Approximation of controls by primal methods

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Craiova, CDPS 2013

joint works with NICOLAE CÎNDEA (Clermont-Ferrand), ENRIQUE FERNÁNDEZ-CARA and DIEGO A. DE SOUZA (Sevilla)

THE TALK BRIEFLY SURVEYS SOME RECENTS WORKS IN COLLABORATION WITH N. CÎNDEA, E. FERNÁNDEZ-CARA, DIEGO DE SOUZA CONCERNING THE NUMERICAL APPROXIMATIONS OF CONTROL FOR DISTRIBUTED SYSTEMS

SO-CALLED "PRIMAL METHODS" ARE USED : THE IDEA IS TO SOLVE DIRECTLY OPTIMALITY CONDITIONS RELATED TO A EXTREMAL PROBLEM LEADING TO STRONG CONVERGENT APPROXIMATIONS

IDEAS CAN BE FOUND IN LIONS'S BOOKS AND IN

FURSIKOV-92 : *Lagrange principle for problems of optimal control of ill-posed or singular distributed systems. J. Math. Pures Appl. (1992)*

WE CONSIDER THE WAVE EQUATION, THE HEAT EQUATION THEN THE STOKES SYSTEM

I - WAVE TYPE EQUATION : BOUNDARY CASE

$Q_T = (0,1) \times (0,T)$; $a \in C^3([0,1])$, $a(x) \geq a_0 > 0$ in $[0,1]$, $b \in L^\infty(Q_T)$,

$$\begin{cases} y_{tt} - (a(x)y_x)_x + b(x,t)y = 0, & (x,t) \in Q_T \\ y(0,\cdot) = 0, \qquad y(1,\cdot) = v, & t \in (0,T) \\ (y(\cdot,0), y_t(\cdot,0)) = (y_0, y_1) \in L^2(0,1) \times H^{-1}(0,1), & x \in (0,1). \end{cases} \tag{1}$$

$v = v(t)$ is the *control* in $L^2(0,T)$ and $y = y(x,t)$ is the associated state.

We associate the extremal problem :

$$\begin{cases} \text{Minimize } \; J(y,v) = \dfrac{1}{2}\displaystyle\iint_{Q_T} \rho^2 |y|^2 \, dx\, dt + \dfrac{1}{2}\int_0^T {\rho_0}^2 |v|^2 \, dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0, y_1; T) \end{cases} \tag{2}$$

$\mathcal{C}(y_0, y_1; T) = \{\, (y,v) : v \in L^2(0,T),\ y \text{ solves (1) and satisfies } y(\cdot,T) = y_t(\cdot,T) = 0 \,\}$.

$\rho \in C(Q_T, \mathbb{R}^+)$, $\rho_0 \in C((0,T), \mathbb{R}^+_\star)$.

For any $x_0 < 0$ and $a_0 > 0$, we assume that the function a belongs to

$$\begin{aligned}\mathcal{A}(x_0, a_0) = \{\, a \in C^3([0,1]) : a(x) \geq a_0 > 0, \\ -\min_{[0,1]} \Big(a(x) + (x - x_0)a_x(x)\Big) < \min_{[0,1]} \Big(a(x) + \frac{1}{2}(x - x_0)a_x(x)\Big) \,\}\end{aligned} \tag{3}$$

and then that

$$T > T^\star(a) := \frac{2}{\beta} \max_{[0,1]} a(x)^{1/2}(x - x_0).$$

for any $\beta > 0$ such that

$$-\min_{[0,1]} \Big(a(x) + (x - x_0)a_x(x)\Big) < \beta < \min_{[0,1]} \Big(a(x) + \frac{1}{2}(x - x_0)a_x(x)\Big)$$

Remark

Constant diffusion $a := a_0 \in \mathcal{A}(x_0, a_0)$ *and leads to* $T^\star(a) = \frac{2(1-x_0)}{\sqrt{a_0}} > \frac{2}{\sqrt{a_0}}$.

[GLOWINSKI-LIONS' 95]
$T > T^\star(a)$. Duality arguments lead to the unconstrained dual problem

$$\left\{\begin{array}{l} \text{Minimize } J^\star(\mu,\phi_0,\phi_1) = \frac{1}{2}\iint_{Q_T} \rho^{-2}|\mu|^2\,dx\,dt + \frac{1}{2}\int_0^T \rho_0^{-2}|a(1)\phi_x(1,t)|^2\,dt \\ \qquad + \int_0^1 y_0(x)\,\phi_t(x,0)\,dx - \langle y_1,\phi(\cdot\,,0)\rangle_{H^{-1},H_0^1} \\ \text{Subject to } (\mu,\phi_0,\phi_1) \in L^2(Q_T)\times H_0^1(\Omega)\times L^2(\Omega), \end{array}\right. \tag{4}$$

where ϕ solves

$$L\phi = \mu \quad \text{in } Q_T, \qquad \phi = 0 \quad \text{on } \Sigma_T, \qquad (\phi(\cdot\,,T),\phi_t(\cdot\,,T)) = (\phi_0,\phi_1) \quad \text{in } (0,1).$$

THE (NUMERICAL) DIFFICULTY IS TO FIND A FINITE CONFORMAL APPROXIMATION OF $L^2(Q_T)\times H_0^1\times L^2$ SATISFYING THE CONSTRAINT $L\phi=\mu$!

THE TRICK IS TO CONTROL A FINITE DIMENSIONAL AND CONSISTENT APPROXIMATION OF THE WAVE EQ. : THIS REQUIRES TO PROVE UNIFORM DISCRETE INEQUALITY OBSERVABILITY, STILL OPEN IN THE GENERAL CASE.

I-1 THE CASE ρ UNIFORMLY POSITIVE

N. Cîndea, E. Fernández-Cara and AM,
Numerical controllability of the wave equation through primal methods and Carleman estimates,
ESAIM:COCV (2013),

Let $T > T^\star(a)$ and P be the completion of $P_0 = \{\, q \in C^\infty(\overline{Q_T}) : q = 0 \text{ on } \Sigma_T \,\}$ with respect to the scalar product

$$(p, q)_P := \iint_{Q_T} \rho^{-2} Lp\, Lq\, dx\, dt + \int_0^T \rho_0^{-2}\, a(1)^2\, p_x(1,t)\, q_x(1,t)\, dt$$

Proposition (Cindea, Fernandez-Cara, M' 13)

Let us assume that $\rho \geq \rho_\star > 0$ on Q_T, $\rho_0 \geq \rho_\star > 0$ on $(0,T)$. Let $(y, v) \in \mathcal{C}(y_0, y_1, T)$ be the solution to (2). Then there exists $p \in P$ such that

$$y = -\rho^{-2} Lp, \qquad v = -(a(x)\rho_0^{-2} p_x)\big|_{x=1}. \tag{5}$$

Moroever, p is the unique solution to the variational equality:

$$(p, q)_P = \int_0^1 y_0(x)\, q_t(x,0)\, dx - \langle y^1, q(\cdot,0)\rangle_{H^{-1},H_0^1} \quad \forall q \in P. \tag{6}$$

Here :

$$\langle y^1, q(\cdot,0)\rangle_{H^{-1},H_0^1} = \int_0^1 \frac{\partial}{\partial x}((-\Delta)^{-1} y_1)(x)\, q_x(x,0)\, dx,$$

where $-\Delta$ is the Dirichlet Laplacian in $(0,1)$.

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Lemma

Let us assume that $a \in \mathcal{A}(x_0, a_0)$ and that $T > T^\star(a)$. Then there exists a constant $C_0 > 0$, only depending on x_0, a_0, $\|a\|_{C^3([0,1])}$, $\|b\|_{L^\infty(Q_T)}$ and T, such that

$$\|p(\cdot\,,0), p_t(\cdot\,,0)\|^2_{H^1_0(0,1)\times L^2(0,1)} \le C_0\,(p,p)_P \quad \forall p \in P. \tag{7}$$

PROOF -

1. via Carleman estimate: technical exponential form for the weight appears :

$$\rho(x,t) := e^{-s\varphi(x,2t-T)}, \quad \rho_0(t) := \rho(1,t)(x,t) \in Q_T,$$

(see PUEL'00, ZHANG'00, IMMANUVILOV'02, BAUDOUIN-DE BUHAN-ERVEDOZA'11, ETC) and

2. via Multipliers technics [YAO' 99]

Remark

The weights ρ, ρ_0 are arbitrary. In particular P does not depend on ρ and ρ_0.

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Remark

The weights ρ, ρ_0 are arbitrary. In particular P does not depend on ρ and ρ_0.

Theorem (Cindea-Fernandez-Cara, M'13)

Let us assume that $x_0 < 0$, $a_0 > 0$ and $a \in \mathcal{A}(x_0, a_0)$. Moreover, let us assume that $T > T^\star(a)$.
Then there exist positive constants s_0 and M, only depending on x_0, a_0, $\|a\|_{C^3([0,1])}$, $\|b\|_{L^\infty(Q_T)}$ and T, such that, for all $s > s_0$, one has

$$s \int_{-T}^{T}\int_0^1 e^{2s\varphi}\left(|w_t|^2 + |w_x|^2\right)\,dx\,dt + s^3\int_{-T}^{T}\int_0^1 e^{2s\varphi}|w|^2\,dx\,dt \leq M\int_{-T}^{T}\int_0^1 e^{2s\varphi}|Lw|^2\,dx\,dt + Ms\int_{-T}^{T} e^{2s\varphi}|w_x(1,t)|^2\,dt$$

for any $w \in L^2(-T,T;H_0^1(0,1))$ satisfying $Lw \in L^2((0,1)\times(-T,T))$ and $w_x(1,\cdot) \in L^2(-T,T)$.

extends PUEL'00, BAUDOUIN'01, BAUDOUIN-DE BUHAN-ERVEDOZA'11 to non constant diffusion a.

Boundary value problem

$$\begin{cases} \displaystyle\iint_{Q_T} \rho^{-2} Lp\, Lq\, dx\, dt + \int_0^T \rho_0^{-2} a^2(1) p_x(1,\cdot)\, q_x(1,\cdot)\, dt \\ \displaystyle = \int_0^1 y_0\, q_t(\cdot,0)\, dx - \langle y^1, q(\cdot,0)\rangle_{H^{-1},H_0^1} \quad \forall q \in P; \quad p \in P. \end{cases}$$

Remark

The function p solves, at least in the distributional sense, the following differential problem, that is of the fourth-order in time and space:

$$\begin{cases} L(\rho^{-2}Lp) = 0, & (x,t) \in Q_T \\ p(0,\cdot) = (\rho^{-2}Lp)(0,\cdot) = 0, & t \in (0,T) \\ p(1,\cdot) = (\rho^{-2}Lp + a\rho_0^{-2}p_x)(1,\cdot) = 0, & t \in (0,T) \\ (\rho^{-2}Lp)(\cdot,0) = y_0, \ \ (\rho^{-2}Lp)(\cdot,T) = 0, & x \in (0,1) \\ (\rho^{-2}Lp)_t(\cdot,0) = y_1, \ \ (\rho^{-2}Lp)_t(\cdot,T) = 0, & x \in (0,1). \end{cases}$$

Notice that the "boundary" conditions at $t = 0$ and $t = T$ are of the Neumann kind.

For any given finite dimensional space $P_h \subset P$ for each $h \in \mathbb{R}_+$, we define $p_h \in P_h$ the unique solution of

$$(p_h, q_h)_P = \langle \ell, q_h \rangle, \quad \forall q_h \in P_h. \tag{8}$$

We define the interpolation operator $\Pi_h : P_0 \to P_h$ and we assume that

$$\|p - \Pi_h p\|_P \to 0 \quad \text{as} \quad h \to 0, \quad \forall p \in P_0$$

From the density of P_0 into P for the $P - norm$,

Theorem

Let $p_h \in P_h$ the unique solution of (18) and let $p \in P$ the solution of the variational formulation. Then,

$$\|p - p_h\|_P \to 0 \quad as \quad h \to 0$$

Moreover, if we set

$$y_h := \rho^{-2} L p_h, \quad v_h := -\rho_0^{-2} a(x) p_{h,x}\big|_{x=1}.$$

Then one has

$$\|y - y_h\|_{L^2(Q_T)} \to 0 \text{ and } \|v - v_h\|_{L^2(0,T)} \to 0 \quad as \quad h \to 0,$$

where (y, v) is the solution to (2).

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where (y, v) is the solution to (2).

C^1 finite element approximation P_h

The spaces P_h must be chosen such that $\rho^{-1}Lp_h \in L^2(Q_T)$ for any $p_h \in P_h$.

A conformal approximation based on a standard quadrangulation of Q_T "requires" spaces of functions continuously differentiable with respect to both variables x and t.

$$\begin{cases} P_h = \{\, z_h \in C^1(\overline{Q_T}) : z_h|_K \in \mathbb{P}(K) \;\; \forall K \in \mathcal{Q}_h,\; z_h = 0 \;\text{ on } \Sigma_T \,\} \subset P \\ \mathcal{Q}_h \text{ a regular triangulation } \quad \overline{Q_T} = \displaystyle\bigcup_{K \in \mathcal{Q}_h} K \\ \mathbb{P}(K) \text{ denotes space of polynomial functions in } x \text{ and } t \end{cases}$$

Bogner-fox C^1 element : $\mathbb{P}(K) = (\mathbb{P}_{3,x} \otimes \mathbb{P}_{3,t})(K)$
Composite C^1 finite element : Reduced Fraeijs de Veubeke-Sanders for rectangle, Reduced Hsieh-Clough-Tocker for triangle

The resolution of the elliptic formulation

$$(p_h, q_h)_P = \langle \ell, q_h \rangle, \quad \forall q_h \in P_h.$$

amounts to solve a symmetric, positive definite, sparse linear system.

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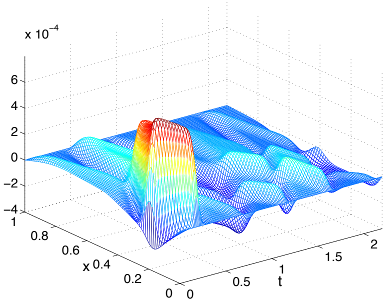
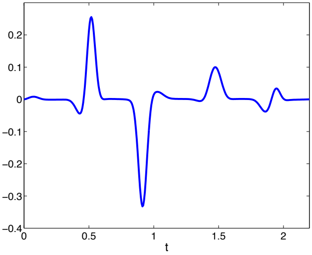
$$(p_h, q_h)_P = \langle \ell, q_h \rangle, \quad \forall q_h \in P_h.$$

amounts to solve a symmetric, positive definite, sparse linear system.

$$T = 2.2; \quad \begin{cases} y_0(x) = e^{-500(x-0.2)^2} \\ y_1(x) = 0; \end{cases} \qquad a(x) = \begin{cases} 1 & x \in [0, 0.45] \\ \in [1., 5.] \quad (a' > 0), & x \in (0.45, 0.55) \\ 5 & x \in [0.55, 1] \end{cases}$$



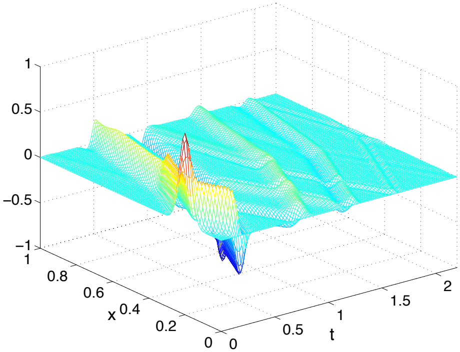


p_h over Q_T and $v_h = -a(1)p_{h,x}(1,\cdot)$ on $(0, T)$ - $h = (1/80, 1/80)$.

One example - Bi-cubic element

$\Delta x, \Delta t$	$1/10$	$1/20$	$1/40$	$1/80$	$1/160$
$\|\hat{p}_h - p\|_{P_h}$	1.25×10^{-1}	5.75×10^{-2}	2.64×10^{-2}	1.01×10^{-2}	-
$\|\hat{v}_h - v\|_{L^2(0,T)}$	5.07×10^{-1}	4.17×10^{-2}	2.03×10^{-2}	4.86×10^{-3}	-
$\|\hat{y}_h(\cdot\,,T)\|_{L^2(0,1)}$	1.09×10^{-1}	7.89×10^{-2}	1.81×10^{-2}	1.16×10^{-2}	1.71×10^{-3}
$\|\hat{y}_{t,h}(\cdot\,,T)\|_{H^{-1}(0,1)}$	1.01×10^{-1}	8.39×10^{-2}	4.81×10^{-2}	7.52×10^{-3}	1.55×10^{-3}

$\|p-\hat{p}_h\|_P = \mathcal{O}(h^{1.91})$ $\|v-\hat{v}_h\|_{L^2(0,T)} = \mathcal{O}(h^{1.56})$ $\|\hat{y}_h(\cdot,T)\|_{L^2(0,1)} = \mathcal{O}(h^{1.71})$ $\|\hat{y}_{t,h}(\cdot,T)\|_{H^{-1}(0,1)} = \mathcal{O}(h^{1.31})$



Approximation y_h of the controlled state.

I-2 Case $\rho := 0$

$$\begin{cases} \text{Minimize } \; J(y,v) = \frac{1}{2}\int_0^T \rho_0^2 |v|^2 \, dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0, y_1; T) \end{cases}$$

The previous approach DOES NOT apply, but we can adapt it !
In the sequel, to simplify, $\rho_0 := 1$

N. Cîndea and AM,
Mixed formulation for the direct approximation of the HUM control for linear wave equation,
Preprint (2013),

$$\min\ J^\star(\varphi_0,\varphi_1) = \frac{1}{2}\int_0^T |a(1)\varphi_x(1,t)|^2\,dt + \int_0^1 y_0\,\varphi_t(\cdot,0)\,dx - \langle y_1,\varphi(\cdot\,,0)\rangle_{H^{-1},H_0^1}$$

$$L\varphi = 0 \quad \text{in } Q_T, \qquad \varphi = 0 \quad \text{on} \quad \Sigma_T, \qquad (\varphi(\cdot\,,T),\varphi_t(\cdot\,,T)) = (\varphi_0,\varphi_1) \quad \text{in } (0,1).$$

Since the variable φ is completely and uniquely determined by (φ_0,φ_1), we keep φ as the main variable and consider the extremal problem:

$$\min_{\varphi\in W} J^{\prime,\star}(\varphi) = \frac{1}{2}\int_0^T |a(1)\varphi_x(1,t)|^2 dt + \int_0^1 y_0\,\varphi_t(\cdot,0)dx - \langle y_1,\varphi(\cdot,0)\rangle_{H^{-1},H_0^1}.$$

$$\begin{cases} W = \left\{\varphi\in L^2(Q_T):\ \varphi = 0 \text{ on } \Sigma_T; \quad L\varphi = 0 \in L^2(Q_T); \quad \varphi_x(1,\cdot)\in L^2(0,T)\right\}, \\ W \text{ -Hilbert space endowed with the inner product} \\ (\varphi,\overline{\varphi})_W = \int_0^T a(1)\varphi_x(1,t)\overline{\varphi}_x(1,t)dt + \eta\iint_{Q_T} L\varphi L\overline{\varphi}dx\,dt, \qquad \forall\varphi,\overline{\varphi}\in W, \eta>0. \end{cases} \tag{9}$$

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$$L\varphi=0\quad\text{in } Q_T,\qquad \varphi=0\quad\text{on}\quad\Sigma_T,\qquad (\varphi(\cdot\,,T),\varphi_t(\cdot\,,T))=(\varphi_0,\varphi_1)\quad\text{in }(0,1).$$

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The main variable is now φ submitted to the constraint equality $L\varphi = 0$. This constraint is addressed introducing a mixed formulation. We define the space Φ larger than W (endowed with the same norm) by

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$$\begin{cases} a_r(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) & = & l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi \\ b(\varphi, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{cases}$$

where ($r > 0$ - augmentation parameter)

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2. *The unique solution $(\varphi, \lambda) \in \Phi \times L^2(Q_T)$ is the unique saddle-point of the Lagrangian $\mathcal{L} : \Phi \times L^2(Q_T) \to \mathbb{R}$ defined by*

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From the direct inequality,

$$\int_0^T |a(1)\varphi_{0,x}(1,t)|^2 dt \leq C_{\Omega,T}\, a^2(1) \|\lambda_0\|^2_{L^2(Q_T)}$$

we get that $\varphi_{0,x}(1,\cdot) \in L^2(0,T)$ and $\varphi_0 \in \Phi$. In particular, $b(\varphi_0, \lambda_0) = \|\lambda_0\|^2_{L^2(Q_T)}$ and

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Combining the above two inequalities, we obtain

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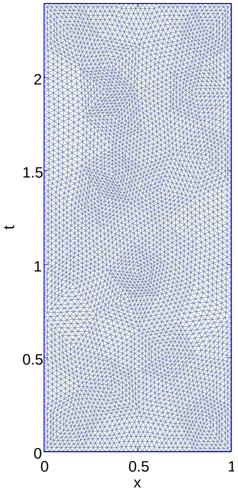
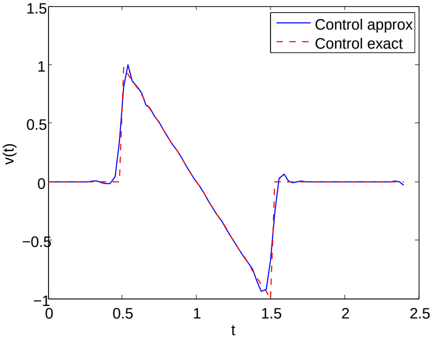
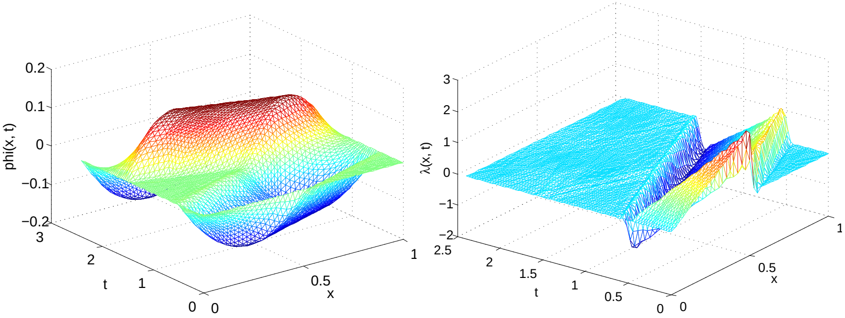
Numerical illustration in a singular case : discontinuous y_0

$$T = 2.4; \quad y_0(x) = 4x\, 1_{[0,1/2]}(x); \quad y_1 = 0; \quad v(t) = 2(1-t)\, 1_{[1/2,3/2]}(t)$$



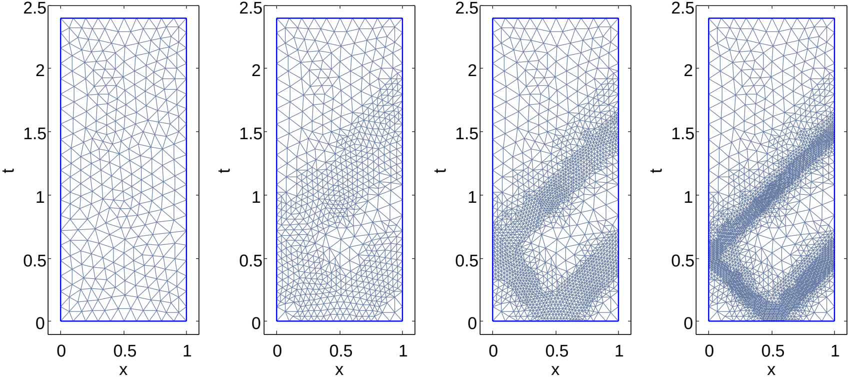


Regular triangulation of Q_T and HUM-Control v_h



φ_h and λ_h on Q_T.

Adaptation of the Q_T mesh



Time-Space Refinement of the mesh according to the gradient of λ_h

Lemma

Let A be the linear operator from $L^2(Q_T)$ into $L^2(Q_T)$ defined by

$$A\lambda := L\varphi, \quad \forall\lambda \in L^2(Q_T) \quad \textit{where} \quad \varphi \in \Phi \quad \textit{solves} \quad a_r(\varphi, \overline{\varphi}) = b(\overline{\varphi}, \lambda), \quad \forall\overline{\varphi} \in \Phi.$$

For any $r > 0$, the operator A is a strongly elliptic, symmetric isomorphism from $L^2(Q_T)$ into $L^2(Q_T)$.

Theorem

$$\sup_{\lambda\in L^2(Q_T)} \inf_{\varphi\in\Phi} \mathcal{L}_r(\varphi, \lambda) = - \inf_{\lambda\in L^2(Q_T)} J^{\star\star}(\lambda) \quad + \mathcal{L}_r(\varphi_0, 0)$$

where $\varphi_0 \in \Phi$ solves $a_r(\varphi_0, \overline{\varphi}) = l(\overline{\varphi}), \forall\overline{\varphi} \in \Phi$ and $J^{\star\star} : L^2(Q_T) \to \mathbb{R}$ defined by

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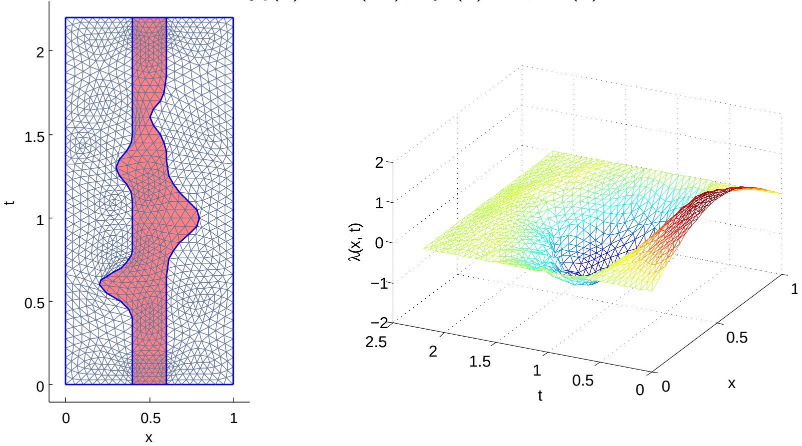
II- WAVE TYPE EQUATION : DISTRIBUTED CASE

$$\begin{cases} \omega \text{ a nonempty subset of } \Omega, \\ Ly := y_{tt} - (a(x)y_x)_x + Ay = v\, 1_\omega \qquad (x,t) \in Q_T \end{cases}$$

$$T = 2.2; \quad y_0(x) = \sin(\pi x), \quad y_1(x) = 0, \quad a(x) := 1;$$



Time dependent control support and corresponding controlled solution y_h

⇒ POSSIBILITY TO OPTIMIZE RIGOROUSLY AND EASILY THE SUPPORT OF THE CONTROL ! (IN PROGRESS)

III- (Linear) Heat type equation : Distributed case ($\rho \neq 0$)

$a \in C^1([0,1], \mathbb{R}_*^+)$, $y_0 \in L^2(0,1)$, $q_T = \omega \times (0,T)$, $v \in L^2(q_T)$, $A \in L^\infty(Q_T)$

$$\begin{cases} L_A y := y_t - (a(x)y_x)_x + Ay = v1_\omega, & Q_T \\ y = 0, \quad \Sigma_T, & y(\cdot, 0) = y_0, \quad \Omega. \end{cases}$$

[Lebeau Robbiano'95] [Fursikov Imanuvilov'95]
Notation : $L^\star p := -p_t - (a(x)p_x)_x + Ap$

E. Fernández-Cara and AM,
Numerical controllability of the wave equation through primal methods and Carleman estimates,
SéMA (2013),

$L^2(0,1)$-norm of the HUM control with respect to time

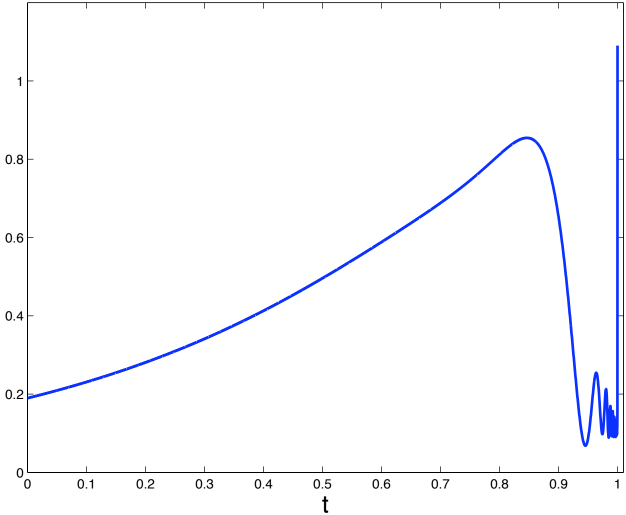


Figure: $y_0(x) = \sin(\pi x)$ - $T = 1$ - $\omega = (0.2, 0.8)$ - $t \rightarrow \|v(\cdot, t)\|_{L^2(0,1)}$ in $[0, T]$

First, let us set $P_0 = \{\, q \in C^2(\overline{Q}_T) : q = 0 \text{ on } \Sigma_T \,\}$. In this linear space, the bilinear form

$$(p, q)_P := \iint_{Q_T} \rho^{-2} L^* p \, L^* q \, dx \, dt + \iint_{q_T} \rho_0^{-2} p \, q \, dx \, dt$$

is a scalar product.

Proposition (Characterization of the optimal pair)

Let ρ and ρ_0 be given by (16). Let (y, v) be the corresponding optimal pair for J. Then there exists $p \in P$ such that

$$y = \rho^{-2} L^* p, \quad v = -\rho_0^{-2} p|_{q_T} \,. \tag{14}$$

The function p is the unique solution in P of

$$(p, q)_P = \int_0^1 y_0 \, q(\cdot, 0) \, dx, \quad \forall q \in P \tag{15}$$

There are “good” weight functions ρ and ρ_0 that blow up at $t = T$ and provide a very suitable solution to the original null controllability problem. They were determined and systematically used by Fursikov and Imanuvilov'96 and are the following:

$$\begin{cases} \rho(x,t) = \exp\left(\dfrac{\beta(x)}{T-t}\right), \; \rho_0(x,t) = (T-t)^{3/2}\rho(x,t), \; \beta(x) = K_1\left(e^{K_2} - e^{\beta_0(x)}\right) \\ \text{the } K_i \text{ are large positive constants (depending on } T,\, a_0,\, \|a\|_{C^1} \text{ and } \|A\|_\infty) \\ \text{and } \beta_0 \in C^\infty([0,1]),\, \beta_0 > 0 \text{ in } (0,1),\, \beta_0(0) = \beta_0(1) = 0,\, |\beta_0'| > 0 \text{ outside } \omega. \end{cases} \tag{16}$$

Lemma (Global Carleman estimate - Fursikov-Imanuvilov'95)

Let ρ and ρ_0 be given by (16). Then, for any $\delta > 0$, $P \hookrightarrow C^0([0, T-\delta]; H_0^1(0,1))$ and the embedding is continuous. In particular, there exists $C_0 > 0$, only depending on ω, T, a_0, $\|a\|_{C^1}$ and $\|A\|_\infty$, such that

$$\|q(\cdot\,,0)\|^2_{H_0^1(0,1)} \le C_0 \left(\iint_{Q_T} \rho^{-2}|L^*q|^2\,dx\,dt + \iint_{q_T} \rho_0^{-2}|q|^2\,dx\,dt \right) \tag{17}$$

for all $q \in P$.

For any dimensional space $P_h \subset P$, we can introduce the following *approximate* problem:

$$(p_h, \overline{p}_h)_P = < l, \overline{p}_h >, \quad \forall \overline{p}_h \in P_h; \quad p_h \in P_h. \tag{18}$$

$$P_h = \{ z_h \in C^{1,0}_{x,t}(\overline{Q_T}) : z_h|_K \in (\mathbb{P}_{3,x} \otimes \mathbb{P}_{1,t})(K) \ \ \forall K \in \mathcal{Q}_h, \ z_h = 0 \ \text{ on } \Sigma_T \}. \tag{19}$$

(Fernández-Cara, AM)

Let $p_h \in P_h$ be the unique solution to (18), where P_h is given by (19). Let us set

$$y_h := \rho^{-2} L_A^\star p_h, \quad v_h := -\rho_0^{-2} p_h 1_{q_T}.$$

Then one has

$$\|y - y_h\|_{L^2(Q_T)} \to 0 \text{ and } \|v - v_h\|_{L^2(q_T)} \to 0, \qquad as \quad h \to 0$$

where (y, v) is the minimizer of J.

Conformal finite element approximation

For any dimensional space $P_h \subset P$, we can introduce the following *approximate* problem:

$$(p_h, \overline{p}_h)_P = < l, \overline{p}_h >, \quad \forall \overline{p}_h \in P_h; \quad p_h \in P_h. \tag{18}$$

$$P_h = \{ z_h \in C^{1,0}_{x,t}(\overline{Q_T}) : z_h|_K \in (\mathbb{P}_{3,x} \otimes \mathbb{P}_{1,t})(K) \ \ \forall K \in \mathcal{Q}_h, \ z_h = 0 \ \text{ on } \Sigma_T \}. \tag{19}$$

Theorem (Fernández-Cara, AM)

Let $p_h \in P_h$ be the unique solution to (18), where P_h is given by (19). Let us set

$$y_h := \rho^{-2} L_A^\star p_h, \quad v_h := -\rho_0^{-2} p_h \, 1_{q_T}.$$

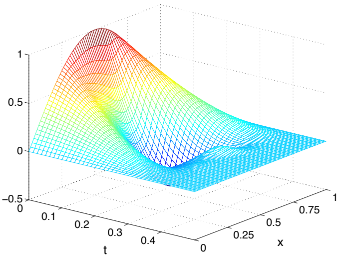
Then one has

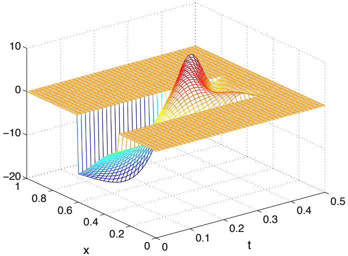
$$\|y - y_h\|_{L^2(Q_T)} \to 0 \text{ and } \|v - v_h\|_{L^2(q_T)} \to 0, \qquad as \quad h \to 0$$

where (y, v) is the minimizer of J.

1D example - Bi-cubic element - Uniform quadrangulation - $y_0(x) = \sin(\pi x)$ - $T = 1/2$ - $a(x) = 1/10$ - $\omega = (0.3, 0.6)$

$\Delta x, \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$	$1/320$
$\|v_h\|_{L^2(q_T)}$	1.597	2.023	2.348	2.58	2.733
$\|y_h\|_{L^2(Q_T)}$	1.879×10^{-1}	1.834×10^{-1}	1.826×10^{-1}	1.827×10^{-1}	1.829×10^{-1}
$\|y_h(\cdot, T)\|_{L^2(0,1)}$	4.96×10^{-3}	1.82×10^{-3}	5.91×10^{-4}	1.71×10^{-4}	4.65×10^{-5}
$\|y - y_h\|_{L^2(Q_T)}$	7.52×10^{-2}	4.82×10^{-2}	2.62×10^{-2}	1.04×10^{-2}	-
$\|v - v_h\|_{L^2(q_T)}$	1.57	1.04	0.59	0.25	-





y_h and v_h over Q_T - $h = (1/80, 1/80)$.

IV- Semi-Linear Heat type equation : Distributed case ($\rho \neq 0$)

$$\begin{cases} y_t - (a(x)y_x)_x + f(y) = v1_\omega, & Q_T \\ y = 0, \quad \Sigma_T, & y(\cdot,0) = y_0, \quad \Omega. \end{cases} \tag{20}$$

$y_0 \in L^\infty$, $f \in C^1(R)$ globally Lipschitz continuous.
$f(0) = 0$. $f(s)/(s\log^{3/2}(1+|s|)) \to 0$ as $|s| \to \infty$.
[Barbu'00] [Fernández-Cara Zuazua'00]

E. Fernández-Cara and AM,
Numerical null controllability of semi-linear 1D heat equations : fixed point, least squares and Newton methods,
Mathematical Control and Related Fields, (2012)

I - Linearization of the equation :

$$y_t - (a(x)y_x)_x + g(z)y = v\,1_\omega, \quad Q_T, \tag{21}$$

with

$$g(s) = \frac{f(s)}{s} \text{ if } s \neq 0, \; g(0) = f'(0) \text{ otherwise.}$$

II - Definition of the operator $\Lambda : L^2(Q_T) \to L^2(Q_T)$ defined by :

$$\begin{cases} \Lambda z := y \\ y \in C(y_0, z, T) \quad \text{such that} \quad (y_z, v_z) \text{ minimize } J(y_z, v_z) \end{cases}$$

III - Approximation of a fixed point iteratively :

- Relaxed Picard iterates :

$$z^0 \in L^2(Q_T), \quad z^{n+1} = \alpha z^n + (1-\alpha)\Lambda z^n, \quad n \geq 0, \qquad \alpha \in (0,1)$$

- Least-Squares type approach :

$$minimize_{z \in L^2(Q_T)} \|z - \Lambda(z)\|^2_{L^2(Q_T)}$$

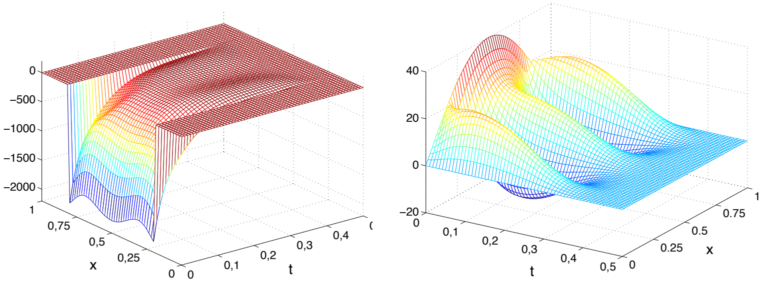
$$f(s) = -5s\ \log^{\frac{7}{5}}(1+|s|) \quad \forall s \in \mathbf{R}, \quad a(x) = 1/10; \quad T = 1/2 \quad y_0(x) = 40\sin(\pi x)$$

without control, blow up time $t_c \approx 0.339 < T$.

A fixed point : a numerical application

$$f(s) = -5s \log^{\frac{7}{5}}(1+|s|) \quad \forall s \in \mathbf{R}, \quad a(x) = 1/10; \quad T = 1/2 \quad y_0(x) = 40\sin(\pi x)$$

without control, blow up time $t_c \approx 0.339 < T$.



Control v_h and corresponding controlled solution y_h

V- STOKES / NS SYSTEM : DISTRIBUTED CASE

$\Omega \subset \mathbb{R}^N$ bounded, connected open set whose boundary $\partial\Omega$ is regular enough (for instance of class C^2; $N = 2$ or $N = 3$)

$$\begin{cases} L\mathbf{y} + \nabla\pi = \mathbf{v}1_\omega, \quad \nabla\cdot\mathbf{y} = 0 \quad \text{in } Q_T \\ \mathbf{y} = \mathbf{0} \quad \text{on } \Sigma_T, \quad \mathbf{y}(\cdot, 0) = \mathbf{y}_0 \quad \text{in } \Omega \end{cases} \tag{22}$$

[FURSIKOV IMANUVILOV'95]
Notations : $L\mathbf{y} := \mathbf{y}_t - \nu\Delta\mathbf{y}$; $L^\star\mathbf{p} := -\mathbf{p}_t - \nu\Delta\mathbf{p}$

D. Araujo de Souza, E. Fernández-Cara and AM,
Numerical null controllability of the Stokes system, In progress .

Optimality

$$\Phi_0 = \left\{ (\mathbf{p}, \sigma) : p_i, \sigma \in C^2(\overline{Q_T}),\ \nabla \cdot \mathbf{p} \equiv 0,\ p_i = 0 \text{ on } \Sigma,\ \int_\Omega \sigma(\mathbf{x}, t) d\mathbf{x} = 0\ \forall t \right\}.$$

Let Φ be the completion of Φ_0 with respect to the scalar product defined by

$$m((\mathbf{p}, \sigma), (\mathbf{p}', \sigma')) := \iint_{Q_T} \left(\rho^{-2} (\mathbf{L}^\star \mathbf{p} + \nabla \sigma) \cdot (\mathbf{L}^\star \mathbf{p}' + \nabla \sigma') + 1_\omega \rho_0^{-2} \mathbf{p} \cdot \mathbf{p}' \right) d\mathbf{x}\, dt$$

Theorem (Characterization of the optimality)

Let the weights ρ and ρ_0 as before and let $(\mathbf{y}, \mathbf{v})$ be the unique minimizer for J. Then one has

$$\mathbf{y} = \rho^{-2} (\mathbf{L}^\star \mathbf{p} + \nabla \sigma), \quad \mathbf{v} = -\rho_0^{-2} \mathbf{p}\big|_{\omega \times (0,T)}, \tag{23}$$

where $(\mathbf{p}, \sigma)$ is the unique solution to the variational equality

$$\left\{ \begin{array}{l} m((\mathbf{p}, \sigma), ((\mathbf{p}', \sigma')) = \langle B_0, (\mathbf{p}', \sigma') \rangle \\ \quad \forall (\mathbf{p}', \sigma') \in \Phi;\ (\mathbf{p}, \sigma) \in \Phi. \end{array} \right. \tag{24}$$

with B_0 given by

$$\langle B_0, (\mathbf{p}, \sigma) \rangle := \int_\Omega \mathbf{y}_0 \cdot \mathbf{p}(\cdot, 0) d\mathbf{x}.$$

$$\Phi_0 = \left\{ (\mathbf{p},\sigma) : p_i, \sigma \in C^2(\overline{Q_T}),\ \nabla\cdot\mathbf{p} \equiv 0,\ p_i = 0 \text{ on } \Sigma,\ \int_\Omega \sigma(\mathbf{x},t)d\mathbf{x} = 0\ \forall t \right\}.$$

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with B_0 given by

$$\langle B_0, (\mathbf{p},\sigma)\rangle := \int_\Omega \mathbf{y}_0\cdot\mathbf{p}(\cdot,0)d\mathbf{x}.$$

The variational equality (24) can be regarded as the weak formulation of a (non-scalar) boundary-value problem for a PDE that is fourth-order in $\mathbf{x}$ and second-order in t. Indeed, taking “test functions” $(\mathbf{p}, \sigma) \in \Phi$ first with $p_i, \sigma \in C_0^\infty(Q_T)$, then with $p_i, \sigma \in C^2(\overline{\Omega} \times (0,T))$ and finally with $p_i, \sigma \in C^2(\overline{Q_T})$, we can easily deduce that $(\mathbf{p}, \sigma)$ satisfies, together with some $\pi \in \mathcal{D}'(Q_T)$, the following:

$$\begin{cases} \mathbf{L}(\rho^{-2}(\mathbf{L}^\star \mathbf{p} + \nabla\sigma)) + \nabla\pi + 1_\omega \rho_0^{-2}\mathbf{p} = 0 & \text{in } Q_T, \\ \nabla\cdot(\rho^{-2}(\mathbf{L}^\star \mathbf{p} + \nabla\sigma)) = 0, \ \nabla\cdot\mathbf{p} = 0 & \text{in } Q_T, \\ \mathbf{p} = \mathbf{0}, \ \ \rho^{-2}(\mathbf{L}^\star \mathbf{p} + \nabla\sigma) = \mathbf{0} & \text{on } \Sigma_T, \\ \rho^{-2}(\mathbf{L}^\star \mathbf{p} + \nabla\sigma)\big|_{t=0} = \mathbf{y}_0, \ \ \rho^{-2}(\mathbf{L}^\star \mathbf{p} + \nabla\sigma)\big|_{t=T} = \mathbf{0} & \text{in } \Omega. \end{cases} \tag{25}$$

Again Navier-Stokes, local ECT:

$$(NS) \qquad \begin{cases} y_t + (y\cdot\nabla)y - \Delta y + \nabla p = v 1_\omega, \quad \nabla\cdot y = 0 \\ y(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T) \\ y(x,0) = y^0(x) \end{cases}$$

Fix a solution $(\overline{y},\overline{p})$, with $\overline{y}\in L^\infty$
Goal: Find v such that $y(T) = \overline{y}(T)$
Strategy:

- Reformulation: NC
- Fixed point

Again Navier-Stokes, local ECT:

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Test 1: Poiseuille flow

$$\overline{y} = (4x_2(1 - x_2), 0), \quad \overline{p} = 4x_1$$

(stationary)

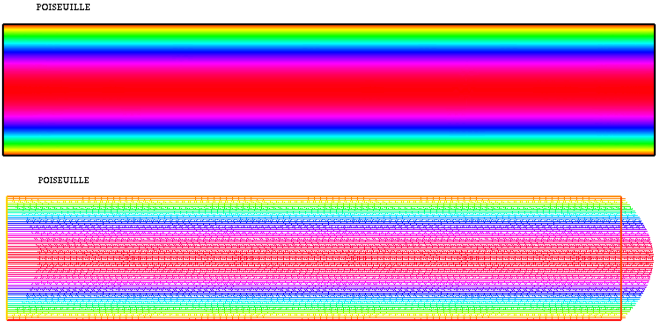


Figure: Poiseuille flow

Test 1: Poiseuille flow $\Omega = (0,5) \times (0,1)$, $\omega = (1,2) \times (0,1)$, $T = 2$
$y_0 = \overline{y} + m\,z,\ \ z = \nabla \times \psi,\ \ \psi = (1-y)^2 y^2 (5-x)^2 x^2\ (m << 1\ )$
Approximation: P_2 in (x_1, x_2, t) + multipliers ... – freefem++

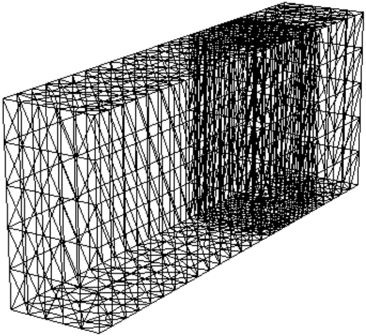


Figure: The Mesh – Nodes: 1830, Elements: 7830, Variables: 7×1830

Test 1: Poiseuille flow

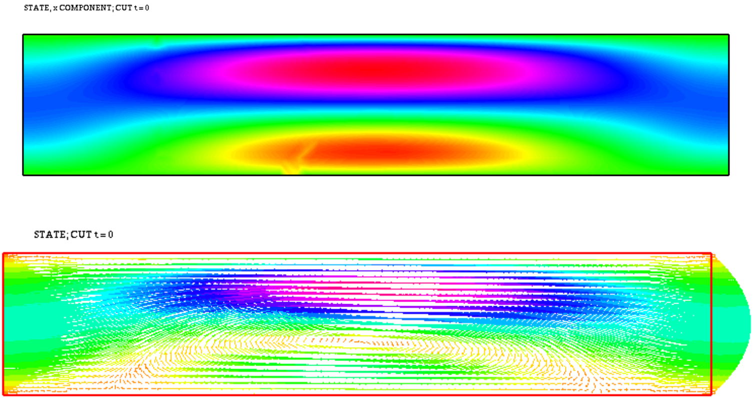


Figure: The initial State

Test 1: Poiseuille flow

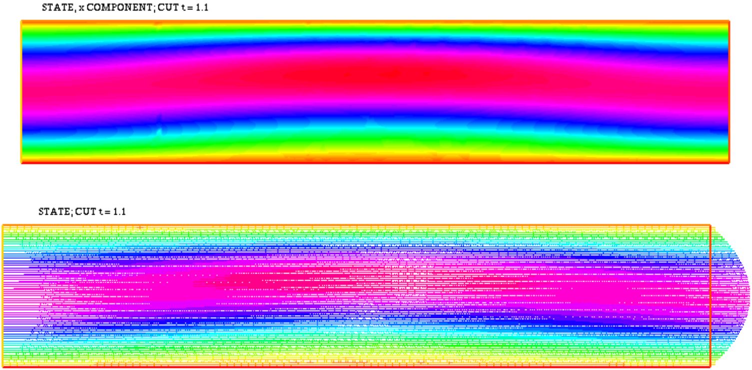
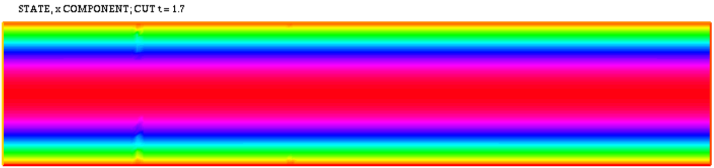


Figure: The State at $t = 1.1$

Test 1: Poiseuille flow



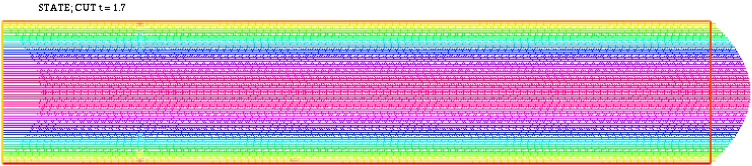


Figure: The State at $t = 1.7$

ZPoisseuille.edp

Final comments

THE VARIATIONAL APPROACH CAN BE USED IN THE CONTEXT OF MANY OTHER CONTROLLABLE SYSTEMS FOR WHICH APPROPRIATE CARLEMAN ESTIMATES ARE AVAILABLE.

THE APPROXIMATION IS ROBUST (WE HAVE TO INVERSE SYMMETRIC DEFINITE POSITIVE AND VERY SPARSE MATRICE WITH DIRECT LU AND CHOLESKY SOLVERS)

WITH CONFORMAL TIME-SPACE FINITE ELEMENTS APPROXIMATIONS, WE OBTAIN STRONG CONVERGENCE RESULTS WITH RESPECT TO $h = (\Delta x, \Delta t)$.

THE PRICE TO PAY IS TO USED C^1 FINITE ELEMENTS (AT LEAST IN SPACE).

IN THAT SPACE-TIME APPROACH, THE SUPPORT OF THE CONTROL MAY VARIES IN TIME (WITHOUT ADDITIONAL DIFFICULTIES).

THIS APPROACH MAY BE APPLIED FOR INVERSE PROBLEMS, OBSERVATION PROBLEMS, RECONSTRUCTION OF DATA, ...

NADA MA(S) !

THANK YOU VERY MUCH FOR YOUR ATTENTION