RELAXATION OF AN OPTIMAL DESIGN PROBLEM IN FRACTURE MECHANIC

ARNAUD MÜNCH * AND PABLO PEDREGAL †

Abstract. We consider a bi-dimensional crack domain Ω submitted to a boundary load and composed of two isotropic elastic materials. In the framework of the linear fracture theory, a common tool used to describe the smooth evolution of any crack is the so-called energy release rate defined as the variation of the mechanical energy with respect to the crack dimension. Precisely, the well-known Griffith's criterion postulates the evolution of the crack if this rate reaches a critical value. In this work, we consider the shape design problem which consists in optimizing the distribution of the two materials in Ω in order to reduce this rate. Since this kind of problem is usually ill-posed, we first derive a relaxation by using the classical non-convex variational method. The computation of the quasi-convexified of the cost is performed by using div-curl Young measures, leads to an explicit relaxed formulation of the original problem, and exhibits fine microstructure in the form of first order laminates. Finally, numerical simulations suggest that the optimal distribution permits to reduce significantly the value of the energy release rate.

Key words: Fracture mechanics, Optimal design problem, Relaxation, Numerical experiments.

Mathematics Subject Classification : 49J45, 65N30.

1. Introduction - Problem Statement. Let Ω be a bounded domain of $\mathbb{R}^2$ (referred to the orthogonal frame $(O; \boldsymbol{e_1}, \boldsymbol{e_2})$) containing a cut γ of extremity the point $\boldsymbol{F} \in \Omega$ and occupied by two constituent media with constant isotropic conductivity α and β such that $0 < \alpha < \beta < \infty$ (see Figure 1.1). The overall conductivity in Ω is denoted by $a_{\mathcal{X}_\omega}$ defined by

$$a_{\mathcal{X}_\omega}(\boldsymbol{x}) = \alpha \mathcal{X}_\omega(\boldsymbol{x}) + \beta(1 - \mathcal{X}_\omega(\boldsymbol{x})), \quad \boldsymbol{x} = (x_1, x_2) \in \Omega \tag{1.1}$$

where $\mathcal{X}_\omega$ denotes the characteristic function of any domain ω included in Ω. We introduce Γ_0 and Γ_g as two non-empty disjoint parts of $\partial\Omega$ so that $\Gamma_0 \cap \gamma = \emptyset$ and $\Gamma_g \cap \gamma = \emptyset$. For any $u_0 \in H^{1/2}(\Gamma_0)$ and $g \in L^2(\Gamma_g)$, we then consider (in a weak sense) the scalar solution u of the following problem

$$\begin{cases} -div(a_{\mathcal{X}_\omega}(\boldsymbol{x})\nabla u) = 0 & \Omega, \\ u = u_0 & \Gamma_0 \subset \partial\Omega, \\ \beta\, \nabla u \cdot \boldsymbol{\nu} = g & \Gamma_g \subset \partial\Omega \end{cases} \tag{1.2}$$

where $\boldsymbol{\nu}$ designates the outward unit normal to Ω. The perfect transmission conditions are supposed to hold on $\partial\omega$ between the phase α and β. The weak solution enjoys the regularity $u \in H^1(\Omega)$ (see [16]). Finally, we associate with u the finite energy

$$E(u, \gamma) = \frac{1}{2} \int_\Omega a_{\mathcal{X}_\omega}(\boldsymbol{x}) |\nabla u|^2 dx - \int_{\Gamma_g} g u d\sigma. \tag{1.3}$$

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As is well-known, the previous system is a simplified *scalar* modelisation of a structure S occupying the domain Ω, fixed on Γ_0, submitted to a normal load g on Γ_g and containing a crack γ. Our motivation in this work is to optimize the distribution of the two materials α and β along the structure S in order to prevent, or at least reduce, the growth of the crack point $\boldsymbol{F}$. In this respect, in the framework of Fracture Mechanics (we refer to [3, 19]), a well-known and still widely used growth criterion is due to A.A. Griffith [15]. This criterion is related to the so-called energy release $g_{\boldsymbol{\psi}}$ (defined as minus the variation of the energy E with respect to variations of $\boldsymbol{F}$)

$$g_{\boldsymbol{\psi}}(u, \mathcal{X}_\omega) = \int_\Omega a_{\mathcal{X}_\omega}(\boldsymbol{x})(A_{\boldsymbol{\psi}}(\boldsymbol{x})\nabla u, \nabla u)dx. \tag{1.4}$$

It postulates the growth of the point $\boldsymbol{F}$ if $g_{\boldsymbol{\psi}}(u, \mathcal{X}_\omega)$ reaches a critical positive value experimentally determined. $A_{\boldsymbol{\psi}}$ is for all $\boldsymbol{x} \in \Omega$ a real 2×2 matrix defined in Section 2 and $(,)$ denotes the scalar product in $\mathbb{R}^2$. We point out that this criterion, associated with E and recently revisited in [14], is global, in contrast with stress criteria such as Von Misses and Tresca criteria. We also point out that this rate is nonnegative for all u.

In order to reduce the growth of $\boldsymbol{F}$ due to the load g, we therefore consider in this work from a mathematical point of view, the following problem

$$(P): \qquad \inf_{\mathcal{X}_\omega \in \mathcal{X}_L} g_{\boldsymbol{\psi}}(u, \mathcal{X}_\omega) \tag{1.5}$$

where, for any $L \in (0, 1)$,

$$\mathcal{X}_L = \{\mathcal{X} \in L^\infty(\Omega, \{0, 1\}), \|\mathcal{X}\|_{L^1(\Omega)} = L|\Omega|\} \tag{1.6}$$

and where u is the solution of (1.2). (P) is a so-called *nonlinear optimal design problem* associated with a functional which depends quadratically on the gradient of u. The relation $\|\mathcal{X}\|_{L^1(\Omega)} = L|\Omega|$ expresses that the amount of material α to be distributed on Ω is fixed and equal to $L|\Omega|$.

To our knowledge, very few works have investigated the control of the crack growth in this context. We mention two preliminary notes by P. Destuynder ([7, 8]). In [7], the author considers the dynamic wave equation posed on a 2D cracked domain and defines a growth criterion based on the stress intensity factor. A formulation for the derivative of this criterion is given with respect to a control defined on the boundary of the domain. The reference [8] considers a stationary loaded structure with a crack, and suggests a computational method for a control law which restricts the crack evolution (we refer to [10] for some numerical treatments). In the more recent work [17], an active control strategy is addressed which consists in minimizing the rate $g_{\boldsymbol{\psi}}$ with respect to the support and amplitude of an additional boundary load. Following this idea, we also mention the note [28] which studies the possibility to annihilate the singularity in a cracked domain using additional (singular) boundary loads. Problem (P) is conceptually different and may be qualified as passive control, assuming that the cracked structure is built once for all. On the mathematical viewpoint, this problem is a prototype of ill-posed problem in the sense that the infimum may be not reached in the class of characteristic functions: the optimal domain ω is then composed of an infinite number of disjoint components. The study of (P) then consists in finding a well-posed relaxation for which there exists an optimal solution. This can be done by mainly two approaches: the Homogenization method (we refer to [1, 34]), and the classical non-convex variational method (we refer to [5, 31] and references therein).

The well-known application in conductivity is the minimization of the compliance for which the matrix $A_{\boldsymbol{\psi}}$ is diagonal, positive definite and independent of $\boldsymbol{x}$. The non diagonal and space dependent case provided by the energy release rate $g_{\boldsymbol{\psi}}$ seems original in this context and presents several difficulties. Moreover, the 2×2 matrix $A_{\boldsymbol{\psi}}$ is not positive definite for all $\boldsymbol{x} \in \Omega$. In this work, following some previous works [24, 25, 26], we address the relaxation of (P) through the variational method using the div-curl tool introduced and analyzed in [33]. The analysis amounts to computing the quasi-convexified functional cost.

The paper is organized as follows. In Section 2, we recall the expression of the energy release rate in terms of a surface integral and some important properties, and then focus on two relevant choices for the matrix $A_{\boldsymbol{\psi}}(\boldsymbol{x})$: the diagonal, and the non diagonal cases (Remark 2.4). Then in Section 3, by using the variational approach and Young measures, we determine a full relaxation (RP) of the original problem (P). The analysis is divided in three steps: i) an equivalent variational reformulation (VP) of (P) (Section 3.1); ii) the computation of a sub-relaxation of (VP) derived from the expression of the poly-convexification of the cost (Section 3.2); iii) the determination of at least one (div-curl) Young measure for which the lower bound is actually attained. We obtain that both the diagonal and non diagonal cases exhibit first-order laminates (Section 3.3). Then, in Section 3.4 by introducing an additional field, we transform the explicit but non standard relaxation (RP) into a new equivalent formulation $(\underline{RP})$ where u appears as a solution of a nonlinear elliptic equation under a divergence form (see eq. 3.71). This final step then permits to address the numerical approximation and discuss some experiments in Section 4. Some remarks and perspectives conclude this work (Section 5).

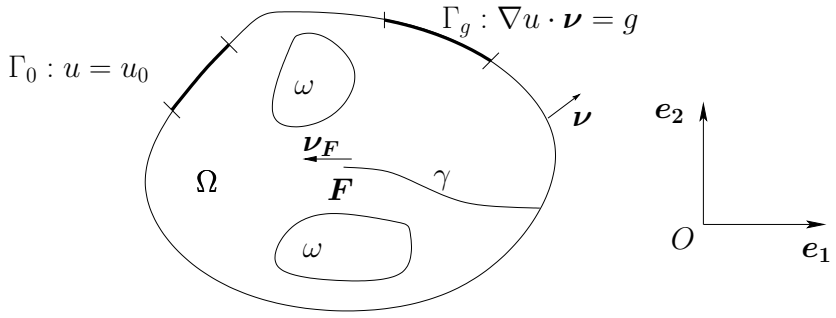

FIG. 1.1. *Illustration of problem (P): Optimization of the location of ω support of the α-material in the crack domain Ω.*

2. Overview about the energy release rate. In this section, we recall the definition of the energy release rate and its expression in terms of a surface integral. We use the notation $\psi_{,i}$ for $\partial\psi/\partial x_i$, $i = 1, 2$ as well as the convention of summation of repeated indices.

We assume that, in the neighborhood of $\boldsymbol{F}$, the crack γ is rectilinear and (without loss of generalities) oriented along $\boldsymbol{e_1}$. We introduce any velocity field $\boldsymbol{\psi} = (\psi_1, \psi_2) \in W \equiv \{\boldsymbol{\psi} \in (W^{1,\infty}(\Omega, \mathbb{R}))^2, \boldsymbol{\psi} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega/\gamma\}$, where $\boldsymbol{\nu}$ designates the unit outward normal to Ω. Moreover, we assume that the support of the function $\boldsymbol{\psi}$ is disjoint from the support Γ_g of the load. Let $\eta > 0$ and the transformation $\mathcal{F}^\eta : \boldsymbol{x} \to \boldsymbol{x} + \eta\boldsymbol{\psi}(\boldsymbol{x})$ so that $\mathcal{F}^\eta(\boldsymbol{F}) = \boldsymbol{F^\eta}$ and $\mathcal{F}^\eta(\gamma) = \gamma^\eta$; we first recall the following definition (see [3, 19]).

DEFINITION 2.1 (Energy release rate). *Let u be the solution of (1.2). The derivative of the functional $-E(u,\gamma)$ with respect to a variation of γ (precisely $\boldsymbol{F}$) in the direction $\boldsymbol{\psi}$ is defined as the Fréchet derivative in W at 0 of the application $\eta \to -E(u,(Id+\eta\boldsymbol{\psi})(\gamma))$, i.e.*

$$E(u,(Id+\eta\boldsymbol{\psi})(\gamma)) = E(u,\gamma) - \eta\frac{\partial E(u,\gamma)}{\partial\gamma}.\boldsymbol{\psi} + o(\eta^2). \tag{2.1}$$

In the sequel, we denote this derivative by $g_{\boldsymbol{\psi}}(u,\mathcal{X}_\omega)$. ■

The procedure to obtain the explicit expression of $g_{\boldsymbol{\psi}}$ is technical but by now well-known (see [11, 22, 23]). Moreover, since the problem is self-adjoint, the derivative may be expressed only in terms of u as follows.

LEMMA 2.2. *The first derivative of $-E$ with respect to γ in the direction $\boldsymbol{\psi} = (\psi_1,\psi_2) \in W$ is given by*

$$g_{\boldsymbol{\psi}}(u,\mathcal{X}_\omega) = \int_\Omega a_{\mathcal{X}_\omega}(\boldsymbol{x})\nabla u\cdot(\nabla\boldsymbol{\psi}\cdot\nabla u)dx - \frac{1}{2}\int_\Omega a_{\mathcal{X}_\omega}(\boldsymbol{x})|\nabla u|^2 div(\boldsymbol{\psi})dx \tag{2.2}$$

where u is the solution of (1.2). ■

Remark that the load g does not appears explicitly in (2.2) since we have assumed for simplicity that $\Gamma_g \cap \text{supp}\ \boldsymbol{\psi} = \emptyset$. Introducing the 2×2 matrix $A_{\boldsymbol{\psi}}(\boldsymbol{x})$ for all $\boldsymbol{x}\in\Omega$ as follows:

$$\begin{aligned} A_{\boldsymbol{\psi}}(\boldsymbol{x}) &= \nabla\boldsymbol{\psi} - \frac{1}{2}div(\boldsymbol{\psi})I_2 = \nabla\boldsymbol{\psi} - \frac{1}{2}Tr(\nabla\boldsymbol{\psi})I_2 \\ &= \frac{1}{2}\begin{pmatrix} \psi_{1,1}-\psi_{2,2} & 2\psi_{1,2} \\ 2\psi_{2,1} & \psi_{2,2}-\psi_{1,1}\end{pmatrix}, \end{aligned} \tag{2.3}$$

the energy release rate takes the form (1.4).

Moreover, since $g_{\boldsymbol{\psi}}$ is a shape derivative (with respect to $\boldsymbol{F}$), $g_{\boldsymbol{\psi}}$ should depend on the function $\boldsymbol{\psi}\in W$ only in a neighborhood of the crack tip $\boldsymbol{F}$. This invariance is true for all $\boldsymbol{\psi}\in W$ in the isotropic case for which $\alpha=\beta$; in our situation, we have to assume that the function $\boldsymbol{\psi}$ is such that $\{\boldsymbol{x}\in\Omega, \boldsymbol{\psi}(\boldsymbol{x})\neq 0\}\cap\partial\omega = \emptyset$: this simply requires to have a uniform material (α or β) in a neighborhood, say $\mathcal{D}\subset\Omega$, of $\boldsymbol{F}$, so that $\overline{\omega}\cap\mathcal{D}=\emptyset$ and $\{\boldsymbol{x}\in\Omega,\boldsymbol{\psi}(\boldsymbol{x})\neq 0\}\subset\mathcal{D}$:

$$\boldsymbol{F}\in\{\boldsymbol{x}\in\Omega,\boldsymbol{\psi}(\boldsymbol{x})\neq 0\}\subset\mathcal{D}\subset\Omega/\overline{\omega}. \tag{2.4}$$

This assumption then permits to link the derivative $g_{\boldsymbol{\psi}}$, which is a mathematical quantity defined on Ω, to the thermo-dynamic strength G (locally defined on $\boldsymbol{F}$).

LEMMA 2.3. *[(Local) Energy release rate] Let $C(\boldsymbol{F},r)$ be the circle of center $\boldsymbol{F}$ and radius $r>0$, $\boldsymbol{\nu_c} = (\nu_{c,1},\nu_{c,2})$ its outward normal and*

$$G_r(u,\mathcal{X}_\omega) = \frac{1}{2}\int_{C(\boldsymbol{F},r)} a_{\mathcal{X}_\omega}(\boldsymbol{x})u_{,j}u_{,j}\psi_k\nu_{c,k}d\sigma - \int_{C(\boldsymbol{F},r)} a_{\mathcal{X}_\omega}(\boldsymbol{x})u_{,j}u_{,k}\psi_k\nu_{c,j}d\sigma,$$

where u is solution of (1.2). The thermo-dynamic strength G is linked to $g_{\boldsymbol{\psi}}$ as follows:

$$g_{\boldsymbol{\psi}}(u,\mathcal{X}_\omega) = \lim_{r\to 0} G_r(u,\mathcal{X}_\omega)\ (\boldsymbol{\psi}\cdot\boldsymbol{\nu})_{|\boldsymbol{F}} \equiv G(u,\mathcal{X}_\omega)\ \boldsymbol{\psi}(\boldsymbol{F})\cdot\boldsymbol{\nu_F}, \quad \forall\boldsymbol{\psi}\in W, \tag{2.5}$$

where $\boldsymbol{\nu_F} = (\nu_{F,1},\nu_{F,2})$ designates the orientation of the crack γ at the point $\boldsymbol{F}$. ■

It follows from (2.5) that the energy release rate $g_{\boldsymbol{\psi}}$ is related to the strength G by

$$G(u,\mathcal{X}_\omega) = g_{\boldsymbol{\psi}}(u,\mathcal{X}_\omega), \quad \forall\boldsymbol{\psi}\in W \text{ such that } \boldsymbol{\psi}(\boldsymbol{F})\cdot\boldsymbol{\nu_F} = 1. \tag{2.6}$$

Summarizing, the function $\boldsymbol{\psi}$ which defines the virtual crack extension should belong to

$$\boldsymbol{W} = \{\boldsymbol{\psi} \in (W^{1,\infty}(\Omega))^2, \boldsymbol{\psi}\cdot\boldsymbol{\nu} = 0 \text{ on } \partial\Omega/\gamma, \{\boldsymbol{x} \in \Omega, \boldsymbol{\psi}(\boldsymbol{x}) \neq 0\} \cap \partial\omega = \emptyset, \boldsymbol{\psi}(\boldsymbol{F})\cdot\boldsymbol{\nu_F} = 1\} \tag{2.7}$$

in order that the (local) strength may be computed through his integral expression $g_{\boldsymbol{\psi}}$.

REMARK 2.4.

- *Since the crack is oriented along the axis $\boldsymbol{e_1}$, the natural choice is to take $\boldsymbol{\psi} = (\psi_1, 0)$ with $\psi_1\nu_1 = 0$ on $\partial\Omega/\gamma$. In this case, $A_{\boldsymbol{\psi}}$ is simply*

$$A_{\boldsymbol{\psi}} = \frac{1}{2}\begin{pmatrix} \psi_{1,1} & 2\psi_{1,2} \\ 0 & -\psi_{1,1} \end{pmatrix}. \tag{2.8}$$

  Moreover, since only the derivative of $\boldsymbol{\psi}$ are involved in $g_{\boldsymbol{\psi}}$ defined by (2.2), it is more accurate from a numerical point of view to consider a function ψ_1 which is constant in a neighborhood of $\boldsymbol{F}$. This permits to obtain the strength G with the relation (2.2) only as a function of the solution $\boldsymbol{u}$ far away from $\boldsymbol{F}$ where it is singular [11]. A simple choice is given by the radial function

$$\psi_1(\boldsymbol{x}) = \zeta(dist(\boldsymbol{x}, \boldsymbol{F}))\nu_{F,1}, \quad \forall \boldsymbol{x} \in \Omega \tag{2.9}$$

  defining the function $\zeta \in C^1(\mathbb{R}^+; [0,1])$ as follows:

$$\zeta(r) = \begin{cases} 1 & r \leq r_1 \\ \dfrac{(r-r_2)^2(3r_1 - r_2 - 2r)}{(r_1 - r_2)^3} & r_1 \leq r \leq r_2 \\ 0 & r \geq r_2 \end{cases} \tag{2.10}$$

  with $0 < r_1 < r_2 < dist(\partial\Omega/\gamma, \boldsymbol{F}) = \inf_{\boldsymbol{x}\in\partial\Omega/\gamma} dist(\boldsymbol{x}, \boldsymbol{F})$. This situation, described on Figure 2.1, leads for some $\boldsymbol{x} \in \Omega$, to a non diagonal and non positive definite matrix $A_{\boldsymbol{\psi}}(\boldsymbol{x})$. In practice, the material is then assumed fixed equal to α or β in the set $\mathcal{D} \equiv \{\boldsymbol{x} \in \Omega, dist(\boldsymbol{x}, \boldsymbol{F}) \leq r_3\}$ for any $r_3 > r_2$.
- *Let us designates by $\Omega_\epsilon = \{\boldsymbol{x} \in \Omega, |x_1 - x_F| \leq \epsilon\}$ and $\partial\Omega_\epsilon = \{\boldsymbol{x} \in \partial\Omega, |x_1 - x_F| \leq \epsilon\}$. If there exists an $\epsilon > 0$ for which $\nu_1 = 0$ on $\partial\Omega_\epsilon$ and $\partial\Gamma_g \cap \partial\Omega_\epsilon = \emptyset$, then one may construct an admissible function ψ_1 independent of x_2. It suffices to take $\psi_1(x_1, x_2) = \zeta(x_1)$*

$$\zeta(x_1) = \begin{cases} 0 & x_1 \leq r_1 \\ \frac{(x_1-r_1)^2(2x_1+r_1-3r_2)}{r_1-r_2} & r_1 \leq x_1 \leq r_2, \\ 1 & r_2 \leq x_1 \leq r_3, \\ \frac{(x_1-r_4)^2(2x_1+r_4-3r_3)}{r_4-r_3} & r_3 \leq x_1 \leq r_4, \\ 0 & x_1 \geq r_4 \end{cases} \tag{2.11}$$

  with

$$r_1 = x_F - \frac{2\epsilon}{3}, \quad r_2 = x_F - \frac{\epsilon}{3}, \quad r_3 = x_F + \frac{\epsilon}{3}, \quad r_4 = x_F + \frac{2\epsilon}{3}. \tag{2.12}$$

  In this case, $\psi_{1,2} = 0$ and the matrix $A_{\boldsymbol{\psi}}$ defined in (2.8) is simply diagonal for all $\boldsymbol{x} \in \Omega$. However, this choice, described on Figure 2.2, enforces the material to be constant in the whole vertical strip $\mathcal{D} \equiv \{\boldsymbol{x} \in \Omega, |x_1 - x_F| \leq r_3\}$ for any $r_3 > \epsilon$ so that $\Omega_\epsilon \subset \mathcal{D}$, which is more restrictive than the previous situation.

- *One may also construct, for any domain, a function* $\boldsymbol{\psi} = (\psi_1, \psi_2)$ *such that* $\psi_{1,2} = \psi_{2,1}$. *In this more general case, the matrix* $A_{\boldsymbol{\psi}}$ *defined by (2.3) is symmetric. It suffices to take* ψ_1 *given by (2.10) and* $\psi_2(x_1, x_2) = \int_0^{x_1} \psi_{1,2}(s, x_2)ds$. *Since* ψ_1 *is radial,* $\psi_{1,2} = 0$ *on* $\Gamma_{x_F} = \{(x_1, x_2) \in \Omega, x_1 = x_F\}$. *Therefore,* $\psi_2 = 0$ *on* Γ_{x_F}, *and the virtual extension* $\boldsymbol{F^\eta} = \boldsymbol{F} + \eta\boldsymbol{\psi}(\boldsymbol{F})$ *remains on* Γ_{x_F} *for all* $\eta > 0$ *small.*
- *From the definition,* $u = 0$ *implies* $g_{\boldsymbol{\psi}}(u, \mathcal{X}_\omega) = 0$. *The converse is not true since for all* $\boldsymbol{x} \in \Omega$ *for which* $\nabla\boldsymbol{\psi}(\boldsymbol{x}) \neq 0$ *(i.e.* $\boldsymbol{\psi}$ *is not constant),* $A_{\boldsymbol{\psi}}$ *is not positive definite.* ■

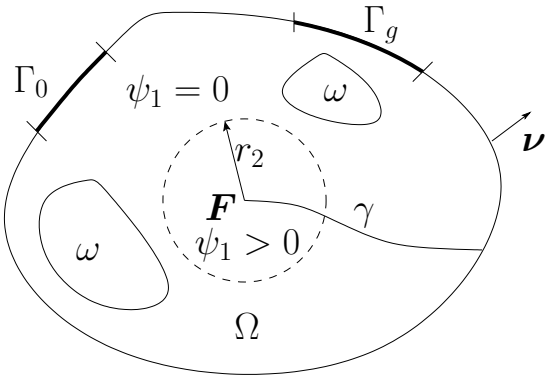

FIG. 2.1. *Choice of a radial function* $\psi_1(\boldsymbol{x})$ *leading to a non diagonal matrix* $A_{\boldsymbol{\psi}}$.

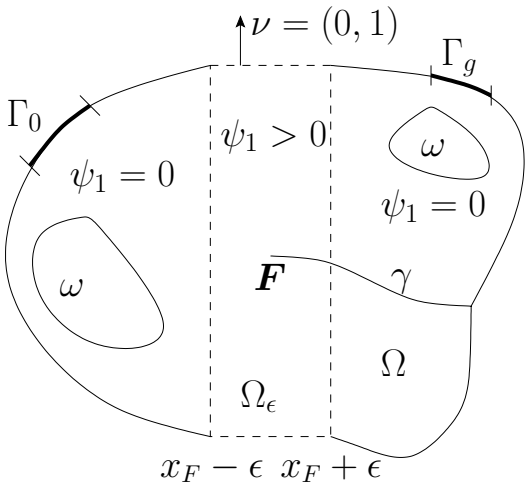

FIG. 2.2. *Choice of a function* $\psi_1(\boldsymbol{x}) = \psi_1(x_1)\mathcal{X}_{\Omega_\epsilon}$ *leading to a diagonal matrix* $A_{\boldsymbol{\psi}}$ *assuming the existence of a domain* Ω_ϵ.

REMARK 2.5. *The condition* $\beta\nabla u \cdot \nu = g$ *on* Γ_g *forces the conductivity to be equal to* β *on* Γ_g. *To simplify the analysis in the next section, we assume that the conductivity is* β *on all the boundary so that* ω *is strictly included in* Ω. *Consequently,*

the conductivity is β in $\mathcal{D}$. Observe that we may relaxe this condition on $\partial\Omega$ if the normal load g is zero (see Section 4). ■

3. Relaxation of (P). Problem (P), which involves a functional depending on the gradient ∇u, typically lacks optimal solution which means that the infimum may only be achieved by a sequence of more and more intricate subset ω_n of Ω (see for instance [27, 34]). The goal of this section is to perform a relaxation of problem (P). It consists in looking for another minimization problem (RP) for which there does exist an optimal solution, this minimum has the same value of the infimum of (P), and more importantly, the optimal solution of the relaxed problem encodes the information about (some) minimizing sequence for the original problem. Following the procedure described in [33], a relaxed problem may be obtained by using Young measures generated by sequence of pairs $\{G_n, H_n\}$, associated with the design $\mathcal{X}_{\omega_n}$ admissible for (P), for which we have the information that the divergence of the first component vanishes while the second component is a gradient. Such class of Young measures, the so-called div-curl Young measures, has been explicitly introduced and studied in [33]. In our situation, we denote by $\nu = \{\nu_x\}_{x\in\Omega}$ the div-curl associated with the sequence where each element has the following components:

$$G_n(\boldsymbol{x}) = (\alpha\mathcal{X}_n + \beta(1-\mathcal{X}_n))\nabla u_n(\boldsymbol{x}), \quad H_n(\boldsymbol{x}) = \nabla u_n(\boldsymbol{x}) \tag{3.1}$$

respectively. Since G_n comes from the state equation, then each individual ν_x is supported in the union of the two linear manifolds

$$\Lambda_\gamma = \{(\lambda,\rho)\in\mathbb{R}^2\times\mathbb{R}^2 : \rho = \gamma\lambda\}, \quad \gamma = \alpha,\beta \tag{3.2}$$

so that $supp(\nu_x) \subset \Lambda_\alpha \cup \Lambda_\beta$. As is usual, the measure ν_x may be written as

$$\nu_x = s(\boldsymbol{x})\nu_{x,\alpha} + (1-s(\boldsymbol{x}))\nu_{x,\beta} \tag{3.3}$$

with $supp(\nu_{x,\gamma}) \subset \Lambda_\gamma$ and $s(\boldsymbol{x}) \in [0,1]$, the weak$-\star$ limit in $L^\infty(\Omega)$ of a subsequence of $\mathcal{X}_{\omega_n}$. Suppose now that $(\mathcal{X}_{\omega_n})_{(n>0)}$ is a minimizing sequence for (P). Then, by the fundamental property of Young measures (see [30], Theorem 6.2), we may represent the limit of the cost associated with $\mathcal{X}_{\omega_n}$ through the measure ν. Precisely, using the relation $(A_{\boldsymbol{\psi}}\nabla u, \nabla u) = A_{\boldsymbol{\psi}} \otimes (\nabla u \nabla u^T)$ where ∇u^T designates the transpose of the vector ∇u, this limit is

$$\begin{aligned}\lim_{n\to\infty} g_{\boldsymbol{\psi}}(u_n, \mathcal{X}_{\omega_n}) = \int_\Omega \Bigg[& \alpha s(\boldsymbol{x}) A_{\boldsymbol{\psi}}(\boldsymbol{x}) \otimes \int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_{x,\alpha}(\lambda) \\ & + \beta(1-s(\boldsymbol{x}))A_{\boldsymbol{\psi}}(\boldsymbol{x}) \otimes \int_{\mathbb{R}^2} \lambda\lambda^T d\nu^{(1)}_{x,\beta}(\lambda)\Bigg] dx\end{aligned} \tag{3.4}$$

where $\nu^{(1)}_{x,\gamma}$, $\gamma = \alpha,\beta$, designates the projection of $\nu_{x,\lambda}$ onto the first copy of $\mathbb{R}^2$. Therefore, with each minimizing sequence of (P), we associate an optimal div-curl Young measure. In this sense, optimizing with respect to $\mathcal{X}_{\omega_n}$ is equivalent to optimizing with respect to ν.

3.1. Variational reformulation. We now proceed to the analysis of problem (P) in a similar fashion as in [33]. We first put (P) in an equivalent variational setting. We introduce the non Carathéodory functions

$$W(\boldsymbol{x},\rho,\lambda) = \begin{cases} \alpha A_{\boldsymbol{\psi}}(\boldsymbol{x}) \otimes \lambda\lambda^T & \text{if} \quad (\rho,\lambda)\in\Lambda_\alpha, \\ \beta A_{\boldsymbol{\psi}}(\boldsymbol{x}) \otimes \lambda\lambda^T & \text{if} \quad (\rho,\lambda)\in\Lambda_\beta, \\ +\infty & \text{else,} \end{cases} \tag{3.5}$$

and

$$V(\rho,\lambda)=\begin{cases}1 & \text{if} \quad (\rho,\lambda)\in\Lambda_\alpha,\\ 0 & \text{if} \quad (\rho,\lambda)\in\Lambda_\beta,\\ +\infty & \text{else.}\end{cases} \tag{3.6}$$

Then we check that (P) is equivalent to the following new problem

$$(VP): \qquad \inf_{G,u}\int_\Omega W(\boldsymbol{x},G(\boldsymbol{x}),\nabla u(\boldsymbol{x}))dx \tag{3.7}$$

subject to

$$\begin{cases} G\in L^2(\Omega;\mathbb{R}^2), \quad u\in H^1(\Omega;\mathbb{R}),\\ div\ G=0 \qquad\qquad \text{in} \quad H^{-1}(\Omega),\\ u=u_0 \qquad\qquad \Gamma_0,\\ \beta\nabla u\cdot\boldsymbol{\nu}=g \qquad\qquad \Gamma_g\subset\partial\Omega\backslash(\gamma\cup\Gamma_0),\\ \displaystyle\int_\Omega V(G(\boldsymbol{x}),\nabla u(\boldsymbol{x}))dx=L|\Omega|. \end{cases} \tag{3.8}$$

This equivalent formulation suffers from the same troubles as the initial problem, so that it is in need of relaxation. The crucial step is the computation of the constrained quasi-convexification CQW of the density W leading to a relaxation (RP) of (VP) (and thus of (P)):

$$(RP): \qquad \min_{s,G,u}\int_\Omega CQW(\boldsymbol{x},s(\boldsymbol{x}),G(\boldsymbol{x}),\nabla u(\boldsymbol{x}))dx \tag{3.9}$$

for $s\in L^\infty(\Omega,[0,1])$, and subject to (3.8), but replacing the integral constraint involving V by $\int_\Omega s(x)\,dx=L|\Omega|$. The constrained quasi-convex density CQW is computed by solving the problem in measures :

$$\begin{aligned} &CQW(\boldsymbol{x},s(\boldsymbol{x}),G(\boldsymbol{x}),\nabla u(\boldsymbol{x}))\\ &=\inf_\nu\left\{\alpha s(\boldsymbol{x})A_{\boldsymbol{\psi}}(\boldsymbol{x})\otimes\int_{\mathbb{R}^2}\lambda\lambda^T d\nu^{(1)}_{x,\alpha}(\lambda)+\beta(1-s(\boldsymbol{x}))A_{\boldsymbol{\psi}}(\boldsymbol{x})\otimes\int_{\mathbb{R}^2}\lambda\lambda^T d\nu^{(1)}_{x,\beta}(\lambda)\right\} \end{aligned} \tag{3.10}$$

for any measure ν subject to

$$\begin{cases} \nu=\{\nu_x\}_{x\in\Omega}, \quad \nu_x=s(\boldsymbol{x})\nu_{x,\alpha}+(1-s(\boldsymbol{x}))\nu_{x,\beta},\ \mathrm{supp}(\nu_{x,\gamma})\subset\Lambda_\gamma,\\ \nu \quad \text{is div-curl Young measure,}\\ \displaystyle G(\boldsymbol{x})=\int_{\mathbb{R}^2}\rho d\nu_x(\lambda,\rho), \quad div\ G=0 \text{ weakly in } \Omega,\\ \displaystyle\nabla u(\boldsymbol{x})=\int_{\mathbb{R}^2}\lambda d\nu_x(\lambda,\rho). \end{cases} \tag{3.11}$$

Moreover, Remark 2.5 enforces the following property on the density:

$$s(\boldsymbol{x})=0 \quad \text{in} \quad \mathcal{D}\cup\partial\Omega. \tag{3.12}$$

3.2. Relaxation. To proceed further with the analysis of this relaxed formulation, we regard $\boldsymbol{x} \in \Omega$ as a parameter and put $\nu_x = \nu$, $G(\boldsymbol{x}) = \rho$, $\nabla u = \lambda$, and $s(\boldsymbol{x}) = s$.

Let us take ν a div-curl Young measure which is supported in the set $\Lambda = \Lambda_\alpha \cup \Lambda_\beta$ where $\Lambda_\gamma = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2; y = \gamma x\}$, a linear manifold in $\mathbb{R}^2 \times \mathbb{R}^2$. We can decompose $\nu = s\nu_\alpha + (1-s)\nu_\beta$ where ν_γ is a probability measure (most likely not a Div-Curl Young measure itself) supported in Λ_γ.

Concerning the first moment of ν, we may write

$$(\lambda, \rho) = \int_\Lambda (x,y) d\nu(x,y) = s \int_{\mathbb{R}^2} (x, \alpha x) d\nu_\alpha^{(1)}(x) + (1-s) \int_{\mathbb{R}^2} (x, \beta x) d\nu_\beta^{(1)}(x) \quad (3.13)$$

where $\nu_\gamma^{(1)}$ is the projection of ν_γ onto the first copy of $\mathbb{R}^2$ of the product $\mathbb{R}^2 \times \mathbb{R}^2$. By introducing

$$\lambda_\gamma = \int_{\mathbb{R}^2} x d\nu_\gamma^{(1)}(x), \quad (3.14)$$

we have $\lambda = s\lambda_\alpha + (1-s)\lambda_\beta$, $\rho = s\alpha\lambda_\alpha + (1-s)\beta\lambda_\beta$, and then

$$\lambda_\alpha = \frac{1}{s(\beta-\alpha)}(\beta\lambda - \rho), \quad \lambda_\beta = \frac{1}{(1-s)(\beta-\alpha)}(\rho - \alpha\lambda). \quad (3.15)$$

Moreover, the commutation with the inner product yields the relation

$$\lambda^T \rho = \int_\Lambda x^T y d\nu(x,y) = \alpha s \int_{\mathbb{R}^2} x^T x d\nu_\alpha^{(1)}(x) + \beta(1-s) \int_{\mathbb{R}^2} x^T x d\nu_\beta^{(1)}(x). \quad (3.16)$$

To find a lower bound of CQW defined by (3.10), we are going to retain just the relevant property expressed in the commutation (3.16), so that we regard feasible measures ν as Young measures which satisfy this commutation property, but are not necessarily a div-curl Young measure. We introduce

$$X_\gamma = \int_{\mathbb{R}^2} x x^T d\nu_\gamma^{(1)}(x), \quad \gamma = \alpha, \beta \quad (3.17)$$

a convex combination of symmetric rank-one matrices. It is well-known that

$$X_\gamma \geq \lambda_\gamma \lambda_\gamma^T, \quad \gamma = \alpha, \beta \quad (3.18)$$

in the usual sense of symmetric matrices, i.e. that $X_\gamma - \lambda_\gamma \lambda_\gamma^T$ is semi-definite positive. The relation (3.16) becomes

$$\lambda^T \rho = \lambda \cdot \rho = \alpha s Tr(X_\alpha) + \beta(1-s) Tr(X_\beta) \quad (3.19)$$

where Tr designates the Trace operator for square matrices. Similarly, the cost may be written in term of the variable X_γ as follows :

$$s\alpha A_{\boldsymbol{\psi}} \otimes X_\alpha + (1-s)\beta A_{\boldsymbol{\psi}} \otimes X_\beta = s\alpha Tr(A_{\boldsymbol{\psi}} X_\alpha) + (1-s)\beta Tr(A_{\boldsymbol{\psi}} X_\beta) \quad (3.20)$$

from the relation $A_{\boldsymbol{\psi}} \otimes X_\gamma = Tr(A_{\boldsymbol{\psi}} X_\gamma)$, $\gamma = \alpha, \beta$. Consequently, in seeking a lower bound of the constrained quasiconvexification, we are led to consider the mathematical programming problem

$$\min_{X_\alpha, X_\beta} C(X_\alpha, X_\beta) = \alpha s Tr(A_{\boldsymbol{\psi}} X_\alpha) + \beta(1-s) Tr(A_{\boldsymbol{\psi}} X_\beta) \quad (3.21)$$

subject to the constraints

$$\lambda^T \rho = \lambda \cdot \rho = \alpha s Tr(X_\alpha) + \beta(1-s) Tr(X_\beta), \quad X_\gamma \geq \lambda_\gamma \lambda_\gamma^T. \tag{3.22}$$

We first realize that the set of vectors for which the constraints yield a non-empty set takes place if

$$\alpha s Tr(\lambda_\alpha \lambda_\alpha^T) + \beta(1-s) Tr(\lambda_\beta \lambda_\beta^T) \leq \lambda \cdot \rho \tag{3.23}$$

i.e. if

$$B \equiv \lambda \cdot \rho - \alpha s |\lambda_\alpha|^2 - \beta(1-s)|\lambda_\beta|^2 \geq 0 \tag{3.24}$$

using that $Tr(\lambda \rho^T) = \lambda \cdot \rho$. This inequality related to the state equation in (1.2) is the usual one obtained, for instance, in the so-called compliance problem (see [32]). Notice that by using (3.15), B may be factorized as follows : $B = (\lambda_\beta - \lambda_\alpha) \cdot (\beta \lambda_\beta - \alpha \lambda_\alpha)$. We now solve the mathematical programming problem (3.21) in the diagonal and non diagonal case respectively, highlighted in Remark 2.4.

3.2.1. Case $A_{\boldsymbol{\psi}}$ diagonal. We first assume the diagonal situation $A_{\boldsymbol{\psi}} = diag(\psi_{1,1}, -\psi_{1,1})$ for which the cost is simply

$$C(X_\alpha, X_\beta) = \frac{1}{2}\psi_{1,1}\Big(\alpha s(X_{\alpha,11} - X_{\alpha,22}) + \beta(1-s)(X_{\beta,11} - X_{\beta,22})\Big) \tag{3.25}$$

under the constraints

$$\begin{cases} s\alpha(X_{\alpha,11} + X_{\alpha,22}) + (1-s)\beta(X_{\beta,11} + X_{\beta,22}) = \lambda \cdot \rho, \\ X_{\gamma,11} + X_{\gamma,22} \geq \lambda_{\gamma,1}^2 + \lambda_{\gamma,2}^2 = |\lambda_\gamma|^2, \quad \gamma = \alpha, \beta, \\ (X_{\gamma,11} - \lambda_{\gamma,1}^2)(X_{\gamma,22} - \lambda_{\gamma,2}^2) \geq (X_{\gamma,12} - \lambda_{\gamma,1}\lambda_{\gamma,2})^2. \end{cases} \tag{3.26}$$

Let us first consider the point $\boldsymbol{x}$ of Ω for which $\psi_{1,1}(\boldsymbol{x}) \geq 0$. Using the first constraint, we write that $s\alpha X_{\alpha,22} + (1-s)\beta X_{\beta,22} = \lambda \cdot \rho - s\alpha X_{\alpha,11} + (1-s)\beta X_{\beta,11}$, so that the cost becomes simply

$$C(X_\alpha, X_\beta) = \psi_{1,1}\Big(s\alpha X_{\alpha,11} + (1-s)\beta X_{\beta,11}\Big) - \frac{1}{2}\psi_{1,1}\lambda \cdot \rho. \tag{3.27}$$

The minimum is then reached for $X_{\gamma,11} = \max(\lambda_{\gamma,1}^2, |\lambda_\gamma|^2 - X_{\gamma,22})$ since $X_{\gamma,11} \geq \lambda_{\gamma,1}^2$. Now, using that $X_{\gamma,22} \geq \lambda_{\gamma,2}^2$, the maximum is $\lambda_{\gamma,1}^2$. Consequently,

$$X_{\gamma,11} = \lambda_{\gamma,1}^2, \quad X_{\gamma,22} \geq \lambda_{\gamma,2}^2, \quad \gamma = \alpha, \beta. \tag{3.28}$$

The last constraint then provides the equality

$$X_{\gamma,12} = \lambda_{\gamma,1}\lambda_{\gamma,2}, \quad \gamma = \alpha, \beta. \tag{3.29}$$

The cost is then, from (3.15)

$$C(X_\alpha, X_\beta) = \psi_{1,1}\left(\frac{\alpha(\beta\lambda_1 - \rho_1)^2}{s(\beta-\alpha)^2} + \frac{\beta(\rho_1 - \alpha\lambda_1)^2}{(1-s)(\beta-\alpha)^2}\right) - \frac{1}{2}\psi_{1,1}\lambda \cdot \rho. \tag{3.30}$$

Similarly, the study of the case $\psi_{1,1}(\boldsymbol{x}) \leq 0$ leads to

$$X_{\gamma,11} \geq \lambda_{\gamma,1}^2, \quad X_{\gamma,22} = \lambda_{\gamma,2}^2, \quad X_{\gamma,12} = \lambda_{\gamma,1}\lambda_{\gamma,2} \quad \gamma = \alpha, \beta, \tag{3.31}$$

and a cost equal to

$$C(X_\alpha, X_\beta) = -\psi_{1,1}\left(\frac{\alpha(\beta\lambda_2 - \rho_2)^2}{s(\beta-\alpha)^2} + \frac{\beta(\rho_2 - \alpha\lambda_2)^2}{(1-s)(\beta-\alpha)^2}\right) + \frac{1}{2}\psi_{1,1}\lambda\cdot\rho. \tag{3.32}$$

We then obtain the following partial result :

PROPOSITION 3.1 (Diagonal case). *For any $s \in L^\infty(\Omega)$ and $(\lambda, \rho) = (\nabla u, G)$ satisfying (3.8), the function*

$$m(s,\lambda,\rho) = \begin{cases} \psi_{1,1}\left(\frac{\alpha(\beta\lambda_1 - \rho_1)^2}{s(\beta-\alpha)^2} + \frac{\beta(\rho_1 - \alpha\lambda_1)^2}{(1-s)(\beta-\alpha)^2}\right) - \frac{1}{2}\psi_{1,1}\lambda\cdot\rho \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if} \quad \psi_{1,1} \geq 0 \textit{ and } (3.23) \\ -\psi_{1,1}\left(\frac{\alpha(\beta\lambda_2 - \rho_2)^2}{s(\beta-\alpha)^2} + \frac{\beta(\rho_2 - \alpha\lambda_2)^2}{(1-s)(\beta-\alpha)^2}\right) + \frac{1}{2}\psi_{1,1}\lambda\cdot\rho \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if} \quad \psi_{1,1} \leq 0 \textit{ and } (3.23) \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{else} \end{cases} \tag{3.33}$$

is a lower bound for the constrained quasi-convexified CQW of W:

$$m(s,\lambda,\rho) \leq CQW(s,\lambda,\rho). \tag{3.34}$$

■

3.2.2. Case $A_{\boldsymbol{\psi}}$ non-diagonal. We now assume that $A_{\boldsymbol{\psi}}$ of the form (2.8):

$$A_{\boldsymbol{\psi}} = \frac{1}{2}\begin{pmatrix} \psi_{1,1} & 2\psi_{1,2} \\ 0 & -\psi_{1,1} \end{pmatrix} \equiv \begin{pmatrix} a & 2b \\ 0 & -a \end{pmatrix} \tag{3.35}$$

so that the mathematical programming problem is now

$$\min_{X_{\gamma,11}, X_{\gamma,22}, X_{\gamma,12}} C(X_\alpha, X_\beta) \tag{3.36}$$

with

$$C(X_\alpha, X_\beta) = \alpha s[a(X_{\alpha,11} - X_{\alpha,22}) + 2bX_{\alpha,12}] + \beta(1-s)[(a(X_{\beta,11} - X_{\beta,22}) + 2bX_{\beta,12})] \tag{3.37}$$

and the constraint

$$\begin{cases} s\alpha(X_{\alpha,11} + X_{\alpha,22}) + (1-s)\beta(X_{\beta,11} + X_{\beta,22}) = \lambda\cdot\rho, \\ X_{\gamma,11} + X_{\gamma,22} \geq \lambda_{\gamma,1}^2 + \lambda_{\gamma,2}^2 = |\lambda_\gamma|^2, \quad \gamma = \alpha, \beta, \\ (X_{\gamma,11} - \lambda_{\gamma,1}^2)(X_{\gamma,22} - \lambda_{\gamma,2}^2) \geq (X_{\gamma,12} - \lambda_{\gamma,1}\lambda_{\gamma,2})^2. \end{cases} \tag{3.38}$$

With the change of variable $Y_\gamma = X_\gamma - \lambda_\gamma\lambda_\gamma^T$, the cost and the constraints are transformed into

$$\min_{Y_{\gamma,11}, Y_{\gamma,22}, Y_{\gamma,12}} \alpha s(a(Y_{\alpha,11} - Y_{\alpha,22}) + 2bY_{\alpha,12}) + \beta(1-s)((a(Y_{\beta,11} - Y_{\beta,22}) + 2bY_{\beta,12})) + A \tag{3.39}$$

and

$$\begin{cases} s\alpha(Y_{\alpha,11} + Y_{\alpha,22}) + (1-s)\beta(Y_{\beta,11} + Y_{\beta,22}) = B, \\ Y_{\gamma,11} + Y_{\gamma,22} \geq 0, \quad Y_{\gamma,11}Y_{\gamma,22} \geq Y_{\gamma,12}^2 \quad \gamma = \alpha, \beta \end{cases} \tag{3.40}$$

where the constants A and B are defined by

$$A = \alpha s(a(\lambda_{\alpha,1}^2 - \lambda_{\alpha,2}^2) + 2b\lambda_{\alpha,1}\lambda_{\alpha,2}) + \beta(1-s)((a(\lambda_{\beta,1}^2 - \lambda_{\beta,2}^2) + 2b\lambda_{\beta,1}\lambda_{\beta,2})) \quad (3.41)$$

and (3.24) respectively. We first realize that the minimum of the linear cost is reached on the boundary of the convex sets

$$\Gamma_\gamma = \left\{ (Y_{\gamma,11}, Y_{\gamma,22}, Y_{\gamma,12}) \in \mathbb{R}^3, Y_{\gamma,11} \geq 0, Y_{\gamma,22} \geq 0, Y_{\gamma,11}Y_{\gamma,22} \geq Y_{\gamma,12}^2 \right\}, \quad \gamma = \alpha, \beta \quad (3.42)$$

which implies necessarily the equality $Y_{\gamma,11}Y_{\gamma,22} = Y_{\gamma,12}^2$. Therefore, we can introduce the new variables $Z_\gamma \equiv (Z_{\gamma,11}, Z_{\gamma,22})^T$ so that $Y_{\gamma,11} = Z_{\gamma,11}^2$, $Y_{\gamma,22} = Z_{\gamma,22}^2$ and $\epsilon_\gamma = \pm 1$ and then $Z_{\gamma,11}Z_{\gamma,22} = \epsilon_\gamma Y_{\gamma,12}$ reducing the problem to

$$\begin{aligned} \min_{Z_{\gamma,11}, Z_{\gamma,22}, \epsilon_\gamma} C(Z_\gamma, \epsilon_\gamma) = \alpha s(a(Z_{\alpha,11}^2 - Z_{\alpha,22}^2) + 2b\epsilon_\alpha Z_{\alpha,11}Z_{\alpha,22}) \\ + \beta(1-s)((a(Z_{\beta,11}^2 - Z_{\beta,22}^2) + 2b\epsilon_\beta Z_{\beta,11}Z_{\beta,22})) + A \end{aligned} \quad (3.43)$$

under the constraint

$$s\alpha(Z_{\alpha,11}^2 + Z_{\alpha,22}^2) + (1-s)\beta(Z_{\beta,11}^2 + Z_{\beta,22}^2) = B. \quad (3.44)$$

Introducing the Lagrangian L and the multiplier p

$$L(Z_\gamma, p) = C(Z_\gamma, \epsilon_\gamma) - p\Big(s\alpha(Z_{\alpha,11}^2 + Z_{\alpha,22}^2) + (1-s)\beta(Z_{\beta,11}^2 + Z_{\beta,22}^2) - B \Big), \quad (3.45)$$

we arrive at the optimality conditions :

$$A_{\psi,\epsilon_\gamma} Z_\gamma = pZ_\gamma, \qquad A_{\psi,\epsilon_\gamma} = \begin{pmatrix} a & b\epsilon_\gamma \\ b\epsilon_\gamma & -a \end{pmatrix}. \quad (3.46)$$

The trivial solution is $(Z_{\gamma,11}, Z_{\gamma,22}) = (0,0)$ leading to the value of the cost $C(Z_\gamma, \epsilon_\gamma) = A$. The other cases lead to the resolution of a spectral problem: we obtain

$$p = -\sqrt{a^2 + b^2}, \quad Z_\gamma = a_\gamma \Big(b\epsilon_\gamma, -(a + \sqrt{a^2 + b^2}) \Big)^T \quad (3.47)$$

and

$$p = \sqrt{a^2 + b^2}, \quad Z_\gamma = a_\gamma \Big(b\epsilon_\gamma, -(a - \sqrt{a^2 + b^2}) \Big)^T \quad (3.48)$$

for any $a_\gamma \in \mathbb{R}^*$. Now, writing that $a(Z_{\gamma,11}^2 - Z_{\gamma,22}^2) + 2b\epsilon_\gamma Z_{\gamma,11}Z_{\gamma,22} = A_{\psi,\epsilon_\gamma} Z_\gamma \cdot Z_\gamma$, we may write from (3.46) that

$$\begin{aligned} C(Z_\gamma, \epsilon_\gamma) =& \alpha s A_{\psi,\epsilon_\alpha} Z_\alpha \cdot Z_\alpha + \beta(1-s) A_{\psi,\epsilon_\gamma} Z_\beta \cdot Z_\beta + A \\ =& p(\alpha s |Z_\alpha|^2 + \beta(1-s)|Z_\beta|^2) + A \end{aligned} \quad (3.49)$$

and then conclude from the constraint (3.44) that the cost given by (3.43) is $C(Z_\gamma, \epsilon_\gamma) = pB + A$. Therefore, the cost, independent of ϵ_γ is obtained for the lowest eigenvalue (independent here of the sign of a) :

$$\min C(Z_\gamma, \epsilon_\gamma) = -\sqrt{a^2 + b^2} B + A \quad (3.50)$$

for $Z_\gamma = a_\gamma(b\epsilon_\gamma, -(a+\sqrt{a^2+b^2}))^T$. The constraint (3.44) then gives the relation

$$(a_\alpha^2 s\alpha + a_\beta^2(1-s)\beta)(b^2 + (a+\sqrt{a^2+b^2})^2) = B. \tag{3.51}$$

We then observe that the cost for this non trivial solution is lower (except in the case $B=0$, i.e. the equality in (3.23)). It is also important for the search of laminates (see Section 3.3) to remark that the value of the cost is unchanged if $Z_\alpha = 0$ or $Z_\beta = 0$. Precisely, (3.49) remains true. At last, we check that for $b=0$, we recover the cost of the diagonal case. Consequently, the partial result is as follows :

PROPOSITION 3.2 (Non diagonal case). *For any $s \in L^\infty(\Omega)$ and $(\lambda,\rho) = (\nabla u, G)$ satisfying (3.8), the function*

$$m(s,\lambda,\rho) = \begin{cases} \frac{1}{2}\Big[-\sqrt{\psi_{1,1}^2+\psi_{1,2}^2}(\rho\cdot\lambda - \alpha s|\lambda_\alpha|^2 - \beta(1-s)|\lambda_\beta|^2) \\ \quad + \psi_{1,1}(\alpha s\lambda_{\alpha,1}^2 + (1-s)\beta\lambda_{\beta,1}^2) - \psi_{1,1}(\alpha s\lambda_{\alpha,2}^2 + (1-s)\beta\lambda_{\beta,2}^2) \\ \quad + 2\psi_{1,2}(\alpha s\lambda_{\alpha,1}\lambda_{\alpha,2} + (1-s)\beta\lambda_{\beta,1}\lambda_{\beta,2})\Big] & \text{if (3.23)} \\ +\infty & \text{else} \end{cases} \tag{3.52}$$

is a lower bound for the constrained quasi-convexified CQW of W:

$$m(s,\lambda,\rho) \le CQW(s,\lambda,\rho). \tag{3.53}$$

$\lambda_\gamma = \lambda_\gamma(s,\lambda,\rho)$, $\gamma = \alpha,\beta$ are defined by (3.15). ■

3.3. (Div-Curl) laminates and relaxed formulation. We now study whether or not the optimal measure may be recovered by laminates. This would imply that the constrained quasi-convexified is reached. Except for special examples (like the compliance case which exhibits first and second order laminates [32]), the search of explicit laminates is difficult. In the case studied here, the situation is actually straightforward because the function m is zero out of the support of the function $\boldsymbol{\psi}$. Let us discuss the non diagonal case. In the set $\mathcal{D}$, the material β is imposed so that the density s is equal to zero. Therefore, the search of laminates is meaningful only in $\Omega/\mathcal{D}$. According to the computation of Section 3.2.2, the optimal second moment are of the form

$$X_\gamma = \lambda_\gamma\lambda_\gamma^T + a_\gamma^2 \begin{pmatrix} \psi_{1,2}^2 & -\psi_{1,2}(\psi_{1,1}+\sqrt{\psi_{1,1}^2+\psi_{1,2}^2}) \\ -\psi_{1,2}(\psi_{1,1}+\sqrt{\psi_{1,1}^2+\psi_{1,2}^2}) & (\psi_{1,1}+\sqrt{\psi_{1,1}^2+\psi_{1,2}^2})^2 \end{pmatrix} \tag{3.54}$$

leading to the cost $-\sqrt{\psi_{1,1}^2+\psi_{1,2}^2}B + A$. But, on $\Omega/\mathcal{D}$, the radial function ψ is zero so that,

$$X_\gamma = \lambda_\gamma\lambda_\gamma^T, \quad \boldsymbol{x} \in \Omega/\mathcal{D} \tag{3.55}$$

i.e. in particular

$$X_{\gamma,ii} = \int_{\mathbb{R}} x_i^2 d\nu_\gamma^{(1,i)}(x_i) = \left(\int_{\mathbb{R}} x_i d\nu_\gamma^{1,i}(x_i)\right)^2 = (\lambda_{\gamma,i})^2, \quad i=1,2 \tag{3.56}$$

where $\nu_\gamma^{(1,i)}$ denotes the projection of $\nu^{(1)}$ onto the i-th copy of $\mathbb{R}^2$. From the strict convexity of the square function, this implies that $\nu_\gamma^{(1,i)} = \delta_{\lambda_{\gamma,i}}$, i.e.

$$\nu_\alpha^{(1,i)} = \delta_{\frac{\beta\lambda_i - \rho_i}{s(\beta-\alpha)}}, \quad \nu_\beta^{(1,i)} = \delta_{\frac{\rho_i - \alpha\lambda_i}{(1-s)(\beta-\alpha)}}. \tag{3.57}$$

Remark that this is compatible with the third equality $X_{\gamma,12} = \lambda_{\gamma,1}\lambda_{\gamma,2}^T$. This also implies (see for instance (3.51)) the equality in (3.23), i.e. that

$$B = \lambda \cdot \rho - \alpha s|\lambda_\alpha|^2 - \beta(1-s)|\lambda_\beta|^2 = 0. \tag{3.58}$$

Consequently, the optimal value $m(s, \lambda, \rho)$ may be recovered by the following measure

$$\nu = s\delta_{(\alpha\lambda_\alpha, \lambda_\alpha)} + (1-s)\delta_{(\beta\lambda_\beta, \lambda_\beta)} \tag{3.59}$$

which is a first order (div-curl) laminate, the div-curl condition $(\beta\lambda_\beta - \alpha\lambda_\alpha) \cdot (\lambda_\beta - \lambda_\alpha) = 0$ (analogous to a rank one condition for H^1-gradient Young measure) being equivalent precisely to $B = 0$. We therefore may establish the following relaxation :

THEOREM 3.3. *The variational problem*

$$(RP): \qquad \min_{s,u,G} \int_\Omega m(s, \nabla u, G)dx \tag{3.60}$$

subject to

$$\begin{cases} s \in L^\infty(\Omega, [0,1]), s = 0 \text{ in } \mathcal{D} \cup \partial\Omega, \int_\Omega s(\boldsymbol{x})dx = L|\Omega|, \\ u \in H^1(\Omega), \quad u = u_0 \text{ on } \Gamma_0, \quad \beta\nabla u \cdot \boldsymbol{\nu} = g \text{ on } \Gamma_g, \\ G \in (L^2(\Omega))^2, \quad div\ G = 0 \text{ weakly in } \Omega, \end{cases} \tag{3.61}$$

where m is defined by (3.52) is a relaxation of (VP) in the sense that the minimum of (RP) exists and equals the minimum of (VP). Moreover, the underlying Young measure associated with (RP) can be found in the form of a first order laminate whose direction of lamination are given explicitly in terms of the optimal solution (u, G): precisely, the normal are orthogonal to $\lambda_\beta - \lambda_\alpha$. ■

The above formulation may be simplified by taking into account that $B = 0$. Precisely, we use (3.15) to express $B = (\beta\lambda_\beta - \alpha\lambda_\alpha)(\lambda_\beta - \lambda_\alpha) = 0$ as follows

$$(\rho - \lambda^-(s)\lambda) \cdot (\rho - \lambda^+(s)\lambda) = 0 \tag{3.62}$$

in terms of the harmonic and arithmetic mean of α, β defined as follows:

$$\lambda^-(s) = \frac{\alpha\beta}{\alpha(1-s) + \beta s}, \qquad \lambda^+(s) = \alpha s + \beta(1-s). \tag{3.63}$$

Therefore, Theorem 3.3 is equivalent to

THEOREM 3.4. *The variational problem*

$$(\overline{RP}): \qquad \min_{s,u,G} \int_\Omega F(s, \nabla u, G)dx \tag{3.64}$$

subject to

$$\begin{cases} s \in L^\infty(\Omega, [0,1]), s = 0 \text{ in } \mathcal{D} \cup \partial\Omega, \int_\Omega s(\boldsymbol{x})dx = L|\Omega|, \\ u \in H^1(\Omega), \quad u = u_0 \text{ on } \Gamma_0, \quad \beta\nabla u \cdot \boldsymbol{\nu} = g \text{ on } \Gamma_g, \\ G \in (L^2(\Omega))^2, \quad div\ G = 0 \text{ weakly in } \Omega, \\ (G - \lambda^-(s)\nabla u) \cdot (G - \lambda^+(s)\nabla u) = 0 \text{ in } L^2(\Omega), \end{cases} \tag{3.65}$$

where F, deduced from m, is defined

$$F(s,\lambda,\rho) = \frac{1}{2}\Big[\psi_{1,1}(\alpha s\lambda_{\alpha,1}^2 + (1-s)\beta\lambda_{\beta,1}^2) - \psi_{1,1}(\alpha s\lambda_{\alpha,2}^2 + (1-s)\beta\lambda_{\beta,2}^2) + 2\psi_{1,2}(\alpha s\lambda_{\alpha,1}\lambda_{\alpha,2} + (1-s)\beta\lambda_{\beta,1}\lambda_{\beta,2})\Big] \tag{3.66}$$

is a relaxation of (VP) in the sense that the minimum of $(\overline{RP})$ exists and equals the minimum of (VP). ■

3.4. A Final transformation. The above analysis provides an explicit relaxation $(\overline{RP})$, in terms of a minimum of a new functional over a convex set. Remark that this formulation is not standard since the state equation (1.2) under a usual divergence form has disappeared. This state equation is incorporated in the constraints (3.65). This fact explains why a direct numerical approximation (*a priori*, the nonlinear problem $(\overline{RP})$ can not be solved analytically) of this problem is difficult. In order to overpass this point, we remark that the relation (3.62) is equivalent to

$$\left|\rho - \frac{\lambda^+(s)+\lambda^-(s)}{2}\lambda\right|^2 = \left(\frac{\lambda^+(s)-\lambda^-(s)}{2}\right)^2 |\lambda|^2. \tag{3.67}$$

Therefore, by introducing the additional variable $t(\boldsymbol{x}) \in \mathbb{R}^2$ such that $|t| = 1$, we may write $\rho = G(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$ under the form (we use that $\lambda^-(s) \leq \lambda^+(s)$ for all $s \in (0,1)$)

$$\rho = \frac{\lambda^+(s)+\lambda^-(s)}{2}\lambda + \frac{\lambda^+(s)-\lambda^-(s)}{2}|\lambda| t \equiv \phi(s,t,\lambda). \tag{3.68}$$

We have

$$\frac{\lambda^+(s)+\lambda^-(s)}{2} = \frac{2\alpha\beta + s(1-s)(\beta-\alpha)^2}{2(\alpha(1-s)+\beta s)} \equiv A(s), \tag{3.69}$$

and

$$\frac{\lambda^+(s)-\lambda^-(s)}{2} = \frac{s(1-s)(\beta-\alpha)^2}{2(\alpha(1-s)+\beta s)} \equiv B(s). \tag{3.70}$$

The relation $div\ G = 0$ then permits to recover u as the solution of a *nonlinear* equation under a divergence form (having in mind that $\lambda = \nabla u$):

$$\begin{cases} div(A(s)\nabla u + B(s)|\nabla u| t) = 0, & \text{in } \Omega, \\ u = u_0, & \text{on } \Gamma_0, \\ \beta\nabla u \cdot \boldsymbol{\nu} = g, & \text{on } \Gamma_g. \end{cases} \tag{3.71}$$

The study - out of the scope of this work - of this apparently non standard elliptic equation would be very interesting. We assume here that (3.71) is well-posed. Note that the nonlinear part vanishes where s takes the values in $\{0,1\}$ since $B(0) = B(1) = 0$. In such a case, (3.71) is nothing but (1.2). The relaxed problem is then equivalent to the following one, easier to solve numerically although nonlinear in u and non standard :

THEOREM 3.5. *Let F and ϕ be defined respectively by (3.66) and (3.68). The following formulation*

$$(\underline{RP}): \qquad \min_{s,t} I(s,t) = \int_\Omega F(s, \nabla u, \phi(s,t,\nabla u))dx \tag{3.72}$$

subject to the constraints

$$\begin{cases} s \in L^\infty(\Omega,[0,1]), s = 0 \text{ in } \mathcal{D} \cup \partial\Omega, \int_\Omega s(\boldsymbol{x})dx = L|\Omega|, \\ t \in L^\infty(\Omega,\mathbb{R}^2), \quad |t| = 1, \\ u \in H^1(\Omega), \quad u = u_0 \text{ on } \Gamma_0, \quad \beta\nabla u \cdot \boldsymbol{\nu} = g \text{ on } \Gamma_g, \\ div\ \phi(s,t,\nabla u) = 0 \text{ weakly in } \Omega \end{cases} \tag{3.73}$$

is equivalent to the relaxation (RP). In particular, $(\underline{RP})$ is a full well-posed relaxation of (VP). ■

REMARK 3.6. *Using the fact that the density s is identically zero on $\{\boldsymbol{x} \in \Omega, \psi(\boldsymbol{x}) \neq 0\}$, one may simplify a bit more the integrand F. Using (3.15) and (3.68), we explicit λ_γ in terms of λ, s and t:*

$$\begin{aligned} \lambda_\alpha &= \frac{2\beta - (1-s)(\beta-\alpha)}{2(\alpha + s(\beta-\alpha))}\lambda - \frac{(1-s)(\beta-\alpha)}{2(\alpha+s(\beta-\alpha))}|\lambda|t, \\ \lambda_\beta &= \frac{s(\beta-\alpha)+2\alpha}{2(\alpha+s(\beta-\alpha))}\lambda + \frac{s(\beta-\alpha)}{2(\alpha+s(\beta-\alpha))}|\lambda|t. \end{aligned} \tag{3.74}$$

and compute that

$$F(0,\lambda,\phi(0,t,\lambda)) = \frac{1}{2}\beta\psi_{1,1}(\lambda_1^2 - \lambda_2^2) + \beta\psi_{1,2}\lambda_1\lambda_2. \tag{3.75}$$

Now since $s = 0$ in $\{\boldsymbol{x} \in \Omega, \psi(\boldsymbol{x}) \neq 0\}$, we deduce that (taking $\lambda = \nabla u$)

$$F(s,\nabla u,\phi(s,t,\nabla u)) = \frac{1}{2}\beta\psi_{1,1}(u_{,1}^2 - u_{,2}^2) + \beta\psi_{1,2}u_{,1}u_{,2} \tag{3.76}$$

which is nothing but the integrand of $g_{\boldsymbol{\psi}}$. Therefore, the relaxation of (P) is simply apparent through the nonlinear state equation (3.71). ■

4. Numerical study. We illustrate in this section our theoretical results by some numerical developments. We first detail the numerical resolution of the relaxed formulation $(\underline{RP})$ and then provides some numerical examples.

4.1. Numerical resolution of the relaxed problem $(\underline{RP})$. Since u is completely determined by s and t, the minimization of the cost is over s and t using a first order gradient method. We compute explicitly the first variation of I with respect to s and t in the direction δs and δt, defined (formally) as :

$$\frac{\partial I(s,t)}{\partial s} \cdot \delta s = \lim_{\eta\to 0} \frac{I(s+\eta\delta s,t) - I(s,t)}{\eta}, \quad \eta \in \mathbb{R}^* \tag{4.1}$$

and

$$\frac{\partial I(s,t)}{\partial t} \cdot \delta t = \lim_{\eta\to 0} \frac{I(s,t+\eta\delta t) - I(s,t)}{\eta}, \quad \eta \in \mathbb{R}^*. \tag{4.2}$$

THEOREM 4.1. *The first variation of I with respect to s and t in the direction δs and δt exist and are given respectively by*

$$\begin{aligned}\frac{d\mathcal{I}(s,t,u,p)}{ds}\cdot\delta s = &\int_\Omega F_{,s}(s,\nabla u,\phi(s,t,\nabla u))\cdot\delta s\,dx\\ &+\int_\Omega\Big(A_{,s}(s)\nabla u\cdot\nabla p + B_{,s}(s)|\nabla u|t\cdot\nabla p\Big)\cdot\delta s\,dx\end{aligned}\tag{4.3}$$

and

$$\frac{d\mathcal{I}(s,t,u,p)}{dt}\cdot\delta t = \int_\Omega F_{,t}(s,\nabla u,\phi(s,t,\nabla u))\cdot\delta t\,dx + \int_\Omega B(s)|\nabla u|\delta t\cdot\nabla p\,dx \tag{4.4}$$

where $p\in H^1_{\Gamma_0}(\Omega)=\{v\in H^1(\Omega), v=0\ on\ \Gamma_0\}$ solves the adjoint problem

$$\int_\Omega F_{,u}(s,\nabla u,\phi(s,t,\nabla u))\cdot v\,dx+\int_\Omega\Big(A(s)\nabla v\cdot\nabla p+B(s)\frac{\nabla u\cdot\nabla v}{|\nabla u|}t\cdot\nabla p\Big)dx=0, \tag{4.5}$$

for all v in $H^1_{\Gamma_0}(\Omega)$. $A_{,s}$ and $B_{,s}$ denote the partial derivative of A and B with respect to s and $F_{,t}$ the partial derivative of F with respect to t. ■

Notice that the adjoint formulation (4.5) is linear in contrast with the formulation associated with u, which reads as follows,

$$\int_\Omega\Big(A(s)\nabla u\cdot\nabla v + B(s)|\nabla u|t\cdot\nabla v\Big)dx=\int_{\Gamma_g} gv\,d\sigma,\quad\forall v\in H^1_{\Gamma_0}(\Omega) \tag{4.6}$$

using that $s=0$ on $\partial\Omega$ and that $A(0)=\beta$, $B(0)=0$.
Proof of Theorem 4.1- The proof is standard. We introduce the lagrangian

$$\begin{aligned}\mathcal{L}(s,t,u,p)=&\int_\Omega F(s,\nabla u,\phi(s,t,\nabla u))dx\\ &+\int_\Omega\Big(A(s)\nabla u\cdot\nabla p+B(s)|\nabla u|t\cdot\nabla p\Big)dx-\int_{\Gamma_g}gpd\sigma\end{aligned}\tag{4.7}$$

and write that

$$\frac{d\mathcal{L}(s,t,u,p)}{d(s,t)}\cdot(\delta s,\delta t)=\frac{d\mathcal{L}(s,t,u,p)}{ds}\cdot\delta s+\frac{d\mathcal{L}(s,t,u,p)}{dt}\cdot\delta t. \tag{4.8}$$

Formally, we have

$$\begin{aligned}\frac{d\mathcal{L}(s,t,u,p)}{ds}\cdot\delta s=&\frac{\partial\mathcal{L}(s,t,u,p)}{\partial s}\cdot\delta s+<\frac{\partial\mathcal{L}(s,t,u,p)}{\partial u},\frac{\partial u}{\partial s}\cdot\delta s>\\ &+<\frac{\partial\mathcal{L}(s,t,u,p)}{\partial p},\frac{\partial p}{\partial s}\cdot\delta s>.\end{aligned}\tag{4.9}$$

As usual, since $\mathcal{L}$ is linear in p and since u is the solution of (4.6), the third term is equal to zero. The solution p is then determined in order that the second term be also equal to zero. We write (for simplicity, we note δu for $(\partial u/\partial s)\cdot\delta s$)

$$\begin{aligned}<\frac{\partial\mathcal{L}(s,t,u,p)}{\partial u},\delta u>=&\int_\Omega F_{,u}(s,\nabla u,\phi(s,t,\nabla u))\cdot\delta u dx\\ &+\int_\Omega\Big(A(s)\nabla\delta u\cdot\nabla p+B(s)\frac{\nabla u\cdot\nabla\delta u}{|\nabla u|}t\cdot\nabla p\Big)dx\end{aligned}\tag{4.10}$$

for all $\delta u \in H^1_{\Gamma_0}(\Omega)$ leading to the linear weak formulation of $p \in H^1_{\Gamma_0}(\Omega)$. In particular, we check that $\nabla p \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega/\Gamma_0$. The variation of $\mathcal{L}$ with respect to s is then given by (4.3). Relation (4.4) is obtained in a similar way. ■

Theorem 4.1 leads to the descent directions

$$\delta s = -F_{,s}(s, \nabla u, \phi(s,t,\nabla u)) - \Big(A_{,s}(s)\nabla u \cdot \nabla p + B_{,s}(s)|\nabla u| t \cdot \nabla p \Big), \quad \text{in} \quad \Omega \tag{4.11}$$

and

$$\delta t = -\Big(F_{,t}(s, \nabla u, \phi(s,t,\nabla u)) + B(s)|\nabla u| \nabla p \Big), \quad \text{in} \quad \Omega. \tag{4.12}$$

Moreover, since $|t| = 1$, we introduce the variable θ and write $t = (\cos(\theta), \sin(\theta))$, in which case (4.12) is replaced by

$$\delta\theta = -\sin(\theta)(\delta t)_1 + \cos(\theta)(\delta t)_2, \quad \text{in} \quad \Omega. \tag{4.13}$$

The volume constraint on s is taken into account through a classical and efficient way by introducing an explicit lagrange multiplier (we refer to [21]). The algorithm for the variable s is therefore

$$\begin{cases} s^{(0)} \in L^\infty(\Omega, [0,1]), \\ s^{(k+1)} = s^{(k)} + \epsilon f_s(s^k)\delta s^{(k)}, \quad f_s(s^{(k)}) = s^k(1 - s^{(k)}), \quad k \geq 0 \end{cases} \tag{4.14}$$

where the positive value $f_s(s^{(k)})$ is introduced in order to enforce $s^{(k+1)}$ to be in $[0,1]$ and ϵ a positive real small enough. The descent algorithm for the field θ is

$$\theta^{(0)} \in L^\infty(\Omega, \mathbb{R}), \quad \theta^{(k+1)} = \theta^{(k)} + \epsilon\delta\theta^{(k)}, \quad k \geq 0. \tag{4.15}$$

At each step k of these two algorithms, the solution u of the nonlinear system (3.71) is solved from (4.6) by using the Newton method :

$$\begin{cases} u^0 \in H^1(\Omega), u^0 = u_0 \text{ on } \Gamma_0, \\ \displaystyle\int_\Omega \Big(A(s^{(k)})\nabla u^{n+1} \cdot \nabla v + B(s^{(k)}) \frac{\nabla u^{n+1} \cdot \nabla u^n}{|\nabla u^n|} \, t^{(k)} \cdot \nabla v \Big) dx \\ \qquad\qquad = \displaystyle\int_{\Gamma_g} gv \, d\sigma, \forall n > 0, \forall v \in H^1_{\Gamma_0}(\Omega). \end{cases} \tag{4.16}$$

This formulation and the linear equation (4.5) for p are solved using continuous finite elements of order one approximating the space $H^1_{\Gamma_0}(\Omega)$ by the following finite dimensional space :

$$H^1_{\Gamma_0,h}(\Omega) = \{v_h, v_h \in C^0(\overline{\Omega}), v_h|_Q \in Q_1(Q), \forall Q \in Q_h, v_h = 0 \text{ on } \Gamma_0\} \tag{4.17}$$

where $Q_1(Q)$ denotes the space of polynomial functions of degree ≤ 1 on Q, the notation $(Q_h)_{h>0}$ stands for a regular family of quadrangulations characterized by the space step h such that $\overline{\Omega} = \cup_{Q \in Q_h} Q$. We highlight that the corresponding stiffness matrix is identical for (4.5) and (4.16). The lips of the crack γ, assumed rectilinear, are composed of edges of elements in Q_h. Besides, as proved in [12] (Theorem 4.2 page 96), (4.17) implies the *a priori* estimation $|g_{\boldsymbol{\psi}} - g_{\boldsymbol{\psi},h}| = O(h^{1-\eta}), \forall \eta > 0$ if $g_{\boldsymbol{\psi},h}$ designates the numerical approximation of the energy release rate $g_{\boldsymbol{\psi}}$.

4.2. Numerical experiments. For simplicity, we consider in the sequel the domain $\Omega = (0,1)^2$ with the crack $\gamma = [0.5,1] \times \{a\}$ $(a \in (0,1))$. Γ_0 is divided into two parts $\Gamma_{0,1} \cup \Gamma_{0,2}$: $\Gamma_{0,1} = \{0\} \times [0,1] = \{\boldsymbol{x} = (x_1,x_2) \in \mathbb{R}^2, x_1 = 0, x_2 \in [0,1]\}$ where $u_0 = 0$ and $\Gamma_{0,2} = \{1\} \times [0.5,0.8]$ where $u_0 = 0.5$. There is not normal load : $\Gamma_g = \emptyset$.

The examples that we describe in the sequel concern the non diagonal case (2.8) which imposes the material to be constant around $\boldsymbol{F}$, $\mathcal{D} = \{\boldsymbol{x} \in \Omega, \|\boldsymbol{x} - \boldsymbol{F}\| \leq r_3\}$ (see Remark 2.4, item i)). In order to limit the measure of this set, we take r_3 small enough with respect to the size of the domain, precisely $r_3 = 0.05$. The radial function $\boldsymbol{\psi} = (\psi_1, 0)$ is then defined by (2.9) with $r_1 = 0.015$ and $r_2 = 0.045 < r_3$. Therefore, ψ_1 is constant equal to $\nu_{F,1} = -1$ on $\{\boldsymbol{x} \in \Omega, \|\boldsymbol{x} - \boldsymbol{F}\| \leq r_1\}$ (we recall that this permits to avoid the singularity of u on $\boldsymbol{F}$). These small values limit the measure of $\mathcal{D}$ but enforces a very fine mesh where $\psi_1 > 0$ in order to have an accurate approximation of $g_{\boldsymbol{\psi}}$. We therefore use a non uniform regular quadrangulation $(Q_h)_{h>0}$ of Ω with a refinement around the crack point. For $\boldsymbol{F} = (1/2, 1/2)$, an example is given on Figure 4.1, corresponding to 52×52 elements (and 2916 degrees of freedom). In practice, 20×20 elements on $\{\boldsymbol{x} \in \Omega, \|\boldsymbol{x} - \boldsymbol{F}\| \leq r_2\}$ are sufficient to obtain an accurate and invariant (with respect to r_1 and r_2) approximation of the energy release rate. In the sequel, a mesh composed of 82×82 quadrangles will be used.

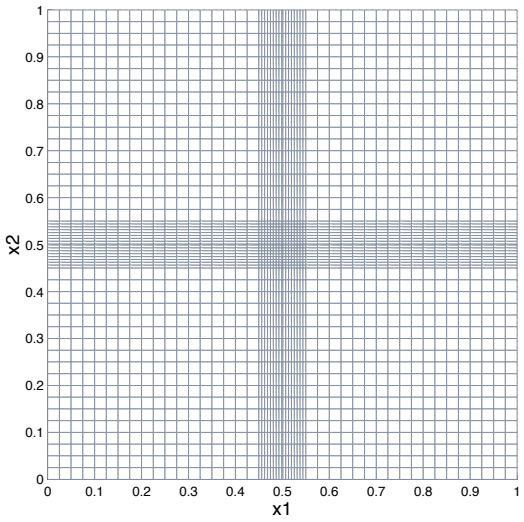

FIG. 4.1. *Example of quadrangulation of the unit square with a refinement on the support of the radial function ψ_1 (52×52 finite elements) around the point $\boldsymbol{F} = (1/2, 1/2)$.*

We discuss the optimal distribution of the two materials with respect to the values of (α, β) and a. Let us first take $(\alpha, \beta) = (1, 2)$ and $L = 2/5$ and $a = 1/2$ so that $\boldsymbol{F} = (1/2, 1/2)$. The iso-value of the density s^{opt} obtained after 1000 iterations is depicted on Figure 4.2. The algorithm is initialized with $\theta^{(0)} = 0$ on Ω and $s^{(0)} = 0$ on $\mathcal{D} \cup \partial\Omega$ and constant elsewhere which does not privilege any location for ω. The constant is determined in order to satisfy the volume constraint. At the convergence, the cost is $I(s^{opt}, t^{opt}) = 2.87 \times 10^{-2}$. This distribution permits to divide by three the cost from the initial guess. The evolution of the cost with respect to the iteration is given on Figure 4.3-Left. At each iteration, around five iterations of the Newton algorithm permit to solve the nonlinear system (3.71) (the Newton algorithm is stopped as soon

as the residual is lower than 10^{-10}). The corresponding solution is given on Figure 4.3-Right. As expected the soft material is located around the part of the boundary where the displacement is imposed. This has the effect to absorb partially the load and reduce his influence on the crack zone. We also observe that the density is almost a characteristic function, i.e. a $(0,1)$ function. We check *a posteriori* that the relation (3.62) holds : we obtain

$$\|(\rho - \lambda^-(s^{opt})\lambda) \cdot (\rho - \lambda^+(s^{opt})\lambda)\|_{L^2(\Omega)} \approx 1.32 \times 10^{-6}. \tag{4.18}$$

Moreover, we obtain

$$\|\rho - \lambda^+(s^{opt})\lambda\|_{L^2(\Omega)} \approx 3.13 \times 10^{-4}, \quad \|\rho - \lambda^-(s^{opt})\lambda\|_{L^2(\Omega)} \approx 4.21 \times 10^{-3}. \tag{4.19}$$

We recall that $\lambda^- = \lambda^+$ for a characteristic function. We also observe that we obtain a similar cost ($I(s^{opt}, t^{opt}) \approx 2.84 \times 10^{-2}$) when the constraint $s = 0$ is relaxed on the boundary, in agreement with Remark 2.5. The corresponding distribution is given on Figure 4.4.

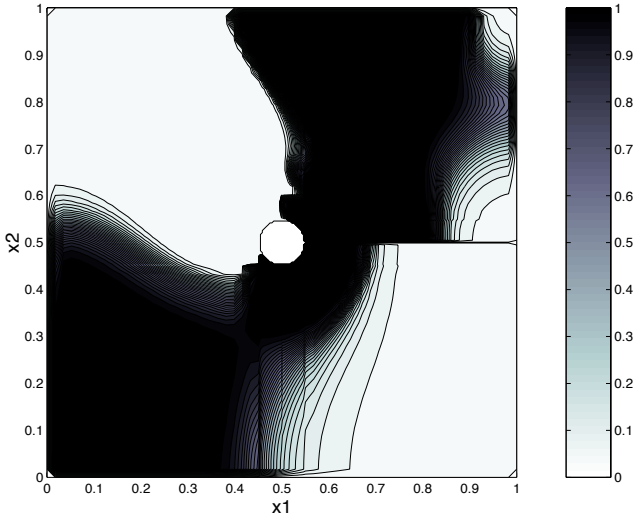

FIG. 4.2. $(\alpha, \beta) = (1, 2)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - *Iso-value of the density* s^{opt} *on the crack domain* Ω *with* $s^{opt} = 0$ *on* $\partial\Omega$.

As already observed in several different situations ([26] for the heat equation and [20, 24, 25] for the wave equation), the results are qualitatively different when the gap $\beta - \alpha$ is greater. Figure 4.5 gives the iso-values of the optimal density for $(\alpha, \beta) = (1, 10)$. The cost is $I(s^{opt}, t^{opt}) \approx 1.15 \times 10^{-2}$. This (local) optimal density is no more a $(0,1)$-function. The soft material α is however mainly concentrated around the point $\boldsymbol{F}$ and the part of the boundary $\Gamma_{0,2}$ where u is imposed. This observation justifies the need of relaxation for this problem. In this case, the convergence of the algorithm is slower (Figure 4.6-Left). Moreover, at each step, the Newton method requires more iterations (around 8): the term $B(s)$ in (3.71) is greater in that case so that the nonlinear term is not negligible with respect to the linear one. Moreover, as expected, the equality (3.62) still holds

$$\|(\rho - \lambda^-(s^{opt})\lambda) \cdot (\rho - \lambda^+(s^{opt})\lambda)\|_{L^2(\Omega)} \approx 1.32 \times 10^{-5} \tag{4.20}$$

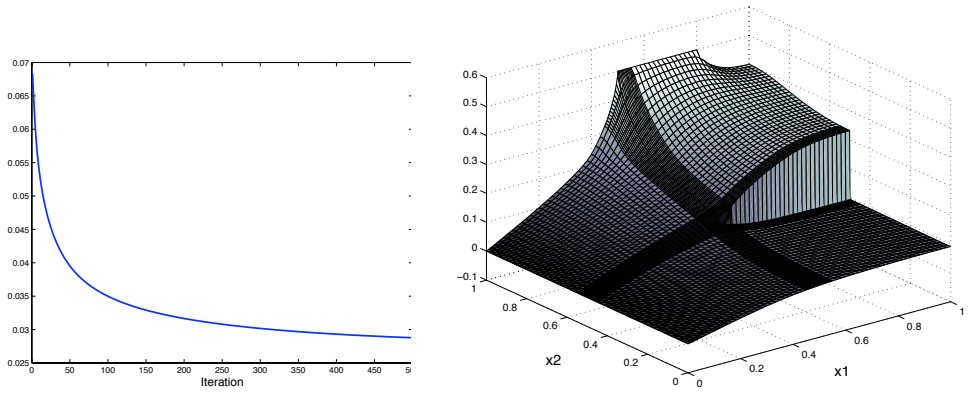

FIG. 4.3. $(\alpha, \beta) = (1, 2)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - *Evolution of the relaxed cost* $I(s^{(k)}, t^{(k)})$ *w.r.t the iteration* (***Left***) *and final solution* u *on* Ω (***Right***).

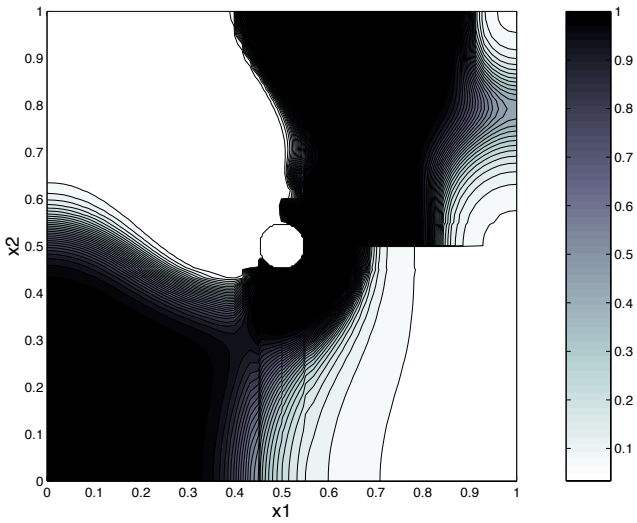

FIG. 4.4. $(\alpha, \beta) = (1, 2)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - *Iso-value of the density* s *on the crack domain with* s *free on* $\partial\Omega$.

but not for the arithmetic nor the harmonic mean :

$$\|\rho - \lambda^+(s^{opt})\lambda\|_{L^2(\Omega)} \approx 8.21 \times 10^{-1}, \quad \|\rho - \lambda^-(s^{opt})\lambda\|_{L^2(\Omega)} \approx 4.09 \times 10^{-1}. \quad (4.21)$$

If we assume *a priori* for instance the arithmetic mean, i.e. if we simply replace in (P) $\mathcal{X}_\omega$ by s and optimize with respect to s, then we obtain a greater cost equal to 4.39×10^{-2}. The corresponding solution u on the crack domain is depicted on Figure 4.6-Right.

It is then necessary to associate with this optimal composite material a workable shape ω, i.e. to construct a sequence of characteristic function, say $\mathcal{X}_{\omega^{(k)}}$, for which $g_\psi(\mathcal{X}_{\omega^{(k)}})$ converges toward $I(s^{opt}, t^{opt})$. A simple approach, using a local mean argument on s is proposed in [20] to approximate such a sequence. Further, it would be interesting to use the information of the normal of the first-order laminate at each

point given by $\lambda_\beta - \lambda_\alpha$ (we refer to [29] for such analysis in the context of Homogeneisation): iso-values of the vector $\lambda_\beta - \lambda_\alpha$ are given in Figure 4.7.

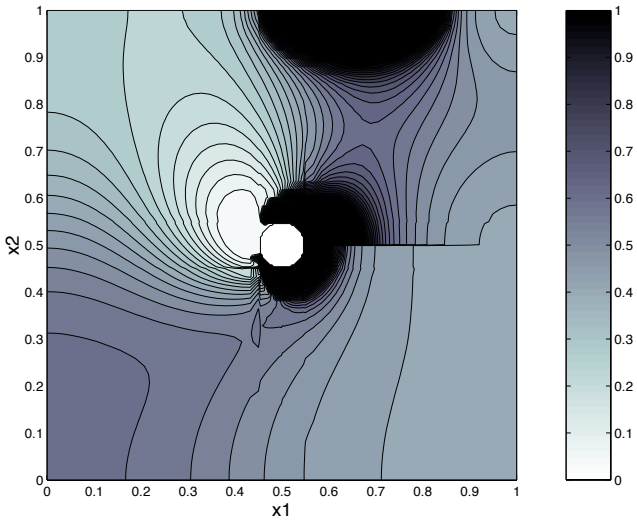

FIG. 4.5. $(\alpha, \beta) = (1, 10)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - *Iso-values of the density s on the crack domain.*

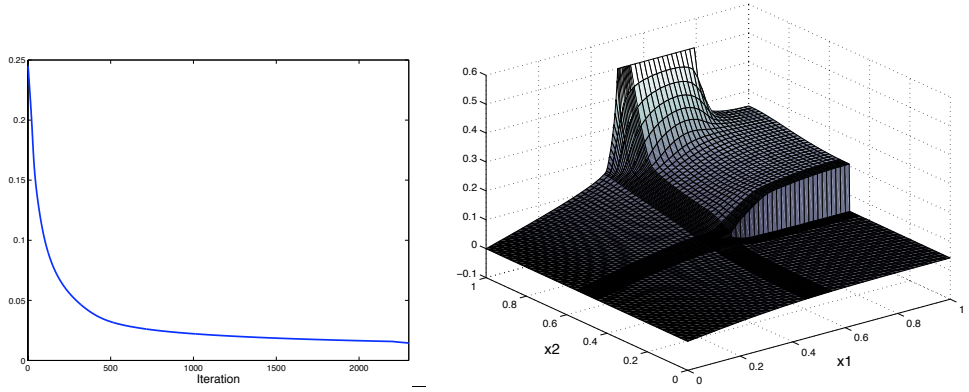

FIG. 4.6. $(\alpha, \beta) = (1, 10)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - *Relaxed cost $I(s^{(k)}, t^{(k)})$ w.r.t. the iteration (**Left**) and final solution u on Ω (**Right**).*

Finally, similar results are observed for different value of a and L vary. For $a = 1/3$ corresponding to $\boldsymbol{F} = (1/2, 1/3)$, Figure 4.8 represents the optimal density obtained for the volume fraction $L = 2/5$ and $L = 1/5$. For $L = 2/5$, we check that the cost $I(s, t) \approx 1.02 \times 10^{-2}$ is lower in that case since the crack point is far away from the load support $\Gamma_{0,2}$.

We also observe that the energy release rate is reduced but not arbitrarily small in spite of the important degree of freedom contained by the shape of ω. Therefore the singularities are not cancelled. We suspect that this is due to our mechanical assumption around the point $\boldsymbol{F}$. This also suggests, quite surprisingly, that the optimal distribution around the crack point is not composed only of the harder material.

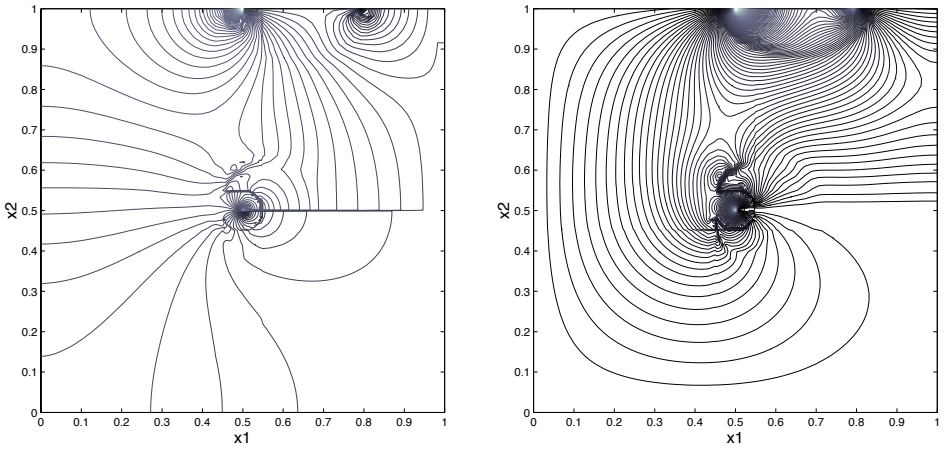

FIG. 4.7. $(\alpha, \beta) = (1, 10)$- $L = 2/5$; $\boldsymbol{F} = (1/2, 1/2)$ - *Iso-values of the components of the vector* $\lambda_\beta - \lambda_\alpha$.

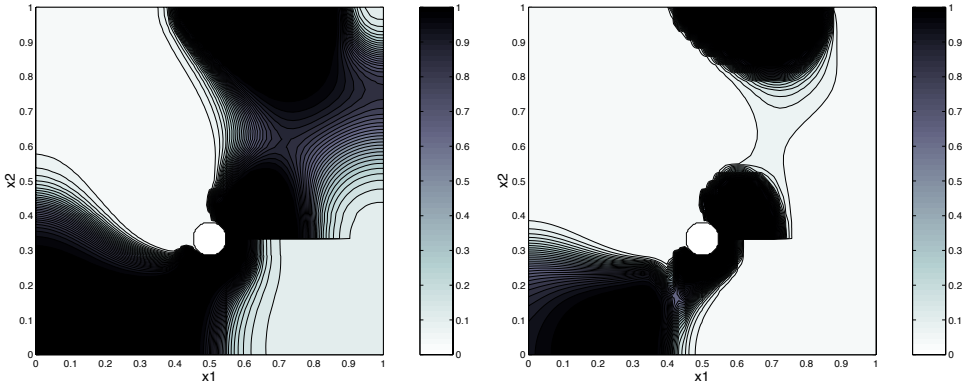

FIG. 4.8. $(\alpha, \beta) = (1, 2)$- $\boldsymbol{F} = (1/2, 1/3)$ - *Iso-values of the density s for* $L = 2/5$ ***(Left)*** *and* $L = 1/5$ ***(Right)***.

5. Concluding remarks. To our knowledge, this work is the first one which attempts to minimize the energy release rate, and therefore to control the crack growth, with respect to the conductivity coefficient. This energy release rate presents the originality to be expressed as a scalar product in terms of a non definite positive matrix, in contrast with usual examples such as the energy itself. This apparent difficulty in terms of relaxation is compensated by the mechanical assumption around the crack point. Thus, the variational non-convex approach coupled with Young measures permits to derive an explicit relaxed formulation (RP) of the optimal design problem, involving a non Caratheodory quasi-convexification. Moreover, the optimal measure is a first order laminate. Then, following [33], (RP) is transformed into an equivalent relaxed formulation $(\underline{RP})$ involving an original nonlinear divergence form system. The numerical experiments suggest that an optimal distribution permits to reduce significantly (with respect to an isotropic one) the cost. However, the optimal cost is not arbitrarily small so that the singularities around the crack tip are not cancelled in contrast with [17] where the control variable is an additional boundary

load. This phenomenon is very likely due to the condition (necessary in our context) which imposes the conductivity to be constant around the crack tip.

This preliminary work would merit to be enriched in several directions. For instance, it is worth to replace the conductivity system by the elastic one (and thus take into account the contact condition on the crack γ as it is done in [17]). For the elasticity operator, the full relaxation process is still an open problem. However, the fact that the integrand of the cost $g_{\boldsymbol{\psi}}$ is non zero only where the material is uniform may be helpful in the search of div-curl laminates as it is in this work. Moreover, it would be interesting to obtain an optimal distribution independent of the normal load $g \in L^2(\Gamma_g)$ and thus consider an inf-sup problem of the type $\inf_{\mathcal{X}_\omega} \sup_{(g,\Gamma_g)} g_{\boldsymbol{\psi}}$. Similarly, in view of the growth of the point $\boldsymbol{F}$, it would be interesting to minimize the rate independently of the length of the crack, assumed straight. Finally, we also plan in the near future to compare the numerical results derived from this relaxation approach with a more direct one based on level set method (we refer [4, 21]). In this direction, we mention the recent work [2] in the framework of damage mechanic.

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