Inverse problems for linear hyperbolic equation via mixed formulations

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Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) and $T > 0$.

$$\begin{cases} Ly := y_{tt} - \nabla \cdot (c(x)\nabla y) + d(x,t)y = f, & (x,t) \in Q_T := \Omega \times (0,T) \\ y = 0, & (x,t) \in \Gamma_T := \partial\Omega \times (0,T) \\ (y(\cdot,0), y_t(\cdot,0)) = (y_0, y_1), & x \in \Omega. \end{cases} \quad (1)$$

$c \in C^1(\overline{\Omega}, \mathbb{R}))$ $c(x) \geq c_0 > 0$ in $\overline{\Omega}$, $d \in L^\infty(Q_T)$, $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$;
$f \in L^2(Q_T)$.
Let $\omega \subset \Omega$ and $q_T := \omega \times (0,T) \subset Q_T$.

(IP)-Given an element $y_{obs} \in L^2(q_T)$, find y the solution of (1) such that
$$y \equiv y_{obs} \quad in \quad q_T.$$

From a "good" *measurement* y_{obs} on q_T, we want to recover y solution of (1).

From the unique continuation property for (1), if q_T satisfies some geometric conditions, then the state y corresponding to y_{obs} is unique.

Objective - Find a convergent approximation of the solution

Problem statement

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Most natural approach: Least-squares method

The most natural (and widely used in practice) approach consists to introduce a least-squares type technic, i.e. consider the extremal problem

$$(\mathcal{IP}) \quad \begin{cases} \text{minimize} \quad J(y_0, y_1) := \dfrac{1}{2}\|y - y_{obs}\|^2_{L^2(q_T)} \\ \text{subject to} \quad (y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(Q_T) \\ \text{where} \quad y \quad \text{solves} \quad (1) \end{cases} \tag{2}$$

A minimizing sequence $(y_0, y_1)_{(k>0)}$ is defined in term of the solution of an adjoint problem.

A difficulty, when one wants to prove the convergence of a discrete approximation : it is not possible to minimize over a discrete subspace of $\{y; Ly - f = 0\}$: If $\dim(Y_h) < \infty$, $\{y_h \in Y_h \subset Y : Ly_h - f = 0\}$ is 0 or empty

The minimization procedure first requires the discretization of J and of the system (1);

This raises the issue of uniform coercivity property of the discrete functional with respect to the approximation parameter h.

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This raises the issue of uniform coercivity property of the discrete functional with respect to the approximation parameter h.

[Auroux-Blum 2005],[Chapelle,Cindea,Moireau,2012], [Ramdani-Tucsnak 2011], etc...

Define a dynamic

$$L\overline{y} = G(y_{obs}, q_T) \quad \overline{y}(\cdot, 0) \quad \text{fixed}$$

such that

$$\|\overline{y}(\cdot, t) - y(\cdot, t)\|_{N(\Omega)} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

The reversibility of the wave equation then allows to recover y for any time.

But, for the same reasons, on a numerically point of view, this method requires to prove uniform discrete observability properties.

[Klibanov, Beilina 20xx], [Bourgeois, Darde 2010]

X, D Hilbert spaces - $P : X \to D$, P linear continuous, $Ker(P) = \{0\}$
$\varepsilon > 0$. For $d \in D$, find $y \in Y$ s.t. $Py = d$:

QR$_\varepsilon$ method : for $d \in D$, find $y_\varepsilon \in Y$ such that

$$(Py_\varepsilon, Py) + \varepsilon(y_\varepsilon, y)_Y = (d, Py), \quad \forall y \in Y$$

Here, $d = (f, y_{obs})$ - $Py = (Ly, y_{q_T})$

$$\inf_{y \in \mathcal{Y}_d} J_\varepsilon(y) := \frac{1}{2}\|Ly - f\|^2_{L^2(Q_T)} + \frac{\varepsilon}{2}\|y\|^2_{\mathcal{A}} \tag{3}$$

- $\mathcal{A}$ denotes a functional space which gives a meaning to the first term
- $\varepsilon > 0$ a Tikhonov parameter which ensures the well-posedness
- $\mathcal{Y}_d$ a subset of $\mathcal{A}$ involving the data of the problem (for instance the observation y_{obs} on q_T, or some Cauchy data on the boundary).

Klibanov and co-workers approach: Quasi-reversibility for ill-posed problem

[Klibanov, Beilina 20xx], [Bourgeois, Darde 2010]

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Main assumption: a generalized obs. inequality

Without loss of generality, $f \equiv 0$.
We consider the vectorial space Z defined by

$$Z := \{y : y \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)), Ly \in L^2(Q_T)\}. \quad (4)$$

and then introduce the following hypothesis :

Hypothesis

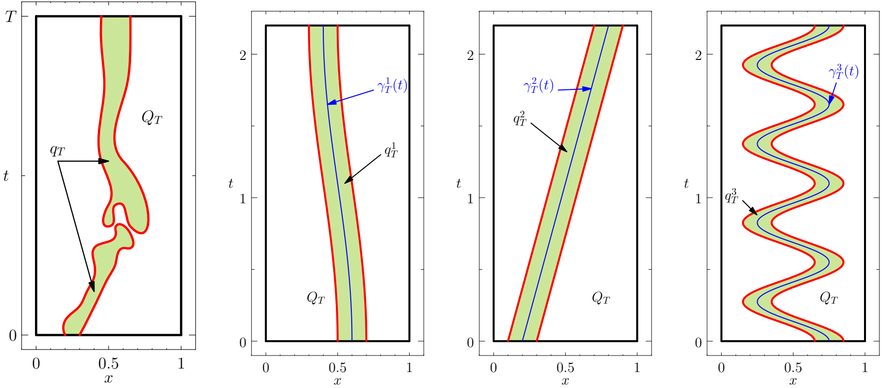
There exists a constant $C_{obs} = C(\omega, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)})$ such that the following estimate holds :

$$(\mathcal{H}) \qquad \|y(\cdot,0), y_t(\cdot,0)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq C_{obs}\Big(\|y\|^2_{L^2(q_T)} + \|Ly\|^2_{L^2(Q_T)}\Big), \quad \forall y \in Z. \quad (5)$$

hold true if (ω, T, Ω) satisfies a geometric optic condition. "Any characteristic line starting at the point $x \in \Omega$ at time $t = 0$ and following the optical geometric laws when reflecting at $\partial\Omega$ must meet q_T".

$$\|z\|^2_{L^2(Q_T)} \leq C_{\Omega,T}\Big(C_{obs}\|z\|^2_{L^2(q_T)} + (1 + C_{obs})\|Lz\|^2_{L^2(Q_T)}\Big) \quad \forall z \in Z. \quad (6)$$

[Castro-Cindea-Münch, SICON 2014],
In 1D, the observability inequality also holds for non cylindrical domains.



Time dependent domains $q_T \subset Q_T = \Omega \times (0, T)$

[Lebeau et al, 20xx] for $N \geq 1$

Generalized Observability inequality: weaker hypothesis

Then, within this hypothesis, for any $\eta > 0$, we define on Z the bilinear form

$$(y,\overline{y})_Z := \iint_{q_T} y\,\overline{y}\,dxdt + \eta \iint_{Q_T} Ly L\overline{y}\,dxdt, \qquad \|y\|_Z := \sqrt{(y,y)_Z} \quad \forall y,\overline{y} \in Z. \tag{7}$$

$(Z, \|\cdot\|)$ is a Hilbert space.
Then, we consider the following extremal problem :

$$(\mathcal{P}) \quad \begin{cases} \inf J(y) := \frac{1}{2}\|y - y_{obs}\|^2_{L^2(q_T)} + \frac{r}{2}\|Ly\|^2_{L^2(Q_T)}, \quad r \geq 0 \\ \text{subject to} \quad y \in W := \{y \in Z;\ Ly = 0 \text{ in } L^2(Q_T)\} \end{cases}$$

$(\mathcal{P})$ is well posed : J is continuous over W, strictly convex and $J(y) \to +\infty$ as $\|y\|_W \to \infty$.

The solution of $(\mathcal{P})$ in W does not depend on η.

From (5), the solution y in Z of $(\mathcal{P})$ satisfies $(y(\cdot,0), y_t(\cdot,0)) \in H^1_0(\Omega) \times L^2(\Omega)$, so that problem $(\mathcal{P})$ is equivalent to the minimization of J w.r.t $(y_0, y_1) \in H^1_0(\Omega) \times L^2(\Omega)$.

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Direct approach

In order to solve $(\mathcal{P})$, we have to deal with the constraint equality which appears W. We introduce a Lagrange multiplier $\lambda \in \Lambda := L^2(Q_T)$ and the following mixed formulation: find $(y, \lambda) \in Z \times \Lambda$ solution of

$$\begin{cases} a_r(y, \overline{y}) + b(\overline{y}), \lambda) & = & l(\overline{y}), & \forall \overline{y} \in Z \\ b(y, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in \Lambda, \end{cases} \tag{8}$$

where

$$a_r : Z \times Z \to \mathbb{R}, \quad a_r(y, \overline{y}) := \iint_{q_T} y\, \overline{y}\, dxdt + r \iint_{Q_T} Ly\, L\overline{y}\, dxdt, \tag{9}$$

$$b : Z \times \Lambda \to \mathbb{R}, \quad b(y, \lambda) := \iint_{Q_T} \lambda\, Ly\, dxdt, \tag{10}$$

$$l : Z \to \mathbb{R}, \quad l(y) := \iint_{q_T} y_{obs}\, y\, dxdt. \tag{11}$$

System (8) is nothing else than the optimality system corresponding to the extremal problem $(\mathcal{P})$.

Theorem

Under the hypothesis $(\mathcal{H})$,

1. *The mixed formulation (8) is well-posed.*
2. *The unique solution* $(y,\lambda) \in Z \times \Lambda$ *is the unique saddle-point of the Lagrangian* $\mathcal{L} : Z \times \Lambda \to \mathbb{R}$ *defined by*

$$\mathcal{L}(y,\lambda) := \frac{1}{2} a_r(y,y) + b(y,\lambda) - l(y).$$

3. *We have the estimate*

$$\|y\|_Y = \|y\|_{L^2(q_T)} \le \|y_{obs}\|_{L^2(q_T)}, \quad \|\lambda\|_{L^2(Q_T)} \le 2\sqrt{C_{\Omega,T} + \eta}\|y_{obs}\|_{L^2(q_T)}. \tag{12}$$

The kernel $\mathcal{N}(b) = \{y \in Z; b(y,\lambda) = 0 \quad \forall \lambda \in \Lambda\}$ coincides with W: we easily get

$$a_r(y,y) = \|y\|_Z^2, \quad \forall y \in \mathcal{N}(b) = W.$$

It remains to check the inf-sup constant property : $\exists \delta > 0$ such that

$$\inf_{\lambda \in \Lambda} \sup_{y \in Z} \frac{b(y,\lambda)}{\|y\|_Z \|\lambda\|_\Lambda} \geq \delta. \tag{13}$$

For any fixed $\lambda \in \Lambda$, we define y as the unique solution of

$$Ly = \lambda \text{ in } Q_T, \quad (y(\cdot,0), y_t(\cdot,0)) = (0,0) \text{ on } \Omega, \quad y = 0 \text{ on } \Sigma_T. \tag{14}$$

We get $b(y,\lambda) = \|\lambda\|_\Lambda^2$ and $\|y\|_Z^2 = \|y\|_{L^2(q_T)}^2 + \eta \|\lambda\|_{L^2(Q_T)}^2$.

The estimate $\|y\|_{L^2(q_T)} \leq \sqrt{C_{\Omega,T}} \|\lambda\|_{L^2(Q_T)}$ implies that $y \in Z$ and that

$$\sup_{y \in Z} \frac{b(y,\lambda)}{\|y\|_Y \|\lambda\|_\Lambda} \geq \frac{1}{\sqrt{C_{\Omega,T} + \eta}} > 0$$

leading to the result with $\delta = (C_{\Omega,T} + \eta)^{-1/2}$.

Remarks

Assuming enough regularity on the solution λ, at the optimality, the Lagrange Multiplier solves

$$\begin{cases} L\lambda = -(y - y_{obs})_{1_{q_T}}, \quad \lambda = 0 \quad \text{in} \quad \Sigma_T, \\ \lambda = \lambda_t = 0 \quad \text{on } \Omega \times \{0, T\}. \end{cases} \tag{15}$$

λ (defined in the weak sense) is a null controlled solution of the wave equation through the control $-(y - y_{obs})\, 1_{\omega}$.

If y_{obs} is the restriction to q_T of a solution of (1), then λ must vanish almost everywhere.

In that case, $\sup_{\lambda \in \Lambda} \inf_{y \in Y} \mathcal{L}_r(y, \lambda) = \inf_{y \in Y} \mathcal{L}_r(y, 0) = \inf_{y \in Y} J_r(y)$ with

$$J_r(y) := \frac{1}{2} \|y - y_{obs}\|^2_{L^2(Q_T)} + \frac{r}{2} \|Ly\|^2_{L^2(Q_T)}. \tag{16}$$

The corresponding variational formulation is then : find $y \in Z$ such that

$$a_r(y, \overline{y}) = \iint_{q_T} y\, \overline{y}\, dxdt + r \iint_{Q_T} Ly\, L\overline{y}\, dxdt = l(\overline{y}), \quad \forall \overline{y} \in Z.$$

$\Longrightarrow$ QR$_\varepsilon$ method with $Py = (\sqrt{r} Ly, y\, 1_{q_T})$, $d = (0, y_{obs})$, $\varepsilon = 0$

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Stabilized mixed formulation

$$\widetilde{\Lambda} := \{\lambda \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), L\lambda \in L^2(Q_T), \lambda(\cdot,0) = \lambda_t(\cdot,0) = 0\}.$$

$$\begin{cases} \sup_{\lambda \in \widetilde{\Lambda}} \inf_{y \in Z} \mathcal{L}_{r,\alpha}(y,\lambda), \quad \alpha \in (0,1) \\ \mathcal{L}_{r,\alpha}(y,\lambda) := \mathcal{L}_r(y,\lambda) - \frac{\alpha}{2} \| L\lambda + (y - y_{obs}) 1_\omega \|^2_{L^2(Q_T)}. \end{cases}$$

Find $(y,\lambda) \in Z \times \widetilde{\Lambda}$ such that

$$\begin{cases} a_{r,\alpha}(y,\overline{y}) + b_\alpha(\overline{y},\lambda) & = & l_{1,\alpha}(\overline{y}), & \forall \overline{y} \in Y \\ b_\alpha(y,\overline{\lambda}) - c_\alpha(\lambda,\overline{\lambda}) & = & l_{2,\alpha}(\overline{\lambda}), & \forall \overline{\lambda} \in \widetilde{\Lambda}, \end{cases} \tag{17}$$

$$a_{r,\alpha} : Y \times Y \to \mathbb{R}, \quad a_{r,\alpha}(y,\overline{y}) := (1-\alpha) \iint_{q_T} y\overline{y}\, dxdt + r \iint_{Q_T} Ly\, L\overline{y}\, dxdt,$$

$$b_\alpha : Y \times \widetilde{\Lambda} \to \mathbb{R}, \quad b_\alpha(y,\lambda) := \iint_{Q_T} \lambda\, Ly\, dxdt - \alpha \iint_{q_T} y\, L\lambda\, dxdt,$$

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$$\begin{cases} a_{r,\alpha}(y,\overline{y}) + b_\alpha(\overline{y},\lambda) = l_{1,\alpha}(\overline{y}), & \forall \overline{y} \in Y \\ b_\alpha(y,\overline{\lambda}) - c_\alpha(\lambda,\overline{\lambda}) = l_{2,\alpha}(\overline{\lambda}), & \forall \overline{\lambda} \in \widetilde{\Lambda}, \end{cases} \tag{17}$$

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Proposition

Under the hypothesis $(\mathcal{H})$, for any $\alpha \in (0,1)$, the corresponding mixed formulation is well-posed. The unique pair (y,λ) in $Z \times \widetilde{\Lambda}$ satisfies

$$\theta_1 \|y\|_Z^2 + \theta_2 \|\lambda\|_{\widetilde{\Lambda}}^2 \leq \left(\frac{(1-\alpha)^2}{\theta_1} + \frac{\alpha^2}{\theta_2} \right) \|y_{obs}\|^2_{L^2(q_T)}. \tag{18}$$

with $\theta_1 := \min\left(1-\alpha, r\,\eta^{-1}\right), \theta_2 := \frac{1}{2}\min\left(\alpha, C_{\Omega,T}^{-1}\right)$.

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If the solution $(y,\lambda) \in Z \times \Lambda$ of (8) enjoys the property $\lambda \in \widetilde{\Lambda}$, then the solutions of (8) and (17) coincide.

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The results apply if the distributed observation on q_T is replaced by a Neumann boundary observation on a sufficiently large subset Σ_T of $\partial\Omega \times (0, T)$ (i.e. assuming $\frac{\partial y}{\partial \nu} = y_{obs} \in L^2(\Sigma_T)$ is known on Σ_T).

If (Q_T, Σ_T, T) satisfy some geometric condition, then there exists a positive constant $C_{obs} = C(\omega, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)})$ such that

$$\|y(\cdot,0), y_t(\cdot,0)\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \leq C_{obs}\left(\left\|\frac{\partial y}{\partial \nu}\right\|^2_{L^2(\Sigma_T)} + \|Ly\|^2_{L^2(Q_T)}\right), \quad \forall y \in Z \qquad (19)$$

It suffices to re-define the form a in by $a(y, y) := \iint_{\Sigma_T} \frac{\partial y}{\partial \nu}\frac{\partial \overline{y}}{\partial \nu}\, d\sigma dx$ and the form l by $l(y) := \iint_{\Sigma_T} \frac{\partial y}{\partial \nu} y_{obs}\, d\sigma dx$ for all $y, \overline{y} \in Z$.

The mixed formulation has a structure very closed to the one we get when we address - using the same approach - the null controllability of (1): the control of minimal $L^2(q_T)$-norm which drives to rest $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ is given by $v = \varphi \, 1_{q_T}$ where $(\varphi, \lambda) \in \Phi \times L^2(0, T; H_0^1(\Omega))$ solves

$$\begin{cases} a(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) = l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi \\ b(\varphi, \overline{\lambda}) = 0, & \forall \overline{\lambda} \in L^2(0, T; H_0^1(0,1)), \end{cases} \tag{20}$$

where

$$a : \Phi \times \Phi \to \mathbb{R}, \quad a(\varphi, \overline{\varphi}) = \iint_{q_T} \varphi(x,t)\overline{\varphi}(x,t)\, dx\, dt$$

$$b : \Phi \times L^2(0, T; H_0^1(0,1)) \to \mathbb{R}, \quad b(\varphi, \lambda) = \int_0^T < L\varphi, \lambda >_{H^{-1}, H_0^1} dt$$

$$l : \Phi \to \mathbb{R}, \quad l(\varphi) = - < \varphi_t(\cdot, 0), y_0 >_{H^{-1}(0,1), H_0^1(0,1)} + \int_0^1 \varphi(\cdot, 0)\, y_1\, dx.$$

with $\Phi = \{\varphi \in L^2(q_T), \varphi = 0 \text{ on } \Sigma_T \text{ such that } L\varphi \in L^2(0, T; H^{-1}(0,1))\}$.
[Cindea - Fernandez-Cara - Münch, COCV 2013] [Cindea- Münch, Calcolo 2014]

"Reversing the order of priority" between the constraint $y - y_{obs} = 0$ in $L^2(q_T)$ and $Ly - f = 0$ in $L^2(Q_T)$, a possibility could be to minimize the functional

$$\begin{cases} \text{minimize} \quad J(y) := \|Ly - f\|^2_{L^2(Q_T)} \\ \text{subject to } y \in Z \quad \text{and to} \quad y - y_{obs} = 0 \quad \text{in} \quad L^2(q_T) \end{cases} \tag{21}$$

via the introduction of a Lagrange multiplier in $L^2(q_T)$.

The proof of the inf-sup property : there exists $\delta > 0$ such that

$$\inf_{\lambda \in L^2(q_T)} \sup_{y \in Z} \frac{\iint_{q_T} \lambda y \, dxdt}{\|\lambda\|_{L^2(q_T)} \|y\|_Y} \geq \delta$$

of the corresponding mixed-formulation is however unclear.

This issue is solved by the introduction of a ε-term in J_ε (Klibanov-Beilina 20xx).

Lemma

Let A_r be the linear operator from $L^2(Q_T)$ into $L^2(Q_T)$ defined by

$$A_r\lambda := Ly, \quad \forall \lambda \in L^2(Q_T) \quad \textit{where} \quad y \in Z \quad \textit{solves} \quad a_r(y,\overline{y}) = b(\overline{y},\lambda), \quad \forall \overline{y} \in Z.$$

For any $r > 0$, the operator A_r is a strongly elliptic, symmetric isomorphism from $L^2(Q_T)$ into $L^2(Q_T)$.

Theorem

$$\sup_{\lambda \in L^2(Q_T)} \inf_{y \in Z} \mathcal{L}_r(y,\lambda) = - \inf_{\lambda \in L^2(Q_T)} J_r^{\star\star}(\lambda) \quad + \mathcal{L}_r(y_0, 0)$$

where $y_0 \in Z$ solves $a_r(y_0,\overline{y}) = l(\overline{y}), \forall \overline{y} \in Y$ and $J_r^{\star\star} : L^2(Q_T) \to \mathbb{R}$ defined by

$$J_r^{\star\star}(\lambda) = \frac{1}{2} \iint_{Q_T} (A_r\lambda)\lambda \, dx \, dt - b(y_0,\lambda)$$

Dual of the mixed problem

Lemma

Let A_r be the linear operator from $L^2(Q_T)$ into $L^2(Q_T)$ defined by

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For any $r > 0$, the operator A_r is a strongly elliptic, symmetric isomorphism from $L^2(Q_T)$ into $L^2(Q_T)$.

Theorem

$$\sup_{\lambda\in L^2(Q_T)} \inf_{y\in Z} \mathcal{L}_r(y,\lambda) = -\inf_{\lambda\in L^2(Q_T)} J_r^{\star\star}(\lambda) + \mathcal{L}_r(y_0, 0)$$

where $y_0 \in Z$ solves $a_r(y_0, \overline{y}) = l(\overline{y}), \forall \overline{y} \in Y$ and $J_r^{\star\star} : L^2(Q_T) \rightarrow \mathbb{R}$ defined by

$$J_r^{\star\star}(\lambda) = \frac{1}{2}\iint_{Q_T} (A_r\lambda)\lambda \, dx\, dt - b(y_0, \lambda)$$

We assume again that $(\mathcal{H})$ holds. We note $Y := Z \times L^2(Q_T)$ and define on Y the bilinear form, for any $\varepsilon, \eta > 0$

$$((y,f),(\overline{y},\overline{f}))_Y := \iint_{q_T} y\,\overline{y}\,dxdt + \eta \iint_{Q_T} (Ly - f)(L\overline{y} - f)\,dxdt + \varepsilon \iint_{Q_T} f\,\overline{f}\,dxdt, \quad \forall (y,f),(\overline{y} \tag{22}$$

$$\|(y,f)\|_Y := \sqrt{((y,f),(y,f))_Y}.$$

Then, for any $\varepsilon > 0$, we consider the following extremal problem :

$$(\mathcal{P}_\varepsilon) \quad \begin{cases} \inf J_\varepsilon(y,f) := \dfrac{1}{2}\|y - y_{obs}\|^2_{L^2(q_T)} + \dfrac{\varepsilon}{2}\|f\|^2_{L^2(Q_T)}, \\ \text{subject to} \quad (y,f) \in W := \{(y,f) \in Y;\ Ly - f = 0 \text{ in } L^2(Q_T)\} \end{cases}$$

$\forall \varepsilon > 0$, $(\mathcal{P}_\varepsilon)$ is well posed.

Recovering the solution and the source f

Find $((y_\varepsilon, f_\varepsilon), \lambda_\varepsilon) \in Y \times \Lambda$ solution of

$$\begin{cases} a_\varepsilon((y_\varepsilon, f_\varepsilon), (\overline{y}, \overline{f})) + b((\overline{y}, \overline{f}), \lambda_\varepsilon) & = & l(\overline{y}, \overline{f}), & \forall (\overline{y}, \overline{f}) \in Y \\ b((y_\varepsilon, f_\varepsilon), \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in \Lambda, \end{cases} \tag{23}$$

where

$$a_\varepsilon : Y \times Y \to \mathbb{R}, \quad a_\varepsilon((y, f), (\overline{y}, \overline{f})) := \iint_{q_T} y\overline{y}\, dxdt + \varepsilon \iint_{Q_T} f\overline{f}\, dxdt, \tag{24}$$

$$b : Y \times \Lambda \to \mathbb{R}, \quad b((y, f), \lambda) := \iint_{Q_T} \lambda(Ly - f)\, dxdt, \tag{25}$$

$$l : Y \to \mathbb{R}, \quad l(y, f) := \iint_{q_T} y_{obs}\, y\, dxdt. \tag{26}$$

Theorem

Under the hypothesis $(\mathcal{H})$, the mixed formulation (23) is well-posed and

$$\|(y_\varepsilon, f_\varepsilon)\|_Y = \left(\|y_\varepsilon\|^2_{L^2(q_T)} + \varepsilon \|f_\varepsilon\|^2_{L^2(Q_T)}\right)^{1/2} \leq \|y_{obs}\|_{L^2(q_T)} \tag{27}$$

and

$$\|\lambda_\varepsilon\|_{L^2(Q_T)} \leq 2\sqrt{C_{\Omega,T} + \eta}\|y_{obs}\|_{L^2(q_T)} \tag{28}$$

Recovering the solution and the source f

- $$\delta_\varepsilon := \inf_{\lambda\in\Lambda} \sup_{(y,f)\in Y} \frac{b((y,f),\lambda)}{\|(y,f)\|_Y\|\lambda\|_\Lambda} \geq \inf_{\lambda\in\Lambda} \frac{b((0,\lambda),\lambda)}{\|(0,\lambda)\|_Y\|\lambda\|_\Lambda} = (\varepsilon+\eta)^{-1/2} \qquad (29)$$

- λ_ε is an exact controlled solution of the wave equation through the control $-(y_\varepsilon - y_{obs})\,1_\omega$

$$\begin{cases} L\lambda_\varepsilon = -(y_\varepsilon - y_{obs})_{1_\omega}, \quad \varepsilon f_\varepsilon - \lambda_\varepsilon = 0 \quad \text{in} \quad Q_T, \\ \lambda_\varepsilon = 0 \quad \text{in} \quad \Sigma_T, \\ \lambda_\varepsilon = \lambda_{\varepsilon,t} = 0 \quad \text{on}\ \Omega\times\{0,T\}. \end{cases}$$

- $\|y_\varepsilon - y_{obs}\|_{L^2(q_T)} \to 0$ as $\varepsilon \to 0 \Longrightarrow \|\lambda_\varepsilon\|_{L^2(Q_T)} \to 0$ as $\varepsilon \to 0$
- $\|\sqrt{\varepsilon} f_\varepsilon\|_{L^2(Q_T)} \leq C$ but not $\|f_\varepsilon\|_{L^2(Q_T)}$

Recovering the solution and the source f when the pair (y, f) is unique

$f(x,t) = \sigma(t)\mu(x)$

$c := 1$, $d(x,t) = d(x) \in L^p(\Omega)$, $\sigma \in C^1([0,T]), \sigma(0) \neq 0$, $\mu \in H^{-1}(\Omega)$

$$Y := \{(y,\mu); y \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)), \frac{\partial y}{\partial \nu} \in L^2(\Sigma_T), Ly - \sigma\,\mu \in L^2(Q_T)\}$$

Using [Yamamoto-Zhang 2001], if $c := 1$, $d(x,t) = d(x)$ and (Σ_T, T, Q_T) satisfies the geometric optic condition, then $\exists C > 0$

$$\|\mu\|^2_{H^{-1}(\Omega)} \leq C\left(\left\|\frac{\partial y}{\partial \nu}\right\|^2_{L^2(\Sigma_T)} + \left\|Ly - \sigma(t)\mu(x)\right\|^2_{L^1((0,T),L^2(\Omega))}\right), \quad \forall (y,\mu) \in Y \quad (30)$$

$$\sup_{\lambda \in L^2(Q_T)} \inf_{(y,\mu)\in Y} \mathcal{L}((y,\mu),\lambda) := \frac{1}{2}\left\|\frac{\partial y}{\partial \nu} - y_{obs}\right\|^2_{L^2(\Sigma_T)} + \int_{Q_T} \lambda(Ly - \sigma\mu)\,dx\,dt$$

$f(x,t) = \sigma(t)\mu(x)$
$c := 1$, $d(x,t) = d(x) \in L^p(\Omega)$, $\sigma \in C^1([0,T]), \sigma(0) \neq 0$, $\mu \in H^{-1}(\Omega)$

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Recovering the solution and the source f when the pair (y, f) is unique

$f(x,t) = \sigma(t)\mu(x)$
$c := 1,\, d(x,t) = d(x) \in L^p(\Omega),\, \sigma \in C^1([0,T]), \sigma(0) \neq 0,\, \mu \in H^{-1}(\Omega)$

$$Y := \{(y,\mu); y \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)), \frac{\partial y}{\partial \nu} \in L^2(\Sigma_T), Ly - \sigma\,\mu \in L^2(Q_T)\}$$

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$f(x,t) = \sigma(t)\mu(x)$
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Using [Yamamoto-Zhang 2001], if $c := 1$, $d(x,t) = d(x)$ and (Σ_T, T, Q_T) satisfies the geometric optic condition, then $\exists C > 0$

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$$\sup_{\lambda \in L^2(Q_T)} \inf_{(y,\mu)\in Y} \mathcal{L}((y,\mu),\lambda) := \frac{1}{2}\left\|\frac{\partial y}{\partial \nu} - y_{obs}\right\|^2_{L^2(\Sigma_T)} + \int_{Q_T} \lambda(Ly - \sigma\mu)\,dx\,dt$$

Conformal approximation of the space-time variational framework

Let Y_h and Λ_h be two finite dimensional spaces parametrized by h such that

$$Y_h \subset Y, \quad \Lambda_h \subset \Lambda, \qquad \forall h > 0.$$

Find $((y_{\varepsilon,h}, f_{\varepsilon,h}), \lambda_{\varepsilon,h}) \in Y_h \times \Lambda_h$ solution of

$$\left\{ \begin{aligned} a_{\varepsilon,r}((y_{\varepsilon,h}, f_{\varepsilon,h}), (\overline{y}_h, \overline{f}_h)) + b((\overline{y}_h, \overline{f}_h), \lambda_\varepsilon) &= l(\overline{y}_h, \overline{f}_h), && \forall (\overline{y}_h, \overline{f}_h) \in Y_h \\ b((y_{\varepsilon,h}, f_{\varepsilon,h}), \overline{\lambda}_h) &= 0, && \forall \overline{\lambda}_h \in \Lambda_h. \end{aligned} \right. \tag{31}$$

- $a_{\varepsilon,r}$ is coercive on $\mathcal{N}_h(b) \subset Y$ thanks to :

$$a_{\varepsilon,r}((y,f),(y,f)) \geq (r/\eta)\|(y,f)\|_Y^2 \quad \forall Y$$

- For any λ_h fixed in Λ_h, taking $y_h = 0$ and $f_h = \lambda_h \in \Lambda_h \subset F_h$, we get

$$\delta_{\varepsilon,h} := \inf_{\lambda_h \in \Lambda_h} \sup_{(y_h, f_h) \in Y_h} \frac{b((y_h, f_h), \lambda_h)}{\|(y_h, f_h)\|_Y \|\lambda_h\|_\Lambda} \geq 1/\sqrt{\varepsilon + \eta} \tag{32}$$

Consequently, for any fixed $h > 0$, there exists a unique couple $(y_{\varepsilon,h}, \lambda_{\varepsilon,h})$ solution of (31).

Let Y_h and Λ_h be two finite dimensional spaces parametrized by h such that

$$Y_h \subset Y, \quad \Lambda_h \subset \Lambda, \qquad \forall h > 0.$$

Find $((y_{\varepsilon,h}, f_{\varepsilon,h}), \lambda_{\varepsilon,h}) \in Y_h \times \Lambda_h$ solution of

$$\begin{cases} a_{\varepsilon,r}((y_{\varepsilon,h}, f_{\varepsilon,h}), (\overline{y}_h, \overline{f}_h)) + b((\overline{y}_h, \overline{f}_h), \lambda_\varepsilon) & = & l(\overline{y}_h, \overline{f}_h), & \forall (\overline{y}_h, \overline{f}_h) \in Y_h \\ b((y_{\varepsilon,h}, f_{\varepsilon,h}), \overline{\lambda}_h) & = & 0, & \forall \overline{\lambda}_h \in \Lambda_h. \end{cases} \tag{31}$$

- $a_{\varepsilon,r}$ is coercive on $\mathcal{N}_h(b) \subset Y$ thanks to :

$$a_{\varepsilon,r}((y,f),(y,f)) \geq (r/\eta)\|(y,f)\|_Y^2 \quad \forall Y$$

- For any λ_h fixed in Λ_h, taking $y_h = 0$ and $f_h = \lambda_h \in \Lambda_h \subset F_h$, we get

$$\delta_{\varepsilon,h} := \inf_{\lambda_h \in \Lambda_h} \sup_{(y_h, f_h) \in Y_h} \frac{b((y_h, f_h), \lambda_h)}{\|(y_h, f_h)\|_Y \|\lambda_h\|_\Lambda} \geq 1/\sqrt{\varepsilon + \eta} \tag{32}$$

Consequently, for any fixed $h > 0$, there exists a unique couple $(y_{\varepsilon,h}, \lambda_{\varepsilon,h})$ solution of (31).

Let Y_h and Λ_h be two finite dimensional spaces parametrized by h such that

$$Y_h \subset Y, \quad \Lambda_h \subset \Lambda, \qquad \forall h > 0.$$

Find $((y_{\varepsilon,h}, f_{\varepsilon,h}), \lambda_{\varepsilon,h}) \in Y_h \times \Lambda_h$ solution of

$$\left\{ \begin{array}{rcll} a_{\varepsilon,r}((y_{\varepsilon,h}, f_{\varepsilon,h}), (\overline{y}_h, \overline{f}_h)) + b((\overline{y}_h, \overline{f}_h), \lambda_\varepsilon) & = & l(\overline{y}_h, \overline{f}_h), & \forall (\overline{y}_h, \overline{f}_h) \in Y_h \\ b((y_{\varepsilon,h}, f_{\varepsilon,h}), \overline{\lambda}_h) & = & 0, & \forall \overline{\lambda}_h \in \Lambda_h. \end{array} \right. \tag{31}$$

- $a_{\varepsilon,r}$ is coercive on $\mathcal{N}_h(b) \subset Y$ thanks to :

$$a_{\varepsilon,r}((y,f),(y,f)) \geq (r/\eta)\|(y,f)\|_Y^2 \quad \forall Y$$

- For any λ_h fixed in Λ_h, taking $y_h = 0$ and $f_h = \lambda_h \in \Lambda_h \subset F_h$, we get

$$\delta_{\varepsilon,h} := \inf_{\lambda_h \in \Lambda_h} \sup_{(y_h,f_h)\in Y_h} \frac{b((y_h, f_h), \lambda_h)}{\|(y_h, f_h)\|_Y \|\lambda_h\|_\Lambda} \geq 1/\sqrt{\varepsilon + \eta} \tag{32}$$

Consequently, for any fixed $h > 0$, there exists a unique couple $(y_{\varepsilon,h}, \lambda_{\varepsilon,h})$ solution of (31).

Proposition

Let $(y_\varepsilon, f_\varepsilon, \lambda_\varepsilon)$ and $(y_{\varepsilon,h}, f_{\varepsilon,h}, \lambda_{\varepsilon,h})$ be the solution of (23) and (31) respectively. The following hold :

$$\|(y_\varepsilon, f_\varepsilon) - (y_{\varepsilon,h}, f_{\varepsilon,h})\|_Y \leq 2\left(1 + \sqrt{\frac{\eta+\varepsilon}{\eta}}\right) d((y_\varepsilon, f_\varepsilon), Y_h) + \frac{1}{\sqrt{\eta}} d(\lambda_\varepsilon, \Lambda_h)$$

$$\|\lambda_\varepsilon - \lambda_{\varepsilon,h}\|_\Lambda \leq \sqrt{\eta+\varepsilon}\left(2 + \sqrt{\frac{\eta+\varepsilon}{\eta}}\right) d((y_\varepsilon, f_\varepsilon), Y_h) + 3\sqrt{\frac{\eta+\varepsilon}{\eta}} d(\lambda_\varepsilon, \Lambda_h),$$

where $d(\lambda_\varepsilon, \Lambda_h) := \inf_{\lambda_h \in \Lambda_h} \|\lambda_\varepsilon - \lambda_h\|_\Lambda = \inf_{\lambda_h \in \Lambda_h} \|\lambda_\varepsilon - \lambda_h\|_{L^2(Q_T)}$ *and*

$$\begin{aligned} d((y_\varepsilon, f_\varepsilon), Y_h) &:= \inf_{(y_h, f_h) \in Y_h} \|(y_\varepsilon, f_\varepsilon) - (y_h, f_h)\|_Y \\ &= \inf_{(y_h, f_h) \in Y_h} \Big(\|y_\varepsilon - y_h\|^2_{L^2(Q_T)} + \varepsilon \|f_\varepsilon - f_h\|^2_{L^2(Q_T)} + \\ &\qquad \eta \|L(y_\varepsilon - y_h) - (f_\varepsilon - f_h)\|^2_{L^2(Q_T)} \Big)^{1/2}. \end{aligned}$$

Let $n_h = \dim Y_h$, $m_h = \dim \Lambda_h$ and let the real matrices $A_{\varepsilon,r,h} \in \mathbb{R}^{n_h,n_h}$, $B_h \in \mathbb{R}^{m_h,n_h}$, $J_h \in \mathbb{R}^{m_h,m_h}$ and $L_h \in \mathbb{R}^{n_h}$ be defined by

$$\begin{cases} a_{\varepsilon,r}((y_h,f_h),(\overline{y_h},\overline{f_h})) = \langle A_{\varepsilon,r,h}(\{y_h\},\{f_h\}),(\{\overline{y_h}\},\{\overline{f_h}\})\rangle_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}} \\ b((y_h,f_h),\lambda_h) = \langle B_h\{y_h\},\{\lambda_h\}\rangle_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}} \\ \iint_{Q_T} \lambda_h\overline{\lambda_h}\,dx\,dt = \langle J_h\{\lambda_h\},\{\overline{\lambda_h}\}\rangle_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}} \\ l(y_h,f_h) = \langle L_h,(\{y_h,f_h\})\rangle_{\mathbb{R}^{n_h}}, \end{cases}$$

for every (y_h,f_h), $(\overline{y_h},\overline{f_h}) \in Y_h$ and for every λ_h, $\overline{\lambda_h} \in \Lambda_h$.
The problem (31) reads as follows : find $\{y_h,f_h\} \in \mathbb{R}^{n_h}$ and $\{\lambda_h\} \in \mathbb{R}^{m_h}$ such that

$$\begin{pmatrix} A_{\varepsilon,r,h} & B_h^T \\ B_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix} (\{y_h,f_h\} \\ \{\lambda_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} L_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}. \tag{33}$$

The matrix of order $m_h + n_h$ in (33) is symmetric but not positive definite.

Choice of the space Y_h and Λ_h

The space Y_h must be chosen such that $Ly_h \in L^2(Q_T)$ for any $y_h \in Y_h$. This is guaranteed for instance as soon as y_h possesses second-order derivatives in $L^2_{loc}(Q_T)$. A conformal approximation based on standard triangulation of Q_T is obtained with spaces of functions continuously differentiable with respect to both x and t.

We introduce a triangulation $\mathcal{T}_h$ such that $\overline{Q_T} = \cup_{K\in\mathcal{T}_h} K$ and we assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family. We note $h := \max\{\mathrm{diam}(K), K \in \mathcal{T}_h\}$.

We introduce the space Φ_h as follows:

$$Z_h = \{y_h \in Z \in C^1(\overline{Q_T}) : z_h|_K \in \mathbb{P}(K) \quad \forall K \in \mathcal{T}_h,\ z_h = 0 \text{ on } \Sigma_T\}$$

where $\mathbb{P}(K)$ denotes an appropriate space of polynomial functions in x and t. We consider for $\mathbb{P}(K)$ the reduced *Hsieh-Clough-Tocher* C^1-element (Composite finite element and involves as degrees of freedom the values of $\varphi_h, \varphi_{h,x}, \varphi_{h,t}$ on the vertices of each triangle K).

We also define the finite dimensional space

$$\Lambda_h = \{\lambda_h \in C^0(\overline{Q_T}), \lambda_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h\}$$

For any $h > 0$, we have $Y_h := Z_h \times \Lambda_h \subset Y$ and $\Lambda_h \subset L^2(Q_T)$.

Choice of the space Y_h and Λ_h

The space Y_h must be chosen such that $Ly_h \in L^2(Q_T)$ for any $y_h \in Y_h$. This is guaranteed for instance as soon as y_h possesses second-order derivatives in $L^2_{loc}(Q_T)$. A conformal approximation based on standard triangulation of Q_T is obtained with spaces of functions continuously differentiable with respect to both x and t.

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$$\Lambda_h = \{\lambda_h \in C^0(\overline{Q_T}), \lambda_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h\}$$

For any $h > 0$, we have $Y_h := Z_h \times \Lambda_h \subset Y$ and $\Lambda_h \subset L^2(Q_T)$.

The space Y_h must be chosen such that $Ly_h \in L^2(Q_T)$ for any $y_h \in Y_h$. This is guaranteed for instance as soon as y_h possesses second-order derivatives in $L^2_{loc}(Q_T)$. A conformal approximation based on standard triangulation of Q_T is obtained with spaces of functions continuously differentiable with respect to both x and t.

We introduce a triangulation $\mathcal{T}_h$ such that $\overline{Q_T} = \cup_{K\in\mathcal{T}_h} K$ and we assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family. We note $h := \max\{\mathrm{diam}(K), K \in \mathcal{T}_h\}$.

We introduce the space Φ_h as follows:

$$Z_h = \{y_h \in Z \in C^1(\overline{Q_T}) : z_h|_K \in \mathbb{P}(K) \quad \forall K \in \mathcal{T}_h,\ z_h = 0 \text{ on } \Sigma_T\}$$

where $\mathbb{P}(K)$ denotes an appropriate space of polynomial functions in x and t. We consider for $\mathbb{P}(K)$ the reduced *Hsieh-Clough-Tocher* C^1-element (Composite finite element and involves as degrees of freedom the values of $\varphi_h, \varphi_{h,x}, \varphi_{h,t}$ on the vertices of each triangle K).

We also define the finite dimensional space

$$\Lambda_h = \{\lambda_h \in C^0(\overline{Q_T}), \lambda_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h\}$$

For any $h > 0$, we have $Y_h := Z_h \times \Lambda_h \subset Y$ and $\Lambda_h \subset L^2(Q_T)$.

Proposition (BFS element for $N = 1$ - Rates of convergence for the norm Y)

Let $h > 0$ and an integer $k \leq 2$. Let $(y_\varepsilon, f_\varepsilon, \lambda_\varepsilon)$ and $(y_{\varepsilon,h}, f_{\varepsilon,h}, \lambda_{\varepsilon,h})$ be the solution of (23) and (31) respectively. If $(y_\varepsilon, f_\varepsilon)$ belongs to $H^{k+2}(Q_T) \times H^k(Q_T)$ and if λ_ε belongs to $H^k(Q_T)$, then there exists two positives constant $K_i = K_i(\|y\|_{H^{k+2}(Q_T)}, \|f\|_{H^k(Q_T)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)}, \varepsilon, \eta)$, $i = 1, 2$, independent of h, such that

$$\|(y_\varepsilon, f_\varepsilon) - (y_{\varepsilon,h}, f_{\varepsilon,h})\|_Y \leq K_1 h^k, \qquad \|\lambda_\varepsilon - \lambda_{\varepsilon,h}\|_\Lambda \leq K_2 h^k. \tag{34}$$

We write that $(y_\varepsilon - y_{\varepsilon,h})$ solves

$$\begin{cases} L(y_\varepsilon - y_{\varepsilon,h}) = (f_\varepsilon - f_{\varepsilon,h}) + (f_{\varepsilon,h} - Ly_{\varepsilon,h}) & \text{in } Q_T \\ ((y_\varepsilon - y_{\varepsilon,h}), (y_\varepsilon - y_{\varepsilon,h})_t)(0) \in H^1(\Omega) \times L^2(\Omega) \\ y_\varepsilon - y_{\varepsilon,h} = 0 \qquad \text{on } \Sigma_T. \end{cases}$$

Therefore using (6), there exists a constant $C(C_{\Omega,T}, C_{obs})$ such that

$$\|y_\varepsilon - y_{\varepsilon_h}\|_{L^2(Q_T)} \leq C(C_{\Omega,T}, C_{obs})\sqrt{3}\max(1, \frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\eta}})\|(y_\varepsilon, f_\varepsilon) - (y_{\varepsilon,h}, f_{\varepsilon,h})\|_Y.$$

Theorem (BFS element for $N = 1$ - Rate of convergence in $L^2(Q_T)$)

Assume that the hypothesis $(\mathcal{H})$ holds. Let $h > 0$ and an integer $k \leq 2$. Let $(y_\varepsilon, f_\varepsilon, \lambda_\varepsilon)$ and $(y_{\varepsilon,h}, f_{\varepsilon,h}, \lambda_{\varepsilon,h})$ be the solution of (23) and (31) respectively. If $(y_\varepsilon, f_\varepsilon)$ belongs to $H^{k+2}(Q_T) \times H^k(Q_T)$ and if λ_ε belongs to $H^k(Q_T)$, then there exists a positive constant $K = K(\|y\|_{H^{k+2}(Q_T)}, \|f\|_{H^k(Q_T)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)}, \varepsilon, \eta)$ independent of h, such that

$$\|y_\varepsilon - y_{\varepsilon,h}\|_{L^2(Q_T)} \leq C(C_{\Omega,T}, C_{obs})\max(1, \frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\eta}})Kh^k, \quad \forall h > 0. \qquad (35)$$

We write that $(y_\varepsilon - y_{\varepsilon,h})$ solves

$$\begin{cases} L(y_\varepsilon - y_{\varepsilon,h}) = (f_\varepsilon - f_{\varepsilon,h}) + (f_{\varepsilon,h} - Ly_{\varepsilon,h}) & \text{in } Q_T \\ ((y_\varepsilon - y_{\varepsilon,h}), (y_\varepsilon - y_{\varepsilon,h})_t)(0) \in H^1(\Omega) \times L^2(\Omega) \\ y_\varepsilon - y_{\varepsilon,h} = 0 \qquad \text{on } \Sigma_T. \end{cases}$$

Therefore using (6), there exists a constant $C(C_{\Omega,T}, C_{obs})$ such that

$$\|y_\varepsilon - y_{\varepsilon_h}\|_{L^2(Q_T)} \leq C(C_{\Omega,T}, C_{obs})\sqrt{3}\max(1, \frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\eta}})\|(y_\varepsilon, f_\varepsilon) - (y_{\varepsilon,h}, f_{\varepsilon,h})\|_Y.$$

Theorem (BFS element for $N = 1$ - Rate of convergence in $L^2(Q_T)$)

Assume that the hypothesis $(\mathcal{H})$ holds. Let $h > 0$ and an integer $k \leq 2$. Let $(y_\varepsilon, f_\varepsilon, \lambda_\varepsilon)$ and $(y_{\varepsilon,h}, f_{\varepsilon,h}, \lambda_{\varepsilon,h})$ be the solution of (23) and (31) respectively. If $(y_\varepsilon, f_\varepsilon)$ belongs to $H^{k+2}(Q_T) \times H^k(Q_T)$ and if λ_ε belongs to $H^k(Q_T)$, then there exists a positive constant $K = K(\|y\|_{H^{k+2}(Q_T)}, \|f\|_{H^k(Q_T)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)}, \varepsilon, \eta)$ independent of h, such that

$$\|y_\varepsilon - y_{\varepsilon,h}\|_{L^2(Q_T)} \leq C(C_{\Omega,T}, C_{obs})\max(1, \frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\eta}})Kh^k, \quad \forall h > 0. \tag{35}$$

The problem (17) becomes : find $(y_h, \lambda_h) \in Z_h \times \widetilde{\Lambda}_h$ solution of

$$\left\{ \begin{array}{rcll} a_{r,\alpha}(y_h, \overline{y}_h) + b_\alpha(\lambda_h, \overline{y}_h) & = & l_{1,\alpha}(\overline{y}_h), & \forall \overline{y}_h \in Z_h \\ b_\alpha(\overline{\lambda}_h, y_h) - c_\alpha(\lambda_h, \overline{\lambda}_h) & = & l_{2,\alpha}(\overline{\lambda}_h), & \forall \overline{\lambda}_h \in \widetilde{\Lambda}_h, \end{array} \right. \tag{36}$$

$$\widetilde{\Lambda}_h = \{\lambda \in Z_h ; \lambda(\cdot, 0) = \lambda_t(\cdot, 0) = 0\}. \tag{37}$$

Proposition (BFS element for $N = 1$ - Rates of convergence - Stabilized mixed formulation)

Let $h > 0$, let $k \leq 2$ be a positive integer and let $\alpha \in (0,1)$. Let (y, λ) and (y_h, λ_h) be the solution of (17) and (36) respectively. If (y, λ) belongs to $H^{k+2}(Q_T) \times H^{k+2}(Q_T)$, then there exists a positive constant $K = K(\|y\|_{H^{k+2}(Q_T)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)}, \alpha, r, \eta)$ independent of h, such that

$$\|y - y_h\|_Z + \|\lambda - \lambda_h\|_{\widetilde{\Lambda}} \leq K h^k. \tag{38}$$

$$(\textbf{EX1}) \quad \begin{cases} y_0(x) = 16x^2(1-x)^2, \\ y_1(x) = (3x - 4x^3)\,1_{(0,0.5)}(x) + (4x^3 - 12x^2 + 9x - 1)\,1_{(0.5,1)}(x), \end{cases} x \in (0,1)$$

and $f = 0$. The corresponding solution of (1) with $c \equiv 1, d \equiv 0$ is given by

$$y(x,t) = \sum_{k>0} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sqrt{2} \sin(k\pi x)$$

with

$$a_k = \frac{32\sqrt{2}(\pi^2 k^2 - 12)}{\pi^5 k^5}((-1)^k - 1), \quad b_k = \frac{48\sqrt{2}\sin(\pi k/2)}{\pi^4 k^4}, \quad k > 0.$$

Numerical illustration - $N = 1$ - $\varepsilon = 0$

$T = 2$ - $r = h^2$ - $\omega = (0.1, 0.3)$ - BFS

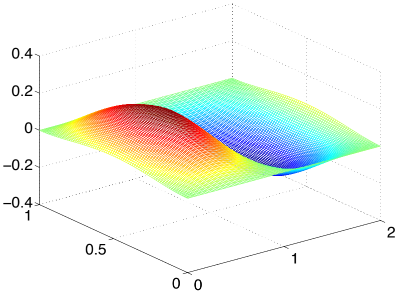
h	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\frac{\Vert y-y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}}$	9.55×10^{-2}	4.58×10^{-2}	2.24×10^{-2}	1.10×10^{-2}	5.52×10^{-3}
$\frac{\Vert y-y_h\Vert_{L^2(q_T)}}{\Vert y\Vert_{L^2(q_T)}}$	8.35×10^{-2}	4.28×10^{-2}	2.16×10^{-2}	1.09×10^{-2}	5.51×10^{-3}
$\Vert Ly_h\Vert_{L^2(Q_T)}$	5.62×10^{-3}	3.21×10^{-3}	1.78×10^{-3}	9.99×10^{-4}	8.54×10^{-4}
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	2.67×10^{-5}	1.37×10^{-5}	6.89×10^{-6}	3.44×10^{-6}	1.76×10^{-6}

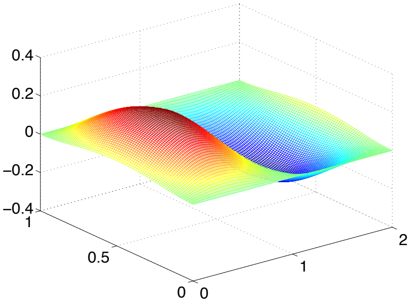
$$\frac{\Vert y - y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}} = \mathcal{O}(h^{1.03}), \quad \frac{\Vert y - y_h\Vert_{L^2(q_T)}}{\Vert y\Vert_{L^2(q_T)}} = \mathcal{O}(h^{0.98}), \quad \Vert\lambda_h\Vert_{L^2(Q_T)} = \mathcal{O}(h^{0.98}). \tag{39}$$

The L^2-norm of Ly_h do also converges to 0 with h, with a lower rate:

$$\Vert Ly_h\Vert_{L^2(Q_T)} = \mathcal{O}(h^{0.71}). \tag{40}$$

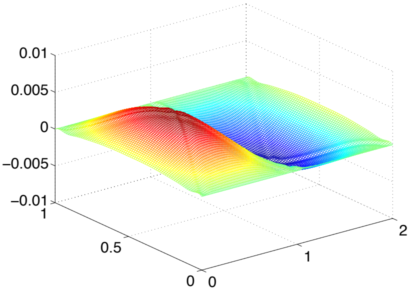
$r = h^2$ - $h = 0.0125$

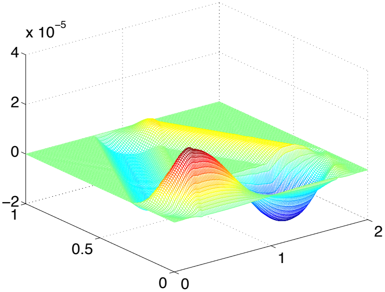




y and y_h in Q_T

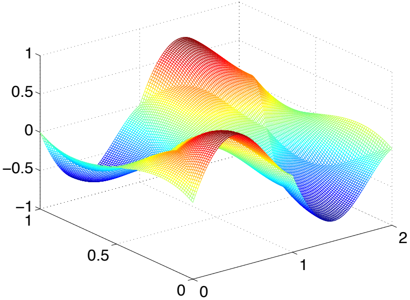
Example 1 - $N = 1$

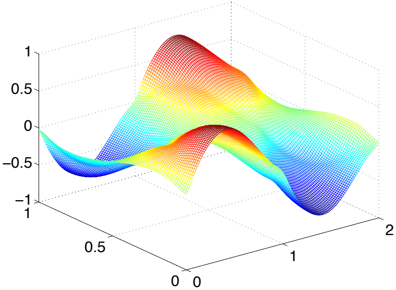




$$y - y_h \quad \text{and} \quad \lambda_h \quad \text{in} \quad Q_T$$

$r = h^2$ - $h = 0.0125$

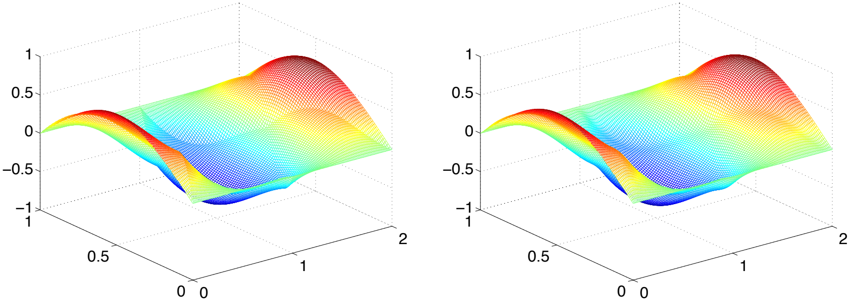




y_x and $(y_x)_h$ in Q_T

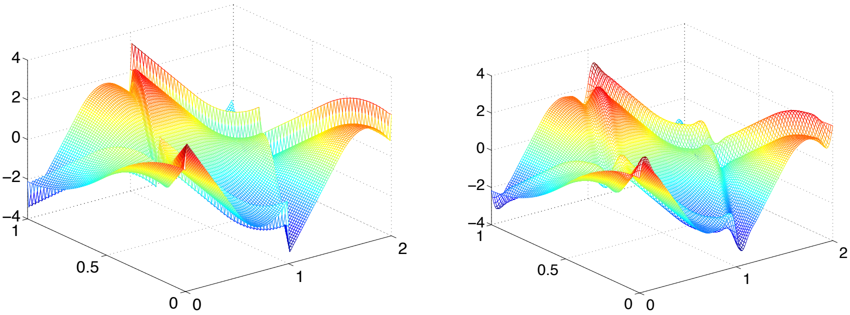
Example 1 - $N = 1$

$r = h^2$ - $h = 0.0125$



$y_t \quad \text{and} \quad (y_t)_h \quad \text{in} \quad Q_T$

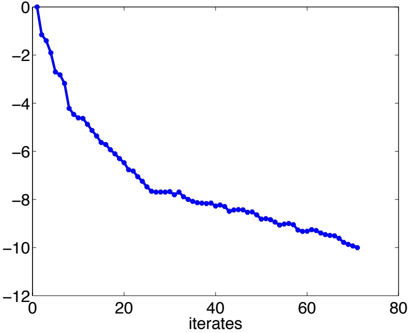
$r = h^2$ - $h = 0.0125$



$$y_{xt} \quad \text{and} \quad (y_{xt})_h \quad \text{in} \quad Q_T$$

Example 1 - Minimization of $J^{\star\star}$

h	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
κ	1.4×10^{10}	4.6×10^{11}	1.3×10^{13}	4.2×10^{14}	1.3×10^{16}
card($\{\lambda_h\}$)	861	3 321	13 041	51 681	205 761
$\sharp$ CG iterates	27	42	70	96	90



log_{10} of the residus w.r.t. iterates

Example 2 - $N = 1$

$$\textbf{(EX2)} \quad y_0(x) = 1 - |2x - 1|, \quad y_1(x) = 1_{(1/3,2/3)}(x), \qquad x \in (0,1)$$

in $H_0^1 \times L^2$ for which the Fourier coefficients are

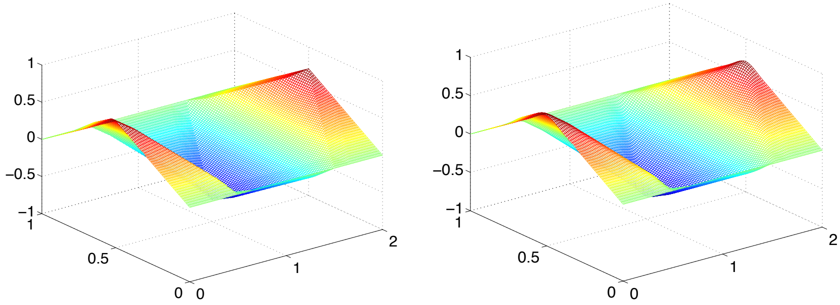
$$a_k = \frac{4\sqrt{2}}{\pi^2 k^2} \sin(\pi k/2), \quad b_k = \frac{1}{\pi k}(\cos(\pi k/3) - \cos(2\pi k/3)), \quad k > 0$$

h	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\frac{\|y-y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}}$	1.01×10^{-1}	4.81×10^{-2}	2.34×10^{-2}	1.15×10^{-2}	5.68×10^{-3}
$\frac{\|y-y_h\|_{L^2(q_T)}}{\|y\|_{L^2(q_T)}}$	1.34×10^{-1}	5.05×10^{-2}	2.37×10^{-2}	1.16×10^{-2}	5.80×10^{-3}
$\|Ly_h\|_{L^2(Q_T)}$	7.18×10^{-2}	6.59×10^{-2}	6.11×10^{-2}	5.55×10^{-2}	5.10×10^{-2}
$\|\lambda_h\|_{L^2(Q_T)}$	1.07×10^{-4}	4.70×10^{-5}	2.32×10^{-5}	1.15×10^{-5}	5.76×10^{-6}
$\sharp$ CG iterates	29	46	83	133	201

$$\|Ly_h\|_{L^2(Q_T)} = \mathcal{O}(h^{0.123}). \tag{41}$$

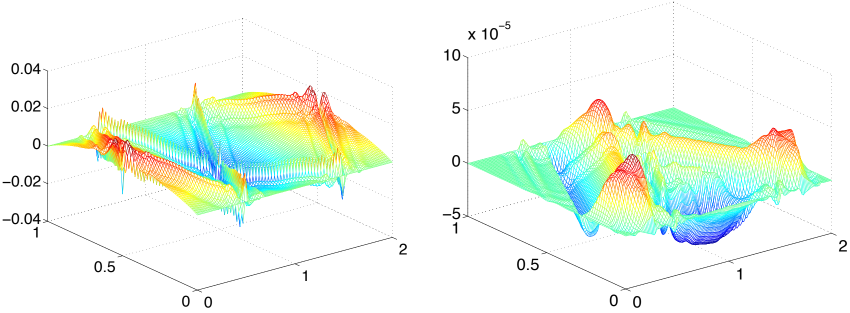
Enough to guarantee the convergence of y_h toward a solution of the wave equation: recall (see Remark **??**) that then $\|Ly_h\|_{L^2(0,T;H^{-1}(0,1))} = \mathcal{O}(h^{1.123})$.

Example 2 - $N = 1$



y and y_h in Q_T

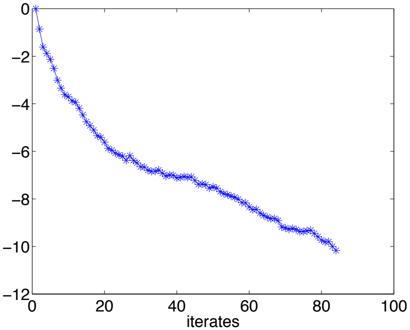
Example 2 - $N = 1$



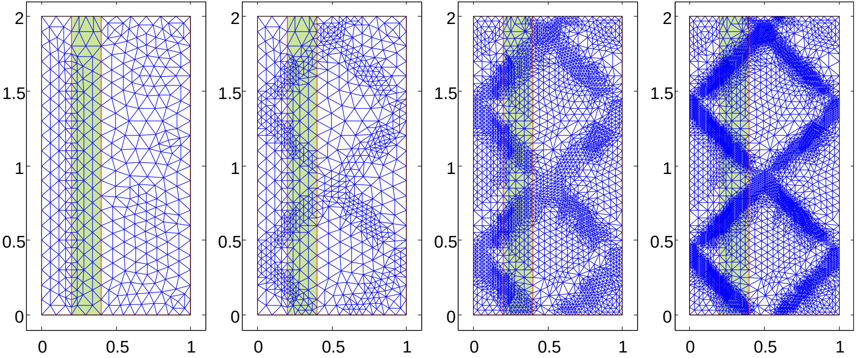
$$y - y_h \quad \text{and} \quad \lambda_h \quad \text{in} \quad Q_T$$

Example 2 - $N = 1$

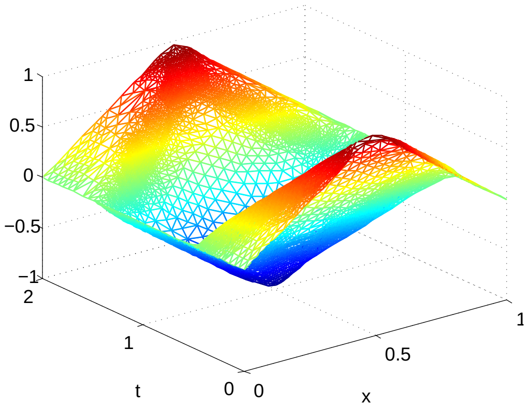
h	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\sharp$ CG iterates	29	46	83	133	201



log_{10} of the residus w.r.t. iterates

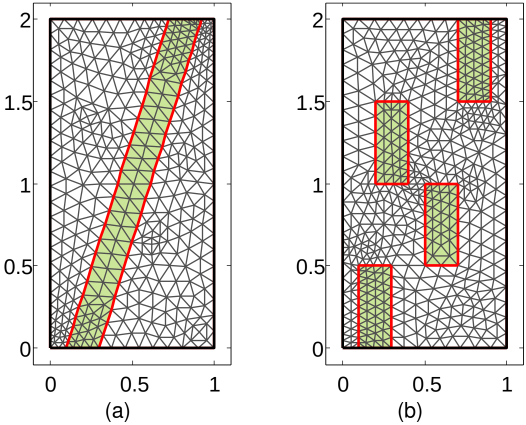


Iterative local refinement of the mesh according to the gradient of y_h

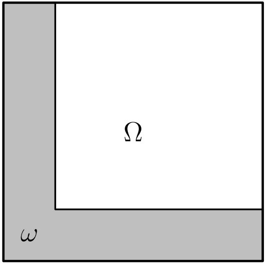


Reconstruct state *yh* on the adapted mesh

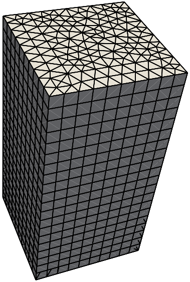
Triangular meshes - reduced HCT elements



Domain q_T^1 (a) and domain q_T^2 (b) triangulated using some coarse meshes.



(a)



(b)

Mesh Number	0	1	2	3
Number of elements	5 320	15 320	31 740	120 160
Number of nodes	3 234	8 799	17 670	64 411

Characteristics of the three meshes associated with Q_T.

$2D$ example: $\Omega = (0,1)^2$

$(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$:

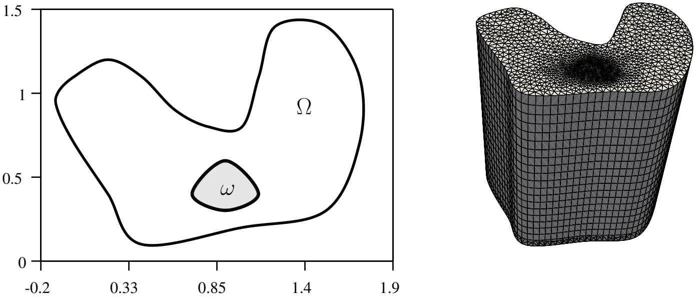
$$\textbf{(EX2–2D)} \quad \begin{cases} y_0(x_1,x_2) = (1-|2x_1-1|)(1-|2x_2-1|) \\ y_1(x_1,x_2) = \mathbf{1}_{(\frac{1}{3},\frac{2}{3})^2}(x_1,x_2) \end{cases} \quad (x_1,x_2) \in \Omega. \qquad (42)$$

The Fourier coefficients of the corresponding solution are

$$a_{kl} = \frac{2^5}{\pi^4 k^2 l^2} \sin\frac{\pi k}{2} \sin\frac{\pi l}{2}$$
$$b_{kl} = \frac{1}{\pi^2 kl}\left(\cos\frac{\pi k}{3} - \cos\frac{2\pi k}{3}\right)\left(\cos\frac{\pi l}{3} - \cos\frac{2\pi l}{3}\right).$$

Mesh number	0	1	2	3
$\frac{\|y-y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}}$	4.74×10^{-2}	3.72×10^{-2}	2.4×10^{-2}	1.35×10^{-2}
$\|Ly_h\|_{L^2(Q_T)}$	1.18	0.89	0.99	0.99
$\|\lambda_h\|_{L^2(Q_T)}$	3.21×10^{-5}	1.46×10^{-5}	1.02×10^{-5}	3.56×10^{-6}

Table: Example **EX2–2D** – $r = h^2$



(a) (b)

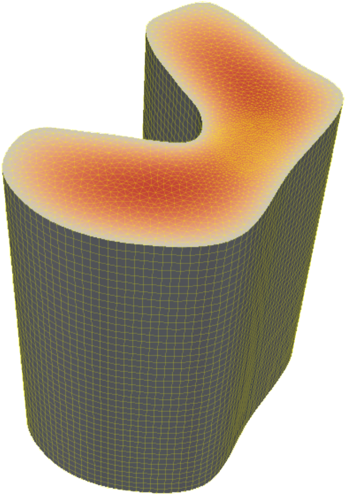
Mesh number	0	1	2
Number of elements	5 730	44 900	196 040
Number of nodes	3 432	24 633	103 566

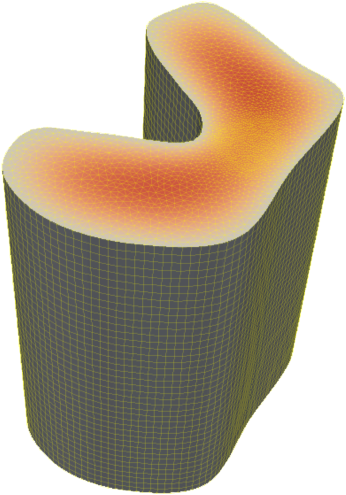
Characteristics of the three meshes associated with Q_T.

$$\begin{cases} -\Delta y_0 = 10, & \text{in } \Omega \\ y_0 = 0, & \text{on } \partial\Omega, \end{cases} \qquad y_1 = 0. \tag{43}$$

Mesh number	0	1	2
$\frac{\lVert \overline{y}_h - y_h \rVert_{L^2(Q_T)}}{\lVert \overline{y}_h \rVert_{L^2(Q_T)}}$	1.88×10^{-1}	8.04×10^{-2}	5.41×10^{-2}
$\lVert L y_h \rVert_{L^2(Q_T)}$	3.21	2.01	1.17
$\lVert \lambda_h \rVert_{L^2(Q_T)}$	8.26×10^{-5}	3.62×10^{-5}	2.24×10^{-5}

$$r = h^2 - T = 2$$


(a)


(b)

y and y_h in Q_T

MIXED FORMULATION ALLOWS TO RECONSTRUCT SOLUTION AND SOURCE

DIRECT AND ROBUST METHOD - STRONG CONVERGENCE

THE MINIMIZATION OF $J_r^{\star\star}(\lambda)$ IS VERY ROBUST AND FAST CONTRARY TO THE MINIMIZATION OF $J(y_0, y_1)$ (INVERSION OF SYMMETRIC DEFINITE POSITIVE AND VERY SPARSE MATRIX WITH DIRECT CHOLESKY SOLVERS)

DIRECT APPROACH CAN BE USED FOR MANY OTHER OBSERVABLE SYSTEMS FOR WHICH A GENERALIZED OBS. ESTIMATE IS AVAILABLE. IN PARTICULAR, HEAT, STOKES

$$\mathcal{L}_r(y,\lambda) := \frac{1}{2}\|\rho_0(y - y_{obs})\|^2_{L^2(q_T)} + \frac{r}{2}\|\rho L y\|^2_{L^2(Q_T)} + \iint_{Q_T} \rho_1 \lambda L y$$

(In progress with D. A. de Souza)

RECONSTRUCTION OF POTENTIAL, COEFFICIENTS