Mixed formulations for the direct approximation of L^2-weighted null controls for the linear heat equation

ARNAUD MÜNCH

Université Blaise Pascal - Clermont-Ferrand - France

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joint work with DIEGO A. DE SOUZA (Sevilla)

$\Omega \subset \mathbb{R}^N$; $Q_T = \Omega \times (0, T)$; $q_T = \omega \times (0, T)$

$$\begin{cases} y_t - \nabla \cdot (c(x)\nabla y) + d(x,t)y = v\, 1_\omega, & \text{in} \quad Q_T, \\ y = 0, & \text{in} \quad \Sigma_T, \\ y(x,0) = y_0(x), & \text{in} \quad \Omega. \end{cases} \tag{1}$$

$c := (c_{i,j}) \in C^1(\overline{\Omega}; \mathcal{M}_N(\mathbb{R}))$; $(c(x)\xi, \xi) \geq c_0|\xi|^2$ in $\overline{\Omega}$ $(c_0 > 0)$,

$d \in L^\infty(Q_T)$, $y_0 \in L^2(\Omega)$;

$v = v(x,t)$ is the *control* $\quad y = y(x,t)$ is the associated state.

RESULTS - For any $\omega \subset \Omega$, $T > 0$, $y_0 \in L^2(\Omega)$, $\exists v \in L^2(q_T)$ s.t. $y(\cdot, T) = 0$ a.e. Ω [FURSIKOV-IMANUVILOV 95, LEBEAU-ROBBIANO 95]

GOAL - Approximate numerically such v's.

NOTATIONS -
$Ly := y_t - \nabla \cdot (c(x)\nabla y) + d(x,t)y; \quad L^\star\varphi := -\varphi_t - \nabla \cdot (c(x)\nabla\varphi) + d(x,t)\varphi$

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Control of minimal L^2-norm - Difficulties

PRIMAL PROBLEM

$$\text{Minimize} \;\; J_1(y,v) := \frac{1}{2}\iint_{q_T} |v(x,t)|^2\, dx\, dt \tag{2}$$

over
$(y,v) \in \mathcal{C}(y_0;T) := \{\, (y,v) : v \in L^2(q_T),\; y \text{ solves (1) and satisfies } y(\cdot,T)=0 \,\}.$

DUAL PROBLEM

$$\min_{\varphi_T \in \mathcal{H}} \; J_1^\star(\varphi_T) := \frac{1}{2}\iint_{q_T} |\varphi(x,t)|^2\, dxdt + \int_\Omega y_0(x)\varphi(x,0)dx$$

where φ solves the backward heat equation :

$$L^\star\varphi = 0 \quad \text{in} \quad Q_T, \qquad \varphi = 0 \quad \text{on} \quad \Sigma_T; \quad \varphi(\cdot,T) = \varphi_T \quad \text{in} \quad \Omega, \tag{3}$$

$\mathcal{H}$ - Hilbert space defined as the completion of $L^2(\Omega)$ w.r.t. $\|\varphi_T\|_{\mathcal{H}} := \|\varphi\|_{L^2(q_T)}$.
Coercivity of $J_1^\star$ over $\mathcal{H}$ is a consequence of the so-called *observability inequality*

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \le C(\omega,T)\iint_{q_T} |\varphi(x,t)|^2\, dx\, dt \quad \forall \varphi_T \in \mathcal{H}.$$

(Numerical) Ill-posedness of the minimization of $J_1^\star$ is due to the hugeness of $\mathcal{H}$.

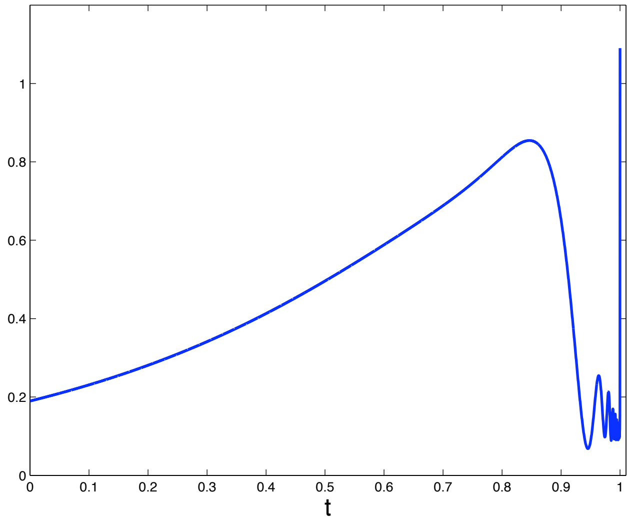

Figure: $y_0(x) = \sin(\pi x)$ - $T = 1$ - $\omega = (0.2, 0.8)$ - $t \rightarrow \|v(\cdot, t)\|_{L^2(0,1)}$ in $[0, T]$

THE MINIMIZATION OF $J_1^\star$ REQUIRES TO FIND A FINITE DIMENSIONAL AND CONFORMAL APPROXIMATION OF $\mathcal{H}$ SUCH THAT THE CORRESPONDING DISCRETE ADJOINT SOLUTION SATISFIES (3), WHICH IS IN GENERAL IMPOSSIBLE FOR POLYNOMIAL PIECEWISE APPROXIMATION.

IN PRACTICE, THE TRICK INITIALLY INTRODUCED BY GLOWINKSI-LIONS-CARTHEL 94, CONSISTS FIRST TO INTRODUCE A DISCRETE AND CONSISTENT APPROXIMATION OF (1) AND THEN TO MINIMIZE THE CORRESPONDING DISCRETE CONJUGATE FUNCTIONAL.

THIS REQUIRES TO PROVE UNIFORM DISCRETE OBSERVABILITY INEQUALITIES (OPEN !)
THIS PROPERTY AND THE HUGENESS OF $\mathcal{H}$ HAS LEAD MANY AUTHORS TO RELAX THE CONTROLLABILITY PROBLEM AND MINIMIZE OVER $L^2(\Omega)$ THE FUNCTIONAL

$$J_1^\star(\varphi_T) + \frac{\varepsilon}{2}\|\varphi_T\|^2_{L^2(\Omega)}$$

[CARTHEL94,BOYER13, ZHENG10, MUNCH-ZUAZUA10, LABBE-TRELAT06, FERNANDEZCARA-MUNCH13, ...]

FOR $\varepsilon = 0$ AND WITHIN DUAL TYPE METHODS, STRONG CONVERGENCE OF SOME APPROXIMATIONS IS STILL OPEN !

In [FERNANDEZCARA-MUNCH 12,13], we consider

$$\text{Minimize } \; J(y,v) := \frac{1}{2}\iint_{Q_T} \rho^2 |y|^2\,dx\,dt + \frac{1}{2}\iint_{q_T} \rho_0^2 |v|^2\,dx\,dt.$$

$\rho, \rho_0 \in C(Q_T, \mathbb{R}^+_\star) \cap L^\infty(Q_{T-\delta})$ for any $\delta > 0$.
Let P the completed space of $P_0 = \{q \in C^\infty(\overline{Q_T}) : q = 0 \text{ on } \Sigma_T\}$ w.r.t.

$$(p,q)_P := \iint_{Q_T} \rho^{-2} L^\star p\, L^\star q\,dx\,dt + \iint_{q_T} \rho_0^{-2}\, p\, q\,dx\,dt,$$

The optimal pair (y, v) is

$$y = \rho^{-2} L^\star p \quad \text{in} \quad Q_T, \qquad v = -\rho_0^{-2} p\, 1_\omega \quad \text{in} \quad Q_T$$

where $p \in P$ solves the variational problem (of order 2 in t and 4 in x):

$$(p,q)_P = (y_0,\, q(\cdot,0))_{L^2(\Omega)} \quad \forall q \in P.$$

Carleman type weights : $\rho(x,t) := \exp(\frac{\beta(x)}{(T-t)})$, $\rho_0(x,t) := (T-t)^{3/2}\rho(x,t)$

For any dimensional space $P_h \subset P$, we can introduce the following *approximate* problem:

$$(p_h, q_h)_P = (y_0, q_h)_{L^2(\Omega)}, \quad \forall q_h \in P_h; \quad p_h \in P_h. \tag{4}$$

$$P_h = \{ z_h \in C^{1,0}_{x,t}(\overline{Q_T}) : z_h|_K \in (\mathbb{P}_{3,x} \otimes \mathbb{P}_{1,t})(K) \ \ \forall K \in \mathcal{Q}_h, \ z_h = 0 \ \text{ on } \Sigma_T \}. \tag{5}$$

Theorem (Fernandez-Cara, M, 13)

Let $p_h \in P_h$ be the unique solution to (4), where P_h is given by (5). Let us set

$$y_h := \rho^{-2} L_A^\star p_h, \quad v_h := -\rho_0^{-2} p_h \, 1_{q_T}.$$

Then one has

$$\|y - y_h\|_{L^2(Q_T)} \to 0 \text{ and } \|v - v_h\|_{L^2(q_T)} \to 0, \qquad as \quad h \to 0$$

where (y, v) is the minimizer of J.

$\Longrightarrow$ NO NEED TO PROVE UNIFORM DISCRETE PROPERTY !!!!!!!

Let $\rho_\star \in \mathbb{R}^+_\star$ and let $\rho_0 \in \mathcal{R}$ defined by

$$\mathcal{R} := \{w : w \in C(Q_T);\, w \geq \rho_\star > 0 \text{ in } Q_T;\, w \in L^\infty(Q_{T-\delta})\ \forall\delta > 0\}$$

We consider the approximate controllability case (for any $\varepsilon > 0$, the problem reads as follows:

$$\begin{cases} \text{Minimize } \ J_\varepsilon(y,v) := \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^2 |v|^2\, dt + \dfrac{1}{2\varepsilon}\|y(\cdot,T)\|^2_{L^2(\Omega)} \\ (y,v) \in \mathcal{A}(y_0;T) := \{\,(y,v) : v \in L^2(q_T),\ y \text{ solves } (1)\,\} \end{cases}$$

The corresponding conjugate and well-posed problem is

$$\begin{cases} \text{Minimize } \ J^\star_\varepsilon(\varphi_T) := \dfrac{1}{2}\displaystyle\iint_{q_T} \rho_0^{-2} |\varphi(x,t)|^2 dx\, dt + \dfrac{\varepsilon}{2}\|\varphi_T\|^2_{L^2(\Omega)} + (y_0, \varphi(\cdot,0))_{L^2(\Omega)} \\ \text{Subject to } \ \varphi_T \in L^2(\Omega). \end{cases}$$

where φ solves

$$L^\star\varphi = 0 \quad \text{in} \quad Q_T, \qquad \varphi = 0 \quad \text{on} \quad \Sigma_T; \quad \varphi(\cdot,T) = \varphi_T \quad \text{in} \quad \Omega,$$

Mixed formulation I: the $\epsilon > 0$ case

Since φ is completely and uniquely determined by the data φ_T, the main idea of the reformulation is to keep φ as main variable.
Let $\Phi_0 := \{\varphi \in C^2(\overline{Q_T}),\, \varphi = 0 \,\text{on}\, \Sigma_T\}$. For any $\eta > 0$, we define the bilinear form

$$(\varphi, \overline{\varphi})_{\Phi_0} := \iint_{q_T} \rho_0^{-2} \varphi\, \overline{\varphi}\, dx\, dt + \varepsilon(\varphi(\cdot, T), \overline{\varphi}(\cdot, T))_{L^2(\Omega)} + \eta \iint_{Q_T} L^\star \varphi\, L^\star \overline{\varphi}\, dx\, dt,$$

For any $\varepsilon > 0$, let Φ_ε be the completion of Φ_0 for this scalar product. We denote the norm over Φ_ε by

$$\|\varphi\|^2_{\Phi_\varepsilon} := \|\rho_0^{-1}\varphi\|^2_{L^2(q_T)} + \varepsilon\|\varphi(\cdot, T)\|^2_{L^2(\Omega)} + \eta\|L^\star\varphi\|^2_{L^2(Q_T)}, \quad \forall \varphi \in \Phi_\varepsilon.$$

Finally, we defined the closed subset W_ε of Φ_ε by

$$W_\epsilon = \{\varphi \in \Phi_\varepsilon : L^\star\varphi = 0 \text{ in } L^2(Q_T)\}.$$

Then, we define the following extremal problem :

$$\min_{\varphi \in W_\varepsilon} \hat{J}^\star_\varepsilon(\varphi) := \frac{1}{2}\iint_{q_T} \rho_0^{-2}|\varphi(x,t)|^2 dx\, dt + \frac{\varepsilon}{2}\|\varphi(\cdot, T)\|^2_{L^2(\Omega)} + (y_0, \varphi(\cdot, 0))_{L^2(\Omega)}.$$

$\varphi \in W_\varepsilon \Rightarrow \varphi(\cdot, 0) \in L^2(\Omega)$ so $\hat{J}^\star_\varepsilon$ is well-defined over W_ε
$\varphi \in W_\varepsilon \Rightarrow \varphi(\cdot, T) \in L^2(\Omega)$, problems are equivalent.

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We consider the following mixed formulation : find $(\varphi_\varepsilon, \lambda_\varepsilon) \in \Phi_\varepsilon \times L^2(Q_T)$ solution of

$$\begin{cases} a_\varepsilon(\varphi_\varepsilon, \overline{\varphi}) + b(\overline{\varphi}, \lambda_\varepsilon) = l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi_\varepsilon \\ b(\varphi_\varepsilon, \overline{\lambda}) = 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{cases} \tag{6}$$

where

$$a_\varepsilon : \Phi_\varepsilon \times \Phi_\varepsilon \to \mathbb{R}, \quad a_\varepsilon(\varphi, \overline{\varphi}) := \iint_{q_T} \rho_0^{-2} \varphi\, \overline{\varphi}\, dx\, dt + \varepsilon(\varphi(\cdot, T), \overline{\varphi}(\cdot, T))_{L^2(\Omega)}$$

$$b : \Phi_\varepsilon \times L^2(Q_T) \to \mathbb{R}, \quad b(\varphi, \lambda) := -\iint_{Q_T} L^\star \varphi\, \lambda\, dx\, dt$$

$$l : \Phi_\varepsilon \to \mathbb{R}, \quad l(\varphi) := -(y_0, \varphi(\cdot, 0))_{L^2(\Omega)}.$$

We have the following result :

Theorem (De souza, M 14)

1. *The mixed formulation (6) is well-posed.*
2. *The unique solution $(\varphi_\varepsilon, \lambda_\varepsilon) \in \Phi_\varepsilon \times L^2(Q_T)$ is the unique saddle-point of the Lagrangian $\mathcal{L}_\varepsilon : \Phi_\varepsilon \times L^2(Q_T) \to \mathbb{R}$ defined by $\mathcal{L}_\varepsilon(\varphi, \lambda) := \frac{1}{2} a_\varepsilon(\varphi, \varphi) + b(\varphi, \lambda) - l(\varphi)$.*
3. *The optimal function φ_ε is the minimizer of $\hat{J}_\varepsilon^\star$ over W_ε while the optimal multiplier $\lambda_\varepsilon \in L^2(Q_T)$ is the state of the heat equation (1) in the weak sense.*

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The bilinear form a_ε is continuous over $\Phi_\varepsilon \times \Phi_\varepsilon$, symmetric and positive. The bilinear form b_ε is continuous over $\Phi_\varepsilon \times L^2(Q_T)$. Furthermore, for any fixed ε, the continuity of the linear form l over Φ_ε can be viewed from the energy estimate :

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leq C \iint_{Q_T} |L^\star \varphi|^2 dx\, dt + \|\varphi(\cdot,T)\|^2_{L^2(\Omega)} \leq \max(C\eta^{-1}, \varepsilon^{-1})\|\varphi\|^2_{\Phi_\varepsilon}, \quad \forall \varphi \in \Phi_\varepsilon.$$

Therefore, the well-posedness is a consequence :

- a_ε is coercive on $\mathcal{N}(b)$, where $\mathcal{N}(b)$ denotes the kernel of b :

$$\mathcal{N}(b) := \{\varphi \in \Phi_\varepsilon \ : \ b(\varphi,\lambda) = 0 \text{ for every } \lambda \in L^2(Q_T)\};$$

- b satisfies the usual “inf-sup" condition over $\Phi_\varepsilon \times L^2(Q_T)$: there exists $\delta > 0$ such that

$$\inf_{\lambda \in L^2(Q_T)} \sup_{\varphi \in \Phi_\varepsilon} \frac{b(\varphi,\lambda)}{\|\varphi\|_{\Phi_\varepsilon} \|\lambda\|_{L^2(Q_T)}} \geq \delta. \quad (7)$$

From the definition of a_ε, for all $\varphi \in \mathcal{N}(b) = W_\varepsilon$, $a_\varepsilon(\varphi,\varphi) = \|\varphi\|^2_{\Phi_\varepsilon}$.

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Mixed formulation I: the $\epsilon > 0$ case

For any fixed $\lambda^0 \in L^2(Q_T)$, we define the (unique) element φ^0 such that

$$L^\star \varphi^0 = -\lambda^0 \quad \text{in} \quad Q_T, \quad \varphi = 0 \quad \text{on} \quad \Sigma_T, \quad \varphi^0(\cdot, T) = 0 \quad \text{in} \quad \Omega.$$

From energy estimates,

$$\iint_{q_T} \rho_0^{-2} |\varphi^0|^2 dx\, dt \leq \rho_\star^{-2} \iint_{q_T} |\varphi^0|^2 dx\, dt \leq \rho_\star^{-2}\, C_{\Omega,T}\, \|\lambda^0\|^2_{L^2(Q_T)}.$$

Consequently, $\varphi^0 \in \Phi_\varepsilon$ and $b(\varphi^0, \lambda^0) = \|\lambda^0\|^2_{L^2(Q_T)}$ and

$$\sup_{\varphi \in \Phi_\varepsilon} \frac{b(\varphi, \lambda^0)}{\|\varphi\|_{\Phi_\varepsilon} \|\lambda^0\|_{L^2(Q_T)}} \geq \frac{b(\varphi^0, \lambda^0)}{\|\varphi^0\|_{\Phi_\varepsilon} \|\lambda^0\|_{L^2(Q_T)}} = \frac{\|\lambda^0\|^2_{L^2(Q_T)}}{\left(\|\rho_0^{-1} \varphi^0\|^2_{L^2(q_T)} + \eta \|\lambda_0\|^2_{L^2(Q_T)}\right)^{\frac{1}{2}} \|\lambda_0\|_{L^2(Q_T)}}$$

Combining the above two inequalities, we obtain

$$\sup_{\varphi_0 \in \Phi_\varepsilon} \frac{b(\varphi_0, \lambda_0)}{\|\varphi_0\|_{\Phi_\varepsilon} \|\lambda_0\|_{L^2(Q_T)}} \geq \frac{1}{\sqrt{\rho_\star^2\, C_{\Omega,T} + \eta}} := \delta$$

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$$\iint_{q_T} \rho_0^{-2} |\varphi^0|^2 dx\, dt \leq \rho_\star^{-2} \iint_{q_T} |\varphi^0|^2 dx\, dt \leq \rho_\star^{-2}\, C_{\Omega,T}\, \|\lambda^0\|^2_{L^2(Q_T)}.$$

Consequently, $\varphi^0 \in \Phi_\varepsilon$ and $b(\varphi^0, \lambda^0) = \|\lambda^0\|^2_{L^2(Q_T)}$ and

$$\sup_{\varphi \in \Phi_\varepsilon} \frac{b(\varphi, \lambda^0)}{\|\varphi\|_{\Phi_\varepsilon} \|\lambda^0\|_{L^2(Q_T)}} \geq \frac{b(\varphi^0, \lambda^0)}{\|\varphi^0\|_{\Phi_\varepsilon} \|\lambda^0\|_{L^2(Q_T)}} = \frac{\|\lambda^0\|^2_{L^2(Q_T)}}{\left(\|\rho_0^{-1} \varphi^0\|^2_{L^2(q_T)} + \eta \|\lambda_0\|^2_{L^2(Q_T)}\right)^{\frac{1}{2}} \|\lambda_0\|_{L^2(Q_T)}}$$

Combining the above two inequalities, we obtain

$$\sup_{\varphi_0 \in \Phi_\varepsilon} \frac{b(\varphi_0, \lambda_0)}{\|\varphi_0\|_{\Phi_\varepsilon} \|\lambda_0\|_{L^2(Q_T)}} \geq \frac{1}{\sqrt{\rho_\star^2\, C_{\Omega,T} + \eta}} := \delta$$

The point (ii) is due to the symmetry and to the positivity of the bilinear form a_ε.
(iii) Concerning the third point, the first equation of the mixed formulation reads as follows:

$$\iint_{q_T} \rho_0^{-2}\varphi_\varepsilon\,\overline{\varphi}\,dx\,dt+\varepsilon(\varphi_\varepsilon(\cdot,T),\overline{\varphi}(\cdot,T))-\iint_{Q_T} L^\star\overline{\varphi}(x,t)\,\lambda_\varepsilon(x,t)\,dx\,dt = l(\overline{\varphi}), \quad \forall\overline{\varphi}\in\Phi_\varepsilon,$$

or equivalently, since the control is given by $v_\varepsilon := \rho_0^{-2}\varphi_\varepsilon\,1_\omega$,

$$\iint_{q_T} v_\varepsilon\,\overline{\varphi}\,dx\,dt+(\varepsilon\varphi_\varepsilon(\cdot,T),\overline{\varphi}(\cdot,T))-\iint_{Q_T} L^\star\overline{\varphi}(x,t)\,\lambda_\varepsilon(x,t)\,dx\,dt = l(\overline{\varphi}), \quad \forall\overline{\varphi}\in\Phi_\varepsilon.$$

But this means that $\lambda_\varepsilon \in L^2(Q_T)$ is solution of the heat equation in the transposition sense. Since $y_0 \in L^2(\Omega)$ and $v_\varepsilon \in L^2(q_T)$, λ_ε must coincide with the unique weak solution to (1) ($y_\varepsilon = \lambda_\varepsilon$) such that $\lambda_\varepsilon(\cdot,T) = -\varepsilon\varphi_\varepsilon(\cdot,T)$.

Theorem 2 reduces the search of the approximated control to the resolution of the mixed formulation (6), or equivalently the search of the saddle point for $\mathcal{L}_\varepsilon$. In general, it is convenient to "augment" the Lagrangian, and consider instead the Lagrangian $\mathcal{L}_{\varepsilon,r}$ defined for any $r > 0$ by

$$\begin{cases} \mathcal{L}_{\varepsilon,r}(\varphi,\lambda) := \dfrac{1}{2} a_{\varepsilon,r}(\varphi,\varphi) + b(\varphi,\lambda) - l(\varphi), \\ a_{\varepsilon,r}(\varphi,\varphi) := a_\varepsilon(\varphi,\varphi) + r \displaystyle\iint_{Q_T} |L^\star \varphi|^2 \, dx\, dt. \end{cases}$$

Since $a_\varepsilon(\varphi,\varphi) = a_{\varepsilon,r}(\varphi,\varphi)$ on W_ε and since the function φ_ε such that $(\varphi_\varepsilon, \lambda_\varepsilon)$ is the saddle point of $\mathcal{L}_\varepsilon$ verifies $\varphi_\varepsilon \in W_\varepsilon$, the lagrangian $\mathcal{L}_\varepsilon$ and $\mathcal{L}_{\varepsilon,r}$ share the same saddle-point.

Proposition

Assume $r > 0$. The following equality holds :

$$\sup_{\lambda\in L^2(Q_T)} \inf_{\varphi\in\Phi_\varepsilon} \mathcal{L}_{\varepsilon,r}(\varphi,\lambda) = -\inf_{\lambda\in L^2(Q_T)} J^{\star\star}_{\varepsilon,r}(\lambda) \; + \mathcal{L}_{\varepsilon,r}(\varphi^0,0).$$

- $J^{\star\star}_{\varepsilon,r} : L^2(Q_T) \to L^2(Q_T)$ *is defined by*

$$J^{\star\star}_{\varepsilon,r}(\lambda) := \frac{1}{2}\iint_{Q_T} (\mathcal{A}_{\varepsilon,r}\lambda)\,\lambda\,dx\,dt - b(\varphi^0,\lambda).$$

- $\varphi^0 \in \Phi_\varepsilon$ *solves* $a_{\varepsilon,r}(\varphi^0,\overline{\varphi}) = l(\overline{\varphi}), \quad \forall\overline{\varphi}\in\Phi_\varepsilon$.
- *The linear operator* $\mathcal{A}_{\varepsilon,r}$ *from* $L^2(Q_T)$ *into* $L^2(Q_T)$ *by*

$$\mathcal{A}_{\varepsilon,r}\lambda := L^\star\varphi, \quad \forall\lambda\in L^2(Q_T)$$

where $\varphi\in\Phi_\varepsilon$ *solves* $a_{\varepsilon,r}(\varphi,\overline{\varphi}) = -b(\overline{\varphi},\lambda), \quad \forall\overline{\varphi}\in\Phi_\varepsilon$.
For any $r > 0$, $\mathcal{A}_{\varepsilon,r}$ *is a strongly elliptic, symmetric isomorphism from* $L^2(Q_T)$ *into* $L^2(Q_T)$.

Remark 3: Relaxation of the constraint

Similar results with

$$L^\star \varphi_\varepsilon = 0 \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)) \Longrightarrow \lambda_\varepsilon \in L^2(0,T;H_0^1(\Omega))$$

$$b:\Phi_\varepsilon \times L^2(0,T;H_0^1(\Omega)) \to \mathbb{R}, \quad b(\varphi,\lambda) := -\int_0^T < L^\star\varphi, \lambda >_{H^{-1},H_0^1} \, dt$$

Let $\rho \in \mathcal{R}$. Let $\widetilde{\Phi}_0 = \{\varphi \in C^2(\overline{Q_T}) : \varphi = 0 \text{ on } \Sigma_T\}$ and, for any $\eta > 0$,

$$(\varphi, \overline{\varphi})_{\widetilde{\Phi}_0} := \iint_{q_T} \rho_0^{-2} \varphi\, \overline{\varphi}\, dx\, dt + \eta \iint_{Q_T} \rho^{-2} L^\star \varphi\, L^\star \overline{\varphi}\, dx\, dt, \quad \forall \varphi, \overline{\varphi} \in \widetilde{\Phi}_0.$$

Let $\widetilde{\Phi}_{\rho_0,\rho}$ be the completion of $\widetilde{\Phi}_0$ for this scalar product. We note the norm over $\widetilde{\Phi}_{\rho_0,\rho}$

$$\|\varphi\|^2_{\widetilde{\Phi}_{\rho_0,\rho}} := \|\rho_0^{-1}\varphi\|^2_{L^2(q_T)} + \eta \|\rho^{-1} L^\star \varphi\|^2_{L^2(Q_T)}, \quad \forall \varphi \in \widetilde{\Phi}_{\rho_0,\rho}.$$

Finally, we defined the closed subset $\widetilde{W}_{\rho_0,\rho}$ of $\widetilde{\Phi}_{\rho_0,\rho}$ by

$$\widetilde{W}_{\rho_0,\rho} = \{\varphi \in \widetilde{\Phi}_{\rho_0,\rho} : \rho^{-1} L^\star \varphi = 0 \text{ in } L^2(Q_T)\}$$

We then define the following extremal problem :

$$\min_{\varphi \in \widetilde{W}_{\rho_0,\rho}} \hat{J}^\star(\varphi) = \frac{1}{2} \iint_{q_T} \rho_0^{-2} |\varphi(x,t)|^2 dx\, dt + (y_0, \varphi(\cdot, 0))_{L^2(\Omega)}. \tag{8}$$

For any $\varphi \in \widetilde{W}_{\rho_0,\rho}$, $L^\star \varphi = 0$ a.e. in Q_T and $\|\varphi\|_{\widetilde{W}_{\rho_0,\rho}} = \|\rho_0^{-1} \varphi\|_{L^2(q_T)}$ so that $\varphi(\cdot, T) \in \mathcal{H}$.

$$\left\{ \begin{array}{rcll} \tilde{a}_r(\varphi,\overline{\varphi}) + \tilde{b}(\overline{\varphi},\lambda) & = & \tilde{l}(\overline{\varphi}), & \forall \overline{\varphi} \in \widetilde{\Phi}_{\rho_0,\rho} \\ \tilde{b}(\varphi,\overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{array} \right. \tag{9}$$

where

$$\tilde{a}_r : \widetilde{\Phi}_{\rho_0,\rho} \times \widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \tilde{a}_r(\varphi,\overline{\varphi}) = \iint_{q_T} \rho_0^{-2} \varphi\, \overline{\varphi}\, dx\, dt + r \iint_{Q_T} |\rho^{-1} L^\star \varphi|^2\, dx\, dt$$

$$\tilde{b} : \widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T) \to \mathbb{R}, \quad \tilde{b}(\varphi,\lambda) = -\iint_{Q_T} \rho^{-1} L^\star \varphi\, \lambda\, dx\, dt$$

$$\tilde{l} : \widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \tilde{l}(\varphi) = -(y_0, \varphi(\cdot,0))_{L^2(\Omega)}.$$

Theorem

Let $\rho_0 \in \mathcal{R}$ and $\rho \in \mathcal{R} \cap L^\infty(Q_T)$ and assume that there exists a positive constant K such that

$$\rho_0 \leq K \rho_0^c, \quad \rho \leq K \rho^c \quad in \quad Q_T. \tag{10}$$

1. *The mixed formulation (9) defined over $\widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T)$ is well-posed.*
2. *The optimal function φ is the minimizer of $\hat{J}^\star$ over $\widetilde{\Phi}_{\rho_0,\rho}$ while $\rho^{-1}\lambda \in L^2(Q_T)$ is the state of the heat equation (1) in the weak sense.*

Mixed formulation II: the limit case $\varepsilon = 0$.

$$\begin{cases} \tilde{a}_r(\varphi,\overline{\varphi}) + \tilde{b}(\overline{\varphi},\lambda) = \tilde{l}(\overline{\varphi}), & \forall \overline{\varphi} \in \widetilde{\Phi}_{\rho_0,\rho} \\ \tilde{b}(\varphi,\overline{\lambda}) = 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{cases} \tag{9}$$

where

$$\tilde{a}_r : \widetilde{\Phi}_{\rho_0,\rho} \times \widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \tilde{a}_r(\varphi,\overline{\varphi}) = \iint_{q_T} \rho_0^{-2} \varphi\, \overline{\varphi}\, dx\, dt + r \iint_{Q_T} |\rho^{-1} L^\star \varphi|^2\, dx\, dt$$

$$\tilde{b} : \widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T) \to \mathbb{R}, \quad \tilde{b}(\varphi,\lambda) = -\iint_{Q_T} \rho^{-1} L^\star \varphi\, \lambda\, dx\, dt$$

$$\tilde{l} : \widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \tilde{l}(\varphi) = -(y_0, \varphi(\cdot,0))_{L^2(\Omega)}.$$

Theorem

Let $\rho_0 \in \mathcal{R}$ and $\rho \in \mathcal{R} \cap L^\infty(Q_T)$ and assume that there exists a positive constant K such that

$$\rho_0 \le K \rho_0^c, \quad \rho \le K \rho^c \quad \mathit{in} \quad Q_T. \tag{10}$$

1. *The mixed formulation* (9) *defined over $\widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T)$ is well-posed.*
2. *The optimal function φ is the minimizer of $\hat{J}^\star$ over $\widetilde{\Phi}_{\rho_0,\rho}$ while $\rho^{-1}\lambda \in L^2(Q_T)$ is the state of the heat equation (1) in the weak sense.*

$$\left\{\begin{array}{rcll} \tilde{a}_r(\varphi,\overline{\varphi}) + \tilde{b}(\overline{\varphi},\lambda) & = & \tilde{l}(\overline{\varphi}), & \forall \overline{\varphi} \in \widetilde{\Phi}_{\rho_0,\rho} \\ \tilde{b}(\varphi,\overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{array}\right. \tag{9}$$

where

$$\tilde{a}_r : \widetilde{\Phi}_{\rho_0,\rho} \times \widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \tilde{a}_r(\varphi,\overline{\varphi}) = \iint_{q_T} \rho_0^{-2} \varphi\,\overline{\varphi}\,dx\,dt + r \iint_{Q_T} |\rho^{-1} L^\star \varphi|^2\,dx\,dt$$

$$\tilde{b} : \widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T) \to \mathbb{R}, \quad \tilde{b}(\varphi,\lambda) = -\iint_{Q_T} \rho^{-1} L^\star \varphi\,\lambda\,dx\,dt$$

$$\tilde{l} : \widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \tilde{l}(\varphi) = -(y_0, \varphi(\cdot,0))_{L^2(\Omega)}.$$

Theorem

Let $\rho_0 \in \mathcal{R}$ and $\rho \in \mathcal{R} \cap L^\infty(Q_T)$ and assume that there exists a positive constant K such that

$$\rho_0 \le K\rho_0^c, \quad \rho \le K\rho^c \quad in \quad Q_T. \tag{10}$$

1. *The mixed formulation* (9) *defined over $\widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T)$ is well-posed.*
2. *The optimal function φ is the minimizer of $\hat{J}^\star$ over $\widetilde{\Phi}_{\rho_0,\rho}$ while $\rho^{-1}\lambda \in L^2(Q_T)$ is the state of the heat equation* (1) *in the weak sense.*

Let then $\Phi_{\varepsilon,h}$ and $M_{\varepsilon,h}$ be two finite dimensional spaces parametrized by the variable h such that, for any $\varepsilon > 0$,

$$\Phi_{\varepsilon,h} \subset \Phi_{\varepsilon}, \quad M_{\varepsilon,h} \subset L^2(Q_T), \quad \forall h > 0.$$

Then, we can introduce the following approximated problems : find $(\varphi_h, \lambda_h) \in \Phi_{\varepsilon,h} \times M_{\varepsilon,h}$ solution of

$$\begin{cases} a_{\varepsilon,r}(\varphi_h, \overline{\varphi}_h) + b(\overline{\varphi}_h, \lambda_h) &= l(\overline{\varphi}_h), & \forall \overline{\varphi}_h \in \Phi_{\varepsilon,h} \\ b(\varphi_h, \overline{\lambda}_h) &= 0, & \forall \overline{\lambda}_h \in M_{\varepsilon,h}. \end{cases} \tag{11}$$

- $$\forall r > 0 \quad a_{\varepsilon,r}(\varphi, \varphi) \geq \frac{r}{\eta} \|\varphi\|^2_{\Phi_\varepsilon}, \quad \forall \varphi \in \Phi_\varepsilon$$

$a_{r,\varepsilon}$ is coercive on $\mathcal{N}_h(b) = \{\varphi_h \in \Phi_{\varepsilon,h}; b(\varphi_h, \lambda_h) = 0 \quad \forall \lambda_h \in M_{\varepsilon,h}\} \subset \Phi_{\varepsilon,h} \subset \Phi_\varepsilon$. ($r\|L^\star\varphi\|^2_{L^2(Q_T)}$ acts as a (numerical) stabilization term)

- $$\forall h > 0, \delta_h := \inf_{\lambda_h \in M_{\varepsilon,h}} \sup_{\varphi_h \in \Phi_{\varepsilon,h}} \frac{b(\varphi_h, \lambda_h)}{\|\varphi_h\|_{\Phi_{\varepsilon,h}} \|\lambda_h\|_{M_{\varepsilon,h}}} > 0. \tag{12}$$

$\Phi_{\varepsilon,h}$ chosen such that $L^\star \varphi_h \in L^2(Q_T) \quad \forall \varphi_h \in \Phi_{\varepsilon,h}$.

$$\Phi_{\varepsilon,h} = \{\varphi_h \in C^1(\overline{Q_T}) : \varphi_h|_K \in \mathbb{P}(K) \quad \forall K \in \mathcal{T}_h,\ \varphi_h = 0 \text{ on } \Sigma_T\}$$

$\mathbb{P}(K)$ - *Bogner-Fox-Schmit* (BFS for short) C^1-element defined for rectangles.

$$M_{\varepsilon,h} = \{\lambda_h \in C^0(\overline{Q_T}) : \lambda_h|_K \in \mathbb{Q}(K) \quad \forall K \in \mathcal{T}_h\},$$

$\mathbb{Q}(K)$- space of affine functions both in x and t on the element K.

Let $n_h = \dim \Phi_h$, $m_h = \dim M_h$. Problem (11) reads as follows : find $\{\varphi_h\} \in \mathbb{R}^{n_h}$ and $\{\lambda_h\} \in \mathbb{R}^{m_h}$ such that

$$\begin{pmatrix} A_{\varepsilon,r,h} & B_h^T \\ B_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix} \{\varphi_h\} \\ \{\lambda_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} L_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}. \qquad (13)$$

$$\begin{aligned}\delta_{\varepsilon,r,h} &:= \inf_{\lambda_h \in M_{\varepsilon,h}} \sup_{\varphi_h \in \Phi_{\varepsilon,h}} \frac{b(\varphi_h,\lambda_h)}{\|\varphi_h\|_{\Phi_{\varepsilon,h}} \|\lambda_h\|_{M_{\varepsilon,h}}} \\ &:= \inf\left\{\sqrt{\delta} : B_h A^{-1}_{\varepsilon,r,h} B_h^T \{\lambda_h\} = \delta\, J_h\{\lambda_h\}, \quad \forall \{\lambda_h\} \in \mathbb{R}^{m_h} \setminus \{0\}\right\}.\end{aligned} \tag{14}$$

We obtain that

$$\delta_{\varepsilon,r,h} \approx C_{\varepsilon,r,h} \times r^{-1}; \quad C_{\varepsilon,r,h} = O(1) \quad (r \text{ large}) \tag{15}$$

$\Longrightarrow$ APPROXIMATIONS $\Phi_{\varepsilon,h}, M_{\varepsilon,h}$ DO PASS THE TEST !

Inf-sup test: $\delta_{\varepsilon,r,h}$ w.r.t. ε and h.

$$\Omega=(0,1),\omega=(0.2,0.5)\quad T=1/2,\quad \rho_0(x,t):=(T-t)^{3/2}\exp\left(\frac{3}{4(T-t)}\right),\tag{16}$$

$r=10^{-2}$:

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\varepsilon=10^{-2}$	8.358	8.373	8.381	8.386
$\varepsilon=10^{-4}$	9.183	9.213	9.229	9.237
$\varepsilon=10^{-8}$	9.263	9.318	9.354	9.383

$r=10^{2}$

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\varepsilon=10^{-2}$	9.933×10^{-2}	9.939×10^{-2}	9.940×10^{-2}	9.941×10^{-2}
$\varepsilon=10^{-4}$	9.933×10^{-2}	9.939×10^{-2}	9.941×10^{-2}	9.942×10^{-2}
$\varepsilon=10^{-8}$	9.933×10^{-2}	9.939×10^{-2}	9.941×10^{-2}	9.942×10^{-2}

$$\delta_{\varepsilon,r,h}\approx C_{\varepsilon,r,h}\times r^{-1};\quad C_{\varepsilon,r,h}=O(1)\quad (r\text{ large})\tag{17}$$

$\Longrightarrow$ SPACE $(\Phi_{\varepsilon,h},M_{\varepsilon,h})$ DO PASS THE TEST !

$\varepsilon = 10^{-2}$

$y_0(x) = \sin(\pi x), \quad \Omega = (0,1) \quad \omega = (0.2, 0.5), T = 1/2, C := 0.1, d = 0$

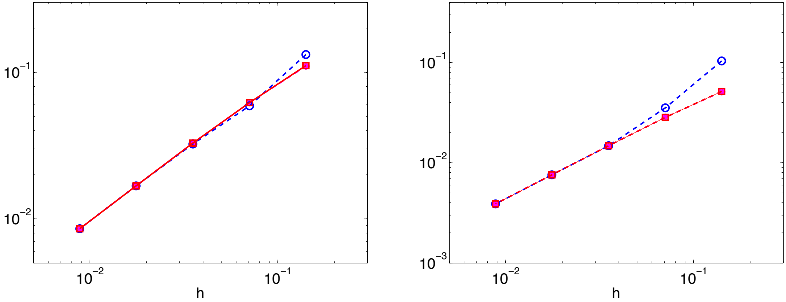

Figure: $\frac{\|\rho_0(v_\varepsilon - v_{\varepsilon,h})\|_{L^2(q_T)}}{\|\rho_0 v_\varepsilon\|_{L^2(q_T)}}$ (**Left**) and $\frac{\|y_\varepsilon - \lambda_{\varepsilon,h}\|_{L^2(Q_T)}}{\|y_\varepsilon\|_{L^2(Q_T)}}$ (**Right**) vs. h for $r = 10^2$ ($\circ$), $r = 1.$ ($\star$) and $r = 10^{-2}$ ($\square$).

$\varepsilon = 10^{-8}$

$y_0(x) = \sin(\pi x), \quad \Omega = (0,1) \quad \omega = (0.2, 0.5), T = 1/2, C := 0.1, d = 0$

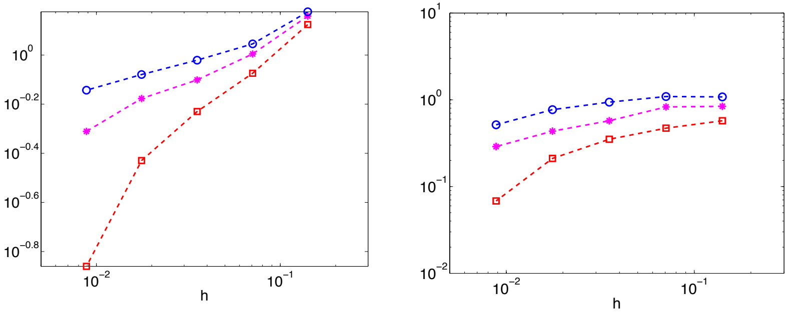

Figure: $\frac{\|\rho_0(v_\varepsilon - v_{\varepsilon,h})\|_{L^2(q_T)}}{\|\rho_0 v_\varepsilon\|_{L^2(q_T)}}$ (**Left**) and $\frac{\|y_\varepsilon - \lambda_{\varepsilon,h}\|_{L^2(Q_T)}}{\|y_\varepsilon\|_{L^2(Q_T)}}$ (**Right**) vs. h for $r = 10^2$ ($\circ$), $r = 1.$ ($\star$) and $r = 10^{-2}$ ($\square$).

$$\|\mathcal{A}_{\varepsilon,r}(\lambda)\|_{L^2(Q_T)}\| \leq r^{-1}\|\lambda\|_{L^2(Q_T)}$$
$$\nu(\mathcal{A}_{\varepsilon,r}) = \|\mathcal{A}_{\varepsilon,r}\|\|\mathcal{A}^{-1}_{\varepsilon,r}\| = \nu(B_h A^{-1}_{\varepsilon,r,h} B_h^T) \leq r^{-1}\delta^{-2}_{\varepsilon,r,h} \approx \sqrt{C_{\varepsilon,r,h}}$$

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\varepsilon = 10^{-2}$	1.431	1.426	1.423	1.423
$\varepsilon = 10^{-4}$	1.185	1.177	1.173	1.171
$\varepsilon = 10^{-8}$	1.165	1.151	1.142	1.135

Table: $r^{-1}\delta^{-2}_{\varepsilon,h}$ w.r.t. ε and h ; $r = 10^{-2}$; $\Omega = (0,1)$, $\omega = (0.2, 0.5)$, $T = 1/2$.

h	7.07×10^{-2}	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}
$\sharp$ iterates - $\varepsilon = 10^{-2}$	9	8	8	8
$\sharp$ iterates - $\varepsilon = 10^{-4}$	8	8	8	8
$\sharp$ iterates - $\varepsilon = 10^{-8}$	8	7	7	7
$\kappa(A_{\varepsilon,r,h})$ - $\varepsilon = 10^{-2}$	1.10×10^{11}	6.81×10^{12}	3.83×10^{14}	1.91×10^{16}

Table: Mixed formulation (6) - $r = 10^{-2}$ - $\omega = (0.2, 0.5)$; Conjugate gradient algorithm.

Mixed formulation II: the $\epsilon = 0$ case

Same features for the limit case, up to a crucial point : make the change of variable

$$\varphi := \rho_0 \psi$$

and solve the mixed formulation w.r.t. (ψ, λ) over $\rho_0^{-1}\widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T)$.

$$\hat{a} : \rho_0^{-1}\widetilde{\Phi}_{\rho_0,\rho} \times \rho_0^{-1}\widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \hat{a}(\psi, \overline{\psi}) = \iint_{q_T} \psi\, \overline{\psi}\, dx\, dt$$

$$\hat{b} : \rho_0^{-1}\widetilde{\Phi}_{\rho_0,\rho} \times L^2(Q_T) \to \mathbb{R}, \quad \hat{b}(\psi, \lambda) = -\iint_{Q_T} \rho^{-1} L^\star(\rho_0\, \psi)\, \lambda\, dx\, dt$$

$$\hat{l} : \rho_0^{-1}\widetilde{\Phi}_{\rho_0,\rho} \to \mathbb{R}, \quad \hat{l}(\varphi) = -(y_0, \rho_0(\cdot, 0)\psi(\cdot, 0))_{L^2(\Omega)}.$$

If $\rho(t) = \exp(\frac{3}{4(T-t)})$ and $\rho_0(t) = (T-t)^{3/2}\rho(t)$:

$$\begin{aligned} \rho^{-1}\, L^\star(\rho_0\psi) &= \rho^{-1}\rho_0 L^\star\psi - \rho^{-1}\rho_{0t}\psi \\ &= (T-t)^{3/2} L^\star\psi + \left(-\frac{3}{2}(T-t)^{1/2} + K_1(T-t)^{-1/2}\right)\psi. \end{aligned} \tag{18}$$

Mixed formulation II: the $\epsilon = 0$ case

h	3.53×10^{-2}	1.76×10^{-2}	8.83×10^{-3}	4.41×10^{-3}	2.2×10^{-3}
$\Vert\rho^{-1}L^\star(\rho_0\psi_h)\Vert_{L^2(Q_T)}$	29.76	24.86	21.12	17.92	15.42
$\frac{\Vert\rho_0(v-v_h)\Vert_{L^2(q_T)}}{\Vert\rho_0 v\Vert_{L^2(q_T)}}$	5.35×10^{-1}	3.34×10^{-1}	2.42×10^{-1}	1.63×10^{-1}	8.45×10^{-2}
$\Vert\rho_0 v_h\Vert_{L^2(q_T)}$	15.20	16.642	17.52	18.07	18.43
$\Vert\rho^{-1}\lambda_h\Vert_{L^2(Q_T)}$	3.15×10^{-1}	3.34×10^{-1}	3.46×10^{-1}	3.52×10^{-1}	3.56×10^{-1}
$\frac{\Vert y - \rho^{-1}\lambda_h\Vert_{L^2(Q_T)}}{\Vert y\Vert_{L^2(Q_T)}}$	1.96×10^{-1}	1.20×10^{-1}	6.97×10^{-2}	3.67×10^{-2}	1.49×10^{-2}
$\sharp$ CG iterates	52	55	56	56	55
$r^{-1}\delta_{r,h}^{-2}$	27.04	29.37	31.73	33.37	–
$\kappa(A_{r,h})$	9.5×10^4	1.4×10^7	3.03×10^9	1.1×10^{12}	–
n_h=size($A_{r,h}$)	3 444	13 284	52 264	206 724	823 044
$\Vert y_h(\cdot,T)\Vert_{L^2(0,1)}$	1.52×10^{-1}	6.109×10^{-2}	2.59×10^{-2}	1.162×10^{-2}	5.41×10^{-3}

Table: Mixed formulation (9) - $r = 10^{-2}$ and $\varepsilon = 0$ with $\omega = (0.2, 0.5)$.

$$\omega = (0.2, 0.5); \quad y_0(x) = \sin(\pi x), \quad \varepsilon = 0$$

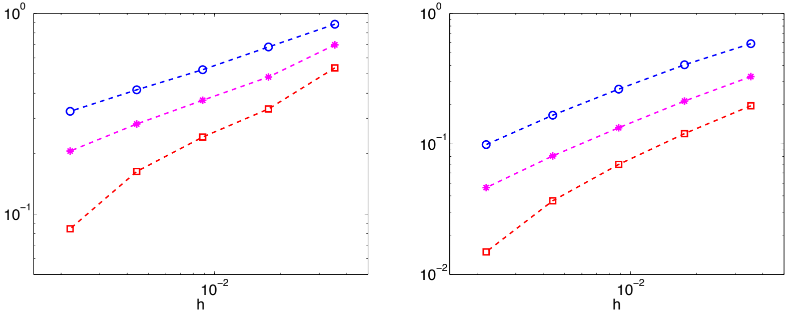

Figure: $\frac{\|\rho_0(v-v_h)\|_{L^2(q_T)}}{\|\rho_0 v\|_{L^2(q_T)}}$ (**Left**) and $\frac{\|y-\rho^{-1}\lambda_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}}$ (**Right**) vs. h for $r = 10^2$ ($\circ$), $r = 1.$ ($\star$) and $r = 10^{-2}$ ($\square$).

$$\omega = (0.2, 0.5); \quad y_0(x) = \sin(\pi x), \quad \varepsilon = 0$$

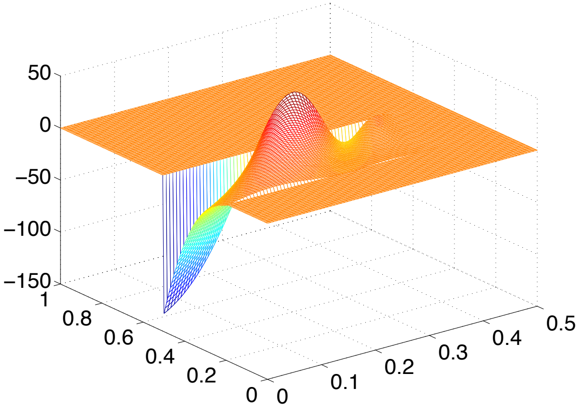

Figure: $\omega = (0.2, 0.5)$; Approximation $v_h = \rho_0^{-1}\psi_h$ of the null control v over Q_T - $r = 1$ and $h = 8.83 \times 10^{-3}$.

Mixed formulation II: the $\epsilon = 0$ case

$$\omega = (0.2, 0.5); \quad y_0(x) = \sin(\pi x), \quad \varepsilon = 0$$

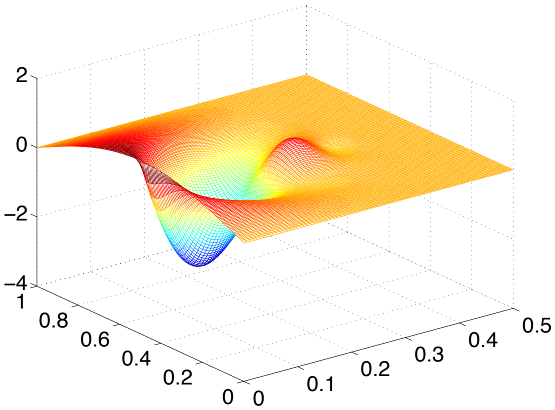

Figure: $\omega = (0.2, 0.5)$; Approximation $\rho^{-1}\lambda_h$ of the controlled state y over Q_T - $r = 1$ and $h = 8.83 \times 10^{-3}$.

MIXED FORMULATION ALLOWS TO APPROXIMATE DIRECTLY WEIGHTED L^2 CONTROLS FOR THE HEAT EQ.

THE MINIMISATION OF $J_r^{\star\star}(\lambda)$ IS VERY ROBUST AND FAST CONTRARY TO THE MINIMISATION OF $J^{\star}(\varphi_T)$ (INVERSION OF SYMMETRIC DEFINITE POSITIVE AND VERY SPARSE MATRICE WITH DIRECT CHOLESKY SOLVERS)

SPACE-TIME FINITE ELEMENT FORMULATION IS VERY WELL-ADAPTED TO MESH ADAPTATION AND TO NON-CYLINDRICAL SITUATION

DIRECT APPROACH CAN BE USED OF MANY OTHER CONTROLLABLE SYSTEMS FOR WHICH APPROPRIATE CARLEMAN ESTIMATES ARE AVAILABLE. [CINDEA,MUNCH 2014 CALCOLO] FOR THE WAVE EQ.

THE PRICE TO PAY IS TO USED C^1 FINITE ELEMENTS (AT LEAST IN SPACE) UNLESS $L^{\star}\varphi = 0$ IS SEEN IN A WEAKER SPACE THAN $L^2(Q_T)$.

A NICE OPEN QUESTION IF THE DISCRETE INF-SUP PROPERTY !?? A SIMPLE STRATEGY IS TO ADD THE LAGRANGIAN THE TERM

$$-\|L\lambda_h + \rho_0^{-2}\varphi_h 1_\omega\|^2_{L^2(Q_T)}$$

THIS APPROACH MAY BE APPLIED FOR INVERSE PROBLEMS, OBSERVATION PROBLEMS, RECONSTRUCTION OF DATA, ...

Given the observation $z \in L^2(q_T)$, find y such that

$$\begin{cases} Ly = 0 & \text{in} \quad Q_T, \\ y = z & \text{in} \quad q_T, \\ y = 0 & \text{on} \quad \Sigma_T \end{cases}$$

Solve the Least-Squares problem :

$$\inf_{y \in Y} \frac{1}{2} \iint_{q_T} (y - z)^2 \, dx \, dt$$

with $Y = \{y \in L^2(q_T), Ly = 0 \,\text{in}\, L^2(Q_T), y = 0 \,\text{on}\, \Sigma_T\}$,

through a mixed formulation ......
In progress !

THANK YOU FOR YOUR ATTENTION

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