Inverse problems for linear PDEs via variational formulations: Robust numerical approximations

ARNAUD MÜNCH

Université Blaise Pascal - Clermont-Ferrand

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with NICOLAE CÎNDEA (CF) and DIEGO ARAUJO DE SOUZA (P6)

General context and purpose

Given a suitable observation $y_{obs} (= B(y))$ of y, unique solution of a linear well-posed PDE

$$\begin{cases} PDE(y, \nabla y, ...) = f, \quad \Omega \times (0, T), \\ + \text{ boundary and initial conditions} \end{cases},$$

find a convergent (numerical) approximation of the following linear inverse problem :

reconstruct the solution y and the source f such that $B(y) = y_{obs}$.

The main aim is to highlight that space-time variational approach of first and second order leads to robust approximation.

We consider hyperbolic (wave eq.) and parabolic (heat eq.) situation.

The approach is inspired from recent works on exact controllability

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Hyperbolic situation

Hyperbolic equation - Problem statement

$\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ - $T > 0$, $c \in C^1(\overline{\Omega}, \mathbb{R})$, $d \in L^\infty(Q_T)$, $(y_0, y_1) \in \boldsymbol{H}$, $f \in X$.

$$\begin{cases} Ly := y_{tt} - \nabla \cdot (c \nabla y) + dy = f, & Q_T := \Omega \times (0, T) \\ y = 0, & \Sigma_T := \partial\Omega \times (0, T) \\ (y(\cdot, 0), y_t(\cdot, 0)) = (y_0, y_1), & \Omega. \end{cases} \tag{1}$$

- Inverse Problem 1: Distributed observation on $q_T = \omega \times (0, T), \omega \subset \Omega$

$$\begin{cases} \boldsymbol{H} = L^2 \times H^{-1}, X = L^2(H^{-1}), \\ \text{Given } (y_{obs}, f) \in L^2(q_T) \times X, \text{ find } y \ s.t. \{(1) \text{ and } y - y_{obs} = 0 \quad \text{on} \quad q_T\} \end{cases}$$

- Inverse Problem 2: Boundary observation on $\Gamma_T \subset \partial\Omega \times (0, T)$

$$\begin{cases} \boldsymbol{H} = H_0^1 \times L^2, X = L^2(L^2) \\ \text{Given } y_{obs,\nu} \in L^2(\Gamma_T), \text{ find } (y, f) \ s.t. \{(1) \text{ and } \partial_\nu y - y_{obs,\nu} = 0 \quad \text{on} \quad \Gamma_T\} \end{cases}$$

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Inverse problem 1

$$Z := \left\{ y : y \in C([0,T], L^2(\Omega)) \cap C^1([0,T], H^{-1}(\Omega)), Ly \in X, y_{|\Sigma_T} = 0 \right\}.$$

Introducing the operator $P : Z \to X \times L^2(q_T)$

$$P\,y := (Ly, y_{|q_T}),$$

Inverse Problem 1 is reformulated as :

$$\textit{find } y \in Z \textit{ solution of } P\,y = (f, y_{obs}). \tag{IP}$$

If unique continuation property holds for (1) and if y_{obs} is a restriction to q_T of a solution of (1), then (IP) is well-posed: the state y corresponding to the pair (y_{obs}, f) is unique.

Most natural approach: Relaxation via Least-squares method

The most natural (and widely used in practice) approach consists in introducing a least-squares type technic, i.e. consider the extremal problem

$$(LS) \quad \begin{cases} \text{minimize} \quad J(y_0, y_1) := \frac{1}{2}\|y - y_{obs}\|^2_{L^2(q_T)} \\ \text{subject to} \quad (y_0, y_1) \in \boldsymbol{H} \\ \text{where} \quad y \quad \text{solves} \quad (1) \end{cases}$$

A minimizing sequence $(y_{0k}, y_{1k})_{(k>0)} \in \boldsymbol{H}$ is defined in term of an adjoint problem.

Drawback : it is difficult to minimize over a finite dimensional subspace of the set of constraints
The minimization procedure first requires the discretization of J and of the system (1);

This raises the issue of uniform coercivity property of the discrete functional w.r.t. the approximation parameter.

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A not so different approach : Luenberger observers type approach

[Auroux-Blum 2005],[Chapelle,Cindea,Moireau,2012], [Ramdani-Haine 2011], etc...

Define a dynamic

$$L\overline{y} = G(y_{obs}, q_T)$$
$$\overline{y}(\cdot, 0) \quad \text{fixed}$$

such that

$$\|\overline{y}(\cdot, t) - y(\cdot, t)\|_{N(\Omega)} \to 0 \quad \text{as} \quad t \to \infty$$

$N(\Omega)$ - appropriate norm
The reversibility of the eq. then allows to recover y for any time.

But again, on a numerically point of view, this method requires to prove uniform discrete observability properties.

Keeping y as the main variable ...

Without loss of generality, $f \equiv 0$.

$$Z := \{y : y \in C([0,T], L^2(\Omega)) \cap C^1([0,T], H^{-1}(\Omega)), Ly \in X, y_{|\Sigma_T} = 0\}.$$

Hypothesis (Generalized Observability Inequality)

Assume that $\exists C_{obs} = C(\omega, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)})$ *s.t. :*

$$(\mathcal{H}) \qquad \|y(\cdot,0), y_t(\cdot,0)\|_{\boldsymbol{H}}^2 \leq C_{obs}\left(\|y\|^2_{L^2(q_T)} + \|Ly\|^2_X\right), \quad \forall y \in Z. \qquad (2)$$

- in 1-D, (2) if $T \geq T^\star(c,d)$ [Fernandez-Cara, Cindea,Münch, COCV 2013],
- in N-D, for $c = 1$ and $d = 0$, (2) if (Ω, ω, T) satisfies geometric optic condition [Bardos, Lebeau, Rauch, 1992]

$$\|y\|^2_{L^2(Q_T)} \leq C_{\Omega,T}\left(C_{obs}\|y\|^2_{L^2(q_T)} + (1 + C_{obs})\|Ly\|^2_X\right) \quad \forall y \in Z. \qquad (3)$$

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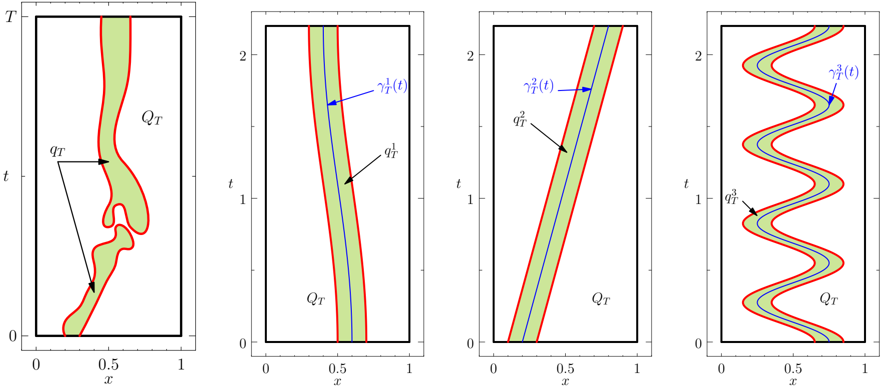
Non cylindrical situation in 1D

[Castro-Cindea-Münch, SICON 2014], [Lebeau, 2015]
In 1D with $c \equiv 1$ and $d \equiv 0$, the observability ineq. also holds for non cylindrical domains.



Time dependent domains $q_T \subset Q_T = \Omega \times (0, T)$

Equivalent formulation of IP

Within this hypothesis, for any $\eta > 0$, we define on Z the bilinear form

$$\langle y, \overline{y} \rangle_Z := \iint_{q_T} y \, \overline{y} \, dxdt + \eta \int_0^T \langle Ly, \, L\overline{y} \rangle_{H^{-1}(\Omega)} \, dt \quad \forall y, \overline{y} \in Z. \tag{4}$$

$(Z, \| \cdot \|)$ is a Hilbert space.
Then, we consider the extremal problem :

$$(\mathcal{P}) \quad \begin{cases} \inf J(y) := \frac{1}{2} \|y - y_{obs}\|^2_{L^2(q_T)} + \frac{r}{2} \|Ly\|^2_X, \quad r \geq 0 \\ \text{subject to} \quad y \in W := \{ y \in Z; \, Ly = 0 \text{ in } X \} \end{cases}$$

$(\mathcal{P})$ is well posed : J is continuous over W, strictly convex and $J(y) \to +\infty$ as $\|y\|_W \to \infty$.

The solution of $(\mathcal{P})$ in W does not depend on η.

From (2), the solution y in Z of $(\mathcal{P})$ satisfies $(y(\cdot, 0), y_t(\cdot, 0)) \in \boldsymbol{H}$, so that problem $(\mathcal{P})$ is equivalent to the minimization of J w.r.t $(y_0, y_1) \in \boldsymbol{H}$.

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Optimality of $(\mathcal{P})$

In order to solve $(\mathcal{P})$, we have to deal with the constraint eq. which appears in W. We introduce a Lagrange multiplier $\lambda \in X'$ and the following mixed formulation: find $(y, \lambda) \in Z \times X'$ solution of

$$\begin{cases} a_r(y, \overline{y}) + b(\overline{y}, \lambda) & = & l(\overline{y}), & \forall \overline{y} \in Z \\ b(y, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in \Lambda, \end{cases} \tag{5}$$

where

$$a_r : Z \times Z \to \mathbb{R}, \quad a_r(y, \overline{y}) := \iint_{q_T} y\, \overline{y}\, dxdt + r \int_0^T \langle Ly,\, L\overline{y} \rangle_{H^{-1}(\Omega)}\, dt,$$

$$b : Z \times X' \to \mathbb{R}, \quad b(y, \lambda) := \int_0^T \langle \lambda,\, Ly \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt,$$

$$l : Z \to \mathbb{R}, \quad l(y) := \iint_{q_T} y_{obs}\, y\, dxdt.$$

System (19) is the optimality system corresponding to the extremal problem $(\mathcal{P})$.

Well-posedness of the mixed formulation

Theorem

Under the hypothesis $(\mathcal{H})$, *for any* $r \geq 0$,

1. *The mixed formulation (19) is well-posed.*
2. *The unique solution* $(y, \lambda) \in Z \times X'$ *is the unique saddle-point of the Lagrangian* $\mathcal{L} : Z \times X' \to \mathbb{R}$ *defined by*

$$\mathcal{L}(y, \lambda) := \frac{1}{2} a_r(y, y) + b(y, \lambda) - l(y).$$

3. *We have the estimate*

$$\|y\|_Y = \|y\|_{L^2(q_T)} \leq \|y_{obs}\|_{L^2(q_T)}, \quad \|\lambda\|_{X'} \leq 2\sqrt{C_{\Omega,T} + \eta}\|y_{obs}\|_{L^2(q_T)}. \tag{6}$$

Well-posedness

The kernel $\mathcal{N}(b) = \{y \in Z; b(y,\lambda) = 0 \quad \forall \lambda \in X'\}$ coincides with W: we easily get

$$a_r(y,y) = \|y\|_Z^2, \quad \forall y \in \mathcal{N}(b) = W.$$

It remains to check the inf-sup constant property : $\exists \delta > 0$ such that

$$\inf_{\lambda \in X'} \sup_{y \in Z} \frac{b(y,\lambda)}{\|y\|_Z \|\lambda\|_{X'}} \geq \delta. \tag{7}$$

For any fixed $\lambda \in X'$, we define $y^0 \in Z$ as the unique solution of

$$Ly^0 = -\Delta\lambda \;\text{ in }\; Q_T, \quad (y^0(\cdot,0), y_t^0(\cdot,0)) = (0,0) \;\text{ on }\; \Omega, \quad y^0 = 0 \;\text{ on }\; \Sigma_T.$$

We get $b(y^0,\lambda) = \|\lambda\|_{X'}^2$ and $\|y^0\|_Z^2 = \|y^0\|_{L^2(q_T)}^2 + \eta\|\lambda\|_{X'}^2$.

The estimate $\|y^0\|_{L^2(q_T)} \leq \sqrt{C_{\Omega,T}}\|\lambda\|_{X'}$ implies that

$$\sup_{y \in Z} \frac{b(y,\lambda)}{\|y\|_Y \|\lambda\|_{X'}} \geq \frac{b(y^0,\lambda)}{\|y^0\|_Y \|\lambda\|_{X'}} \geq \frac{1}{\sqrt{C_{\Omega,T} + \eta}} > 0$$

leading to (7) with $\delta = (C_{\Omega,T} + \eta)^{-1/2}$.

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Remark 1

From the first eq.,

$$\iint_{q_T} (y - y_{obs})\overline{y}dtdx + \int_0^T < \lambda, L\overline{y} >_{H_0^1, H^{-1}} dt = 0, \quad \forall \overline{y} \in Z$$

the multiplier $\lambda \in X'$ solves in the sense of transposition

$$\begin{cases} L\lambda = -(y - y_{obs})\, 1_{q_T}, \quad \lambda = 0 \quad \text{in} \quad \Sigma_T, \\ \lambda(\cdot, T) = \lambda_t(\cdot, T) = 0 \quad \text{in}\, \Omega \end{cases} \tag{8}$$

If y_{obs} is the restriction to q_T of a solution of (1), then λ vanishes almost everywhere.

In that case, $\sup_{\lambda \in \Lambda} \inf_{y \in Y} \mathcal{L}_r(y, \lambda) = \inf_{y \in Y} \mathcal{L}_r(y, 0)$

The mixed formulation is reduced to : find $y \in Z$ such that

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The mixed formulation is reduced to : find $y \in Z$ such that

$$a_r(y, \overline{y}) = \iint_{q_T} y\, \overline{y}\, dxdt + r \int_0^T \langle Ly,\, L\overline{y} \rangle_{H^{-1}, H^{-1}(\Omega)} dt = l(\overline{y}), \quad \forall \overline{y} \in Z.$$

Remark 2

In the general case, the mixed formulation can be rewritten as follows: find $(z, \lambda) \in Z \times X'$ solution of

$$\begin{cases} \langle P_r y, P_r \overline{y} \rangle_{X \times L^2(q_T)} + \langle L\overline{y}, \lambda \rangle_{X,X'} = \langle (0, y_{obs}), P_r \overline{y} \rangle_{X \times L^2(q_T)}, & \forall \overline{y} \in Z, \\ \langle Ly, \overline{\lambda} \rangle_{X,X'} = 0, & \forall \overline{\lambda} \in X' \end{cases}$$

with $P_r y := (\sqrt{r} L\, y, y_{|q_T})$.

Analogy with the quasi-reversibility method [Klibanov-Beilina 08, Bourgeois-Darde 10]: for any $\varepsilon > 0$, find $y_\varepsilon \in Z$ such that

$$\langle P y_\varepsilon, P \overline{y} \rangle_{X \times L^2(q_T)} + \varepsilon \langle y_\varepsilon, \overline{y} \rangle_Z = \langle (f, y_{obs}), P\overline{y} \rangle_{X \times L^2(q_T), X \times L^2(q_T)}, \quad \forall \overline{y} \in Z, \quad (QR)$$

equivalent to the minimization over Z of

$$\begin{aligned} y \rightarrow & \| Py - (f, y_{obs}) \|^2_{X \times L^2(q_T)} + \varepsilon \|y\|^2_Z \\ & = \|Ly - f\|^2_X + \|y - y_{obs}\|^2_{L^2(q_T)} + \varepsilon \|y\|^2_Z \end{aligned} \qquad (9)$$

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Remark 3: Stabilized mixed formulation

$$\Lambda := \{\lambda \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), L\lambda \in L^2(Q_T), \lambda(\cdot,T) = \lambda_t(\cdot,T) = 0\}.$$

$$\begin{cases} \sup_{\lambda\in\Lambda} \inf_{y\in Z} \mathcal{L}_{r,\alpha}(y,\lambda) \\ \mathcal{L}_{r,\alpha}(y,\lambda) := \mathcal{L}_r(y,\lambda) - \frac{\alpha}{2}\|L\lambda + (y - y_{obs})1_\omega\|^2_{L^2(Q_T)}, \qquad \alpha > 0. \end{cases}$$

For $\alpha \geq 0$, find $(y,\lambda) \in Z \times \Lambda$ such that

$$\begin{cases} a_{r,\alpha}(y,\overline{y}) + b_\alpha(\overline{y},\lambda) & = & l_{1,\alpha}(\overline{y}), & \forall \overline{y} \in Y \\ b_\alpha(y,\overline{\lambda}) - c_\alpha(\lambda,\overline{\lambda}) & = & l_{2,\alpha}(\overline{\lambda}), & \forall \overline{\lambda} \in \widetilde{\Lambda}, \end{cases} \tag{10}$$

$$a_{r,\alpha} : Z \times Z \to \mathbb{R}, \quad a_{r,\alpha}(y,\overline{y}) := (1-\alpha)\iint_{q_T} y\overline{y}\,dxdt + r\int_0^T (Ly, L\overline{y})_{H^{-1}(\Omega)}dt,$$

$$b_\alpha : Z \times \Lambda \to \mathbb{R}, \quad b_\alpha(y,\lambda) := \int_0^T \langle \lambda, Ly\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}dt - \alpha\iint_{q_T} y\, L\lambda\, dxdt,$$

$$c_\alpha : \Lambda \times \Lambda \to \mathbb{R}, \quad c_\alpha(\lambda,\overline{\lambda}) := \alpha\iint_{Q_T} L\lambda\, L\overline{\lambda},\ dxdt$$

$$l_{1,\alpha} : Z \to \mathbb{R}, \quad l_{1,\alpha}(y) := (1-\alpha)\iint_{q_T} y_{obs}\, y\, dxdt,$$

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Remark 3: Stabilized mixed formulation

Proposition

Under the hypothesis $(\mathcal{H})$, for any $\alpha \in (0,1)$, the corresponding mixed formulation is well-posed. The unique pair (y,λ) in $Z \times \Lambda$ satisfies

$$\theta_1 \|y\|_Z^2 + \theta_2 \|\lambda\|_\Lambda^2 \leq \left(\frac{(1-\alpha)^2}{\theta_1} + \frac{\alpha^2}{\theta_2} \right) \|y_{obs}\|^2_{L^2(q_T)}. \tag{11}$$

with $\theta_1 := \min\left(1-\alpha, r\,\eta^{-1}\right), \theta_2 := \frac{1}{2}\min\left(\alpha, C^{-1}_{\Omega,T}\right)$.

Proposition

If the solution $(y,\lambda) \in Z \times X'$ of (19) enjoys the property $\lambda \in \Lambda$, then the solutions of (19) and (10) coincide.

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Remark 4 - Link with controllability

The mixed formulation has a structure very closed to the one we get when we address - using the same approach - the null controllability of (1): the control of minimal $L^2(q_T)$-norm which drives to rest $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ is given by $v = \varphi\, 1_{q_T}$ where $(\varphi, \lambda) \in \Phi \times L^2(0, T; H_0^1(\Omega))$ solves

$$\begin{cases} a(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) = l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi \\ b(\varphi, \overline{\lambda}) = 0, & \forall \overline{\lambda} \in L^2(0, T; H_0^1(\Omega)), \end{cases} \tag{12}$$

where

$$a : \Phi \times \Phi \to \mathbb{R}, \quad a(\varphi, \overline{\varphi}) = \iint_{q_T} \varphi(x,t)\overline{\varphi}(x,t)\, dx\, dt$$

$$b : \Phi \times L^2(0, T; H_0^1(0,1)) \to \mathbb{R}, \quad b(\varphi, \lambda) = \int_0^T < L\varphi, \lambda >_{H^{-1}, H_0^1} dt$$

$$l : \Phi \to \mathbb{R}, \quad l(\varphi) = - < \varphi_t(\cdot, 0), y_0 >_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_0^1 \varphi(\cdot, 0)\, y_1\, dx.$$

with $\Phi = \{\varphi \in L^2(q_T),\ \varphi = 0 \text{ on } \Sigma_T \text{ such that } L\varphi \in L^2(0, T; H^{-1}(0,1))\}$.
[Cîndea- Münch, Calcolo 2015]

Remark 6 : Dual of the mixed problem - Minimization over λ

Theorem

$$\sup_{\lambda\in X'}\inf_{y\in Z}\mathcal{L}_r(y,\lambda) = -\inf_{\lambda\in X'} J_r^{\star\star}(\lambda) \quad + \mathcal{L}_r(y_0,0)$$

where $y_0\in Z$ solves $a_r(y_0,\overline{y}) = l(\overline{y}), \forall \overline{y}\in Y$ and

$$J_r^{\star\star}: X'\to\mathbb{R}, \qquad J_r^{\star\star}(\lambda) = \frac{1}{2}\int_0^T \langle \mathcal{P}_r\lambda,\lambda\rangle_{X'}dt - b(y_0,\lambda).$$

Lemma

Let $\mathcal{P}_r$ be the linear operator from X' into X' defined by

$$\mathcal{P}_r\lambda := -\Delta^{-1}(Ly), \quad \forall\lambda\in X' \quad \textit{where} \quad y\in Z \quad \textit{solves} \quad a_r(y,\overline{y}) = b(\overline{y},\lambda), \quad \forall\overline{y}\in Z.$$

i.e.

$$\iint_{Q_T} y\overline{y}\,dxdt + r\iint_{Q_T} Ly L\overline{y}\,dxdt = \int_0^T < Ly,\lambda >_{X,X'} dt, \forall\overline{y}\in Z \qquad (13)$$

For any $r>0$, the operator $\mathcal{P}_r$ is a strongly elliptic, symmetric isomorphism from X' into X'.

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where $y_0\in Z$ *solves* $a_r(y_0,\overline{y}) = l(\overline{y}), \forall\overline{y}\in Y$ *and*

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Remark 7 - Boundary observation

$(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ - Ω of class C^2

The results apply if the distributed observation on q_T is replaced by a Neumann boundary observation on a sufficiently large subset Σ_T of $\partial\Omega \times (0, T)$ (i.e. assuming $\frac{\partial y}{\partial \nu} = y_{\nu,obs} \in L^2(\Sigma_T)$ is known on Σ_T).

If (Q_T, Σ_T, T) satisfy some geometric condition, then there exists a positive constant $C_{obs} = C(\omega, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)})$ such that

$$\|y(\cdot,0), y_t(\cdot,0)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq C_{obs}\left(\left\|\frac{\partial y}{\partial \nu}\right\|^2_{L^2(\Sigma_T)} + \|Ly\|^2_{L^2(Q_T)}\right), \quad \forall y \in Z \qquad (14)$$

It suffices to re-define the form a in by $a(y, y) := \iint_{\Sigma_T} \frac{\partial y}{\partial \nu}\frac{\partial \overline{y}}{\partial \nu}\, d\sigma dx$ and the form l by $l(y) := \iint_{\Sigma_T} \frac{\partial y}{\partial \nu} y_{obs}\, d\sigma dx$ for all $y, \overline{y} \in Z$.

Inverse problem 2: Simultaneous reconstruction of y and the source from $\partial_\nu y$

$f(x,t) = \sigma(t)\mu(x)$
$c := 1$, $d(x,t) = d(x) \in L^p(\Omega)$, $\sigma \in C^1([0,T]), \sigma(0) \neq 0$, $\mu \in H^{-1}(\Omega)$

Theorem (Yamamoto-Zhang 2001)

Assume that (Γ_T, T, Q_T) satisfies the geometric optic condition. Let $y = y(\mu) \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ be the weak solution of (1) with $c := 1$ and $(y_0, y_1) = (0,0)$. $\exists C > 0$ s.t.

$$C^{-1}\|\mu\|_{H^{-1}(\Omega)} \leq \|c(x)\,\partial_\nu y\|_{L^2(\Gamma_T)} \leq C\|\mu\|_{H^{-1}(\Omega)}, \quad \forall \mu \in H^{-1}(\Omega).$$

This leads to the extremal problem :

$$\begin{cases} \inf J(y,\mu) := \frac{1}{2}\|c(x)(\partial_\nu y - y_{\nu,obs})\|^2_{L^2(\Gamma_T)} + \frac{r}{2}\iint_{Q_T}(Ly - \sigma\mu)^2\,dxdt, \\ \text{subject to } (y,\mu) \in W := \Big\{(y,\mu);\ y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \\ \qquad \mu \in H^{-1}(\Omega), Ly - \sigma\mu = 0 \text{ in } Q_T, y(\cdot,0) = y_t(\cdot,0) = 0\Big\}. \end{cases} \quad (\mathcal{P}_{y,\mu})$$

Attached to $\|(y,\mu)\|_W := \|c(x)\partial_\nu y\|_{L^2(\Gamma_T)}$, W is a Hilbert space.

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Recovering the solution and the source f when the pair (y, f) is unique

$$Y := \Big\{ (y,\mu); y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \mu \in H^{-1}(\Omega), \\ Ly - \sigma\mu \in L^2(Q_T), y(\cdot,0) = y_t(\cdot,0) = 0 \Big\}. \tag{15}$$

Hypothesis

$\exists C_{obs} = C(\Gamma_T, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)}) > 0$ *s.t.* :

$$\|\mu\|^2_{H^{-1}(\Omega)} \leq C_{obs} \Big(\|c(x)\partial_\nu y\|^2_{L^2(\Gamma_T)} + \|Ly - \sigma\mu\|^2_{L^2(Q_T)} \Big), \quad \forall (y,\mu) \in Y. \tag{$\mathcal{H}_2$}$$

Then, $\forall \eta > 0$, we define on Y the bilinear form

$$\langle (y,\mu), (\overline{y},\overline{\mu}) \rangle_Y := \iint_{\Gamma_T} (c(x))^2 \, \partial_\nu y \, \partial_\nu \overline{y} \, d\sigma dt + \eta \iint_{Q_T} (Ly - \sigma\mu)(L\overline{y} - \sigma\overline{\mu}) \, dxdt \quad \forall y, \overline{y} \in Z. \tag{16}$$

$$\|(y,z)\|_Y := \sqrt{< (y,\mu), (y,\mu) >_Y}.$$

Lemma

Under the hypotheses $(\mathcal{H}_2)$, the space $(Y, \|\cdot\|_Y)$ is a Hilbert space.

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Hypothesis

$\exists C_{obs} = C(\Gamma_T, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)}) > 0$ *s.t. :*

$$\|\mu\|^2_{H^{-1}(\Omega)} \leq C_{obs}\Big(\|c(x)\partial_\nu y\|^2_{L^2(\Gamma_T)} + \|Ly - \sigma\mu\|^2_{L^2(Q_T)}\Big), \quad \forall (y,\mu) \in Y. \tag{$\mathcal{H}_2$}$$

Then, $\forall \eta > 0$, we define on Y the bilinear form

$$\langle (y,\mu), (\overline{y},\overline{\mu})\rangle_Y := \iint_{\Gamma_T} (c(x))^2\, \partial_\nu y\, \partial_\nu \overline{y}\, d\sigma dt + \eta \iint_{Q_T} (Ly - \sigma\mu)\,(L\overline{y} - \sigma\overline{\mu})\, dxdt \quad \forall y, \overline{y} \in Z.$$
$$\|(y,z)\|_Y := \sqrt{< (y,\mu),(y,\mu) >_Y}. \tag{16}$$

Lemma

Under the hypotheses $(\mathcal{H}_2)$, the space $(Y, \|\cdot\|_Y)$ is a Hilbert space.

Recovering the solution and the source f: mixed formulation

Find $((y,\mu),\lambda) \in Y \times L^2(Q_T)$ solution of

$$\begin{cases} a_r((y,\mu),(\overline{y},\overline{\mu})) + b((\overline{y},\overline{\mu}),\lambda) & = & l(\overline{y},\overline{\mu}), & \forall(\overline{y},\overline{\mu}) \in Y \\ b((y,\mu),\overline{\lambda}) & = & 0, & \forall\overline{\lambda} \in L^2(Q_T), \end{cases} \tag{17}$$

where

$$a_r : Y \times Y \to \mathbb{R}, \quad a_r((y,\mu),(\overline{y},\overline{\mu})) := \iint_{\Gamma_T} c^2(x)\partial_\nu y \partial_\nu \overline{y}\, d\sigma dt$$
$$+ r \iint_{Q_T} (Ly - \sigma\mu)(L\overline{y} - \sigma\overline{\mu})\, dxdt, \; r \geq 0$$

$$b : Y \times L^2(Q_T) \to \mathbb{R}, \quad b((y,\mu),\lambda) := \iint_{Q_T} \lambda(Ly - \sigma\mu) dx\, dt,$$

$$l : Y \to \mathbb{R}, \quad l(y,\mu) := \iint_{\Gamma_T} c^2(x)\, \partial_\nu y\, y_{\nu,obs}\, d\sigma dt.$$

Parabolic situation

$\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ - $T > 0$, $c \in C^1(\overline{\Omega}, \mathbb{R}))$, $d \in L^\infty(Q_T)$, $y_0 \in \boldsymbol{H}$

$$\left\{ \begin{array}{ll} Ly := y_t - \nabla \cdot (c \nabla y) + dy = f, & Q_T := \Omega \times (0, T) \\ y = 0, & \Sigma_T := \partial\Omega \times (0, T) \\ y(\cdot, 0) = y_0, & \Omega. \end{array} \right. \tag{18}$$

- Inverse Problem : Distributed observation on $q_T = \omega \times (0, T), \omega \subset \Omega$

$$\left\{ \begin{array}{l} X = L^2(q_T), \\ \text{Given } (y_{obs}, f) \in (L^2(q_T), X), \text{find } y \; s.t. \; \{(18) \text{ and } y - y_{obs} = 0 \quad \text{on} \quad q_T\} \end{array} \right.$$

Well-known Difficulty:

$$\left(Ly \in L^2(Q_T), y \in L^2(q_T), y_{|\Sigma_T} = 0 \right) \Longrightarrow y \in C([\delta, T], H_0^1(\Omega)), \quad \forall \delta > 0$$

Second order mixed formulation .... as in the previous part

We then define the following extremal problem :

$$\begin{cases} \text{Minimize } J(y) := \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^{-2}|y(x,t) - y_{obs}(x,t)|^2\,dx\,dt + r\iint_{Q_T} (\rho^{-1} L\,y)^2\,dx\,dt \\ \text{Subject to } y \in \mathcal{W} := \left\{ y \in \mathcal{Y} : \rho^{-1} Ly = 0 \text{ in } L^2(Q_T) \right\} \end{cases} \qquad (P)$$

with $\rho_0, \rho \in \mathcal{R}$ where ($\rho_\star \in \mathbb{R}_\star^+$)

$$\mathcal{R} := \{w : w \in C(Q_T); w \geq \rho_\star > 0 \text{ in } Q_T; w \in L^\infty(\Omega \times (\delta, T))\ \forall \delta > 0\}$$

Let $\mathcal{Y}_0 := \left\{ y \in C^2(\overline{Q}_T) : y = 0 \text{ on } \Sigma_T \right\}$ and for $\eta > 0$, $\rho \in \mathcal{R}$, the bilinear form by

$$(y, \overline{y})_{\mathcal{Y}_0} := \iint_{q_T} \rho_0^{-2} y\,\overline{y}\,dx\,dt + \eta \iint_{Q_T} \rho^{-2} L\,y L\,\overline{y}\,dx\,dt, \quad \forall y, \overline{y} \in \mathcal{Y}_0.$$

Let $\mathcal{Y}$ be the completion of $\mathcal{Y}_0$ for this scalar product endowed with the norm

$$\|y\|_{\mathcal{Y}}^2 := \|\rho_0^{-1} y\|_{L^2(q_T)}^2 + \eta\|\rho^{-1} L\,y\|_{L^2(Q_T)}^2, \quad \forall y \in \mathcal{Y}.$$

Second order mixed formulation .... as in the previous part

We then define the following extremal problem :

$$\left\{\begin{array}{l}\text{Minimize } \; J(y) := \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^{-2} |y(x,t) - y_{obs}(x,t)|^2 \, dx\, dt + r \iint_{Q_T} (\rho^{-1} L\, y)^2 \, dx\, dt \\ \text{Subject to } \; y \in \mathcal{W} := \Big\{ y \in \mathcal{Y} : \rho^{-1} L y = 0 \text{ in } L^2(Q_T) \Big\} \end{array}\right. \qquad (P)$$

with $\rho_0, \rho \in \mathcal{R}$ where $(\rho_\star \in \mathbb{R}_\star^+)$

$$\mathcal{R} := \{ w : w \in C(Q_T); w \geq \rho_\star > 0 \text{ in } Q_T; w \in L^\infty(\Omega \times (\delta, T)) \; \forall \delta > 0 \}$$

Let $\mathcal{Y}_0 := \Big\{ y \in C^2(\overline{Q}_T) : y = 0 \text{ on } \Sigma_T \Big\}$ and for $\eta > 0$, $\rho \in \mathcal{R}$, the bilinear form by

$$(y, \overline{y})_{\mathcal{Y}_0} := \iint_{q_T} \rho_0^{-2} y \, \overline{y} \, dx\, dt + \eta \iint_{Q_T} \rho^{-2} L\, y L \overline{y} \, dx\, dt, \quad \forall y, \overline{y} \in \mathcal{Y}_0.$$

Let $\mathcal{Y}$ be the completion of $\mathcal{Y}_0$ for this scalar product endowed with the norm

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Mixed formulation

Find $(y, \lambda) \in \mathcal{Y} \times L^2(Q_T)$ solution of

$$\begin{cases} a_r(y, \overline{y}) + b(\overline{y}, \lambda) & = & l(\overline{y}) & \forall \overline{y} \in \mathcal{Y}, \\ b(y, \overline{\lambda}) & = & 0 & \forall \overline{\lambda} \in L^2(Q_T), \end{cases} \tag{19}$$

where

$$a_r : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}, \quad a(y, \overline{y}) := \iint_{q_T} \rho_0^{-2} y \, \overline{y} \, dx \, dt + r \iint_{Q_T} \rho^{-2} Ly \, L\overline{y} \, dx \, dt$$

$$b : \mathcal{Y} \times L^2(Q_T) \to \mathbb{R}, \quad b(y, \lambda) := \iint_{Q_T} \rho^{-1} Ly \, \lambda \, dx \, dt$$

$$l : \mathcal{Y} \to \mathbb{R}, \quad l(y) := \iint_{q_T} \rho_0^{-2} y \, y_{obs} \, dx \, dt.$$

Theorem

Let $\rho_0 \in \mathcal{R}$ and $\rho \in \mathcal{R} \cap L^\infty(Q_T)$.

1. *The mixed formulation* (19) *is well-posed.*
2. *The unique solution $(y, \lambda) \in \mathcal{Y} \times L^2(Q_T)$ is the unique saddle-point of the Lagrangian $\mathcal{L}_r : \mathcal{Y} \times L^2(Q_T) \to \mathbb{R}$ defined by*

$$\mathcal{L}_r(y, \lambda) := \frac{1}{2} a_r(y, y) + b(y, \lambda) - l(y).$$

3. *The solution (y, λ) satisfies the estimates*

$$\|y\|_{\mathcal{Y}} \leq \|\rho_0^{-1} y_{obs}\|_{L^2(q_T)}, \quad \|\lambda\|_{L^2(Q_T)} \leq 2\sqrt{\rho_\star^{-2} \|\rho\|^2_{L^\infty(Q_T)} + \eta}\, \|\rho_0^{-1} y_{obs}\|_{L^2(q_T)}.$$

Corollary

Let $\rho_0 \in \mathcal{R}, \rho \in \mathcal{R} \cap L^\infty(Q_T)$ and assume $\exists K$ s.t.

$$\rho_0 \leq K \rho_{c,0}, \quad \rho \leq K \rho_c \quad in \quad Q_T.$$

If (y, λ) is the solution of the mixed formulation (19), then $\exists C > 0$ such that

$$\|\rho_{c,0}^{-1} y\|_{L^2(Q_T)} \leq C \|y\|_{\mathcal{Y}}.$$

Mixed formulation

Theorem

Let $\rho_0 \in \mathcal{R}$ and $\rho \in \mathcal{R} \cap L^\infty(Q_T)$.

1. *The mixed formulation* (19) *is well-posed.*
2. *The unique solution* $(y, \lambda) \in \mathcal{Y} \times L^2(Q_T)$ *is the unique saddle-point of the Lagrangian* $\mathcal{L}_r : \mathcal{Y} \times L^2(Q_T) \to \mathbb{R}$ *defined by*

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3. *The solution* (y, λ) *satisfies the estimates*

$$\|y\|_{\mathcal{Y}} \leq \|\rho_0^{-1} y_{obs}\|_{L^2(q_T)}, \quad \|\lambda\|_{L^2(Q_T)} \leq 2\sqrt{\rho_\star^{-2} \|\rho\|^2_{L^\infty(Q_T)} + \eta}\, \|\rho_0^{-1} y_{obs}\|_{L^2(q_T)}.$$

Corollary

Let $\rho_0 \in \mathcal{R}$, $\rho \in \mathcal{R} \cap L^\infty(Q_T)$ and assume $\exists K$ s.t.

$$\rho_0 \leq K \rho_{c,0}, \quad \rho \leq K \rho_c \quad in \quad Q_T.$$

If (y, λ) is the solution of the mixed formulation (19), then $\exists C > 0$ such that

$$\|\rho_{c,0}^{-1} y\|_{L^2(Q_T)} \leq C \|y\|_{\mathcal{Y}}.$$

Stabilization

The first equation of the mixed formulation (19) reads as follows:

$$\iint_{q_T} \rho_0^{-2}\, y\, \overline{y}\, dx\, dt + \iint_{Q_T} \rho^{-1} L\overline{y}\, \lambda\, dx\, dt = \iint_{q_T} \rho_0^{-2}\, y_{obs}\, \overline{y}\, dx\, dt \quad \forall \overline{y} \in \mathcal{Y}.$$

$\rho^{-1}\lambda \in L^2(Q_T)$ solves the parabolic equation in the transposition sense, i.e. $\rho^{-1}\lambda$ solves the problem :

$$\left\{ \begin{array}{lll} L^\star\,(\rho^{-1}\lambda) = -\rho_0^{-2}\,(y - y_{obs})1_{q_T} & \text{in} & Q_T, \\ \rho^{-1}\lambda = 0 & \text{on} & \Sigma_T, \\ (\rho^{-1}\lambda)(\cdot, T) = 0 & \text{in} & \Omega. \end{array} \right. \tag{20}$$

Therefore, $\rho^{-1}\lambda$ belongs to $C^0([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$.

$$\begin{aligned} \Lambda := \{\lambda : \rho^{-1}\lambda \in C^0([0,T]; L^2(\Omega)), \rho_0\, L^\star(\rho^{-1}\lambda) \in L^2(Q_T), \\ \rho^{-1}\lambda = 0 \text{ on } \Sigma_T,\ (\rho^{-1}\lambda)(\cdot, T) = 0\}. \end{aligned} \tag{21}$$

$$\left\{ \begin{array}{l} \sup\limits_{\lambda \in \Lambda} \inf\limits_{y \in \mathcal{Y}} \mathcal{L}_{r,\alpha}(y,\lambda), \\ \mathcal{L}_{r,\alpha}(y,\lambda) := \mathcal{L}_r(y,\lambda) - \dfrac{\alpha}{2} \left\| \rho_0 \left(L^\star(\rho^{-1}\lambda) + \rho_0^{-2}(y - y_{obs})1_\omega \right) \right\|^2_{L^2(Q_T)}. \end{array} \right.$$

Dual formulation

For any $r > 0$, let us define the linear operator $\mathcal{T}_r$ from $L^2(Q_T)$ into $L^2(Q_T)$ by

$$\mathcal{T}_r\lambda := \rho^{-1}Ly, \quad \forall\lambda \in L^2(Q_T)$$

where $y \in \mathcal{Y}$ is the unique solution to

$$a_r(y, \overline{y}) = b(\overline{y}, \lambda), \quad \forall\overline{y} \in \mathcal{Y}. \tag{22}$$

Lemma

For any $r > 0$, the operator $\mathcal{T}_r$ is a strongly elliptic, symmetric isomorphism from $L^2(Q_T)$ into $L^2(Q_T)$.

Proposition

For any $r > 0$, let $y_0 \in \mathcal{Y}$ be the unique solution of

$$a_r(y_0, \overline{y}) = l(\overline{y}), \quad \forall\overline{y} \in \mathcal{Y}$$

and let $J_r^{\star\star} : L^2(Q_T) \to L^2(Q_T)$ be the functional defined by

$$J_r^{\star\star}(\lambda) := \frac{1}{2}\iint_{Q_T} (\mathcal{T}_r\lambda)\lambda \, dx\, dt - b(y_0, \lambda).$$

The following equality holds :

$$\sup_{\lambda\in L^2(Q_T)} \inf_{y\in\mathcal{Y}} \mathcal{L}_r(y, \lambda) = - \inf_{\lambda\in L^2(Q_T)} J_r^{\star\star}(\lambda) \quad + \mathcal{L}_r(y_0, 0).$$

Dual formulation

For any $r > 0$, let us define the linear operator $\mathcal{T}_r$ from $L^2(Q_T)$ into $L^2(Q_T)$ by

$$\mathcal{T}_r \lambda := \rho^{-1} L y, \quad \forall \lambda \in L^2(Q_T)$$

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The following equality holds :

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$H_0^1 - L^2$ first order formulation

First order formulation involving y and the flux $\mathbf{p} = c(x)\nabla y$.

$$\begin{cases} \mathcal{I}(y,\mathbf{p}) := y_t - \nabla\cdot\mathbf{p} + d\,y = f, \quad \mathcal{J}(y,\mathbf{p}) := c(x)\nabla y - \mathbf{p} = \mathbf{0} & \text{in} \quad Q_T, \\ y = 0 & \text{on} \quad \Sigma_T, \\ y(x,0) = y_0(x) & \text{in} \quad \Omega. \end{cases} \tag{23}$$

$$(y_0, f) \in L^2(\Omega)\times L^2(Q_T) \Longrightarrow p \in \mathbf{L}^2(Q_T), y \in L^2(0,T,H_0^1(\Omega)), y_t \in L^2(0,T,H^{-1}(\Omega))$$

- Inverse Problem : Distributed observation on $q_T = \omega\times(0,T), \omega\subset\Omega$

$$\begin{cases} X = L^2(q_T), \\ \text{Given } (y_{obs}, f) \in (L^2(q_T), X), \text{find } (y,\mathbf{p}) \ s.t. \ \{(23) \text{ and } y - y_{obs} = 0 \quad \text{on} \quad q_T\} \end{cases}$$

Parabolic case: $H_0^1 - L^2$ first order formulation of the parabolic

The extremal problem is then :

$$\begin{cases} \text{Minimize } \; J(y,\mathbf{p}) := \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^{-2}|y(x,t)-y_{obs}(x,t)|^2\,dx\,dt + \mathbf{r}\ldots. \\ (y,\mathbf{p}) \in \mathcal{V} := \Big\{(y,\mathbf{p}) \in \mathcal{U} \;:\; \rho_1^{-1}\mathcal{J}(y,\mathbf{p}) = 0 \text{ in } \mathbf{L}^2(Q_T), \quad \rho^{-1}\mathcal{I}(y,\mathbf{p}) = 0 \text{ in } L^2(Q_T)\Big\} \end{cases}$$

$\mathcal{U}$ - completion of $\mathcal{U}_0 := \Big\{(y,\mathbf{p}) \in C^1(\overline{Q}_T) \times \mathbf{C}^1(\overline{Q}_T) : y = 0 \text{ on } \Sigma_T\Big\}$ for

$$\begin{aligned} ((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}))_{\mathcal{U}_0} = &\iint_{q_T} \rho_0^{-2} y\,\overline{y}\,dx\,dt + \eta_1 \iint_{Q_T} \rho_1^{-2}\mathcal{J}(y,\mathbf{p})\cdot\mathcal{J}(\overline{y},\overline{\mathbf{p}})\,dx\,dt \\ &+ \eta_2 \iint_{Q_T} \rho^{-2}\mathcal{I}(y,\mathbf{p})\mathcal{I}(\overline{y},\overline{\mathbf{p}})\,dx\,dt \quad \forall (y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}) \in \mathcal{U}_0. \end{aligned}$$

for any $\eta_1,\eta_2 > 0$ and any $\rho,\ \rho_0,\ \rho_1 \in \mathcal{R}$

$$\|(y,\mathbf{p})\|_{\mathcal{U}}^2 := \|\rho_0^{-1}y\|_{L^2(q_T)}^2 + \eta_1\|\rho_1^{-1}\mathcal{J}(y,\mathbf{p})\|_{\mathbf{L}^2(Q_T)}^2 + \eta_2\|\rho^{-1}\mathcal{I}(y,\mathbf{p})\|_{L^2(Q_T)}^2.$$

Parabolic case: $H_0^1 - L^2$ first order formulation of the parabolic

The extremal problem is then :

$$\begin{cases} \text{Minimize } \; J(y,\mathbf{p}) := \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^{-2}|y(x,t) - y_{obs}(x,t)|^2 \, dx\, dt + \mathbf{r}.... \\ (y,\mathbf{p}) \in \mathcal{V} := \Big\{ (y,\mathbf{p}) \in \mathcal{U} \;:\; \rho_1^{-1}\mathcal{J}(y,\mathbf{p}) = 0 \text{ in } \mathbf{L}^2(Q_T), \quad \rho^{-1}\mathcal{I}(y,\mathbf{p}) = 0 \text{ in } L^2(Q_T) \Big\} \end{cases}$$

$\mathcal{U}$ - completion of $\mathcal{U}_0 := \Big\{ (y,\mathbf{p}) \in C^1(\overline{Q}_T) \times \mathbf{C}^1(\overline{Q}_T) : y = 0 \text{ on } \Sigma_T \Big\}$ for

$$((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}))_{\mathcal{U}_0} = \iint_{q_T} \rho_0^{-2} y\,\overline{y}\, dx\, dt + \eta_1 \iint_{Q_T} \rho_1^{-2} \mathcal{J}(y,\mathbf{p}) \cdot \mathcal{J}(\overline{y},\overline{\mathbf{p}})\, dx\, dt$$
$$+ \eta_2 \iint_{Q_T} \rho^{-2} \mathcal{I}(y,\mathbf{p}) \mathcal{I}(\overline{y},\overline{\mathbf{p}})\, dx\, dt \quad \forall (y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}}) \in \mathcal{U}_0.$$

for any $\eta_1, \eta_2 > 0$ and any $\rho,\ \rho_0,\ \rho_1 \in \mathcal{R}$

$$\|(y,\mathbf{p})\|_{\mathcal{U}}^2 := \|\rho_0^{-1} y\|_{L^2(q_T)}^2 + \eta_1 \|\rho_1^{-1}\mathcal{J}(y,\mathbf{p})\|_{\mathbf{L}^2(Q_T)}^2 + \eta_2 \|\rho^{-1}\mathcal{I}(y,\mathbf{p})\|_{L^2(Q_T)}^2.$$

Parabolic case: $H_0^1 - L^2$ first order formulation - Mixed formulation

Precisely, we set $\mathcal{X} := L^2(Q_T) \times \mathbf{L}^2(Q_T)$ and then we consider the following mixed formulation : find $((y,\mathbf{p}),(\lambda,\boldsymbol{\mu})) \in \mathcal{U} \times \mathcal{X}$ solution of

$$\left\{ \begin{aligned} a_{\boldsymbol{r}}((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}})) + b((\overline{y},\overline{\mathbf{p}}),(\lambda,\boldsymbol{\mu})) &= l(\overline{y},\overline{\mathbf{p}}) && \forall(\overline{y},\overline{\mathbf{p}}) \in \mathcal{U}, \\ b((y,\mathbf{p}),(\overline{\lambda},\overline{\boldsymbol{\mu}})) &= 0 && \forall(\overline{\lambda},\overline{\boldsymbol{\mu}}) \in \mathcal{X}, \end{aligned} \right. \tag{24}$$

where

$$a_{\boldsymbol{r}} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}, \quad a_{\boldsymbol{r}}((y,\mathbf{p}),(\overline{y},\overline{\mathbf{p}})) := \iint_{q_T} \rho_0^{-2} y\, \overline{y}\, dx\, dt$$

$$+ r_1 \iint_{Q_T} \rho_1^{-2} \mathcal{J}(y,\mathbf{p}) \cdot \mathcal{J}(\overline{y},\overline{\mathbf{p}})\, dx\, dt + r_2 \iint_{Q_T} \rho^{-2} \mathcal{I}(y,\mathbf{p}) \mathcal{I}(\overline{y},\overline{\mathbf{p}})\, dx\, dt$$

$$b : \mathcal{U} \times \mathcal{X} \to \mathbb{R}, \quad b((y,\mathbf{p}),(\lambda,\boldsymbol{\mu})) := \iint_{Q_T} \rho_1^{-1} \mathcal{J}(y,\mathbf{p}) \cdot \boldsymbol{\mu}\, dx\, dt + \iint_{Q_T} \rho^{-1} \mathcal{I}(y,\mathbf{p}) \lambda\, dx\, dt$$

$$l : \mathcal{U} \to \mathbb{R}, \quad l(y,\mathbf{p}) := \iint_{q_T} \rho_0^{-2} y\, y_{obs}\, dx\, dt.$$

$\forall \boldsymbol{r} = (r_1, r_2) \in (\mathbb{R}^+)^2$

Proposition (Imanuvilov-Puel-Yamamoto, 2010)

$$\rho_p(x,t) := \exp\left(\frac{\beta(x)}{t^2}\right), \quad \beta(x) := K_1\left(e^{K_2} - e^{\beta_0(x)}\right),$$

$$\rho_{p,0}(x,t) := t\rho_p(x,t), \quad \rho_{p,1}(x,t) := t^{-1}\rho_p(x,t), \quad \rho_{p,2}(x,t) := t^{-2}\rho_p(x,t)$$

$\exists C = C(\omega, T) > 0$ *s.t.*

$$\|\rho_{p,0}^{-1} y\|^2_{L^2(Q_T)} + \|\rho_{p,1}^{-1}\nabla y\|^2_{L^2(Q_T)} \leq C\left(\|\rho_p^{-1}\mathbf{G}\|^2_{\mathbf{L}^2(Q_T)} + \|\rho_{p,2}^{-1} g\|^2_{L^2(Q_T)} + \|\rho_{p,0}^{-1} y\|^2_{L^2(q_T)}\right),$$

for any

$$\begin{cases} y \in \mathcal{K} := \left\{y \in L^2(0,T; H_0^1(\Omega)) : y_t \in L^2(0,T; H^{-1}(\Omega))\right\}, \\ \qquad Ly = g + \nabla\cdot\mathbf{G} \text{ in } Q_T, \quad (g,\mathbf{G}) \in L^2(Q_T)\times\mathbf{L}^2(Q_T). \end{cases}$$

$$\begin{cases} Ly = \mathcal{I}(y,\mathbf{p}) - \nabla\cdot\mathcal{J}(y,\mathbf{p}), \\ \mathcal{J}(y,\mathbf{p}) := c(x)\nabla y - \mathbf{p}, \quad \mathcal{I}(y,\mathbf{p}) := y_t - \nabla\cdot\mathbf{p} + dy \end{cases}$$

FINITE DIMENSIONAL APPROXIMATION

Conformal approximation of the mixed formulation

(boundary observation case, to fix idea)

Let $Z_h \subset Z$ and $\Lambda_h \subset L^2(Q_T)$, $\dim(Z_h), \dim(\Lambda_h) < \infty$, $\forall h > 0$. Find $(y_h, \lambda_h) \in Z_h \times \Lambda_h$ solution of

$$\begin{cases} a_r(y_h, \overline{y}_h) + b(\overline{y}_h, \lambda_h) = l(\overline{y}_h), & \forall \overline{y}_h \in Z_h \\ b(y_h, \overline{\lambda}_h) = 0, & \forall \overline{\lambda}_h \in \Lambda_h. \end{cases} \quad (25)$$

if $r > 0$, a_r is coercive on Z: $a_r(y, y) \geq \frac{r}{\eta} \|y\|_Z^2 \quad \forall y \in Z$.

If there $\delta > 0$ such that

$$\forall h > 0 \qquad \delta_h := \inf_{\lambda_h \in \Lambda_h} \sup_{y_h \in Z_h} \frac{b(y_h, \lambda_h)}{\|\lambda_h\|_{L^2(Q_T)} \|y_h\|_Z} > \delta. \quad (26)$$

then, $\forall h > 0$ fixed, if $r > 0$, there exists a unique couple (y_h, λ_h) solution of (25).

Conformal approximation of the mixed formulation

(boundary observation case, to fix idea)

Let $Z_h \subset Z$ and $\Lambda_h \subset L^2(Q_T)$, $\dim(Z_h), \dim(\Lambda_h) < \infty$, $\forall h > 0$. Find $(y_h, \lambda_h) \in Z_h \times \Lambda_h$ solution of

$$\begin{cases} a_r(y_h, \overline{y}_h) + b(\overline{y}_h, \lambda_h) = l(\overline{y}_h), & \forall \overline{y}_h \in Z_h \\ b(y_h, \overline{\lambda}_h) = 0, & \forall \overline{\lambda}_h \in \Lambda_h. \end{cases} \quad (25)$$

if $r > 0$, a_r is coercive on Z: $a_r(y, y) \geq \frac{r}{\eta}\|y\|_Z^2 \quad \forall y \in Z$.

If there $\delta > 0$ such that

$$\forall h > 0 \qquad \delta_h := \inf_{\lambda_h \in \Lambda_h} \sup_{y_h \in Z_h} \frac{b(y_h, \lambda_h)}{\|\lambda_h\|_{L^2(Q_T)}\|y_h\|_Z} > \delta. \quad (26)$$

then, $\forall h > 0$ fixed, if $r > 0$, there exists a unique couple (y_h, λ_h) solution of (25).

Conformal approximation of the mixed formulation

(boundary observation case, to fix idea)

Let $Z_h \subset Z$ and $\Lambda_h \subset L^2(Q_T)$, $\dim(Z_h), \dim(\Lambda_h) < \infty$, $\forall h > 0$. Find $(y_h, \lambda_h) \in Z_h \times \Lambda_h$ solution of

$$\begin{cases} a_r(y_h, \overline{y}_h) + b(\overline{y}_h, \lambda_h) = l(\overline{y}_h), & \forall \overline{y}_h \in Z_h \\ b(y_h, \overline{\lambda}_h) = 0, & \forall \overline{\lambda}_h \in \Lambda_h. \end{cases} \tag{25}$$

if $r > 0$, a_r is coercive on Z: $a_r(y, y) \geq \frac{r}{\eta}\|y\|_Z^2 \quad \forall y \in Z$.

If there $\delta > 0$ such that

$$\forall h > 0 \qquad \delta_h := \inf_{\lambda_h \in \Lambda_h} \sup_{y_h \in Z_h} \frac{b(y_h, \lambda_h)}{\|\lambda_h\|_{L^2(Q_T)}\|y_h\|_Z} > \delta. \tag{26}$$

then, $\forall h > 0$ fixed, if $r > 0$, there exists a unique couple (y_h, λ_h) solution of (25).

First estimate

Proposition

Let $h > 0$. Let (y, λ) and (y_h, λ_h) be the solution of (19) and of (25) respectively. Let δ_h the discrete inf-sup constant defined by (26). Then,

$$\|y - y_h\|_Z \leq 2\left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right) d(y, Z_h) + \frac{1}{\sqrt{\eta}} d(\lambda, \Lambda_h),$$
$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq \left(2 + \frac{1}{\sqrt{\eta}\delta_h}\right) \frac{1}{\delta_h} d(y, Z_h) + \frac{3}{\sqrt{\eta}\delta_h} d(\lambda, \Lambda_h)$$

$d(\lambda, \Lambda_h) := \inf_{\lambda_h \in \Lambda_h} \|\lambda - \lambda_h\|_{L^2(Q_T)}$

$\mathcal{T}_h$-triangulation s.t. $\overline{Q_T} = \cup_{K \in \mathcal{T}_h} K$. $h := \max\{\mathrm{diam}(K), K \in \mathcal{T}_h\}$.

$$Z_h = \{y_h \in Z \in C^1(\overline{Q_T}) : z_h|_K \in \mathbb{P}(K) \quad \forall K \in \mathcal{T}_h,\ z_h = 0 \text{ on } \Sigma_T\},$$
$$\Lambda_h = \{\lambda_h \in C^0(\overline{Q_T}), \lambda_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h\}$$

Convergence rate in Z and in $L^2(Q_T)$

Proposition (BFS element for $N=1$ - Convergence in Z)

Let $h>0$, let $k\le 2$. If $(y,\lambda)\in H^{k+2}(Q_T)\times H^k(Q_T)$, $\exists K>0$

$$\|y-y_h\|_Z \le K\left(1+\frac{1}{\sqrt{\eta}\delta_h}+\frac{1}{\sqrt{\eta}}\right)h^k, \quad \|\lambda-\lambda_h\|_{L^2(Q_T)} \le K\left(\left(1+\frac{1}{\sqrt{\eta}\delta_h}\right)\frac{1}{\delta_h}+\frac{1}{\sqrt{\eta}\delta_h}\right)h^k.$$

Writing the ineq. obs. for $y-y_h\in Z$ and using that $L(y-y_h)=-Ly_h$, we get

$$\begin{aligned}\|y-y_h\|^2_{L^2(Q_T)} &\le C_{\Omega,T}(C_{obs}+1)(\|\partial_\nu(y-y_h)\|^2_{L^2(\Gamma_T)}+\|Ly_h\|^2_{L^2(Q_T)})\\ &\le C_{\Omega,T}(C_{obs}+1)\max(1,\frac{2}{\sqrt{\eta}})\|y-y_h\|_Z\end{aligned}$$

Theorem (BFS element for $N=1$ - Convergence in $L^2(Q_T)$)

Let $h>0$, let $k\le 2$. If $(y,\lambda)\in H^{k+2}(Q_T)\times H^k(Q_T)$,

$$\|y-y_h\|_{L^2(Q_T)} \le K\max(1,\frac{2}{\sqrt{\eta}})\left(1+\frac{1}{\sqrt{\eta}\delta_h}+\frac{1}{\sqrt{\eta}}\right)h^k.$$

Convergence rate in Z and in $L^2(Q_T)$

Proposition (BFS element for $N=1$ - Convergence in Z)

Let $h>0$, let $k\le 2$. If $(y,\lambda)\in H^{k+2}(Q_T)\times H^k(Q_T)$, $\exists K>0$

$$\|y-y_h\|_Z\le K\left(1+\frac{1}{\sqrt{\eta}\delta_h}+\frac{1}{\sqrt{\eta}}\right)h^k,\quad \|\lambda-\lambda_h\|_{L^2(Q_T)}\le K\left(\left(1+\frac{1}{\sqrt{\eta}\delta_h}\right)\frac{1}{\delta_h}+\frac{1}{\sqrt{\eta}\delta_h}\right)h^k.$$

Writing the ineq. obs. for $y-y_h\in Z$ and using that $L(y-y_h)=-Ly_h$, we get

$$\begin{aligned}\|y-y_h\|^2_{L^2(Q_T)}&\le C_{\Omega,T}(C_{obs}+1)(\|\partial_\nu(y-y_h)\|^2_{L^2(\Gamma_T)}+\|Ly_h\|^2_{L^2(Q_T)})\\&\le C_{\Omega,T}(C_{obs}+1)\max(1,\frac{2}{\sqrt{\eta}})\|y-y_h\|_Z\end{aligned}$$

Theorem (BFS element for $N=1$ - Convergence in $L^2(Q_T)$)

Let $h>0$, let $k\le 2$. If $(y,\lambda)\in H^{k+2}(Q_T)\times H^k(Q_T)$,

$$\|y-y_h\|_{L^2(Q_T)}\le K\max(1,\frac{2}{\sqrt{\eta}})\left(1+\frac{1}{\sqrt{\eta}\delta_h}+\frac{1}{\sqrt{\eta}}\right)h^k.$$

Convergence rate in Z and in $L^2(Q_T)$

Proposition (BFS element for $N=1$ - Convergence in Z)

Let $h>0$, let $k\le 2$. If $(y,\lambda)\in H^{k+2}(Q_T)\times H^k(Q_T)$, $\exists K>0$

$$\|y-y_h\|_Z \le K\left(1+\frac{1}{\sqrt{\eta}\delta_h}+\frac{1}{\sqrt{\eta}}\right)h^k, \quad \|\lambda-\lambda_h\|_{L^2(Q_T)} \le K\left(\left(1+\frac{1}{\sqrt{\eta}\delta_h}\right)\frac{1}{\delta_h}+\frac{1}{\sqrt{\eta}\delta_h}\right)h^k.$$

Writing the ineq. obs. for $y-y_h\in Z$ and using that $L(y-y_h)=-Ly_h$, we get

$$\begin{aligned}\|y-y_h\|^2_{L^2(Q_T)} &\le C_{\Omega,T}(C_{obs}+1)(\|\partial_\nu(y-y_h)\|^2_{L^2(\Gamma_T)}+\|Ly_h\|^2_{L^2(Q_T)})\\ &\le C_{\Omega,T}(C_{obs}+1)\max(1,\frac{2}{\sqrt{\eta}})\|y-y_h\|_Z\end{aligned}$$

Theorem (BFS element for $N=1$ - Convergence in $L^2(Q_T)$)

Let $h>0$, let $k\le 2$. If $(y,\lambda)\in H^{k+2}(Q_T)\times H^k(Q_T)$,

$$\|y-y_h\|_{L^2(Q_T)} \le K\max(1,\frac{2}{\sqrt{\eta}})\left(1+\frac{1}{\sqrt{\eta}\delta_h}+\frac{1}{\sqrt{\eta}}\right)h^k.$$

Choice of r versus δ_h

$(\eta = r)$

$$\delta_{r,h} \approx C_r \frac{h}{\sqrt{r}} \quad \text{as} \quad h \to 0^+, \qquad C_r > 0 \tag{27}$$

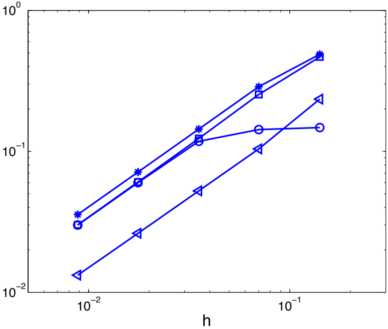


Figure: BFS finite element - Evolution of $\sqrt{r}\delta_{h,r}$ with respect to h for $r = 1$ ($\square$), $r = 10^{-2}$ ($\circ$), $r = h$ ($\star$) and $r = h^2$ ($<$).

Choice of r versus δ_h

$$\|y - y_h\|_{L^2(Q_T)} \le K \max(1, \frac{2}{\sqrt{r}})\left(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}\right) h^k.$$

Optimal parameter: $r \approx 1$ leading to $\|y - y_h\|_{L^2(Q_T)} \le K h^{k-1}$.

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \le K_2 \frac{\sqrt{r}}{h}(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}) h^k.$$

Optimal parameter: $r \approx h^2$ leading to $\|\lambda - \lambda_h\|_{L^2(Q_T)} \le K_2 h^{k-1}$.

Choice of r versus δ_h

$$\|y - y_h\|_{L^2(Q_T)} \leq K \max(1, \frac{2}{\sqrt{r}})\Big(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}\Big) h^k.$$

Optimal parameter: $r \approx 1$ leading to $\|y - y_h\|_{L^2(Q_T)} \leq K h^{k-1}$.

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 \frac{\sqrt{r}}{h}(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}) h^k.$$

Optimal parameter: $r \approx h^2$ leading to $\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 h^{k-1}$.

$\alpha \in (0,1)$ - Stabilized mixed formulation

The problem (10) becomes : find $(y_h, \lambda_h) \in Z_h \times \Lambda_h$ solution of

$$\begin{cases} a_{r,\alpha}(y_h, \overline{y}_h) + b_\alpha(\lambda_h, \overline{y}_h) & = & l_{1,\alpha}(\overline{y}_h), & \forall \overline{y}_h \in Z_h \\ b_\alpha(\overline{\lambda}_h, y_h) - c_\alpha(\lambda_h, \overline{\lambda}_h) & = & l_{2,\alpha}(\overline{\lambda}_h), & \forall \overline{\lambda}_h \in \widetilde{\Lambda}_h, \end{cases} \qquad (28)$$

$$\Lambda_h = \{\lambda \in Z_h; \lambda(\cdot, T) = \lambda_t(\cdot, T) = 0\}.$$

Proposition (BFS element for $N = 1$ - Rates of convergence)

Let $h > 0$, let $k \in \{0, 2\}$. If the solution $(y, \lambda) \in H^{k+2}(Q_T) \times H^k(Q_T)$, $\exists K > 0$

$$\|y - y_h\|_Z + \|\lambda - \lambda_h\|_\Lambda \leq K h^k.$$

Recovering the solution and the source $\mu \in H^{-1}(\Omega)$

$$\left\{ \begin{array}{rcll} a_r((y_h,\mu_h),(\overline{y}_h,\overline{\mu}_h)) + b(\overline{y}_h,\lambda_h) & = & l(\overline{y}_h), & \forall(\overline{y}_h,\overline{\mu}_h)\in Y_h \\ b((y_h,\mu_h),\overline{\lambda}_h) & = & 0, & \forall\overline{\lambda}_h \in \Lambda_h. \end{array} \right. \tag{29}$$

Theorem (BFS element for $N = 1$ - Rate of convergence $L^2(Q_T)$)

Let $h > 0$, let $k, q \in \{0, 2\}$ be two nonnegative integers. If $((y,\mu),\lambda) \in H^{k+2}(Q_T) \times H^q(\Omega) \times H^k(Q_T)$, $\exists$

$$K = K(\|y\|_{H^{k+2}(Q_T)}, \|\mu\|_{H^k(\Omega)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)}),$$

independent of h, such that

$$\|y - y_h\|_{L^2(Q_T)} \leq K C_{\Omega,T}(1 + \|\sigma\|_{L^2(0,T)}\sqrt{C_{obs}})\max(1,\frac{1}{\sqrt{\eta}}) \left[\left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right)h^k + \left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right)(\Delta x)^q\right].$$

First order versus second order - Heat case

Heat eq. ; Observation on q_T

- Second order formulation in (y, λ) ;

  $\mathbb{Q}_3 \times \mathbb{Q}_1$ approximation $\Longrightarrow \delta_h \approx \frac{C_r}{\sqrt{r}}$

$$(y,\lambda) \in H^3(Q_T) \times H^1(Q_T) \Longrightarrow \|\rho_{1,c}^{-1}(y - y_h)\|_{L^2(Q_T)} \leq K \frac{h}{\sqrt{r}}$$

- First order formulation in $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu}))$;

  $(\mathbb{Q}_1 \times \mathbb{Q}_1) \times (\mathbb{Q}_1 \times \mathbb{Q}_1)$ approximation; $\Longrightarrow \delta_h = 0 \ \forall r, h > 0$ $(Ker(B_h^\star) \neq \{0\})$

- First order stabilized formulation in $((y, \mathbf{p}), (\lambda, \mu))$ is needed

  If $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu})) \in (H^2(Q_T) \times \boldsymbol{H}^2(Q_T)) \times (H^1(Q_T) \times \boldsymbol{H}^1(Q_T))$

$$\|\rho_{1,p}^{-1}(y - y_h)\|_{L^2(Q_T)} \leq Kh$$

First order versus second order - Heat case

Heat eq. ; Observation on q_T

- Second order formulation in (y, λ) ;

  $\mathbb{Q}_3 \times \mathbb{Q}_1$ approximation $\Longrightarrow \delta_h \approx \frac{C_r}{\sqrt{r}}$

  $$(y, \lambda) \in H^3(Q_T) \times H^1(Q_T) \Longrightarrow \|\rho_{1,c}^{-1}(y - y_h)\|_{L^2(Q_T)} \leq K \frac{h}{\sqrt{r}}$$

- First order formulation in $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu}))$;

  $(\mathbb{Q}_1 \times \mathbb{Q}_1) \times (\mathbb{Q}_1 \times \mathbb{Q}_1)$ approximation; $\Longrightarrow \delta_h = 0 \; \forall r, h > 0 \; (Ker(B_h^\star) \neq \{0\})$

- First order stabilized formulation in $((y, \mathbf{p}), (\lambda, \mu))$ is needed

  If $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu})) \in (H^2(Q_T) \times \boldsymbol{H}^2(Q_T)) \times (H^1(Q_T) \times \boldsymbol{H}^1(Q_T))$

  $$\|\rho_{1,p}^{-1}(y - y_h)\|_{L^2(Q_T)} \leq Kh$$

First order versus second order - Heat case

Heat eq. ; Observation on q_T

- Second order formulation in (y, λ) ;

  $\mathbb{Q}_3 \times \mathbb{Q}_1$ approximation $\Longrightarrow \delta_h \approx \frac{C_r}{\sqrt{r}}$

  $$(y, \lambda) \in H^3(Q_T) \times H^1(Q_T) \Longrightarrow \|\rho_{1,c}^{-1}(y - y_h)\|_{L^2(Q_T)} \leq K \frac{h}{\sqrt{r}}$$

- First order formulation in $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu}))$;

  $(\mathbb{Q}_1 \times \mathbb{Q}_1) \times (\mathbb{Q}_1 \times \mathbb{Q}_1)$ approximation; $\Longrightarrow \delta_h = 0 \; \forall r, h > 0 \; (Ker(B_h^\star) \neq \{0\})$

- First order stabilized formulation in $((y, \mathbf{p}), (\lambda, \mu))$ is needed

  If $((y, \mathbf{p}), (\lambda, \boldsymbol{\mu})) \in (H^2(Q_T) \times \boldsymbol{H}^2(Q_T)) \times (H^1(Q_T) \times \boldsymbol{H}^1(Q_T))$

  $$\|\rho_{1,p}^{-1}(y - y_h)|_{L^2(Q_T)} \leq Kh$$

EXPERIMENTS

Numerical illustration - $N = 1$

$$\textbf{(EX1)} \quad y_0(x) = 1 - |2x - 1|, \quad y_1(x) = 1_{(1/3,2/3)}(x), \qquad x \in (0,1)$$

in $H_0^1 \times L^2$ for which the Fourier coefficients are

$$a_k = \frac{4\sqrt{2}}{\pi^2 k^2} \sin(\pi k/2), \quad b_k = \frac{1}{\pi k}(\cos(\pi k/3) - \cos(2\pi k/3)), \quad k > 0$$

$f = 0$. $T = 2$ - The corresponding solution of (1) with $c \equiv 1, d \equiv 0$ is given by

$$y(x,t) = \sum_{k>0} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sqrt{2} \sin(k\pi x)$$

Example 1 - $N = 1$ - Observation on q_T

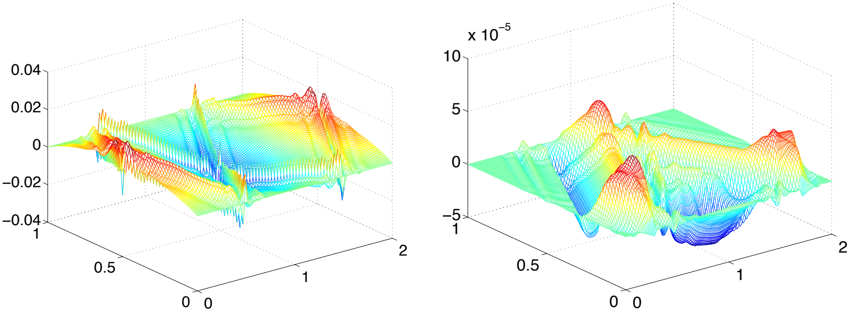
$q_T = (0.1, 0.3) \times (0, T)$

h	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\frac{\lVert y-y_h\rVert_{L^2(Q_T)}}{\lVert y\rVert_{L^2(Q_T)}}$	1.01×10^{-1}	4.81×10^{-2}	2.34×10^{-2}	1.15×10^{-2}	5.68×10^{-3}
$\frac{\lVert y-y_h\rVert_{L^2(q_T)}}{\lVert y\rVert_{L^2(q_T)}}$	1.34×10^{-1}	5.05×10^{-2}	2.37×10^{-2}	1.16×10^{-2}	5.80×10^{-3}
$\lVert Ly_h\rVert_{L^2(Q_T)}$	7.18×10^{-2}	6.59×10^{-2}	6.11×10^{-2}	5.55×10^{-2}	5.10×10^{-2}
$\lVert \lambda_h\rVert_{L^2(Q_T)}$	1.07×10^{-4}	4.70×10^{-5}	2.32×10^{-5}	1.15×10^{-5}	5.76×10^{-6}
$\sharp$ CG iterates	29	46	83	133	201

$$\frac{\lVert y - y_h\rVert_{L^2(Q_T)}}{\lVert y\rVert_{L^2(Q_T)}} = \mathcal{O}(h^{0.574}), \quad \frac{\lVert y - y_h\rVert_{L^2(q_T)}}{\lVert y\rVert_{L^2(q_T)}} = \mathcal{O}(h^{0.94}). \tag{30}$$

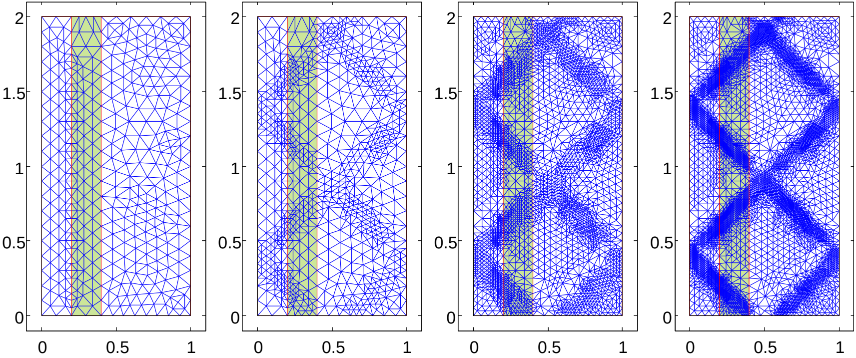
$$\lVert Ly_h\rVert_{L^2(Q_T)} = \mathcal{O}(h^{0.123}). \tag{31}$$

Example 2 - $N = 1$ - Observation on q_T



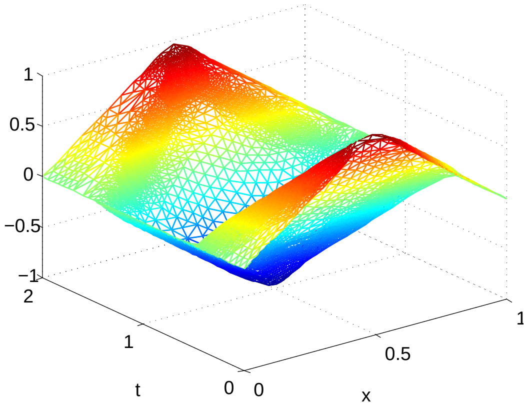
$y - y_h$ and λ_h in Q_T

Example 1 - $N = 1$ - Mesh adaptation



Iterative local refinement of the mesh according to the gradient of y_h

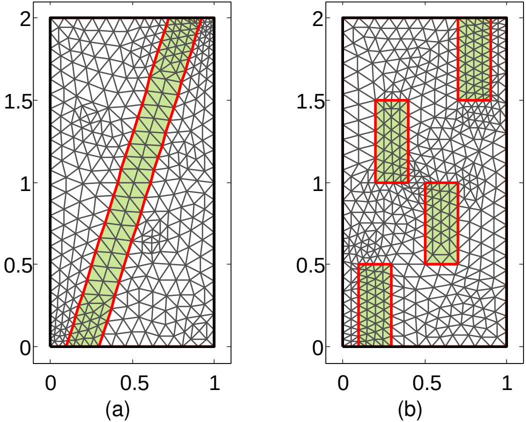
Example 1 - $N = 1$ - Mesh adaptation



Reconstructed state y_h on the adapted mesh

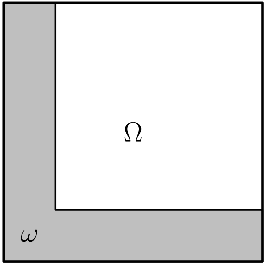
Exemple 2 : $N = 1$ - Non cylindrical domain q_T

Triangular meshes - reduced HCT elements

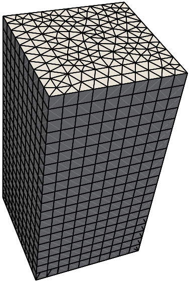


Domain q_T^1 (a) and domain q_T^2 (b) triangulated using some coarse meshes.

$2D$ example: $\Omega = (0,1)^2$ - Observation on q_T



(a)



(b)

Mesh Number	0	1	2	3
Number of elements	5 320	15 320	31 740	120 160
Number of nodes	3 234	8 799	17 670	64 411

Characteristics of the three meshes associated with Q_T.

$2D$ example: $\Omega = (0,1)^2$ - Observation on q_T

$(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$:

$$(\mathbf{EX2\text{–}2D}) \quad \begin{cases} y_0(x_1,x_2) = (1-|2x_1-1|)(1-|2x_2-1|) \\ y_1(x_1,x_2) = \mathbf{1}_{(\frac{1}{3},\frac{2}{3})^2}(x_1,x_2) \end{cases} \quad (x_1,x_2) \in \Omega. \qquad (32)$$

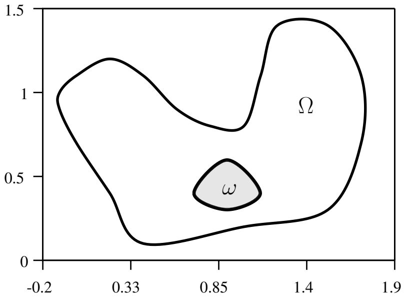
The Fourier coefficients of the corresponding solution are

$$a_{kl} = \frac{2^5}{\pi^4 k^2 l^2} \sin\frac{\pi k}{2} \sin\frac{\pi l}{2}$$
$$b_{kl} = \frac{1}{\pi^2 kl}\left(\cos\frac{\pi k}{3} - \cos\frac{2\pi k}{3}\right)\left(\cos\frac{\pi l}{3} - \cos\frac{2\pi l}{3}\right).$$

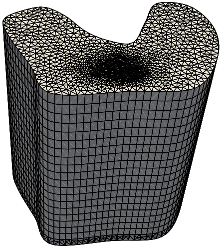
Mesh number	0	1	2	3
$\frac{\|y-y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}}$	4.74×10^{-2}	3.72×10^{-2}	2.4×10^{-2}	1.35×10^{-2}
$\|Ly_h\|_{L^2(Q_T)}$	1.18	0.89	0.99	0.99
$\|\lambda_h\|_{L^2(Q_T)}$	3.21×10^{-5}	1.46×10^{-5}	1.02×10^{-5}	3.56×10^{-6}

Table: Example **EX2–2D** – $r = h^2$

$2D$ example - Observation on q_T



(a)



(b)

Mesh number	0	1	2
Number of elements	5 730	44 900	196 040
Number of nodes	3 432	24 633	103 566

Characteristics of the three meshes associated with Q_T.

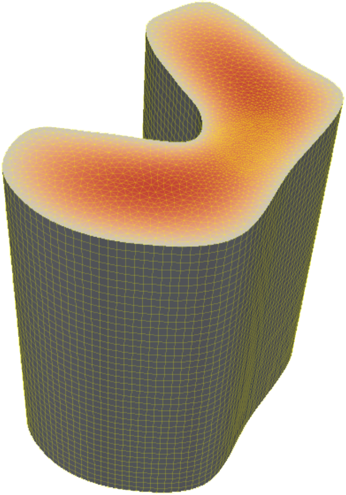
$2D$ example - Observation on q_T

$$\begin{cases} -\Delta y_0 = 10, & \text{in } \Omega \\ y_0 = 0, & \text{on } \partial\Omega, \end{cases} \qquad y_1 = 0. \tag{33}$$

Mesh number	0	1	2
$\frac{\Vert\overline{y}_h - y_h\Vert_{L^2(Q_T)}}{\Vert\overline{y}_h\Vert_{L^2(Q_T)}}$	1.88×10^{-1}	8.04×10^{-2}	5.41×10^{-2}
$\Vert Ly_h\Vert_{L^2(Q_T)}$	3.21	2.01	1.17
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	8.26×10^{-5}	3.62×10^{-5}	2.24×10^{-5}

$$r = h^2 - T = 2$$

$2D$ example - Observation on q_T



(a) (b)

y and y_h in Q_T

Numerical illustration - $N = 1$ - Observation on Γ_T

$f = 0$ - $T = 2$

$$\textbf{(EX2)} \quad y_0(x) = 1 - |2x - 1|, \quad y_1(x) = 1_{(1/3,2/3)}(x), \qquad x \in (0,1)$$

in $H_0^1 \times L^2$ for which the Fourier coefficients are

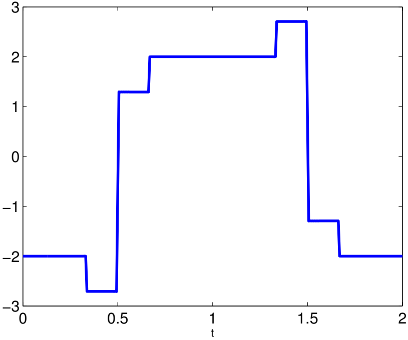


Figure: The observation $y_{\nu,obs}$ on $\{1\} \times (0, T)$ associated to initial data **EX1**.

Numerical illustration - $N = 1$ - Observation on Γ_T

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\frac{\Vert y-y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}}$	1.63×10^{-2}	6.63×10^{-3}	2.78×10^{-3}	1.29×10^{-3}	5.72×10^{-4}
$\frac{\Vert \partial_\nu(y-y_h)\Vert_{L^2(\Gamma_T)}}{\Vert \partial_\nu y\Vert_{L^2(\Gamma_T)}}$	7.67×10^{-3}	4.95×10^{-3}	3.24×10^{-3}	2.16×10^{-3}	1.48×10^{-3}
$\Vert Ly_h\Vert_{L^2(Q_T)}$	0.937	1.204	1.496	1.798	2.135
$\Vert \lambda_h\Vert_{L^2(Q_T)}$	7.74×10^{-3}	3.74×10^{-3}	1.72×10^{-3}	7.90×10^{-4}	3.60×10^{-4}
card($\{\lambda_h\}$)	861	3 321	13 041	51 681	205 761
♯ CG iterates	57	103	172	337	591

$$r = h^2 : \quad \frac{\Vert y - y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}} = \mathcal{O}(h^{1.20}), \quad \frac{\Vert \partial_\nu(y - y_h)\Vert_{L^2(\Gamma_T)}}{\Vert \partial_\nu y\Vert_{L^2(\Gamma_T)}} = \mathcal{O}(h^{0.59}),$$
$$\Vert \lambda_h\Vert_{L^2(Q_T)} = \mathcal{O}(h^{1.11}), \quad \Vert Ly_h\Vert_{L^2(Q_T)} = \mathcal{O}(h^{-0.29}). \tag{34}$$

Example 2 - $N = 2$ - The stadium

$T = 3$

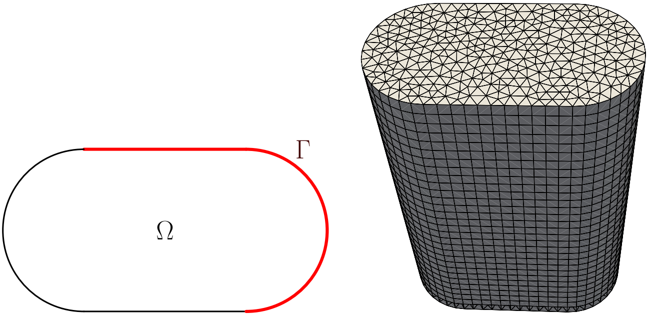


Figure: Bunimovich's stadium and the subset Γ of $\partial\Omega$ on which the observations are available. Example of mesh of the domain Q_T.

Example 2 - $N = 2$ - Recovering of the initial data

$T = 3$

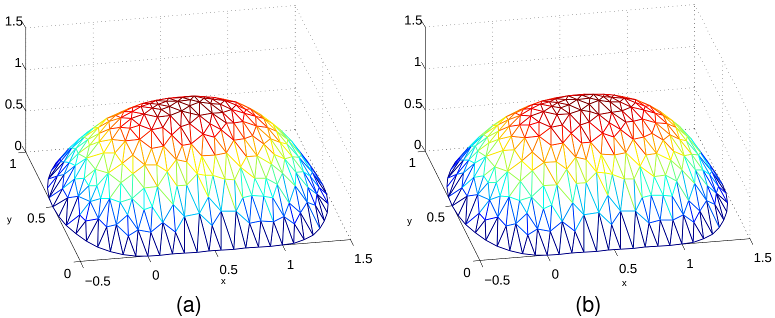


Figure: (a) Initial data y_0 given by (33). (b) Reconstructed initial data $y_h(\cdot, 0)$.

$N = 1$ - Reconstruction of y and μ from the boundary

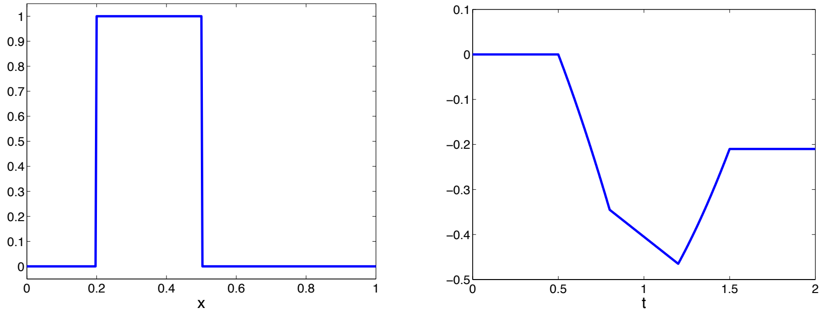
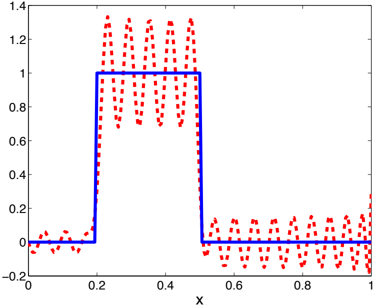


Figure: $\mu(x)$ and corresponding $\partial_\nu y|_{q_T} = y_x(1, t)$ on $(0, T)$.

$N = 1$ - Reconstruction of y and μ from the boundary

$\Delta x = \Delta t = 1/160$



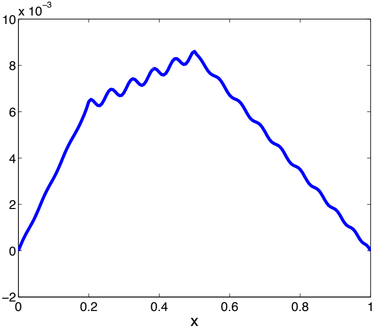


Figure: μ_h, μ and $\frac{-\Delta^{-1}(\mu-\mu_h)}{\|-\Delta^{-1}(\mu)\|_{H_0^1}}$.

$$\frac{\|\mu-\mu_h\|_{H^{-1}(\Omega)}}{\|\mu\|_{H^{-1}(\Omega)}} \approx 7.18 \times 10^{-2}, \qquad \|y - y_h\|_{L^2(Q_T)} \approx 8.68 \times 10^{-4}$$

$N = 1$ - Reconstruction of y and μ from the boundary

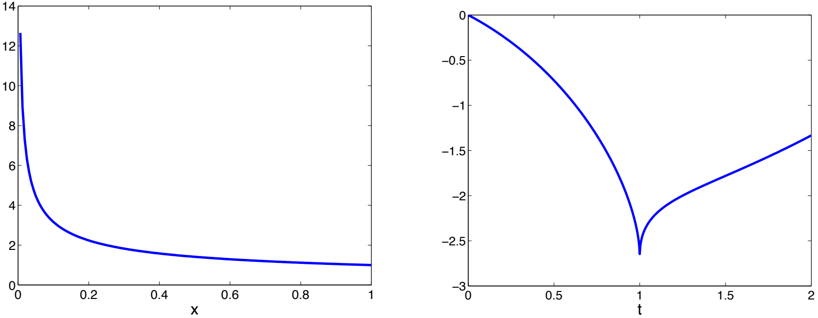


Figure: $\mu(x) = \frac{1}{\sqrt{x}}$ and corresponding $\partial_\nu y|_{q_T} = y_x(1, t)$ on $(0, T)$.

$N = 1$ - Reconstruction of y and μ from the boundary

$\Delta x = \Delta t = \frac{1}{160}$

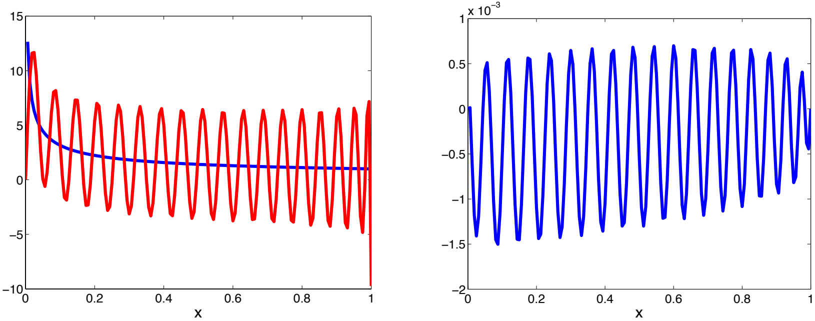


Figure: μ_h, μ and $\frac{-\Delta^{-1}(\mu-\mu_h)}{\|-\Delta^{-1}(\mu)\|_{H_0^1}}$.

$$\frac{\|\mu-\mu_h\|_{H^{-1}(\Omega)}}{\|\mu\|_{H^{-1}(\Omega)}} \approx 2.21 \times 10^{-2}, \qquad \|y - y_h\|_{L^2(Q_T)} \approx 3.56 \times 10^{-5}$$

$N = 1$ - Reconstruction of y and μ from the boundary

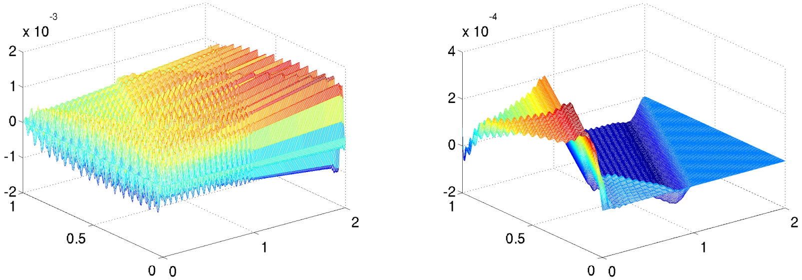
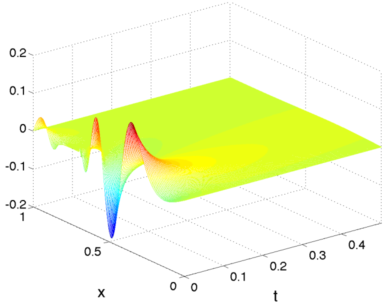


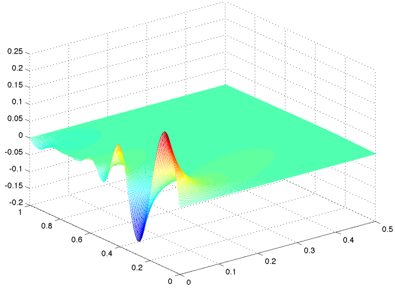
Figure: $y - y_h$ and λ_h

$N = 1$ - Heat eq. Comparison with the standard method

$$y_0(x) = \sin(\pi x)^{20}, \quad Q_T = (0,1)\times(0,T), \quad q_T = (0.7, 0.8)\times(0,T), \quad T = 1/2$$

$$\min_{y_{0h}}\left(J_h(y_{0h}) + \frac{h^2}{2}\|y_{0h}\|^2_{L^2(\Omega)}\right) \quad \textit{vs.} \quad \min_{\lambda_h} J^{\star\star}(\lambda_h) \quad \text{over} \quad \Lambda_h \tag{35}$$

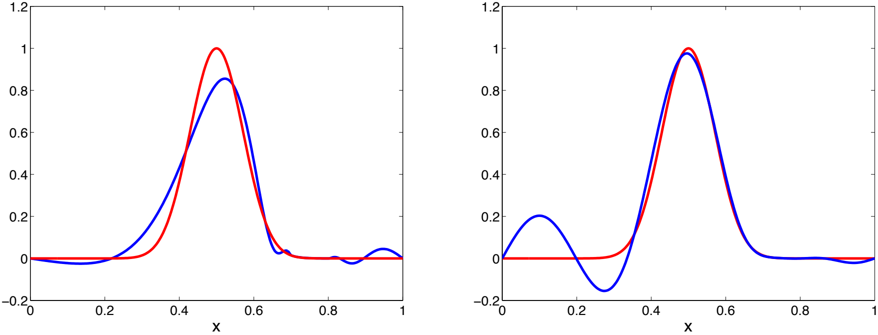




$$\frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} \approx 5.86 \times 10^{-2}, \qquad \frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} \approx 7.70 \times 10^{-2}$$

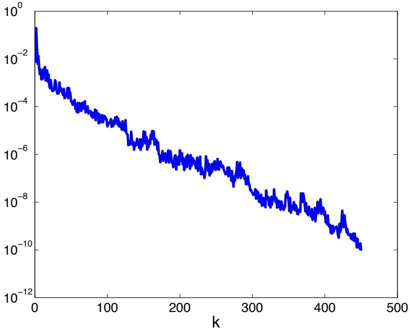
$N = 1$ - Comparison with the standard method

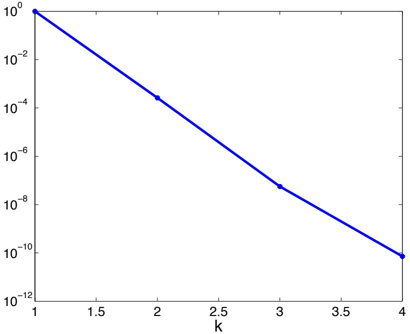
$$y_0(x) = \sin(\pi x)^{20}, \quad Q_T = (0,1) \times (0,T), \quad q_T = (0.7, 0.8) \times (0,T), \quad T = 1/2$$



Restriction at $(0,1) \times \{0\}$

$N = 1$ - Comparison with the standard method





Evolution of the relative residu $\frac{\|g^k\|}{\|g^0\|}$ w.r.t. iterate k

Concluding remarks

MIXED FORMULATION ALLOWS TO RECONSTRUCT SOLUTION AND SOURCE

DIRECT AND ROBUST METHOD - STRONG CONVERGENCE

NO NEED TO PROVE UNIFORM DISCRETE OBSERVABILITY ESTIMATE

THE MINIMIZATION OF $J_r^{\star\star}(\lambda)$ SEEMS VERY ROBUST AND FAST CONTRARY TO THE MINIMIZATION OF $J(y_0, y_1)$ (INVERSION OF SYMMETRIC DEFINITE POSITIVE AND VERY SPARSE MATRIX WITH DIRECT CHOLESKY SOLVERS)

DIRECT APPROACH CAN BE USED FOR MANY OTHER OBSERVABLE SYSTEMS FOR WHICH A GENERALIZED OBS. ESTIMATE IS AVAILABLE.

PERSPECTIVE: RECONSTRUCTION OF POTENTIAL AND COEFFICIENT

THANK YOU FOR YOUR ATTENTION

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