Long time behavior of a two-phase optimal design for the heat equation

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joint work with G. ALLAIRE (X-CMAP, Palaiseau) and F. PERIAGO (UPCT, Carthagena)

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Related to the work [*Relaxation of an optimal design problem for the heat equation*, J. Math. Pures Appl. (2008) AM-Pedregal-Periago] where the following optimal design problem is analyzed :

$$(\mathrm{P}_T) \quad \text{Minimize in } \mathcal{X} \in \mathbf{CD} : \quad J_T(\mathcal{X}) = \frac{1}{T}\int_0^T\int_\Omega K(x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt$$

$\Omega \subset \mathbb{R}^N$, where the state variable $u = u(t,x)$ is the solution of the system

$$\begin{cases} \beta(x)\,u'(t,x) - \operatorname{div}\,(K(x)\nabla u(t,x)) = f(t,x) & \text{in} \quad (0,T)\times\Omega \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1}$$

with

$$\begin{cases} \beta(x) = \mathcal{X}(x)\,\beta_1 + (1-\mathcal{X}(x))\,\beta_2 \\ K(x) = \mathcal{X}(x)\,k_1 I_N + (1-\mathcal{X}(x))\,k_2 I_N. \end{cases}$$

$k_i > 0$ - thermal conductivity , $\beta_i = \rho_i c_i$ ($\rho_i > 0$ mass density - $c_i > 0$ specific heat)

The design variable $\mathcal{X}$ indicates the region occupied by the material (β_1, k_1) and is subjected to belong to the class of *classical designs* **CD** defined as

$$\mathbf{CD} = \left\{ \mathcal{X} \in L^\infty(\Omega;\{0,1\}) : \int_\Omega \mathcal{X}(x)dx = L|\Omega| \right\}, \tag{2}$$

Goal of the work

$\Longrightarrow$ Study the asymptotic behavior as $T \to \infty$ of the solution $(\theta_T, K_T^\star)$ of the relaxed problem (RP_T)

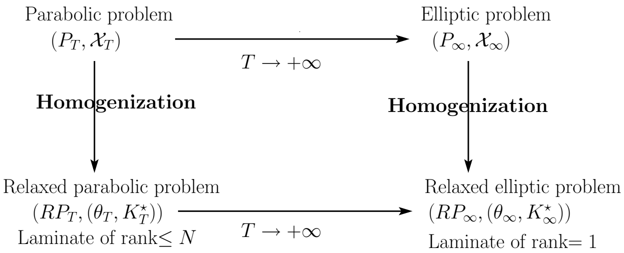

Figure: Commutation between Homogenization process and limit of the heat system as $T \to \infty$???

$\Longrightarrow$ We assume that $\mathcal{X}$ is time independent and use tools from Homogenization theory.

Outline

1. Overview of the relaxed formulation (RP_T) and (RP_∞)
2. H-convergence of optimal effective tensors $K_T^\star$ toward $K_\infty^\star$
3. Structure of the optimal effective tensor $K_T^\star$ in term of sequential laminates
4. (Formal) Analysis of the micro-structure of $K_T^\star$ for T arbitrarily large (and small)
5. Numerical experiments for $N = 2$
6. A word about the open case where $\mathcal{X}$ is time-dependent.

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The relaxed formulation (RP_T) (T fixed) - Overview

The relaxed formulation (RP_T) involves the space of *relaxed designs*

$$\mathbf{RD} = \left\{(\theta, K^\star) \in L^\infty\left(\Omega; [0,1] \times \mathcal{M}_N^s(k_1, k_2)\right) : K^\star(x) \in G_{\theta(x)} \text{ a.e. } x \in \Omega, \|\theta\|_{L^1(\Omega)} = L|\Omega|\right\},$$

where $\mathcal{M}_N^s(k_1, k_2)$ is the space of real symmetric squared matrices M of order N satisfying, for all $\xi \in \mathbb{R}^N$, $k_1 |\xi|^2 \leq M\xi \cdot \xi$ and $k_2 |\xi|^2 \leq M^{-1}\xi \cdot \xi$.

For a given $\theta \in L^\infty(\Omega; [0,1])$, the so-called G_θ-closure is the set of all symmetric matrices with eigenvalues $\lambda_1, \cdots, \lambda_N$ satisfying

$$\begin{cases} \lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+, \qquad 1 \leq j \leq N, \\ \sum_{j=1}^N \frac{1}{\lambda_j - k_1} \leq \frac{1}{\lambda_\theta^- - k_1} + \frac{N-1}{\lambda_\theta^+ - k_1}, \\ \sum_{j=1}^N \frac{1}{k_2 - \lambda_j} \leq \frac{1}{k_2 - \lambda_\theta^-} + \frac{N-1}{k_2 - \lambda_\theta^+}, \end{cases}$$

where $\lambda_\theta^- = \left(\frac{\theta}{k_1} + \frac{1-\theta}{k_2}\right)^{-1}$ is the harmonic mean and $\lambda_\theta^+ = \theta k_1 + (1-\theta) k_2$ the arithmetic mean of (k_1, k_2).

The relaxed formulation (RP_T) (T fixed) - Overview

Theorem (PARABOLIC CASE - AM-Pedregal-Periago, JMPA 2008)

The following problem

$$(RP_T) \quad \text{Minimize in } (\theta, K^\star) \in \mathbf{RD} : \quad J_T^\star(\theta, K^\star) = \frac{1}{T}\int_0^T\int_\Omega K^\star(x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt$$

where u solves

$$\begin{cases} \beta^\star(x)\,u'(t,x) - div\,(K^\star(x)\nabla u(t,x)) = f(t,x) & in \quad (0,T)\times\Omega \\ u = 0 & on \quad (0,T)\times\partial\Omega \\ u(0,x) = u_0(x) & in \quad \Omega, \end{cases} \tag{3}$$

with $\beta^\star(x) = \theta(x)\beta_1 + (1-\theta(x))\beta_2$ is a relaxation of (P_T) in the following sense:

(i) *there exists at least one minimizer for (RP_T) in the space* **RD**,

(ii) *up to a subsequence, every minimizing sequence of classical designs $\mathcal{X}_n$ converges, weak-$\star$ in $L^\infty(\Omega;[0,1])$, to a relaxed density θ, and its associated sequence of tensors*

$$K_n = \mathcal{X}_n k_1 I_N + (1-\mathcal{X}_n)k_2 I_N$$

$H-$converges to an effective tensor $K^\star$ such that $(\theta, K^\star)$ is a minimizer for (RP_T), and

(iii) *conversely, every relaxed minimizer $(\theta, K^\star) \in$ **RD** of (RP_T) is attained by a minimizing sequence $\mathcal{X}_n$ of (P_T) in the sense that*

$$\begin{cases} \mathcal{X}_n \rightharpoonup \theta & weak \star \text{ in } L^\infty(\Omega), \\ K_n \xrightarrow{H} K^\star. \end{cases}$$

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Assuming that the heat source f depends only on the space variable, the unique solution of (1) converges as $t \to \infty$ to $\overline{u} \in H_0^1(\Omega)$, solution of the stationary equation

$$\begin{cases} -\text{div}\ (K(x)\nabla \overline{u}(x)) = f(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4}$$

Associated with this PDE we consider the design problem

$$(\mathrm{P}_\infty) \quad \text{Minimize in } \mathcal{X} \in \mathbf{CD}: \quad J_\infty(\mathcal{X}) = \int_\Omega K(x)\nabla\overline{u}(x)\cdot\nabla\overline{u}(x)dx.$$

(ELLIPTIC CASE)

Consider the following problem

$$(RP_\infty) \quad \textit{Minimize in } (\theta, K^\star) \in \mathbf{RD}: \quad J^\star_\infty(\theta, K^\star) = \int_\Omega K^\star(x)\nabla\overline{u}(x)\cdot\nabla\overline{u}(x)dx$$

where $\overline{u} \in H_0^1(\Omega)$ solves

$$\begin{cases} -div\ (K^\star\nabla\overline{u}) = f & in \quad \Omega \\ \overline{u} = 0 & on \quad \partial\Omega. \end{cases}$$

(RP_∞) is a relaxation of (P_∞) is the sense of the previous theorem. Moreover, the optimal effective tensor for (RP_∞) is obtained in the form of a first-order laminate in any direction orthogonal to $\nabla\overline{u}$.

The limit (P_∞) of (P_T) as $T \to \infty$ and its relaxation (RP_∞) - Overview

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where $\overline{u} \in H_0^1(\Omega)$ *solves*

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We assume henceforth that $f \in L^2(\Omega)$ is time independent and that $u_0 \in L^2(\Omega)$.

Let $\{T_n\}_{n\in\mathbb{N}}$ be an increasing sequence of positive times converging to infinity. For each T_n, problem (RP$_{T_n}$) has (at least) a minimizer $\left(\theta_{T_n}, K_{T_n}^\star\right) \in$ **RD**.

Since $\left(\theta_{T_n}, K_{T_n}^\star\right)$ is bounded in $L^\infty\left(\Omega; [0,1] \times \mathcal{M}_N^s(k_1, k_2)\right)$, up to subsequences still labeled by n, we have

$$\left\{\begin{array}{lll} \theta_{T_n} & \rightharpoonup & \theta_{T_\infty} \text{ weak-* in } L^\infty(\Omega; [0,1]) \\ K_{T_n}^\star & \xrightarrow{H} & K_{T_\infty}^\star \end{array}\right. \qquad \text{as} \quad n \to \infty$$

Theorem (Allaire-AM-Periago)

If $\left(\theta_{T_n}, K_{T_n}^\star\right)$ is an optimal solution of (RP$_{T_n}$), then any weak limit $\left(\theta_{T_\infty}, K_{T_\infty}^\star\right)$ of a converging subsequence of $(\theta_{T_n}, K_{T_n}^\star)$ is an optimal solution of (RP$_\infty$).

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Lemma

Let u_n be the solution of

$$\begin{cases} \beta_n^\star(x)\, u_n'(t,x) - div\left(K_{T_n}^\star(x)\,\nabla u_n(t,x)\right) = f(x) & in \quad (0,\infty)\times\Omega \\ u_n = 0 & on \quad (0,\infty)\times\partial\Omega \\ u_n(0,x) = u_0(x) & in \quad \Omega, \end{cases} \tag{5}$$

with $\beta_n^\star(x) = \theta_{T_n}(x)\,\beta_1 + \left(1 - \theta_{T_n}(x)\right)\beta_2$. *Then,*

$$\lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n}\int_\Omega K_{T_n}^\star(x)\,\nabla u_n(t,x)\cdot\nabla u_n(t,x)\,dxdt = \int_\Omega K_{T_\infty}^\star(x)\,\nabla\overline{u}_\infty(x)\cdot\nabla\overline{u}_\infty(x)\,dx, \tag{6}$$

where $\overline{u}_\infty(x) \in H_0^1(\Omega)$ *is the solution of*

$$\begin{cases} -\,div\left(K_{T_\infty}^\star(x)\nabla\overline{u}_\infty(x)\right) = f(x) & in \quad \Omega \\ \overline{u}_\infty = 0 & on \quad \partial\Omega. \end{cases} \tag{7}$$

We decompose

$$\frac{1}{T_n}\int_0^{T_n}\int_\Omega K^\star_{T_n}(x)\,\nabla u_n(t,x)\cdot\nabla u_n(t,x)\,dxdt - \int_\Omega K^\star_{T_\infty}(x)\,\nabla\overline{u}_\infty(x)\cdot\nabla\overline{u}_\infty(x)\,dx = I_1^n + I_2^n$$

where

$$I_1^n = \frac{1}{T_n}\int_0^{T_n}\int_\Omega K^\star_{T_n}\nabla u_n(t,x)\cdot\nabla u_n(t,x)\,dxdt - \int_\Omega K^\star_{T_n}\nabla\overline{u}_n(x)\cdot\nabla\overline{u}_n(x)\,dx$$

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where $\overline{u}_n$ solves

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To show that $I_1^n \to 0$, we prove that there exist $C_1, C_2 > 0$, independent of n, such that

$$\|u_n(t) - \overline{u}_n\|_{L^2(\Omega)} \le C_1 e^{-C_2 t}, \quad t > 0, \tag{9}$$

The function $v_n(t,x) = u_n(t,x) - \overline{u}_n(x)$ solves

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$$\|u_n(t) - \overline{u}_n\|_{L^2(\Omega)} \leq C_1 e^{-C_2 t}, \quad t > 0, \tag{9}$$

The function $v_n(t,x) = u_n(t,x) - \overline{u}_n(x)$ solves

$$\begin{cases} \beta^\star_n(x)\,v_n'(t,x) - \operatorname{div}\left(K^\star_{T_n}(x)\,\nabla v_n(t,x)\right) = 0 & \text{in} \quad (0,\infty)\times\Omega \\ v_n = 0 & \text{on} \quad (0,\infty)\times\partial\Omega \\ v_n(0,x) = u_0(x) - \overline{u}_n(x) & \text{in} \quad \Omega. \end{cases}$$

Convergence of $I_1^n \to 0$ as $T_n \to \infty$

Using the Fourier method,

$$v_n(t,x) = \sum_{k=1}^{\infty} a_n^k e^{-\lambda_n^k t} \omega_n^k(x), \quad \omega_n^k \in H_0^1(\Omega), \quad \left\| \omega_n^k \right\|^2_{L^2_{\beta_n^\star}(\Omega)} = \int_\Omega \beta_n^\star \left| \omega_n^k \right|^2 dx = 1$$

$$\begin{cases} -\operatorname{div}\left(K_{T_n}^\star \nabla \omega_n^k \right) = \lambda_n^k \beta_n^\star \omega_n^k & \text{in} \quad \Omega \\ \omega_n^k = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $0 < \lambda_n^1 < \lambda_n^2 \leq \lambda_n^3 \leq \cdots$, its associated eigenvalues, and

$$a_n^k = \int_\Omega \beta_n^\star(x) \left(u_0(x) - \overline{u}_n(x) \right) \omega_n^k(x)\, dx, \quad k, n \in \mathbb{N}.$$

Using that $\beta_1 \leq \beta_n^\star(x)$ a.e. $x \in \Omega$ and Parseval's identity, we have

$$\beta_1 \left\| v_n(t) \right\|^2_{L^2(\Omega)} \leq \left\| v_n(t) \right\|^2_{L^2_{\beta_n^\star}(\Omega)} = \sum_{k=1}^{\infty} e^{-2\lambda_n^k t} \left| a_n^k \right|^2 \leq e^{-2\lambda_n^1 t} \left\| u_0 - \overline{u}_n \right\|^2_{L^2_{\beta_n^\star}(\Omega)}.$$

Since $K_{T_n}^\star \xrightarrow{H} K_{T_\infty}^\star$ and $0 < \beta_1 \leq \beta_n^\star(x) \leq \beta_2$ a.e. $x \in \Omega$, the term $\left\| u_0 - \overline{u}_n \right\|^2_{L^2_{\beta_n^\star}(\Omega)}$ is uniformly bounded. Moreover, the uniform ellipticity of the sequence of tensors $K_{T_n}^\star$ lead to

$$\lambda_n^1 = \min_{\varphi \neq 0,\, \varphi \in H_0^1} \frac{\int_\Omega K_{T_n}^\star \nabla\varphi \cdot \nabla\varphi}{\left\| \varphi \right\|^2_{L^2_{\beta_n^\star}(\Omega)}} \geq \frac{k_1}{\beta_2} \min_{\varphi \neq 0,\, \varphi \in H_0^1} \frac{\int_\Omega \nabla\varphi \cdot \nabla\varphi}{\left\| \varphi \right\|^2_{L^2(\Omega)}} = \frac{k_1}{\beta_2} \lambda_1,$$

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$$\beta_1 \|v_n(t)\|^2_{L^2(\Omega)} \leq \|v_n(t)\|^2_{L^2_{\beta_n^\star}(\Omega)} = \sum_{k=1}^{\infty} e^{-2\lambda_n^k t}\left|a_n^k\right|^2 \leq e^{-2\lambda_n^1 t}\|u_0 - \overline{u}_n\|^2_{L^2_{\beta_n^\star}(\Omega)}.$$

Since $K^\star_{T_n} \xrightarrow{H} K^\star_{T_\infty}$ and $0 < \beta_1 \leq \beta_n^\star(x) \leq \beta_2$ a.e. $x \in \Omega$, the term $\|u_0 - \overline{u}_n\|^2_{L^2_{\beta_n^\star}(\Omega)}$ is uniformly bounded. Moreover, the uniform ellipticity of the sequence of tensors $K^\star_{T_n}$ lead to

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Using the weak form of (8), and multiplying the heat equation in system (5) by $u_n(t,x)$ and integrating by parts,

$$I_1^n = \frac{1}{2}\frac{1}{T_n}\int_\Omega \beta_n^\star \left(u_0^2(x) - u_n^2(T_n, x)\right) dx + \frac{1}{T_n}\int_0^{T_n}\int_\Omega f(x)\left(u_n(t,x) - \overline{u}_n(x)\right) dxdt.$$

By (9) and the boundedness of $\|\overline{u}_n\|_{L^2(\Omega)}$, the first term in the right-hand side of this expression converges to zero as $T_n \to \infty$. Using once again (9) and the Cauchy-Schwartz inequality,

$$\begin{aligned}
\left|\frac{1}{T_n}\int_0^{T_n}\int_\Omega f(x)\left(u_n(t,x) - \overline{u}_n(x)\right) dxdt\right| &\leq \|f\|_{L^2(\Omega)} \frac{1}{T_n}\int_0^{T_n} \|u_n(t) - \overline{u}_n\|_{L^2(\Omega)}\, dt \\
&\leq \|f\|_{L^2(\Omega)} \frac{1}{T_n}\int_0^{T_n} C_1 e^{-C_2 t} dt \\
&\to 0 \quad \text{as } T_n \to \infty.
\end{aligned}$$

Assume that $\left(\theta_{T_\infty}, K^\star_{T_\infty}\right)$ is not a solution of (RP_∞). Then, there exists another $\left(\widehat{\theta},\ \widehat{K}^\star\right) \in \textbf{RD}$ and $\varepsilon > 0$ such that

$$\int_\Omega K^\star_{T_\infty}(x)\,\nabla\overline{u}_\infty(x)\cdot\nabla\overline{u}_\infty(x)\,dx = \int_\Omega \widehat{K}^\star(x)\,\nabla\widehat{u}(x)\cdot\nabla\widehat{u}(x)\,dx + \varepsilon,$$

where $\widehat{u}(x)$ is the solution of the elliptic equation with conductivity $\widehat{K}^\star$. By (6), there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\frac{1}{T_n}\int_0^{T_n}\int_\Omega K^\star_{T_n}(x)\,\nabla u_n(t,x)\cdot\nabla u_n(t,x)\,dx > \int_\Omega K^\star_{T_\infty}(x)\,\nabla\overline{u}_\infty(x)\cdot\nabla\overline{u}_\infty(x)\,dx - \frac{\varepsilon}{3}.$$

Now let $u(t,x)$ solve

$$\begin{cases} \widehat{\beta}^\star(x)\,u'(t,x) - \text{div}\left(\widehat{K}^\star(x)\,\nabla u(t,x)\right) = f(x) & \text{in} \quad (0,T)\times\Omega \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

with $\widehat{\beta}^\star(x) = \widehat{\theta}(x)\,\beta_1 + \left(1-\widehat{\theta}(x)\right)\beta_2$. Then, multiplying this equation by $u(t,x)$ and integrating by parts, we get the convergence

$$\frac{1}{T_n}\int_0^{T_n}\int_\Omega \widehat{K}^\star(x)\,\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt \to \int_\Omega \widehat{K}^\star(x)\,\nabla\widehat{u}(x)\cdot\nabla\widehat{u}(x)\,dx \quad \text{as } n\to\infty.$$

Assume that $\left(\theta_{T_\infty}, K_{T_\infty}^\star\right)$ is not a solution of (RP$_\infty$). Then, there exists another $\left(\widehat{\theta},\ \widehat{K}^\star\right) \in$ **RD** and $\varepsilon > 0$ such that

$$\int_\Omega K_{T_\infty}^\star (x) \nabla \overline{u}_\infty (x) \cdot \nabla \overline{u}_\infty (x)\, dx = \int_\Omega \widehat{K}^\star (x) \nabla \widehat{u} (x) \cdot \nabla \widehat{u} (x)\, dx + \varepsilon,$$

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$$\frac{1}{T_n}\int_0^{T_n}\int_\Omega \widehat{K}^\star (x) \nabla u (t,x) \cdot \nabla u (t,x)\, dxdt \to \int_\Omega \widehat{K}^\star (x) \nabla \widehat{u} (x) \cdot \nabla \widehat{u} (x)\, dx \quad \text{as } n \to \infty.$$

Therefore, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$

$$\frac{1}{T_n}\int_0^{T_n}\int_\Omega \widehat{K}^\star(x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt < \int_\Omega \widehat{K}^\star(x)\nabla\widehat{u}(x)\cdot\nabla\widehat{u}(x)\,dx + \frac{\varepsilon}{3}.$$

Hence, for $n \geq \max(n_0, n_1)$ we have

$$\begin{aligned}\frac{1}{T_n}\int_0^{T_n}\int_\Omega \widehat{K}^\star(x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt &< \int_\Omega \widehat{K}^\star(x)\nabla\widehat{u}(x)\cdot\nabla\widehat{u}(x)\,dx + \frac{\varepsilon}{3}\\ &= \int_\Omega K_{T_\infty}^\star(x)\nabla\overline{u}_\infty(x)\cdot\nabla\overline{u}_\infty(x)\,dx - \varepsilon + \frac{\varepsilon}{3}\\ &< \frac{1}{T_n}\int_0^{T_n}\int_\Omega K_{T_n}^\star(x)\nabla u_n(t,x)\cdot\nabla u_n(t,x)\,dxdt - \frac{\varepsilon}{3}\end{aligned}$$

which contradicts the fact that $\left(\theta_{T_n}, K_{T_n}^\star\right)$ is an optimal solution of (RP_{T_n}). ■

Therefore, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$

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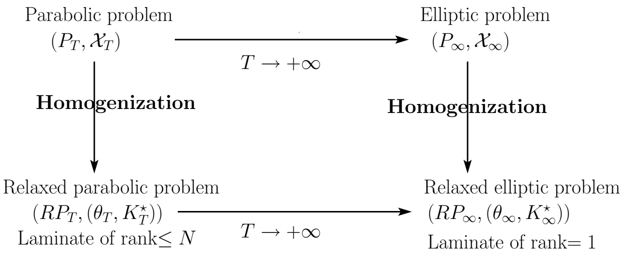

Figure: Commutation between Homogenization process and limit of the heat system as $T \to \infty$???

What about the structure of the optimal effective tensor $K_T^\star$ and its behavior w.r.t. T ?

$$\overline{J}_T^\star(\theta, K^\star) = \frac{1}{T}\int_0^T\int_\Omega K^\star \nabla u \cdot \nabla u \, dxdt + l\int_\Omega \theta(x)\, dx. \tag{10}$$

Theorem

*The objective function $\overline{J}_T^\star(\theta, K^\star)$ is Gâteaux differentiable on the space of admissible relaxed designs **RD** and*

$$\delta\overline{J}_T^\star(\theta, K^\star) = \int_\Omega \left[l - 2(\beta_2 - \beta_1)\frac{1}{T}\int_0^T u' p dt \right] \delta\theta \, dx + \frac{1}{T}\int_0^T\int_\Omega \delta K^\star \nabla u \cdot (2\nabla p + \nabla u) \, dxdt \tag{11}$$

*where $\delta\theta$ and $\delta K^\star$ are admissible increments in **RD** and p the solution of the adjoint equation*

$$\begin{cases} -\beta^\star p' - div(K^\star \nabla p) = div(K^\star \nabla u) & in \quad (0,T)\times\Omega \\ p = 0 & on \quad (0,T)\times\partial\Omega \\ p(T) = 0 & in \quad \Omega. \end{cases} \tag{12}$$

Consequently, if $(\theta, K^\star)$ is a minimizer of the function $\overline{J}_T^\star$, it must satisfy $\delta\overline{J}_T^\star(\theta, K^\star) \geq 0$ for any admissible increments $\delta\theta$, $\delta K^\star$.

Definition

*Let $(\theta, K^\star) \in$ **RD** satisfy the optimality condition $\delta \overline{J}_T^\star(\theta, K^\star) \geq 0$. For any fixed $T > 0$, we introduce the symmetric matrix of order N*

$$M_T = -\frac{1}{T}\int_0^T \nabla u \odot (2\nabla p + \nabla u)\, dt \tag{13}$$

where $\odot$ denotes the symmetrized tensor product of two vectors, with entries

$$(M_T)_{ij} = -\frac{1}{2T}\int_0^T \left[(\nabla u)_i\,(2\nabla p + \nabla u)_j + (\nabla u)_j\,(2\nabla p + \nabla u)_i\right] dt, \quad 1 \leq i, j \leq N$$

where u and p are its associated state and adjoint state, respectively.

Remark

If $(f, u_0) \in (L^2(\Omega))^2$, then $u \in L^2(0, T; H_0^1(\Omega)) \cap C^0(0, T; L^2(\Omega))$ and then $p \in L^2(0, T; H_0^1(\Omega))$. This implies that $M_T \in L^1(\Omega)$.

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Characterization of the effective optimal tensor $K_T^\star$ in term of sequential laminates

Theorem (ORDER OF LAMINATION)

Let $(\theta_T, K_T^\star)$ be a minimizer of $\overline{J}_T^\star$ and let u and p be its associated state and adjoint state, respectively.

1. $(\theta_T, K_T^\star)$ satisfies the following characterization

$$K_T^\star : M_T = \max_{K^0 \in G_{\theta_T}} K^0 : M_T \quad a.e. \quad x \in \Omega$$

   where $M_T \in L^1(\Omega; R^{N\times N})$ is given by (13) and $:$ the full contraction of two matrices.

2. $K_T^\star$ is a tensor corresponding to a sequential laminate of rank at most N with lamination directions given by the eigenvectors of M_T.

3. The function

$$\theta_T \longmapsto f(\theta_T, M_T) \equiv \max_{K^0 \in G_{\theta_T}} K^0 : M_T$$

   is $C^1([0,1])$ and the optimal density θ_T satisfies

$$\begin{cases} \theta_T(x) = 0 & \text{if and only if} & Q_T(x) > 0 \\ \theta_T(x) = 1 & \text{if and only if} & Q_T(x) < 0 \\ 0 \leq \theta_T(x) \leq 1 & \text{if} & Q_T(x) = 0 \end{cases} \tag{14}$$

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$$Q_T(x) = l - 2(\beta_0 - \beta_1)\frac{1}{\cdots}\int^T u' p dt + \frac{\partial f(\theta_T, M_T)}{\cdots} \tag{15}$$

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The case where $x \in \Omega$ is such that $M_T(x) \neq 0$

We fix θ and consider the path $K_T^\star(s) = sK^0 + (1-s)K_T^\star$ for any $K^0 \in G_\theta$, $0 \leq s \leq 1$ and for $K_T^\star$ an optimal tensor for (RP$_T$). Consequently, $\delta\theta = 0$ and $\delta K_T^\star = K^0 - K_T^\star$. The optimality condition $\delta \overline{J}_T^\star(\theta, K_T^\star) \geq 0$ implies that

$$\int_\Omega K_T^\star : M_T dx \geq \int_\Omega K^0 : M_T dx \quad \forall K^0 \in G_\theta. \tag{16}$$

Since M_T is well-defined and it belongs to $L^1(\Omega)$, we may therefore apply the Localization principle to conclude that (16) is equivalent a.e. $x \in \Omega$ to the following characterization of the optimal tensor $K_T^\star$

$$K_T^\star : M_T = \max_{K^0 \in G_\theta} K^0 : M_T, \quad \text{a.e.} \quad x \in \Omega. \tag{17}$$

It is known that the optimal tensor $K_T^\star$ of (17) must be simultaneously diagonalizable with M_T. Consequently, if $(e_j)_{1\leq j\leq N}$ is a basis of eigenvectors of M_T with associated eigenvalues $(\mu_j)_{1\leq j\leq N}$, then (17) transforms into

$$K_T^\star : M_T = \max_{(\lambda_j)\in G_\theta} \sum_{j=1}^N \lambda_j \mu_j, \quad (\lambda_j)_{1\leq j\leq N} \in \sigma(K^0), K^0 \in G_\theta$$

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It is known that the optimal tensor $K_T^\star$ of (17) must be simultaneously diagonalizable with M_T. Consequently, if $\left(e_j\right)_{1\leq j\leq N}$ is a basis of eigenvectors of M_T with associated eigenvalues $\left(\mu_j\right)_{1\leq j\leq N}$, then (17) transforms into

$$K_T^\star : M_T = \max_{(\lambda_j)\in G_\theta} \sum_{j=1}^N \lambda_j\mu_j, \quad (\lambda_j)_{1\leq j\leq N} \in \sigma(K^0), K^0 \in G_\theta$$

The case where $x \in \Omega$ is such that $M_T(x) \neq 0$

We fix θ and consider the path $K_T^\star(s) = sK^0 + (1-s)K_T^\star$ for any $K^0 \in G_\theta$, $0 \leq s \leq 1$ and for $K_T^\star$ an optimal tensor for (RP$_T$). Consequently, $\delta\theta = 0$ and $\delta K_T^\star = K^0 - K_T^\star$. The optimality condition $\delta \overline{J}_T^\star(\theta, K_T^\star) \geq 0$ implies that

$$\int_\Omega K_T^\star : M_T dx \geq \int_\Omega K^0 : M_T dx \quad \forall K^0 \in G_\theta. \tag{16}$$

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Assume that $x \in \Omega$ is such that $M_T(x) \neq 0$. Since the cost function in (18) is linear and the set G_θ convex, the solution belongs to the boundary of G_θ. This implies that $K_T^\star$ corresponds to a sequential laminate of rank at most N with lamination directions given by the eigenvectors of M_T.

Assume that $x \in \Omega$ is such that $M_T(x) = 0$. one can not conclude directly from the relation (18) which degenerates. However, in that case and in dimension $N = 2$, the optimal tensor $K_T^\star \in G_\theta$ may be replaced by a tensor which belongs to the boundary of G_θ without changing the value of the objective function. Indeed, assume that $K_T^\star$ belongs to the interior of G_θ

$$\frac{1}{\lambda_1 - k_1} + \frac{1}{\lambda_2 - k_1} < g^-(\theta) \equiv \frac{1}{\lambda^-(\theta) - k_1} + \frac{1}{\lambda^+(\theta) - k_1}$$

and that

$$\frac{1}{k_2 - \lambda_1} + \frac{1}{k_2 - \lambda_2} < g^+(\theta) \equiv \frac{1}{k_2 - \lambda^-(\theta)} + \frac{1}{k_2 - \lambda^+(\theta)}.$$

Since the continuous function g^- is strictly increasing and satisfies $g^-(0) = 2/(k_2 - k_1) \le (\lambda_1 - k_1)^{-1} + (\lambda_2 - k_1)^{-1}$, there exists $\theta^- \in (0, \theta)$ such that

$$\frac{1}{\lambda_1 - k_1} + \frac{1}{\lambda_2 - k_1} = g^-(\theta^-).$$

$$K_T^\star : M_T = \max_{(\lambda_j)\in G_\theta} \sum_{j=1}^{N} \lambda_j \mu_j, \quad (\lambda_j)_{1\le j\le N} \in \sigma(K^0), K^0 \in G_\theta \tag{18}$$

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Since the continuous function g^- is strictly increasing and satisfies $g^-(0) = 2/(k_2 - k_1) \leq (\lambda_1 - k_1)^{-1} + (\lambda_2 - k_1)^{-1}$, there exists $\theta^- \in (0, \theta)$ such that

$$\frac{1}{\lambda_1 - k_1} + \frac{1}{\lambda_2 - k_1} = g^-(\theta^-).$$

Similarly, since the continuous function g^+ is strictly decreasing and satisfies $g^+(1) = 2/(k_2 - k_1) \leq (k_2 - \lambda_1)^{-1} + (k_2 - \lambda_2)^{-1}$, there exists $\theta^+ \in (\theta, 1)$ such that

$$\frac{1}{k_2 - \lambda_1} + \frac{1}{k_2 - \lambda_2} = g^+(\theta^+).$$

Consequently, at the point x where $M_T(x) = 0$, we may consider the composite with materials k_1 and k_2 in proportions θ^- and $(1 - \theta^-)$, respectively, or the composite with materials k_1 and k_2 in proportions $1 - \theta^+$ and θ^+. Notice that this choice allows us to ensure that the volume constraint $\|\theta\|_{L^1(\Omega)} = L|\Omega|$ holds. In both cases, the eigenvalues λ_1, λ_2 of $K_T^\star$ remain unchanged and so the value of the cost $\overline{J}_T^\star$.

Lemma

For any T, we note by μ_1^T, μ_2^T, $\mu_1^T \le \mu_2^T$ the eigenvalues of the matrix M_T of order $N = 2$ defined by (13). The solution of the linear problem

$$\max_{K^0 \in G_{\theta_T}} K^0 : M_T = \max_{(v_1^T, v_2^T) \in G_{\theta_T}} v_1^T \mu_1^T + v_2^T \mu_2^T \qquad \textit{is given by}$$

$$\begin{cases} (v_1^T, v_2^T) = (k_2, k_2) + \dfrac{\sqrt{\mu_1^T} + \sqrt{\mu_2^T}}{(\lambda^+_{\theta_T} - k_2)^{-1} + (\lambda^-_{\theta_T} - k_2)^{-1}} \left(\dfrac{1}{\sqrt{\mu_1^T}}, \dfrac{1}{\sqrt{\mu_2^T}} \right) \\ \qquad \textit{if} \quad \mu_1^T \ge 0 \quad \textit{and} \quad \sqrt{\mu_1^T}(k_2 - \lambda^-_{\theta_T}) > \sqrt{\mu_2^T}(k_2 - \lambda^+_{\theta_T}) [\textit{Second order laminate}] \\ (v_1^T, v_2^T) = (k_1, k_1) + \dfrac{\sqrt{-\mu_1^T} + \sqrt{-\mu_2^T}}{(\lambda^+_{\theta_T} - k_1)^{-1} + (\lambda^-_{\theta_T} - k_1)^{-1}} \left(\dfrac{1}{\sqrt{-\mu_1^T}}, \dfrac{1}{\sqrt{-\mu_2^T}} \right) \\ \qquad \textit{if} \quad \mu_2^T \le 0 \quad \textit{and} \quad \sqrt{-\mu_1^T}(\lambda^-_{\theta_T} - k_1) < \sqrt{-\mu_2^T}(\lambda^+_{\theta_T} - k_1) [\textit{Second order laminate}] \\ (v_1^T, v_2^T) = (\lambda^-_{\theta_T}, \lambda^+_{\theta_T}) \quad \textit{else}. [\textit{First order laminate}] \end{cases} \tag{19}$$

Structure of the matrix M_T as $T \to \infty$ for $N = 2$ (Formal analysis)

For any T fixed, we consider the normalized eigenfunctions $(w_m^T)_{m>0}$ and corresponding eigenvalues $(\lambda_m^T)_{m>0}$ of

$$\begin{cases} -\operatorname{div}(K_T^\star \nabla w_m^T) = \lambda_m^T \beta_T^\star w_m^T & \text{in } \Omega, \\ w_m^T = 0 & \text{on } \partial\Omega \end{cases}$$

where $K_T^\star$ is the optimal tensor for (RP_T). Since $K_T^\star \in \partial G_{\theta_T}$, v_1^T, v_2^T are uniformly bounded with respect to T as well as $\{\lambda_m^T\}_m$ and $(w_m^T)_{m>0}$ in $H_0^1(\Omega)$.
Assume that the source f and the initial datum u_0 are expanded as follows:

$$\begin{aligned} &f(x) = \sum_{m>0} f_m^T w_m^T(x), \quad u_0(x) = \sum_{m>0} a_m^T w_m^T(x), \quad \{a_m^T\}_{m>0}, \{f_m^T\}_{m>0} \in l^2(\mathbb{N}), \\ &u(t,x) = \sum_{m>0} a_m^T(t) w_m^T(x), \quad p(t,x) = \sum_{m>0} b_m^T(t) w_m^T(x) \end{aligned} \tag{20}$$

We rewrite the symmetric matrix M_T as follows:

$$-M_T(x) = \begin{pmatrix} \sum_{m,n>0} c_{mn}^T (w_m^T)_{x_1} (w_n^T)_{x_1} & \frac{1}{2} \sum_{m,n>0} c_{mn}^T \Big((w_m^T)_{x_1} (w_n^T)_{x_2} + (w_m^T)_{x_2} (w_n^T)_{x_1} \Big) \\ sym. & \sum_{m,n>0} c_{mn}^T (w_m^T)_{x_2} (w_n^T)_{x_2} \end{pmatrix}.$$

with $c_{mn}^T = \frac{1}{T} \int_0^T a_m^T(t)(a_n^T(t) + 2b_n^T(t))dt, \quad m,n>0$

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Structure of the matrix M_T - T large

$$c_{mn}^T = -\frac{f_m^T}{\lambda_m^T}\frac{f_n^T}{\lambda_n^T} + \frac{1}{T}\frac{f_n^T}{(\lambda_m^T)^2(\lambda_n^T)^2}(\lambda_n^T f_m^T - \lambda_m^T a_m^T \lambda_n^T) + O(e^{-\lambda_m T}, e^{-\lambda_n^T T})$$

where the coefficients $c_{mn}^{i,T}$ are bounded with respect to T. Notice that only the coefficients of the heat source f are involved in the first order terms $c_{mn}^{0,T}$. We put

$$M_T(x) = M_T^0(x) + \frac{M_T^1(x)}{T} + M^2(T,x), \quad x \in \Omega.$$

Now, we observe that the symmetric matrix $M_T^0(x)$, which is given by

$$M_T^0(x) = -\begin{pmatrix} \left(\sum_{m>0} \frac{f_m^T}{\lambda_m^T}(w_m^T)_{x_1}\right)^2 & \left(\sum_{m>0} \frac{f_m^T}{\lambda_m^T}(w_m^T)_{x_1}\right)\left(\sum_{m>0} \frac{f_m^T}{\lambda_m^T}(w_m^T)_{x_2}\right) \\ SYM & \left(\sum_{m>0} \frac{f_m^T}{\lambda_m^T}(w_m^T)_{x_2}\right)^2 \end{pmatrix}$$

is singular: $det(M_T^0(x)) = 0$ so that $arg(\max_{K^0 \in G_{\theta_T}} K^0 : M_T^0) = (\lambda_{\theta_T}^-, \lambda_{\theta_T}^+)$

Proposition (T large)

For any $T \geq \sup_{x\in\Omega} T^+(x)$, the solution of the problem $\max_{K^0 \in G_{\theta_T}} K^0 : M_T$ is $(\lambda_{\theta_T}^-, \lambda_{\theta_T}^+)$ so that the optimal tensor $K_T^\star$ is a first order laminate.

Structure of the matrix M_T - T large

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$$c_{mn}^T = -\frac{f_m^T}{\lambda_m^T}\frac{f_n^T}{\lambda_n^T} + \frac{1}{T}\frac{f_n^T}{(\lambda_m^T)^2(\lambda_n^T)^2}(\lambda_n^T f_m^T - \lambda_m^T a_m^T \lambda_n^T) + O(e^{-\lambda_m T}, e^{-\lambda_n^T T})$$

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Proposition (T large)

For any $T \geq \sup_{x\in\Omega} T^+(x)$, the solution of the problem $\max_{K^0 \in G_{\theta_T}} K^0 : M_T$ is $(\lambda_{\theta_T}^-, \lambda_{\theta_T}^+)$ so that the optimal tensor $K_T^\star$ is a first order laminate.

The analysis for T arbitrarily small is similar. Precisely, we obtain

$$c_{mn}^T = a_m^T a_n^T - \frac{1}{2} a_m^T a_n^T (\lambda_m^T + 3\lambda_n^T) T + \frac{1}{2} (a_m^T f_n^T + a_n^T f_m^T) T + O(T^2), \quad m, n > 0 \quad (21)$$

so that we decompose the matrix M_T as $M_T(x) = M_T^0(x) + TM_T^1(x) + M_T^2(T, x)$ in Ω, where the symmetric matrix M_T^0 depends only on the coefficients a_m^T of the initial condition u_0, assumed different from zero. By arguing as before, we obtain that $M_T^0(x)$ is singular so that for T small enough, says $T \leq T^-(x)$, the determinant of $M_T(x)$ is negatif.

Proposition (T small)

For any $T \leq \inf_{x \in \Omega} T^-(x)$, the solution of the problem $\max_{K^0 \in G_{\theta_T}} K^0 : M_T$ is $(\lambda_{\theta_T}^-, \lambda_{\theta_T}^+)$ so that the micro-structure is recovered by first order laminates.

If $u_0 = 0$, $M_T^0 = M_T^1 = 0$ and $M_T(x) = T^2 M_T^2 + \dots$ with $det(M_T^2(x)) \leq 0$ for all $x \in \Omega$ $T > 0$.

Algorithm of minimization of $\overline{J}^{\star}_{T}(\theta_T, K^{\star}_{T})$

$\Longrightarrow$ Gradient method for the variable θ_T and optimality condition for the variable $K^{\star}_{T}$:

Given $T > 0$, $L \in (0,1)$, $\Omega \subset \mathbb{R}^2$, $u_0, f \in L^2(\Omega)$ and $0 < \varepsilon << 1$,

- Initialization: $\theta^0_T = L$, $(v^0_1, v^0_2, \cdots, v^0_N) \in \partial G_\theta$, $P^0 = Id_N$, Λ^0_T the diagonal matrix such that $(\Lambda^0_T)_{ii} = v^0_i$ and finally put $K^{\star,0}_T = P^0 \Lambda^0_T (P^0)^t$.
- For $n \geq 1$, iteration until convergence $(|\overline{J}^{\star}_{T}(\theta^n_T, K^{\star,n-1}) - \overline{J}^{\star}_{T}(\theta^n_T, K^{\star,n})| \leq \varepsilon |\overline{J}^{\star}_{T}(\theta^0_T, K^{\star,0})|)$:
  1. Compute the state u^n and the adjoint state p^n.
  2. Compute the descent direction $\delta\theta(u^n, p^n)$ given by

Algorithm of minimization of $\overline{J}_T^\star(\theta_T, K_T^\star)$

$\Longrightarrow$ Gradient method for the variable θ_T and optimality condition for the variable $K_T^\star$:
Given $T > 0$, $L \in (0,1)$, $\Omega \subset \mathbb{R}^2$, $u_0, f \in L^2(\Omega)$ and $0 < \varepsilon << 1$,

- Initialization: $\theta_T^0 = L$, $(v_1^0, v_2^0, \cdots, v_N^0) \in \partial G_\theta$, $P^0 = Id_N$, Λ_T^0 the diagonal matrix such that $(\Lambda_T^0)_{ii} = v_i^0$ and finally put $K_T^{\star,0} = P^0 \Lambda_T^0 (P^0)^t$.
- For $n \geq 1$, iteration until convergence $(|\overline{J}_T^\star(\theta_T^n, K^{\star,n-1}) - \overline{J}_T^\star(\theta_T^n, K^{\star,n})| \leq \varepsilon |\overline{J}_T^\star(\theta_T^0, K^{\star,0})|)$:
  1. Compute the state u^n and the adjoint state p^n.
  2. Compute the descent direction $\delta\theta(u^n, p^n)$ given by
  $$\delta\theta = \frac{1}{T}\int_0^T K_{T,\theta}^\star \nabla u \cdot (\nabla u + 2\nabla p) dt - 2(\beta_2 - \beta_1)\frac{1}{T}\int_0^T u' p \, dt + l \quad \text{in} \quad \Omega,$$
  and then update the density $\theta_T^n := \theta_T^{n-1} + \eta\delta\theta(u^n, p^n)$, where $\eta \in L^\infty(\Omega, \mathbb{R}_+^\star)$ denotes a function, which depends on the multiplier l and chosen so that θ^n satisfies the volume constraint.
  3. Compute the matrix $M_T^n = M_T^n(u^n, p^n)$ in Ω, its eigenvalues $\mu_1^n, \mu_2^n, \cdots, \mu_N^n$ and corresponding eigenvectors $e_1^n, e_2^n, \cdots, e_N^n$, and set $P^n = (e_1^n, e_2^n, \cdots, e_N^n)$.
  4. Solve the linear problem $\max_{K^0 \in G_{\theta_T}} K^0 : M_T^n$ leading to $(v_1^n, v_2^n, \cdots, v_N^n) \in \partial G_{\theta_T^n}$. Consider the matrix $\Lambda_T^n = (v_1^n, v_2^n, \cdots, v_N^n)$ and then put $K_T^{\star,n} = P^n \Lambda_T^n (P^n)^t$.

$\Longrightarrow$ Gradient method for the variable θ_T and optimality condition for the variable $K^\star_T$:
Given $T > 0$, $L \in (0,1)$, $\Omega \subset \mathbb{R}^2$, $u_0, f \in L^2(\Omega)$ and $0 < \varepsilon << 1$,

- Initialization: $\theta^0_T = L$, $(v^0_1, v^0_2, \cdots, v^0_N) \in \partial G_\theta$, $P^0 = Id_N$, Λ^0_T the diagonal matrix such that $(\Lambda^0_T)_{ii} = v^0_i$ and finally put $K^{\star,0}_T = P^0 \Lambda^0_T (P^0)^t$.
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Exemple 1: Uniform heat source $f = 1$ and $u_0 = 0$

$$N = 2, \quad \Omega = (0,1)^2, \quad (\beta_1, k_1) = (1, 0.07), \quad (\beta_2, k_2) = (1, 0.14), \quad L = 0.5, \quad f = 1, u_0 = 0$$

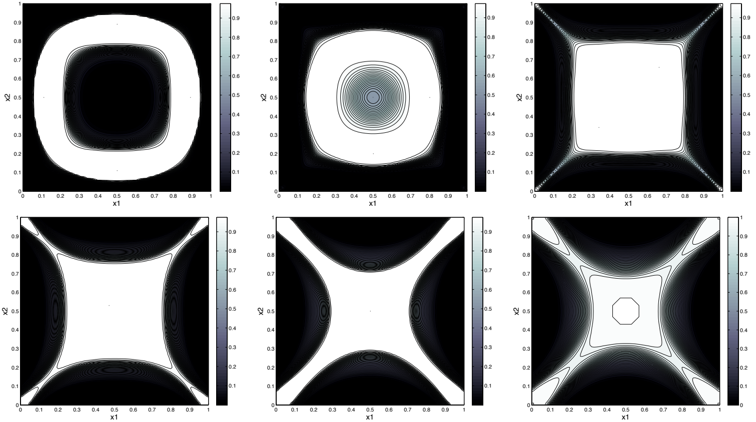

Figure: Isovalues of θ_T in Ω for $T = 0.5$ (**Top left**), $T = 1$, $T = 1.5$, $T = 2$, $T = 4$ and the limit elliptic case "$T = \infty$" (**Bottom right**). The white zones correspond to the weaker conductor phase (β_1, k_1).

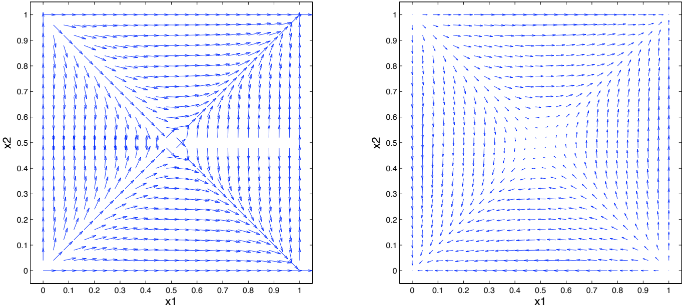

Figure: Direction of lamination for $T = 2$ (**Left**) and $T = \infty$ (**Right**).

$\Longrightarrow$ We observe first order laminates for all $x \in \Omega$ and all $T > 0$.

Exemple 2: Non uniform heat source and $u_0 = 0$

$$f(x) = \mathcal{X}_{(0.05,0.15)\times(0.1,0.9)}(x) - \mathcal{X}_{(0.85,0.95)\times(0.1,0.9)}(x)$$

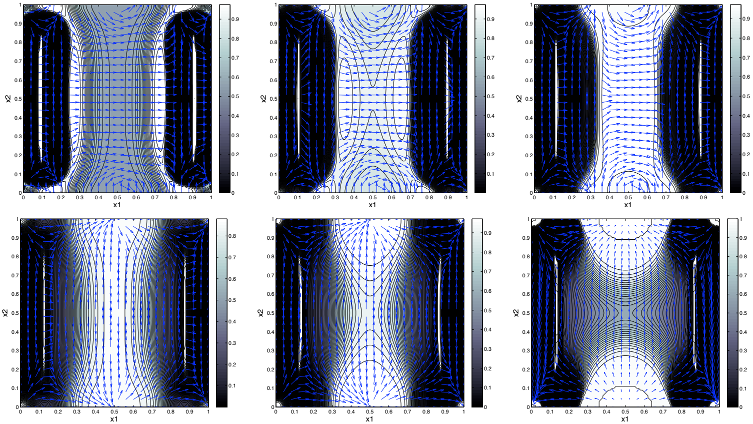

Figure: Isovalues of θ_T and direction of lamination in Ω for $T = 0.25$ (**Top left**), $T = 0.5$, $T = 1$, $T = 2$, $T = 4$ and the limit elliptic case "$T = \infty$" (**Bottom right**).

$$u_0(x) = \frac{1}{4}\mathcal{X}_{(0.2,0.8)\times(0.1,0.2)}(x) - \frac{1}{4}\mathcal{X}_{(0.2,0.8)\times(0.8,0.9)}(x)$$

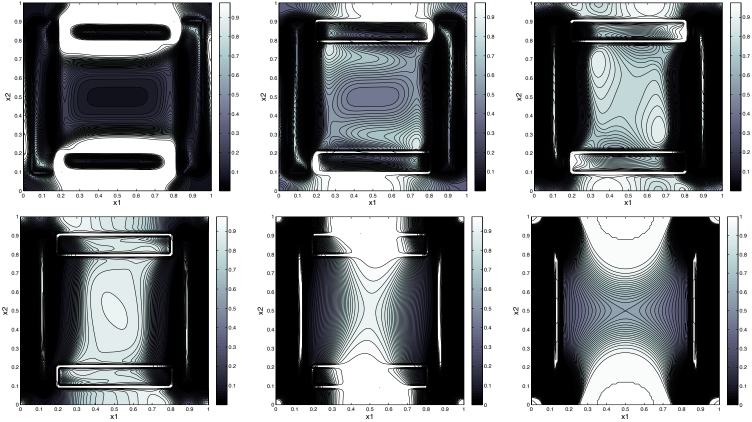

Figure: Isovalues of θ_T in Ω for $T = 0.125, 0.25, 0.5$, $T = 1$, $T = 4$ and "$T = \infty$".

$\Longrightarrow$ We observe second order laminates for $T \in \simeq [0.125, 1]$

Exemple 3: Interplay between f and u_0

$$u_0(x) = \frac{1}{4}\mathcal{X}_{(0.2,0.8)\times(0.1,0.2)}(x) - \frac{1}{4}\mathcal{X}_{(0.2,0.8)\times(0.8,0.9)}(x)$$

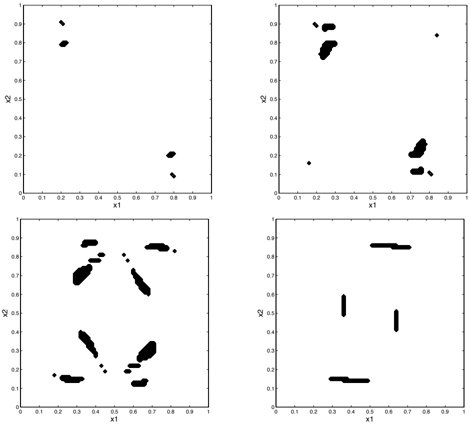

Figure: Second order laminate zone in Ω for $T = 0.125, 0.25, 0.5$ and $T = 1$.

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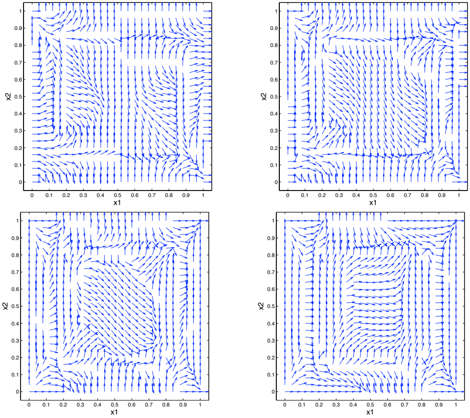

Figure: First eigenvector of the matrix M_T for $T = 0.125, 0.25, 0.5$ and $T = 1$.

Conclusions

- The minimizers $K_T^\star$ H-converge toward the minimizers of $K_\infty^\star$
- Minimizers $K_T^\star$ are represented with at MOST N order (sequential) laminates
- We strongly suspect that minimizers $K_T^\star$ are represented with first order laminates is T is large enough
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Theorem (AM-Pedregal-Periago, JMPA 2008)

Assume that the solution of of the heat system has the regularity $u \in L^2\left(0, T; H^2(\Omega)\right)$ and depends continuously on the initial datum in the corresponding norms. Then the variational problem

$$(RP_t) \quad \text{Minimize in } \left(\theta, \overline{G}, u\right): \quad \overline{J_t}(\theta, \overline{G}, u) = \frac{1}{2}\int_0^T\int_\Omega \left[k_1 \frac{\left|\overline{G} - k_2\nabla u\right|^2}{\theta\,(k_1 - k_2)^2} + k_2 \frac{\left|\overline{G} - k_1\nabla u\right|^2}{(1-\theta)\,(k_2 - k_1)^2}\right] dxdt$$

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is a relaxation of (VP_t) in the sense that

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More details on www.math.univ-bpclermont.fr/~*munch*/

THANK YOU FOR YOUR ATTENTION