Inverse problems for linear hyperbolic equation via mixed formulations

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Besançon, March 5 , 2015

joint work with NICOLAE CÎNDEA (Clermont-Ferrand)

$\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ - $T > 0$.

$$\begin{cases} Ly := y_{tt} - \nabla \cdot (c(x)\nabla y) + d(x,t)y = f, & (x,t) \in Q_T := \Omega \times (0,T) \\ y = 0, & (x,t) \in \Sigma_T := \partial\Omega \times (0,T) \\ (y(\cdot,0), y_t(\cdot,0)) = (y_0, y_1), & x \in \Omega. \end{cases} \quad (1)$$

$c \in C^1(\overline{\Omega}, \mathbb{R}))$ $c(x) \geq c_0 > 0$ in $\overline{\Omega}$, $d \in L^\infty(Q_T)$, $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega) \equiv \boldsymbol{H}$;
$f \in L^2(H^{-1}) = X$.
Let $\omega \subset \Omega$ and $q_T := \omega \times (0,T) \subset Q_T$.

(IP)-Given $y_{obs} \in L^2(q_T)$, find y the solution of (1) such that $y \equiv y_{obs} \quad \text{on} \quad q_T$.

From a "good" *measurement* y_{obs} on q_T, we want to recover y solution of (1).

Introducing the operator $P : L^2(Q_T) \to X \times L^2(q_T)$ defined by $P\, y := (Ly, y_{|q_T})$, the problem is reformulated as :

$$\textit{find } y \in L^2(Q_T) \textit{ solution of } P\, y = (f, y_{obs}). \qquad (IP)$$

From the unique continuation property for (1), if the set q_T satisfies some geometric conditions and if y_{obs} is a restriction to q_T of a solution of (1), then the problem is well-posed in the sense that the state y corresponding to the pair (y_{obs}, f) is unique.

Objective - Find a convergent (numerical) approximation of the solution

Problem statement (bis)

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Most natural approach: Least-squares method

The most natural (and widely used in practice) approach consists in introducing a least-squares type technic, i.e. consider the extremal problem

$$(LS) \quad \begin{cases} \text{minimize} \quad J(y_0, y_1) := \frac{1}{2}\|y - y_{obs}\|^2_{L^2(q_T)} \\ \text{subject to} \quad (y_0, y_1) \in \boldsymbol{H} \\ \text{where} \quad y \quad \text{solves} \quad (1) \end{cases} \tag{2}$$

A minimizing sequence $(y_0, y_1)_{(k>0)}$ is defined in term of the solution of an adjoint problem.

A difficulty, when one wants to prove the convergence of a discrete approximation : it is not possible to minimize over a discrete subspace of $\{y \in Y; Ly - f = 0\}$: If $\dim(Y_h) < \infty$, $\{y_h \in Y_h \subset Y : Ly_h - f = 0\}$ is 0 or empty

The minimization procedure first requires the discretization of J and of the system (1);

This raises the issue of uniform coercivity property of the discrete functional with respect to the approximation parameter h.

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[Auroux-Blum 2005],[Chapelle,Cindea,Moireau,2012], [Ramdani-Tucsnak 2011], etc...

Define a dynamic

$$L\overline{y} = G(y_{obs}, q_T) \quad \overline{y}(\cdot, 0) \quad \text{fixed}$$

such that

$$\|\overline{y}(\cdot, t) - y(\cdot, t)\|_{N(\Omega)} \to 0 \quad \text{as} \quad t \to \infty$$

$N(\Omega)$ - appropriate norm
The reversibility of the wave equation then allows to recover y for any time.

But, for the same reasons, on a numerically point of view, this method requires to prove uniform discrete observability properties.

Klibanov and co-workers approach: Quasi-reversibility for ill-posed problem

[Klibanov, Beilina 20xx], [Bourgeois, Darde 2010]

QR$_\varepsilon$ method (Quasi-Reversibility): for any $\varepsilon > 0$, find $y_\varepsilon \in \mathcal{A}$ such that

$$\langle Py_\varepsilon, P\overline{y}\rangle_{X\times L^2(q_T)} + \varepsilon\langle y_\varepsilon, \overline{y}\rangle_{\mathcal{A}} = \langle (f, y_{obs}), P\overline{y}\rangle_{X\times L^2(q_T), X\times L^2(q_T)}, \qquad (QR)$$

for all $\overline{y} \in \mathcal{A}$,

- $\mathcal{A}$ denotes a functional space which gives a meaning to the first term
- $\varepsilon > 0$ a Tikhonov parameter which ensures the well-posedness

equivalent to the minimization over $\mathcal{A}$ of

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QR_ε method (Quasi-Reversibility): for any $\varepsilon > 0$, find $y_\varepsilon \in \mathcal{A}$ such that

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Main assumption: a generalized obs. inequality

Without loss of generality, $f \equiv 0$.
We consider the vectorial space Z defined by

$$Z := \{y : y \in C([0,T], L^2(\Omega)) \cap C^1([0,T], H^{-1}(\Omega)), Ly \in X\}. \tag{3}$$

and then introduce the following hypothesis :

Hypothesis

There exists a constant $C_{obs} = C(\omega, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)})$ such that the following estimate holds :

$$(\mathcal{H}) \qquad \|y(\cdot,0), y_t(\cdot,0)\|^2_{\boldsymbol{H}} \le C_{obs}\left(\|y\|^2_{L^2(q_T)} + \|Ly\|^2_X\right), \quad \forall y \in Z. \tag{4}$$

hold true if (ω, T, Ω) satisfies a geometric optic condition. "Any characteristic line starting at the point $x \in \Omega$ at time $t = 0$ and following the optical geometric laws when reflecting at $\partial\Omega$ must meet q_T".

$$\|z\|^2_{L^2(Q_T)} \le C_{\Omega,T}\left(C_{obs}\|z\|^2_{L^2(q_T)} + (1 + C_{obs})\|Lz\|^2_X\right) \quad \forall z \in Z. \tag{5}$$

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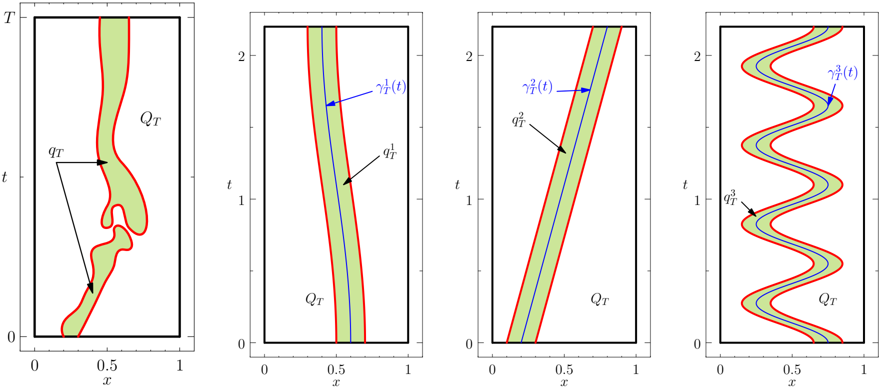
[Castro-Cindea-Münch, SICON 2014],
In 1D with $c \equiv 1$ and $d \equiv 0$, the observability inequality also holds for non cylindrical domains.



Time dependent domains $q_T \subset Q_T = \Omega \times (0, T)$

Then, within this hypothesis, for any $\eta > 0$, we define on Z the bilinear form

$$\langle y, \overline{y} \rangle_Z := \iint_{q_T} y\,\overline{y}\,dxdt + \eta \int_0^T \langle Ly,\, L\overline{y} \rangle_{H^{-1}(\Omega)}\,dt \quad \forall y, \overline{y} \in Z. \tag{6}$$

$(Z, \|\cdot\|)$ is a Hilbert space.
Then, we consider the following extremal problem :

$$(\mathcal{P}) \quad \begin{cases} \inf J(y) := \dfrac{1}{2}\|y - y_{obs}\|^2_{L^2(q_T)} + \dfrac{r}{2}\|Ly\|^2_X, \quad r \geq 0 \\ \text{subject to} \quad y \in W := \{y \in Z;\ Ly = 0 \text{ in } X\} \end{cases}$$

$(\mathcal{P})$ is well posed : J is continuous over W, strictly convex and $J(y) \to +\infty$ as $\|y\|_W \to \infty$.

The solution of $(\mathcal{P})$ in W does not depend on η.

From (4), the solution y in Z of $(\mathcal{P})$ satisfies $(y(\cdot,0), y_t(\cdot,0)) \in \boldsymbol{H}$, so that problem $(\mathcal{P})$ is equivalent to the minimization of J w.r.t $(y_0, y_1) \in \boldsymbol{H}$.

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In order to solve $(\mathcal{P})$, we have to deal with the constraint equality which appears W. We introduce a Lagrange multiplier $\lambda \in X'$ and the following mixed formulation: find $(y, \lambda) \in Z \times X'$ solution of

$$\begin{cases} a_r(y, \overline{y}) + b(\overline{y}), \lambda) & = & l(\overline{y}), & \forall \overline{y} \in Z \\ b(y, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in \Lambda, \end{cases} \tag{7}$$

where

$$a_r : Z \times Z \to \mathbb{R}, \quad a_r(y, \overline{y}) := \iint_{q_T} y\,\overline{y}\, dxdt + r \int_0^T \langle Ly,\, L\overline{y}\rangle_{H^{-1}(\Omega)}\, dt, \tag{8}$$

$$b : Z \times X' \to \mathbb{R}, \quad b(y, \lambda) := \int_0^T \langle \lambda,\, Ly\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt, \tag{9}$$

$$l : Z \to \mathbb{R}, \quad l(y) := \iint_{q_T} y_{obs}\, y\, dxdt. \tag{10}$$

System (7) is nothing else than the optimality system corresponding to the extremal problem $(\mathcal{P})$.

Theorem

Under the hypothesis $(\mathcal{H})$, *for any* $r \geq 0$,

1. *The mixed formulation (7) is well-posed.*
2. *The unique solution* $(y, \lambda) \in Z \times X'$ *is the unique saddle-point of the Lagrangian* $\mathcal{L} : Z \times X' \to \mathbb{R}$ *defined by*

$$\mathcal{L}(y, \lambda) := \frac{1}{2} a_r(y, y) + b(y, \lambda) - l(y).$$

3. *We have the estimate*

$$\|y\|_Y = \|y\|_{L^2(q_T)} \leq \|y_{obs}\|_{L^2(q_T)}, \quad \|\lambda\|_{X'} \leq 2\sqrt{C_{\Omega,T} + \eta}\|y_{obs}\|_{L^2(q_T)}. \quad (11)$$

The kernel $\mathcal{N}(b) = \{y \in Z; b(y,\lambda) = 0 \quad \forall \lambda \in X'\}$ coincides with W: we easily get

$$a_r(y,y) = \|y\|_Z^2, \quad \forall y \in \mathcal{N}(b) = W.$$

It remains to check the inf-sup constant property : $\exists \delta > 0$ such that

$$\inf_{\lambda \in X'} \sup_{y \in Z} \frac{b(y,\lambda)}{\|y\|_Z \|\lambda\|_{X'}} \geq \delta. \tag{12}$$

For any fixed $\lambda \in X'$, we define y as the unique solution of

$$Ly = -\Delta\lambda \text{ in } Q_T, \quad (y(\cdot,0), y_t(\cdot,0)) = (0,0) \text{ on } \Omega, \quad y = 0 \text{ on } \Sigma_T. \tag{13}$$

We get $b(y,\lambda) = \|\lambda\|_{X'}^2$ and $\|y\|_Z^2 = \|y\|_{L^2(q_T)}^2 + \eta\|\lambda\|_{X'}^2$.

The estimate $\|y\|_{L^2(q_T)} \leq \sqrt{C_{\Omega,T}}\|\lambda\|_{X'}$ implies that $y \in Z$ and that

$$\sup_{y \in Z} \frac{b(y,\lambda)}{\|y\|_Y \|\lambda\|_{X'}} \geq \frac{1}{\sqrt{C_{\Omega,T} + \eta}} > 0$$

leading to the result with $\delta = (C_{\Omega,T} + \eta)^{-1/2}$.

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Assuming enough regularity on the solution λ, at the optimality, the Lagrange Multiplier solves

$$\begin{cases} L\lambda = -(y - y_{obs})_{1_{q_T}}, \quad \lambda = 0 \quad \text{in} \quad \Sigma_T, \\ \lambda = \lambda_t = 0 \quad \text{on}\ \Omega \times \{0, T\}. \end{cases} \tag{14}$$

λ (defined in the weak sense) is a null controlled solution of the wave equation through the control $-(y - y_{obs})\,1_\omega$.

If y_{obs} is the restriction to q_T of a solution of (1), then λ must vanish almost everywhere.

In that case, $\sup_{\lambda\in\Lambda} \inf_{y\in Y} \mathcal{L}_r(y,\lambda) = \inf_{y\in Y} \mathcal{L}_r(y,0) = \inf_{y\in Y} J_r(y)$ with

$$J_r(y) := \frac{1}{2}\|y - y_{obs}\|^2_{L^2(Q_T)} + \frac{r}{2}\|Ly\|^2_X. \tag{15}$$

The corresponding variational formulation is then : find $y \in Z$ such that

$$a_r(y,\overline{y}) = \iint_{q_T} y\,\overline{y}\,dxdt + r\int_0^T \langle \lambda,\, Ly\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} dt = l(\overline{y}), \quad \forall \overline{y} \in Z.$$

Remark 1

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Remark 2

In the general case, the mixed formulation can be rewritten as follows: find $(z, \lambda) \in Z \times X'$ solution of

$$\begin{cases} \langle P_r y, P_r \overline{y} \rangle_{X \times L^2(q_T)} + \langle L\overline{y}, \lambda \rangle_{X,X'} = \langle (0, y_{obs}), P_r \overline{y} \rangle_{X \times L^2(q_T)}, & \forall \overline{y} \in Z, \\ \langle Ly, \overline{\lambda} \rangle_{X,X'} = 0, \quad \forall \overline{\lambda} \in X' \end{cases} \tag{16}$$

with $P_r y := (\sqrt{r} L\, y, y_{|q_T})$.

This approach may be seen as generalization of the (QR) problem (see (QR)), where the variable λ is adjusted automatically (while the choice of the parameter ε in (QR) is in general a delicate issue).

Remark 3: Stabilized mixed formulation

$$\Lambda := \{\lambda \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), L\lambda \in L^2(Q_T), \lambda(\cdot,0) = \lambda_t(\cdot,0) = 0\}.$$

$$\begin{cases} \sup\limits_{\lambda\in\Lambda} \inf\limits_{y\in Z} \mathcal{L}_{r,\alpha}(y,\lambda) \\ \mathcal{L}_{r,\alpha}(y,\lambda) := \mathcal{L}_r(y,\lambda) - \frac{\alpha}{2}\|L\lambda + (y - y_{obs})1_\omega\|^2_{L^2(Q_T)}. \end{cases}$$

For $\alpha \in (0,1)$, find $(y,\lambda) \in Z \times \Lambda$ such that

$$\begin{cases} a_{r,\alpha}(y,\overline{y}) + b_\alpha(\overline{y},\lambda) = l_{1,\alpha}(\overline{y}), & \forall \overline{y} \in Y \\ b_\alpha(y,\overline{\lambda}) - c_\alpha(\lambda,\overline{\lambda}) = l_{2,\alpha}(\overline{\lambda}), & \forall \overline{\lambda} \in \widetilde{\Lambda}, \end{cases} \tag{17}$$

$$a_{r,\alpha} : Z \times Z \to \mathbb{R}, \quad a_{r,\alpha}(y,\overline{y}) := (1-\alpha)\iint_{q_T} y\overline{y}\,dxdt + r\int_0^T (Ly, L\overline{y})_{H^{-1}(\Omega)}dt,$$

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Remark 3: Stabilized mixed formulation

Proposition

Under the hypothesis $(\mathcal{H})$, for any $\alpha \in (0,1)$, the corresponding mixed formulation is well-posed. The unique pair (y,λ) in $Z \times \Lambda$ satisfies

$$\theta_1 \|y\|_Z^2 + \theta_2 \|\lambda\|_\Lambda^2 \leq \left(\frac{(1-\alpha)^2}{\theta_1} + \frac{\alpha^2}{\theta_2} \right) \|y_{obs}\|^2_{L^2(q_T)}. \tag{18}$$

with $\theta_1 := \min\left(1-\alpha, r\,\eta^{-1}\right), \theta_2 := \frac{1}{2}\min\left(\alpha, C_{\Omega,T}^{-1}\right)$.

Proposition

If the solution $(y,\lambda) \in Z \times X'$ of (7) enjoys the property $\lambda \in \Lambda$, then the solutions of (7) and (17) coincide.

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Proposition

Under the hypothesis $(\mathcal{H})$, for any $\alpha \in (0,1)$, the corresponding mixed formulation is well-posed. The unique pair (y,λ) in $Z \times \Lambda$ satisfies

$$\theta_1 \|y\|_Z^2 + \theta_2 \|\lambda\|_\Lambda^2 \leq \left(\frac{(1-\alpha)^2}{\theta_1} + \frac{\alpha^2}{\theta_2} \right) \|y_{obs}\|^2_{L^2(q_T)}. \tag{18}$$

with $\theta_1 := \min\left(1-\alpha, r\,\eta^{-1}\right), \theta_2 := \frac{1}{2}\min\left(\alpha, C_{\Omega,T}^{-1}\right)$.

Proposition

If the solution $(y,\lambda) \in Z \times X'$ of (7) enjoys the property $\lambda \in \Lambda$, then the solutions of (7) and (17) coincide.

The mixed formulation has a structure very closed to the one we get when we address - using the same approach - the null controllability of (1): the control of minimal $L^2(q_T)$-norm which drives to rest $(y_0, y_1) \in H^1_0(\Omega) \times L^2(\Omega)$ is given by $v = \varphi\, 1_{q_T}$ where $(\varphi, \lambda) \in \Phi \times L^2(0, T; H^1_0(\Omega))$ solves

$$\begin{cases} a(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) = l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi \\ b(\varphi, \overline{\lambda}) = 0, & \forall \overline{\lambda} \in L^2(0, T; H^1_0(\Omega)), \end{cases} \qquad (19)$$

where

$$a : \Phi \times \Phi \to \mathbb{R}, \quad a(\varphi, \overline{\varphi}) = \iint_{q_T} \varphi(x,t)\overline{\varphi}(x,t)\, dx\, dt$$

$$b : \Phi \times L^2(0, T; H^1_0(0,1)) \to \mathbb{R}, \quad b(\varphi, \lambda) = \int_0^T < L\varphi, \lambda >_{H^{-1}, H^1_0} dt$$

$$l : \Phi \to \mathbb{R}, \quad l(\varphi) = - < \varphi_t(\cdot, 0), y_0 >_{H^{-1}(\Omega), H^1_0(\Omega)} + \int_0^1 \varphi(\cdot, 0)\, y_1\, dx.$$

with $\Phi = \{\varphi \in L^2(q_T),\ \varphi = 0 \text{ on } \Sigma_T \text{ such that } L\varphi \in L^2(0, T; H^{-1}(0,1))\}$.
[Cîndea- Münch, Calcolo 2015]

Remark 5

"Reversing the order of priority" between the constraint $y - y_{obs} = 0$ in $L^2(q_T)$ and $Ly - f = 0$ in X, a possibility could be to minimize the functional

$$\begin{cases} \text{minimize} \quad J(y) := \|Ly - f\|_X^2 + \varepsilon\|y\|_{\mathcal{A}}^2 \\ \text{subject to } y \in Z \quad \text{and to} \quad y - y_{obs} = 0 \quad \text{in} \quad L^2(q_T) \end{cases} \tag{20}$$

via the introduction of a Lagrange multiplier in $L^2(q_T)$.

The proof of the inf-sup property : there exists $\delta > 0$ such that

$$\inf_{\lambda \in L^2(q_T)} \sup_{y \in Z} \frac{\iint_{q_T} \lambda y \, dxdt}{\|\lambda\|_{L^2(q_T)} \|y\|_Y} \geq \delta$$

of the corresponding mixed-formulation is however unclear.

This issue is solved by the introduction of a ε-term in J_ε (Klibanov-Beilina 20xx).

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Lemma

Let $\mathcal{P}_r$ be the linear operator from X' into X' defined by

$$\mathcal{P}_r\lambda := -\Delta^{-1}(Ly), \quad \forall\lambda \in X' \quad \textit{where} \quad y \in Z \quad \textit{solves} \quad a_r(y,\overline{y}) = b(\overline{y},\lambda), \quad \forall\overline{y} \in Z.$$

For any $r > 0$, the operator $\mathcal{P}_r$ is a strongly elliptic, symmetric isomorphism from X' into X'.

Theorem

$$\sup_{\lambda\in X'}\inf_{y\in Z}\mathcal{L}_r(y,\lambda) = -\inf_{\lambda\in X'} J_r^{\star\star}(\lambda) \quad + \mathcal{L}_r(y_0, 0)$$

where $y_0 \in Z$ solves $a_r(y_0,\overline{y}) = l(\overline{y}), \forall\overline{y} \in Y$ and $J_r^{\star\star} : X' \to \mathbb{R}$ defined by

$$J_r^{\star\star}(\lambda) = \frac{1}{2}\int_0^T \langle\mathcal{P}_r\lambda,\lambda\rangle_{H_0^1(\Omega)}dt - b(y_0,\lambda).$$

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Remark 7 - Boundary observation

$(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ - Ω of class C^2

The results apply if the distributed observation on q_T is replaced by a Neumann boundary observation on a sufficiently large subset Σ_T of $\partial\Omega \times (0, T)$ (i.e. assuming $\frac{\partial y}{\partial \nu} = y_{\nu,obs} \in L^2(\Sigma_T)$ is known on Σ_T).

If (Q_T, Σ_T, T) satisfy some geometric condition, then there exists a positive constant $C_{obs} = C(\omega, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)})$ such that

$$\|y(\cdot,0), y_t(\cdot,0)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq C_{obs}\left(\left\|\frac{\partial y}{\partial \nu}\right\|^2_{L^2(\Sigma_T)} + \|Ly\|^2_{L^2(Q_T)}\right), \quad \forall y \in Z \qquad (21)$$

It suffices to re-define the form a in by $a(y, y) := \iint_{\Sigma_T} \frac{\partial y}{\partial \nu}\frac{\partial \overline{y}}{\partial \nu}\, d\sigma dx$ and the form l by $l(y) := \iint_{\Sigma_T} \frac{\partial y}{\partial \nu} y_{obs}\, d\sigma dx$ for all $y, \overline{y} \in Z$.

$f(x,t) = \sigma(t)\mu(x)$

$c := 1$, $d(x,t) = d(x) \in L^p(\Omega)$, $\sigma \in C^1([0,T]), \sigma(0) \neq 0$, $\mu \in H^{-1}(\Omega)$

Theorem (Yamamoto-Zhang 2001)

Let us assume that the triplet (Γ_T, T, Q_T) satisfies the geometric optic condition. Let $y = y(\mu) \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ be the weak solution of (1) with $c := 1$ and $(y_0, y_1) = (0,0)$. Then, there exists a positive constant C such that

$$C^{-1}\|\mu\|_{H^{-1}(\Omega)} \leq \|c(x)\,\partial_\nu y\|_{L^2(\Gamma_T)} \leq C\|\mu\|_{H^{-1}(\Omega)}, \quad \forall \mu \in H^{-1}(\Omega). \tag{22}$$

We consider the following extremal problem :

$$\begin{cases} \inf J(y,\mu) := \dfrac{1}{2}\|c(x)(\partial_\nu y - y_{\nu,obs})\|^2_{L^2(\Gamma_T)}, \\ \text{subject to} \quad (y,\mu) \in W \end{cases} \tag{$\mathcal{P}_{y,\mu}$}$$

where W is the space defined by

$$W := \Big\{(y,\mu); y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \mu \in H^{-1}(\Omega), \\ Ly - \sigma\mu = 0 \text{ in } Q_T, y(\cdot,0) = y_t(\cdot,0) = 0\Big\}. \tag{23}$$

Attached to the norm $\|(y,\mu)\|_W := \|c(x)\partial_\nu y\|_{L^2(\Gamma_T)}$, W is a Hilbert space.

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Attached to the norm $\|(y,\mu)\|_W := \|c(x)\partial_\nu y\|_{L^2(\Gamma_T)}$, W is a Hilbert space.

Recovering the solution and the source f when the pair (y, f) is unique

$f(x,t) = \sigma(t)\mu(x)$
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$$
\begin{aligned}
Y := \Big\{ &(y,\mu); y \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)), \mu \in H^{-1}(\Omega), \\
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\end{aligned}
\tag{24}
$$

Hypothesis

There exists a constant $C_{obs} = C(\Gamma_T, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)})$ such that the following estimate holds :

$$
\|\mu\|^2_{H^{-1}(\Omega)} \le C_{obs}\Big(\|c(x)\partial_\nu y\|^2_{L^2(\Gamma_T)} + \|Ly - \sigma\mu\|^2_{L^2(Q_T)} \Big), \quad \forall (y,\mu) \in Y. \qquad (\mathcal{H}_2)
$$

Then, for any $\eta > 0$, we define on Y the bilinear form

$$
\begin{aligned}
&\langle (y,\mu),(\overline{y},\overline{\mu})\rangle_Y := \iint_{\Gamma_T} (c(x))^2\, \partial_\nu y\, \partial_\nu \overline{y}\, d\sigma dt + \eta \iint_{Q_T} (Ly - \sigma\mu)\,(L\overline{y} - \sigma\overline{\mu})\, dxdt \quad \forall y, \overline{y} \in Z. \\
&\|(y,z)\|_Y := \sqrt{<(y,\mu),(y,\mu)>_Y}.
\end{aligned}
\tag{25}
$$

Lemma

Under the hypotheses $(\mathcal{H}_2)$, the space $(Y, \|\cdot\|_Y)$ is a Hilbert space.

Recovering the solution and the source f when the pair (y, f) is unique

$$Y := \Big\{(y,\mu); y \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \mu \in H^{-1}(\Omega), \\ Ly - \sigma\mu \in L^2(Q_T), y(\cdot,0) = y_t(\cdot,0) = 0\Big\}. \tag{24}$$

Hypothesis

There exists a constant $C_{obs} = C(\Gamma_T, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)})$ such that the following estimate holds :

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Lemma

Under the hypotheses $(\mathcal{H}_2)$, the space $(Y, \|\cdot\|_Y)$ is a Hilbert space.

Find $((y,\mu),\lambda) \in Y \times L^2(Q_T)$ solution of

$$\left\{ \begin{array}{rcll} a_r((y,\mu),(\overline{y},\overline{\mu})) + b((\overline{y},\overline{\mu}),\lambda) & = & l(\overline{y},\overline{\mu}), & \forall (\overline{y},\overline{\mu}) \in Y \\ b((y,\mu),\overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{array} \right. \tag{26}$$

where

$$a_r : Y \times Y \to \mathbb{R}, \quad a_r((y,\mu),(\overline{y},\overline{\mu})) := \iint_{\Gamma_T} c^2(x) \partial_\nu y \partial_\nu \overline{y} \, d\sigma dt \tag{27}$$

$$+ r \iint_{Q_T} (Ly - \sigma\mu)(L\overline{y} - \sigma\overline{\mu}) \, dxdt, \; r \geq 0$$

$$b : Y \times L^2(Q_T) \to \mathbb{R}, \quad b((y,\mu),\lambda) := \iint_{Q_T} \lambda(Ly - \sigma\mu) dx \, dt,$$

$$l : Y \to \mathbb{R}, \quad l(y,\mu) := \iint_{\Gamma_T} c^2(x) \, \partial_\nu y \, y_{\nu,obs} \, d\sigma dt.$$

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(boundary observation case, to fix idea)

Let Z_h and Λ_h be two finite dimensional spaces parametrized by the variable h such that $Z_h \subset Z, \Lambda_h \subset L^2(Q_T)$ for every $h > 0$. Find the $(y_h, \lambda_h) \in Z_h \times \Lambda_h$ solution of

$$\begin{cases} a_r(y_h, \overline{y}_h) + b(\overline{y}_h, \lambda_h) = l(\overline{y}_h), & \forall \overline{y}_h \in Z_h \\ b(y_h, \overline{\lambda}_h) = 0, & \forall \overline{\lambda}_h \in \Lambda_h. \end{cases} \tag{28}$$

if $r > 0$, a_r is coercive on Z: $a_r(y, y) \geq \frac{r}{\eta}\|y\|_Z^2 \quad \forall y \in Z$.

$$\forall h > 0 \qquad \delta_h := \inf_{\lambda_h \in \Lambda_h} \sup_{y_h \in Z_h} \frac{b(y_h, \lambda_h)}{\|\lambda_h\|_{L^2(Q_T)} \|y_h\|_Z} > 0. \tag{29}$$

Consequently, $\forall h > 0$ fixed, if $r > 0$, there exists a unique couple (y_h, λ_h) solution of (28).

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Consequently, $\forall h > 0$ fixed, if $r > 0$, there exists a unique couple (y_h, λ_h) solution of (28).

First estimate

Proposition

Let $h > 0$. Let (y, λ) and (y_h, λ_h) be the solution of (7) and of (28) respectively. Let δ_h the discrete inf-sup constant defined by (29). Then,

$$\|y - y_h\|_Z \leq 2\left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right) d(y, Z_h) + \frac{1}{\sqrt{\eta}} d(\lambda, \Lambda_h), \tag{30}$$

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq \left(2 + \frac{1}{\sqrt{\eta}\delta_h}\right) \frac{1}{\delta_h} d(y, Z_h) + \frac{3}{\sqrt{\eta}\delta_h} d(\lambda, \Lambda_h) \tag{31}$$

where $d(\lambda, \Lambda_h) := \inf_{\lambda_h \in \Lambda_h} \|\lambda - \lambda_h\|_{L^2(Q_T)}$ *and*

$$\begin{aligned} d(y, Z_h) &:= \inf_{y_h \in Z_h} \|y - y_h\|_Z \\ &= \inf_{y_h \in Z_h} \left(\|\partial_\nu y - \partial_\nu y_h\|^2_{L^2(\Gamma_T)} + \eta \|L(y - y_h)\|^2_{L^2(Q_T)} \right)^{1/2}. \end{aligned} \tag{32}$$

Let $n_h = \dim Z_h$, $m_h = \dim \Lambda_h$ and let the real matrices $A_{r,h} \in \mathbb{R}^{n_h,n_h}$, $B_h \in \mathbb{R}^{m_h,n_h}$, $J_h \in \mathbb{R}^{m_h,m_h}$ and $L_h \in \mathbb{R}^{n_h}$ be defined by

$$\begin{cases} a_r(y_h,\overline{y_h}) = \langle A_{r,h}\{y_h\},\{\overline{y_h}\}\rangle_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}} & \forall y_h,\overline{y_h} \in Z_h, \\ b(y_h,\lambda_h) = \langle B_h\{y_h\},\{\lambda_h\}\rangle_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}} & \forall y_h \in Z_h, \lambda_h \in \Lambda_h, \\ \displaystyle\iint_{Q_T} \lambda_h\overline{\lambda_h}\,dx\,dt = \langle J_h\{\lambda_h\},\{\overline{\lambda_h}\}\rangle_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}} & \forall \lambda_h,\overline{\lambda_h} \in \Lambda_h, \\ l(y_h) = \langle L_h,\{y_h\}\rangle_{\mathbb{R}^{n_h}} & \forall y_h \in Z_h, \end{cases} \tag{33}$$

where $\{y_h\} \in \mathbb{R}^{n_h}$ denotes the vector associated to y_h and $\langle\cdot,\cdot\rangle_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}$ the usual scalar product over $\mathbb{R}^{n_h}$. With these notations, the problem (28) reads as follows: find $\{y_h\} \in \mathbb{R}^{n_h}$ and $\{\lambda_h\} \in \mathbb{R}^{m_h}$ such that

$$\begin{pmatrix} A_{r,h} & B_h^T \\ B_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix} \{y_h\} \\ \{\lambda_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} L_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}. \tag{34}$$

The matrix of order $m_h + n_h$ is symmetric but not positive definite.

Choice of the space Y_h and Λ_h

We introduce a regular triangulation $\mathcal{T}_h$ such that $\overline{Q_T} = \cup_{K\in\mathcal{T}_h} K$. We note $h := \max\{\mathrm{diam}(K), K \in \mathcal{T}_h\}$.

We introduce the space Φ_h as follows:

$$Z_h = \{y_h \in Z \in C^1(\overline{Q_T}) : z_h|_K \in \mathbb{P}(K) \quad \forall K \in \mathcal{T}_h,\ z_h = 0 \text{ on } \Sigma_T\}$$

where $\mathbb{P}(K)$ denotes an appropriate space of functions in x and t.

- The *Bogner-Fox-Schmit* (BFS for short) C^1 element defined for rectangles. Therefore $\mathbb{P}(K) = \mathbb{P}_{3,x} \otimes \mathbb{P}_{3,t}$
- The reduced *Hsieh-Clough-Tocher* (HCT for short) C^1 element defined for triangles. This is a so-called composite finite element.

We also define the finite dimensional space

$$\Lambda_h = \{\lambda_h \in C^0(\overline{Q_T}), \lambda_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h\}$$

For any $h > 0$, we have $Y_h := Z_h \times \Lambda_h \subset Y$ and $\Lambda_h \subset L^2(Q_T)$.

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For any $h > 0$, we have $Y_h := Z_h \times \Lambda_h \subset Y$ and $\Lambda_h \subset L^2(Q_T)$.

Proposition (BFS element for $N = 1$ - Rate of convergence for the norm Z)

Let $h > 0$, let $k \leq 2$ be a nonnegative integer. Let (y, λ) and (y_h, λ_h) be the solution of (7) and (28) respectively. If the solution (y, λ) belongs to $H^{k+2}(Q_T) \times H^k(Q_T)$, then there exists two positives constants

$$K_i = K_i(\|y\|_{H^{k+2}(Q_T)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)}), \qquad i \in \{1, 2\},$$

independent of h, such that

$$\|y - y_h\|_Z \leq K_1 \left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right) h^k, \tag{35}$$

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 \left(\left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right)\frac{1}{\delta_h} + \frac{1}{\sqrt{\eta}\delta_h}\right) h^k. \tag{36}$$

Precisely, we write that $(y - y_h)$ solves the hyperbolic equation

$$\begin{cases} L(y - y_h) = -Ly_h & \text{in } Q_T \\ ((y - y_h), (y - y_h)_t)(0) \in \boldsymbol{V} \\ y - y_h = 0 & \text{on } \Sigma_T. \end{cases}$$

The continuous dependance combined with the observability inequality applied to $(y - y_h)$ lead to

$$\|y - y_h\|^2_{L^2(Q_T)} \leq C_{\Omega,T}(C_{obs} + 1)(\|\partial_\nu (y - y_h)\|^2_{L^2(\Gamma_T)} + \|Ly_h\|^2_{L^2(Q_T)})$$

from which we deduce, in view of the definition of the norm Y, that

$$\|y - y_h\|_{L^2(Q_T)} \leq C_{\Omega,T}(C_{obs} + 1) \max(1, \frac{2}{\sqrt{\eta}}) \|y - y_h\|_Z. \qquad (37)$$

Theorem (BFS element for $N = 1$ - Rate of convergence for the norm $L^2(Q_T)$)

Assume that the hypothesis (4) holds. Let $h > 0$, let $k \leq 2$ be a positive integer. Let (y, λ) and (y_h, λ_h) be the solution of (7) and (28) respectively. If the solution (y, λ) belongs to $H^{k+2}(Q_T) \times H^k(Q_T)$, then there exists two positives constant $K = K(\|y\|_{H^{k+2}(Q_T)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)}, C_{\Omega,T}, C_{obs})$, independent of h, such that

$$\|y - y_h\|_{L^2(Q_T)} \leq K \max(1, \frac{2}{\sqrt{\eta}}) \left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right) h^k. \qquad (38)$$

Precisely, we write that $(y - y_h)$ solves the hyperbolic equation

$$\begin{cases} L(y-y_h) = -Ly_h & \text{in } Q_T \\ ((y-y_h),(y-y_h)_t)(0) \in \boldsymbol{V} \\ y - y_h = 0 & \text{on } \Sigma_T. \end{cases}$$

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Theorem (BFS element for $N = 1$ - Rate of convergence for the norm $L^2(Q_T)$)

Assume that the hypothesis (4) *holds. Let $h > 0$, let $k \leq 2$ be a positive integer. Let (y, λ) and (y_h, λ_h) be the solution of (7) and (28) respectively. If the solution (y, λ) belongs to $H^{k+2}(Q_T) \times H^k(Q_T)$, then there exists two positives constant $K = K(\|y\|_{H^{k+2}(Q_T)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)}, C_{\Omega,T}, C_{obs})$, independent of h, such that*

$$\|y-y_h\|_{L^2(Q_T)} \leq K\max(1,\frac{2}{\sqrt{\eta}})\left(1+\frac{1}{\sqrt{\eta}\delta_h}+\frac{1}{\sqrt{\eta}}\right)h^k. \tag{38}$$

Choice of r versus δ_h

$(\eta = r)$

$$\delta_h = \inf\left\{ \sqrt{\delta} : B_h A_{r,h}^{-1} B_h^T \{\lambda_h\} = \delta\, J_h\{\lambda_h\}, \quad \forall \{\lambda_h\} \in \mathbb{R}^{m_h} \setminus \{0\} \right\} \tag{39}$$

$$\delta_{r,h} \approx C_r \frac{h}{\sqrt{r}} \quad \text{as} \quad h \to 0^+, \qquad C_r > 0 \tag{40}$$

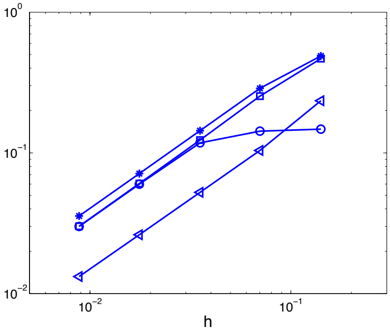


Figure: BFS finite element - Evolution of $\sqrt{r}\delta_{h,r}$ with respect to h for $r = 1$ ($\square$), $r = 10^{-2}$ ($\circ$), $r = h$ ($\star$) and $r = h^2$ ($<$).

$$\|y - y_h\|_{L^2(Q_T)} \le K \max(1, \frac{2}{\sqrt{r}})\left(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}\right)h^k.$$

The right hand side is minimal for r of the order one leading to $\|y - y_h\|_{L^2(Q_T)} \le Kh^{k-1}$.

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \le K_2 \frac{\sqrt{r}}{h}(1 + \frac{1}{h} + \frac{1}{\sqrt{r}})h^k.$$

The optimal value of the augmentation parameter is now $r = h^2$ leading to $\|\lambda - \lambda_h\|_{L^2(Q_T)} \le K_2 h^{k-1}$.

Choice of r versus δ_h

$$\|y - y_h\|_{L^2(Q_T)} \leq K \max(1, \frac{2}{\sqrt{r}})\left(1 + \frac{1}{h} + \frac{1}{\sqrt{r}}\right)h^k.$$

The right hand side is minimal for r of the order one leading to $\|y - y_h\|_{L^2(Q_T)} \leq Kh^{k-1}$.

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 \frac{\sqrt{r}}{h}(1 + \frac{1}{h} + \frac{1}{\sqrt{r}})h^k.$$

The optimal value of the augmentation parameter is now $r = h^2$ leading to $\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 h^{k-1}$.

The problem (17) becomes : find $(y_h, \lambda_h) \in Z_h \times \Lambda_h$ solution of

$$\left\{ \begin{array}{rcll} a_{r,\alpha}(y_h, \overline{y}_h) + b_\alpha(\lambda_h, \overline{y}_h) & = & l_{1,\alpha}(\overline{y}_h), & \forall \overline{y}_h \in Z_h \\ b_\alpha(\overline{\lambda}_h, y_h) - c_\alpha(\lambda_h, \overline{\lambda}_h) & = & l_{2,\alpha}(\overline{\lambda}_h), & \forall \overline{\lambda}_h \in \widetilde{\Lambda}_h, \end{array} \right. \tag{41}$$

$$\Lambda_h = \{\lambda \in Z_h; \lambda(\cdot, 0) = \lambda_t(\cdot, 0) = 0\}. \tag{42}$$

Proposition (BFS element for $N = 1$ - Rates of convergence - Stabilized mixed formulation)

Assume that the hypothesis (4) *holds. Let* $h > 0$*, let* $k \leq 2$ *be a positive integer. Let* (y, λ) *and* (y_h, λ_h) *be the solution of (7) and (28) respectively. If the solution* (y, λ) *belongs to* $H^{k+2}(Q_T) \times H^k(Q_T)$*, then there exists two positives constant* $K = K(\|y\|_{H^{k+2}(Q_T)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)}, C_{\Omega,T}, C_{obs})$*, independent of h, such that*

$$\|y - y_h\|_Z + \|\lambda - \lambda_h\|_\Lambda \leq Kh^k. \tag{43}$$

$$\left\{ \begin{array}{rcll} a_r((y_h,\mu_h),(\overline{y}_h,\overline{\mu}_h)) + b(\overline{y}_h,\lambda_h) & = & l(\overline{y}_h), & \forall (\overline{y}_h,\overline{\mu}_h) \in Y_h \\ b((y_h,\mu_h),\overline{\lambda}_h) & = & 0, & \forall \overline{\lambda}_h \in \Lambda_h. \end{array} \right. \tag{44}$$

Theorem (BFS element for $N = 1$ - Rate of convergence for the $L^2(Q_T)$-norm)

Let $h > 0$, let $k, q \leq 2$ be two nonnegative integers. Let (y,λ) and (y_h,λ_h) be the solution of (26) and (44) respectively. If the solution $((y,\mu),\lambda)$ belongs to $H^{k+2}(Q_T) \times H^q(\Omega) \times H^k(Q_T)$, then there exists a positive constant

$$K = K(\|y\|_{H^{k+2}(Q_T)}, \|\mu\|_{H^k(\Omega)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)}),$$

independent of h, such that

$$\|y - y_h\|_{L^2(Q_T)} \leq K C_{\Omega,T}(1 + \|\sigma\|_{L^2(0,T)}\sqrt{C_{obs}}) \max(1, \frac{1}{\sqrt{\eta}}) \left[\left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right)h^k + \left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right)(\Delta x)^q\right]. \tag{45}$$

$$\textbf{(EX1)} \quad y_0(x) = 1 - |2x - 1|, \quad y_1(x) = 1_{(1/3,2/3)}(x), \qquad x \in (0,1)$$

in $H_0^1 \times L^2$ for which the Fourier coefficients are

$$a_k = \frac{4\sqrt{2}}{\pi^2 k^2} \sin(\pi k/2), \quad b_k = \frac{1}{\pi k}(\cos(\pi k/3) - \cos(2\pi k/3)), \quad k > 0$$

$f = 0$. $T = 2$ - The corresponding solution of (1) with $c \equiv 1, d \equiv 0$ is given by

$$y(x,t) = \sum_{k>0} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sqrt{2} \sin(k\pi x)$$

Example 1 - $N = 1$ - Observation on q_T

$q_T = (0.1, 0.3) \times (0, T)$

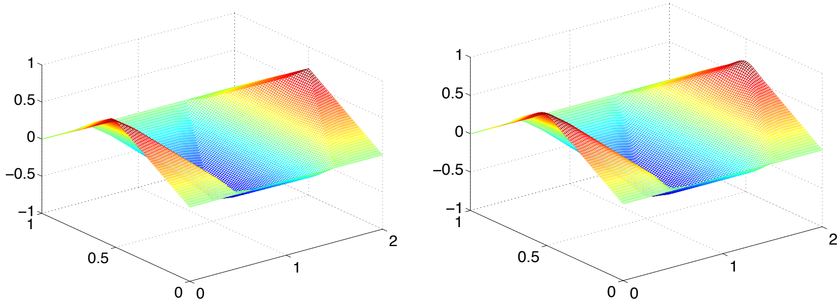
h	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\frac{\Vert y-y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}}$	1.01×10^{-1}	4.81×10^{-2}	2.34×10^{-2}	1.15×10^{-2}	5.68×10^{-3}
$\frac{\Vert y-y_h\Vert_{L^2(q_T)}}{\Vert y\Vert_{L^2(q_T)}}$	1.34×10^{-1}	5.05×10^{-2}	2.37×10^{-2}	1.16×10^{-2}	5.80×10^{-3}
$\Vert Ly_h\Vert_{L^2(Q_T)}$	7.18×10^{-2}	6.59×10^{-2}	6.11×10^{-2}	5.55×10^{-2}	5.10×10^{-2}
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	1.07×10^{-4}	4.70×10^{-5}	2.32×10^{-5}	1.15×10^{-5}	5.76×10^{-6}
♯ CG iterates	29	46	83	133	201

$$\frac{\Vert y - y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}} = \mathcal{O}(h^{0.574}), \quad \frac{\Vert y - y_h\Vert_{L^2(q_T)}}{\Vert y\Vert_{L^2(q_T)}} = \mathcal{O}(h^{0.94}). \tag{46}$$

$$\Vert Ly_h\Vert_{L^2(Q_T)} = \mathcal{O}(h^{0.123}). \tag{47}$$

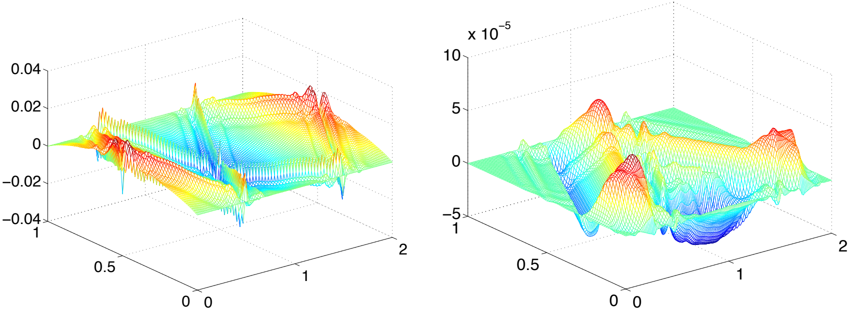
Enough to guarantee the convergence of y_h toward a solution of the wave equation: recall that then $\Vert Ly_h\Vert_{L^2(0,T;H^{-1}(0,1))} = \mathcal{O}(h^{1.123})$.

Example 2 - $N = 1$ - Observation on q_T



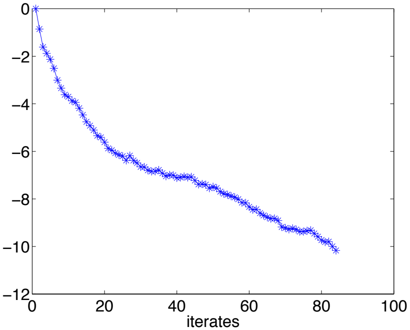
y and y_h in Q_T

Example 2 - $N = 1$ - Observation on q_T

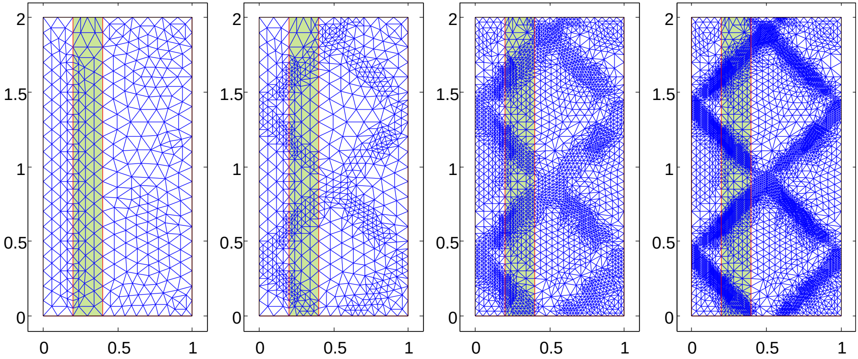


$$y - y_h \quad \text{and} \quad \lambda_h \quad \text{in} \quad Q_T$$

h	7.01×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\sharp$ CG iterates	29	46	83	133	201

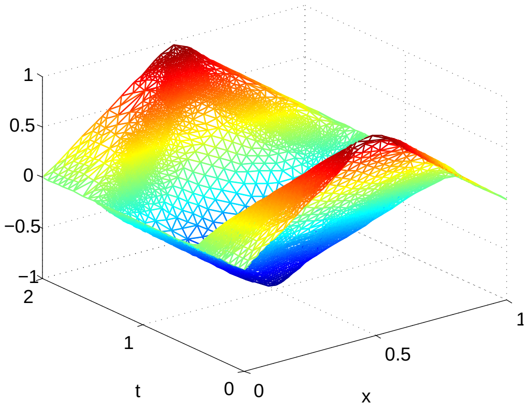


log_{10} of the residus w.r.t. iterates



Iterative local refinement of the mesh according to the gradient of y_h

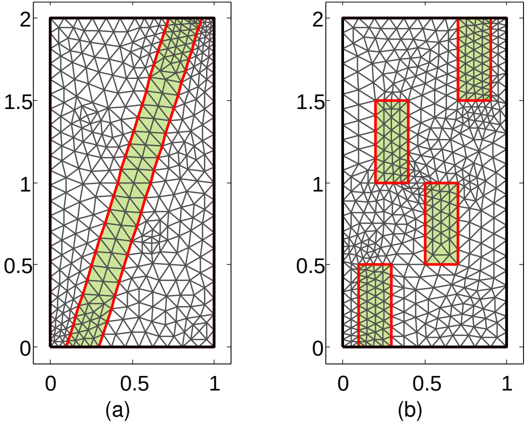
Example 2 - $N = 1$ - Mesh adaptation



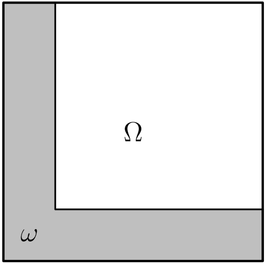
Reconstructed state y_h on the adapted mesh

Non cylindrical domain q_T

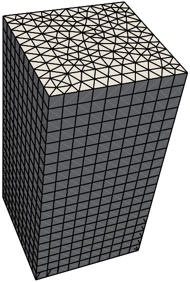
Triangular meshes - reduced HCT elements



Domain q_T^1 (a) and domain q_T^2 (b) triangulated using some coarse meshes.



(a)


(b)

Mesh Number	0	1	2	3
Number of elements	5 320	15 320	31 740	120 160
Number of nodes	3 234	8 799	17 670	64 411

Characteristics of the three meshes associated with Q_T.

$(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$:

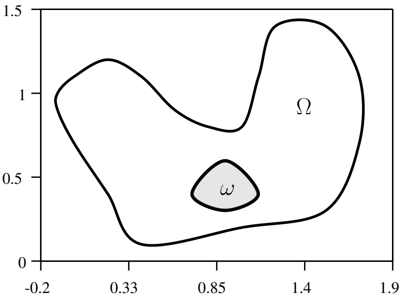
$$\textbf{(EX2–2D)} \quad \begin{cases} y_0(x_1,x_2) = (1-|2x_1-1|)(1-|2x_2-1|) \\ y_1(x_1,x_2) = \mathbf{1}_{(\frac{1}{3},\frac{2}{3})^2}(x_1,x_2) \end{cases} \quad (x_1,x_2) \in \Omega. \qquad (48)$$

The Fourier coefficients of the corresponding solution are

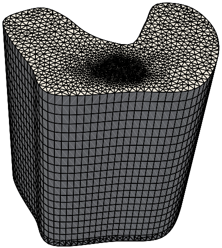
$$a_{kl} = \frac{2^5}{\pi^4 k^2 l^2} \sin\frac{\pi k}{2} \sin\frac{\pi l}{2}$$
$$b_{kl} = \frac{1}{\pi^2 kl}\left(\cos\frac{\pi k}{3} - \cos\frac{2\pi k}{3}\right)\left(\cos\frac{\pi l}{3} - \cos\frac{2\pi l}{3}\right).$$

Mesh number	0	1	2	3
$\frac{\Vert y-y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}}$	4.74×10^{-2}	3.72×10^{-2}	2.4×10^{-2}	1.35×10^{-2}
$\Vert Ly_h\Vert_{L^2(Q_T)}$	1.18	0.89	0.99	0.99
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	3.21×10^{-5}	1.46×10^{-5}	1.02×10^{-5}	3.56×10^{-6}

Table: Example **EX2–2D** – $r = h^2$



(a)



(b)

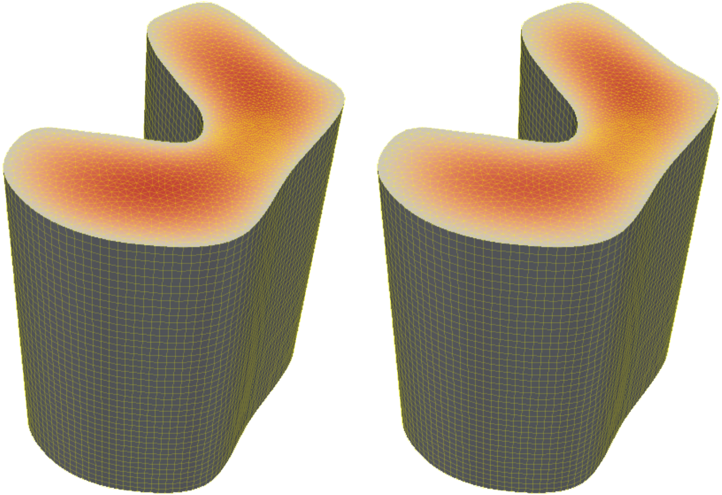
Mesh number	0	1	2
Number of elements	5 730	44 900	196 040
Number of nodes	3 432	24 633	103 566

Characteristics of the three meshes associated with Q_T.

$$\left\{ \begin{array}{ll} -\Delta y_0 = 10, & \text{in } \Omega \\ y_0 = 0, & \text{on } \partial\Omega, \end{array} \right. \qquad y_1 = 0. \tag{49}$$

Mesh number	0	1	2
$\frac{\Vert\overline{y}_h - y_h\Vert_{L^2(Q_T)}}{\Vert\overline{y}_h\Vert_{L^2(Q_T)}}$	1.88×10^{-1}	8.04×10^{-2}	5.41×10^{-2}
$\Vert Ly_h\Vert_{L^2(Q_T)}$	3.21	2.01	1.17
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	8.26×10^{-5}	3.62×10^{-5}	2.24×10^{-5}

$$r = h^2 - T = 2$$



(a) (b)

y and y_h in Q_T

$f = 0$ - $T = 2$

$$(\textbf{EX2}) \quad y_0(x) = 1 - |2x - 1|, \quad y_1(x) = 1_{(1/3,2/3)}(x), \qquad x \in (0,1)$$

in $H_0^1 \times L^2$ for which the Fourier coefficients are

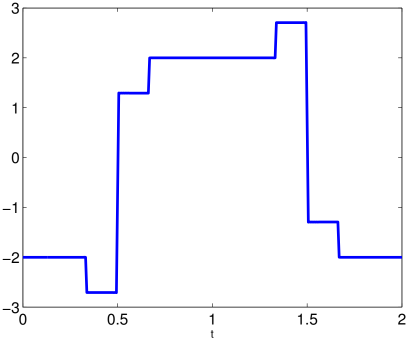
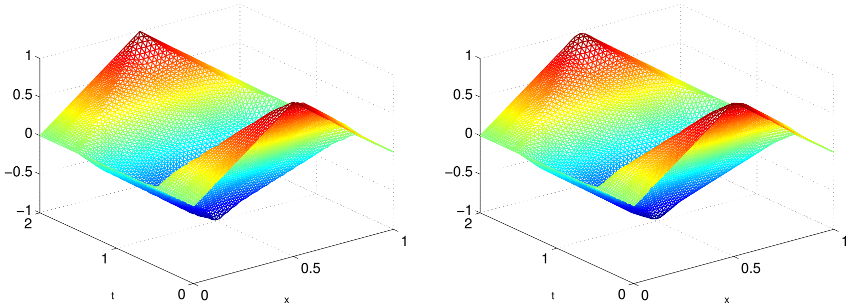


Figure: The observation $y_{\nu,obs}$ on $\{1\} \times (0, T)$ associated to initial data **EX1**.

Numerical illustration - $N = 1$ - Observation on Γ_T

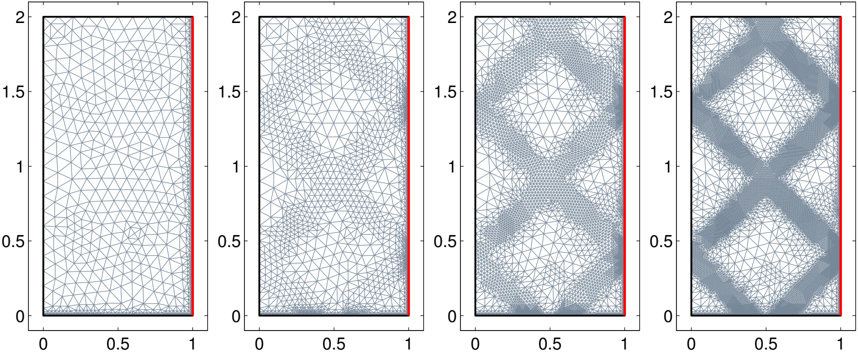
h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.42×10^{-3}
$\frac{\Vert y-y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}}$	1.63×10^{-2}	6.63×10^{-3}	2.78×10^{-3}	1.29×10^{-3}	5.72×10^{-4}
$\frac{\Vert \partial_\nu(y-y_h)\Vert_{L^2(\Gamma_T)}}{\Vert \partial_\nu y\Vert_{L^2(\Gamma_T)}}$	7.67×10^{-3}	4.95×10^{-3}	3.24×10^{-3}	2.16×10^{-3}	1.48×10^{-3}
$\Vert Ly_h\Vert_{L^2(Q_T)}$	0.937	1.204	1.496	1.798	2.135
$\Vert \lambda_h\Vert_{L^2(Q_T)}$	7.74×10^{-3}	3.74×10^{-3}	1.72×10^{-3}	7.90×10^{-4}	3.60×10^{-4}
card($\{\lambda_h\}$)	861	3 321	13 041	51 681	205 761
$\sharp$ CG iterates	57	103	172	337	591

$$r = h^2 : \quad \frac{\Vert y - y_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}} = \mathcal{O}(h^{1.20}), \quad \frac{\Vert \partial_\nu(y - y_h)\Vert_{L^2(\Gamma_T)}}{\Vert \partial_\nu y\Vert_{L^2(\Gamma_T)}} = \mathcal{O}(h^{0.59}),$$
$$\Vert \lambda_h\Vert_{L^2(Q_T)} = \mathcal{O}(h^{1.11}), \quad \Vert Ly_h\Vert_{L^2(Q_T)} = \mathcal{O}(h^{-0.29}). \tag{50}$$



y and y_h in Q_T

Example 2 - $N = 1$ - Mesh adaptation



Iterative local refinement of the mesh according to the gradient of y_h (reduced HCT element)

$T = 3$

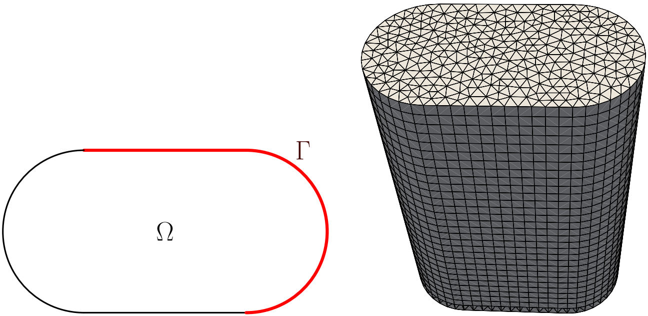
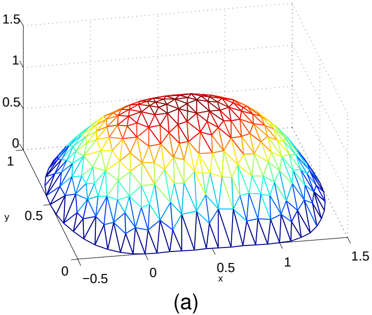


Figure: Bunimovich's stadium and the subset Γ of $\partial\Omega$ on which the observations are available. Example of mesh of the domain Q_T.

$T = 3$



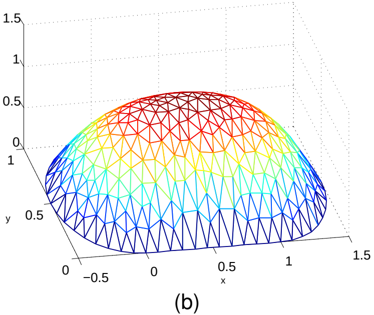


Figure: (a) Initial data y_0 given by (49). (b) Reconstructed initial data $y_h(\cdot, 0)$.

$N = 1$ - Reconstruction of y and μ from the boundary

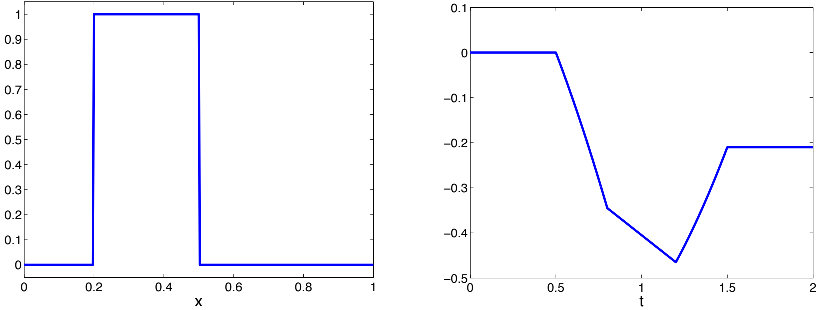


Figure: $\mu(x)$ and corresponding $\partial_\nu y|_{q_T} = y_x(1,t)$ on $(0,T)$.

$\Delta x = \Delta t = 1/160$

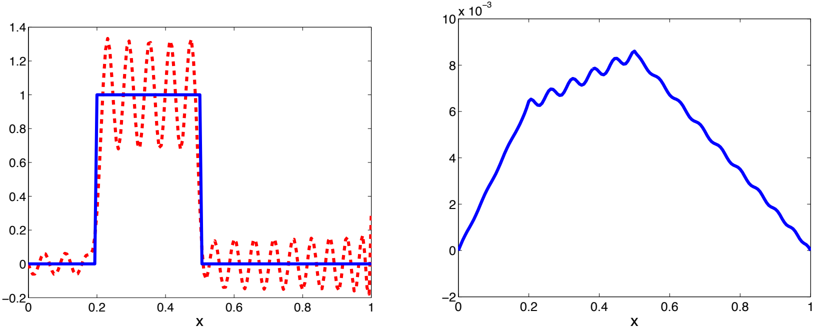


Figure: μ_h, μ and $\frac{-\Delta^{-1}(\mu - \mu_h)}{\|-\Delta^{-1}(\mu)\|_{H_0^1}}$.

$$\frac{\|\mu - \mu_h\|_{H^{-1}(\Omega)}}{\|\mu\|_{H^{-1}(\Omega)}} \approx 7.18 \times 10^{-2}, \qquad \|y - y_h\|_{L^2(Q_T)} \approx 8.68 \times 10^{-4}$$

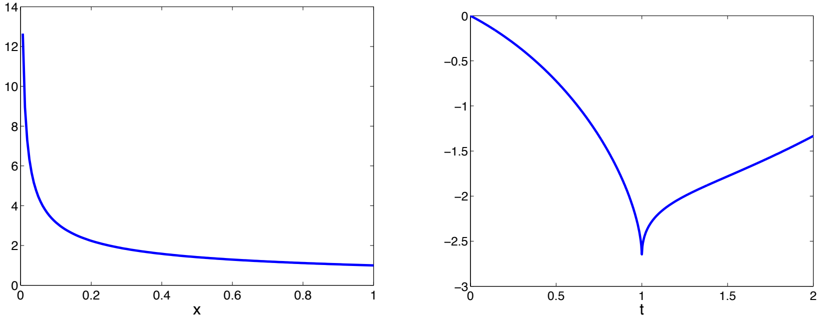


Figure: $\mu(x) = \frac{1}{\sqrt{x}}$ and corresponding $\partial_\nu y|_{q_T} = y_x(1,t)$ on $(0,T)$.

$N = 1$ - Reconstruction of y and μ from the boundary

$\Delta x = \Delta t = \frac{1}{160}$

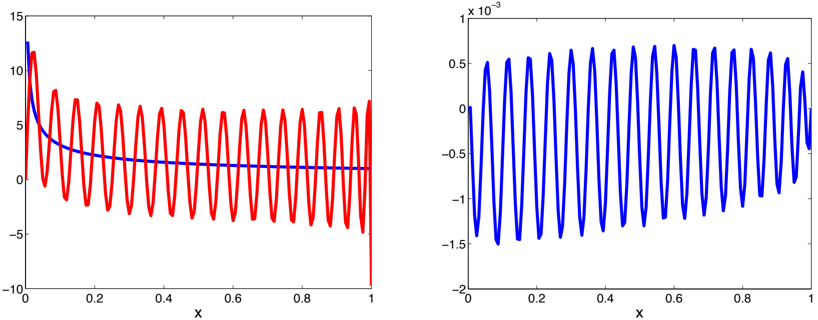


Figure: μ_h, μ and $\frac{-\Delta^{-1}(\mu - \mu_h)}{\|-\Delta^{-1}(\mu)\|_{H_0^1}}$.

$$\frac{\|\mu - \mu_h\|_{H^{-1}(\Omega)}}{\|\mu\|_{H^{-1}(\Omega)}} \approx 2.21 \times 10^{-2}, \qquad \|y - y_h\|_{L^2(Q_T)} \approx 3.56 \times 10^{-5}$$

$N = 1$ - Reconstruction of y and μ from the boundary

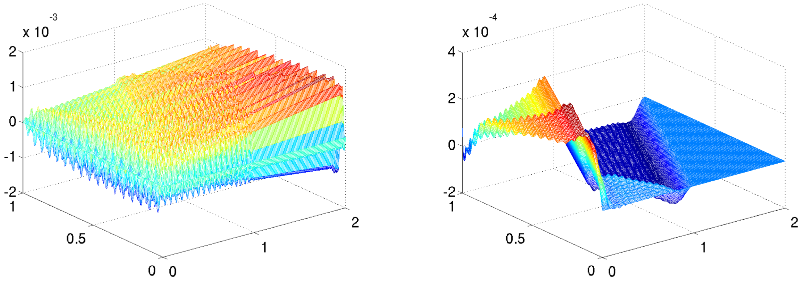


Figure: $y - y_h$ and λ_h

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$$\|y_h(\cdot,0),y_{h,t}(\cdot,0)\|^2_{\boldsymbol{H}} \leq C_{obs}\Big(\|y_h\|^2_{L^2(q_T)}+\|Ly_h\|^2_X\Big),\quad \forall y_h\in Z_h\subset Z$$

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Concluding remarks

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RECONSTRUCTION OF POTENTIAL, COEFFICIENTS

THANK YOU FOR YOUR ATTENTION

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