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BOUNDARY STABILIZATION OF A NONLINEAR SHALLOW BEAM: THEORY AND NUMERICAL APPROXIMATION

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Abstract. We consider a dynamical one-dimensional nonlinear Marguerre-Vlaslov model for an elastic arch depending on one parameter $\varepsilon > 0$ and study its asymptotic behavior for large time as $\varepsilon \to 0$. Introducing appropriate boundary feedbacks, we prove that the corresponding energy decays exponentially uniformly with respect to ε and the curvature. The analysis highlights the importance of the damping mechanism - assumed to be proportional to ε^{α}, $0 \leq \alpha \leq 1$ - on the longitudinal deformation of the arch. The limit as $\varepsilon \to 0$, first exhibits a linear and a nonlinear arch model, for $\alpha > 0$ and $\alpha = 0$ respectively and then, permits to obtain exponential decay properties. Some numerical experiments confirm the theoretical results, analyze the cases $\alpha \notin [0, 1]$ and evaluate the influence of the curvature on the stabilization.

1. **Introduction - Problem statement.** We are concerned in this work with the stabilization of a nonlinear shallow arch model by boundary feedbacks. More precisely, we are devoted to proving how standard arch models may be obtained as singular limit of a 1-D Marguerre-Vlasov system - with respect to a small parameter ε -, so that the exponential decay rate of the energy remains uniform as this parameter goes to zero. A similar issue was addressed in [14] with internal damping mechanism and Dirichlet boundary condition. Here, we carry out the control strategy which is given in [13] to study the model when the dissipation acts on the boundary, a situation which add important technical difficulties. Our analysis shows how crucial are the boundary conditions on the nature of the limit system and on the decay properties.

A widely accepted dynamical model to describe large deflections of an elastic shallow arch is the so-called Marguerre-Vlasov system (see [1, 2, 19] and the references therein). Precisely, for $\Omega = (0, L)$ where L designates the length of the arch,

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the following system is considered :

$$\begin{cases} \varepsilon u^\varepsilon_{tt} - h(u^\varepsilon_x, w^\varepsilon)_x = 0 \qquad x \in \Omega, \quad t > 0, \\ w^\varepsilon_{tt} - w^\varepsilon_{xxtt} + w^\varepsilon_{xxxx} - (w^\varepsilon_x h(u^\varepsilon_x, w^\varepsilon))_x + k(x)h(u^\varepsilon_x, w^\varepsilon) = 0 \qquad x \in \Omega, \quad t > 0, \end{cases} \tag{1}$$

where h denotes the longitudinal strain of the arch

$$h(u^\varepsilon_x, w^\varepsilon) = \left(u^\varepsilon_x + \frac{1}{2}(w^\varepsilon_x)^2 + k(x) w^\varepsilon \right). \tag{2}$$

For simplicity, we put here the physical constants equal to one. The quantities $u^\varepsilon = u^\varepsilon(x,t)$ and $w^\varepsilon = w^\varepsilon(x,t)$ represent, respectively, the longitudinal and transversal displacement of the arch at point $x \in \Omega = (0, L)$ and at time $t > 0$. $k = k(x)$ represents the curvature of the arch and is assumed to be small enough. Finally, ε denotes a real positive parameter, introduced by Berger in [1] in the context of plates in order to make the link between Von Kármán system (see [4]) for which $\varepsilon = 1$ and Kirchhoff-Love's system (see for instance [8]) for which $\varepsilon = 0$. Thus, the nonlinear model above reflects the effects of stretching on bending, a necessary consideration for arch that undergoes large deflections from the equilibrium point $u = w = 0$.

We assume that the arch is clamped at $x = 0$

$$u^\varepsilon(0,t) = w^\varepsilon(0,t) = w^\varepsilon_x(0,t) = 0, \quad \forall t > 0. \tag{3}$$

On the other edge $x = L$ of the arch, assuming that the velocities $u^\varepsilon_t(L,t)$ and $w^\varepsilon_t(L,t)$ and the rate of bending $w^\varepsilon_{xt}(L,t)$ can be measured for all $t > 0$, we prescribe the following boundary feedback terms in the form of moments and shears :

$$\begin{cases} h(u^\varepsilon_x, w^\varepsilon)(L,t) = -\gamma\, u^\varepsilon_t(L,t), \quad t > 0, \\ (w^\varepsilon_{xxx} - w^\varepsilon_{xtt} - h(u^\varepsilon_x, w^\varepsilon) w^\varepsilon_x)(L,t) = w^\varepsilon_t(L,t), \quad t > 0, \\ w^\varepsilon_{xx}(L,t) = -w^\varepsilon_{xt}(L,t), \quad t > 0, \end{cases} \tag{4}$$

with $\gamma \in \mathbb{R}$. In the sequel, in order to have only one parameter in the model, we assume that γ may be expressed in term of ε as follows :

$$\gamma(\varepsilon) = \varepsilon^\alpha, \quad \alpha \in \mathbb{R}. \tag{5}$$

At last, the initial condition for (1)-(4) are

$$(u^\varepsilon(x,0), u^\varepsilon_t(x,0), w^\varepsilon(x,0), w^\varepsilon_t(x,0)) = (u^0, u^1, w^0, w^1), \quad 0 \le x \le L \tag{6}$$

with (u^0, u^1, w^0, w^1) independent of ε.

Then, the energy associated to (1)-(6) is defined by

$$E_\varepsilon(t) = \frac{1}{2} \int_0^L \left\{ \varepsilon (u^\varepsilon_t)^2 + (w^\varepsilon_t)^2 + (w^\varepsilon_{xt})^2 + (w^\varepsilon_{xx})^2 + h(u^\varepsilon_x, w^\varepsilon)^2 \right\} dx, \forall t > 0 \tag{7}$$

and according to the boundary conditions (4) and (5), it satisfies (formally) the following dissipation law

$$\frac{dE_\varepsilon(t)}{dt} = -\varepsilon^\alpha (u^\varepsilon_t)^2(L,t) - (w^\varepsilon_t)^2(L,t) - (w^\varepsilon_{xt})^2(L,t), \quad \forall t > 0. \tag{8}$$

Therefore, the boundary terms play the role of feedback damping mechanisms and one can wonder if the energy decays to zero as time goes to infinity. In this paper, we analyze both theoretical and numerically the following two questions with respect to the value of α :

1. Uniform stabilization of E_ε with respect to ε and the curvature k;

2. Convergence of the solution $\{u^\varepsilon, w^\varepsilon\}$ as ε goes to zero.

In a first part, we show that, for $\alpha \in [0,1]$, the energy E_ε decays exponentially in time, uniformly with respect to the parameter ε and the curvature k. In this way, we make use of multipliers as in [13] to obtain non standard energy identities. These identities and the assumption on the smallness of the curvature then allow to construct an appropriate Lyapunov function leading to the decay property.

In a second part, we show that the asymptotic limit of the system (1) permits to recover two modelizations of the transversal displacement of a beam partially clamped. For $\alpha > 0$, we obtain that the limit w of the transversal displacement w^ε is solution of the classical linear beam equation of fourth order

$$w_{xx} - w_{xxtt} + w_{xxxx} = 0 \tag{9}$$

whereas for $\alpha = 0$, we obtain the nonlinear equation

$$w_{xx} - w_{xxtt} + w_{xxxx} - \frac{1}{L}\left[\zeta + \int_\Omega \left(\frac{w_x^2}{2} + k(x)w\right)dx\right](w_{xx} - k(x)) = 0 \tag{10}$$

where $\zeta = \zeta(t)$ is a time-scalar function, solution of a first order ODE and related to the limit u of u^ε via the initial condition $\zeta(0) = u^\varepsilon(L,0) = u^0(L)$. In this respect, results of this paper may also be considered as a contribution in the context of vibration modelling. The connections between the various models available for a given mechanical problem are often described by means of singular perturbation problems. We refer the reader to the monograph of Ciarlet [4] in which various shell models are derived as singular limits of the 3-D elasticity system and [5] for an asymptotic analysis of nonlinear beam models.

The outline of the paper is as follows. In section 2, we apply a general result of Lagnese-Leugering [13] to show the well-posedness of the nonlinear boundary value problem (1-6) in the space of finite energy. Section 3 is then devoted to the proof of the uniform exponential decay of E_ε. In Section 4, combining energy estimates and compactness arguments, we derive the two asymptotic limits of $(u^\varepsilon, w^\varepsilon)$ in precise senses, depending on the positive value of α (see Theorem 4.1 and Theorem 4.2). The main difficulty is the identification of the nonlinear term. This is done by using ad'hoc test functions which depend on the boundary conditions in a sensitive way. We also discuss the cases $\alpha < 0$ and $\alpha > 1$. Finally, in Section 5, we numerically check and specify the theoretical convergence. We conclude with some remarks and perspectives. Our analysis improves the earlier work [16] and gives a satisfactory answer to a problem suggested in [14].

2. **Existence and uniqueness for the Marguerre-Vlaslov model.** For any $\varepsilon > 0$ fixed, we first prove the well-posedness of the system (1-6) in the space of finite energy. Taking the boundary conditions (4) and the energy dissipation law (8) into account we introduce the following Hilbert space

$$H = V \times L^2(\Omega) \times W \times V \tag{11}$$

where

$$V = \{v \in H^1(\Omega) : v(0) = 0\}, \quad W = \{v \in H^2(\Omega) : v(0) = v_x(0) = 0\}. \tag{12}$$

The Hilbert space H is endowed with the natural norm

$$\| (v,y,w,z) \|_H = \left(\| v \|^2_{H^1(\Omega)} + \varepsilon \| y \|^2_{L^2(\Omega)} + \| w_{xx} \|^2_{L^2(\Omega)} + \| z \|^2_{H^1(\Omega)} \right)^{1/2}. \tag{13}$$

Then, the following result holds.

Theorem 2.1. *Let $\varepsilon > 0$, $\alpha \in \mathbb{R}$, $k \in H^1(\Omega)$ and $(u^0, u^1, w^0, w^1) \in H$. Then, problem (1)-(6) has a unique global weak solution*

$$(u^\varepsilon, u_t^\varepsilon, w^\varepsilon, w_t^\varepsilon) \in C([0, +\infty); H) \tag{14}$$

and the total energy E_ε given by (7) satisfies (8) for all $t > 0$.

Proof. The proof is obtained following closely the arguments developed in [13]. For the sake of completeness, we sketch the basic arguments. In order to study the well-posedness, we formulate the system (1)-(6) as an abstract evolution equation in H. Then, local (in time) existence is obtained using standard semigroup theory. Global existence is a consequence of the energy dissipation law (8).

First, let us introduce the following variables

$$z_0^\varepsilon = h(u_x^\varepsilon, w^\varepsilon); \quad z_1^\varepsilon = u_t^\varepsilon; \quad w_0^\varepsilon = w_t^\varepsilon; \quad w_1^\varepsilon = w^\varepsilon. \tag{15}$$

After some integrations by parts using the boundary conditions, we arrive at the variational system

$$\begin{cases} (z_{0t}^\varepsilon, \psi_0) - (z_{1x}^\varepsilon, \psi_0) = 0, \quad \forall\ \psi_0 \in L^2(\Omega), \\ \varepsilon(z_{1t}^\varepsilon, \psi_1) + (z_0^\varepsilon, \psi_{1x}) + \varepsilon^\alpha z_1^\varepsilon(L,t)\psi_1(L) = 0, \quad \forall\ \psi_1 \in V, \\ (w_{0t}^\varepsilon, \phi_1)_V + (w_1^\varepsilon, \phi_1)_W + w_0^\varepsilon(L,t)\phi_1(L) + w_{0x}^\varepsilon(L,t)\phi_{1x}(L), \\ \qquad\qquad = -(z_0^\varepsilon w_{1x}^\varepsilon, \phi_{1x}) - (k(x) z_0^\varepsilon, \phi_1), \quad \forall\ \phi_1 \in W, \\ (w_{1t}^\varepsilon, \phi_2)_W - (w_0^\varepsilon, \phi_2)_W = 0, \quad \forall\ \phi_2 \in W, \end{cases} \tag{16}$$

where $(,)$ denotes the scalar product in $L^2(\Omega)$. Then, following the same steps developed in [13], we can rewrite (16) as a first order differential system in H, namely

$$\begin{cases} \dfrac{dU^\varepsilon}{dt} = AU^\varepsilon + F(U^\varepsilon), \quad t > 0, \\ U^\varepsilon(0) = U^0, \end{cases} \tag{17}$$

where $U^\varepsilon = (z_0^\varepsilon, z_1^\varepsilon, w_0^\varepsilon, w_1^\varepsilon)$ and $U^0 = (u^0, u^1, w^0, w^1)$. The operator A is maximal and dissipative and therefore the underlying linear system is governed by a semigroup of contractions in H (see [13]). To deal with the nonlinear term arising both on the equation and on the boundary, we can also proceed as in [13] to prove that F is locally Lipschitz continuous in H. Indeed, the additional difficulty here could be the presence of the term $(k(x)z_0^\varepsilon, \phi_1)$ in the structure of F. But this latter does not affect the result since it defines a Lipschitz function as well. Therefore, for every $(u^0, u^1, w^0, w^1) \in H$, the initial-value problem (17) has a unique local (in time) weak solution. In order to obtain global existence, we need an a priori estimate, which in this case, is given by the energy dissipation law. It implies that $||U^\varepsilon(t)||_H$ is bounded in each time interval where the solution exists, since $E_\varepsilon(t)$ is equivalent to $||U^\varepsilon(t)||_H$. Uniqueness is proved in the usual way using Gronwall's inequality. □

3. **Exponential decay for E_ε.** In order to simplify the notations we write in this section u for u^ε and w for w^ε. Moreover, we denote by C a constant that may change from line to line but is independent of ε.

For $\alpha \in [0,1]$, we prove the uniform exponential decay of E_ε. Using multiplier techniques and introducing a suitable perturbation F_ε of the energy, we show that the functional $\mathcal{L}_\varepsilon \equiv E_\varepsilon + \delta F_\varepsilon$ verifies for δ small enough,

$$\frac{d\mathcal{L}_\varepsilon(t)}{dt} \leq -C\, E_\varepsilon(t), \qquad \frac{1}{2}\, E_\varepsilon(t) \leq \mathcal{L}_\varepsilon(t) \leq \frac{3}{2}\, E_\varepsilon(t), \quad t > 0 \tag{18}$$

This will lead to the following result.

Theorem 3.1. *Let $\{u, w\}$ be the global weak solution of problem (1)-(6) given in Theorem 2.1. Assume that $\alpha \in [0,1]$ and that the curvature $k = k(x)$ satisfies*

$$||k||_\infty + ||k_x||_\infty \text{ is small.} \tag{19}$$

Then, there exist positive constants C and μ, independent of ε, such that

$$E_\varepsilon(t) \leq C E_\varepsilon(0) e^{-\frac{\mu}{2+\varepsilon^\alpha(E_\varepsilon(0)+||k||^2_\infty)} t}, \quad \forall\, t > 0. \tag{20}$$

In order to obtain (18), we first derive a differential inequality for the perturbation F_ε defined in the following lemma.

Lemma 3.2. *Let $\{u, w\}$ be the global weak solution of problem (1)-(6), $a, b > 0$ and F_ε be given by*

$$F_\varepsilon(t) = \int_\Omega \Big[x(\varepsilon u_x u_t + w_x w_t) + (w_{xt}(x w_x)_x) - b(w w_t + w_x w_{xt}) - a\varepsilon u u_t \Big] dx. \tag{21}$$

Under the conditions of Theorem 3.1, there exist constants C, C_1, C_2 and η, independent of ε, such that

$$\begin{aligned} \frac{dF_\varepsilon(t)}{dt} \leq& (C_1 + C\varepsilon^\alpha \eta) \int_\Omega h^2(u_x, w) dx \\ &+ \Big(2\eta C + C\varepsilon^\alpha \eta (||\, k\, ||^2_\infty + E_\varepsilon(0)) - (3/2 - b - C_2) \Big) \int_\Omega w_{xx}^2 dx \\ &- (1/2 + b) \int_\Omega w_t^2 dx + (1/2 - b) \int_\Omega w_{xt}^2 dx - (1/2 + a) \int_\Omega \varepsilon u_t^2 dx \\ &+ (\eta^{-1} + L/2) w_t^2(L,t) + (C\eta^{-1} + L) w_{xt}^2(L,t) \\ &+ \Big(C\varepsilon^\alpha (\eta^{-1} + \eta\varepsilon^\alpha) + L(\varepsilon - \varepsilon^{2\alpha})/2 \Big) u_t^2(L,t), \qquad \forall t > 0. \end{aligned} \tag{22}$$

Proof. The idea of the proof follows the main steps given in [13] which we adapt in our context. Observe first that the derivative in time of F_ε is given by

$$\begin{aligned} \frac{dF_\varepsilon(t)}{dt} =& \int_\Omega x[\varepsilon u_{xt} u_t + \varepsilon u_x u_{tt} + w_{xt} w_t + w_x w_{tt}] dx \\ &+ \int_\Omega [w_{xtt}(x w_x)_x + w_{xt}(x w_{xt})_x] dx - b \int_\Omega (w_t^2 + w w_{tt}) dx \\ &- b \int_\Omega (w_{xt}^2 + w_x w_{xtt}) dx - a \int_\Omega \varepsilon (u_t^2 + u u_{tt}) dx. \end{aligned} \tag{23}$$

Then, integrating by parts over Ω, we can replace the terms εu_{tt} and w_{tt} to obtain

$$\begin{aligned}\frac{dF_\varepsilon(t)}{dt} &= \int_\Omega h(u_x,w)_x(xu_x-au)dx\\ &+\int_\Omega\Big(-w_{xxxx}+(w_xh(u_x,w))_x-k(x)h(u_x,w)\Big)(xw_x-bw)dx\\ &+\varepsilon\int_\Omega xu_{xt}u_tdx+\int_\Omega xw_{xt}w_tdx+\int_\Omega w_{xt}(xw_{xt})_xdx\\ &-b\int_\Omega(w_t^2+w_{xt}^2)dx-a\varepsilon\int_\Omega u_t^2dx+w_{xtt}(L,t)(Lw_x(L,t)-bw(L,t)).\end{aligned}\tag{24}$$

Using the boundary condition (6), the terms in h_x in (24) are transformed respectively as follows :

$$\begin{aligned}\int_\Omega h(u_x,w)_x(xu_x-au)dx &= -\varepsilon^\alpha u_t(L,t)(Lu_x(L,t)-au(L,t))\\ &\quad-\int_\Omega h(u_x,w)(xu_{xx}+(1-a)u_x)dx\end{aligned}\tag{25}$$

and

$$\begin{aligned}\int_\Omega(w_xh(u_x,w))_x(xw_x-bw)dx =&h(u_x,w)(L,t)w_x(L,t)(Lw_x(L,t)-bw(L,t))\\ &-\int_\Omega h(u_x,w)w_x(xw_{xx}+(1-b)w_x)dx.\end{aligned}\tag{26}$$

We now estimate the integrals in h in (24) as follows:

$$\begin{aligned}&-\int_\Omega h(u_x,w)\Big(xu_{xx}+(1-a)u_x+w_x(xw_{xx}+(1-b)w_x)+k(xw_x-bw)\Big)dx\\ &\quad=-\int_\Omega h(u_x,w)x(u_{xx}+w_xw_{xx}+(kw)_x)dx\\ &\qquad\qquad-\int_\Omega h(u_x,w)\Big((1-a)u_x+2(1-b)\frac{w_x^2}{2}-bkw-xk_xw\Big)dx\\ &\quad=-\int_\Omega xh(u_x,w)h(u_x,w)_xdx-\int_\Omega h(u_x,w)\Big((1-a)u_x+2(1-b)\frac{w_x^2}{2}+ckw\Big)dx\\ &\qquad\qquad+\int_\Omega h(u_x,w)((b+c)kw+xk_xw)dx\\ &\quad=\frac{1}{2}(1-d)\int_\Omega h^2(u_x,w)dx+\int_\Omega h(u_x,w)((b+c)kw+xk_xw)dx-\frac{1}{2}Lh^2(u_x,w)(L,t)\end{aligned}\tag{27}$$

where $c>0$ and d (precised in the sequel) are such that

$$(1-a)u_x+2(1-b)\frac{w_x^2}{2}+ck(x)w=\frac{d}{2}h(u_x,w).\tag{28}$$

Then, for any $\eta_1 > 0$ (independent of ε), Young inequality leads to

$$\int_\Omega h(u_x,w)((b+c)kw + xk_x w)dx \le \frac{\eta_1}{2}((b+c)^2+L^2)\int_\Omega h^2(u_x,w)dx + \frac{1}{2\eta_1}(||k||_\infty^2 + ||k_x||_\infty^2)\int_\Omega w^2 dx. \tag{29}$$

Summarizing, from (27)-(29) and using the boundary condition, we get

$$\begin{aligned} -\int_\Omega h(u_x,w)\Big(xu_{xx} + (1-a)u_x) + w_x(xw_{xx} + (1-b)w_x) + k(xw_x - bw)\Big)dx \\ \le C_1\int_\Omega h^2(u_x,w)dx + \frac{1}{2\eta_1}(||k||_\infty^2 + ||k_x||_\infty^2)\int_\Omega w^2 dx - \frac{1}{2}Lh^2(u_x,w)(L,t) \\ \le C_1\int_\Omega h^2(u_x,w)dx + C_2\int_\Omega w_{xx}^2 dx - \frac{L}{2}\varepsilon^{2\alpha}u_t^2(L,t) \end{aligned} \tag{30}$$

with C_1 and C_2 defined by

$$C_1 = \frac{1}{2}(1-d) + \frac{\eta_1}{2}((b+c)^2+L^2) \quad \text{and} \quad C_2 = C\frac{1}{2\eta_1}(||k||_\infty^2 + ||k_x||_\infty^2). \tag{31}$$

Thus, returning to (24), we obtain

$$\begin{aligned} \frac{dF_\varepsilon(t)}{dt} \le & -\int_\Omega w_{xxxx}(xw_x - bw)dx + \varepsilon\int_\Omega xu_{xt}u_t dx + \int_\Omega xw_{xt}w_t dx + \int_\Omega w_{xt}(xw_{xt})_x dx \\ & - b\int_\Omega (w_t^2 + w_{xt}^2)dx - a\varepsilon\int_\Omega u_t^2 dx + C_1\int_\Omega h^2(u,w)dx + C_2\int_\Omega w_{xx}^2 dx \\ & + \Big(w_{xtt}(L,t) + h(u,w)(L,t)w_x(L,t)\Big)\Big(Lw_x(L,t) - bw(L,t)\Big) \\ & - \varepsilon^\alpha u_t(L,t)\Big(Lu_x(L,t) - au(L,t)\Big) - L/2\varepsilon^{2\alpha}u_t^2(L,t), \qquad \forall t > 0. \end{aligned} \tag{32}$$

Integrations by parts in the first term of the right hand side of (32) give that

$$\begin{aligned} -\int_\Omega w_{xxxx}(xw_x - bw)dx = & - w_{xxx}(L,t)(Lw_x(L,t) - bw(L,t)) \\ & + \int_\Omega w_{xxx}(xw_{xx} + (1-b)w_x)dx \\ = & - w_{xxx}(L,t)(Lw_x(L,t) - bw(L,t)) + L/2w_{xt}^2(L,t) \\ & + (1-b)(w_{xx}w_x)(L,t) - (3/2-b)\int_\Omega w_{xx}^2 dx. \end{aligned} \tag{33}$$

Analogously, the other integral terms in (32) are estimated using Young and Poincaré inequalities :

$$\begin{cases} \varepsilon\displaystyle\int_\Omega xu_{xt}u_t dx = \frac{\varepsilon}{2}Lu_t^2(L,t) - \frac{1}{2}\int_\Omega \varepsilon u_t^2 dx, \\ \displaystyle\int_\Omega xw_{xt}w_t dx = \frac{L}{2}w_t^2(L,t) - \frac{1}{2}\int_\Omega w_t^2 dx, \\ \displaystyle\int_\Omega w_{xt}(xw_{xt})_x dx = \frac{L}{2}w_{xt}^2(L,t) + \frac{1}{2}\int_\Omega w_{xt}^2 dx. \end{cases} \tag{34}$$

From (32), using (33) and (34), we then deduce the following inequality

$$\begin{aligned}\frac{dF_\varepsilon(t)}{dt} \leq & - w_t(L,t)(Lw_x(L,t) - bw(L,t)) + (1-b)(w_{xx}w_x)(L,t) \\ & - \varepsilon^\alpha u_t(L,t)(Lu_x(L,t) - au(L,t)) \\ & + C_1\int_\Omega h^2(u,w)dx - (3/2 - b - C_2)\int_\Omega w_{xx}^2 dx \\ & - (1/2+a)\int_\Omega \varepsilon u_t^2 dx - (1/2+b)\int_\Omega w_t^2 dx + (1/2-b)\int_\Omega w_{xt}^2 dx \\ & + L/2 w_t^2(L,t) + L/2 u_t^2(L,t)(\varepsilon - \varepsilon^{2\alpha}) + Lw_{xt}^2(L,t).\end{aligned} \tag{35}$$

Now, using the Sobolev embedding theorem together with Young and Poincaré inequalities, we can deduce that

$$| -w_t(L,t)(Lw_x(L,t) - bw(L,t)) | \leq \eta^{-1} w_t^2(L,t) + \eta C\int_\Omega w_{xx}^2 dx,$$

$$| (1-b)(w_{xx}w_x)(L,t) | \leq (1-b)^2\eta^{-1} \mid w_{xt}^2(L,t) \mid + \eta C\int_\Omega w_{xx}^2 dx,$$

and

$$\varepsilon^\alpha u_t(L,t)\Big(Lu_x(L,t) - au(L,t)\Big) \leq \varepsilon^\alpha C\Big(\eta^{-1}u_t(L,t)^2 + \eta u_x^2(L,t) + \eta\int_\Omega u_x^2 dx\Big), \tag{36}$$

for any $\eta > 0$ (independent of ε). It remains to estimate the terms $u_x^2(L,t)$ and $\int_\Omega u_x^2 dx$ in (36) which are not part of dE_ε/dt and E_ε, respectively. In this respect, we observe that

$$\begin{aligned}u_x^2(L,t) =& (h(u_x,w)(L,t) - \frac{1}{2}w_x^2(L,t) - kw(L,t))^2 \\ & \leq C(h^2(u_x,w)(L,t) + w_x^4(L,t) + k^2w^2(L,t))\end{aligned}$$

leading to

$$u_x^2(L,t) \leq C\Big(\varepsilon^{2\alpha}u_t^2(L,t) + (\| k \|_\infty^2 + E_\varepsilon(0))\int_\Omega w_{xx}^2 dx\Big).$$

Analogously, we get

$$\int_\Omega u_x^2(x,t)dx \leq C\Big(\int_\Omega h^2(u_x,w)dx + (\| k \|_\infty^2 + E_\varepsilon(0))\int_\Omega w_{xx}^2 dx\Big).$$

Thus, the above estimates and (35) allow finally to obtain the desired inequality. □

Assuming the curvature and its derivative small enough, one may now choose the constants a, b, c, d and η, η_1 in order to bound the derivative of F_ε in term of $E_\varepsilon(t)$ and $dE_\varepsilon(t)/dt$.

Lemma 3.3. *Under the conditions of Theorem 3.1, there exists a constant $\beta > 0$, independent of ε, such that*

$$\begin{aligned}\frac{dF_\varepsilon(t)}{dt} \leq & - \beta E_\varepsilon(t) + (\eta^{-1} + L/2)w_t^2(L,t) + (C\eta^{-1} + L)w_{xt}^2(L,t) \\ & + \Big(C\varepsilon^\alpha(\eta^{-1} + \eta\varepsilon^\alpha) + L(\varepsilon - \varepsilon^{2\alpha})/2\Big)u_t^2(L,t)\end{aligned} \tag{37}$$

for any $\eta > 0$ sufficiently small.

Proof. From (22), we need to enforce that

$$\begin{cases} 1/2 - b < 0, \quad 1/2 + b > 0, \quad 1/2 + a > 0, \\ C_1 + C\varepsilon^\alpha \eta < 0, \quad \eta C\Big(2 + \varepsilon^\alpha(||k||_\infty^2 + E_\varepsilon(0))\Big) < (3/2 - b - C_2), \end{cases} \tag{38}$$

$C_1 = C_1(b, c, d, \eta_1)$ and $C_2 = C_2(\eta_1, ||k||_\infty^2 + ||k_x||_\infty^2)$ being defined by (31). We first observe that the conditions $(38)_1$ and (28) hold if we take $a = 1/4$, $b = 5/8$, $c = 3/4$ and $d = 3/2$. Let us then choose

$$\eta_1 = \frac{1}{2}\frac{d-1}{(b+c)^2 + L^2} = \frac{1}{4}\frac{8^2}{11^2 + 8^2L^2},$$

so that $C_1 = -1/8 < 0$. The second condition in $(38)_2$ is then equivalent to

$$C_2 \leq \frac{7}{8} - \eta C\Big(2 + \varepsilon^\alpha(||\ k\ ||_\infty^2 + E_\varepsilon(0))\Big). \tag{39}$$

If we now introduce $\lambda \in (0,1)$ (small enough but independent of ε and $E_\varepsilon(0)$) and fixe the function η as follows

$$\eta = \frac{7}{8C}\frac{\lambda}{2 + \varepsilon^\alpha(||\ k\ ||_\infty^2 + E_\varepsilon(0))} \tag{40}$$

condition (39) becomes $C_2 < 7/8(1-\lambda)$ and requires a geometrical restriction on the curvature k and its derivative

$$C(||\ k\ ||_\infty^2 + ||\ k_x\ ||_\infty^2) \leq \frac{7}{4}\eta_1(1-\lambda) < \frac{7}{4}\eta_1 \tag{41}$$

which was assumed in Theorem 3.1. Finally, if $\alpha > 0$, the first condition $C_1 + C\varepsilon^\alpha\eta = -1/8 + C\varepsilon^\alpha\eta < 0$ always holds if ε (devoted to go to zero in the sequel) is small enough. If $\alpha = 0$, it suffices to take λ small enough. Consequently, for all $t > 0$, we have

$$\begin{aligned} \frac{dF_\varepsilon(t)}{dt} \leq & (-1/8 + C\varepsilon^\alpha\eta)\int_\Omega h^2(u_x, w)dx - (7(1-\lambda)/8 - C_2)\int_\Omega w_{xx}^2 dx \\ & - 9/8\int_\Omega w_t^2 dx - 1/8\int_\Omega w_{xt}^2 dx - 3/4\int_\Omega \varepsilon u_t^2 dx \\ & + (\eta^{-1} + L/2)w_t^2(L,t) + (C\eta^{-1} + L)w_{xt}^2(L,t) \\ & + \Big(C\varepsilon^\alpha(\eta^{-1} + \eta\varepsilon^\alpha) + L(\varepsilon - \varepsilon^{2\alpha})/2\Big)u_t^2(L,t) \\ \leq & -\beta E_\varepsilon(t) + (\eta^{-1} + L/2)w_t^2(L,t) + (C\eta^{-1} + L)w_{xt}^2(L,t) \\ & + \Big(C\varepsilon^\alpha(\eta^{-1} + \eta\varepsilon^\alpha) + L(\varepsilon - \varepsilon^{2\alpha})/2\Big)u_t^2(L,t) \end{aligned} \tag{42}$$

with

$$\beta = 2\min\Big(1/8 - C\varepsilon^\alpha\eta, 7(1-\lambda)/8 - C_2\Big). \tag{43}$$

The term $7(1-\lambda)/8 - C_2$ is independent of ε and strictly positive thanks to the geometrical assumption on the curvature. The term $1/8 - C\varepsilon^\alpha\eta$ is also bounded by below uniformly with respect to ε, writing that

$$\frac{1}{8} - C\varepsilon^\alpha\eta = \frac{1}{8} - \frac{7\lambda}{8}\frac{\varepsilon^\alpha}{2 + \varepsilon^\alpha(||k||_\infty^2 + E_\varepsilon(0))} > \frac{1}{8}(1 - 7\lambda) > 0, \tag{44}$$

for all $\varepsilon < 1$ and λ small enough. Consequently, Lemma 3.3 is proved. □

We are now in position to prove Theorem 3.1:

Proof. We introduce $\delta > 0$ and set

$$\mathcal{L}_\varepsilon(t) = E_\varepsilon(t) + \delta F_\varepsilon(t), \quad t \geq 0. \tag{45}$$

Combining (8) and Lemma 3.3, we obtain for all $t > 0$

$$\begin{aligned}\frac{d\mathcal{L}_\varepsilon(t)}{dt} \leq & -\delta\beta E_\varepsilon(t) \\ & - \Big(\varepsilon^\alpha - \delta\Big(C\varepsilon^\alpha(\eta^{-1} + \eta\varepsilon^\alpha) + L(\varepsilon - \varepsilon^{2\alpha})/2\Big)\Big) u_t^2(L,t) \\ & - \Big(1 - \delta\Big(\eta^{-1} + L/2\Big)\Big) w_t^2(L,t) - \Big(1 - \delta\Big(C\eta^{-1} + L\Big)\Big) w_{xt}^2(L,t).\end{aligned} \tag{46}$$

Now, we choose $\delta > 0$ satisfying

$$\begin{cases} 1 - \delta\Big(C\eta^{-1} + L\Big) \geq 0, \quad 1 - \delta\Big(\eta^{-1} + L/2\Big) \geq 0, \\ 1 - \delta\Big(C(\eta^{-1} + \eta\varepsilon^\alpha) + (\varepsilon^{1-\alpha} - \varepsilon^\alpha)L/2\Big) \geq 0 \end{cases} \tag{47}$$

in order to obtain

$$\frac{d\mathcal{L}_\varepsilon(t)}{dt} \leq -\delta\beta E_\varepsilon(t). \tag{48}$$

In view of (40), (47) will be satisfied if we choose $\delta > 0$ of the form

$$\delta = \frac{C}{2 + \varepsilon^\alpha(\| k \|_\infty^2 + E_\varepsilon(0))}. \tag{49}$$

On the other hand, usual arguments permit to compare F_ε and E_ε (see for instance [16]) in the following manner

$$| F_\varepsilon(t) | \leq C[E_\varepsilon(t) + \varepsilon E_\varepsilon(t)^2] \leq C[1 + \varepsilon E_\varepsilon(0)]E_\varepsilon(t) \tag{50}$$

leading to

$$\underbrace{\Big(1 - \delta C[1 + \varepsilon E_\varepsilon(0)]\Big)}_{c_1} E_\varepsilon(t) \leq \mathcal{L}_\varepsilon(t) \leq \underbrace{\Big(1 + \delta C[1 + \varepsilon E_\varepsilon(0)]\Big)}_{c_2} E_\varepsilon(t) \tag{51}$$

and then to

$$E_\varepsilon(t) \leq \frac{c_2}{c_1} E_\varepsilon(0) e^{-\frac{\beta}{c_1}\delta t}, \quad \forall t > 0. \tag{52}$$

Finally, we take δ small enough and independent of ε so that $\delta C(1 + \varepsilon E_\varepsilon(0)) < 1/2$ (we recall that $E_\varepsilon(0)$ is bounded in ε); this implies that $c_1 > 1/2$, $c_2 < 3/2$ and finally the uniform estimation (20) of Theorem 3.1. □

4. **Singular limit as $\varepsilon \to 0$.** In this section, we analyze the limit of the solution $(u^\varepsilon, w^\varepsilon)$ of (1)-(6) as $\varepsilon \to 0$.

The first step is devoted to obtain some uniform bounds in ε. According to (8) and the fact that the initial energy is bounded by a constant independent of ε, the following sequences remain bounded in $L^\infty(0, \infty; L^2(\Omega))$:

$$\{\sqrt{\varepsilon} u_t^\varepsilon\}, \ \{h(u_x^\varepsilon, w^\varepsilon)\}, \ \{w_t^\varepsilon\}, \{w_{xt}^\varepsilon\}, \ \{w_{xx}^\varepsilon\}. \tag{53}$$

The boundedness of the sequences $\{u^\varepsilon\}$ and $\{w^\varepsilon\}$ implies that we can extract subsequences, (still denoted by the same index ε) and functions ζ, η and w such that

$$\sqrt{\varepsilon} u_t^\varepsilon \rightharpoonup \zeta \quad \text{weakly} - \star \quad \text{in} \quad L^\infty(0,\infty; L^2(\Omega)), \tag{54}$$

$$h(u_x^\varepsilon, w^\varepsilon) \rightharpoonup \eta \quad \text{weakly} - \star \quad \text{in} \quad L^\infty(0,\infty; L^2(\Omega)), \tag{55}$$

$$w^\varepsilon \rightharpoonup w \quad \text{weakly} - \star \quad \text{in} \quad L^\infty(0,\infty; W) \cap W^{1,\infty}(0,\infty; V) \tag{56}$$

as $\varepsilon \to 0$. Moreover, we can use the Aubin-Lions compactness criteria (see e.g. [20]) to deduce that

$$w^\varepsilon \to w \quad \text{strongly} \quad \text{in} \quad L^\infty(0,T; H^{2-\delta}(\Omega)) \tag{57}$$

for any $\delta > 0$ and $T < \infty$. Then, it follows from (55) and (57) that

$$w_x^\varepsilon h(u_x^\varepsilon, w^\varepsilon) \rightharpoonup w_x \eta \quad \text{weakly} \quad \text{in} \quad L^2(\Omega \times (0,T)) \tag{58}$$

for any $T < \infty$.

The above weak convergences suffice to pass the limit in the linear terms of (1). The difficulty is to identify the weak limit of the nonlinear term $(w_x^\varepsilon h(u_x^\varepsilon, w^\varepsilon))_x$. From (53) we can deduce that $\{u_x^\varepsilon\}$ is bounded in $L^2(\Omega \times (0,T))$ and, therefore, we can obtain a subsequence such that

$$u_x^\varepsilon \rightharpoonup \rho \quad \text{weakly} \quad \text{in} \quad L^2(\Omega \times (0,T)) \tag{59}$$

for some $\rho = \rho(x,t)$. Then, combining (57) and (59), we deduce that

$$h(u_x^\varepsilon, w^\varepsilon) \rightharpoonup h(\rho, w) \quad \text{weakly} \quad \text{in} \quad L^2(\Omega \times (0,T)) \tag{60}$$

which, together with (54), implies that

$$\eta = h(\rho, w). \tag{61}$$

Now, we claim that η is independent of x. In fact, due to (54), we have that

$$\varepsilon u_{tt}^\varepsilon \rightharpoonup 0 \quad \text{weakly} \quad \text{in} \quad H^{-1}(0,T; L^2(\Omega)) \tag{62}$$

and from the first equation in (1), (61) and (62) it follows that

$$\eta_x = h(\rho, w)_x = 0$$

which implies that the function η is independent of $x : \eta(x,t) = \eta(t)$.

In addition, we can also prove that the weak limit w takes the initial data w^0 and w^1: due to (57) we know that $w^\varepsilon \to w$ in $C([0,T]; L^2(\Omega))$. Then, $w^\varepsilon(x,0) = w^0(x) \to w(x,0)$ in $L^2(\Omega)$. Consequently, $w(x,0) = w^0(x)$. To prove that $w_t^\varepsilon(x,0) = w^1(x)$, we proceed in a similar way obtaining a bound for $\{w_{tt}^\varepsilon\}$ in $L^2(0,T; L^2(\Omega))$ as follows:

$$w_{tt}^\varepsilon = -\left(I - \frac{d^2}{dt^2}\right)^{-1} \left(w_{xxxx}^\varepsilon - (w_x^\varepsilon h(u_x^\varepsilon, w^\varepsilon))_x + k(x) h(u_x^\varepsilon, w^\varepsilon) \right). \tag{63}$$

The next steps are devoted to identify the function η analyzing its relation with α and the boundary conditions.

4.1. **The case** $\alpha \in (0,1]$**.** Multiplying the first equation in (1) by $a(x) \equiv x/L$ and integrating over Ω, the following holds

$$\varepsilon \frac{d^2}{dt^2} \int_0^L u^\varepsilon a(x)dx = \int_\Omega h(u_x^\varepsilon, w^\varepsilon)a(x)dx = -\varepsilon^\alpha u_t^\varepsilon(L,t) - \frac{1}{L}\int_\Omega h(u_x^\varepsilon, w^\varepsilon)dx. \quad (64)$$

As $\varepsilon \to 0$, the left-hand side tends to zero in $\mathcal{D}'(0,T)$. On the other hand, the right-hand side converges to $-\eta(t)$ because the boundary term tends to zero. Indeed, thanks to the energy dissipation, we have

$$\varepsilon^\alpha \int_0^T \mid u_t^\varepsilon \mid^2 dt \leq E_\varepsilon(0), \quad \forall T > 0 \quad (65)$$

and therefore

$$\varepsilon^{\alpha/2} u_t^\varepsilon(L,t) \text{ is bounded in } L^2(0,T). \quad (66)$$

Consequently,

$$\varepsilon^\alpha u_t^\varepsilon(L,t) \to 0 \text{ in } \mathcal{D}'(0,T) \text{ as } \varepsilon \to 0, \quad (67)$$

since $\alpha \in (0,1]$. Thus, combining (61), (64) and (67), we deduce that $\eta = 0$. Summarizing, we have proved the following result

Theorem 4.1. *Let* $(u^0, u^1, w^0, w^1) \in H$, $\alpha > 0$ *and* $k \in H^1(\Omega)$. *Consider the global solution* $(u^\varepsilon, w^\varepsilon)$ *of system (1)-(6) obtained in Theorem 2.1. Then, as* $\varepsilon \to 0$,

$$w^\varepsilon \rightharpoonup w \quad \textit{weakly} \quad \textit{in} \quad L^\infty([0,\infty; W) \cap W^{1,\infty}([0,\infty); V),$$

where w *is the weak solution of*

$$\begin{cases} w_{tt} - w_{xxtt} + w_{xxxx} = 0 \qquad x \in \Omega, \quad t > 0 \\ w(0,t) = w_x(0,t) = 0, \quad t > 0 \\ (w_{xxx} - w_{xtt})(L,t) = w_t(L,t), \quad w_{xx}(L,t) = -w_{xt}(L,t), \quad t > 0 \\ (w(x,0), w_t(x,0)) = (w^0, w^1), \quad x \in \Omega. \end{cases} \quad (68)$$

Remark 1. We observe that the curvature $k = k(x)$ does not appear anymore in the limit system and therefore, (68) is the system obtained in [16] where the 1-D version of the nonlinear von Kármán system is considered. We also remark that this property does not hold in the internal stabilization case studied in [15] where the curvature remains for $\alpha > 0$. Moreover, as discussed in [16], system (68) has a unique global weak solution in $C([0,\infty); W) \cap C^1([0,\infty); V)$. As last, the total energy associated to (68) is given by

$$E(t) = \frac{1}{2}\int_\Omega (w_t^2 + w_{xt}^2 + w_{xx}^2)dx \quad (69)$$

and obeys the energy dissipation law

$$\frac{dE(t)}{dt} = -w_t^2(L,t) - w_{xt}^2(L,t). \quad (70)$$

It is well-known that $E(t)$ tends exponentially uniformly to zero as t goes to infinity (we refer for instance to [12]). Indeed, according to Theorem 3.1, this property may be recovered by the limiting process, writing that

$$E(t) \leq \frac{1}{2}\int_\Omega \zeta^2 dx + E(t) \leq \text{liminf}_{\varepsilon\to 0} E_\varepsilon(t) \leq CE(0)e^{-\frac{\mu}{2}t}, \quad \forall t > 0. \quad (71)$$

4.2. **The case $\alpha = 0$.** As previously, the difficulty is to identify the weak limit of the nonlinear term $w_x^\varepsilon h(u^\varepsilon, w^\varepsilon)$. Once again, multiplying the first equation in (1) by $a(x) = x/L$, and integrating by parts, we obtain

$$\begin{aligned}\varepsilon\frac{d^2}{dt^2}\int_\Omega u^\varepsilon a(x)dx &= \int_\Omega h(u^\varepsilon, w^\varepsilon)_x a(x)dx\\ &= -u_t^\varepsilon(L,t) - \frac{1}{L}\int_\Omega h(u^\varepsilon, w^\varepsilon)dx\\ &= -u_t^\varepsilon(L,t) - \frac{1}{L}u^\varepsilon(L,t) - \frac{1}{2L}\int_0^L (w_x^\varepsilon)^2 dx - \frac{1}{L}\int_\Omega k(x)w^\varepsilon dx.\end{aligned} \tag{72}$$

According to the energy dissipation (see also the previous subsection), the sequence

$$\{u^\varepsilon(L,t)\} \text{ is bounded in } H^1(0,T),\ \forall\, T > 0.$$

Therefore, we can extract a subsequence of $\{u^\varepsilon(L,t)\}$, such that

$$u^\varepsilon(L,t) \rightharpoonup \zeta = \zeta(t) \text{ weakly in } H^1(0,T) \tag{73}$$

and

$$u^\varepsilon(L,t) \to \zeta = \zeta(t) \text{ in } C([0,T]), \tag{74}$$

for all $T > 0$, as $\varepsilon \to 0$. Now, passing (72) to the limit and taking (56), (62) and (73) into account, we conclude that ζ satisfies the first order ordinary differential equation

$$\zeta_t + \frac{1}{L}\left[\zeta + \int_\Omega \left(\frac{w_x^2}{2} + k(x)w\right)dx\right] = 0, \quad \forall t > 0. \tag{75}$$

Moreover, due to (74), we have

$$\zeta(0) = u^0(L). \tag{76}$$

It remains to identify $\eta = \eta(t)$. Integrating the relation (61) over $\Omega = (0,L)$ and writing that

$$\begin{aligned}\int_0^L \rho(x,t)dx =& lim_{\varepsilon\to 0}\int_0^L h(u_x^\varepsilon, w^\varepsilon)dx\\ =& lim_{\varepsilon\to 0}(u^\varepsilon(L,t) - u^\varepsilon(0,t)) + \int_\Omega\left(\frac{w_x^2}{2} + k(x)w\right)dx\\ =& \zeta(t) + \int_\Omega\left(\frac{w_x^2}{2} + k(x)w\right)dx,\end{aligned} \tag{77}$$

we get

$$L\eta(t) = \zeta(t) + \int_\Omega\left(\frac{w_x^2}{2} + k(x)w\right)dx. \tag{78}$$

Summarizing, we have the following result :

Theorem 4.2. *Let $(u^0, u^1, w^0, w^1) \in H$, $\alpha = 0$ and $k \in H^1(\Omega)$. Consider the global solution $(u^\varepsilon, w^\varepsilon)$ of system (1)-(6) obtained in Theorem 2.1. Then, as $\varepsilon \to 0^+$, the solution w^ε converges to w solution of*

$$\begin{cases}\zeta_t + \frac{1}{L}\left[\zeta + \int_\Omega\left(\frac{w_x^2}{2} + k(x)w\right)dx\right] = 0, \quad t > 0,\\ w_{tt} + w_{xxxx} - w_{xxtt} - \frac{1}{L}\left[\zeta + \int_\Omega\left(\frac{w_x^2}{2} + k(x)w\right)dx\right](w_{xx} - k(x)) = 0, \quad in \quad \Omega\times(0,T),\end{cases} \tag{79}$$

with boundary conditions

$$\begin{cases} w(0,t) = w_x(0,t) = 0, \quad w_{xx}(L,t) = -w_{xt}(L,t), \quad t > 0, \\ \left[w_{xxx} - w_{xtt} - \dfrac{1}{L}\left[\zeta(t) + \displaystyle\int_\Omega \left(\frac{1}{2} w_x^2 + k(x) w \right) \right] w_x \right](L,t) = w_t(L,t), \end{cases} \tag{80}$$

and initial conditions

$$\zeta(0) = u^0(L), \quad w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x), \quad x \in \Omega. \tag{81}$$

Remark 2. Contrary to the previous case, the curvature and the nonlinearity still occur in the limit system (79)-(81). On the other hand, integrating the first equation in (79), we can deduce that

$$\zeta(t) = u^0(L)\, e^{-\frac{t}{L}} - \frac{1}{L} \int_0^t e^{\frac{(s-t)}{L}} \int_\Omega \left(\frac{w_x^2}{2} + k(x) w \right) dx ds.$$

Substituting the above expression in the second equation of (79) we conclude that $w = w(x,t)$ satisfies

$$w_{tt} + w_{xxxx} - w_{xxtt} - M(t, w, w_x)(w_{xx} - k(x)) = 0$$

where

$$M(t,w,w_x) = \frac{1}{L}\left[u^0(L)\, e^{-\frac{t}{L}} - \frac{1}{L}\int_0^t e^{\frac{(s-t)}{L}} \int_\Omega \left(\frac{w_x^2}{2} + k(x)w \right) dxds + \int_\Omega \left(\frac{w_x^2}{2} + k(x) w \right) dx \right].$$

Thus, the limit system (79)-(81) can be viewed as a linear arch model "perturbed" by the nonlinear term $M(t,w,w_x)(w_{xx} - k(x))$. Performing as [13] (see also Theorem 2.1), we can prove the existence and uniqueness of global weak solution. Moreover, the energy of the limit system is given by

$$E(t) = \frac{1}{2} \int_\Omega (w_t^2 + w_{xt}^2 + w_{xx}^2) dx + \frac{1}{2L} \left[\zeta + \int_\Omega \left(\frac{1}{2} w_x^2 + k(x) w \right) dx \right]^2, \tag{82}$$

which is the natural limit of the ε-system energy. From the following relation

$$\frac{dE(t)}{dt} = -w_t^2(L,t) - w_{xt}^2(L,t) - \zeta_t^2(L), \tag{83}$$

we observe that the limit system is dissipative and a rate of decay is expected. Indeed, once again the exponential decay in time for E defined by (82) is a consequence of the uniform (w.r.t. to ε) exponential decay of E_ε, obtained in the previous section (see also Remark 1). As far as we know, the proof of the exponential decay by direct methods, i.e., without introducing a limiting process in ε, remains open.

4.3. **Cases $\alpha < 0$ and $\alpha > 1$.** The analysis developed here allows us to address partially the remaining cases $\alpha > 1$ and $\alpha < 0$. Following the proof of Theorem 3.1, it is possible to obtain that E_ε decays exponentially in both cases, but any information on how the decay rate depends on ε is provided. The numerical experiments detailed in Section 5 will suggest that in these cases, the decay rate is not bounded by below uniformly with respect to ε.

Concerning the limit as $\varepsilon \to 0$, we have two situations :

- for $\alpha > 1$, we may proceed as in the case $0 < \alpha \leq 1$ to prove that the limit model is the linear model (68) of Theorem 4.1 (case $\alpha \in (0,1]$).

- In the case $\alpha < 0$, we multiply the first equation in (1) by $a(x) = x/L$ to get
$$\varepsilon^{-\alpha/2+1}\frac{d^2}{dt^2}\int_\Omega u^\varepsilon a(x)dx = -\varepsilon^{\alpha/2}u_t^\varepsilon(L,t) - \frac{\varepsilon^{-\alpha/2}}{L}\int_\Omega h(u_x^\varepsilon, w^\varepsilon)dx. \tag{84}$$
According to the dissipation law, we know that $\{\varepsilon^{\alpha/2}u_t^\varepsilon(L,t)\}$ is bounded in $L^2(0,T)$. Therefore, we can extract a subsequence such that $\varepsilon^{\alpha/2}u_t^\varepsilon(L,t) \rightharpoonup \zeta$ in $L^2(0,T)$. Then, passing to the limit in (84) and taking the previous convergence into account, we obtain $\zeta(t) = 0$, for all $t \geq 0$. Once again, it remains to identify the function $\eta = \eta(t)$. Due to (59) and the Sobolev embedding, the sequence $\{u^\varepsilon(L,t)\}$ is bounded in $L^2(0,T)$
$$\int_0^T \mid u^\varepsilon(L,t)\mid^2 dt \leq C\int_0^T \mid\mid u^\varepsilon \mid\mid_V^2 dt \leq C. \tag{85}$$
Consequently, we extract a subsequence satisfying $u^\varepsilon(L,t) \rightharpoonup \beta$ in $L^2(0,T)$. Now, integrating (61) and writing that
$$L\eta(t) = \int_\Omega \rho(x,t)dx = lim_{\varepsilon\to 0}\int_\Omega h(u_x^\varepsilon, w^\varepsilon)dx = \beta(t) + \int_\Omega\left(\frac{1}{2}w_x^2 + k(x)w\right)dx \tag{86}$$
we express the function η in term of β and w. Summarizing, as $\varepsilon \to 0$, the limit w of w^ε is solution the nonlinear system (similarly to the case $\alpha = 0$)
$$\begin{cases} w_{tt} + w_{xxxx} - w_{xxtt} - \dfrac{1}{L}\left[\beta(t) + \displaystyle\int_\Omega\left(\frac{w_x^2}{2} + k(x)w\right)dx\right](w_{xx} - k(x)) = 0, \quad \Omega\times(0,T), \\ w(0,t) = w_x(0,t) = 0, w_{xx}(L,t) = -w_{xt}(L,t), \quad t > 0 \\ \left[w_{xxx} - w_{xtt} - \dfrac{1}{L}\left[\zeta(t) + \displaystyle\int_\Omega\left(\frac{1}{2}w_x^2 + k(x)w\right)\right]w_x\right](L,t) = w_t(L,t), \quad t > 0 \\ w(x,0) = w^0(x), w_t(x,0) = w^1(x), \quad x \in \Omega \end{cases} \tag{87}$$
where β is a non explicit function of $L^2(0,T)$.

In the next section, we study numerically the behavior of $(u^\varepsilon, w^\varepsilon)$ and the corresponding decay of energy for the various cases w.r.t. α discussed above.

5. **Numerical experiments.** In this section, we check numerically the asymptotic results as ε goes to zero for several values of α and $k(x)$. For simplicity, we consider the case of a cylindrical arch for which the curvature is constant : $k(x) = k > 0$ for all $x \in \Omega$.

The initial system (1) and the limit ones (68) and (79) are solved in space using a $C^0 - C^1$-finite element method with mass lumping and Newmark scheme (we refer to [6, 18]). Precisely, introducing a triangulation $\mathcal{T}_h$ of Ω ($h = \max_{T\in\mathcal{T}_h}|T|$), we approximate $L^2(\Omega)$ and $H^1(\Omega)$ by the finite-dimensional space $V_h = \{u_h | u_h \in C^0(\overline{\Omega}), u_{h|T} \in \mathbb{P}_1, \forall T \in \mathcal{T}_h\}$ and $H^2(\Omega)$ by the finite-dimensional space $W_h = \{w_h | w_h \in C^1(\overline{\Omega}), w_{h|T} \in \mathbb{P}_3, \forall T \in \mathcal{T}_h\}$. $\mathbb{P}_k$, $k \in \mathbb{N}$, designates the space of the polynomials of degree $\leq k$. The time discretization is performed in a standard way using implicit centered finite difference schemes of order two. Moreover, due to the term ε in front of u_{tt}^ε, we take a small ratio $dt/h = 1/100$ between the time and space parameter, in order to capture precisely the variation of u^ε. Finally, the ordinary differential equation (75) in ζ is solved using the implicit Euler scheme.

We consider the following initial condition $(u^0, u^1, w^0, w^1) \in H$
$$(u^0(x), u^1(x), w^0(x), w^1(x)) = (\sin(\pi x), 0, \sin^2(\pi x), 0), \quad x \in \Omega \tag{88}$$

and take $T=2$.

5.1. **Case** $\alpha \in (0,1]$**.** We first comment the results we observe in the case $\alpha=1$. Tables 1 and 2 collect the numerical results for $k=1/5$ and $k=4$ respectively. In agreement with the theoretical part, we first check the strong convergence of w^ε in $L^2(\Omega)$ and $H^1(\Omega)$ toward w, solution of the limit system (68). For $k=1/5$, we obtain

$$\frac{\| w^\varepsilon - w \|_{L^2(\Omega\times(0,T))}}{\| w \|_{L^2(\Omega\times(0,T))}} \approx e^{-2.98}\varepsilon^{0.76}, \quad \frac{\| w_x^\varepsilon - w_x \|_{L^2(\Omega\times(0,T))}}{\| w_x \|_{L^2(\Omega\times(0,T))}} \approx e^{-4.16}\varepsilon^{0.55} \tag{89}$$

whereas for $k=4$, we observe

$$\frac{\| w^\varepsilon - w \|_{L^2(\Omega\times(0,T))}}{\| w \|_{L^2(\Omega\times(0,T))}} \approx e^{-0.28}\varepsilon^{0.51}, \quad \frac{\| w_x^\varepsilon - w_x \|_{L^2(\Omega\times(0,T))}}{\| w_x \|_{L^2(\Omega\times(0,T))}} \approx e^{-0.93}\varepsilon^{0.38} \tag{90}$$

highlighting the influence of the curvature on the rate of convergence. Similarly, we verify the weak convergence of $h(u_x^\varepsilon, w^\varepsilon)-\eta$ as ε goes to zero. Furthermore, we observe the exponential decay of the energies (7) and (69) uniformly with respect to ε. We remark however that the energy E_ε does not converge toward E (for instance in the $L^2(0,T)$-norm), because the part of the energy $E_{\varepsilon,u^\varepsilon} \equiv 1/2\int_0^T\int_\Omega \varepsilon(u_t^\varepsilon)^2dx$, which is bounded uniformly with respect to ε, does not converge toward zero. We obtain that only the part $E_{\varepsilon,w^\varepsilon}$ of E_ε defined by

$$E_{\varepsilon,w^\varepsilon}(t) \equiv \frac{1}{2}\int_0^L \Big\{ (w_t^\varepsilon)^2 + (w_{xt}^\varepsilon)^2 + (w_{xx}^\varepsilon)^2 + h(u_x^\varepsilon, w^\varepsilon)^2 \Big\} dx \tag{91}$$

converges toward the limit energy E : for $k=1/5$, we observe that

$$\frac{\| E_{\varepsilon,w^\varepsilon} - E \|_{L^2(0,T)}}{\| E \|_{L^2(0,T)}} \approx e^{-6.21}\varepsilon^{0.203}. \tag{92}$$

As a consequence, we observe that only the exponential decay rate associated to $E_{\varepsilon,w^\varepsilon}$ converges to the exponential decay associated to E, approximatively equal to -4.037. Of course, this is not in contradiction with the uniform exponential decay in time of E_ε.

	$\varepsilon=10^{-1}$	$\varepsilon=10^{-2}$	$\varepsilon=10^{-3}$
$\frac{\|\|w^\varepsilon-w\|\|_{L^2(\Omega\times(0,T))}}{\|\|w\|\|_{L^2(\Omega\times(0,T))}}$	9.33×10^{-3}	1.31×10^{-3}	2.76×10^{-4}
$\frac{\|\|w_x^\varepsilon-w_x\|\|_{L^2(\Omega\times(0,T))}}{\|\|w_x\|\|_{L^2(\Omega\times(0,T))}}$	4.76×10^{-3}	1.05×10^{-3}	3.77×10^{-4}
$\mid\int_0^T\int_\Omega(h(u_x^\varepsilon,w^\varepsilon)-\eta)dxdt\mid$	6.55×10^{-3}	6.56×10^{-4}	1.86×10^{-3}
$\|\| u_t^\varepsilon(L,\cdot) \|\|_{L^2(0,T)}$	4.762	15.392	49.66
$\|\| \varepsilon^{\alpha/2}u_t^\varepsilon(L,\cdot) \|\|_{L^2(0,T)}$	1.505	1.539	1.570
$\frac{\|\|E_\varepsilon-E\|\|_{L^2(0,T)}}{\|\|E\|\|_{L^2(0,T)}}$	3.72×10^{-2}	3.43×10^{-2}	3.43×10^{-2}
$\frac{\|\|E_{\varepsilon,w}-E\|\|_{L^2(0,T)}}{\|\|E\|\|_{L^2(0,T)}}$	1.31×10^{-3}	7.23×10^{-4}	5.13×10^{-4}
$\frac{1}{2}\int_0^T\int_\Omega\varepsilon(u_t^\varepsilon)^2dtdx$	1.255	1.1822	1.2225
Decay rate for E_ε	-3.6691	-3.5705	-3.6469
Decay rate for $E_{\varepsilon,w^\varepsilon}$	-4.0304	-4.0368	-4.0367
Decay rate for $E_{\varepsilon,u^\varepsilon}$	-2.1100	-2.0261	-2.0181

TABLE 1. Estimations in the case $\boldsymbol{\alpha=1}$ and $k=1/5$ - $dt=h/100$.

	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$
$\frac{\|\|w^\varepsilon - w\|\|_{L^2(\Omega\times(0,T))}}{\|\|w\|\|_{L^2(\Omega\times(0,T))}}$	2.51×10^{-1}	5.98×10^{-2}	2.38×10^{-2}
$\frac{\|\|w_x^\varepsilon - w_x\|\|_{L^2(\Omega\times(0,T))}}{\|\|w_x\|\|_{L^2(\Omega\times(0,T))}}$	1.81×10^{-1}	5.40×10^{-2}	3.08×10^{-2}
$\mid \int_0^T\int_\Omega (h(u_x^\varepsilon, w^\varepsilon) - \eta)dxdt \mid$	4.69×10^{-3}	1.81×10^{-2}	7.77×10^{-2}
$\|\| u_t^\varepsilon(L,\cdot) \|\|_{L^2(0,T)}$	5.861	17.768	88.89
$\|\| \varepsilon^{\alpha/2} u_t^\varepsilon(L,\cdot) \|\|_{L^2(0,T)}$	1.8533	1.7768	2.811
$\frac{\|\|E_\varepsilon - E\|\|_{L^2(0,T)}}{\|\|E\|\|_{L^2(0,T)}}$	9.65×10^{-2}	7.20×10^{-2}	8.60×10^{-2}
$\frac{\|\|E_{\varepsilon,w} - E\|\|_{L^2(0,T)}}{\|\|E\|\|_{L^2(0,T)}}$	4.23×10^{-2}	3.98×10^{-2}	3.20×10^{-2}
$\frac{1}{2}\int_0^T\int_\Omega \varepsilon(u_t^\varepsilon)^2 dtdx$	3.157	1.735	4.421
Decay rate for E_ε	-3.33	-3.64	-3.22
Decay rate for $E_{\varepsilon,w^\varepsilon}$	-3.65	-3.98	-4.01

TABLE 2. Estimations in the case $\boldsymbol{\alpha = 1}$ and $k = 4$ - $dt = h/100$.

On the other hand, for $\alpha \in (0,1)$ the term $\sqrt{\varepsilon} u_t^\varepsilon(L,t)$ converges toward zero in $L^2(0,T)$ (writing that $\sqrt{\varepsilon} u_t^\varepsilon = \varepsilon^{(1-\alpha)/2}(\varepsilon^{\alpha/2} u_t^\varepsilon)$ and using (64)). Numerical results are collected in Tables 3 and 4 for $\alpha = 1/2$. We have

$$|| \varepsilon^{1/2} u_t^\varepsilon(L,t) ||_{L^2(0,T)} \approx e^{0.40}\varepsilon^{0.2471} \approx e^{0.40}\varepsilon^{\frac{1}{2}(1-\alpha)} \tag{93}$$

and we observe now that the full energy E_ε defined by (7) converges to the limit energy defined by (69)

$$\frac{|| E_\varepsilon - E ||_{L^2(0,T)}}{|| E ||_{L^2(0,T)}} \approx e^{-2.89}\varepsilon^{0.2494} \approx e^{-2.89}\varepsilon^{\frac{1}{2}(1-\alpha)}. \tag{94}$$

This highlights the influence of the boundary term on the convergence toward zero (w.r.t. ε) of the term $E_{\varepsilon,u^\varepsilon}$. In this case, the exponential decay rate associated to the full energy E_ε converges toward the exponential decay rate value -4.037. We remark that this value is the same for $k = 1/5$ and $k = 4$ which confirms the independence of the limit system with respect to the curvature in the case $\alpha > 0$. Finally, in agreement with the analytical expression (20), the decay rate for E_ε, $\varepsilon > 0$ is decreasing with the curvature : as expected, the curvature acts as a limiting factor for the stabilization of the beam.

Figures 1 depict the function $t \to u^\varepsilon(L,t)$, $t \in [0,T]$ for $\alpha = 1$ (Left) and $\alpha = 1/2$ (Right). In both case, due to exponential decay of E_ε, the displacement u^ε is damped in time. The case $\alpha = 1$ highlights an oscillating phenomenon in time while the case $\alpha = 1/2$ exhibits a faster decay of the displacement in time, as ε tends toward zero.

5.2. **Case** $\alpha = 0$**.** Let us know comment the result we obtain in the case $\alpha = 0$ which exhibits at the limit a nonlinear behavior and a dependence with respect to the curvature. Results, for $k = 1/5$ and $k = 4$, are collected in Tables 5 and 6, respectively. Once again, we recover the theoretical convergence of w^ε toward w in H_0^1 and also of $u^\varepsilon(L)$ toward the function ζ solution of the ODE (75)-(76). For $k = 1/5$, we observe that

$$\frac{||u^\varepsilon(L,\cdot) - \zeta||_{L^\infty(0,T)}}{||\zeta||_{L^\infty(0,T)}} \approx e^{4.40}\varepsilon^{0.43}. \tag{95}$$

	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$
$\frac{\|\|w^\varepsilon - w\|\|_{L^2(\Omega\times(0,T))}}{\|\|w\|\|_{L^2(\Omega\times(0,T))}}$	6.83×10^{-3}	1.02×10^{-3}	1.83×10^{-4}
$\frac{\|\|w_x^\varepsilon - w_x\|\|_{L^2(\Omega\times(0,T))}}{\|\|w_x\|\|_{L^2(\Omega\times(0,T))}}$	3.46×10^{-3}	9.60×10^{-4}	1.79×10^{-4}
$\lvert \int_0^T\int_\Omega (h(u_x^\varepsilon, w^\varepsilon) - \eta)dxdt \rvert$	2.48×10^{-4}	7.24×10^{-4}	7.46×10^{-4}
$\|\| u_t^\varepsilon(L,t) \|\|_{L^2(0,T)}$	2.705	4.934	8.885
$\|\| \varepsilon^{\alpha/2} u_t^\varepsilon(L,\cdot) \|\|_{L^2(0,T)}$	1.521	1.560	1.579
$\|\| \varepsilon^{1/2} u_t^\varepsilon(L,\cdot) \|\|_{L^2(0,T)}$	8.55×10^{-1}	4.93×10^{-1}	2.80×10^{-1}
$\frac{\|\|E_\varepsilon - E\|\|_{L^2(0,T)}}{\|\|E\|\|_{L^2(0,T)}}$	3.12×10^{-2}	1.76×10^{-2}	9.89×10^{-3}
$\frac{\|\|E_{\varepsilon,w} - E\|\|_{L^2(0,T)}}{\|\|E\|\|_{L^2(0,T)}}$	1.20×10^{-3}	2.85×10^{-4}	1.56×10^{-4}
$\frac{1}{2}\int_0^T\int_\Omega \varepsilon (u_t^\varepsilon)^2 dtdx$	7.53×10^{-1}	2.43×10^{-1}	7.92×10^{-2}
Decay rate for E_ε	-4.056	-4.042	-4.038
Decay rate for $E_{\varepsilon,w^\varepsilon}$	-4.0359	-4.0377	-4.0372

TABLE 3. Estimations in the case $\boldsymbol{\alpha = 1/2}$ and $k = 1/5$ - $dt = h/100$.

	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$
$\frac{\|\|w^\varepsilon - w\|\|_{L^2(\Omega\times(0,T))}}{\|\|w\|\|_{L^2(\Omega\times(0,T))}}$	2.19×10^{-1}	1.24×10^{-1}	4.40×10^{-2}
$\frac{\|\|w_x^\varepsilon - w_x\|\|_{L^2(\Omega\times(0,T))}}{\|\|w_x\|\|_{L^2(\Omega\times(0,T))}}$	1.66×10^{-1}	7.55×10^{-2}	2.48×10^{-2}
$\lvert \int_0^T\int_\Omega (h(u_x^\varepsilon, w^\varepsilon) - \eta)dxdt \rvert$	2.11×10^{-2}	1.81×10^{-2}	1.63×10^{-2}
$\|\| u_t^\varepsilon(L,\cdot) \|\|_{L^2(0,T)}$	3.0986	8.959	15.656
$\|\| \varepsilon^{\alpha/2} u_t^\varepsilon(L,\cdot) \|\|_{L^2(0,T)}$	1.742	2.801	2.784
$\|\| \varepsilon^{1/2} u_t^\varepsilon(L,\cdot) \|\|_{L^2(0,T)}$	9.79×10^{-1}	8.85×10^{-1}	4.95×10^{-1}
$\frac{\|\|E_\varepsilon - E\|\|_{L^2(0,T)}}{\|\|E\|\|_{L^2(0,T)}}$	7.63×10^{-2}	4.75×10^{-2}	2.59×10^{-2}
$\frac{\|\|E_{\varepsilon,w} - E\|\|_{L^2(0,T)}}{\|\|E\|\|_{L^2(0,T)}}$	4.04×10^{-2}	3.12×10^{-2}	1.75×10^{-2}
$\frac{1}{2}\int_0^T\int_\Omega \varepsilon (u_t^\varepsilon)^2 dtdx$	1.853	0.815	0.549
Decay rate for E_ε	-3.70	-4.29	-4.11
Decay rate for $E_{\varepsilon,w^\varepsilon}$	-3.76	-4.29	-4.10
Decay rate for $E_{\varepsilon,u^\varepsilon}$	-6.23	-4.88	-4.14

TABLE 4. Estimations in the case $\boldsymbol{\alpha = 1/2}$ and $k = 4$ - $dt = h/100$.

Figure 2 depicts the function $u^\varepsilon(L,\cdot)$ with respect to $t \in [0,T]$ for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$. The figure exhibits the oscillating behavior of u^ε and the convergence toward ζ as ε goes to zero: in this respect, the contrast between the smoothness of the limit ζ and the oscillations of u^ε illustrates the singular character of the term $\varepsilon u_{tt}^\varepsilon$ in (1). Moreover, with respect to the convergence of the energy and exponential decay, we observe for $\alpha = 0$ the same phenomenon than for $\alpha = 1$: only the energy $E_{\varepsilon,w^\varepsilon}$ converges to the limit energy E defined by (82), which does not contradict the uniform exponential decay. Finally, the exponential decay rates we obtain confirm the dependence in the case $\alpha = 0$ of the limit system with respect to the curvature : for $k = 1/5$, the decay associated with E_ε converges as ε toward approximatively -3.3471 whereas for $k = 4$, the decay converges toward approximatively -3.1302.

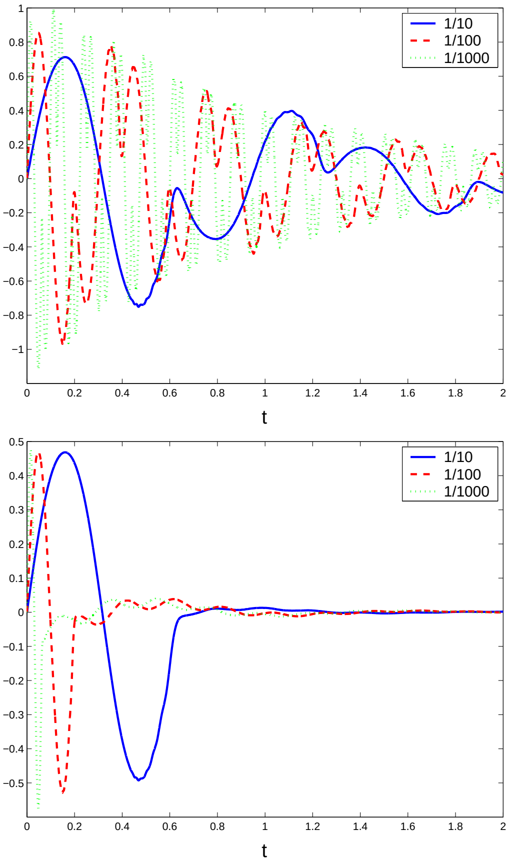

FIGURE 1. $T = 2$ - $k = 1/5$ - $u^\varepsilon(L, \cdot)$ for $\varepsilon = 10^{-1}$ (solid line), $\varepsilon = 10^{-2}$ (dashed line) and $\varepsilon = 10^{-3}$ (dotted line) vs. $t \in [0, T]$: $\boldsymbol{\alpha = 1}$ (Left) and $\boldsymbol{\alpha = 1/2}$ (Right).

	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$
$\frac{\|\|w^\varepsilon - w\|\|_{L^2(\Omega\times(0,T))}}{\|\|w\|\|_{L^2(\Omega\times(0,T))}}$	8.53×10^{-3}	3.75×10^{-3}	1.20×10^{-3}
$\frac{\|\|w^\varepsilon_x - w_x\|\|_{L^2(\Omega\times(0,T))}}{\|\|w_x\|\|_{L^2(\Omega\times(0,T))}}$	4.05×10^{-3}	1.93×10^{-3}	6.39×10^{-4}
$\|\int_0^T \int_\Omega (h(u^\varepsilon_x, w^\varepsilon) - \eta) dxdt\|$	1.60×10^{-3}	5.67×10^{-4}	9.40×10^{-5}
$\|\|u^\varepsilon_t(L, \cdot)\|\|_{L^2(0,T)}$	1.5109	1.6031	1.5581
$\|\|\varepsilon^{1/2} u^\varepsilon_t(L, \cdot)\|\|_{L^2(0,T)}$	4.77×10^{-1}	1.60×10^{-1}	4.97×10^{-2}
$\frac{\|\|u^\varepsilon(L,\cdot) - \zeta\|\|_{L^\infty(0,T)}}{\|\|\zeta\|\|_{L^\infty(0,T)}}$	29.203	11.195	3.861
$\frac{\|\|E_\varepsilon - E\|\|_{L^2(0,T)}}{\|\|E\|\|_{L^2(0,T)}}$	3.71×10^{-2}	3.47×10^{-2}	3.44×10^{-2}
$\frac{\|\|E_{\varepsilon,w} - E\|\|_{L^2(0,T)}}{\|\|E\|\|_{L^2(0,T)}}$	1.21×10^{-3}	8.21×10^{-4}	2.83×10^{-4}
$\frac{1}{2} \int_0^T \int_\Omega \varepsilon (u^\varepsilon_t)^2 dxdt$	2.5583	2.5949	2.4323
Decay rate for E_ε	-3.21	-3.31	-3.33
Decay rate for $E_{\varepsilon,w^\varepsilon}$	-3.67	-3.74	-3.76

TABLE 5. Estimations in the case $\boldsymbol{\alpha = 0}$ and $k = 1/5$ - $dt = h/100$.

	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$
$\frac{\|\|w^\varepsilon - w\|\|_{L^2(\Omega\times(0,T))}}{\|\|w\|\|_{L^2(\Omega\times(0,T))}}$	4.12×10^{-2}	2.10×10^{-2}	1.01×10^{-2}
$\frac{\|\|w^\varepsilon_x - w_x\|\|_{L^2(\Omega\times(0,T))}}{\|\|w_x\|\|_{L^2(\Omega\times(0,T))}}$	3.48×10^{-2}	1.39×10^{-2}	5.42×10^{-3}
$\lvert\int_0^T\int_\Omega (h(u^\varepsilon_x, w^\varepsilon) - \eta)dxdt\rvert$	4.21×10^{-2}	8.21×10^{-3}	4.71×10^{-3}
$\|\|u^\varepsilon_t(L,\cdot)\|\|_{L^2(0,T)}$	4.9237	3.0321	5.4012
$\|\|\varepsilon^{1/2}u^\varepsilon_t(L,\cdot)\|\|_{L^2(0,T)}$	8.41×10^{-1}	3.97×10^{-1}	8.12×10^{-2}
$\frac{\|\|u^\varepsilon(L,\cdot)-\zeta\|\|_{L^\infty(0,T)}}{\|\|\zeta\|\|_{L^\infty(0,T)}}$	47.11	33.12	12.11
$\frac{\|\|E_\varepsilon - E\|\|_{L^2(0,T)}}{\|\|E\|\|_{L^2(0,T)}}$	8.64×10^{-2}	4.81×10^{-2}	2.01×10^{-2}
$\frac{\|\|E_{\varepsilon,w^\varepsilon} - E\|\|_{L^2(0,T)}}{\|\|E\|\|_{L^2(0,T)}}$	3.12×10^{-2}	6.51×10^{-3}	1.91×10^{-3}
$\frac{1}{2}\int_0^T\int_\Omega \varepsilon(u^\varepsilon_t)^2dxdt$	3.1991	5.0712	4.8021
Decay rate for E_ε	-2.9711	-3.0310	-3.1146
Decay rate for $E_{\varepsilon,w^\varepsilon}$	-3.2930	-3.3701	-3.4100

TABLE 6. Estimations in the case $\boldsymbol{\alpha = 0}$ and $k = 4$ - $dt = h/100$.

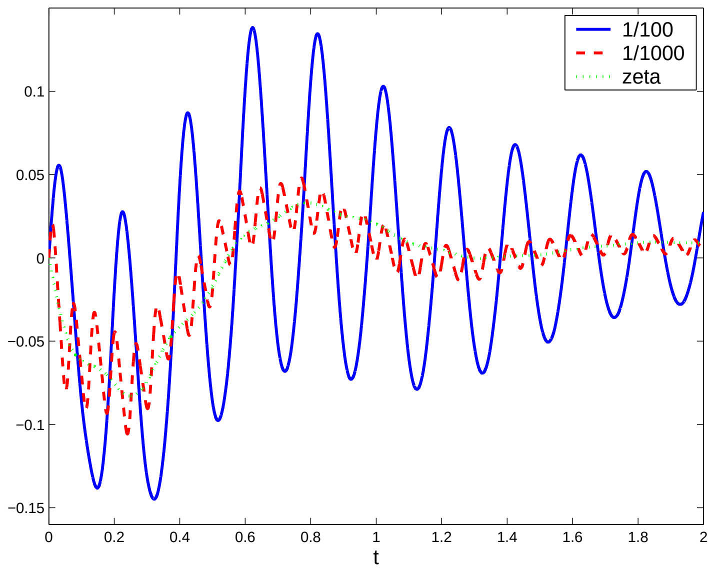

FIGURE 2. $T = 2$, $k = 1/5$, $\boldsymbol{\alpha = 0}$, $u^\varepsilon(L,\cdot)$ for $\varepsilon = 10^{-2}$ (solid line), $\varepsilon = 10^{-3}$ (dashed line) and the limit ζ (dotted line) vs. $t \in [0,T]$.

5.3. **Cases** $\alpha < 0$ **and** $\alpha > 1$**.** To end this section, we give the decay rates obtained for $\alpha = -1$ and $\alpha = 2$. As briefly discussed in Section 4.3, the limit solution w associated to these cases is respectively solution of a linear and a nonlinear system. However, Theorem 3.1 does not apply here and does not provide information on the decay rate. To our knowledge, this question is open in the literature. Tables 7 and 8 collect the decay rates for $E_{\varepsilon,u^\varepsilon}$ and $E_{\varepsilon,w^\varepsilon}$ for $\alpha = -1$ and $\alpha = 2$ respectively. In both case, it appears that the decay rate for $E_{\varepsilon,u^\varepsilon}$ (and therefore for E_ε) is not bounded by below uniformly with respect to ε. At the limit in ε, only the transversal displacement of the beam is stabilized in time. Figure 3 illustrates the non-dissipation in time of $u^\varepsilon(L)$ in these cases. These observations also highlight the nonlinearity (see Figure 4) of the response of the beam with respect to the amplitude of the boundary dissipation. The case $\alpha < 0$ highlights the over-damping phenomenon, well-known for instance for the wave equation. The case $\alpha > 1$ highlights the lack of boundary dissipation for the longitudinal displacement u^ε. We therefore conjecture that Theorem 3.1 does not hold in these cases, whatever the value of the curvature.

	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$
Decay rate for $E_{\varepsilon,u^\varepsilon}$	-2.15×10^{-1}	-6.76×10^{-2}	-2.19×10^{-3}
Decay rate for $E_{\varepsilon,w^\varepsilon}$	-3.6219	-3.6821	-3.6902

TABLE 7. Exponential decay rate in the case $\boldsymbol{\alpha = -1}$ and $k = 1/5$ - $dt = h/100$.

	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$
Decay rate for $E_{\varepsilon,u^\varepsilon}$	-2.17×10^{-1}	-2.38×10^{-2}	-2.27×10^{-3}
Decay rate for $E_{\varepsilon,w^\varepsilon}$	-4.0405	-4.2187	-4.0367

TABLE 8. Exponential decay rate in the case $\boldsymbol{\alpha = 2}$ and $k = 1/5$ - $dt = h/100$.

6. **Concluding remarks.** The analysis performed in this work highlights the sensibility of the value of the longitudinal deformation $h(u^\varepsilon_x, w^\varepsilon)(L,t) = -\varepsilon^\alpha u^\varepsilon_t(L,t)$ imposed at one extremity of a nonlinear elastic beam on its stabilization in time. The different behavior as ε goes to zero are summarized in Table 9. We observe that $\alpha = 0$ is the only value for which the decay rate is uniformly bounded and leading to nonlinear asymptotic system, dependent on the curvature. In this respect, the corresponding boundary condition is less rigid, from a mechanical viewpoint, than the one obtained for $\alpha > 0$, a case which do not retain the nonlinear terms, as ε goes to zero. We also observe that the curvature of the beam is a limiting factor of the stabilization process in time. Moreover, the restriction on the curvature imposed for the proof of the uniform decay is only a weak mechanical assumption.

Adapting some technics used in [14], it seems interesting to extend this analysis to the two dimensional case. It is also worth to investigate similar commuting property in the context of exact controllability (see [7, 17]).

$\alpha < 0$	$\alpha = 0$	$\alpha \in (0,1)$	$\alpha = 1$	$\alpha > 1$
Non Linear	Non Linear	Linear	Linear	Linear
$\mu^\varepsilon \to 0$ $(*)$	$\mu^\varepsilon \geq \mu > 0$	$\mu^\varepsilon \geq \mu > 0$	$\mu^\varepsilon \geq \mu > 0$	$\mu^\varepsilon \to 0$ $(*)$
$E_{\varepsilon,w^\varepsilon} \to E$ $(*)$	$E_{\varepsilon,w^\varepsilon} \to E$ $(*)$	$E_\varepsilon \to E$ $(*)$	$E_{\varepsilon,w^\varepsilon} \to E$ $(*)$	$E_{\varepsilon,w^\varepsilon} \to E$ $(*)$

TABLE 9. Summary of the behavior of the system with respect to ε in function of α ($*$: Numerical observation) - μ^ε designates the exponential decay rate.

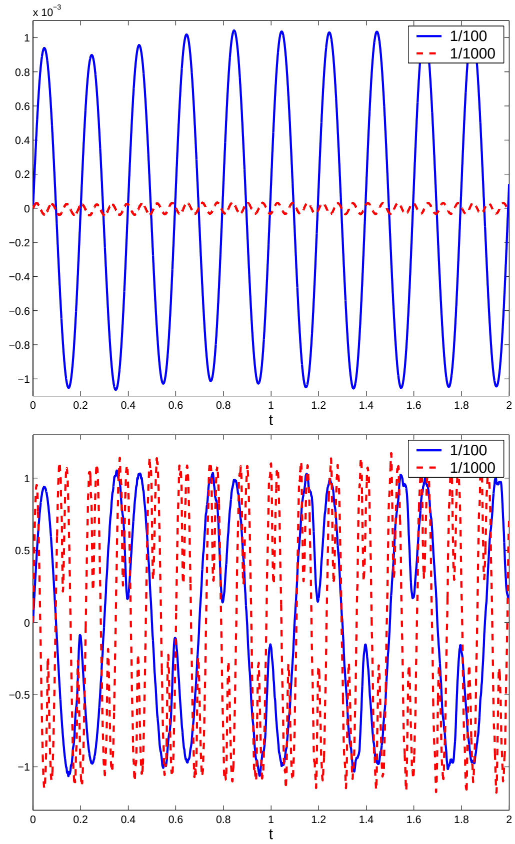

FIGURE 3. $T = 2$ - $k = 1/5$ - $u^\varepsilon(L,\cdot)$ for $\varepsilon = 10^{-2}$ (solid line), $\varepsilon = 10^{-3}$ (dashed line) for $\boldsymbol{\alpha = -1}$ (Left) and $\boldsymbol{\alpha = 2}$ (Right).

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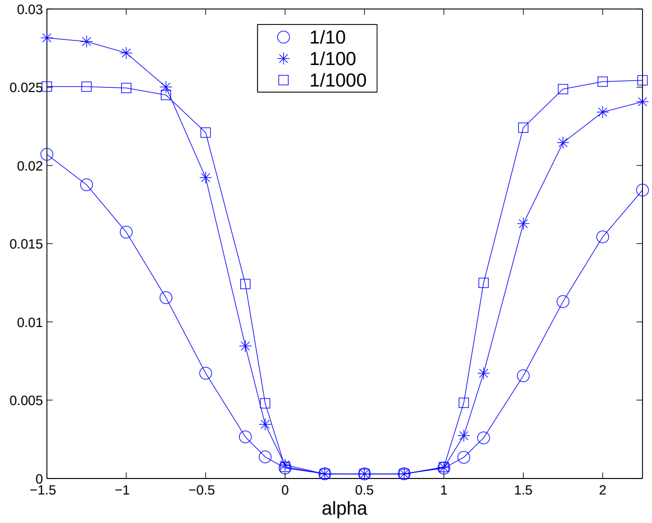

FIGURE 4. $T = 2$, $k = 1/5$, Ratio of the energy $E_\varepsilon(T)/E_\varepsilon(0)$ for $\varepsilon = 10^{-1}$ ($\circ$), $\varepsilon = 10^{-2}$ ($\star$) and $\varepsilon = 10^{-3}$ ($\square$) vs. α

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