbol{\phi^0}$ for all $t>0$.

We now introduce the orthogonal of the subspace $Ker\boldsymbol{A_M}$ in $\boldsymbol{H}$:

$$\boldsymbol{H}^\perp=\left\{\boldsymbol{\psi}=(\psi_1,\psi_3)\in\boldsymbol{H},\int_\omega(\psi_1\phi_1+\psi_3\phi_3)d\xi=0,\ \forall(\phi_1,\phi_3)\in Ker\boldsymbol{A_M}\right\}.$$

From the definition of $Ker\boldsymbol{A_M}$, we get that $\boldsymbol{H}^\perp=\{(\psi_1,\psi_3)\in\boldsymbol{H},\ c\psi_1+\psi_{3,1}=0$ in $H^{-1}(\omega)\}$ and then

$$\boldsymbol{H}^\perp=\{\boldsymbol{v}=(\psi_{,1},-c\psi)\in\boldsymbol{H},\psi\in H^1(\omega)\}.\tag{8}$$

The subspace $\boldsymbol{H}^\perp$ is spanned by $\{\boldsymbol{v_k},k\geq0\}$. We denote by $\boldsymbol{V}^\perp$ the orthogonal in $\boldsymbol{V}$ of $Ker\boldsymbol{A_M}$; it is also spanned, in $\boldsymbol{V}$, by $\{\boldsymbol{v_k},k\geq0\}$. At last, let us remark that the energy $E(t,\boldsymbol{\phi})=E(0,\boldsymbol{\phi})$ defines a norm over $\boldsymbol{V}^\perp\times\boldsymbol{H}^\perp$. Moreover, if $(\boldsymbol{\phi^0},\boldsymbol{\phi^1})\in\boldsymbol{V}^\perp\times\boldsymbol{H}^\perp$ then $\boldsymbol{\phi}(\cdot,t)\in\boldsymbol{V}^\perp$ for all t.

2.2. **Observability inequality and controllability results.** From standard dual arguments [21], the control property of (5) is related to the existence of two constants $C_1, C_2 > 0$ such that, for $T > 0$ large enough, the solution of (6) satisfies

$$C_1\|(\boldsymbol{\phi^0}, \boldsymbol{\phi^1})\|^2_{\boldsymbol{V}\times\boldsymbol{H}} \leq \int_0^T b_M(\boldsymbol{\phi}, \boldsymbol{\phi})(1,t)dt \leq C_2\|(\boldsymbol{\phi^0}, \boldsymbol{\phi^1})\|^2_{\boldsymbol{V}\times\boldsymbol{H}} \quad \forall(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \in \boldsymbol{V}\times\boldsymbol{H}.$$

Since $0 \in \sigma(\boldsymbol{A_M})$, the left inequality (called the observability inequality) can not hold for all $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \in \boldsymbol{V} \times \boldsymbol{H}$. It suffices to take $\boldsymbol{\phi^0}, \boldsymbol{\phi^1} \in Ker\boldsymbol{A_M}$ so that $b_M(\boldsymbol{\phi}, \boldsymbol{\phi}) = 0$ for all $t > 0$. We have the following result.

Proposition 2. *Let $c > 0$, $\gamma^\star(c) = \min(2c, \sqrt{c^2 + \pi^2} - c)$. For all $T > T^\star(c) \equiv 2\pi/\gamma^\star(c)$, there exist two strictly positive constants $C_1(c)$ and $C_2(c)$ such that*

$$C_1(c)\|(\boldsymbol{\phi^0}, \boldsymbol{\phi^1})\|^2_{\boldsymbol{V}\times\boldsymbol{H}} \leq \int_0^T |(\phi_{1,1} + c\phi_3)(1,t)|^2 dt \leq C_2(c)\|(\boldsymbol{\phi^0}, \boldsymbol{\phi^1})\|^2_{\boldsymbol{V}\times\boldsymbol{H}} \tag{9}$$

for all $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \in \boldsymbol{V}^\perp \times \boldsymbol{H}^\perp$.

Relation (9) is equivalent to the existence of two positive constants C_1, C_2 such that

$$C_1(c)E_0(\boldsymbol{\phi}) \leq \int_0^T |(\phi_{1,1} + c\phi_3)(1,t)|^2 dt \leq C_2(c)E_0(\boldsymbol{\phi}), \quad \forall(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \in \boldsymbol{V}^\perp \times \boldsymbol{H}^\perp. \tag{10}$$

These inequalities may be obtained from a direct application of the following Ingham's theorem on Nonharmonic series (see [16] for recent developments) :

Theorem 2.1. *Let $K \in \mathbb{Z}$ and $(w_k)_{k\in K}$ be a family of real numbers satisfying the uniform gap condition $\gamma = \inf_{k\neq n} |w_k - w_n| > 0$. If I is a bounded interval of length $|I| > 2\pi/\gamma$, then there exist two positives constants C_1, C_2 such that*

$$C_1 \sum_{k\in K} |x_k|^2 \leq \int_I |x(t)|^2 dt \leq C_2 \sum_{k\in K} |x_k|^2$$

for all functions given by the sum $x(t) = \sum_{k\in K} x_k e^{iw_k t}$ with square-summable complex coefficients x_k.

Proof of Proposition 2. One finds $E(0, \boldsymbol{\phi}) = \frac{c^2}{2}(A_0^2 + B_0^2) + \frac{1}{2}\sum_{k=1}^\infty \lambda_k(A_k^2 + B_k^2)$. On the other hand, one has

$$\begin{aligned}
(\phi_{1,1} + c\phi_3)(1,t) &= c(A_0\cos(\mu_0 t) + B_0\sin(\mu_0 t)) \\
&\qquad + \sum_{k\geq 1}(-1)^k \frac{\mu_k^2}{k\pi}(A_k\cos(\mu_k t) + B_k\sin(\mu_k t)) \\
&= \frac{1}{2}\sum_{k\geq 1}(-1)^k \frac{\mu_k^2}{k\pi}(A_k + iB_k)e^{-i\mu_k t} + \frac{c}{2}(A_0 + iB_0)e^{-i\mu_0 t} \\
&\qquad + \frac{c}{2}(A_0 - iB_0)e^{i\mu_0 t} + \frac{1}{2}\sum_{k\geq 1}(-1)^k \frac{\mu_k^2}{k\pi}(A_k - iB_k)e^{i\mu_k t}
\end{aligned}$$

so that $\phi_{1,1}(1,t) + c\phi_3(1,t) = \sum_{k\in\mathbb{Z}, k\neq 0} x_k e^{i\Lambda_k t}$ with

$$\Lambda_k = \begin{cases} -\mu_{-(k+1)} & \text{for} \quad k < -1 \\ -\mu_0 & \text{for} \quad k = -1 \\ \mu_0 & \text{for} \quad k = 1 \\ \mu_{k-1} & \text{for} \quad k > 1 \end{cases}$$

and

$$x_k = \begin{cases} \frac{1}{2}(-1)^{-(k+1)}\frac{\mu^2_{-(k+1)}}{-(k+1)\pi}(A_{-(k+1)} + iB_{-(k+1)}) & \text{for} \quad k < -1 \\ \frac{c}{2}(A_0 + iB_0) & \text{for} \quad k = -1 \\ \frac{c}{2}(A_0 - iB_0) & \text{for} \quad k = 1 \\ \frac{1}{2}(-1)^{(k-1)}\frac{\mu^2_{(k-1)}}{(k-1)\pi}(A_{(k-1)} - iB_{(k-1)}) & \text{for} \quad k > 1. \end{cases}$$

We then apply Theorem 2.1 with $I = (0, T)$, $K = \mathbb{Z}\backslash\{0\}$ and the sequence $(w_k)_k = (\Lambda_k)_k = (\cdots, -\mu_2, -\mu_1, -\mu_0, \mu_0, \mu_1, \mu_2, \cdots)$ to obtain that there exist two positives constants $C_1(c)$ and $C_2(c)$ such that (10) holds for all $c > 0$, under the condition $T > 2\pi/\gamma$ with $\gamma = \min(\mu_0 - (-\mu_0), \inf_{k\in\mathbb{N}} |\mu_k - \mu_{k-1}|)$. From the concavity of the square root function, we deduce that $|\mu_1 - \mu_0| \le |\mu_{k+1} - \mu_k|$ for all $k \ge 0$ leading to $T^\star$ defined in Proposition 2.

The lower bound value $T^\star$ of controllability may be specified as follows

$$T^\star(c) = \frac{\pi}{c}\mathcal{X}_{(c\le\pi/\sqrt{8})} + \frac{2\pi}{\sqrt{c^2+\pi^2} - c}\mathcal{X}_{(c>\pi/\sqrt{8})}$$

and reaches its minimum when $c = \pi/\sqrt{8}$ for which $T^\star(c) = \sqrt{8}$. We observe that the minimal time of controllability $T^\star$ goes to infinity, almost linearly, as c goes to infinity. Remark that in practice, c is not greater than 2π for which the arch is a closed circle. Moreover, the time of controllability blows up as the curvature c of the arch goes to zero. This is due to the eigenvalue λ_0 which vanishes with c. Therefore, one can not expect a uniform convergence of the control with respect to c. Precisely, if the initial condition is $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) = (\boldsymbol{v_0}, \mu_0\boldsymbol{v_0})$ so that $\phi_{1,1}(1,t) + c\phi_3(1,t) = c(A_0\cos(\mu_0 t) + B_0, \sin(\mu_0 t))$ we obtain explicitly

$$\begin{aligned} \int_0^T & |(\phi_{1,1} + c\phi_3)(1,t)|^2\, dt \\ &= \mu_0^2\bigg(A_0^2\frac{\cos(\mu_0 T)\sin(\mu_0 T) + \mu_0 T}{2\mu_0} \\ &\qquad + B_0^2\frac{\mu_0 T - \cos(\mu_0 T)\sin(\mu_0 T)}{2\mu_0} + A_0B_0\frac{\sin^2(\mu_0 T)}{\mu_0}\bigg) \\ &= \mu_0^2\bigg(A_0^2\Big[T + O(\mu_0^2)\Big] + B_0^2\Big[\frac{T^3}{3}\mu_0^2 + O(\mu_0^4)\Big] + A_0B_0\Big[T^2\mu_0 + O(\mu_0^3)\Big]\bigg) \end{aligned}$$

and $E_0(\boldsymbol{\phi}) = \mu_0^2(A_0^2 + B_0^2)/2$, so that

$$2\min(T, T^3c^2/3)\, E_0(\boldsymbol{\phi}) \le \int_0^T |(\phi_{1,1} + c\phi_3)(1,t)|^2\, dt \le 2\max(T, T^3c^2/3)\, E_0(\boldsymbol{\phi}).$$

The constant $C_1(c)$ goes to zero as c goes to zero unless $T = \mathcal{O}(c^{-1})$. Consequently, the observability inequality is not uniform with respect to c for an arbitrarily shallow arch. The observability is uniform if and only if $B_0 = 0$, i.e. in this case if $\phi_3^1 = 0$.

If we denote by $\boldsymbol{H_K}$, resp. $\boldsymbol{V_K}$, the closed subspace of $\boldsymbol{H}$, resp. $\boldsymbol{V}$, spanned by $\{\boldsymbol{v_k}, k \ge 1\}$, we have the following result.

Proposition 3. *Let $c>0$ and $\gamma^{\star\star}(c)=\sqrt{c^2+4\pi^2}-\sqrt{c^2+\pi^2}$. For all $T> T^{\star\star}(c)\equiv 2\pi/\gamma^{\star\star}(c)$, there exist two positive constants C_1, C_2 independent of c such that (9) holds for all $(\boldsymbol{\phi^0},\boldsymbol{\phi^1})\in \boldsymbol{V_K}\times\boldsymbol{H_K}$.*

The lower bound $T^{\star\star}$ is now a monotonous increasing function of c such that $\lim_{c\to 0}T^{\star\star}(c)=2$, lower bound for the wave equation controlled at one extremity. We also have that $2<T^{\star\star}(c)<T^{\star}(c)$ for all $c>0$.

We are now in a position to give the corresponding controllability results assuming, without loss of generality since the problem is linear, that the target is zero, i.e. $(\boldsymbol{y_T^0},\boldsymbol{y_T^1})=(0,0)$. Integrations by part show that v is a control for the system (5) if and only if

$$\int_0^T(\phi_{1,1}+c\phi_3)(1,t)\,v(t)dt=<(\boldsymbol{\phi^0},\boldsymbol{\phi^1}),(\boldsymbol{y^1},-\boldsymbol{y^0})>_{\boldsymbol{V}\times\boldsymbol{H},\boldsymbol{V'}\times\boldsymbol{H}}. \tag{11}$$

with $<(\boldsymbol{\phi^0},\boldsymbol{\phi^1}),(\boldsymbol{y^1},-\boldsymbol{y^0})>_{\boldsymbol{V}\times\boldsymbol{H},\boldsymbol{V'}\times\boldsymbol{H}}:=<\boldsymbol{\phi^0},\boldsymbol{y^1}>_{\boldsymbol{V},\boldsymbol{V'}}-<\boldsymbol{\phi^1},\boldsymbol{y^0}>_{\boldsymbol{H},\boldsymbol{H}}$.

We then introduce the continuous and convex functional $\mathcal{J}:\boldsymbol{V}\times\boldsymbol{H}\to\mathbb{R}$ defined by

$$\mathcal{J}(\boldsymbol{\phi^0},\boldsymbol{\phi^1})=\frac{1}{2}\int_0^T|(\phi_{1,1}+c\phi_3)(1,t)|^2\,dt-<(\boldsymbol{\phi^0},\boldsymbol{\phi^1}),(\boldsymbol{y^1},-\boldsymbol{y^0})>_{\boldsymbol{V}\times\boldsymbol{H},\boldsymbol{V'}\times\boldsymbol{H}}. \tag{12}$$

If $\mathcal{J}$ is coercive, then $\mathcal{J}$ admits a unique minimum and the control of minimal L^2-norm is given by $v(t)=(\phi_{1,1}+c\phi_3)(1,t)$, for all $t\in(0,T)$. Remark first that $\mathcal{J}$ is only coercive in $\boldsymbol{V}^\perp\times\boldsymbol{H}^\perp$ for $T>T^\star(c)$. Furthermore, if $(\boldsymbol{y^0},\boldsymbol{y^1})$ belongs to $Ker\boldsymbol{A_M}$, then

$$<(\boldsymbol{\phi^0},\boldsymbol{\phi^1}),(\boldsymbol{y^1},-\boldsymbol{y^0})>_{\boldsymbol{V}\times\boldsymbol{H},\boldsymbol{V'}\times\boldsymbol{H}}=0,\quad\forall(\boldsymbol{\phi^0},\boldsymbol{\phi^1})\in\boldsymbol{V}^\perp\times\boldsymbol{H}^\perp$$

and from the characterization (11), the control is zero; in this case, the solution $\boldsymbol{y}$ remains in $Ker\boldsymbol{A_M}$ for all $t>0$ but is not controlled ! Hence, we consider $\boldsymbol{y^0}$ (resp. $\boldsymbol{y^1}$) in a dual of $\boldsymbol{H}^\perp$ (resp. $\boldsymbol{V}^\perp$). Precisely, we take $\boldsymbol{y^0}\in\boldsymbol{H}^\perp$ and $\boldsymbol{y^1}\in\boldsymbol{V}^{\perp\prime}$ where $\boldsymbol{V}^{\perp\prime}\subset\boldsymbol{V'}$ is the orthogonal of $Ker\boldsymbol{A_M}$ in the duality $<\cdot,\cdot>_{\boldsymbol{V},\boldsymbol{V'}}$. Let us remark that $\boldsymbol{V}^{\perp\prime}$ is the closure of $\boldsymbol{V}^\perp$ in $\boldsymbol{V}'$. As a consequence of Proposition 2, we have the following result.

Theorem 2.2. *Let $c>0$. For any $T>T^\star(c)$ and any initial data $(\boldsymbol{y^0},\boldsymbol{y^1})\in\boldsymbol{H}^\perp\times\boldsymbol{V}^{\perp\prime}$, there exists a control function $v\in L^2(0,T)$ which drives to rest at time T the solution $\boldsymbol{y}$ of (5) associated with $(\boldsymbol{y^0},\boldsymbol{y^1})$. Moreover, the control of minimal L^2-norm is given by $v=(\phi_{1,1}+c\phi_3)(1,\cdot)$ where $\boldsymbol{\phi}$ is solution of (6) and associated with $(\boldsymbol{\phi^0},\boldsymbol{\phi^1})$ minimum of $\mathcal{J}$ defined by (12) over $\boldsymbol{V}^\perp\times\boldsymbol{H}^\perp$.*

Remark 1. The non controllable modes $\boldsymbol{w_k}\in Ker\boldsymbol{A_M}$, $k\geq 1$, do not correspond to modes of arbitrarily small energy. Precisely, for $(\boldsymbol{y^0},\boldsymbol{y^1})=\sum_{k\geq1}(a_k,b_k)\boldsymbol{w_k}$, the norm of the solution at time $t=T$ is (since the control v has no effect on $\boldsymbol{w_k}$)

$$\|\boldsymbol{y}(T)\|^2_{\boldsymbol{V}}=\sum_{k\geq1}(a_k+b_kT)^2\left(c^2+\frac{\lambda_k}{(k\pi)^2}\right),\quad \|\boldsymbol{y'}(T)\|^2_{\boldsymbol{H}}=\frac{1}{2}\sum_{k\geq1}b_k^2\frac{\lambda_k}{(k\pi)^2}$$

and different from zero. Consequently, approximate controllability for the system (5) does not hold anymore.

Let us denote by $\boldsymbol{V'_K}$ the closure of $\boldsymbol{V_K}$ in $\boldsymbol{V'}$. As a consequence of Proposition 3, we have :

Theorem 2.3. *Let $c > 0$. For any $T > T^{\star\star}(c)$ and any initial data $(\boldsymbol{y^0}, \boldsymbol{y^1}) \in \boldsymbol{H_K} \times \boldsymbol{V'_K}$, there exists a control function $v \in L^2(0,T)$ which drives to rest at time T the solution $\boldsymbol{y}$ of (5) associated with $(\boldsymbol{y^0}, \boldsymbol{y^1})$. Moreover, the control of minimal L^2-norm is given by $v = (\phi_{1,1} + c\phi_3)(1,\cdot)$ where $\boldsymbol{\phi}$ is solution of (6) and associated with $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1})$ minimum of $\mathcal{J}$ defined by (12) over $\boldsymbol{V_K} \times \boldsymbol{H_K}$. Finally, this control converges weakly in $L^2(0,T)$ as c goes to zero toward the control of minimal L^2-norm which drives to rest the solution $\overline{y}$ of the wave equation: $\overline{y}'' - \overline{y}_{,11} = 0$ in q_T, $(\overline{y}(\xi,0), \overline{y}'(\xi,0)) = (y_1^0, y_1^1)$.*

As $c \to 0$, the state equation of (5) degenerates into two uncoupled equations : the wave equation $y_1'' - y_{1,11} = 0$ controlled by v and $y_3'' = 0$. The uniform controllability holds here, because, as $c \to 0$, the initial condition $(\boldsymbol{y^0}, \boldsymbol{y^1}) \in \boldsymbol{H_K} \times \boldsymbol{V'_K}$ degenerates into $((y_1^0, 0), (y_1^1, 0))$, so that at the limit the component y_3 solution of the Cauchy problem $y_3''(\xi,t) = 0$, $t > 0$, $(y_3(\xi,0), y_3'(\xi,0)) = (0,0)$ vanishes.

Remark 2. Another natural control for (5) consists in acting on the longitudinal strain $y_{1,1} + cy_3$ of the arch. Thus, in the case $v(t) = (y_{1,1} + cy_3)(1,t)$ (corresponding to a Neumann control), we get similar controllability results. In particular, $\sigma(\boldsymbol{A_M}) = \{0, \lambda_k, k \geq 1\}$ with $0 \in \sigma_{ess}(\boldsymbol{A_M})$ (we recall that the essential spectrum does not depend on the boundary condition).

2.3. **Uniform partial controllability.** We now emphasize that the partial controllability do hold uniformly with respect to the data. For any $T > 0$ large enough, $(\boldsymbol{y^0}, \boldsymbol{y^1}) \in \boldsymbol{H} \times \boldsymbol{V'}$, (y_0^T, y_1^T) in $L^2(\omega) \times H^{-1}(\omega)$, we study the existence of a control function $v \in L^2(0,T)$ such that the tangential displacement y_1 of (5) satisfies the condition

$$(y_1(\cdot,T), y_1'(\cdot,T)) = (y_T^0, y_T^1) \quad \text{in} \quad \omega. \tag{13}$$

We want to control only y_1, the normal component y_3 being free. We refer to [20] for a similar - but different - treatment in a case where $\sigma_{ess}(\boldsymbol{A_M}) \in \mathbb{R}_+^\star$. Eliminating the variable y_3, one may reformulate the problem; y_3 solution of the ordinary differential equation $y_3'' + c^2 y_3 = -cy_{1,1}$ takes the expression

$$y_3(\xi,t) = y_3^0(\xi)\cos(c\,t) + \frac{y_3^1(\xi)}{c}\sin(c\,t) - \int_0^t \sin(c(t-u))y_{1,1}(\xi,u)du. \tag{14}$$

Assuming to simplify the notation that $y_3^0 = y_3^1 = 0$ in ω, and reporting (14) in the first equation of (5), we obtain that y_1 is solution of the following integro-differential equation

$$y_1''(\xi,t) - y_{1,11}(\xi,t) + c\int_0^t \sin(c(t-u))y_{1,11}(\xi,u)du = 0, \quad (\xi,t) \in q_T \tag{15}$$

submitted to the boundary and initial condition

$$y_1(0,t) = 0,\ y_1(1,t) = v(t),\ t \in (0,T), \quad (y_1(\xi,0), y_1'(\xi,0)) = (y_1^0(\xi), y_1^1(\xi)),\ \xi \in \omega. \tag{16}$$

For any $v \in L^2(0,T)$, the well-posedness of (15)-(16) is a consequence of the well-posedness of the full system considered in the previous section. This system, not reversible in time, is reminiscent of systems with memory (see [21], tome 2, chapter 7). The crucial difference is the presence of an essential spectrum.

We consider the homogeneous adjoint problem

$$\begin{cases} z''(\xi,t) - z_{,11}(\xi,t) + c\displaystyle\int_t^T \sin(c(u-t))z_{,11}(\xi,u)du = 0, & (\xi,t)\in q_T \\ z(0,t) = z(1,t) = 0, & t\in(0,T) \\ (z(\cdot,T), z'(\cdot,T)) = (z^0, z^1), & \xi\in\omega \end{cases} \tag{17}$$

and put

$$G_z^\star(\xi,t) = \int_t^T \sin(c(u-t))z(\xi,u)du.$$

Then, proceeding as in the previous section, the controls are formally characterized by the following equality

$$\int_0^1 [y_1'(\cdot,T)z(\cdot,T) - y_1(\cdot,T)z'(\cdot,T)]d\xi + \int_0^T v(t)\Big(z_{,1} - c\,G^\star_{z,1}\Big)(1,t)dt = 0. \tag{18}$$

Defining the application $\mu : H_0^1(\omega)\times L^2(\omega) \to H^{-1}(\omega)\times L^2(\omega)$, $\mu\{z^0,z^1\} = \{y_T^1, -y_T^0\}$, (18) is equivalent to

$$< \mu\{z^0,z^1\}, \{z^0,z^1\} >_{(H^{-1}\times L^2),(H_0^1\times L^2)} = \int_0^T y(1,t)\Big(z_{,1} - c\,G^\star_{z,1}\Big)(1,t)dt$$

and the controllability problem is reduced to the injectivity of the application μ, i.e. the existence of $C>0$ such that

$$\|(z^0,z^1)\|^2_{H_0^1\times L^2} \le C\int_0^T |(z_{,1} - c\,G^\star_{z,1})(1,t)|^2\,dt, \quad \forall (z^0,z^1)\in H_0^1(\omega)\times L^2(\omega).$$

If this inequality holds, the control of minimal L^2-norm is given by

$$v(t) = (z_{,1} - c\,G^\star_{z,1})(1,t), \quad t\in(0,T).$$

Once again, this question may be addressed via a spectral analysis. Since $z \in C([0,T]; H_0^1(\omega))$, we assume that the solution z of the adjoint problem (17) can be written as

$$z(\xi,t) = \sum_{k\ge1} f_k(t)\sin(k\pi\xi)$$

with $(z(\xi,T), z'(\xi,T)) = (z^0(\xi), z^1(\xi)) = \sum_{k\ge1}(f_k^T, f_k^{T\prime})\sin(k\pi\xi) \in H_0^1(\omega)\times L^2(\omega)$ so that $\sum_{k\ge0}(k\pi)^2(f_k^T)^2 < \infty$ and $\sum_{k\ge0}(f_k^{T\prime})^2 < \infty$. From (17), f_k solves the equation

$$f_k''(t) + (k\pi)^2 f_k(t) - c(k\pi)^2\int_t^T \sin(c(u-t))f_k(u)du = 0, \quad \forall k\ge1. \tag{19}$$

Then, derivating twice in time this equality, we get

$$f_k^{(4)}(t) + (k\pi)^2 f_k''(t) + r^{-3}(k\pi)^2\int_t^T \sin(c(u-t))f_k(u)du - c^2(k\pi)^2 f_k(t) = 0, \quad k\ge1$$

and then using (19), we eliminate the integral term to obtain the fourth order ODE:

$$f_k^{(4)}(t) + ((k\pi)^2 + c^2)f_k^{(2)}(t) = 0, \quad \forall k\ge1$$

so that

$$f_k(t) = C_k^{(1)} + C_k^{(2)}(t-T) + C_k^{(3)}\cos(\mu_k(t-T)) + C_k^{(4)}\sin(\mu_k(t-T)), \quad \forall k\ge1.$$

We recall that $\mu_k = \sqrt{c^2 + (k\pi)^2}$. Reporting this expression in the second order differential equation (19), we obtain that

$$C_k^{(3)} = C_k^{(1)} \frac{(k\pi)^2}{c^2}, \quad C_k^{(4)} = C_k^{(2)} \frac{(k\pi)^2}{c^2 \mu_k}, \quad \forall k \geq 1.$$

Finally, from $f_k(T) = f_k^T$ and $f_k'(T) = f_k^{T\prime}$, we obtain the following expression

$$f_k(t) = \frac{c^2}{\lambda_k} f_k^T + \frac{c^2}{\lambda_k} f_k^{T\prime}(t-T) + \frac{(k\pi)^2}{\lambda_k} f_k^T \cos(\mu_k(t-T)) + \frac{(k\pi)^2}{\lambda_k^{3/2}} f_k^{T\prime} \sin(\mu_k(t-T))$$

and the expression of the adjoint solution

$$\begin{aligned} z(\xi,t) = \sum_{k\geq 1} \biggl(\frac{c^2}{\lambda_k} f_k^T + \frac{c^2}{\lambda_k} f_k^{T\prime}(t-T) + \frac{(k\pi)^2}{\lambda_k} f_k^T \cos(\mu_k(t-T)) \\ + \frac{(k\pi)^2}{\lambda_k^{3/2}} f_k^{T\prime} \sin(\mu_k(t-T)) \biggr) \sin(k\pi\xi). \end{aligned} \tag{20}$$

As in the previous section, the Fourier expansion of the adjoint solution is expressed in terms of two sums: the first one, linear in time, is associated with the sequence $\{\lambda_k^-\}_{k\geq 0}$, $\lambda_k^- = 0 \in \sigma_{ess}(\boldsymbol{A_M})$, for all $k \geq 1$. The second one, periodic and bounded in time, corresponds to the discrete spectrum $\{\lambda_k^+\}_{k\geq 1} = \{\lambda_k\}_{k\geq 1}$. The fundamental difference here, with respect to the expansion (7), comes from the fact that the coefficient of these sums are related each other.

Using (20), we now evaluate the quantity $v(t) = z_{,1}(1,t) - c\, G^\star_{z,1}(1,t)$. We have

$$v(t) = \sum_{k\geq 1} (-1)^k (k\pi) \biggl(f_k(t) - c \int_t^T \sin(c(u-t)) f_k(u) du \biggr)$$

and from (19)

$$v(t) = \sum_{k\geq 1} (-1)^{k+1} \frac{f_k''(t)}{(k\pi)} = \sum_{k\geq 1} (-1)^k k\pi \biggl(f_k^T \cos(\mu_k(t-T)) + \frac{f_k^{T\prime}}{\mu_k} \sin(\mu_k(t-T)) \biggr). \tag{21}$$

Therefore

$$\begin{aligned} \int_0^T |v(t)|^2 dt &= \frac{1}{4} \int_0^T \biggl(\sum_{k\geq 1} \Bigl[a_k e^{i\mu_k(t-T)} + \overline{a_k} e^{-i\mu_k(t-T)} \Bigr] \biggr)^2 dt \\ &= \frac{1}{4} \int_0^T \biggl(\sum_{k\geq 1} \Bigl[a_k e^{-i\mu_k s} + \overline{a_k} e^{i\mu_k s} \Bigr] \biggr)^2 ds \end{aligned}$$

with $a_k = (-1)^k (k\pi)(f_k^T - i\mu_k^{-1} f_k^{T\prime})$ for all $k \geq 1$ and then

$$\int_0^T |v(t)|^2 dt = \frac{1}{4} \int_0^T \biggl(\sum_{k\in\mathbb{Z}, k\neq 0} A_k e^{i\Lambda_k s} \biggr)^2 ds$$

with $(A_k, \Lambda_k) = (\overline{a_k}, \mu_k)$ if $k > 0$ and $(A_k, \Lambda_k) = (\overline{a_{-k}}, -\mu_k)$ if $k < 0$. We then can apply the Ingham type Theorem 2.1 with $K = \mathbb{Z}\backslash\{0\}$ to get the equivalence

$$\int_0^T |v(t)|^2 dt \approx \sum_{k\in\mathbb{Z}, k\neq 0} |A_k|^2$$

under the condition $T > 2\pi/\gamma$ with $\gamma = \inf_{k,p\in\mathbb{Z},k\neq p} |\Lambda_p - \Lambda_k|$. We easily obtain that $\gamma = \sqrt{c^2+4\pi^2} - \sqrt{c^2+\pi^2} = \gamma^{\star\star}(c)$ so that $2\pi/\gamma = T^{\star\star}(c)$. Finally, the sum

$$\sum_{k\in\mathbb{Z}} |A_k|^2 = 2\sum_{k>0}(k\pi)^2\left((f_k^T)^2 + \frac{(f_k^{T\prime})^2}{\lambda_k}\right)$$

is equivalent to the $H_0^1 \times L^2$ norm of the data (z^0, z^1) :

$$\|(z^0,z^1)\|^2_{H_0^1(\omega)\times L^2(\omega)} \approx \frac{1}{2}\sum_{k\geq 1}\left((k\pi)^2(f_k^T)^2 + (f_k^{T\prime})^2\right).$$

We therefore conclude that the map μ is an isomorphism from $H_0^1(\omega) \times L^2(\omega)$ into $L^2(\omega) \times H^{-1}(\omega)$. Thus, in the partial controllability case, the property holds uniformly with respect to the initial data (y_1^0, y_1^1) and c.

Theorem 2.4. *For any $T > T^{\star\star}(c)$, and any $(y^0, y^1), (y_T^0, y_T^1) \in L^2(\omega) \times H^{-1}(\omega)$, there exists a control function $v \in L^2(0,T)$ which drives the solution (y^0, y^1) to the target (y_T^0, y_T^1) at time T. The control of minimal L^2-norm is given by*

$$v(t) = (z_{1,1} - c\,G^{\star}_{z,1})(1,t), \quad t \in (0,T)$$

where z is the solution of the adjoint system (17) who initial state (z^0, z^1) minimizes the coercive and convex functional $\mathcal{I} : H_0^1(\omega) \times L^2(\omega) \to \mathbb{R}$:

$$\mathcal{I}(z^0,z^1) = \frac{1}{2}\int_0^T |(z_{1,1} - c\,G^{\star}_{z,1})(1,t)|^2 dt - < (z^0,z^1),(y_T^1, -y_T^0) >_{(H_0^1,L^2)\times(H^{-1},L^2)} .$$

3. **Numerical experiments I.** From duality arguments, the determination of controls for (5) is reduced to the minimization of the functional $\mathcal{J}$ - defined in (12)- over $\boldsymbol{V}^{\perp} \times \boldsymbol{H}^{\perp}$ (resp. $\boldsymbol{V_K} \times \boldsymbol{H_K}$), for any initial data $(\boldsymbol{y^0}, \boldsymbol{y^1}) \in \boldsymbol{H}^{\perp} \times \boldsymbol{V'}^{\perp}$ and $T > T^{\star}(c)$ (resp. $(\boldsymbol{y^0}, \boldsymbol{y^1}) \in \boldsymbol{H_K} \times \boldsymbol{V'_K}$ and $T > T^{\star\star}(c)$). The minimization provides the control of minimal $L^2(\Sigma_T)$-norm: indeed, we have

$$\inf_{(\boldsymbol{\phi^0},\boldsymbol{\phi^1})} \mathcal{J}(\boldsymbol{\phi^0},\boldsymbol{\phi^1}) = -\inf_{v\in\mathcal{C}} \frac{1}{2}\|v\|^2_{L^2(\Sigma_T)}$$

with $\mathcal{C} = \{v \in L^2(\Sigma_T), v\text{-null control for system (5)}\}$. Since $\mathcal{J}$ is coercive over $\boldsymbol{V}^{\perp} \times \boldsymbol{H}^{\perp}$, the conjugate gradient algorithm is commonly used in such situation (see [13] and [25] for details). At each iteration, a projection of the descent direction on the orthogonal of $Ker\,\boldsymbol{A_M}$ is needed (in order to enforce that the minimizer belongs to $\boldsymbol{V}^{\perp} \times \boldsymbol{H}^{\perp}$). Without this projection (which requires a precise characterization of the orthogonal of the kernel), the algorithm does not converge, even when $(\boldsymbol{y^0}, \boldsymbol{y^1}) \in \boldsymbol{H}^{\perp} \times \boldsymbol{V'}^{\perp}$. This is partially due to the high sensibility of controllability with respect to discretization (see [13] chapter 6, [24] and the references therein).

Now, if we start with an initial condition which is not in $\boldsymbol{V}^{\perp} \times \boldsymbol{H}^{\perp}$, then the residual of the algorithm (provided a careful projection just mentioned) first decreases (the method stabilizes the controllable part) then diverge. Since the Fourier decomposition of the solution is explicitly known here, it is simpler to solve directly the optimal condition associated to $\mathcal{J}$ with respect to the coefficient A_k, B_k of $\boldsymbol{\phi^0}, \boldsymbol{\phi^1}$ expressed as follows:

$$(\boldsymbol{\phi^0_N}, \boldsymbol{\phi^1_N}) = (A_0, \mu_0 B_0)\boldsymbol{v_0} + \sum_{k=1}^{N}(A_k, \mu_k B_k)\boldsymbol{v_k} \tag{22}$$

with $N \in \mathbb{N}$ large enough. This reduces the controllability problem to the inversion of a positive symmetric matrix of order $2N$. Up to the truncation of the Fourier series, this method (used in [10] for hemispherical cap) is exact. The corresponding control permits to stabilize at time T large enough the components of the data $(\boldsymbol{y^0}, \boldsymbol{y^1})$ which are in $\boldsymbol{H}^{\perp} \times \boldsymbol{V}^{\perp\prime}$ and leaves uncontrolled the remaining components. If such components are presents, the solution at time T is not stabilized but fulfills the relation $y_{1,1}(\xi, T) + cy_3(\xi, T) = 0$ (i.e. $\boldsymbol{y}(\cdot, T) \in Ker\boldsymbol{A_M}$).

3.1. **Non uniform exact controllability.** We consider arbitrarily the following initial condition

$$\boldsymbol{y^0} = \alpha \boldsymbol{v_0} + \boldsymbol{v_1} \quad \boldsymbol{y^1} = \alpha \mu_0 \boldsymbol{v_0} + \mu_2 \boldsymbol{v_2} \tag{23}$$

with $\alpha \in \{0, 1\}$. Note that we consider only low frequencies modes in order to avoid some spurious numerical effect (analyzed in [24] in the case of the wave equation). The control obtained with $\alpha = 1$, $c = \pi/2$ and $T = 3.5 > T^\star(c) \approx 3.2552$ is given on Figure 1-left. In order to have a smooth control at time $t = 0$ and at time $t = T$, we impose that the control v is of the form $v(\xi, t) = \rho(t)(\phi_{1,1}(\xi, t) + c\phi_3(\xi, t))$ where $\rho \in C^1_c([0, T]; [0, 1])$ is a compact support function. We take $\rho(t) = \sin^2(\pi t/T)$. The corresponding energy and kinetic energy are given on Figure 1-right. The controlled and uncontrolled solution $\boldsymbol{y}$ are depicted on Figure 2 and 3 respectively. The evolution of the arch in the cartesian [1] plane (O, x_1, x_2) with respect to the time is finally described on Figure 4.

Let us now analyze the behavior of the control with respect to c. For $\alpha = 1$, Table 1 gives the value of the L^2-norm of v for several values of c. As expected from the definition of $T^\star(c)$, these values suggest that the norm of the control is not uniformly bounded with respect to c. Similarly, the behavior of the ratio $\|v\|^2_{L^2(0,T)}/\|(\boldsymbol{\phi^0}, \boldsymbol{\phi^1})\|^2_{\boldsymbol{V}\times\boldsymbol{H}}$ of order $O(c^2)$ which appears in the observability inequality indicates that the functional $\mathcal{J}$ is not uniformly coercive. On the contrary, when the

	$c = \pi$	$c = \pi/4$	$c = \pi/16$	$c = \pi/64$	$c = \pi/256$
♯ Iteration	8	8	9	7	8
$\|v\|_{L^2(0,T)}$	1.415	1.601	4.823	15.791	59.220
$E(0)/E(T)$	2.51×10^{-5}	4.47×10^{-7}	3.12×10^{-6}	3.26×10^{-5}	4.49×10^{-4}
$\frac{\|v\|^2_{L^2(0,T)}}{\|(\boldsymbol{\phi^0}, \boldsymbol{\phi^1})\|^2_{\boldsymbol{V}\times\boldsymbol{H}}}$	5.33×10^{-1}	7.95×10^{-2}	1.67×10^{-3}	9.86×10^{-5}	6.18×10^{-6}

TABLE 1. $\alpha = 1$ - Evolution of the L^2-norm of the control vs. the curvature c.

constant are eliminated in y^0_3 and y^1_3 (corresponding to $\alpha = 0$ in (23)), a uniform behavior with respect to c is recovered in agreement with Theorem 2.3 (see Table 2). Moreover, we then check as c goes to 0, the convergence of v toward the control $v_{c=0}$ associated with the wave equation (see Table 3).

We insist on the fact that the practical computation of controls by the standard dual approach is feasible only thank to the precise characterization of the subspace of controllable initial data. For general operator for which the spectrum is in general unknown, such approach is ineffective.

[1] We recall that the displacement $\boldsymbol{Y}$ in the global cartesian frame $(O, \boldsymbol{e_1}, \boldsymbol{e_2})$ is given by $Y(\boldsymbol{x}, t) = y_1(\xi, t)\boldsymbol{\tau} + y_3(\xi, t)\boldsymbol{\nu}$ where the tangential and normal vector are defined by $\boldsymbol{\tau} = (\cos(c\xi), -\sin(c\xi)$ and $\boldsymbol{\nu} = \sin(c\xi), \cos(c\xi)$ (see [5, 25]).

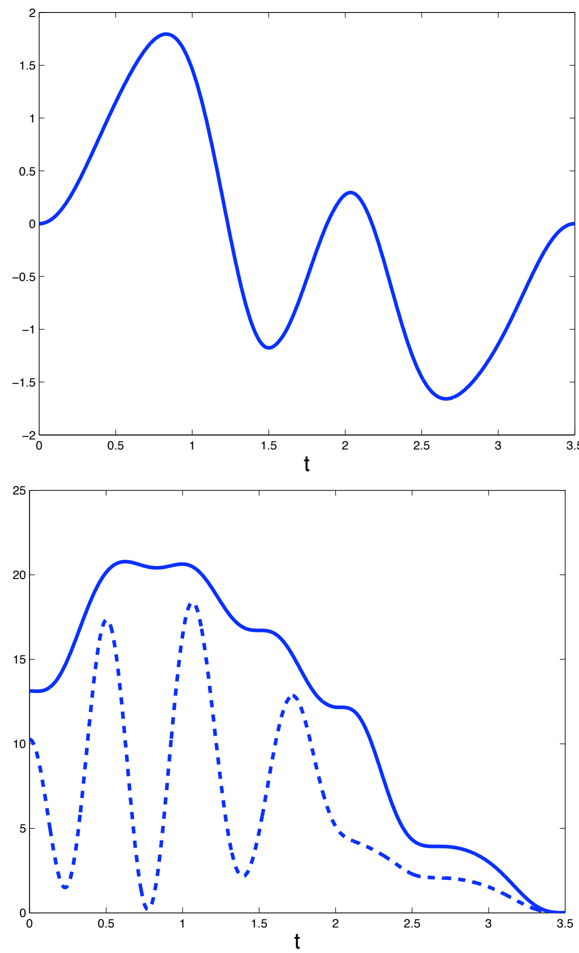

FIGURE 1. **Top**: Control of minimal $L^2(\Sigma_T)$-norm v vs. $t \in (0,T)$; **Bottom**: Energy (solid line) and kinetic energy (dashed line) vs. $t \in (0,T)$.

	$c=\pi$	$c=\pi/4$	$c=\pi/16$	$c=\pi/64$	$c=\pi/256$
$\sharp$ Iteration	5	4	4	4	4
$\Vert v\Vert_{L^2(0,T)}$	0.823	0.692	0.679	0.678	0.678
$E(0)/E(T)$	1.12×10^{-6}	8.47×10^{-8}	4.28×10^{-6}	2.83×10^{-7}	1.05×10^{-7}
$\frac{\Vert v\Vert^2_{L^2(0,T)}}{\Vert(\boldsymbol{\phi^0},\boldsymbol{\phi^1})\Vert^2_{\boldsymbol{V}\times\boldsymbol{H}}}$	2.019	1.565	1.515	1.5126	1.5124

TABLE 2. $\alpha = 0$ - Evolution of the L^2-norm of the control vs. the curvature c.

3.2. **Uniform partial controllability.** We now minimize the functional $\mathcal{I}$ defined in Theorem 2.4. Since the controllability is uniform, one may use here the conjugate gradient algorithm (without any use of projection). We assume that the initial condition are zero and use as a target for the tangential displacement y_1, the initial

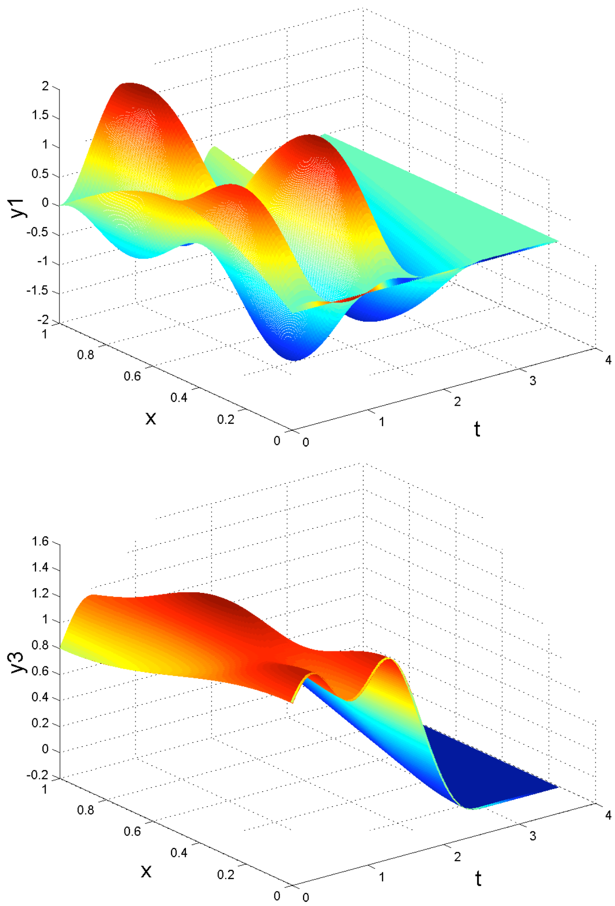

FIGURE 2. Controlled solution $\boldsymbol{y} = (y_1, y_3)$

	$c=\pi$	$c=\pi/4$	$c=\pi/16$	$c=\pi/64$	$c=\pi/256$
$\frac{\Vert v-v_{c=0}\Vert_{L^2(0,T)}}{\Vert v_{c=0}\Vert_{L^2(0,T)}}$	1.679	1.24×10^{-1}	7.87×10^{-3}	5.77×10^{-4}	4.28×10^{-5}

TABLE 3. $\alpha = 0$ - Evolution of $\Vert v - v_{c=0}\Vert_{L^2(0,T)}$ vs. the curvature c - $\Vert v - v_{c=0}\Vert_{L^2(0,T)}/\Vert v_{c=0}\Vert_{L^2(0,T)} \approx O(c^2)$.

condition in (23):

$$(y_T^0(\xi), y_T^1(\xi)) = (\sin(\pi\xi), \mu_2 \sin(2\pi\xi)), \quad \xi \in \omega. \tag{24}$$

According to (14), the state of y_3 at time $t = T$ is then

$$(y_3(\xi,T), y_3'(\xi,T)) = \Big(-\int_0^T \sin(c(T-u)) y_{1,1}(\xi,u)du, -c\int_0^T \cos(c(T-u)) y_{1,1}(\xi,u)du\Big). \tag{25}$$

From the analysis performed in the previous section, the partial control v is such that both $(y_T^0, y_3(\cdot,T)), (y_T^1, y_3'(\cdot,T))$ belong to $(Ker\boldsymbol{A_M})^{\perp}$. We take $T = 3.5$ and $c = \pi/2$. Table 4 reports the L^2-norm of the control with respect to c. We check that

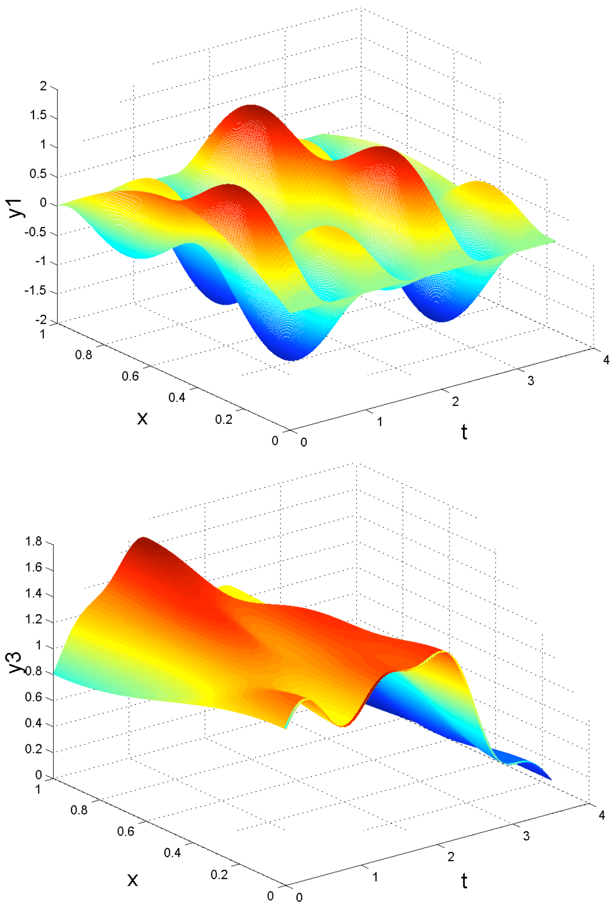

FIGURE 3. Free (uncontrolled) solution $\boldsymbol{y} = (y_1, y_3)$.

	$c = \pi$	$c = \pi/4$	$c = \pi/16$	$c = \pi/64$	$c = \pi/256$
$\Vert v\Vert_{L^2(0,T)}$	7.83×10^{-1}	6.17×10^{-1}	6.08×10^{-1}	6.07×10^{-1}	6.07×10^{-1}

TABLE 4. Evolution of the L^2-norm of the partial control vs. the curvature c.

the minimal L^2- norm of the control which drives the solution of (5) from $(\mathbf{0}, \mathbf{0})$ to $(y_T^0, y_3(\cdot, T)), (y_T^1, y_3'(\cdot, T))$ is lower than the control which drives the solution from $(\boldsymbol{y^0}, \boldsymbol{y^1})$ to $(\mathbf{0}, \mathbf{0})$. Moreover, as c goes to zero, the limit value is different from the one in Table 2. The corresponding control and energy are depicted on Figure 5. The evolution of the controlled component y_1 is described on Figure 6-Left. We check the simulation by computing the norm $\Vert y_1(T) - y_T^0\Vert_{L^2(\omega)} \approx 4.31 \times 10^{-5}$ and that $\Vert y_1'(T) - y_T^1\Vert_{H^{-1}(\omega)} \approx 7.32 \times 10^{-4}$ (we remind that, for any $f \in H^{-1}(\omega)$, $\Vert f\Vert_{H^{-1}(\omega)} = \Vert\tilde{f}\Vert_{H_0^1(\omega)}$ where $\tilde{f} = \tilde{f}(f)$ is the solution of $-\Delta\tilde{f} = f$ in ω, $\tilde{f} = 0$ on $\partial\omega$). The corresponding evolution of the (free) component y_3 is described on Figure 6-Right.

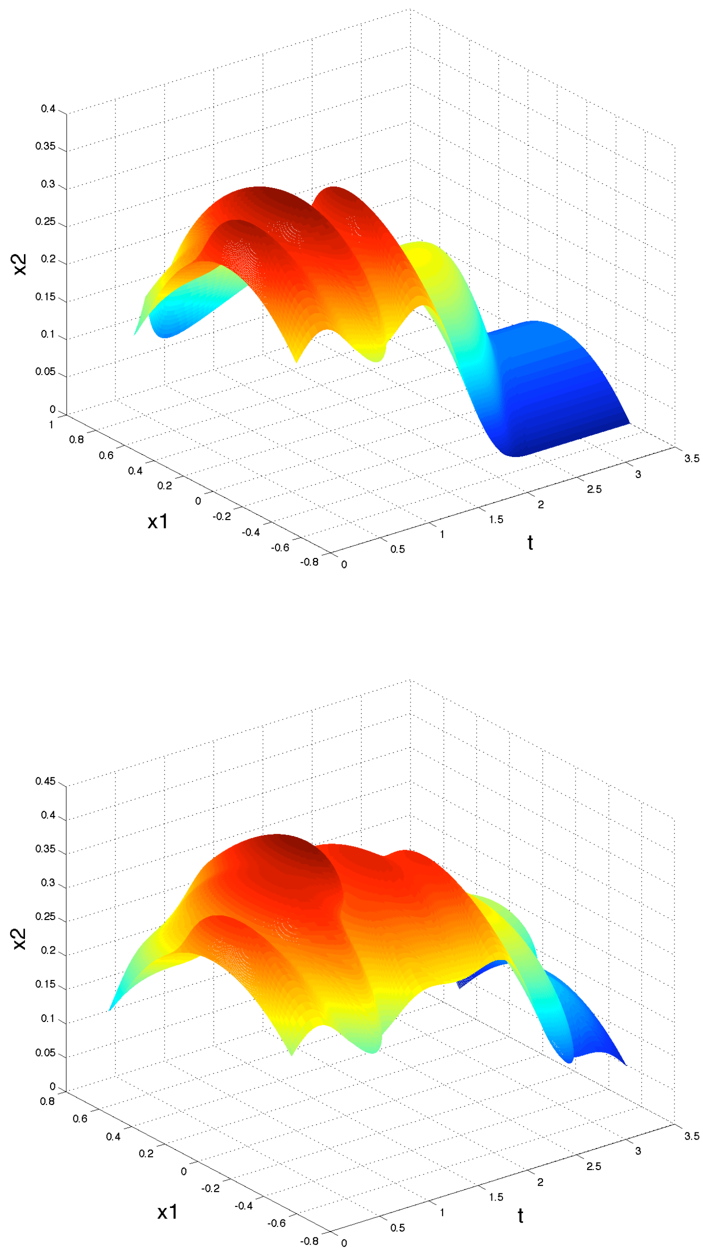

FIGURE 4. Evolution of the controlled (**top**) and uncontrolled (**bottom**) arch in the plane (O, x_1, x_2) vs. $t \in (0, T)$.

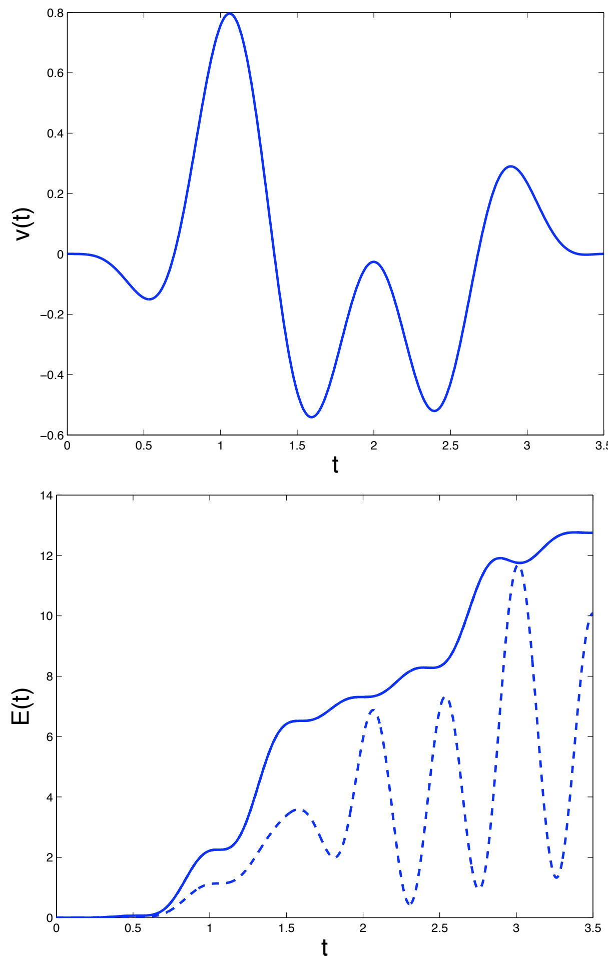

FIGURE 5. **Top**: Partial control v vs. $t \in (0,T)$; **Bottom**: Energy (solid line) and kinetic energy (dashed line) vs. $t \in (0,T)$.

4. **A variational approach.** We now describe an approach which does not make use of the spectral representation of the operator but still allows to approximate exact or partial controls for system (5). The main idea of this (variational) method, as introduced in [27], consists in setting up an error functional which measures the deviation of functions from being a solution of the underlying system, and minimizing such error over the class of feasible functions that comply with initial, boundary, and final conditions.

4.1. **Variational approach.** Consider the convex class of functions

$$\mathcal{A} = \Big\{ \boldsymbol{y} \in H^2(0,T;H^1(\omega)) \times H^2(0,T;L^2(\omega)) : (\boldsymbol{y}(\xi,0), \boldsymbol{y'}(\xi,0)) = (\boldsymbol{y^0}, \boldsymbol{y^1}), \\ (\boldsymbol{y}(\xi,T), \boldsymbol{y'}(\xi,T)) = (\boldsymbol{0},\boldsymbol{0}), \xi \in \omega, y_1(0,t) = 0, t \in (0,T) \Big\} \tag{26}$$

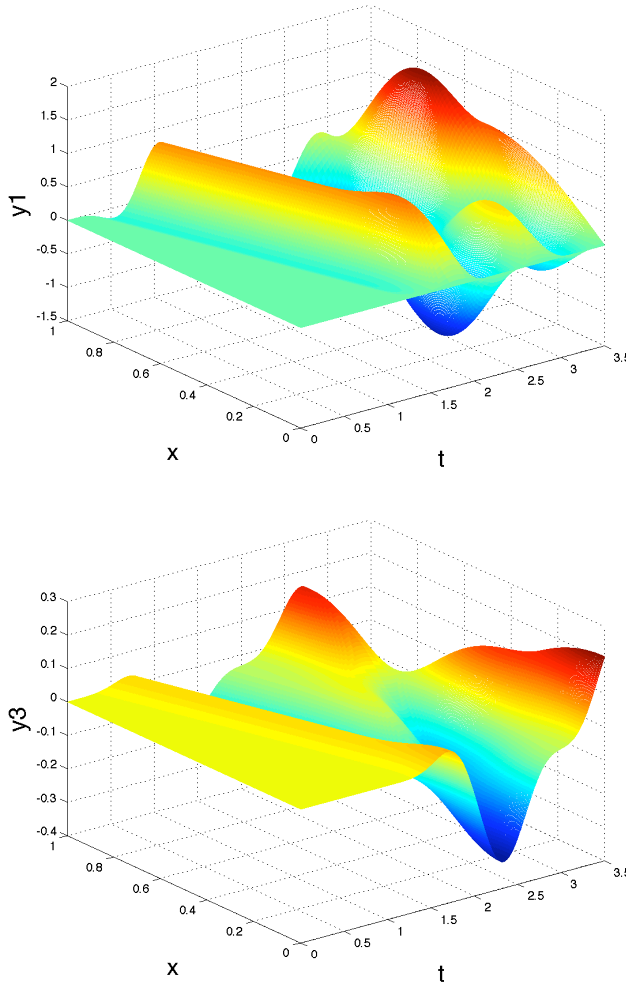

FIGURE 6. Evolution of the controlled component y_1 vs. t (**top**) and corresponding evolution of the component y_3 (**bottom**), starting from $(\mathbf{0},\mathbf{0})$.

assumed non empty. This requirement simply demands some compatibility with the vanishing boundary data for $\xi = 0$, precisely, $y_1^0(0) = 0$, and that $\boldsymbol{y}^{\mathbf{0}}$, as the trace of an $H^2(H^1)$ function over q_T be slightly more regular than $L^2(\omega)$. For any $\boldsymbol{y} \in \mathcal{A}$, we define its corrector $\boldsymbol{v}$ over q_T as the unique solution in $H^1(q_T)$ of the elliptic system

$$\begin{cases} -u_1'' - u_{1,11} = -y_1'' + (y_{1,1} + cy_3)_{,1}, & (\xi,t) \in q_T \\ -u_3'' - u_{3,11} + c^2 u_3 = -y_3'' - c(y_{1,1} + cy_3), & (\xi,t) \in q_T \\ \boldsymbol{u}(0,t) = \boldsymbol{u}(1,t) = 0, & t \in (0,T) \\ \boldsymbol{u'}(\xi,0) = \boldsymbol{u'}(\xi,T) = 0, & t \in (0,T). \end{cases} \tag{27}$$

Then, we define the following quadratic minimization problem :

$$\inf_{\boldsymbol{y}\in\mathcal{A}} E(\boldsymbol{y}) = \frac{1}{2}\iint_{q_T} (|\boldsymbol{u'}|^2 + |\boldsymbol{u}_{,1}|^2 + c^2|u_3|^2) d\xi\, dt \tag{28}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{y})$ is the corrector function associated with $\boldsymbol{y}$. It turns that if (and only if) the infimum is attained in $\mathcal{A}$ and is equal to 0, then the corresponding minimizer $\boldsymbol{y} \in \mathcal{A}$ is a solution of the system (5) (since then $\boldsymbol{u} = \boldsymbol{0}$). Therefore, such solution $\boldsymbol{y}$ is a controlled solution of the system and a boundary control v is simply obtained by taking the trace of y_1 at $\xi = 1$: $v(t) = y_1(t), t \in (0, T)$. In that case, the control, as a trace of an $H^2(H^1)$ function is slightly more regular than $L^2(0, T)$.

Therefore, it amounts proving that, first, the infimum of the error E functional is attained, and second, that it vanishes. The first can be proved in a general framework using the quadratic nature of E and the linear dependance of $\boldsymbol{u}$ with respect to $\boldsymbol{y}$ (we refer to [26, 27]). The second may be obtained, as we will see, by writing down the optimality conditions of E. Note also that these two points are also the consequence of the controllability property proved in Section 2.2, when the initial data $(\boldsymbol{y^0}, \boldsymbol{y^1})$ to be controlled belongs to the orthogonal of $Ker(\boldsymbol{A_M})$ and are the restriction at $t = 0$ of $H^2(H^1) \times H^2(L^2)$ functions.

Let us turn to optimality. We introduce the set of admissible variations of $\boldsymbol{y}$ as follows:

$$\mathcal{A}_0 = \Big\{ \boldsymbol{Y} \in H^2(0, T; H^1(\omega)) \times H^2(0, T; L^2(\omega)) : (\boldsymbol{Y}(\xi, 0), \boldsymbol{Y'}(\xi, 0)) = (0, 0), (\boldsymbol{Y}(\xi, T), \boldsymbol{Y'}(\xi, T)) = (\boldsymbol{0}, \boldsymbol{0}), \xi \in \omega, Y_1(0, t) = 0, t \in (0, T) \Big\} \tag{29}$$

and then define the variation of $E(\boldsymbol{y})$, $\boldsymbol{y} \in \mathcal{A}$ in the direction $\boldsymbol{Y} \in \mathcal{A}_0$

$$< E'(\boldsymbol{y}), \boldsymbol{Y} >= \lim_{\eta\to 0} \frac{E(\boldsymbol{y} + \eta\boldsymbol{Y}) - E(\boldsymbol{y})}{\eta}.$$

We obtain that

$$< E'(\boldsymbol{y}), \boldsymbol{Y} >= \iint_{q_T} (u'\boldsymbol{U'} + \boldsymbol{u_{,1}} \boldsymbol{U_{,1}} + c^2 u_3 U_3)\, dx\, dt \tag{30}$$

where $\boldsymbol{U} \in H^1_{0,x}(q_T)$ is the corrector function associated with $\boldsymbol{Y} \in \mathcal{A}_0$, that is, the solution of

$$\begin{cases} -U_1'' - U_{1,11} = -Y_1'' + (Y_{1,1} + cY_3)_{,1}, & (\xi, t) \in q_T \\ -U_3'' - U_{3,11} + c^2 U_3 = -Y_3'' - c(Y_{1,1} + cY_3), & (\xi, t) \in q_T \\ \boldsymbol{U}(0, t) = \boldsymbol{U}(1, t) = 0, & t \in (0, T) \\ \boldsymbol{U'}(\xi, 0) = \boldsymbol{U'}(\xi, T) = 0, & t \in (0, T). \end{cases} \tag{31}$$

Multiplying the state equations (31) by $\boldsymbol{u}$, integrating by parts, and taking into account the boundary conditions on $\boldsymbol{v}$ and $\boldsymbol{Y}$, we transform (30) into

$$< E'(\boldsymbol{y}), \boldsymbol{Y} >= -\int_{q_T} \Big(\boldsymbol{Y''}\boldsymbol{u} + (Y_{1,1} + cY_3)(u_{1,1} + cu_3) \Big)\, d\xi\, dt, \ \forall \boldsymbol{Y} \in \mathcal{A}_0. \tag{32}$$

Now, let us assume that $\boldsymbol{y} \in \mathcal{A}$ is a minimizer for E, so that $< E'(\boldsymbol{y}), \boldsymbol{Y} >= 0$ for all $\boldsymbol{Y} \in \mathcal{A}_0$. This equality implies that $\boldsymbol{u}$ satisfies the equation

$$\begin{cases} u_1'' - (u_{1,1} + cu_3)_{,1} = 0, & (\xi, t) \in q_T \\ u_3'' + c(u_{1,1} + cu_3) = 0, & (\xi, t) \in q_T \\ u_{1,1} + cu_3 = 0, & (\xi, t) \in \Sigma_T, \end{cases} \tag{33}$$

in addition to the boundary conditions $\boldsymbol{u}(0,t) = \boldsymbol{u}(1,t) = 0$ for $t \in (0,T)$ and $\boldsymbol{u'}(\xi,0) = \boldsymbol{u'}(\xi,T) = 0$ for $\xi \in (0,1)$. For $T > T^\star(c)$, this implies that $\boldsymbol{u} = 0$ in q_T, as a consequence of Proposition 2. Notice that this argument, within the framework of Proposition 2, implies that critical points of E can only occur at zero error.

We insist on the fact that this approach relies on minimization of the error functional, and does not make use of duality argument nor introduce any dual variable. Actually, this variational approach introduced in [27] is a least squares type method, as deeply discussed for instance in [12], chapter VII where the search of solution(s) for $F(u) = 0$, given any $F : H_0^1(\Omega) \to H^{-1}(\Omega)$ and a bounded domain $\Omega \in \mathbb{R}^N$, is replaced by the extremal problem:

$$\min_{u \in H_0^1(\Omega)} \|v\|^2_{H_0^1(\Omega)}$$

where $v = v(u)$ solves the elliptic problem : $-\Delta v = F(u)$ in Ω, $v = 0$ on $\partial\Omega$.

We refer to [6] where a different variational approach, based on Carleman estimates, leading to an elliptic problem defined on q_T has been introduced and analyzed. We refer to [26] where this approach has been used for the heat equation.

Remark 3. There are many ways to define the corrector $\boldsymbol{u}$. For instance, one may replace the state equation of (27) by

$$\begin{cases} y_1'' - u_1'' - (y_{1,1} + u_{1,1} + cy_3)_{,1} = 0, & (\xi, t) \in q_T \\ y_3'' - u_3'' + c(y_{1,1} + cy_3 + cu_3) = 0, & (\xi, t) \in q_T \\ u_1(0,t) = u_1(1,t) = 0, & t \in (0,T) \\ u'(\xi,0) = u'(\xi,T) = 0, & \xi \in \omega. \end{cases} \tag{34}$$

The procedure is the same, with now $\boldsymbol{u} \in H^1(q_T) \times H^1(0,T;L^2(\omega))$, u_3 being simply solution of an ODE of second order in time.

Remark 4. The partial controllability considered in Section 2.3 may be addressed in a similar way. It suffices to relax the conditions on y_3 at time T in $\mathcal{A}$ and work with:

$$\mathcal{A}_p = \Big\{ \boldsymbol{y} \in H^2(0,T;H^1(\omega)) \times H^2(0,T;L^2(\omega)) : (\boldsymbol{y}(\xi,0), \boldsymbol{y'}(\xi,0)) = (\boldsymbol{y^0}, \boldsymbol{y^1}), \\ (y_1(\xi,T), y_1'(\xi,T)) = (0,0), \xi \in \omega, y_1(0,t) = 0, t \in (0,T) \Big\}. \tag{35}$$

Remark 5. Let us assume that the initial datum $(\boldsymbol{y^0}, \boldsymbol{y^1})$ has a component in $Ker \boldsymbol{A_M}$. In that case, the infimum $E(\boldsymbol{y})$ is reached over $\mathcal{A}$ but is not zero and the corresponding corrector $\overline{\boldsymbol{u}}$ does not vanish identically over q_T. We note $\boldsymbol{y} \in \mathcal{A}$ the corresponding minimizer and $v(t) = y_1(1,t)$. Now, we note $\overline{\boldsymbol{y}}$ the solution of the backward problem

$$\begin{cases} \overline{y}_1'' - \overline{u}_1'' - (\overline{y}_{1,1} + \overline{u}_{1,1} + c\overline{y}_3)_{,1} = 0, & (\xi,t) \in q_T \\ \overline{y}_3'' - \overline{u}_3'' + c(\overline{y}_{1,1} + c\overline{y}_3 + c\overline{u}_3) = 0, & (\xi,t) \in q_T \\ \overline{y}_1 = 0, & (\xi,t) \in \partial\omega \times (0,T) \\ (\overline{\boldsymbol{y}}(\xi,T), \overline{\boldsymbol{y}}'(\xi,T)) = (\boldsymbol{0},\boldsymbol{0}), & \xi \in \omega. \end{cases}$$

Then, by linearity of system (5), the solution $(\boldsymbol{y} - \overline{\boldsymbol{y}})$ of

$$\begin{cases} (\boldsymbol{y} - \overline{\boldsymbol{y}})'' + \boldsymbol{A_M}(\boldsymbol{y} - \overline{\boldsymbol{y}}) = 0, & (\xi,t) \in q_T \\ (y_1 - \overline{y}_1)(0,t) = 0, (y_1 - \overline{y}_1)(1,t) = v(t), & (\xi,t) \in \partial\omega \times (0,T) \\ ((\boldsymbol{y} - \overline{\boldsymbol{y}})(\cdot,0), (\boldsymbol{y} - \overline{\boldsymbol{y}}')(\cdot,0)) = (\boldsymbol{y^0} - \overline{\boldsymbol{y}}(\cdot,0), \boldsymbol{y^1} - \overline{\boldsymbol{y}}(\cdot,0)), & \xi \in \omega \end{cases}$$

satisfies $((\boldsymbol{y} - \overline{\boldsymbol{y}})(\cdot,T), (\boldsymbol{y} - \overline{\boldsymbol{y}}')(\cdot,T)) = (\boldsymbol{0},\boldsymbol{0})$. Therefore, we have extract the controllable component $(\boldsymbol{y^0} - \overline{\boldsymbol{y}}(\cdot,0), \boldsymbol{y^1} - \overline{\boldsymbol{y}}(\cdot,0))$ from the initial datum $(\boldsymbol{y^0}, \boldsymbol{y^1})$. It is important to note that this does not require any spectral information on the operator $\boldsymbol{A_M}$: precisely, at the practical level, the lack of controllability is related to the property $\inf_{\boldsymbol{y}\in\mathcal{A}} E(\boldsymbol{y}) \neq 0$.

4.2. **Numerical resolution - Conjugate gradient algorithm.** Let us describe the procedure to approximate numerically the variational problem: $\min_{\boldsymbol{y}\in\mathcal{A}} E(\boldsymbol{y})$. Preliminary, since $\mathcal{A}$ is not an Hilbert space, we consider for any $\overline{\boldsymbol{y}} \in \mathcal{A}$ the equivalent problem $\min_{\boldsymbol{z}\in\mathcal{A}_0} E(\overline{\boldsymbol{y}} + \boldsymbol{z})$ and we endow the Hilbert space $\mathcal{A}_0$ with the scalar product

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathcal{A}_0} = \iint_{q_T} (\boldsymbol{u''}\,\boldsymbol{v''} + \boldsymbol{u'}\,\boldsymbol{v'} + u_{1,1}\,v_{1,1} + c^2 u_3\,v_3)\,d\xi dt, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}_0$$

and note that $\|\boldsymbol{z}\|_{\mathcal{A}_0} = \sqrt{(\boldsymbol{z},\boldsymbol{z})_{\mathcal{A}_0}}$ for all $\boldsymbol{z} \in \mathcal{A}_0$. We note $E(\overline{\boldsymbol{y}} + \boldsymbol{z}) = E_{\overline{\boldsymbol{y}}}(\boldsymbol{z})$ so that $\min_{\boldsymbol{z}\in\mathcal{A}_0} E(\overline{\boldsymbol{y}} + \boldsymbol{z}) = \min_{\boldsymbol{z}\in\mathcal{A}_0} E_{\overline{\boldsymbol{y}}}(\boldsymbol{z})$.

The Polak-Ribière version of the conjugate gradient (CG) algorithm to minimize $E_{\overline{\boldsymbol{y}}}$ over $\mathcal{A}_0$ is as follows (see [12]): for any $\overline{\boldsymbol{y}} \in \mathcal{A}$

- *Step 0: Initialization* - Given any $\varepsilon > 0$ and any $\boldsymbol{z^0} \in \mathcal{A}_0$, compute the residual $\boldsymbol{g^0} \in \mathcal{A}_0$ solution of

$$(\boldsymbol{g^0}, \boldsymbol{Y})_{\mathcal{A}_0} = < E'_{\overline{\boldsymbol{y}}}(\boldsymbol{z^0}), \boldsymbol{Y} > \quad \forall \boldsymbol{Y} \in \mathcal{A}_0.$$

  If $\|\boldsymbol{g^0}\|_{\mathcal{A}_0}/\|\boldsymbol{z^0}\|_{\mathcal{A}_0} \le \varepsilon$ take $\boldsymbol{z} = \boldsymbol{z^0}$ as an approximation of a minimum of $E_{\overline{\boldsymbol{y}}}$. Otherwise, set $\boldsymbol{w^0} = \boldsymbol{g^0}$.

  For $n \geq 0$, assuming $\boldsymbol{z^n}$, $\boldsymbol{g^n}$, $\boldsymbol{w^n}$ being known with $\boldsymbol{g^n}$ and $\boldsymbol{w^n}$ both different from zero, compute $\boldsymbol{z^{n+1}}, \boldsymbol{g^{n+1}}$, and if necessary $\boldsymbol{w^{n+1}}$ as follows:

- *Step 1: Steepest descent* - Set $\boldsymbol{z^{n+1}} = \boldsymbol{z^n} - \lambda_n \boldsymbol{w^n}$ where $\lambda_n \in \mathbb{R}$ is the solution of the one-dimensional minimization problem

$$\text{minimize} \quad E_{\overline{\boldsymbol{y}}}(\boldsymbol{z^n} - \lambda \boldsymbol{w^n}), \quad \text{over}\, \lambda \in \mathbb{R}. \tag{36}$$

  Then, compute the residual $\boldsymbol{g^{n+1}} \in \mathcal{A}_0$ from the relation

$$(\boldsymbol{g^{n+1}}, \boldsymbol{Y})_{\mathcal{A}_0} = < E'_{\overline{\boldsymbol{y}}}(\boldsymbol{z^{n+1}}), \boldsymbol{Y} > \quad \forall \boldsymbol{Y} \in \mathcal{A}_0.$$

- *Step 2: Convergence testing and construction of the new descent direction* - If $\|\boldsymbol{g^{n+1}}\|_{\mathcal{A}_0}/\|\boldsymbol{g^0}\|_{\mathcal{A}_0} \leq \varepsilon$ take $\boldsymbol{z} = \boldsymbol{z^{n+1}}$; otherwise compute

$$\gamma_n = \frac{(\boldsymbol{g^{n+1}}, \boldsymbol{g^{n+1}} - \boldsymbol{g^n})_{\mathcal{A}_0}}{(\boldsymbol{g^n}, \boldsymbol{g^n})_{\mathcal{A}_0}}, \qquad \boldsymbol{w^{n+1}} = \boldsymbol{g^{n+1}} + \gamma_n \boldsymbol{w^n}. \tag{37}$$

  Then do $n = n+1$, and return to step 1.

The following two remarks are in order:

- Since $E_{\overline{\boldsymbol{y}}}$ is a quadratic functional, one may write

$$E_{\overline{\boldsymbol{y}}}(\boldsymbol{z^n} - \lambda \boldsymbol{w^n}) = E_{\overline{\boldsymbol{y}}}(\boldsymbol{z^n}) - \lambda \iint_{q_T} (\boldsymbol{u^{n\prime}} \boldsymbol{W^{n\prime}} + \boldsymbol{u^n_{,1}} \boldsymbol{W^n_{,1}} + c^2 u_3^n W_{,3})\, d\xi\, dt$$
$$+ \frac{\lambda^2}{2} \iint_{q_T} (|\boldsymbol{W^n_t}|^2 + |\boldsymbol{W^n_{,1}}|^2 + c^2 |W_3|)\, d\xi dt$$

  where $\boldsymbol{u^n}$ is the corrector of $\overline{\boldsymbol{y}} + \boldsymbol{z^n}$ and $\boldsymbol{W^n}$ is the corrector of $\boldsymbol{w^n}$, and solve explicitly the problem (36).
- The computation of the residual $\boldsymbol{g^n}$ is performed as follows. According to the equality

$$< E'(\boldsymbol{y}^n), \boldsymbol{Y} >= - \int_{q_T} \left(\boldsymbol{Y''} \boldsymbol{u^n} + (Y_{1,1} + cY_3)(u^n_{1,1} + cu^n_3) \right) d\xi\, dt, \forall \boldsymbol{Y} \in \mathcal{A}_0.$$

  $E'(\boldsymbol{y^n}) \in H^{-2}(q_T))$ may be identified with the linear functional on $\mathcal{A}_0$ defined by

$$\boldsymbol{Y} \to - \int_{q_T} \left(\boldsymbol{Y''} \boldsymbol{u^n} + (Y_{1,1} + cY_3)(u^n_{1,1} + cu^n_3) \right) d\xi\, dt,$$

  It then follows that $\boldsymbol{g^n}$ is the solution of the following linear variational problem : find $\boldsymbol{g^n} \in \mathcal{A}_0$ such that

$$\iint_{q_T} (\boldsymbol{g^{n\prime\prime}} \boldsymbol{Y''} + \boldsymbol{g^{n\prime}} \boldsymbol{Y'} + \boldsymbol{g^n_{,1}} \boldsymbol{Y_{,1}} + c^2 g^n_3 Y_3)\, d\xi\, dt =$$
$$- \int_{q_T} \left(\boldsymbol{Y''} \boldsymbol{u^n} + (Y_{1,1} + cY_3)(u^n_{1,1} + cu^n_3) \right) d\xi\, dt,$$

  for all $\boldsymbol{Y} \in \mathcal{A}_0$, where $\boldsymbol{u^n} \in H^1_{0,\xi}(q_T)$ is the corrector associated with $\boldsymbol{y^n}$.

Remark 6. The parameter γ_n given by (37) corresponds to the *Polak-Ribière* version of the conjugate gradient algorithm. In the present quadratic-linear situation, this one should coincide with the *Fletcher-Reeves conjugate algorithm* for which $\gamma_n = \|\boldsymbol{g^{n+1}}\|^2_{\mathcal{A}_0}/\|\boldsymbol{g^n}\|^2_{\mathcal{A}_0}$ since gradients are conjugate to each other $((\boldsymbol{g^m}, \boldsymbol{g^n})_{\mathcal{A}_0} = 0\ \forall m \neq n)$. As in the parabolic situation described in [26] the *Polak-Ribière* version (mainly used in nonlinear situations) allows to reduce the numerical loss of the orthogonality.

Once the convergence of the algorithm is reached, up to the threshold ε, we take the trace of u on Σ_T to define an approximation of the control v of (5): $v(t) = y_1(1,t)$, $t \in (0,T)$. We next compute an approximation of the controlled solution y by solving (5): the norm $\|(\boldsymbol{y}(\cdot,T), \boldsymbol{y'}(\cdot,T))\|_{\boldsymbol{H}\times\boldsymbol{V'}}$, that may be seen as an *a posteriori* error, allows to evaluate the efficiency of the approach.

4.3. **Numerical approximations.** For "large" integers N_x and N_t , we set $\Delta x = 1/N_x$, $\Delta t = T/N_t$, and $h = (\Delta x, \Delta t)$. Let us denote by $\mathcal{P}_{\Delta x}$ the uniform partition of $\overline{\omega}$ associated with Δx, and let us denote by $\mathcal{Q}_h$ the uniform quadrangulation of q_T associated with h, so that $\overline{q_T} = \bigcup_{K\in\mathcal{Q}_h} K$. The following (conformal) finite element approximation of the space $H^2(0,T;H^1(\omega))$ (appearing in $\mathcal{A}$) is introduced :

$$X_h = \{\varphi_h \in C^{0,1}_{\xi,t}(\overline{q_T}) : \varphi_h|_K \in (\mathbb{P}_{1,\xi}\otimes\mathbb{P}_{3,t})(K) \quad \forall K \in \mathcal{Q}_h\,\}.$$

Here, $\mathbb{P}_{m,x}$ denotes the space of polynomial functions of order m in the variable x and $C^{0,1}_{\xi,t}(\overline{q_T})$ is the space of the functions in $C^0(\overline{q_T})$ that are continuously differentiable with respect to t in $\overline{q_T}$. The space X_h is also a conformal approximation of $H^2(0,T;L^2(\omega))$. We are thus considering finite elements that are C^1 in the variable t. In each quadrangle $K\in\mathcal{Q}_h$, functions are approximated by polynomials of the form $A+B\xi+Ct+D\xi t+Et^2+Ft^2\xi+Gt^3+Ht^3\xi$ involving 8 degrees of freedom. This is necessary to enforce the boundary conditions at $t=0$ and $t=T$ appearing in $\mathcal{A}$ and $\mathcal{A}_0$. The space $\mathcal{A}$ is approximated by the finite dimensional space :

$$\mathcal{A} = \Big\{\boldsymbol{y_h} = (y_{1h}, y_{3h}) \in X_h\times X_h : (\boldsymbol{y}_h(\xi,0), \boldsymbol{y_h'}(\xi,0)) = (\boldsymbol{y_h^0}, \boldsymbol{y_h^1}),$$
$$(\boldsymbol{y}_h(\xi,T), \boldsymbol{y_h'}(\xi,T)) = (\boldsymbol{0},\boldsymbol{0}), \xi\in\omega, y_{1h}(0,t)=0, t\in(0,T)\Big\}.$$

Therefore, for any h, we consider the following problem

$$\begin{cases} \text{Minimize} \quad E_h(\boldsymbol{y_h}) = \dfrac{1}{2}\displaystyle\iint_{q_T} (|\boldsymbol{u_h'}|^2 + |\boldsymbol{u_{h,1}}|^2 + c^2|u_{3h}|^2)\,d\xi\,dt, \\ \text{subject to} \quad \boldsymbol{y_h}\in\mathcal{A}_h. \end{cases} \tag{38}$$

According to the conjugate gradient algorithm, this minimization problem is reduced to the resolution of well-posed elliptic problems defined on q_T in order to compute corrector functions $\boldsymbol{u_h}$ that we also approximate in $X_h\times X_h$ for the sake of the implementation.

5. **Numerical experiments II.** We now present some numerical experiments, and highlights the capability of the variational approach to provide controls for system with mixed order operator. We take for simplicity that $\Delta x = \Delta t$, that is we only consider uniform meshes $\mathcal{Q}_h$.

5.1. **Exact controllability.** We consider the following initial condition

$$\boldsymbol{y^0} = \beta(5\boldsymbol{w_1} - 2\boldsymbol{w_2}) + \boldsymbol{v_0} + \boldsymbol{v_1}, \quad \boldsymbol{y^1} = \beta(4\boldsymbol{w_3} - 2\boldsymbol{w_1}) + \mu_0\boldsymbol{v_0} + \mu_2\boldsymbol{v_2} \tag{39}$$

with $\beta\in\{0,1\}$. For $\beta=0$, $(\boldsymbol{y^0},\boldsymbol{y^1})$ is controllable and coincides with the datum given by (23) (for $\alpha=1$). For $\beta=1$, the datum is not controllable due to the components $\boldsymbol{w_1},\boldsymbol{w_2},\boldsymbol{w_3}\in Ker\boldsymbol{A_M}$. We then define the functions $F_1(t) = \frac{(T-t)^2(T+2t)}{T^3}$, $F_2(t) = \frac{(T-t)^2 t}{T^2}$ and the function

$$\overline{\boldsymbol{y}}(\xi,t) = F_1(t)\boldsymbol{y^0}(\xi) + F_2(t)\boldsymbol{y^1}(\xi), \quad (\xi,t)\in q_T. \tag{40}$$

We easily check that $\overline{\boldsymbol{y}}$ belongs to $\mathcal{A}$, in particular $(\overline{\boldsymbol{y}}(\cdot,0),\overline{\boldsymbol{y}}'(\cdot,0)) = (\boldsymbol{y^0},\boldsymbol{y^1})$ and $(\overline{\boldsymbol{y}}(\cdot,T),\overline{\boldsymbol{y}}'(\cdot,T)) = (\boldsymbol{0},\boldsymbol{0})$.

We take $T=3.5$, $c=\pi/5$ and use the conjugate gradient algorithm to construct a minimizing sequence for E of the form $\boldsymbol{y^n} = \overline{\boldsymbol{y}} + \boldsymbol{z^n}\in\mathcal{A}$ with $\boldsymbol{z^n}\in\mathcal{A}_0$. We start with $\boldsymbol{z^0}\equiv\boldsymbol{0}$. We consider the value $\Delta x = \Delta t = 1/80$.

We first give some results in the case $\beta = 0$. Figure 7-Left depicts the evolution (in $\log_{10}$ scale) of the residual $\|\boldsymbol{g^n}\|_{\mathcal{A}_0}$ and of the cost $E(\boldsymbol{y^n})$ with respect to the iteration. After 3000 iterations, we obtain the value $E(\boldsymbol{y_h^{3000}}) \approx 6.94 \times 10^{-5}$ and $\|\boldsymbol{g_h^n}\|_{\mathcal{A}_0} \approx 3.16 \times 10^{-5}$. More precisely, we get $(\iint_{q_T}(|u'_{1h}|^2 + |u_{1h,1}|^2)\, d\xi\, dt)^{1/2} \approx 1.17 \times 10^{-2}$ and $(\iint_{q_T}(|u'_{3h}|^2 + |u_{3h,1}|^2 + c^2|u_{3h}|^2)\, d\xi\, dt)^{1/2} \approx 9.88 \times 10^{-4}$.

Figure 8 and Figure 9 depict the corresponding minimizer (y_{1h}, y_{3h}) and corrector (u_{1h}, u_{3h}) respectively while Figure 7 depicts the trace of the component y_{1h} along $\Sigma_T = \{1\} \times (0, T)$. We obtain $\|y_{1h}\|_{L^2(\Sigma_T)} \approx 1.119$. We check that this value is bigger than the one obtained by the dual method, in Section 3, that is, for $c = \pi/2$, $\|v_h\|_{L^2(\Sigma_T)} \approx 0.753$.

As in the parabolic situation described in [26], the convergence of the algorithm is rather low: however, it provides a satisfactory approximation of a control for the system (5). Precisely, if we compute the solution, say $\boldsymbol{y}^\star$, of the forward system (5) with control the trace $y_{1,h}(1,t)$, we have just obtained, we observe that $\|\boldsymbol{y_h^\star}(\cdot,T), \boldsymbol{y_h^{\star\prime}}(\cdot,T)\|_{\boldsymbol{V}\times\boldsymbol{H}} \approx 1.40 \times 10^{-3}$.

We now consider the case $\beta = 1$, the other data being unchanged. Figure 10 depicts the evolution of $E_h(\boldsymbol{y_h^n})$ and $\|\boldsymbol{g_h^n}\|_{\mathcal{A}_0}$ (in $\log_{10}$ scale) with respect to n. After 500 iterations, we obtain $\|\boldsymbol{g_h^{500}}\|_{\mathcal{A}_0} \approx 1.30 \times 10^{-4}$ which indicates that the solution $\boldsymbol{y_h^{500}}$ obtained is closed to a local minima for E: we recall that the vector $\boldsymbol{g}$ corresponds to the derivative of E. On the other hand, the value of the cost is $E(\boldsymbol{y_h^{500}}) \approx 2.28 \times 10^{-2}$ to be compared with the value 6.94×10^{-5} obtained in the case $\beta = 0$. Thus, for this local minimum, the cost does not vanish and $(y_{1h}, y_{3h}) \in \mathcal{A}_h$ is not an approximation of the solution of (5). This illustrates the lack of controllability of the data (39) for (5) and is once again in full agreement with the controllability property of (5). Figure 11 gives the corresponding corrector (u_{1h}, u_{3h}) in q_T: we obtain $(\iint_{q_T}(|u'_{1h}|^2 + |u_{1h,1}|^2)\, d\xi\, dt)^{1/2} \approx 5.01 \times 10^{-2}$ and $(\iint_{q_T}(|u'_{3h}|^2 + |u_{3h,1}|^2 + c^2|u_{3h}|^2)\, d\xi\, dt)^{1/2} \approx 2.07 \times 10^{-1}$.

5.2. **Partial uniform controllability.** Finally, we illustrate Remark 4. We consider the data (39) with $\beta = 1$ and work with the relaxed space $\mathcal{A}_p$ defined by (35) and larger than $\mathcal{A}$. The conjugate gradient algorithm remains unchanged.

Figure 12-left depicts the evolution of the residu $\|\boldsymbol{g_h^n}\|_{\mathcal{A}_0}$ and of the cost $E(\boldsymbol{y_h^n})$. We observe that after 5000 iterations, these two quantities take small values, precisely $E(\boldsymbol{y_h^{5000}}) \approx 1.71 \times 10^{-4}$ and $\|\boldsymbol{g_h^{5000}}\|_{\mathcal{A}_0} \approx 4.44 \times 10^{-5}$. This is in agreement with the partial controllability of (5), even for data with components in $Ker\boldsymbol{A_M}$. Figure 12-right gives the trace of the controlled component y_{1h} on Σ_T: we get $\|y_{1h}\|_{L^2(\Sigma_T)} \approx 1.124 \times 10^1$. Figure 13 gives the corresponding controlled component y_{1h} and the free component y_{3h}, minimizer for E. At last, Figure 14 gives the corresponding corrector u_{1h} and u_{3h} in q_T: we obtain $(\iint_{q_T}(|u'_{1h}|^2 + |u_{1h,1}|^2)\, d\xi\, dt)^{1/2} \approx 1.16 \times 10^{-2}$ and $(\iint_{q_T}(|u'_{3h}|^2 + |u_{3h,1}|^2 + c^2|u_{3h}|^2)\, d\xi\, dt)^{1/2} \approx 5.81 \times 10^{-3}$.

6. **Final remarks.** The chart which describes the mid-surface of a cylindrical arch of curvature c, as considered along this paper, is given by $\boldsymbol{\phi}(\xi) = (c^{-1}\sin(c\xi), c^{-1}\cos(c\xi))$, $\xi \in \omega$. The corresponding operator is then given by (4). For a general chart $\boldsymbol{\phi}(\xi) = (\phi_1(\xi), \phi_2(\xi))$, the operator $\boldsymbol{A_M}$ is

$$\boldsymbol{A_M y} = \begin{pmatrix} -\gamma_{11,1}(\boldsymbol{y}) - 2\gamma_{11}(\boldsymbol{y})\Gamma_{11}^1 \\ -\gamma_{11}(\boldsymbol{y})b_{11} \end{pmatrix}, \quad \boldsymbol{y} = (y_1, y_3) \tag{41}$$

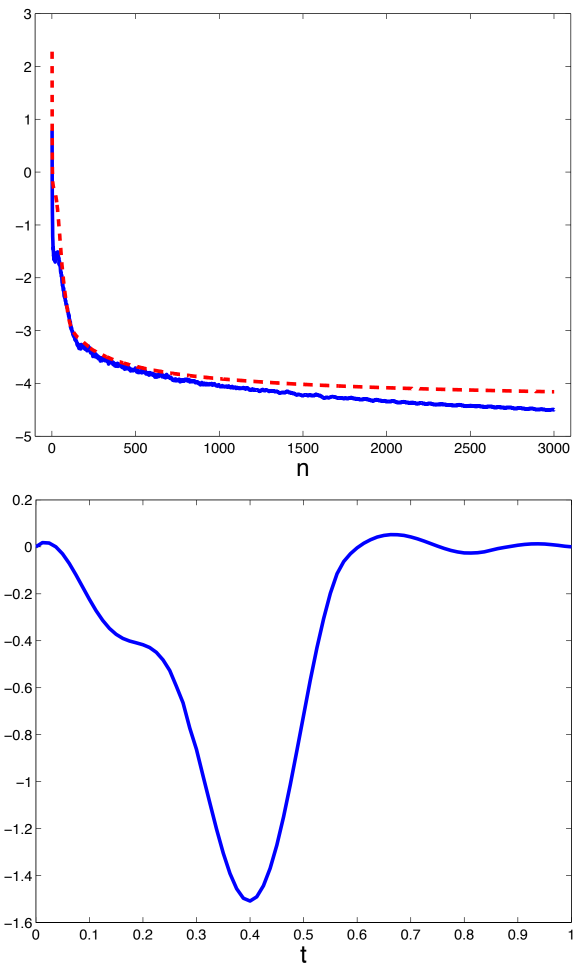

FIGURE 7. $\beta = 0$: $T = 3.5$, $\Delta x = \Delta t = 1/80$ - **Top** : $\log_{10}(E_h(\boldsymbol{y_h^n})$ (**dashed line**) and $\log_{10}(\|\boldsymbol{g_h^n}\|_{\mathcal{A}_0})$ (**full line**) vs. the iteration n of the conjugate gradient algorithm ; **Bottom** : Trace $y_{1h}(\xi = 1, t)$ vs. $t \in (0, T)$

with

$$\begin{cases} \gamma_{11}(y) = y_{1,1} - \Gamma^1_{11} y_1 - b_{11} y_3, \quad \Gamma^1_{11} = t^{-1}(\phi_{1,1}\phi_{1,11} + \phi_{2,1}\phi_{2,11}), \\ t = \phi^2_{1,1} + \phi^2_{2,1}, \quad b_{11} = t^{-3/2}(-\phi_{2,1}\phi_{1,11} + \phi_{1,1}\phi_{2,11}). \end{cases} \tag{42}$$

Γ^1_{11} designates the Christoffel symbol and γ_{11} the longitudinal strain (see [29]). Once again, the kernel is not empty and of infinite dimension - $Ker\boldsymbol{A_M} = \{v_\zeta = (\zeta, b_{11}^{-1}(\zeta_{,1} - \Gamma^1_{11}\zeta)) \in \boldsymbol{H}, \zeta \in H^1_0(\omega)\}$ - so that $0 \in \sigma_{ess}(\boldsymbol{A_M})$ and the controllability does not hold uniformly with respect to the initial data. But, in this general setting, the spectrum is not explicit so that one can not, *a priori*, fully characterized the set of initial controllable data and therefore compute controls by duality arguments, as in Section 3. On the other hand, there is no restriction here to apply the variational approach of Section 4.

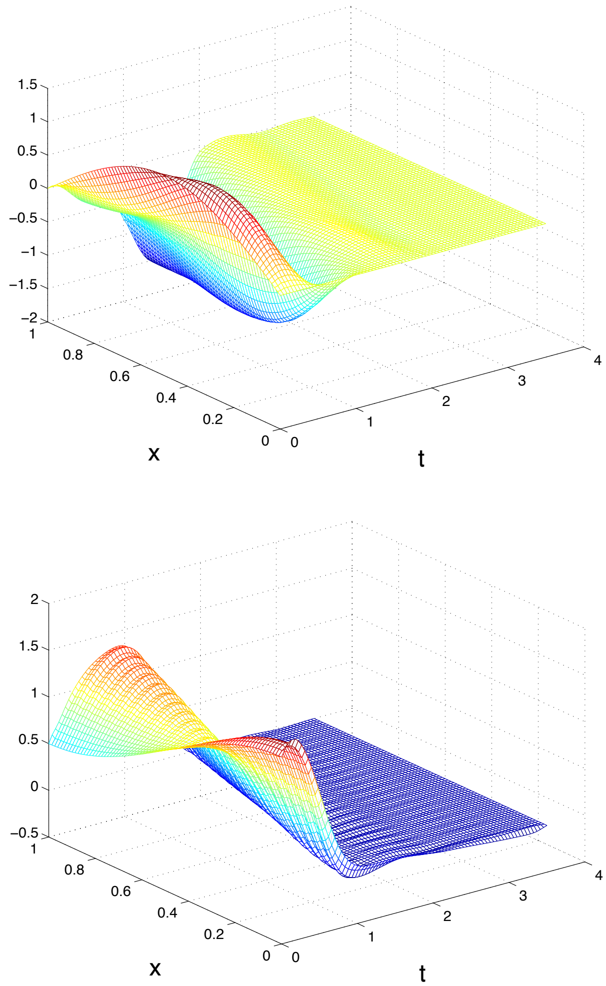

FIGURE 8. $\beta = 0$: Minimizer y_{1h} (**left**) and y_3 (**right**) over q_T.

Similarly, the 3D dimensional situation remains challenging and may be addressed by the variational approach. More precisely, the membranal operator that enters in the modelisation of an elastic cylindrical shell - with constant curvature c in one direction and zero curvature in the other direction - is given by

$$\boldsymbol{A_M} = \begin{pmatrix} -a_1\partial^2_{\xi_1\xi_1} - c\partial^2_{\xi_2\xi_2} & -(a_2+a_3)\partial^2_{\xi_1\xi_2} & -c\,a_1\partial_{\xi_1} \\ -(a_2+a_3)\partial^2_{\xi_1\xi_2} & -a_3\partial^2_{\xi_1\xi_1} - a_1\partial^2_{\xi_2\xi_2} & -c\,a_2\partial_{\xi_2} \\ c\,a_1\partial_{\xi_1} & c\,a_2\partial_{\xi_2} & c^2 a_1 \end{pmatrix} \tag{43}$$

with $a_1 = 8\mu(\lambda+\mu)/(\lambda+2\mu)$, $a_2 = 4\lambda\mu/(\lambda+2\mu)$ and $a_3 = 2\mu$. $\lambda, \mu > 0$ denotes the Lamé coefficient. This mixed and self-adjoint operator enters in the framework of [14] and we obtain $\sigma_{ess}(A) = \left[0, \frac{2(3\lambda+2\mu)}{\lambda+\mu}c^2\right]$. Moreover $Ker\boldsymbol{A_M} = \emptyset$ and $(0,0,1)$ is an eigenfunction associated to $\lambda_0 = c^2 \notin \sigma_{ess}(\boldsymbol{A_M})$. The remaining part of the

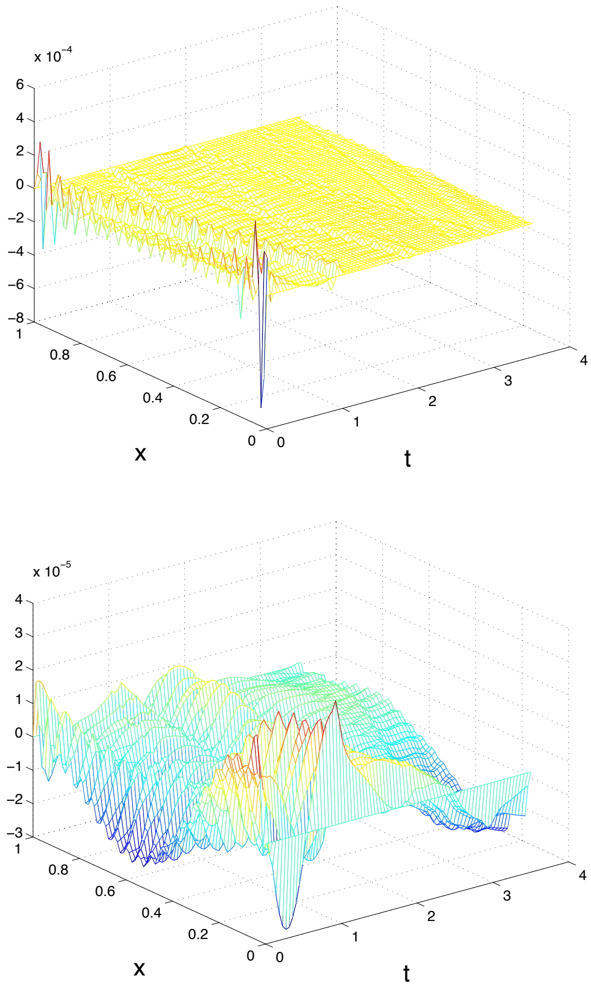

FIGURE 9. $\beta = 0$; Minimizer v_{1h} (**left**) and v_3 (**right**) over q_T.

discrete spectrum is unknown (contrary to the operator defined in (3)), so that once again the standard method of computation is ineffective.

Let us also note that we may use this variational approach to determine the control of minimal L^2-norm. It suffices to introduce the Lagrangian

$$L(\boldsymbol{y}, \lambda) = \|y_1(1,t)\|^2_{L^2(0,T)} + \lambda E(\boldsymbol{y})$$

defined over $\mathcal{A} \times \mathbb{R}$, and apply an Uzawa type method, arguing that the set $\{\boldsymbol{y} \in \mathcal{A}, E(\boldsymbol{y}) = 0\}$ is convex.

The variational approach may be used to address the inner controllability case as well. Set any subset ω_1 of ω and assume that a control v acts on $q_T^1 = \omega_1 \times (0,T)$ on the first equation of (5). Then, the variational approach leads to the minimization

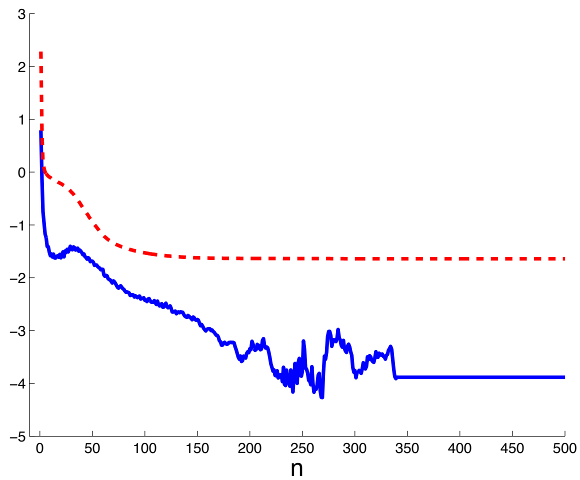

FIGURE 10. $\beta = 1$ - $T = 3.5$, $\Delta x = \Delta t = 1/80$ - **Left** : $\log_{10}$ of $E_h(\boldsymbol{y_h^n})$ (**dashed line**) and $\log_{10}(\|\boldsymbol{g_h^n}\|_{\mathcal{A}_0})$ (**full line**) vs. the iteration n of the conjugate gradient algorithm.

of the functional

$$E(\boldsymbol{y}) = \frac{1}{2}\iint_{q_T\setminus q_T^1}(|u_1'|^2 + |u_{1,1}|^2)d\xi\, dt + \frac{1}{2}\iint_{q_T}(|u_3'|^2 + |u_{3,1}|^2 + c^2|u_3|^2)d\xi\, dt$$

over

$$\mathcal{A} = \Big\{\boldsymbol{y}\in H^2(0,T;H^1(\omega))\times H^2(0,T;L^2(\omega)) : (\boldsymbol{y}(\xi,0),\boldsymbol{y'}(\xi,0)) = (\boldsymbol{y^0},\boldsymbol{y^1}),$$
$$(\boldsymbol{y}(\xi,T),\boldsymbol{y'}(\xi,T)) = (\boldsymbol{0},\boldsymbol{0}), \xi\in\omega, \quad y_1 = 0 \text{ on } \partial\omega\times(0,T)\Big\}$$

where the corrector $\boldsymbol{u}$ still solves (27). Proceeding as in Section 4, we obtain that the infimum of E is attained in $\mathcal{A}$ and is equal to zero: the corrector u_3 vanishes on q_T while the corrector u_1 vanishes on $q_T\setminus q_T^1$, that is out of the subset ω_1. From (27), the corresponding minimizer $\boldsymbol{y}\in\mathcal{A}$ satisfies

$$\begin{cases} \boldsymbol{y''} + \boldsymbol{A_M}\boldsymbol{y} = \begin{pmatrix} u_1'' + u_{1,11} \\ 0 \end{pmatrix} 1_{\omega_1}(\xi), & (\xi,t)\in q_T \\ (\boldsymbol{y}(\xi,0),\boldsymbol{y'}(\xi,0)) = (\boldsymbol{y^0},\boldsymbol{y^1}), & \xi\in\omega. \end{cases}$$

A null control is then defined by $v = (u_1'' + u_{1,11})\, 1_{\omega_1}$ in q_T. We refer to [26] where the inner controllability for the heat equation is addressed through this procedure. More generally, this variational approach introduced in [27] and which sound like a least square method, is very versatile and may be applied to many dynamical systems for which a unique continuation property is available.

Eventually, we mention that the numerical analysis of the method is also feasible since the corrector is solution of an elliptic system (over the space-time domain) for which finite element theory is well adapted.

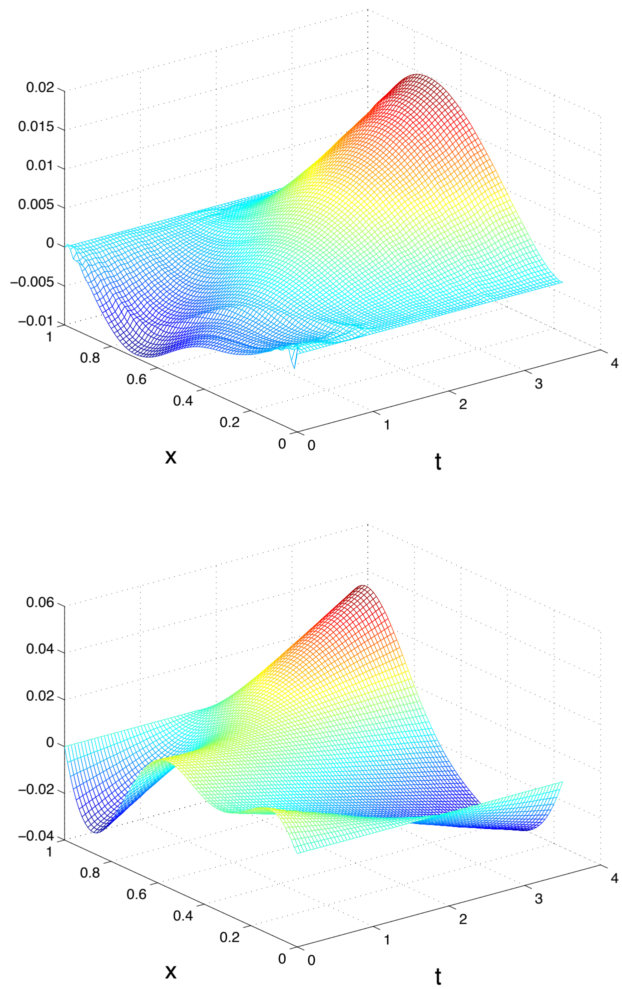

FIGURE 11. $\beta = 1$; Minimizer u_{1h} (**left**) and u_{3h} (**right**) over q_T.

REFERENCES

[1] F. Ammar-Khodja, G. Geymonat and A. Münch, *On the exact controllability of a system of mixed order with essential spectrum*, C.R.Acad. Sci. Paris Série I, **346** (2008), 629–634.

[2] F. Ammar-Khodja, K. Mauffrey and A. Münch, *Exact controllability of a system of mixed order with essential spectrum*, SIAM J. of Control, **49** (2011), 1857–1879.

[3] C. Castro, S. Micu and A. Münch, *Numerical approximation of the boundary control for the wave equation with mixed finite elements in a square*, IMA J. Numerical Analysis, **28(1)** (2008), 186–214.

[4] D. Chapelle and K.J. Bathe, "The finite element analysis of shell - Fundamentals", Springer-Verlag, 2003.

[5] P.G. Ciarlet, "Mathematical elasticity. Vol. III. Theory of shells", In Studies in Mathematics and its Applications, **29.**, North-Holland Publishing Co., Amsterdam, 2000.

[6] N. Cîndea, E. Fernández-Cara and A. Münch, *Numerical controllability of the wave equation through a primal method and Carleman estimates*, Preprint. Available from `http://hal.archives-ouvertes.fr/hal-00668951`.

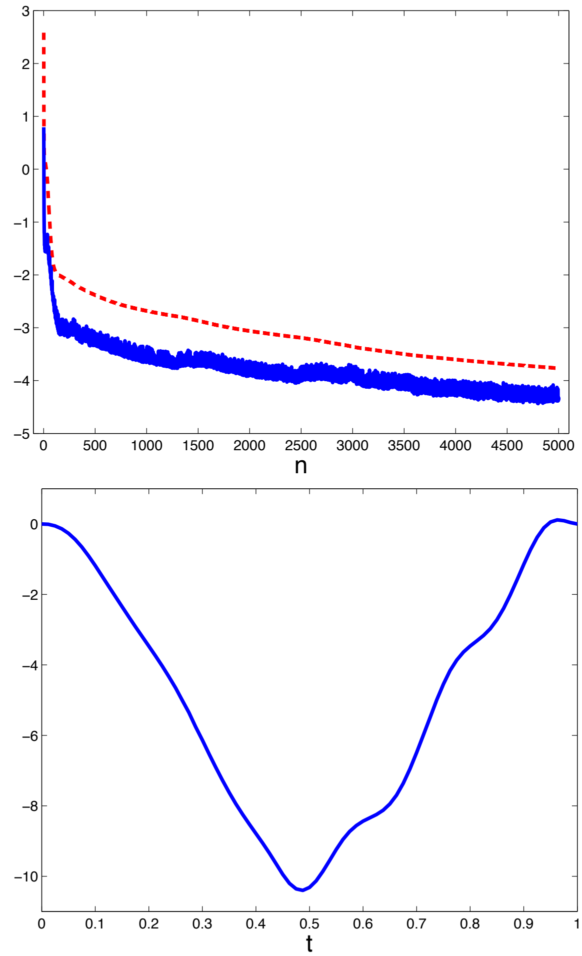

FIGURE 12. $\beta = 1$: $T = 3.5$, $\Delta x = \Delta t = 1/80$ - **Left** : $\log_{10}$ of $E_h(\boldsymbol{y_h^n})$ (**dashed line**) and $\log_{10}(\|\boldsymbol{g_h^n}\|_{\mathcal{A}_0})$ (**full line**) vs. the iteration n of the conjugate gradient algorithm ; **Right** : Trace $y_{1h}(\xi = 1, t)$ vs. $t \in (0, T)$.

[7] Ph. Destuynder, *A classification of thin shell theories*, Act. Appl. Math., **1** (1985), 15–63.

[8] Ph. Destuynder, "Modélisation des coques élastiques minces", Masson, Paris, 1990.

[9] G. Geymonat, P. Loreti and V. Valente, *Contrôlabilité exacte d'un modèle de coque mince*, C.R.Acad.Sci Série I, **313** (1991), 81–86.

[10] G. Geymonat, P. Loreti and V. Valente, *Exact controllability of thin elastic hemispherical shell via harmonic analysis*, In "Boundary value problems for partial differential equations and applications" (eds. J.L. Lions and C. Baiocchi), Masson, (1993), 379–386.

[11] G. Geymonat and V. Valente, *A noncontrollability result for systems of mixed order*, SIAM J. Control Optim, **39(3)** (2000), 661–672.

[12] R. Glowinski, "Numerical methods for nonlinear variational problems", In Springer Series in computational physics, New York, 1984.

[13] R. Glowinski, J.L. Lions and J. He, "Exact and approximate controllability for distributed parameter systems: a numerical approach", Encyclopedia of Mathematics and its applications, **117**, Cambridge, 2008.

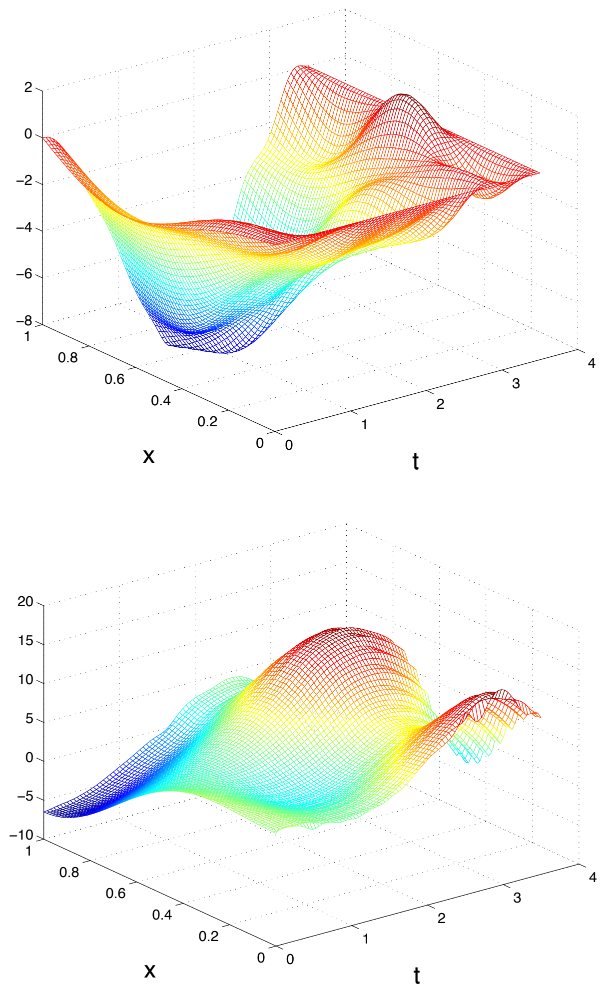

FIGURE 13. $\beta = 0$: Minimizer y_{1h} (**left**) and y_{3h} (**right**) over q_T.

[14] G. Grubb and G. Geymonat, *The essential spectrum of elliptic systems of mixed order*, Math. Ann., **227** (1977), 247–276.

[15] G. Grubb and G. Geymonat, *Eigenvalue asymptotics for selfadjoint elliptic mixed order systems with nonempty essential spectrum*, Boll. Un. Mat. Ital. , **16** (1979), 1032–1048.

[16] V. Komornik and P. Loreti, "Fourier series in control theory", Springer Monographs in Mathematics, Springer-Verlag, New York, 2005.

[17] I. Lasiecka, R. Triggiani and V. Valente, *Uniform stabilization of spherical shells by boundary dissipation*, Advances in differential equations, **4** (1996), 635–674.

[18] I. Lasiecka and R. Triggiani, "Control theory for partial differential equations: continuous and approximation theories. II. Abstract hyperbolic-like systems over a finite time horizon," Encyclopedia of Mathematics and its Applications, 75. Cambridge University Press, Cambridge, 2000.

[19] C. Lebiedzik, *Exact boundary controllability of a shallow intrinsic shell model*, J. Math. Anal. Appl., **335** (2007), 584–614.

[20] P. Loreti and V. Valente, *Partial exact controllability for spherical membranes*, SIAM J. Control Optim, **35(2)** (1997), 641–653.

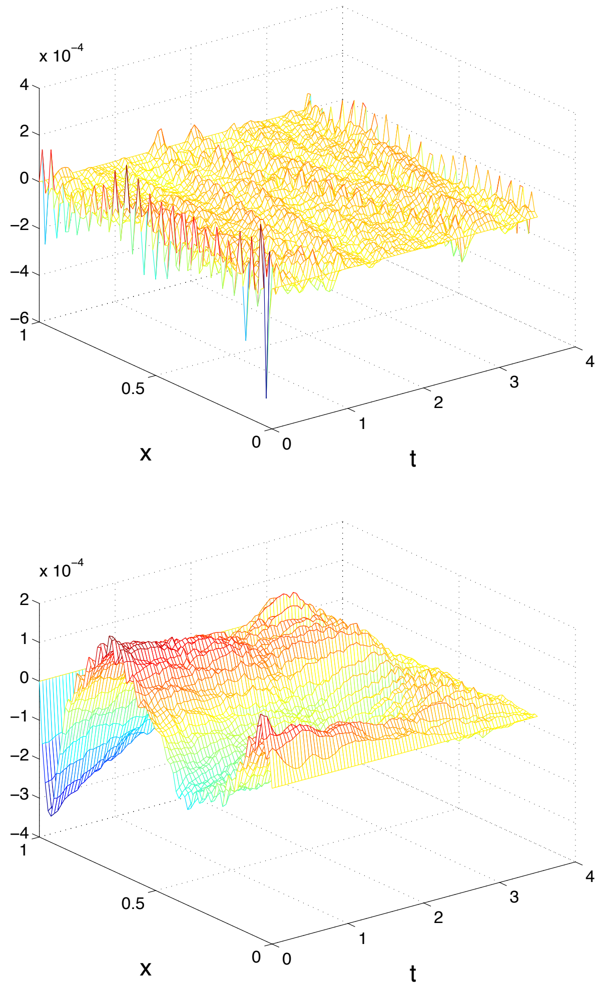

FIGURE 14. $\beta = 0$: Minimizer y_{1h} (**left**) and y_{3h} (**right**) over q_T.

[21] J.L. Lions, "Contrôlabilité exacte, stabilisation et perturbations de systèmes distribués, Tome 1", Masson, RMA 8, Paris (1988).

[22] J.L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications, vol. 1*, Dunod, Paris, 1968.

[23] B. Miara and V. Valente, *Exact controllability of a Koiter Shell by a Boundary Action*, J. Elasticity, **52** (1999), 267–287.

[24] A. Münch, *A uniformly controllable and implicit scheme for the 1-D wave equation*, Mathematical Modeling and Numerical Analysis, **39(2)** (2005), 377–418.

[25] A. Münch, *Null boundary controllability of a circular elastic arch*, IMA J. Mathematical Control and Information, **27** (2010), 119–114.

[26] A. Münch and P. Pedregal, *Numerical null controllability of the heat equation through a least square and variational approach.* Preprint. Available from `http://hal.archives-ouvertes.fr/hal-00724035`

[27] P. Pedregal, *A variational perspective on controllability*, Inverse Problems, **26**(2010), 015004 17p.

[28] J. Sanchez Hubert and E. Sanchez-Palencia, " Vibration and coupling of continuous systems. Asymptotic theory", Springer-Verlag, Berlin, 1989.

[29] J. Sanchez Hubert and E. Sanchez-Palencia, "Coques élastiques minces : propriétés asymptotiques", Masson, Paris, 1993.

[30] R. Triggiani, *Regularity theory, exact controllability, and optimal quadratic cost problem for spherical shells with physical boundary controls*, Control Cybernet., **25(3)** (1996), 553–568.

[31] L. Xiao, *Asymptotic analysis of dynamic problems for linearly elastic shells - Justification of equations for dynamic Koiter shells*, Chin. Ann. of Math., **22(3)** (2001), 267–274.

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