Remarks about the numerical approximation of controls for a semi-linear heat equation

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joint work with ENRIQUE FERNANDEZ-CARA (Sevilla)

$\omega \subset (0,1)$, $a \in C^1([0,1], \mathbb{R}^+_*)$, $y_0 \in L^2(0,1)$, $Q_T = (0,1) \times (0,T)$, $q_T = \omega \times (0,T)$, $v \in L^\infty(q_T)$

$$\begin{cases} y_t - (a(x)y_x)_x + f(y) = v1_\omega, & (x,t) \in Q_T \\ y(x,t) = 0, & (x,t) \in \{0,1\} \times (0,T) \\ y(x,0) = y_0(x), & x \in (0,1). \end{cases} \tag{1}$$

$f : \mathbb{R} \to \mathbb{R}$ is, at least, locally Lipschitz-continuous.

$$|f'(s)| \leq C(1 + |s|^p) \quad \text{a.e., with } \; p \leq 5. \tag{2}$$

Under this condition, (1) possesses exactly one local in time solution.
Under the growth condition [Cazenave-Haraux'89]

$$|f(s)| \leq C(1 + |s| \log(1 + |s|)) \quad \forall s \in \mathbb{R}, \tag{3}$$

the solutions to (1) are globally defined in $[0, T]$ and one has

$$y \in C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_0(0,1)). \tag{4}$$

Statement

$\omega \subset (0,1)$, $a \in C^1([0,1],\mathbb{R}_*^+)$, $y_0 \in L^2(0,1)$, $Q_T = (0,1)\times(0,T)$, $q_T = \omega \times (0,T)$, $v \in L^\infty(q_T)$

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The goal is to analyze numerically the null controllability properties of (1), in particular when blow-up occurs.
Without a growth condition of the kind (3), the solutions to (1) can blow up before $t = T$; in general, the blow-up time depends on the sizes of $\|y_0\|_{L^2(0,1)}$ and $\|a\|_{L^\infty}$.

Assume $f(0) = 0$. The system (1) is said to be *"null-controllable"* at time T if, for any $y_0 \in L^2(0,1)$, there exist controls $v \in L^2(q_T)$ and associated states y that are again globally defined in $[0,T]$ and satisfy (4) and

$$y(x,T) = 0, \quad x \in (0,1). \tag{5}$$

Controllability results

The first one states that, if f is “too super-linear” at infinity, then the control cannot compensate the blow-up phenomena occurring in $(0,1)\backslash\overline{\omega}$:

Theorem (Fernandez-Cara and Zuazua'00)

There exist locally Lipschitz-continuous functions f with $f(0)=0$ and

$$|f(s)| \sim |s| \log^{p}(1+|s|) \quad as \quad |s| \to \infty, \quad p>2, \tag{6}$$

such that (1) fails to be null-controllable for all $T>0$.

The second result provides conditions under which (1) is null-controllable:

Theorem (Fernandez-Cara and Zuazua'00, Barbu'00)

Let $T>0$ be given. Assume that $f:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz-continuous and satisfies (2) and

$$\frac{f(s)}{|s| \log^{3/2}(1+|s|)} \to 0 \quad as \quad |s| \to \infty. \tag{7}$$

Then (1) is null-controllable at time T.

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The proof in [Fernandez-Cara & Zuazua, 2000] is based on

- a linearization of the eq.

$$y_t - (a(x)y_x)_x + g(z)y = v\,1_\omega, \quad Q_T \tag{8}$$

with

$$g(z) = \frac{f(z)}{z} \tag{9}$$

- a fixed point argument : it is shown that the operator $\Lambda_0 : z \rightarrow y$ is continuous compact from $L^2(Q_T)$ to $L^2(Q_T)$ and maps the closed ball $B(0, M) \subset L^2(Q_T)$ into itself. Then, Schauder Theorem provides the existence of at least one fixed point for Λ_0.

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Step 1: Numerical approximation of controls for linear heat eq.

Step 1: A linear control problem

First, we deal with the controllability properties of the following linear system

$$\begin{cases} L_A y := y_t - (a(x)y_x)_x + A(x,t)y = v\,1_\omega + B(x,t), & (x,t) \in Q_T \\ y(x,t) = 0, & (x,t) \in \Sigma_T \\ y(x,0) = y_0(x), & x \in (0,1) \end{cases} \tag{10}$$

that arises naturally after linearization of (1). From Lebeau-Robbiano'95 and Fursikov-Imanuvilov'96, (10) is null-controllable.
We give some numerical methods to address the extremal problem

$$\begin{cases} \text{Minimize } \; J(y,v) = \frac{1}{2}\iint_{Q_T} \rho^2 |y|^2\,dx\,dt + \frac{1}{2}\iint_{q_T} \rho_0^2 |v|^2\,dx\,dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}_{lin}(y_0,T) \end{cases} \tag{11}$$

where $\mathcal{C}_{lin}(y_0,T)$ is the linear manifold

$$\mathcal{C}_{lin}(y_0,T) = \{\,(y,v) : v \in L^2(q_T),\; y \text{ solves (10) and satisfies } y(T) = 0\,\}.$$

We assume that $A \in L^\infty(Q_T)$ and $B \in L^2(Q_T)$ and, also, that B vanishes at $t = T$ in an appropriate sense (i.e. $\iint_{Q_T} \rho_0^2 |B|^2\,dx\,dt < +\infty$).

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$$\inf_{(y,v)\in\mathcal{C}(y_0,T)} J(y,v) = -\inf_{\phi_T\in H} J^\star(\phi_T),\ J^\star(\phi_T) = \frac{1}{2}\int_{q_T} \phi^2 dxdt + \int_\Omega \phi(0,\cdot)y_0 dx$$

where ϕ solves the backward system

$$\begin{cases} L_A^\star\phi := -\phi' - (a(x)\phi_x)_x + A\phi = 0 & Q_T \\ \phi = 0 \quad \Sigma_T = (0,T)\times\partial\Omega, & \phi(T,\cdot) = \phi_T \quad \Omega. \end{cases}$$

The Hilbert space H is defined as the completion of $\mathcal{D}(0,1)$ with respect to the norm

$$\|\phi_T\|_H = \left(\int_{q_T} \phi^2(t,x)dxdt\right)^{1/2}.$$

From the observability inequality

$$C(T,\omega)\|\phi(0,\cdot)\|^2_{L^2(\Omega)} \le \|\phi_T\|^2_H, \quad \forall\phi_T \in L^2(\Omega),$$

$J^\star$ is coercive on H and control of minimal L^2-norm is given by $v = \phi\mathcal{X}_\omega$ on Q_T.

The completed space H is huge: $H^{-s} \subset H \quad \forall s > 0$! and the minimization is severally ill-posed !!

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Micu recently proved in [Micu'11, SCL], using moment theory, that

the set of initial data y_0, for which the corresponding ϕ_T, minimizer of $J^\star$, does not belong to any negative Sobolev spaces, is dense in $L^2(0,1)$!!!

$L^2(0,1)$-norm of the HUM control with respect to time

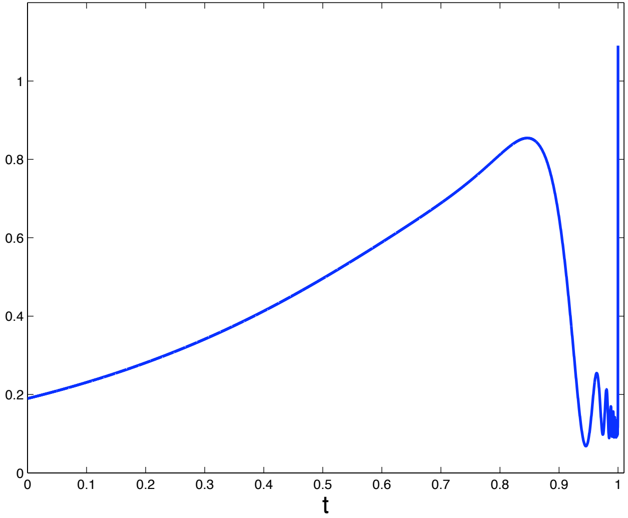

Figure: $y_0(x) = \sin(\pi x)$ - $T = 1$ - $\omega = (0.2, 0.8)$ - $t \rightarrow \|v(\cdot, t)\|_{L^2(0,1)}$ in $[0, T]$

L^2-norm of the HUM control with respect to time: Zoom near T

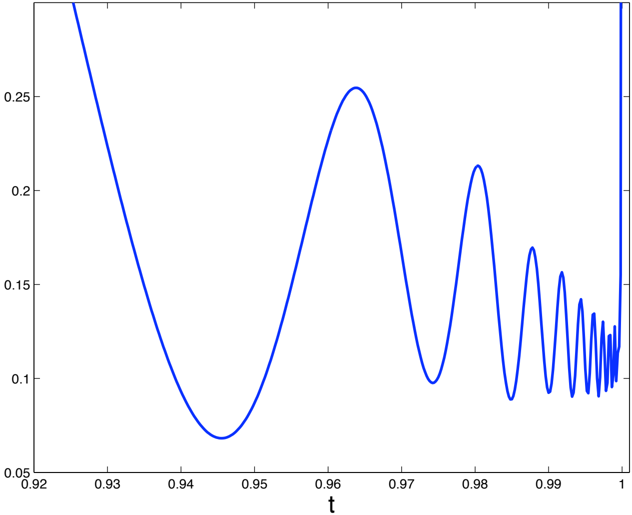

Figure: $y_0(x) = \sin(\pi x)$ - $T = 1$ - $\omega = (0.2, 0.8)$ - $t \to \|v(\cdot, t)\|_{L^2(0,1)}$ in $[0.92T, T]$

$$T = 1, \quad y_0(x) = \sin(\pi x), \quad a(x) = a_0 = 1/10, \quad \omega = (0.2, 0.8)$$

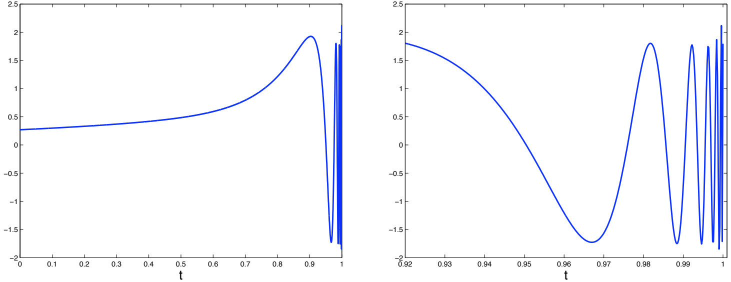

Figure: $T = 1$ - $\omega = (0.2, 0.8)$ - $\phi^N(\cdot, 0.8)$ for $N = 80$ on $[0, T]$ (**Left**) and on $[0.92T, T]$ (**Right**).

[Carthel-Glowinski-Lions'94, JOTA], [AM-Zuazua'11, Inverse Problems]

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There are “good” weight functions ρ and ρ_0 that blow up at $t = T$ and provide a very suitable solution to the original null controllability problem. They were determined and systematically used by Fursikov and Imanuvilov'96 and are the following:

$$\begin{cases} \rho(x,t) = \exp\left(\dfrac{\beta(x)}{T-t}\right), \ \rho_0(x,t) = (T-t)^{3/2}\rho(x,t), \ \beta(x) = K_1\left(e^{K_2} - e^{\beta_0(x)}\right) \\ \text{the } K_i \text{ are large positive constants (depending on } T, a_0, \|a\|_{C^1} \text{ and } \|A\|_\infty) \\ \text{and } \beta_0 \in C^\infty([0,1]), \beta_0 > 0 \text{ in } (0,1), \beta_0(0) = \beta_0(1) = 0, |\beta_0'| > 0 \text{ outside } \omega. \end{cases} \tag{13}$$

Primal (direct) approach with appropriate weights

$$\begin{cases} \text{Minimize } \; J(y,v) = \dfrac{1}{2} \displaystyle\iint_{Q_T} \rho^2 |y|^2 \, dx \, dt + \dfrac{1}{2} \iint_{q_T} \rho_0^2 |v|^2 \, dx \, dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}_{lin}(y_0, T) \end{cases} \quad (12)$$

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Numerical solution via a primal method

The roles of ρ and ρ_0 are clarified by the following arguments and results, which are mainly due to Fursikov and Imanuvilov. First, let us set $P_0 = \{\, q \in C^2(\overline{Q}_T) : q = 0 \ \text{ on } \ \Sigma_T \,\}$. In this linear space, the bilinear form

$$(p, q)_P := \iint_{Q_T} \rho^{-2} L_A^* p \, L_A^* q \, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q \, dx\, dt$$

is a scalar product. Let P be the completion of P_0 for this scalar product. Then P is a Hilbert space and the usual global Carleman estimates for the solutions to parabolic equations lead to the following result (see EFC-AM'10).

Lemma

Let ρ and ρ_0 be given by (13). Then, for any $\delta > 0$, one has $P \hookrightarrow C^0([0, T-\delta]; H_0^1(0,1))$ and the embedding is continuous. In particular, there exists $C_0 > 0$, only depending on ω, T, a_0, $\|a\|_{C^1}$ and $\|A\|_\infty$, such that

$$\|q(\cdot, 0)\|^2_{H_0^1(0,1)} \le C_0 \left(\iint_{Q_T} \rho^{-2} |L_A^* q|^2 \, dx\, dt + \iint_{q_T} \rho_0^{-2} |q|^2 \, dx\, dt \right) \qquad (14)$$

for all $q \in P$.

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Proposition

Let ρ and ρ_0 be given by (13). Let (y, v) be the corresponding optimal pair for J. Then there exists $p \in P$ such that

$$y = \rho^{-2} L_A^* p, \quad v = -\rho_0^{-2} p|_{q_T} . \tag{15}$$

The function p is the unique solution in P of

$$(p, q)_P = \int_0^1 y_0(x)\, q(x,0)\, dx + \iint_{Q_T} Bq\, dx\, dt, \quad \forall q \in P \tag{16}$$

Remark

p solves, at least in $\mathcal{D}'$, the following differential problem, that is second order in time and fourth order in space:

$$\begin{cases} L_A(\rho^{-2} L_A^* p) + \rho_0^{-2} p\, 1_\omega = B, & (x,t) \in (0,1) \times (0,T) \\ p(x,t) = 0, \ \ (\rho^{-2} L^* p)(x,t) = 0 & (x,t) \in \{0,1\} \times (0,T) \\ (\rho^{-2} L_A^* p)(x,0) = y_0(x), \ \ (\rho^{-2} L_A^* p)(x,T) = 0, & x \in (0,1). \end{cases} \tag{17}$$

The "boundary" conditions at $t = 0$ and $t = T$ appear as Neumann conditions.

Primal (direct) approach

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For any dimensional space $P_h \subset P$, we can introduce the following *approximate* problem:

$$(p_h, \overline{p}_h)_P = < l, \overline{p}_h >, \quad \forall \overline{p}_h \in P_h; \quad p_h \in P_h. \tag{18}$$

$$P_h = \{\, z_h \in C^{1,0}_{x,t}(\overline{Q_T}) : z_h|_K \in (\mathbb{P}_{3,x} \otimes \mathbb{P}_{1,t})(K) \;\; \forall K \in \mathcal{Q}_h, \; z_h = 0 \;\text{ on } \Sigma_T \,\}. \tag{19}$$

Theorem (Fernandez-Cara, AM)

Let $p_h \in P_h$ be the unique solution to (18)*, where P_h is given by* (19)*. Let us set*

$$y_h := \rho^{-2} L_A^\star p_h, \quad v_h := -\rho_0^{-2} p_h 1_{q_T}.$$

Then one has

$$\|y - y_h\|_{L^2(Q_T)} \to 0 \textit{ and } \|v - v_h\|_{L^2(q_T)} \to 0, \qquad \textit{as} \quad h \to 0$$

where (y, v) is the minimizer of J.

Conformal finite element approximation

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E. Fernández-Cara and A. Münch,
Numerical null controllability of the 1-d heat equation: primal algorithms,
To appear in SICON, Available at
http://hal.archives-ouvertes.fr/hal-00687884.

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Numerical null controllability of the 1-d heat equation: Carleman weights and duality,
To appear in JOTA - Available at
http://hal.archives-ouvertes.fr/hal-00687887.

A mixed formulation to solve the primal (direct) approach

In order to avoid the use of C^1- finite element and to keep explicit the variable y, we rewrote (16) as an equivalent mixed variational problem: find $(y, p, \lambda) \in Z \times P \times Z$ such that

$$\begin{cases} \displaystyle\iint_{Q_T} \rho^2 y\, z\, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q\, dx\, dt + \iint_{Q_T} (L_A^\star q - \rho^2 z)\, \lambda\, dx\, dt \\ \qquad\qquad = \displaystyle\int_0^1 y_0(x)\, q(x,0)\, dx + \iint_{Q_T} B\, q\, dx\, dt \quad \forall (z,q) \in Z \times P \\ \displaystyle\iint_{Q_T} (L_A^\star p - \rho^2 y)\, \mu\, dx\, dt = 0 \quad \forall \mu \in Z \end{cases} \tag{20}$$

where

$$Z = L^2(\rho^2; Q_T) := \{\, z \in L^1_{\text{loc}}(Q_T) : \iint_{Q_T} \rho^2 |z|^2\, dx\, dt < +\infty \,\}. \tag{21}$$

This mixed formulation is well-posed over $Z \times P \times Z$.

Let us now present a second mixed formulation, equivalent to (20), that does not use unbounded weights. This will be particularly important at the numerical level.
The key idea is to perform the following change of variables :

$$\eta = \rho_0^{-1} p, \quad m = \rho y = \rho^{-1} L_A^\star(\rho_0 \eta).$$

Let us introduce the space $P_* := \rho_0^{-1} P$. Then, (20) is rewritten as follows: find $(m, \eta, \mu) \in L^2(Q_T) \times P_* \times L^2(Q_T)$ such that

$$\begin{cases} \displaystyle\iint_{Q_T} m\overline{m} dx\, dt + \iint_{q_T} \eta\,\overline{\eta}\, dx\, dt + \iint_{Q_T} \Big(\rho^{-1} L_A^\star(\rho_0 \overline{\eta}) - \overline{m}\Big)\mu\, dx\, dt \\ \qquad \displaystyle = \int_0^1 \rho_0(x,0) y_0(x) \overline{\eta}(x,0) dx + \iint_{Q_T} \rho_0 B\,\overline{\eta}\, dx\, dt \quad \forall (\overline{m}, \overline{\eta}) \in L^2(Q_T) \times P_* \\ \displaystyle\iint_{Q_T} \Big(\rho^{-1} L_A^\star(\rho_0 \eta) - m\Big)\overline{\mu}\, dx\, dt = 0, \quad \forall \overline{\mu} \in L^2(Q_T) \end{cases} \tag{22}$$

Proposition

There exists a unique solution $(m, \eta, \mu) \in L^2(Q_T) \times P_* \times L^2(Q_T)$ to (22). Moreover, $y = \rho^{-1} m$ is, together with $v = \rho^{-1} \eta\, 1_\omega$, the unique solution to (12).

Step 2: Fixed points

For simplicity, we will assume that $y_0 \in L^\infty(0,1)$ and $f \in C^1(\mathbf{R})$ and is globally Lipschitz-continuous. Let us introduce the function g, with

$$g(s) = \frac{f(s)}{s} \ \text{ if } \ s \neq 0, \quad g(0) = f'(0) \ \text{ otherwise.}$$

Then $g \in C^0_b(\mathbf{R})$ and $f(s) = g(s)\, s$ for all s (recall that $f(0) = 0$). We will set $G_0 = \|g\|_{L^\infty(\mathbf{R})}$.

For any $z \in L^1(Q_T)$, let us introduce the bilinear form

$$m(z;p,q) = \iint_{Q_T} \rho^{-2} L^*_{g(z)} p \, L^*_{g(z)} q \, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q \, dx\, dt \quad \forall p,q \in P_0. \tag{23}$$

Then $m(z;\cdot\,,\cdot)$ is a scalar product in P_0 and can be used to construct a Hilbert space P that, in principle, may depend on z. We will use the following result, which is a direct consequence of the Carleman estimates :

Lemma

Under the previous conditions, if the constants K_i in (13) are large enough (depending on ω, T, a_0, $\|a\|_{C^1}$ and G_0), then there exist $C_1, C_2 > 0$ such that

$$C_1\, m(0;p,p) \leq m(z;p,p) \leq C_2\, m(0;p,p) \quad \forall p \in P_0 \tag{24}$$

for all $z \in L^1(Q_T)$.

Step 2 : Back to the nonlinear problem

For simplicity, we will assume that $y_0 \in L^\infty(0,1)$ and $f \in C^1(\mathbf{R})$ and is globally Lipschitz-continuous. Let us introduce the function g, with

$$g(s) = \frac{f(s)}{s} \text{ if } s \neq 0, \quad g(0) = f'(0) \text{ otherwise.}$$

Then $g \in C^0_b(\mathbf{R})$ and $f(s) = g(s)\,s$ for all s (recall that $f(0) = 0$). We will set $G_0 = \|g\|_{L^\infty(\mathbf{R})}$.
For any $z \in L^1(Q_T)$, let us introduce the bilinear form

$$m(z; p, q) = \iint_{Q_T} \rho^{-2} L^*_{g(z)} p\, L^*_{g(z)} q \, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q\, dx\, dt \quad \forall p, q \in P_0. \qquad (23)$$

Then $m(z; \cdot, \cdot)$ is a scalar product in P_0 and can be used to construct a Hilbert space P that, in principle, may depend on z. We will use the following result, which is a direct consequence of the Carleman estimates :

Lemma

Under the previous conditions, if the constants K_i in (13) are large enough (depending on ω, T, a_0, $\|a\|_{C^1}$ and G_0), then there exist $C_1, C_2 > 0$ such that

$$C_1\, m(0; p, p) \leq m(z; p, p) \leq C_2\, m(0; p, p) \quad \forall p \in P_0 \qquad (24)$$

for all $z \in L^1(Q_T)$.

Accordingly, all the spaces P provided by the bilinear forms $m(z;\cdot,\cdot)$ are the same and, in fact, (24) holds for all $p \in P$:

$$C_1\, m(0;p,p) \le m(z;p,p) \le C_2\, m(0;p,p) \quad \forall p \in P. \tag{25}$$

We will fix the following norm in P:

$$\|p\|_P = m(0;p,p)^{1/2} \quad \forall p \in P. \tag{26}$$

The operator Λ_0

Let us introduce the mapping $\Lambda_0 : L^2(Q_T) \mapsto L^2(Q_T)$ where, for any $z \in L^2(Q_T)$, $y_z = \Lambda_0(z)$ is, together with v_z, the unique solution to the linear extremal problem

$$\text{Minimize } \; J(z; y, v) := \frac{1}{2} \iint_{Q_T} \rho^2 |y|^2 \, dx\, dt + \frac{1}{2} \iint_{q_T} \rho_0^2 |v|^2 \, dx\, dt \tag{27}$$

subject to $v \in L^2(q_T)$ and

$$\begin{cases} y_t - (a(x) y_x)_x + g(z)\, y = v\, 1_\omega & (x,t) \in Q_T \\ y(x,t) = 0, & (x,t) \in \Sigma_T \\ y(x,0) = y_0(x), & x \in (0,1) \end{cases} \tag{28}$$

such that $y(\cdot, T) = 0$. $\Lambda_0 : L^2(Q_T) \mapsto L^2(Q_T)$ is well defined. Furthermore, applying proposition 1 with $A = g(z)$ and $B = 0$, we obtain that y_z and v_z are characterized as follows :

$$y_z = \Lambda_0(z) = \rho^{-2} L^\star_{g(z)} p_z, \quad v_z = -\rho_0^{-2} p_z|_{q_T}, \tag{29}$$

where $p_z \in P$ is the unique solution to the linear problem

$$m(z; p_z, q) = \int_0^1 y_0(x)\, q(x,0)\, dx \quad \forall q \in P. \tag{30}$$

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subject to $v \in L^2(q_T)$ and

$$\begin{cases} y_t - (a(x) y_x)_x + g(z) \, y = v \, 1_\omega & (x,t) \in Q_T \\ y(x,t) = 0, & (x,t) \in \Sigma_T \\ y(x,0) = y_0(x), & x \in (0,1) \end{cases} \tag{28}$$

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$$m(z; p_z, q) = \int_0^1 y_0(x) \, q(x,0) \, dx \quad \forall q \in P. \tag{30}$$

In order to solve the null controllability problem for (1), it suffices to find a solution to the fixed point equation

$$y = \Lambda_0(y), \quad y \in L^2(Q_T). \tag{31}$$

ALG 1 (fixed point):

$$y^0 \in L^2(Q_T), \quad y^{n+1} = \Lambda_0(y^n),\ n \geq 0 \tag{32}$$

If $(y^n, v^n) \rightharpoonup (y, v)$ in $L^2(Q_T) \times L^2(q_T)$, then (y, v) solves the nonlinear null controllability problem. Indeed, since the $g(y^n)$ are uniformly bounded in $L^\infty(Q_T)$, after extraction of a subsequence it can be assumed that y^n (resp. y^n_t) converges weakly in $L^2(0, T; H^1_0(0, 1))$ (resp. $L^2(0, T; H^{-1}(0, 1))$). Therefore, y^n converges strongly in $L^2(Q_T)$ and a.e., $g(y^n)$ converges to $g(y)$ weakly-$*$ in $L^\infty(Q_T)$ and we can take limits and deduce that y solves, together with v, the nonlinear system.

This fixed point formulation has been used in [Fernandez-Cara, Zuazua, 2000] to prove Theorem 2. Precisely, it is shown there that $\Lambda_0 : L^2(Q_T) \mapsto L^2(Q_T)$ is continuous and compact and, also that there exists $M > 0$ such that Λ_0 maps the whole space $L^2(Q_T)$ inside the ball $B(0; M)$. Then, Schauder's Theorem provides the existence of at least one fixed point for Λ_0.

It is however important to note that this does not imply the convergence of the sequence $\{y^n\}$ defined by $y^{n+1} = \Lambda_0(y^n)$.

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It is however important to note that this does not imply the convergence of the sequence $\{y^n\}$ defined by $y^{n+1} = \Lambda_0(y^n)$.

$$f(s) = C_f s \log^p(1+|s|) \quad \forall s \in \mathbf{R}. \tag{33}$$

We consider the following data:

$$a(x) = 1/10, \;\; p = 1.4, \;\; C_f = -5, \;\; T = 1/2, \;\; y_0(x) = \alpha \sin(\pi x)$$

In the uncontrolled situation, these data lead to the blow-up of the solution of (1) at time $t_c \approx 0.406, 0.367, 0.339, 0.318$ for $\alpha = 10, 20, 40$ and 80, respectively.

We first take $\omega = (0.2, 0.8)$ and initialize **ALG 1** with

$$y^0(x,t) = y_0(x)(1 - t/T)^2.$$

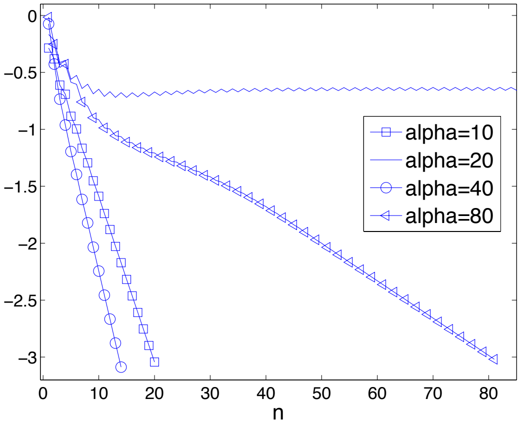

Figure: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) = \alpha \sin(\pi x)$ - Evolution of $log_{10}(\|\Lambda_0(y_h^n) - y_h^n\|_{L^2(Q_T)}/\|y_h^n\|_{L^2(Q_T)})$ vs. n for $\alpha = 10, 20, 40$ and 80.

	$\|v_h\|_{L^2(q_T)}$	$\|v_h\|_{L^\infty(q_T)}$	$\|y_h\|_{L^2(Q_T)}$	$\sharp$ iterates
$\alpha = 10$	3.531×10^1	2.542×10^2	1.742	20
$\alpha = 40$	2.142×10^2	2.053×10^3	6.654	14
$\alpha = 80$	5.109×10^2	7.021×10^3	14.410	81

Table: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) = \alpha \sin(\pi x)$ - Norms for $\alpha = 10, 40$ and $\alpha = 80$.

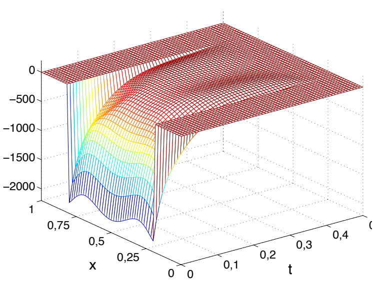

Figure: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) = 40\sin(\pi x)$ - Control v_h (**Left**) and corresponding controlled solution y_h (**Right**) in Q_T.

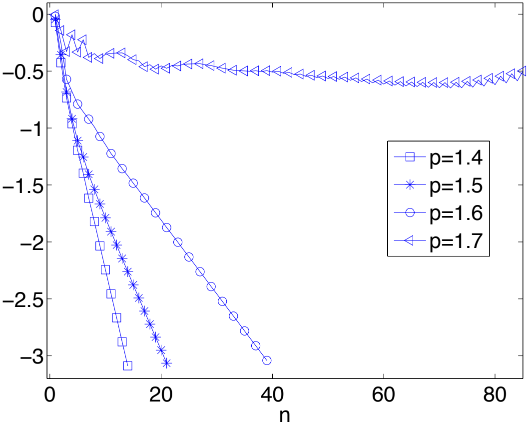

Figure: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) \equiv 40 \sin(\pi x)$ - $p = 1.4$ - Evolution of $\log_{10}(\|\Lambda_0(y_h^n) - y_h^n\|_{L^2(Q_T)} / \|y_h^n\|_{L^2(Q_T)})$ for $p = 1.4, 1.5, 1.6$ and $p = 1.7$.

We now introduce the function $\zeta(t) = (T-t)^{-1/2}$ for all t in $(0,T)$ and the space $Z := L^2(\zeta^2, Q_T)$. We will denote by Λ the restriction to Z of the mapping Λ_0. Obviously, $\Lambda(z) \in Z$ for all $z \in Z$.

Let us consider the following least squares reformulation of (31):

$$\begin{cases} \text{Minimize} \;\; R(z) := \dfrac{1}{2}\|z - \Lambda(z)\|_Z^2 \\ \text{Subject to} \;\; z \in Z. \end{cases} \tag{34}$$

Any solution to (31) solves (34). Conversely, if y solves (34), we necessarily have $R(y) = 0$ (because (1) is null controllable with control-states (y, v) such that $J(z; y, v) < +\infty$); hence, y also solves (31). This shows that (31) and (34) are, in the present context, equivalent.

Proposition

Let us assume that $g \in C^1_b(\mathbf{R})$*. Then* $R \in C^1(Z)$*. Moreover, for any* $z \in Z$*, the gradient of* R *with respect to the inner product of* Z *is given by*

$$DR(z) = (1 - \rho^{-2} g'(z) p_z)(z - y_z) + \zeta^{-2} g'(z)(y_z \lambda_z + p_z \mu_z), \tag{35}$$

where p_z *is the unique solution to* (30)*,* $y_z = \rho^{-2} L^\star_{g(z)} p_z$*,* λ_z *is the unique solution to the linear (adjoint) problem*

$$m(z; q, \lambda_z) = (z - y_z, \rho^{-2} L^\star_{g(z)} q)_Z \quad \forall q \in P; \quad \lambda_z \in P \tag{36}$$

and, finally, $\mu_z = \rho^{-2} L^\star_{g(z)} \lambda_z$*.*

For the proof, we will need the following lemma:

Lemma

For any $q \in P$ *one has* $(\zeta\rho)^{-1} q \in L^\infty(Q_T)$*. Furthermore, there exists* $C > 0$*, only depending on* ω*,* T*,* a_0*,* $\|a\|_{C^1}$ *and* G_0*, such that*

$$\|(\zeta\rho)^{-1} q\|^2_{L^\infty(Q_T)} \leq C \|q\|^2_P \quad \forall q \in P. \tag{37}$$

Proposition

Let us assume that $g \in C^1_b(\mathbf{R})$. Then $R \in C^1(Z)$. Moreover, for any $z \in Z$, the gradient of R with respect to the inner product of Z is given by

$$DR(z) = (1 - \rho^{-2} g'(z) p_z)(z - y_z) + \zeta^{-2} g'(z)(y_z \lambda_z + p_z \mu_z), \tag{35}$$

where p_z is the unique solution to (30), $y_z = \rho^{-2} L^*_{g(z)} p_z$, λ_z is the unique solution to the linear (adjoint) problem

$$m(z; q, \lambda_z) = (z - y_z, \rho^{-2} L^*_{g(z)} q)_Z \quad \forall q \in P; \quad \lambda_z \in P \tag{36}$$

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For the proof, we will need the following lemma:

Lemma

For any $q \in P$ one has $(\zeta\rho)^{-1} q \in L^\infty(Q_T)$. Furthermore, there exists $C > 0$, only depending on ω, T, a_0, $\|a\|_{C^1}$ and G_0, such that

$$\|(\zeta\rho)^{-1} q\|^2_{L^\infty(Q_T)} \leq C \|q\|^2_P \quad \forall q \in P. \tag{37}$$

Proposition

Let the assumptions in proposition 3 be satisfied and let us introduce $G_1 := \|g'\|_{L^\infty(\mathbf{R})}$. *There exists a constant* K *that depends on* ω, T, a_0, $\|a\|_{C^1}$ *and* G_0 *but is independent of* z *and* y_0, *such that the following holds for all* $z \in Z$:

$$\|DR(z)\|_Z \geq (1 - K\, G_1\, \|y_0\|_{L^\infty})\, \|z - \Lambda(z)\|_Z\,. \tag{38}$$

Lemma

With the notation of proposition 3, one has:

$$\|p_z\|_P \leq C\|y_0\|_{L^\infty} \quad \forall z \in Z \tag{39}$$

and

$$\|\lambda_z\|_P \leq C\|\zeta\rho^{-1}\|_\infty \|z - y_z\|_Z \quad \forall z \in Z, \tag{40}$$

where C *depends on* ω, T, a_0, $\|a\|_{C^1}$ *and* G_0.

Proposition

Let the assumptions in proposition 3 be satisfied and let us introduce $G_1 := \|g'\|_{L^\infty(\mathbf{R})}$. There exists a constant K that depends on ω, T, a_0, $\|a\|_{C^1}$ and G_0 but is independent of z and y_0, such that the following holds for all $z \in Z$:

$$\|DR(z)\|_Z \geq \left(1 - K\, G_1\, \|y_0\|_{L^\infty}\right) \|z - \Lambda(z)\|_Z \, . \tag{38}$$

Lemma

With the notation of proposition 3, one has:

$$\|p_z\|_P \leq C\|y_0\|_{L^\infty} \quad \forall z \in Z \tag{39}$$

and

$$\|\lambda_z\|_P \leq C\|\zeta\rho^{-1}\|_\infty \|z - y_z\|_Z \quad \forall z \in Z, \tag{40}$$

where C depends on ω, T, a_0, $\|a\|_{C^1}$ and G_0.

Let $z \in Z$ be given and let us introduce $f := z - y_z$. In view of proposition 3, one has

$$
\begin{aligned}
\|DR(z)\|_Z &\geq \frac{1}{\|f\|_Z}(DR(z), f) \\
&= \frac{1}{\|f\|_Z}\iint_{Q_T}\Big(\zeta^2(1+\rho^{-2}g'(z)\,p_z)\,|f|^2 + g'(z)(y_z\lambda_z + p_z\mu_z)\,f\Big)\,dx\,dt \\
&\geq \|f\|_Z - \frac{1}{\|f\|_Z}\iint_{Q_T}\zeta^2\rho^{-2}|g'(z)|\,|p_z|\,|f|^2\,dx\,dt \\
&\quad - \frac{1}{\|f\|_Z}\iint_{Q_T}|g'(z)|\,(|y_z||\lambda_z| + |p_z||\mu_z|)\,|f|\,dx\,dt
\end{aligned}
$$

In view of lemmas 3 and 4,

$$
\begin{aligned}
\iint_{Q_T}\zeta^2\rho^{-2}|g'(z)|\,|p_z|\,|f|^2\,dx\,dt &\leq \|\rho^{-2}g'(z)\,p_z\|_\infty\left(\iint_{Q_T}\zeta^2|f|^2\,dx\,dt\right) \\
&\leq \|\zeta\rho^{-1}\|_\infty\|g'(z)\|_\infty\,\|(\zeta\rho)^{-1}p_z\|_\infty\|f\|_Z^2 \\
&\leq C\,G_1\,\|p_z\|_P\|f\|_Z^2 \\
&\leq C\,G_1\,\|y_0\|_{L^\infty}\|f\|_Z^2
\end{aligned}
$$

On the other hand, from lemma 3 we also have

$$
\begin{aligned}
&\iint_{Q_T} |g'(z)| \left(|y_z||\lambda_z| + |p_z||\mu_z|\right) |f| \, dx\, dt \\
&\quad = \iint_{Q_T} |g'(z)| \left(|\rho^{-2} L^*_{g(z)} p_z||\lambda_z| + |p_z||\rho^{-2} L^*_{g(z)} \lambda_z|\right) |f| \, dx\, dt \\
&\quad \leq \|g'(z)\|_\infty \left[\left(\iint_{Q_T} \rho^{-2} |L^*_{g(z)} p_z|^2 \, dx\, dt \right)^{1/2} \|(\zeta\rho)^{-1}\lambda_z\|_\infty \right. \\
&\qquad \left. + \left(\iint_{Q_T} \rho^{-2} |L^*_{g(z)} \lambda_z|^2 \, dx\, dt \right)^{1/2} \|(\zeta\rho)^{-1} p_z\|_\infty \right] \left(\iint_{Q_T} \zeta^2 |f|^2 \, dx\, dt \right)^{1/2} \\
&\quad \leq C\, G_1 \left[\|y_0\|_{L^\infty} \|\lambda_z\|_P + \|p_z\|_P \|\zeta\rho^{-1}\|_\infty \|f\|_Z \right] \|f\|_Z \\
&\quad \leq C\, G_1 \, \|y_0\|_{L^\infty} \|f\|_Z^2
\end{aligned}
$$

Consequently,

$$
\|R'(z)\|_Z \geq \|f\|_Z - K\, G_1 \, \|y_0\|_{L^\infty} \|f\|_Z
$$

and we get (38) for some K.

ALG 2 (Least square):

$$z^0 \in L^2(Q_T), \quad (z^{n+1}, h)_Z = (z^n, h)_Z - \eta\,(DR(z^n), h)_Z, \quad n \geq 0$$

Least squares : a numerical application

$$f_\eta(s) = C_f s \ \log^p(1 + |s|_n) \quad \forall s \in \mathbf{R}, \quad |s|_\eta := \sqrt{s^2 + \eta^2} - \eta \tag{41}$$

so that, for all $\eta > 0$, $g_\eta := C_f \log^p(1 + |s|_\eta)$ belongs to $C^1_b(\mathbf{R})$. We have $f_\eta(0) = 0$ and Theorem applies for f_η, since f_η and f are equivalent at infinity. We take $\eta = 10^{-1}$.

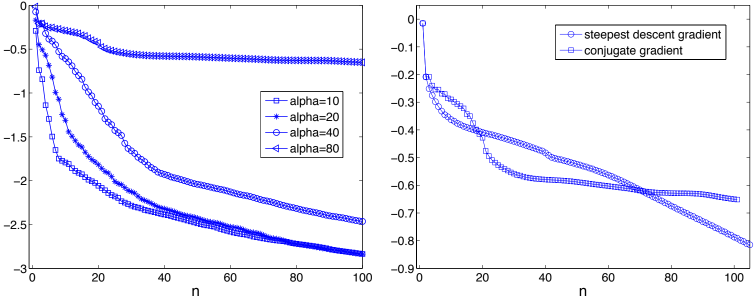

Figure: Least squares method - $h = (1/60, 1/60)$ - Evolution of $\log_{10}(\|\Lambda(z_h^n) - z_h^n\|_{L^2(Q_T)}/\|z_h^n\|_{L^2(Q_T)})$ - Left: $\alpha = 10, 20, 40, 80$ and algorithm **ALG 2'** - Right: $\alpha = 80$ and algorithms **ALG 2** and **ALG 2'**.

$$f_\eta(s) = C_f s \ \log^p(1 + |s|_n) \quad \forall s \in \mathbf{R}, \quad |s|_\eta := \sqrt{s^2 + \eta^2} - \eta \qquad (42)$$

so that, for all $\eta > 0$, $g_\eta := C_f \log^p(1 + |s|_\eta)$ belongs to $C^1_b(\mathbf{R})$. Moreover, we have $f_\eta(0) = 0$ and Theorem applies for f_η, since f_η and f are equivalent at infinity. We shall take $\eta = 10^{-1}$.

	$\|v_h\|_{L^2(q_T)}$	$\|v_h\|_{L^\infty(q_T)}$	$\|z_h\|_{L^2(Q_T)}$	$\|R'(z_h)\|_{L^2(Q_T)}$	e^n
$\alpha = 10$	3.507×10^1	2.532×10^2	1.753	1.27×10^{-3}	1.43×10^{-3}
$\alpha = 20$	8.781×10^1	7.323×10^2	3.180	1.44×10^{-3}	1.54×10^{-3}
$\alpha = 40$	2.137×10^2	2.048×10^3	6.651	5.42×10^{-3}	3.39×10^{-3}
$\alpha = 80$	2.526×10^2	3.299×10^3	14.73	2.23×10^{-1}	7.89×10^{-1}

Table: Least squares method approach after 100 iterates - $h = (1/60, 1/60)$ - $y_0(x) \equiv \alpha \sin(\pi x)$ - $p = 1.4$ - Norms for $\alpha = 10, 20, 40, 80$. Here, $e^n = \|\Lambda(z_h^n) - z_h^n\|_{L^2(Q_T)} / \|z_h^n\|_{L^2(Q_T)}$.

Let us introduce the spaces

$$Y := \{ (y,v) : y \in L^2(\rho^2; Q_T),\ y_x \in L^2(\rho_1^2; Q_T),\ y_t - (a y_x)_x \in L^2(\rho_0^2; Q_T),\\ y(0,t) = y(1,t) = 0 \ \text{a.e.},\ v \in L^2(\rho^2, Q_T) \}$$

and

$$W := L^2(\rho_0^2; Q_T) \times L^2(0,1)$$

and the mapping $F : Y \mapsto W$, with

$$F(y,v) = (y_t - (a y_x)_x + f(y) - v 1_\omega, y(\cdot,0) - y_0) \quad \forall (y,v) \in Y. \tag{43}$$

Obviously, any solution to the nonlinear equation

$$F(y,v) = (0,0), \quad (y,v) \in Y \tag{44}$$

solves the null controllability problem for (1).

For their "natural" norms, Y and W are Hilbert spaces. On the other hand, since $f \in C_b^1(\mathbf{R})$, $F : Y \mapsto W$ is well defined and C^1.

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ALG 3' (Newton):

1. Choose $(y^0, z^0) \in Y$.
2. Then, given $n \geq 0$ and $(y^n, v^n) \in Y$, solve in $(y^{n+1}, v^{n+1}) \in Y$ the linear problem

$$F'(y^n, v^n) \cdot (y^{n+1} - y^n, v^{n+1} - v^n) = -F(y^n, v^n),$$

i.e. find y^{n+1} and v^{n+1} such that $(y^{n+1}, v^{n+1}) \in Y$ and

$$\begin{cases} y_t^{n+1} - (a(x)y_x^{n+1})_x + f'(y^n)\, y^{n+1} = v^{n+1}\, 1_\omega + f'(y^n)\, y^n - f(y^n), & Q_T \\ y^{n+1} = 0, & \Sigma_T \\ y^{n+1}(\cdot, 0) = y_0, & (0,1). \end{cases} \quad (45)$$

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	$\|v_h\|_{L^2(q_T)}$	$\|v_h\|_{L^\infty(q_T)}$	$\|y_h\|_{L^2(Q_T)}$	$\sharp$ iterates
$\alpha = 10$	3.489×10^1	3.12×10^2	1.467	58
$\alpha = 40$	2.110×10^2	2.587×10^3	5.248	18
$\alpha = 80$	5.033×10^2	8.589×10^3	10.976	> 500

Table: Newton-Raphson method - $h = (1/60, 1/60)$ - $y_0(x) \equiv \alpha \sin(\pi x)$ - $p = 1.4$.

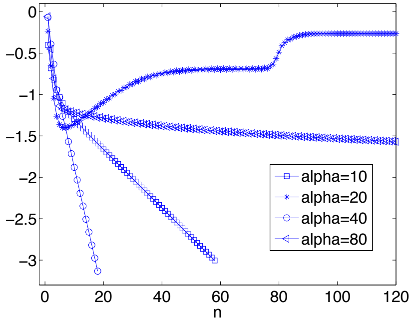

Figure: Newton-Raphson method - $h = (1/60, 1/60)$ - $y_0(x) \equiv \alpha \sin(\pi x)$ - $p = 1.4$ - Evolution of $\log_{10}(\|y_h^{n+1} - y_h^n\|_{L^2(Q_T)}/\|y_h^n\|_{L^2(Q_T)})$ for $\alpha = 10, 20, 40, 80$.

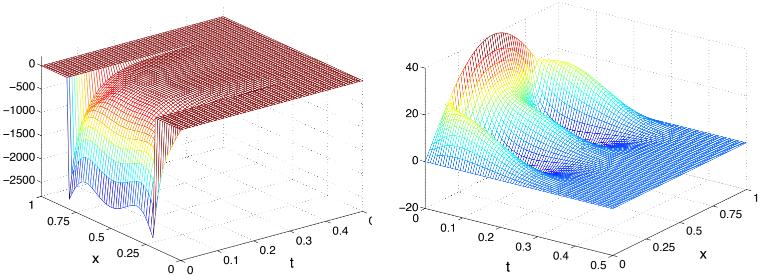

Figure: Newton-Raphson method - $h = (1/60, 1/60)$ - $y_0(x) \equiv 40\sin(\pi x)$ - $p = 1.4$ - Control v_h (Left) and corresponding controlled solution y_h (Right) in Q_T.

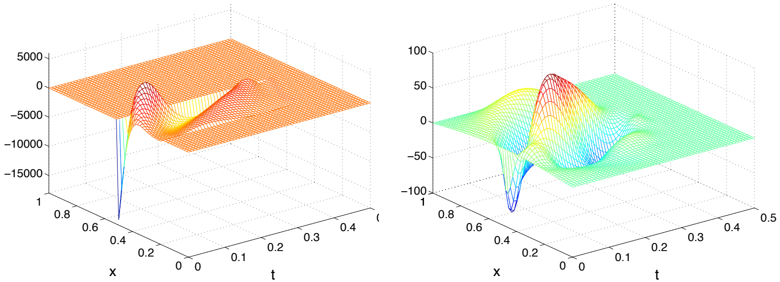

Figure: Fixed point method - $h = (1/60, 1/60)$ - $y_0(x) \equiv 10\sin(\pi x)$ - $p = 1.4$ - $\omega = (0.2, 0.5)$ - Control v_h (Left) and corresponding controlled solution y_h (Right) in Q_T.

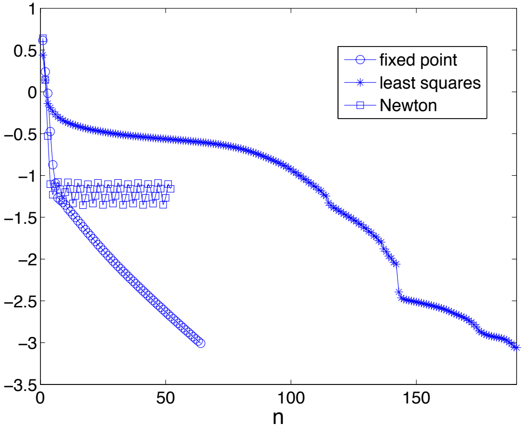

Figure: $h = (1/60, 1/60)$ - $y_0(x) \equiv 10\sin(\pi x)$ - $p = 1.4$ - $\omega = (0.2, 0.5)$ - Evolution of $\log_{10}(\|y_h^{n+1} - y_h^n\|_{L^2(Q_T)}/\|y_h^n\|_{L^2(Q_T)})$.

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Numerical null controllability of semi-linear 1D heat equations : fixed point, least squares and Newton methods,
Mathematical Control and Related Fields, 3(2), 217-246 (2012)

The LS approach can be extended to higher spatial dimensions and can be used in the context of many other controllable systems for which appropriate Carleman estimates are available :

- Stokes and Navier-Stokes system

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + \nabla \pi = \mathbf{v} 1_\omega & \text{dans } Q \\ \nabla \cdot \mathbf{y} = 0 & \text{dans } Q \\ \mathbf{y} = \mathbf{0} & \text{sur } \Sigma \\ \mathbf{y}(\mathbf{x}, 0) = \mathbf{y}_0(\mathbf{x}) & \text{dans } \Omega. \end{cases} \tag{46}$$

  (in progress with Fernandez-Cara and Diego de. Souza)

- Wave-like equations :

$$\begin{cases} y_{tt} - (a(x) y_x)_x + f(y) = 0 & Q_T \\ v = y & \Sigma_T \end{cases} \tag{47}$$

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The End

THANK YOU VERY MUCH FOR YOUR ATTENTION