# Relaxation of optimal design problems

Arnaud MÜNCH

Laboratoire de Mathématiques
Université de Franche-Comté
Besançon, France

TOURS - December 2008

works in collaboration with F. Maestre (Sevilla), P. Pedregal (Ciudad Real) and F. Periago (Cartagena)

# Problem I: Optimal design and stabilization of the wave equation

[Fahroo-Ito, 97], [Freitas, 98], [Hebard-Henrot, 03, 05], [Henrot-Maillot, 05], [AM, Pedegral, Periago, JDE 06], [AM, AMCS 09]

Let $\Omega \subset \mathbb{R}^N$, $N = 1, 2$, $a \in L^\infty(\Omega, \mathbb{R}^+)$, $L \in (0, 1)$, $T > 0$, $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ [1],

$$(P^1_\omega): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + |\nabla u|^2) dxdt \tag{1}$$

subject to

$$\begin{cases} u_{tt} - \Delta u + a(\boldsymbol{x}) \mathcal{X}_\omega u_t = 0 & (0, T) \times \Omega, \\ u = 0 & (0, T) \times \partial\Omega, \\ u(0, \cdot) = u^0, \quad u_t(0, \cdot) = u^1 & \{0\} \times \Omega, \\ \mathcal{X}_\omega \in L^\infty(\Omega; \{0, 1\}), & \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} \leq L \|\mathcal{X}_\Omega\|_{L^1(\Omega)} & \end{cases} \tag{2}$$

[1] AM, P. Pedregal, F. Periago, *Optimal design of the damping set for the stabilization of the wave equation*, JDE (2006)

[Maestre-AM-Pedegral, IFB 08][2]

- Let $\Omega \subset \mathbb{R}$, $0 < \alpha < \beta < \infty$, $L \in (0,1)$, $T > 0$, $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

$$(P^2_\omega): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + a(t, \boldsymbol{x}, \mathcal{X}_\omega)|\nabla u|^2) dxdt \tag{3}$$

with for instance

$$a(t, \boldsymbol{x}, \mathcal{X}_\omega) = 1 \quad \text{(quadratic)} \quad \text{or} \quad a(t, \boldsymbol{x}, \mathcal{X}_\omega) = \alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega) \quad \text{(compliance)} \tag{4}$$

subject to

$$\begin{cases} u_{tt} - div\Big([\alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega)]\nabla u\Big) = 0 & (0, T) \times \Omega, \\ u = 0 & (0, T) \times \partial\Omega, \\ u(0, \cdot) = u^0, \quad u_t(0, \cdot) = u^1 & \Omega, \\ \mathcal{X}_\omega \in L^\infty((0, T) \times \Omega; \{0, 1\}), & \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} \leq L \|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0, T) \end{cases} \tag{5}$$

- $\omega$ depends on $\boldsymbol{x}$ AND on $t$: *Dynamical material* [Lurie 99, 00, 02].

[2] F. Maestre, AM, P. Pedregal, *Optimal design under the one-dimensional wave equation*, Interfaces and Free Boundaries (2008)

[Maestre-AM-Pedegral, IFB 08][2]

- Let $\Omega \subset \mathbb{R}$, $0 < \alpha < \beta < \infty$, $L \in (0,1)$, $T > 0$, $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

$$(P^2_\omega): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + a(t, \boldsymbol{x}, \mathcal{X}_\omega)|\nabla u|^2) dxdt \tag{3}$$

with for instance

$$a(t, \boldsymbol{x}, \mathcal{X}_\omega) = 1 \quad \text{(quadratic)} \quad \text{or} \quad a(t, \boldsymbol{x}, \mathcal{X}_\omega) = \alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega) \quad \text{(compliance)} \tag{4}$$

subject to

$$\begin{cases} u_{tt} - div\Big([\alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega)]\nabla u\Big) = 0 & (0,T) \times \Omega, \\ u = 0 & (0,T) \times \partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \Omega, \\ \mathcal{X}_\omega \in L^\infty((0,T) \times \Omega; \{0,1\}), & \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} \leq L \|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0,T) \end{cases} \tag{5}$$

- $\omega$ depends on $\boldsymbol{x}$ AND on $t$: *Dynamical material* [Lurie 99, 00, 02].

[2] F. Maestre, AM, P. Pedregal, *Optimal design under the one-dimensional wave equation*, Interfaces and Free Boundaries (2008)

[Maestre, AM, Pedregal, SIAM Appl. Math. 07][3]

- Simultaneous optimization w.r.t. to $\omega_1 \subset (0,T) \times \Omega$ et $\omega_2 \subset \Omega$

$$(P^3_\omega): \quad \inf_{\mathcal{X}_{\omega_1}, \mathcal{X}_{\omega_2}} I(\mathcal{X}_{\omega_1}, \mathcal{X}_{\omega_2}) = \int_0^T \int_\Omega (|u_t|^2 + a(t, \boldsymbol{x}, \mathcal{X}_{\omega_1})|\nabla u|^2) dxdt \tag{6}$$

subject to

$$\begin{cases} u_{tt} - div\Big([\alpha \mathcal{X}_{\omega_1} + \beta(1 - \mathcal{X}_{\omega_1})]\nabla u\Big) + a(\boldsymbol{x})\mathcal{X}_{\omega_2} u_t = 0 & (0,T) \times \Omega, \\ u = 0 & (0,T) \times \partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \{0\} \times \Omega, \\ \mathcal{X}_{\omega_1} \in L^\infty(\Omega \times (0,T); \{0,1\}), & \\ \mathcal{X}_{\omega_2} \in L^\infty(\Omega; \{0,1\}), & \\ \|\mathcal{X}_{\omega_1}(t,\cdot)\|_{L^1(\Omega)} \leq L_{des} \|\mathcal{X}_\Omega\|_{L^1(\Omega)}, & (0,T) \\ \|\mathcal{X}_{\omega_2}\|_{L^1(\Omega)} \leq L_{dam} \|\mathcal{X}_\Omega\|_{L^1(\Omega)}, & \end{cases} \tag{7}$$

$L_{dam}, L_{des} \in (0,1)$.

[3] F. Maestre, AM, P. Pedregal *A spatio-temporal design problem for a damped wave equation*, SIAM Appl. Math (2007)

# Formal resolution of $(P^1_\omega)$ using the level-set method

[Allaire-Jouve-Toader 03], [Wang-Wang-Zuo 03], [Burger-Osher 05], ...

$$(u^0(\boldsymbol{x}), u^1(\boldsymbol{x})) = (\sin(\pi x_1)\sin(\pi x_2), 0), \ \Omega = (0,1)^2, \ T = 1, \ L = 1/10, \ a(\boldsymbol{x}) = \boldsymbol{a}\mathcal{X}_\omega(\boldsymbol{x}). \tag{8}$$

$$E(\omega, \boldsymbol{a}, T) - E(\omega, 0, T) = -\frac{\boldsymbol{a}\alpha}{4}(2\alpha T - \sin(2\alpha T))\int_\omega (u_0(\boldsymbol{x}))^2 dx + o(\boldsymbol{a}), \ \forall T \geq 0. \tag{9}$$

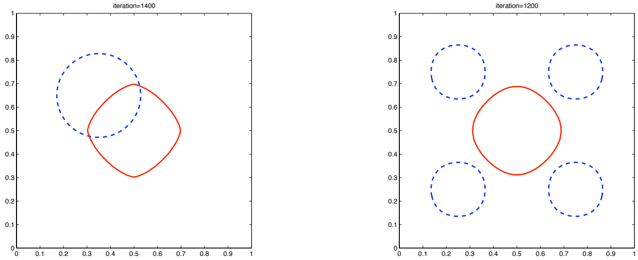



Figure: $\boldsymbol{a} = \mathbf{10}$. - Invariance of $\{\boldsymbol{x} \in \Omega, \psi(\boldsymbol{x}) = 0\}$ w.r.t. initialization $\{\boldsymbol{x} \in \Omega, \psi(\boldsymbol{x}) = 0\}$.

# Formal resolution of $(P^1_\omega)$ using the level-set method

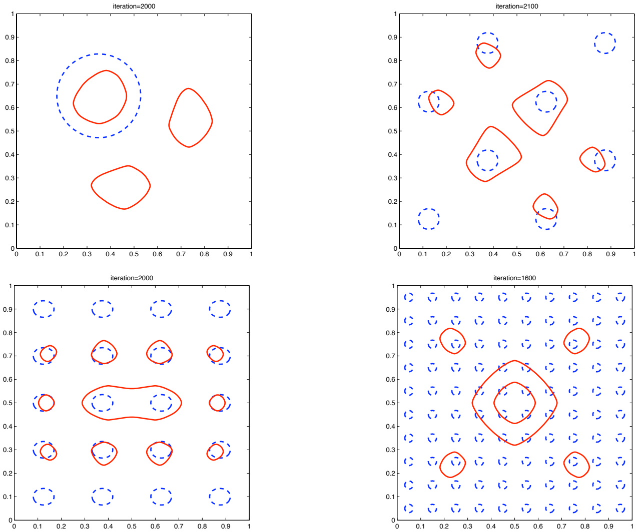



Figure: $\boldsymbol{a = 25.}$ - Loss of invariance of $\{\boldsymbol{x} \in \Omega, \psi(\boldsymbol{x}) = 0\}$.

# Ill-posed problem

- Such optimal design problems are usually not well-posed (Murat counter's example in the elliptic case)
- Infima are not reached in $L^\infty(\Omega \times (0,T), \{0,1\})$
- Minimizing sequences exhibit finer and finer scale.

- How to compute a relaxed well-posed reformulation, says $(RP_\omega)$, of these problems ?
- How to extract from a minimizer of the relaxed problem $(RP_\omega)$ a minimizing sequence of $(P_\omega)$ ?
  - Approach I: Homogeneization (G-convergence, Γ- limit, ....)[Tartar, Murat, ....]
  - Approach II: Vectorial reformulation + Young Measure [Dacorogna, Michaille, Pedregal[4], ...]

---

[4] Pedregal, P., *Vector variational problems and applications to optimal design*, COCV (2005)

# Ill-posed problem

- Such optimal design problems are usually not well-posed (Murat counter's example in the elliptic case)
- Infima are not reached in $L^\infty(\Omega \times (0, T), \{0, 1\})$
- Minimizing sequences exhibit finer and finer scale.

- How to compute a relaxed well-posed reformulation, says $(RP_\omega)$, of these problems ?
- How to extract from a minimizer of the relaxed problem $(RP_\omega)$ a minimizing sequence of $(P_\omega)$ ?
  - Approach I: Homogeneization (G-convergence, Γ- limit, ....)[Tartar, Murat, ....]
  - Approach II: Vectorial reformulation + Young Measure [Dacorogna, Michaille, Pedregal[4], ...]

[4] Pedregal, P., *Vector variational problems and applications to optimal design*, COCV (2005)

# Ill-posed problem

- Such optimal design problems are usually not well-posed (Murat counter's example in the elliptic case)
- Infima are not reached in $L^\infty(\Omega \times (0,T), \{0,1\})$
- Minimizing sequences exhibit finer and finer scale.

- How to compute a relaxed well-posed reformulation, says $(RP_\omega)$, of these problems ?
- How to extract from a minimizer of the relaxed problem $(RP_\omega)$ a minimizing sequence of $(P_\omega)$ ?
  - Approach I: Homogeneization (G-convergence, Γ- limit, ....)[Tartar, Murat, ....]
  - Approach II: Vectorial reformulation + Young Measure [Dacorogna, Michaille, Pedregal[4], ...]

[4] Pedregal, P., *Vector variational problems and applications to optimal design*, COCV (2005)

# Ill-posed problem

- Such optimal design problems are usually not well-posed (Murat counter's example in the elliptic case)
- Infima are not reached in $L^\infty(\Omega \times (0,T), \{0,1\})$
- Minimizing sequences exhibit finer and finer scale.

- How to compute a relaxed well-posed reformulation, says $(RP_\omega)$, of these problems ?
- How to extract from a minimizer of the relaxed problem $(RP_\omega)$ a minimizing sequence of $(P_\omega)$ ?
  - Approach I: Homogeneization (G-convergence, Γ- limit, ....)[Tartar, Murat, ....]
  - Approach II: Vectorial reformulation + Young Measure [Dacorogna, Michaille, Pedregal[4], ...]

[4] Pedregal, P., *Vector variational problems and applications to optimal design*, COCV (2005)

# Ill-posed problem

- Such optimal design problems are usually not well-posed (Murat counter's example in the elliptic case)
- Infima are not reached in $L^{\infty}(\Omega \times (0,T), \{0,1\})$
- Minimizing sequences exhibit finer and finer scale.

- How to compute a relaxed well-posed reformulation, says $(RP_{\omega})$, of these problems ?
- How to extract from a minimizer of the relaxed problem $(RP_{\omega})$ a minimizing sequence of $(P_{\omega})$ ?
  - Approach I: Homogeneization (G-convergence, Γ- limit, ....)[Tartar, Murat, ....]
  - Approach II: Vectorial reformulation + Young Measure [Dacorogna, Michaille, Pedregal[4], ...]

[4] Pedregal, P., *Vector variational problems and applications to optimal design*, COCV (2005)

# Ill-posed problem

- Such optimal design problems are usually not well-posed (Murat counter's example in the elliptic case)
- Infima are not reached in $L^\infty(\Omega \times (0, T), \{0, 1\})$
- Minimizing sequences exhibit finer and finer scale.

- How to compute a relaxed well-posed reformulation, says $(RP_\omega)$, of these problems ?
- How to extract from a minimizer of the relaxed problem $(RP_\omega)$ a minimizing sequence of $(P_\omega)$ ?
  - Approach I: Homogeneization (G-convergence, Γ- limit, ....)[Tartar, Murat, ....]
  - Approach II: Vectorial reformulation + Young Measure [Dacorogna, Michaille, Pedregal[4], ...]

[4] Pedregal, P., *Vector variational problems and applications to optimal design*, COCV (2005)

# Ill-posed problem

- Such optimal design problems are usually not well-posed (Murat counter's example in the elliptic case)
- Infima are not reached in $L^{\infty}(\Omega \times (0,T), \{0,1\})$
- Minimizing sequences exhibit finer and finer scale.

- How to compute a relaxed well-posed reformulation, says $(RP_\omega)$, of these problems ?
- How to extract from a minimizer of the relaxed problem $(RP_\omega)$ a minimizing sequence of $(P_\omega)$ ?
  - Approach I: Homogeneization (G-convergence, Γ- limit, ....)[Tartar, Murat, ....]
  - Approach II: Vectorial reformulation + Young Measure [Dacorogna, Michaille, Pedregal[4], ...]

[4] Pedregal, P., *Vector variational problems and applications to optimal design*, COCV, (2005)

$$(RP^1_\omega): \quad inf_{s\in L^\infty(\Omega)} \int_0^T \int_\Omega (u_t^2 + |\nabla u|^2)\, dx\, dt \tag{10}$$

subject to

$$\begin{cases} u_{tt} - \Delta u + a(x)s(x)u_t = 0 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \text{in } \Omega, \\ 0 \le s(x) \le 1, \quad \int_\Omega s(x)\,dx \le L\,|\Omega| & \text{in } \Omega. \end{cases} \tag{11}$$

**Theorem** (AM - Pedregal - Periago (06))

*Problem $(RP^1_\omega)$ is a full relaxation of $(P^1_\omega)$ in the sense that*

- *there are optimal solutions for $(RP^1_\omega)$;*
- *the infimum of $(P^1_\omega)$ equals the minimum of $(RP^1_\omega)$;*
- *if s is optimal for $(RP^1_\omega)$, then optimal sequences of damping subsets $\omega_j$ for $(P^1_\omega)$ are exactly those for which the Young measure associated with the sequence of their characteristic functions $\mathcal{X}_{\omega_j}$ is precisely*

$$s(x)\delta_1 + (1 - s(x))\delta_0. \tag{12}$$

$$(RP^1_\omega): \quad inf_{s\in L^\infty(\Omega)} \int_0^T\int_\Omega (u_t^2+|\nabla u|^2)\,dx\,dt \tag{10}$$

subject to

$$\begin{cases} u_{tt}-\Delta u + a(x)s(x)u_t = 0 & \text{in } (0,T)\times\Omega,\\ u=0 & \text{on } (0,T)\times\partial\Omega,\\ u(0,\cdot)=u^0, \quad u_t(0,\cdot)=u^1 & \text{in } \Omega,\\ 0\le s(x)\le 1, \quad \int_\Omega s(x)\,dx \le L\,|\Omega| & \text{in } \Omega. \end{cases} \tag{11}$$

**Theorem** (AM - Pedregal - Periago (06))

*Problem $(RP^1_\omega)$ is a full relaxation of $(P^1_\omega)$ in the sense that*

- *there are optimal solutions for $(RP^1_\omega)$;*
- *the infimum of $(P^1_\omega)$ equals the minimum of $(RP^1_\omega)$;*
- *if s is optimal for $(RP^1_\omega)$, then optimal sequences of damping subsets $\omega_j$ for $(P^1_\omega)$ are exactly those for which the Young measure associated with the sequence of their characteristic functions $\mathcal{X}_{\omega_j}$ is precisely*

$$s(x)\delta_1 + (1-s(x))\delta_0. \tag{12}$$

# Relaxation for the problem $(P^1_\omega)$

$$(RP^1_\omega): \quad inf_{s\in L^\infty(\Omega)} \int_0^T \int_\Omega (u_t^2 + |\nabla u|^2)\, dx\, dt \tag{10}$$

subject to

$$\begin{cases} u_{tt} - \Delta u + a(x)s(x)u_t = 0 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \text{in } \Omega, \\ 0 \le s(x) \le 1, \quad \int_\Omega s(x)\,dx \le L\,|\Omega| & \text{in } \Omega. \end{cases} \tag{11}$$

**Theorem** (AM - Pedregal - Periago (06))

*Problem $(RP^1_\omega)$ is a full relaxation of $(P^1_\omega)$ in the sense that*

- *there are optimal solutions for $(RP^1_\omega)$;*
- *the infimum of $(P^1_\omega)$ equals the minimum of $(RP^1_\omega)$;*
- *if s is optimal for $(RP^1_\omega)$, then optimal sequences of damping subsets $\omega_j$ for $(P^1_\omega)$ are exactly those for which the Young measure associated with the sequence of their characteristic functions $\mathcal{X}_{\omega_j}$ is precisely*

$$s(x)\delta_1 + (1 - s(x))\delta_0. \tag{12}$$

# Relaxation for the problem $(P^1_\omega)$

$$(RP^1_\omega): \quad inf_{s\in L^\infty(\Omega)} \int_0^T \int_\Omega (u_t^2 + |\nabla u|^2)\, dx\, dt \tag{10}$$

subject to

$$\begin{cases} u_{tt} - \Delta u + a(x)s(x)u_t = 0 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \text{in } \Omega, \\ 0 \le s(x) \le 1, \quad \int_\Omega s(x)\,dx \le L\,|\Omega| & \text{in } \Omega. \end{cases} \tag{11}$$

**Theorem** (AM - Pedregal - Periago (06))

*Problem $(RP^1_\omega)$ is a full relaxation of $(P^1_\omega)$ in the sense that*

- *there are optimal solutions for $(RP^1_\omega)$;*
- *the infimum of $(P^1_\omega)$ equals the minimum of $(RP^1_\omega)$;*
- *if $s$ is optimal for $(RP^1_\omega)$, then optimal sequences of damping subsets $\omega_j$ for $(P^1_\omega)$ are exactly those for which the Young measure associated with the sequence of their characteristic functions $\mathcal{X}_{\omega_j}$ is precisely*

$$s(x)\delta_1 + (1 - s(x))\delta_0. \tag{12}$$

- Assuming $\omega$ time independent, we have (we note $Div = (\partial_t, \partial_x)$)

$$u_{tt} - \Delta u + a(x)\mathcal{X}_\omega u_t = 0 \Longleftrightarrow Div(u_t + a(x)\mathcal{X}_\omega u, -u_x) = 0 \tag{13}$$

$\Longrightarrow \exists v \in H^1((0,T) \times \Omega)$ such that $u_t + a(x)\mathcal{X}_\omega u = v_x$ and $-u_x = -v_t$

-

$$A\nabla u + B\nabla v = -a\mathcal{X}_\omega \overline{u} \tag{14}$$

where $\nabla u = \begin{pmatrix} u_t \\ u_x \end{pmatrix}$, $\nabla v = \begin{pmatrix} v_t \\ v_x \end{pmatrix}$, $\overline{u} = \begin{pmatrix} u \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

-

$$\omega = \{x \in \Omega, A\nabla u + B\nabla v = -a(x)\overline{u}\} \quad \text{and} \quad \Omega\backslash\omega = \{x \in \Omega, A\nabla u + B\nabla v = 0\} \tag{15}$$

- Let the vector field $U(t,x) = (u(t,x), v(t,x)) \in (H^1((0,T) \times (0,1)))^2$ and the two sets of matrices

$$\begin{cases} \Lambda_0 = \left\{M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = 0\right\} \\ \Lambda_{1,\lambda} = \left\{M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = \lambda e_1\right\} \end{cases} \tag{16}$$

where $M^{(i)}$, $i = 1, 2$ stands for the $i$-th row of the matrix $M$, $\lambda \in \mathbb{R}$ and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

-

$$\omega = \{x \in \Omega, \nabla U \in \Lambda_{1,-a(x)U^{(1)}}\}, \quad \Omega\backslash\omega = \{x \in \Omega, \nabla U \in \Lambda_0\} \tag{17}$$

- Assuming $\omega$ time independent, we have (we note $Div = (\partial_t, \partial_x)$)

$$u_{tt} - \Delta u + a(x)\mathcal{X}_\omega u_t = 0 \Longleftrightarrow Div(u_t + a(x)\mathcal{X}_\omega u, -u_x) = 0 \quad (13)$$

$\Longrightarrow \exists v \in H^1((0,T) \times \Omega)$ such that $u_t + a(x)\mathcal{X}_\omega u = v_x$ and $-u_x = -v_t$

- $$A\nabla u + B\nabla v = -a\mathcal{X}_\omega \overline{u} \quad (14)$$

where $\nabla u = \begin{pmatrix} u_t \\ u_x \end{pmatrix}$, $\nabla v = \begin{pmatrix} v_t \\ v_x \end{pmatrix}$, $\overline{u} = \begin{pmatrix} u \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

- $$\omega = \{x \in \Omega, A\nabla u + B\nabla v = -a(x)\overline{u}\} \quad \text{and} \quad \Omega\backslash\omega = \{x \in \Omega, A\nabla u + B\nabla v = 0\} \quad (15)$$

- Let the vector field $U(t,x) = (u(t,x), v(t,x)) \in (H^1((0,T) \times (0,1)))^2$ and the two sets of matrices

$$\begin{cases} \Lambda_0 = \left\{M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = 0\right\} \\ \Lambda_{1,\lambda} = \left\{M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = \lambda e_1\right\} \end{cases} \quad (16)$$

where $M^{(i)}$, $i = 1, 2$ stands for the $i$-th row of the matrix $M$, $\lambda \in \mathbb{R}$ and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

- $$\omega = \{x \in \Omega, \nabla U \in \Lambda_{1,-a(x)U^{(1)}}\}, \quad \Omega\backslash\omega = \{x \in \Omega, \nabla U \in \Lambda_0\} \quad (17)$$

- Assuming $\omega$ time independent, we have (we note $Div = (\partial_t, \partial_x)$)

$$u_{tt} - \Delta u + a(x)\mathcal{X}_\omega u_t = 0 \Longleftrightarrow Div(u_t + a(x)\mathcal{X}_\omega u, -u_x) = 0 \quad (13)$$

$\Longrightarrow \exists v \in H^1((0,T)\times\Omega)$ such that $u_t + a(x)\mathcal{X}_\omega u = v_x$ and $-u_x = -v_t$

- $$A\nabla u + B\nabla v = -a\mathcal{X}_\omega \overline{u} \quad (14)$$

where $\nabla u = \begin{pmatrix} u_t \\ u_x \end{pmatrix}$, $\nabla v = \begin{pmatrix} v_t \\ v_x \end{pmatrix}$, $\overline{u} = \begin{pmatrix} u \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

- $$\omega = \{x \in \Omega, A\nabla u + B\nabla v = -a(x)\overline{u}\} \quad \text{and} \quad \Omega\backslash\omega = \{x \in \Omega, A\nabla u + B\nabla v = 0\} \quad (15)$$

- Let the vector field $U(t,x) = (u(t,x), v(t,x)) \in (H^1((0,T)\times(0,1)))^2$ and the two sets of matrices

$$\begin{cases} \Lambda_0 = \left\{M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = 0\right\} \\ \Lambda_{1,\lambda} = \left\{M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = \lambda e_1\right\} \end{cases} \quad (16)$$

where $M^{(i)}$, $i = 1, 2$ stands for the $i$-th row of the matrix $M$, $\lambda \in \mathbb{R}$ and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

- $$\omega = \{x \in \Omega, \nabla U \in \Lambda_{1,-a(x)U^{(1)}}\}, \quad \Omega\backslash\omega = \{x \in \Omega, \nabla U \in \Lambda_0\} \quad (17)$$

- Assuming $\omega$ time independent, we have (we note $Div = (\partial_t, \partial_x)$)

$$u_{tt} - \Delta u + a(x)\mathcal{X}_\omega u_t = 0 \Longleftrightarrow Div(u_t + a(x)\mathcal{X}_\omega u, -u_x) = 0 \tag{13}$$

$\Longrightarrow \exists v \in H^1((0,T)\times\Omega)$ such that $u_t + a(x)\mathcal{X}_\omega u = v_x$ and $-u_x = -v_t$

- $$A\nabla u + B\nabla v = -a\mathcal{X}_\omega \overline{u} \tag{14}$$

where $\nabla u = \begin{pmatrix} u_t \\ u_x \end{pmatrix}$, $\nabla v = \begin{pmatrix} v_t \\ v_x \end{pmatrix}$, $\overline{u} = \begin{pmatrix} u \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

- $$\omega = \{x \in \Omega, A\nabla u + B\nabla v = -a(x)\overline{u}\} \quad \text{and} \quad \Omega\backslash\omega = \{x \in \Omega, A\nabla u + B\nabla v = 0\} \tag{15}$$

- Let the vector field $U(t,x) = (u(t,x), v(t,x)) \in (H^1((0,T)\times(0,1)))^2$ and the two sets of matrices

$$\begin{cases} \Lambda_0 = \left\{M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = 0\right\} \\ \Lambda_{1,\lambda} = \left\{M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = \lambda e_1\right\} \end{cases} \tag{16}$$

where $M^{(i)}$, $i = 1,2$ stands for the $i$-th row of the matrix $M$, $\lambda \in \mathbb{R}$ and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

- $$\omega = \{x \in \Omega, \nabla U \in \Lambda_{1,-a(x)U^{(1)}}\}, \quad \Omega\backslash\omega = \{x \in \Omega, \nabla U \in \Lambda_0\} \tag{17}$$

- Assuming $\omega$ time independent, we have (we note $Div = (\partial_t, \partial_x)$)

$$u_{tt} - \Delta u + a(x)\mathcal{X}_\omega u_t = 0 \iff Div(u_t + a(x)\mathcal{X}_\omega u, -u_x) = 0 \qquad (13)$$

$\Longrightarrow \exists v \in H^1((0,T)\times\Omega)$ such that $u_t + a(x)\mathcal{X}_\omega u = v_x$ and $-u_x = -v_t$

-

$$A\nabla u + B\nabla v = -a\mathcal{X}_\omega \overline{u} \qquad (14)$$

where $\nabla u = \begin{pmatrix} u_t \\ u_x \end{pmatrix}$, $\nabla v = \begin{pmatrix} v_t \\ v_x \end{pmatrix}$, $\overline{u} = \begin{pmatrix} u \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

-

$$\omega = \{x \in \Omega, A\nabla u + B\nabla v = -a(x)\overline{u}\} \quad \text{and} \quad \Omega\backslash\omega = \{x \in \Omega, A\nabla u + B\nabla v = 0\} \qquad (15)$$

- Let the vector field $U(t,x) = (u(t,x), v(t,x)) \in (H^1((0,T)\times(0,1)))^2$ and the two sets of matrices

$$\begin{cases} \Lambda_0 = \left\{M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = 0\right\} \\ \Lambda_{1,\lambda} = \left\{M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = \lambda e_1\right\} \end{cases} \qquad (16)$$

where $M^{(i)}$, $i = 1, 2$ stands for the $i$-th row of the matrix $M$, $\lambda \in \mathbb{R}$ and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

-

$$\omega = \{x \in \Omega, \nabla U \in \Lambda_{1,-a(x)U^{(1)}}\}, \quad \Omega\backslash\omega = \{x \in \Omega, \nabla U \in \Lambda_0\} \qquad (17)$$

- Then considering the two following functions $W, V : \mathcal{M}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$

$$W(x,U,M) = \begin{cases} \left|M^{(1)}\right|^2, & M \in \Lambda_0 \cup \Lambda_{1,-a(x)U^{(1)}} \\ +\infty, & \text{else} \end{cases} \quad V(x,U,M) = \begin{cases} 1, & M \in \Lambda_{1,-a(x)U^{(1)}} \\ 0, & M \in \Lambda_0 \setminus \Lambda_{1,-a(x)U^{(1)}} \\ +\infty, & \text{else} \end{cases} \tag{18}$$

- the optimization problem $(P^1_\omega)$ is equivalent to the following vector variational problem

$$\left(VP^1_\omega\right) \quad m \equiv \inf_U \int_0^T \int_0^1 W\left(x, U(t,x), \nabla U(t,x)\right)\, dx\, dt \tag{19}$$

subject to

$$\begin{cases} U = (u,v) \in \left(H^1\left((0,T)\times(0,1)\right)\right)^2 \\ U^{(1)}(t,0) = U^{(1)}(t,1) = 0, & t \in (0,T) \\ U^{(1)}(0,x) = u^0(x), \quad U^{(1)}_t(0,x) = u^1(x), & x \in \Omega \\ \int_0^1 V\left(x, U(t,x), \nabla U(t,x)\right)\, dx \le L \mid \Omega \mid, & t \in (0,T). \end{cases} \tag{20}$$

- This procedure transforms the scalar optimization problem $\left(P^1_\omega\right)$, with differentiable, integrable and pointwise constraints, into a non-convex, vector variational problem $\left(VP^1_\omega\right)$ with only pointwise and integral constraints.

- Then considering the two following functions $W, V : \mathcal{M}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$

$$W(x,U,M) = \begin{cases} \left|M^{(1)}\right|^2, & M \in \Lambda_0 \cup \Lambda_{1,-a(x)U^{(1)}} \\ +\infty, & \text{else} \end{cases} \quad V(x,U,M) = \begin{cases} 1, & M \in \Lambda_{1,-a(x)U^{(1)}} \\ 0, & M \in \Lambda_0 \setminus \Lambda_{1,-a(x)U^{(1)}} \\ +\infty, & \text{else} \end{cases} \tag{18}$$

- the optimization problem $(P^1_\omega)$ is equivalent to the following vector variational problem

$$\left(VP^1_\omega\right) \quad m \equiv \inf_U \int_0^T\int_0^1 W\left(x, U(t,x), \nabla U(t,x)\right) dx\, dt \tag{19}$$

subject to

$$\begin{cases} U = (u,v) \in \left(H^1\left((0,T)\times(0,1)\right)\right)^2 \\ U^{(1)}(t,0) = U^{(1)}(t,1) = 0, & t \in (0,T) \\ U^{(1)}(0,x) = u^0(x), \quad U^{(1)}_t(0,x) = u^1(x), & x \in \Omega \\ \int_0^1 V\left(x, U(t,x), \nabla U(t,x)\right) dx \le L \mid \Omega \mid, & t \in (0,T). \end{cases} \tag{20}$$

- This procedure transforms the scalar optimization problem $\left(P^1_\omega\right)$, with differentiable, integrable and pointwise constraints, into a non-convex, vector variational problem $\left(VP^1_\omega\right)$ with only pointwise and integral constraints.

- Then considering the two following functions $W, V : \mathcal{M}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$

$$W(x,U,M) = \begin{cases} \left|M^{(1)}\right|^2, & M \in \Lambda_0 \cup \Lambda_{1,-a(x)U^{(1)}} \\ +\infty, & \text{else} \end{cases} \quad V(x,U,M) = \begin{cases} 1, & M \in \Lambda_{1,-a(x)U^{(1)}} \\ 0, & M \in \Lambda_0 \setminus \Lambda_{1,-a(x)U^{(1)}} \\ +\infty, & \text{else} \end{cases} \tag{18}$$

- the optimization problem $(P^1_\omega)$ is equivalent to the following vector variational problem

$$\left(VP^1_\omega\right) \quad m \equiv \inf_U \int_0^T \int_0^1 W(x, U(t,x), \nabla U(t,x)) \, dx \, dt \tag{19}$$

subject to

$$\begin{cases} U = (u,v) \in \left(H^1((0,T)\times(0,1))\right)^2 \\ U^{(1)}(t,0) = U^{(1)}(t,1) = 0, & t \in (0,T) \\ U^{(1)}(0,x) = u^0(x), \quad U^{(1)}_t(0,x) = u^1(x), & x \in \Omega \\ \int_0^1 V(x,U(t,x),\nabla U(t,x)) \, dx \le L \mid \Omega \mid, & t \in (0,T). \end{cases} \tag{20}$$

- This procedure transforms the scalar optimization problem $\left(P^1_\omega\right)$, with differentiable, integrable and pointwise constraints, into a non-convex, vector variational problem $\left(VP^1_\omega\right)$ with only pointwise and integral constraints.

# Young measures (basic property)

- A Young measure is a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ associated with a sequence of functions $f_j : \Omega \subset \mathbb{R}^N \longrightarrow A$, such that $supp(\nu_x) \subset A$, depending measurably on $x \in \Omega$, i.e. for any continuous $\phi : A \longrightarrow R$, the function

$$x \longrightarrow \overline{\phi}(x) = \int_A \phi(\lambda) d\nu_x(\lambda) \quad \text{is measurable} \tag{21}$$

- Example : let $f(x) = 2\mathcal{X}_{[0,1/2[} - 1$ for $x \in [0,1]$ 1-periodic and $f_j(x) = f(jx)$, $j \in \mathbb{N}$. For any $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ continuous

$$\lim_j \int_0^1 \phi(f_j(x))dx = \frac{1}{2}(\phi(1) + \phi(-1)), \quad \nu = \frac{1}{2}(\delta_1 + \delta_{-1}) \tag{22}$$

- For any sequel $\{\phi(f_j)\}$ ($\phi : A \longrightarrow R$) weakly convergent in $L^\infty(\Omega) - \star$, the weak-limit is expressed in terms of $\nu$:

$$\lim_j \int_\Omega \phi(f_j)h(x)dx = \int_\Omega h(x) \int_A \boldsymbol{\phi(\lambda)d\nu_x(\lambda)}dx \quad \forall h \in L^1(\Omega). \tag{23}$$

# Young measures (basic property)

- A Young measure is a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ associated with a sequence of functions $f_j : \Omega \subset \mathbb{R}^N \longrightarrow A$, such that $supp(\nu_x) \subset A$, depending measurably on $x \in \Omega$, i.e. for any continuous $\phi : A \longrightarrow R$, the function

$$x \longrightarrow \overline{\phi}(x) = \int_A \phi(\lambda) d\nu_x(\lambda) \quad \text{is measurable} \tag{21}$$

- Example : let $f(x) = 2\mathcal{X}_{[0,1/2[} - 1$ for $x \in [0,1]$ 1-periodic and $f_j(x) = f(jx), j \in \mathbb{N}$. For any $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ continuous

$$\lim_j \int_0^1 \phi(f_j(x))dx = \frac{1}{2}(\phi(1) + \phi(-1)), \quad \nu = \frac{1}{2}(\delta_1 + \delta_{-1}) \tag{22}$$

- For any sequel $\{\phi(f_j)\}$ ($\phi : A \longrightarrow R$) weakly convergent in $L^\infty(\Omega) - \star$, the weak-limit is expressed in terms of $\nu$:

$$\lim_j \int_\Omega \phi(f_j)h(x)dx = \int_\Omega h(x) \int_A \boldsymbol{\phi(\lambda)d\nu_x(\lambda)}dx \quad \forall h \in L^1(\Omega). \tag{23}$$

# Young measures (basic property)

- A Young measure is a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ associated with a sequence of functions $f_j : \Omega \subset \mathbb{R}^N \longrightarrow A$, such that $supp(\nu_x) \subset A$, depending measurably on $x \in \Omega$, i.e. for any continuous $\phi : A \longrightarrow R$, the function

$$x \longrightarrow \overline{\phi}(x) = \int_A \phi(\lambda) d\nu_x(\lambda) \quad \text{is measurable} \tag{21}$$

- Example : let $f(x) = 2\mathcal{X}_{[0,1/2[} - 1$ for $x \in [0,1]$ 1-periodic and $f_j(x) = f(jx), j \in \mathbb{N}$. For any $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ continuous

$$\lim_j \int_0^1 \phi(f_j(x))dx = \frac{1}{2}(\phi(1) + \phi(-1)), \quad \nu = \frac{1}{2}(\delta_1 + \delta_{-1}) \tag{22}$$

- For any sequel $\{\phi(f_j)\}$ $(\phi : A \longrightarrow R)$ weakly convergent in $L^\infty(\Omega) - \star$, the weak-limit is expressed in terms of $\nu$:

$$\lim_j \int_\Omega \phi(f_j)h(x)dx = \int_\Omega h(x) \int_A \boldsymbol{\phi(\lambda)d\nu_x(\lambda)}dx \quad \forall h \in L^1(\Omega). \tag{23}$$

# Young measures

**Theorem** (Fundamental theorem of the Young measures)

*Let $\Omega \subset \mathbb{R}^N$ be a measurable set and let $z_j : \Omega \to \mathbb{R}^m$ be measurable functions such that $\sup_j \int_\Omega g(|z_j|)dx < \infty$, where $g : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function such that $\lim_{t\to\infty} g(t) = \infty$. There exist a subsequence, not relabeled, and a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ depending measurably on $x$ with the property that whenever the sequence $\{\psi(x, z_j(x))\}$ is weakly convergent in $L^1(\Omega)$ for any Carathéodory function $\psi(x, \lambda) : \Omega \times \mathbb{R}^m \to \mathbb{R}^\star$, the weak limit is the function*

$$\int_{\mathbb{R}^m} \psi(x, \lambda)d\nu_x(\lambda). \tag{24}$$

Assume that $\{W(\nabla U_j)\}$ is weakly convergent in $L^1((0, T) \times \Omega)$ where $U_j$ is a minimizing sequence for the cost, we have

$$lim_{j\to\infty} \int_0^T \int_\Omega W(x, t, \nabla U_j(x, t))dxdt = \int_0^T \int_\Omega \int_{\mathcal{M}^{2\times 2}} W(x, t, A)d\nu_{x,t}(A)dxdt \tag{25}$$

where $\nu = \{\nu_{x,t}\}$ is the Young measure associated with $\{\nabla U_j\}$. [Kinderleher-Pedregal, 92].

Morever, if $\int_{\mathcal{M}^{2\times 2}} W(x, t, A)d\nu_{x,t}(A) \geq W\left(x, t, \int_{\mathcal{M}^{2\times 2}} Ad\nu_{x,t}(A)\right)$ then

$$\lim_j \int_0^T \int_\Omega W(x, t, \nabla U_j)dxdt \geq \int_0^T \int_\Omega W\left(x, t, \int_{\mathcal{M}^{2\times 2}} Ad\nu_{x,t}(A)\right) = \int_0^T \int_\Omega W(x, t, \nabla U)dxdt \tag{26}$$

since ($\nabla U = \int_{\mathcal{M}^{2\times 2}} Ad\nu_{x,t}(A)$ is the weak limit of $\{\nabla U_j\}$).

**Theorem** (Fundamental theorem of the Young measures)

*Let $\Omega \subset \mathbb{R}^N$ be a measurable set and let $z_j : \Omega \to \mathbb{R}^m$ be measurable functions such that $\sup_j \int_\Omega g(|z_j|)dx < \infty$, where $g : [0,\infty) \to [0,\infty)$ is a continuous, nondecreasing function such that $\lim_{t\to\infty} g(t) = \infty$. There exist a subsequence, not relabeled, and a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ depending measurably on $x$ with the property that whenever the sequence $\{\psi(x, z_j(x))\}$ is weakly convergent in $L^1(\Omega)$ for any Carathéodory function $\psi(x,\lambda) : \Omega \times \mathbb{R}^m \to \mathbb{R}^\star$, the weak limit is the function*

$$\int_{\mathbb{R}^m} \psi(x,\lambda)d\nu_x(\lambda). \tag{24}$$

Assume that $\{W(\nabla U_j)\}$ is weakly convergent in $L^1((0,T)\times\Omega)$ where $U_j$ is a minimizing sequence for the cost, we have

$$lim_{j\to\infty} \int_0^T \int_\Omega W(x,t,\nabla U_j(x,t))dxdt = \int_0^T \int_\Omega \int_{\mathcal{M}^{2\times 2}} W(x,t,A)d\nu_{x,t}(A)dxdt \tag{25}$$

where $\nu = \{\nu_{x,t}\}$ is the Young measure associated with $\{\nabla U_j\}$. [Kinderleher-Pedregal, 92].

Morever, if $\int_{\mathcal{M}^{2\times 2}} W(x,t,A)d\nu_{x,t}(A) \geq W\left(x,t,\int_{\mathcal{M}^{2\times 2}} Ad\nu_{x,t}(A)\right)$ then

$$\lim_j \int_0^T\int_\Omega W(x,t,\nabla U_j)dxdt \geq \int_0^T\int_\Omega W\left(x,t,\int_{\mathcal{M}^{2\times 2}} Ad\nu_{x,t}(A)\right) = \int_0^T\int_\Omega W(x,t,\nabla U)dxdt \tag{26}$$

since ($\nabla U = \int_{\mathcal{M}^{2\times 2}} Ad\nu_{x,t}(A)$ is the weak limit of $\{\nabla U_j\}$).

**Theorem** (Fundamental theorem of the Young measures)

*Let $\Omega \subset \mathbb{R}^N$ be a measurable set and let $z_j : \Omega \to \mathbb{R}^m$ be measurable functions such that $\sup_j \int_\Omega g(|z_j|)dx < \infty$, where $g : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function such that $\lim_{t\to\infty} g(t) = \infty$. There exist a subsequence, not relabeled, and a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ depending measurably on $x$ with the property that whenever the sequence $\{\psi(x, z_j(x))\}$ is weakly convergent in $L^1(\Omega)$ for any Carathéodory function $\psi(x, \lambda) : \Omega \times \mathbb{R}^m \to \mathbb{R}^\star$, the weak limit is the function*

$$\int_{\mathbb{R}^m} \psi(x, \lambda)d\nu_x(\lambda). \tag{24}$$

Assume that $\{W(\nabla U_j)\}$ is weakly convergent in $L^1((0, T) \times \Omega)$ where $U_j$ is a minimizing sequence for the cost, we have

$$lim_{j\to\infty} \int_0^T \int_\Omega W(x, t, \nabla U_j(x, t))dxdt = \int_0^T \int_\Omega \int_{\mathcal{M}^{2\times 2}} W(x, t, A)d\nu_{x,t}(A)dxdt \tag{25}$$

where $\nu = \{\nu_{x,t}\}$ is the Young measure associated with $\{\nabla U_j\}$. [Kinderleher-Pedregal, 92].

Morever, if $\int_{\mathcal{M}^{2\times 2}} W(x, t, A)d\nu_{x,t}(A) \geq W\left(x, t, \int_{\mathcal{M}^{2\times 2}} Ad\nu_{x,t}(A)\right)$ then

$$\lim_j \int_0^T \int_\Omega W(x, t, \nabla U_j)dxdt \geq \int_0^T \int_\Omega W\left(x, t, \int_{\mathcal{M}^{2\times 2}} Ad\nu_{x,t}(A)\right) = \int_0^T \int_\Omega W(x, t, \nabla U)dxdt \tag{26}$$

since $(\nabla U = \int_{\mathcal{M}^{2\times 2}} Ad\nu_{x,t}(A)$ is the weak limit of $\{\nabla U_j\})$.

**Theorem** (Fundamental theorem of the Young measures)

*Let $\Omega \subset \mathbb{R}^N$ be a measurable set and let $z_j : \Omega \to \mathbb{R}^m$ be measurable functions such that $\sup_j \int_\Omega g(|z_j|)dx < \infty$, where $g : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function such that $\lim_{t\to\infty} g(t) = \infty$. There exist a subsequence, not relabeled, and a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ depending measurably on $x$ with the property that whenever the sequence $\{\psi(x, z_j(x))\}$ is weakly convergent in $L^1(\Omega)$ for any Carathéodory function $\psi(x, \lambda) : \Omega \times \mathbb{R}^m \to \mathbb{R}^\star$, the weak limit is the function*

$$\int_{\mathbb{R}^m} \psi(x, \lambda)d\nu_x(\lambda). \tag{24}$$

Assume that $\{W(\nabla U_j)\}$ is weakly convergent in $L^1((0, T) \times \Omega)$ where $U_j$ is a minimizing sequence for the cost, we have

$$lim_{j\to\infty} \int_0^T \int_\Omega W(x, t, \nabla U_j(x, t))dxdt = \int_0^T \int_\Omega \int_{\mathcal{M}^{2\times 2}} W(x, t, A)d\nu_{x,t}(A)dxdt \tag{25}$$

where $\nu = \{\nu_{x,t}\}$ is the Young measure associated with $\{\nabla U_j\}$. [Kinderleher-Pedregal, 92].

Morever, if $\int_{\mathcal{M}^{2\times 2}} W(x, t, A)d\nu_{x,t}(A) \geq W\left(x, t, \int_{\mathcal{M}^{2\times 2}} Ad\nu_{x,t}(A)\right)$ then

$$\lim_j \int_0^T \int_\Omega W(x, t, \nabla U_j)dxdt \geq \int_0^T \int_\Omega W\left(x, t, \int_{\mathcal{M}^{2\times 2}} Ad\nu_{x,t}(A)\right) = \int_0^T \int_\Omega W(x, t, \nabla U)dxdt \tag{26}$$

since ($\nabla U = \int_{\mathcal{M}^{2\times 2}} Ad\nu_{x,t}(A)$ is the weak limit of $\{\nabla U_j\}$).

**Theorem** (Kinderleher-Pedregal, 92)

*Let $\{u_j\}$ a sequel in $W^{1,p}(\Omega)$, $p > 1$ and $\nu = \{\nu_x\}_{x\in\Omega}$ a family of probability measures such that $supp(\nu_x) \in \mathbb{R}^{n\times m}$. $\nu$ is a Young measure generated by the sequel $\{\nabla u_j\}$ if and and only if:*

- $\nabla u(x) = \int_{\mathbb{R}^{n\times m}} A d\nu_x(A)$ *for some* $u \in W^{1,p}(\Omega)$;
- $\int_{\mathbb{R}^{n\times m}} \phi(A) d\nu_x(A) \geq \phi(\nabla u(x))$ *a.e.* $x \in \Omega$ *and any quasi-convexe function* $\phi$ *with a polynomial growth of order p ;*
- $\int_\Omega \int_{\mathbb{R}^{n\times m}} |A|^p d\nu_x(A) dx < \infty$.

∎

This caracterization allows to express the quasi-convex hull of any $\phi$ in term of $\nu$ :

$$Q\phi(Y) = \inf_{\nu}\left\{ \int_{\mathbb{R}^{n\times m}} \phi(A) d\nu(A);\ \ \nu \text{ is an homogeneous gradient Young measure; } \int_{\mathbb{R}^{n\times m}} A d\nu(A) = Y \right\}. \quad (27)$$

**Theorem** (Kinderleher-Pedregal, 92)

*Let $\{u_j\}$ a sequel in $W^{1,p}(\Omega)$, $p > 1$ and $\nu = \{\nu_x\}_{x\in\Omega}$ a family of probability measures such that $supp(\nu_x) \in \mathbb{R}^{n\times m}$. $\nu$ is a Young measure generated by the sequel $\{\nabla u_j\}$ if and and only if:*

- $\nabla u(x) = \int_{\mathbb{R}^{n\times m}} A d\nu_x(A)$ *for some* $u \in W^{1,p}(\Omega)$;
- $\int_{\mathbb{R}^{n\times m}} \phi(A) d\nu_x(A) \geq \phi(\nabla u(x))$ *a.e.* $x \in \Omega$ *and any quasi-convexe function* $\phi$ *with a polynomial growth of order* $p$ ;
- $\int_\Omega \int_{\mathbb{R}^{n\times m}} |A|^p d\nu_x(A) dx < \infty$.

∎

This caracterization allows to express the quasi-convex hull of any $\phi$ in term of $\nu$ :

$$Q\phi(Y) = \inf_\nu \left\{ \int_{\mathbb{R}^{n\times m}} \phi(A) d\nu(A);\ \nu \text{ is an homogeneous gradient Young measure; } \int_{\mathbb{R}^{n\times m}} A d\nu(A) = Y \right\}. \quad (27)$$

# Characterization of the gradient Young measures

**Theorem** (Kinderleher-Pedregal, 92)

*Let $\{u_j\}$ a sequel in $W^{1,p}(\Omega)$, $p > 1$ and $\nu = \{\nu_x\}_{x\in\Omega}$ a family of probability measures such that supp$(\nu_x) \in \mathbb{R}^{n\times m}$. $\nu$ is a Young measure generated by the sequel $\{\nabla u_j\}$ if and and only if:*

- $\nabla u(x) = \int_{\mathbb{R}^{n\times m}} A d\nu_x(A)$ *for some* $u \in W^{1,p}(\Omega)$;
- $\int_{\mathbb{R}^{n\times m}} \phi(A) d\nu_x(A) \geq \phi(\nabla u(x))$ *a.e.* $x \in \Omega$ *and any quasi-convexe function* $\phi$ *with a polynomial growth of order p ;*
- $\int_\Omega \int_{\mathbb{R}^{n\times m}} |A|^p d\nu_x(A) dx < \infty$.

This caracterization allows to express the quasi-convex hull of any $\phi$ in term of $\nu$ :

$$Q\phi(Y) = \inf_{\nu}\left\{ \int_{\mathbb{R}^{n\times m}} \phi(A) d\nu(A);\ \nu \text{ is an homogeneous gradient Young measure; } \int_{\mathbb{R}^{n\times m}} A d\nu(A) = Y \right\}. \quad (27)$$

# Characterization of the gradient Young measures

**Theorem** (Kinderleher-Pedregal, 92)

*Let $\{u_j\}$ a sequel in $W^{1,p}(\Omega)$, $p > 1$ and $\nu = \{\nu_x\}_{x\in\Omega}$ a family of probability measures such that $supp(\nu_x) \in \mathbb{R}^{n\times m}$. $\nu$ is a Young measure generated by the sequel $\{\nabla u_j\}$ if and and only if:*

- $\nabla u(x) = \int_{\mathbb{R}^{n\times m}} A d\nu_x(A)$ *for some* $u \in W^{1,p}(\Omega)$;
- $\int_{\mathbb{R}^{n\times m}} \phi(A) d\nu_x(A) \geq \phi(\nabla u(x))$ *a.e.* $x \in \Omega$ *and any quasi-convexe function* $\phi$ *with a polynomial growth of order p ;*
- $\int_\Omega \int_{\mathbb{R}^{n\times m}} |A|^p d\nu_x(A) dx < \infty$.

■

This caracterization allows to express the quasi-convex hull of any $\phi$ in term of $\nu$ :

$$Q\phi(Y) = \inf_\nu \Big\{ \int_{\mathbb{R}^{n\times m}} \phi(A) d\nu(A);\ \ \nu \text{ is an homogeneous gradient Young measure; } \int_{\mathbb{R}^{n\times m}} A d\nu(A) = Y \Big\}. \qquad (27)$$

# Characterization of the gradient Young measures

**Theorem** (Kinderleher-Pedregal, 92)

*Let $\{u_j\}$ a sequel in $W^{1,p}(\Omega)$, $p > 1$ and $\nu = \{\nu_x\}_{x\in\Omega}$ a family of probability measures such that $supp(\nu_x) \in \mathbb{R}^{n\times m}$. $\nu$ is a Young measure generated by the sequel $\{\nabla u_j\}$ if and and only if:*

- $\nabla u(x) = \int_{\mathbb{R}^{n\times m}} A d\nu_x(A)$ *for some* $u \in W^{1,p}(\Omega)$;
- $\int_{\mathbb{R}^{n\times m}} \phi(A) d\nu_x(A) \geq \phi(\nabla u(x))$ *a.e.* $x \in \Omega$ *and any quasi-convexe function* $\phi$ *with a polynomial growth of order p ;*
- $\int_\Omega \int_{\mathbb{R}^{n\times m}} |A|^p d\nu_x(A) dx < \infty$.

■

This caracterization allows to express the quasi-convex hull of any $\phi$ in term of $\nu$ :

$$Q\phi(Y) = \inf_\nu \left\{ \int_{\mathbb{R}^{n\times m}} \phi(A) d\nu(A);\ \nu \text{ is an homogeneous gradient Young measure; } \int_{\mathbb{R}^{n\times m}} A d\nu(A) = Y \right\}. \quad (27)$$

# Characterization of the gradient Young measures

**Theorem** (Kinderleher-Pedregal, 92)

*Let $\{u_j\}$ a sequel in $W^{1,p}(\Omega)$, $p > 1$ and $\nu = \{\nu_x\}_{x\in\Omega}$ a family of probability measures such that $supp(\nu_x) \in \mathbb{R}^{n\times m}$. $\nu$ is a Young measure generated by the sequel $\{\nabla u_j\}$ if and and only if:*

- $\nabla u(x) = \int_{\mathbb{R}^{n\times m}} A d\nu_x(A)$ *for some* $u \in W^{1,p}(\Omega)$;
- $\int_{\mathbb{R}^{n\times m}} \phi(A) d\nu_x(A) \geq \phi(\nabla u(x))$ *a.e.* $x \in \Omega$ *and any quasi-convexe function* $\phi$ *with a polynomial growth of order p ;*
- $\int_\Omega \int_{\mathbb{R}^{n\times m}} |A|^p d\nu_x(A) dx < \infty$.

∎

This caracterization allows to express the quasi-convex hull of any $\phi$ in term of $\nu$ :

$$Q\phi(Y) = \inf_\nu \left\{ \int_{\mathbb{R}^{n\times m}} \phi(A) d\nu(A);\ \nu \text{ is an homogeneous gradient Young measure; } \int_{\mathbb{R}^{n\times m}} A d\nu(A) = Y \right\}. \quad (27)$$

- From Dacorogna [5], a relaxed formulation of $(VP^1_\omega)$ is where

$$\overline{m} = \min_{U,s} \left\{ \int_0^T \int_\Omega CQW\left(t, x, \nabla U\left(t, x\right), s\left(x\right)\right) dxdt \right\} \quad (= m) \tag{28}$$

where the minimum is taken over the fields $U \in \left(H^1\left((0, T) \times (0, 1)\right)\right)^2$ which satisfy the initial and boundary conditions and the function $s$ verifies the constraints

$$0 \leq s(x) \leq 1 \quad \forall x \in \Omega, \quad \text{and} \quad \int_\Omega s(x)\, dx \leq L\,|\Omega|\,. \tag{29}$$

- The expression $CQW\left(t, x, \nabla U\left(t, x\right), s\left(x\right)\right)$ stands for the constrained quasi-convexification of the density $W$ and for a fixed $(F, s) \in \mathcal{M}^{2\times 2} \times \mathbb{R}$ is defined as

$$CQW(t, x, F, s) = \inf_\nu \left\{ \int_{\mathcal{M}^{2\times 2}} W(t, x, M)\, d\nu(M) : \ \nu \in \mathcal{A}(F, s) \right\}, \tag{30}$$

where

$$\begin{aligned} \mathcal{A}(F, s) \quad = \quad & \Big\{ \nu : \nu \text{ is a homogeneous } H^1 - \text{Young measure,} \\ & F = \int_{\mathcal{M}^{2\times 2}} M d\nu(M) \quad \text{and} \quad \int_{\mathcal{M}^{2\times 2}} V(M)\, d\nu(M) = s \Big\}. \end{aligned}$$

[5] Dacorogna, B., Direct method in the calculus of variations, 1989

- From Dacorogna [5], a relaxed formulation of $(VP^1_\omega)$ is where

$$\overline{m} = \min_{U,s} \left\{ \int_0^T \int_\Omega CQW\left(t, x, \nabla U\left(t, x\right), s\left(x\right)\right) dxdt \right\} \quad (= m) \tag{28}$$

where the minimum is taken over the fields $U \in \left(H^1\left((0, T) \times (0, 1)\right)\right)^2$ which satisfy the initial and boundary conditions and the function $s$ verifies the constraints

$$0 \leq s(x) \leq 1 \quad \forall x \in \Omega, \quad \text{and} \quad \int_\Omega s(x)\, dx \leq L\left|\Omega\right|. \tag{29}$$

- The expression $CQW\left(t, x, \nabla U\left(t, x\right), s\left(x\right)\right)$ stands for the constrained quasi-convexification of the density $W$ and for a fixed $(F, s) \in \mathcal{M}^{2\times 2} \times \mathbb{R}$ is defined as

$$CQW\left(t, x, F, s\right) = \inf_\nu \left\{ \int_{\mathcal{M}^{2\times 2}} W\left(t, x, M\right) d\nu\left(M\right) : \ \nu \in \mathcal{A}\left(F, s\right) \right\}, \tag{30}$$

where

$$\begin{aligned} \mathcal{A}\left(F, s\right) = & \left\{ \nu : \nu \text{ is a homogeneous } H^1 - \text{Young measure}, \right. \\ & \left. F = \int_{\mathcal{M}^{2\times 2}} M d\nu\left(M\right) \quad \text{and} \quad \int_{\mathcal{M}^{2\times 2}} V\left(M\right) d\nu\left(M\right) = s \right\}. \end{aligned}$$

[5] Dacorogna, B., Direct method in the calculus of variations, 1989

The class $\mathcal{A}(F,s)$ of Gradient Young Measure is NOT explicit. The strategy is as follows : [Kohn-Strang 86], [Fonseca-Muller 00], [Pedregal 05] :

- Minimize over $\nu \in \mathcal{A}^\star$, the class of polyconvex measures such that

$$\mathcal{A}(F,s) \subset \mathcal{A}^\star(F,s), \forall F, s \tag{31}$$

$$\mathcal{A}^\star(F,s) = \Big\{\nu : \nu \text{ is an homogeneous Young measure }, \nu \text{ commute with det}, \\ F = \int_{\mathcal{M}^{2\times 2}} M d\nu(M), s = \int_{\mathcal{M}^{2\times 2}} V(U,M) d\nu(M)\Big\}. \tag{32}$$

$$CPW(F,s) = \min_{\nu} \Big\{ \int_{\mathcal{M}^{2\times 2}} W(M) d\nu(M) : \nu \in \mathcal{A}^\star \Big\} \leq CQW(F,s) \tag{33}$$

- Study if the optimal measure $\nu_{opt} \in \mathcal{A}^\star$ satisfies a rank one condition, in which case, $\nu_{opt}$ belongs to the class of laminates $\mathcal{A}_\star$ such that

$$\mathcal{A}_\star(F,s) \subset \mathcal{A}(F,s), \quad \forall F, s \tag{34}$$

- This implies that $\nu_{opt} \in \mathcal{A}$ and gives $CQW = CPW$ and $m$.

# Computation of $CQW$

The class $\mathcal{A}(F,s)$ of Gradient Young Measure is NOT explicit. The strategy is as follows : [Kohn-Strang 86], [Fonseca-Muller 00], [Pedregal 05] :

- Minimize over $\nu \in \mathcal{A}^\star$, the class of polyconvex measures such that

$$\mathcal{A}(F,s) \subset \mathcal{A}^\star(F,s), \forall F, s \tag{31}$$

$$\begin{aligned}\mathcal{A}^\star(F,s) = \Big\{\nu : &\nu \text{ is an homogeneous Young measure }, \nu \text{ commute with det},\\ &F = \int_{\mathcal{M}^{2\times 2}} M d\nu(M), s = \int_{\mathcal{M}^{2\times 2}} V(U,M) d\nu(M)\Big\}.\end{aligned} \tag{32}$$

$$CPW(F,s) = \min_{\nu} \Big\{ \int_{\mathcal{M}^{2\times 2}} W(M) d\nu(M) : \nu \in \mathcal{A}^\star \Big\} \le CQW(F,s) \tag{33}$$

- Study if the optimal measure $\nu_{opt} \in \mathcal{A}^\star$ satisfies a rank one condition, in which case, $\nu_{opt}$ belongs to the class of laminates $\mathcal{A}_\star$ such that

$$\mathcal{A}_\star(F,s) \subset \mathcal{A}(F,s), \quad \forall F, s \tag{34}$$

- This implies that $\nu_{opt} \in \mathcal{A}$ and gives $CQW = CPW$ and $m$.

# Computation of $CQW$

The class $\mathcal{A}(F, s)$ of Gradient Young Measure is NOT explicit. The strategy is as follows : [Kohn-Strang 86], [Fonseca-Muller 00], [Pedregal 05] :

- Minimize over $\nu \in \mathcal{A}^\star$, the class of polyconvex measures such that

$$\mathcal{A}(F, s) \subset \mathcal{A}^\star(F, s), \forall F, s \tag{31}$$

$$\mathcal{A}^\star(F, s) = \Big\{\nu : \nu \text{ is an homogeneous Young measure }, \nu \text{ commute with det}, \\ F = \int_{\mathcal{M}^{2\times 2}} M d\nu(M), s = \int_{\mathcal{M}^{2\times 2}} V(U, M) d\nu(M)\Big\}. \tag{32}$$

$$CPW(F, s) = \min_{\nu} \Big\{ \int_{\mathcal{M}^{2\times 2}} W(M) d\nu(M) : \nu \in \mathcal{A}^\star \Big\} \leq CQW(F, s) \tag{33}$$

- Study if the optimal measure $\nu_{opt} \in \mathcal{A}^\star$ satisfies a rank one condition, in which case, $\nu_{opt}$ belongs to the class of laminates $\mathcal{A}_\star$ such that

$$\mathcal{A}_\star(F, s) \subset \mathcal{A}(F, s), \quad \forall F, s \tag{34}$$

- This implies that $\nu_{opt} \in \mathcal{A}$ and gives $CQW = CPW$ and $m$.

- $$CPW(F,s) = \min_{\nu} \left\{ \int_{\mathcal{M}^{2\times 2}} W(M) d\nu(M) : \nu \in \mathcal{A}^\star \right\} \leq CQW(F,s) \quad (35)$$

  where

  $$\mathcal{A}^\star(F,s) = \Big\{ \nu : \nu \text{ is a homogeneous Young measure}, \ \nu \text{ commutes with the determinant}, \\ F = \int_{\mathcal{M}^{2\times 2}} M d\nu(M), \ s = \int_{\mathcal{M}^{2\times 2}} V(U,M) d\nu(M) \Big\}. \quad (36)$$

- From the volume constraint ($s = \int_{\mathcal{M}^{2\times 2}} V(U,M) d\nu(M)$), the measure has the form

  $$\nu = s\nu_1 + (1-s)\,\nu_0, \quad \text{with supp}\left(\nu_j\right) \subset \Lambda_j, \ j = 0,1, \quad (37)$$

  and hence for each pair $(F,s)$, the constrained polyconvexification $CPW\,(F,s)$ is computed by solving

  $$CPW(F,s) = \min_{\nu} \left\{ s \int_{\Lambda_1} \left| M^{(1)} \right|^2 d\nu_1(M) + (1-s) \int_{\Lambda_0} \left| M^{(1)} \right|^2 d\nu_0(M) \right\} \quad (38)$$

  subject to

  $$\begin{cases} \nu = s\nu_1 + (1-s)\,\nu_0 \text{ commutes with det}, \\ \text{supp}\left(\nu_j\right) \subset \Lambda_j, \ j = 0,1, \\ F = s \int_{\Lambda_1} M d\nu_1(M) + (1-s) \int_{\Lambda_0} M d\nu_0(M). \end{cases} \quad (39)$$

- $$CPW(F,s)=\min_{\nu}\left\{\int_{\mathcal{M}^{2\times 2}} W(M)d\nu(M) : \nu\in\mathcal{A}^\star\right\}\leq CQW(F,s) \quad (35)$$

  where

  $$\mathcal{A}^\star(F,s)=\Big\{\nu : \nu \text{ is a homogeneous Young measure}, \nu \text{ commutes with the determinant}, \\ F=\int_{\mathcal{M}^{2\times 2}} Md\nu(M),\ s=\int_{\mathcal{M}^{2\times 2}} V(U,M)d\nu(M)\Big\}. \quad (36)$$

- From the volume constraint ($s=\int_{\mathcal{M}^{2\times 2}} V(U,M)d\nu(M)$), the measure has the form

  $$\nu = s\nu_1+(1-s)\,\nu_0, \quad \text{with supp}\left(\nu_j\right)\subset\Lambda_j,\ j=0,1, \quad (37)$$

  and hence for each pair $(F,s)$, the constrained polyconvexification $CPW(F,s)$ is computed by solving

  $$CPW(F,s)=\min_{\nu}\left\{s\int_{\Lambda_1}\left|M^{(1)}\right|^2 d\nu_1(M)+(1-s)\int_{\Lambda_0}\left|M^{(1)}\right|^2 d\nu_0(M)\right\} \quad (38)$$

  subject to

  $$\begin{cases}\nu = s\nu_1+(1-s)\,\nu_0 \text{ commutes with } \det, \\ \text{supp}\left(\nu_j\right)\subset\Lambda_j,\ j=0,1, \\ F=s\int_{\Lambda_1} Md\nu_1(M)+(1-s)\int_{\Lambda_0} Md\nu_0(M).\end{cases} \quad (39)$$

# Step 2: Minimization over $\mathcal{A}^\star$ - Computation of *CPW*

- Let us introduce the following variables

$$S_i = \int_{\mathbb{R}} (M_{1i})^2 \, d\nu^{(1i)}, \quad i = 1, 2, \tag{40}$$

where $\nu^{(1i)}$ stands for the projection of $\nu$ onto the $(1i)$ −th component, and

$$F^j = \int_{\Lambda_j} M d\nu_j (M) \,, \quad j = 0, 1. \tag{41}$$

Since $F^j \in \Lambda_j$, we have

$$F^0_{11} = F^0_{22}, F^0_{12} = F^0_{21} \quad \text{and} \quad F^1_{11} = F^1_{22} + \lambda, F^1_{12} = F^1_{21} \tag{42}$$

On the other hand, from the third condition in (39) it follows that

$$\begin{cases} F_{11} = sF^1_{11} + (1-s)\, F^0_{11}, & F_{12} = sF^1_{12} + (1-s)\, F^0_{12} \\ F_{21} = sF^1_{21} + (1-s)\, F^0_{21}, & F_{22} = sF^1_{22} + (1-s)\, F^0_{22} \end{cases} \tag{43}$$

Substituting (42) into (43) we obtain the system

$$\begin{cases} F_{11} = sF^1_{11} + (1-s)\, F^0_{11}, & F_{12} = sF^1_{12} + (1-s)\, F^0_{12} \\ F_{21} = sF^1_{12} + (1-s)\, F^0_{12}, & F_{22} + s\lambda = sF^1_{11} + (1-s)\, F^0_{11} \end{cases} \tag{44}$$

which has a solution if and only if the compatibility condition

$$F_{12} = F_{21}, \quad F_{11} = F_{22} + s\lambda \tag{45}$$

holds. In this case, the solution is given by

$$F^0_{11} = \alpha, \quad F^0_{12} = \beta \quad F^1_{11} = \frac{1}{s}\left(F_{11} - (1-s)\,\alpha\right) \quad F^1_{12} = \frac{1}{s}\left(F_{12} - (1-s)\,\beta\right) \tag{46}$$

where $(\alpha, \beta) \in \mathbb{R}^2$ are two parameters. Notice then that there is no restriction on $F^1_{11}$, as it can take on

- Let us introduce the following variables

$$S_i = \int_{\mathbb{R}} (M_{1i})^2 \, d\nu^{(1i)}, \quad i = 1, 2, \tag{40}$$

where $\nu^{(1i)}$ stands for the projection of $\nu$ onto the $(1i)$ $-$th component, and

$$F^j = \int_{\Lambda_j} M d\nu_j (M) , \quad j = 0, 1. \tag{41}$$

Since $F^j \in \Lambda_j$, we have

$$F^0_{11} = F^0_{22}, F^0_{12} = F^0_{21} \quad \text{and} \quad F^1_{11} = F^1_{22} + \lambda, F^1_{12} = F^1_{21} \tag{42}$$

On the other hand, from the third condition in (39) it follows that

$$\left\{ \begin{array}{ll} F_{11} = sF^1_{11} + (1-s) F^0_{11}, & F_{12} = sF^1_{12} + (1-s) F^0_{12} \\ F_{21} = sF^1_{21} + (1-s) F^0_{21}, & F_{22} = sF^1_{22} + (1-s) F^0_{22} \end{array} \right. \tag{43}$$

Substituting (42) into (43) we obtain the system

$$\left\{ \begin{array}{ll} F_{11} = sF^1_{11} + (1-s) F^0_{11}, & F_{12} = sF^1_{12} + (1-s) F^0_{12} \\ F_{21} = sF^1_{12} + (1-s) F^0_{12}, & F_{22} + s\lambda = sF^1_{11} + (1-s) F^0_{11} \end{array} \right. \tag{44}$$

which has a solution if and only if the compatibility condition

$$F_{12} = F_{21}, \quad F_{11} = F_{22} + s\lambda \tag{45}$$

holds. In this case, the solution is given by

$$F^0_{11} = \alpha, \quad F^0_{12} = \beta \quad F^1_{11} = \frac{1}{s} (F_{11} - (1-s)\alpha) \quad F^1_{12} = \frac{1}{s} (F_{12} - (1-s)\beta) \tag{46}$$

where $(\alpha, \beta) \in \mathbb{R}^2$ are two parameters. Notice then that there is no restriction on $F^1_{11}$, as it can take on

# Step 2: Minimization over $\mathcal{A}^\star$ - Computation of $CPW$

- Let us introduce the following variables

$$S_i = \int_{\mathbb{R}} (M_{1i})^2 \, d\nu^{(1i)}, \quad i = 1, 2, \tag{40}$$

where $\nu^{(1i)}$ stands for the projection of $\nu$ onto the $(1i)$ $-$th component, and

$$F^j = \int_{\Lambda_j} M d\nu_j(M), \quad j = 0, 1. \tag{41}$$

Since $F^j \in \Lambda_j$, we have

$$F^0_{11} = F^0_{22}, F^0_{12} = F^0_{21} \quad \text{and} \quad F^1_{11} = F^1_{22} + \lambda, F^1_{12} = F^1_{21} \tag{42}$$

On the other hand, from the third condition in (39) it follows that

$$\begin{cases} F_{11} = sF^1_{11} + (1-s)F^0_{11}, & F_{12} = sF^1_{12} + (1-s)F^0_{12} \\ F_{21} = sF^1_{21} + (1-s)F^0_{21}, & F_{22} = sF^1_{22} + (1-s)F^0_{22} \end{cases} \tag{43}$$

Substituting (42) into (43) we obtain the system

$$\begin{cases} F_{11} = sF^1_{11} + (1-s)F^0_{11}, & F_{12} = sF^1_{12} + (1-s)F^0_{12} \\ F_{21} = sF^1_{12} + (1-s)F^0_{12}, & F_{22} + s\lambda = sF^1_{11} + (1-s)F^0_{11} \end{cases} \tag{44}$$

which has a solution if and only if the compatibility condition

$$F_{12} = F_{21}, \quad F_{11} = F_{22} + s\lambda \tag{45}$$

holds. In this case, the solution is given by

$$F^0_{11} = \alpha, \quad F^0_{12} = \beta \quad F^1_{11} = \frac{1}{s}(F_{11} - (1-s)\alpha) \quad F^1_{12} = \frac{1}{s}(F_{12} - (1-s)\beta) \tag{46}$$

where $(\alpha, \beta) \in \mathbb{R}^2$ are two parameters. Notice then that there is no restriction on $F^1_{11}$, as it can take on

# Step 2: Minimization over $\mathcal{A}^\star$ - Computation of *CPW*

- Let us introduce the following variables

$$S_i = \int_{\mathbb{R}} (M_{1i})^2 \, d\nu^{(1i)}, \quad i = 1, 2, \tag{40}$$

where $\nu^{(1i)}$ stands for the projection of $\nu$ onto the $(1i)$ $-$th component, and

$$F^j = \int_{\Lambda_j} M d\nu_j (M) \, , \quad j = 0, 1. \tag{41}$$

Since $F^j \in \Lambda_j$, we have

$$F^0_{11} = F^0_{22}, F^0_{12} = F^0_{21} \quad \text{and} \quad F^1_{11} = F^1_{22} + \lambda, F^1_{12} = F^1_{21} \tag{42}$$

On the other hand, from the third condition in (39) it follows that

$$\begin{cases} F_{11} = sF^1_{11} + (1-s) F^0_{11}, & F_{12} = sF^1_{12} + (1-s) F^0_{12} \\ F_{21} = sF^1_{21} + (1-s) F^0_{21}, & F_{22} = sF^1_{22} + (1-s) F^0_{22} \end{cases} \tag{43}$$

Substituting (42) into (43) we obtain the system

$$\begin{cases} F_{11} = sF^1_{11} + (1-s) F^0_{11}, & F_{12} = sF^1_{12} + (1-s) F^0_{12} \\ F_{21} = sF^1_{12} + (1-s) F^0_{12}, & F_{22} + s\lambda = sF^1_{11} + (1-s) F^0_{11} \end{cases} \tag{44}$$

which has a solution if and only if the compatibility condition

$$F_{12} = F_{21}, \quad F_{11} = F_{22} + s\lambda \tag{45}$$

holds. In this case, the solution is given by

$$F^0_{11} = \alpha, \quad F^0_{12} = \beta \quad F^1_{11} = \frac{1}{s} (F_{11} - (1-s)\alpha) \quad F^1_{12} = \frac{1}{s} (F_{12} - (1-s)\beta) \tag{46}$$

where $(\alpha, \beta) \in \mathbb{R}^2$ are two parameters. Notice then that there is no restriction on $F^1_{11}$, as it can take on

- Let us introduce the following variables

$$S_i = \int_{\mathbb{R}} (M_{1i})^2 \, d\nu^{(1i)}, \quad i = 1, 2, \tag{40}$$

where $\nu^{(1i)}$ stands for the projection of $\nu$ onto the $(1i)$ −th component, and

$$F^j = \int_{\Lambda_j} M d\nu_j (M) \,, \quad j = 0, 1. \tag{41}$$

Since $F^j \in \Lambda_j$, we have

$$F^0_{11} = F^0_{22}, F^0_{12} = F^0_{21} \quad \text{and} \quad F^1_{11} = F^1_{22} + \lambda, F^1_{12} = F^1_{21} \tag{42}$$

On the other hand, from the third condition in (39) it follows that

$$\begin{cases} F_{11} = sF^1_{11} + (1-s)\, F^0_{11}, & F_{12} = sF^1_{12} + (1-s)\, F^0_{12} \\ F_{21} = sF^1_{21} + (1-s)\, F^0_{21}, & F_{22} = sF^1_{22} + (1-s)\, F^0_{22} \end{cases} \tag{43}$$

Substituting (42) into (43) we obtain the system

$$\begin{cases} F_{11} = sF^1_{11} + (1-s)\, F^0_{11}, & F_{12} = sF^1_{12} + (1-s)\, F^0_{12} \\ F_{21} = sF^1_{12} + (1-s)\, F^0_{12}, & F_{22} + s\lambda = sF^1_{11} + (1-s)\, F^0_{11} \end{cases} \tag{44}$$

which has a solution if and only if the compatibility condition

$$F_{12} = F_{21}, \quad F_{11} = F_{22} + s\lambda \tag{45}$$

holds. In this case, the solution is given by

$$F^0_{11} = \alpha, \quad F^0_{12} = \beta \quad F^1_{11} = \frac{1}{s}\left(F_{11} - (1-s)\,\alpha\right) \quad F^1_{12} = \frac{1}{s}\left(F_{12} - (1-s)\,\beta\right) \tag{46}$$

where $(\alpha, \beta) \in \mathbb{R}^2$ are two parameters. Notice then that there is no restriction on $F^1_{11}$, as it can take on

- Moreover, the constraint on the commutation with det yields to

$$\begin{aligned} \det F &= s\int_{\Lambda_1} \det M d\nu_1(M) + (1-s)\int_{\Lambda_0} \det M d\nu_0(M) \\ &= S_1 - S_2 - s\lambda F^1_{11} \end{aligned}$$

since

$$\det M = \begin{cases} (M_{11})^2 - (M_{12})^2 & \text{if} \quad M \in \Lambda_0 \\ (M_{11})^2 - \lambda M_{11} - (M_{12})^2 & \text{if} \quad M \in \Lambda_1 \end{cases} \tag{47}$$

Finally, from Jensen's inequality we obtain the conditions

$$S_i \geq |F_{1i}|^2, \quad i = 1, 2. \tag{48}$$

To sum up, we have to solve the mathematical programming problem

$$\text{Minimize in } \left(S_j, F^1_{11}\right): \quad (S_1 + S_2) \tag{49}$$

subject to

$$\begin{cases} S_1 - S_2 - s\lambda F^1_{11} = \det F \\ S_i \geq |F_{1i}|^2, \quad i = 1, 2. \end{cases} \tag{50}$$

We obtain easily that the solution is

$$S_i = |F_{1i}|^2, \quad i = 1, 2. \tag{51}$$

This implies that

$$CPW(F, s) = \begin{cases} |F^{(1)}|^2 & \text{if (45) holds} \\ +\infty & \text{else.} \end{cases} \tag{52}$$

- Moreover, the constraint on the commutation with det yields to

$$\begin{aligned} \det F &= s\int_{\Lambda_1} \det M d\nu_1(M) + (1-s)\int_{\Lambda_0} \det M d\nu_0(M) \\ &= S_1 - S_2 - s\lambda F_{11}^1 \end{aligned}$$

since

$$\det M = \begin{cases} (M_{11})^2 - (M_{12})^2 & \text{if} \quad M \in \Lambda_0 \\ (M_{11})^2 - \lambda M_{11} - (M_{12})^2 & \text{if} \quad M \in \Lambda_1 \end{cases} \tag{47}$$

Finally, from Jensen's inequality we obtain the conditions

$$S_i \geq |F_{1i}|^2, \quad i = 1, 2. \tag{48}$$

To sum up, we have to solve the mathematical programming problem

$$\text{Minimize in } \left(S_j, F_{11}^1\right) : \quad (S_1 + S_2) \tag{49}$$

subject to

$$\begin{cases} S_1 - S_2 - s\lambda F_{11}^1 = \det F \\ S_i \geq |F_{1i}|^2, \quad i = 1, 2. \end{cases} \tag{50}$$

We obtain easily that the solution is

$$S_i = |F_{1i}|^2, \quad i = 1, 2. \tag{51}$$

This implies that

$$CPW(F, s) = \begin{cases} |F^{(1)}|^2 & \text{if (45) holds} \\ +\infty & \text{else.} \end{cases} \tag{52}$$

- Moreover, the constraint on the commutation with det yields to

$$\begin{aligned} \det F &= s\int_{\Lambda_1} \det M d\nu_1(M) + (1-s)\int_{\Lambda_0} \det M d\nu_0(M) \\ &= S_1 - S_2 - s\lambda F_{11}^1 \end{aligned}$$

since

$$\det M = \begin{cases} (M_{11})^2 - (M_{12})^2 & \text{if} \quad M \in \Lambda_0 \\ (M_{11})^2 - \lambda M_{11} - (M_{12})^2 & \text{if} \quad M \in \Lambda_1 \end{cases} \tag{47}$$

Finally, from Jensen's inequality we obtain the conditions

$$S_i \geq |F_{1i}|^2, \quad i = 1, 2. \tag{48}$$

To sum up, we have to solve the mathematical programming problem

$$\text{Minimize in } \left(S_j, F_{11}^1\right): \quad (S_1 + S_2) \tag{49}$$

subject to

$$\begin{cases} S_1 - S_2 - s\lambda F_{11}^1 = \det F \\ S_i \geq |F_{1i}|^2, \quad i = 1, 2. \end{cases} \tag{50}$$

We obtain easily that the solution is

$$S_i = |F_{1i}|^2, \quad i = 1, 2. \tag{51}$$

This implies that

$$CPW(F, s) = \begin{cases} |F^{(1)}|^2 & \text{if (45) holds} \\ +\infty & \text{else.} \end{cases} \tag{52}$$

$$CPW(F,s) = \min_{\nu} \left\{ \int_{\mathcal{M}^{2\times 2}} W(M)d\nu(M) : \nu \in \mathcal{A}^\star \right\} \leq CQW(F,s) \tag{53}$$

where

$$\mathcal{A}^\star(F,s) = \Big\{ \nu : \nu \text{ is a homogeneous Young measure, } \nu \text{ commutes with the determinant,}$$
$$F = \int_{\mathcal{M}^{2\times 2}} M d\nu(M),\, s = \int_{\mathcal{M}^{2\times 2}} V(U,M)d\nu(M) \Big\}. \tag{54}$$

- We have

$$CPW(F,s) = \begin{cases} |F^{(1)}|^2 & \text{if} \quad F_{21} = F_{12}, F_{11} = F_{22} + s\lambda \\ +\infty & \text{else.} \end{cases} \tag{55}$$

- The optimal , unique, measure $\nu$ is

$$\nu = (1-s)\delta_{G^0} + s\delta_{G^1}, \tag{56}$$

where

$$G^0 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11} \end{pmatrix} \quad \text{and} \quad G^1 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11}+\lambda \end{pmatrix}. \tag{57}$$

$$CPW(F,s) = \min_{\nu} \left\{ \int_{\mathcal{M}^{2\times 2}} W(M)d\nu(M) : \nu \in \mathcal{A}^\star \right\} \leq CQW(F,s) \tag{53}$$

where

$$\begin{aligned} \mathcal{A}^\star(F,s) = \Big\{\nu :& \nu \text{ is a homogeneous Young measure, } \nu \text{ commutes with the determinant,} \\ & F = \int_{\mathcal{M}^{2\times 2}} M d\nu(M), s = \int_{\mathcal{M}^{2\times 2}} V(U,M) d\nu(M) \Big\}. \end{aligned} \tag{54}$$

- We have

$$CPW(F,s) = \begin{cases} |F^{(1)}|^2 & \text{if } \; F_{21} = F_{12}, F_{11} = F_{22} + s\lambda \\ +\infty & \text{else.} \end{cases} \tag{55}$$

- The optimal , unique, measure $\nu$ is

$$\nu = (1-s)\delta_{G^0} + s\delta_{G^1}, \tag{56}$$

where

$$G^0 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11} \end{pmatrix} \text{ and } G^1 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11}+\lambda \end{pmatrix}. \tag{57}$$

- From

$$\nu_{opt} = (1-s)\delta_{G^0} + s\delta_{G^1},$$

where

$$G^0 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11} \end{pmatrix} \quad \text{and} \quad G^1 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11}+\lambda \end{pmatrix}. \tag{58}$$

we get that

$$G^1 - G^0 = b \otimes n, \quad \text{with} \quad b = (0,\lambda) \quad \text{and} \quad n = (0,1) \tag{59}$$

-
$$Rank(G^1 - G^0) = 1 \tag{60}$$

$\Longrightarrow \nu_{opt}$ satisfies a rank one condition.

- The optimal measure $\nu_{opt}$ belongs to $\mathcal{A}_\star$, and $\nu_{opt}$ is a first order laminate with normal $n$
- Conclusion: $\nu_{opt} \in \mathcal{A}$ and $CQW(F,s) = CPW(F,s)$

## Step 3 : Rank-one condition on $\nu_{opt}$ ? - $\nu_{opt} \in \mathcal{A}_\star$ ?

- From
$$\nu_{opt} = (1-s)\delta_{G^0} + s\delta_{G^1},$$
where
$$G^0 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11} \end{pmatrix} \quad \text{and} \quad G^1 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11}+\lambda \end{pmatrix}. \tag{58}$$
we get that
$$G^1 - G^0 = b \otimes n, \quad \text{with} \quad b = (0,\lambda) \quad \text{and} \quad n = (0,1) \tag{59}$$
- $$Rank(G^1 - G^0) = 1 \tag{60}$$
$$\Longrightarrow \nu_{opt} \text{ satisfies a rank one condition.}$$
- The optimal measure $\nu_{opt}$ belongs to $\mathcal{A}_\star$, and $\nu_{opt}$ is a first order laminate with normal $n$
- Conclusion: $\nu_{opt} \in \mathcal{A}$ and $CQW(F,s) = CPW(F,s)$

- From

$$\nu_{opt} = (1-s)\delta_{G^0} + s\delta_{G^1},$$

where

$$G^0 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11} \end{pmatrix} \quad \text{and} \quad G^1 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11}+\lambda \end{pmatrix}. \tag{58}$$

we get that

$$G^1 - G^0 = b \otimes n, \quad \text{with} \quad b = (0,\lambda) \quad \text{and} \quad n = (0,1) \tag{59}$$

- $$Rank(G^1 - G^0) = 1 \tag{60}$$

$$\Longrightarrow \nu_{opt} \text{ satisfies a rank one condition.}$$

- The optimal measure $\nu_{opt}$ belongs to $\mathcal{A}_\star$, and $\nu_{opt}$ is a first order laminate with normal $n$
- Conclusion: $\nu_{opt} \in \mathcal{A}$ and $CQW(F,s) = CPW(F,s)$

# Step 3 : Rank-one condition on $\nu_{opt}$ ? - $\nu_{opt} \in \mathcal{A}_\star$ ?

- From
$$\nu_{opt} = (1-s)\delta_{G^0} + s\delta_{G^1},$$
where
$$G^0 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11} \end{pmatrix} \quad \text{and} \quad G^1 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11}+\lambda \end{pmatrix}. \tag{58}$$
we get that
$$G^1 - G^0 = b \otimes n, \quad \text{with} \quad b = (0,\lambda) \quad \text{and} \quad n = (0,1) \tag{59}$$
- $$Rank(G^1 - G^0) = 1 \tag{60}$$
$\Longrightarrow \nu_{opt}$ satisfies a rank one condition.
- The optimal measure $\nu_{opt}$ belongs to $\mathcal{A}_\star$, and $\nu_{opt}$ is a first order laminate with normal $n$
- Conclusion: $\nu_{opt} \in \mathcal{A}$ and $CQW(F,s) = CPW(F,s)$

- From $\lambda = -a(x)U^{(1)}(t,x)$ and

$$F = \begin{pmatrix} F_{11} & F_{21} \\ F_{12} & F_{22} \end{pmatrix} = \nabla U = \begin{pmatrix} u_t & v_t \\ u_x & v_x \end{pmatrix}. \quad (61)$$

$$CQW(F,s) = \begin{cases} |F^{(1)}|^2 & \text{if} \quad F_{21} = F_{12}, F_{11} = F_{22} + s\lambda \\ +\infty & \text{else}. \end{cases} \quad (62)$$

becomes

$$CQW(\nabla U, s) = \begin{cases} u_t^2(t,x) + u_x^2(t,x) & \text{if} \quad u_x = v_t, u_t = v_x - a(x)s(x)u(t,x) \\ +\infty & \text{else}. \end{cases} \quad (63)$$

equivalently

$$CQW(\nabla U, s) = \begin{cases} u_t^2(t,x) + u_x^2(t,x) & \text{if} \quad u_{tt} - u_{xx} + a(x)s(x)u_t = 0 \\ +\infty & \text{else}. \end{cases} \quad (64)$$

- $min_{U,s} \int_0^T \int_\Omega CQW(\nabla U, s)dxdt$ then leads to $(RP^1_\omega)$

- From $\lambda = -a(x)U^{(1)}(t,x)$ and

$$F = \begin{pmatrix} F_{11} & F_{21} \\ F_{12} & F_{22} \end{pmatrix} = \nabla U = \begin{pmatrix} u_t & v_t \\ u_x & v_x \end{pmatrix}. \quad (61)$$

$$CQW(F,s) = \begin{cases} |F^{(1)}|^2 & \text{if} \quad F_{21} = F_{12}, F_{11} = F_{22} + s\lambda \\ +\infty & \text{else.} \end{cases} \quad (62)$$

becomes

$$CQW(\nabla U, s) = \begin{cases} u_t^2(t,x) + u_x^2(t,x) & \text{if} \quad u_x = v_t, u_t = v_x - a(x)s(x)u(t,x) \\ +\infty & \text{else.} \end{cases} \quad (63)$$

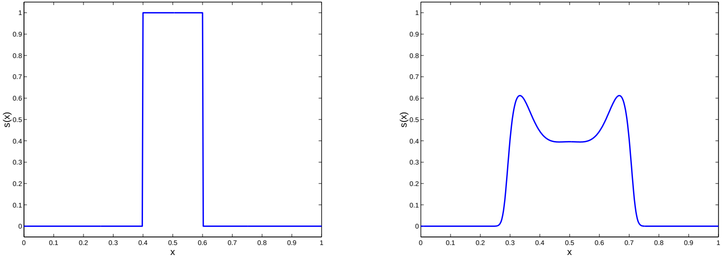
equivalently

$$CQW(\nabla U, s) = \begin{cases} u_t^2(t,x) + u_x^2(t,x) & \text{if} \quad u_{tt} - u_{xx} + a(x)s(x)u_t = 0 \\ +\infty & \text{else.} \end{cases} \quad (64)$$

- $min_{U,s} \int_0^T \int_\Omega CQW(\nabla U, s)dxdt$ then leads to $(RP^1_\omega)$

# Some numerical results for $(RP^1_\omega)$

$$\Omega = (0,1), \quad (u^0(x), u^1(x)) = (\sin(\pi x), 0), \quad L = 1/5, \quad T = 1 \tag{65}$$

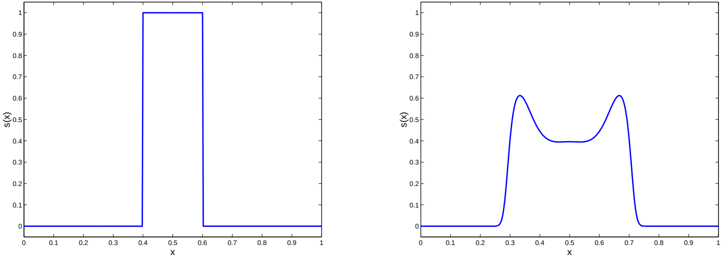



Optimal density for $a(x) = 1$ (Left) and $a(x) = 10$ (Right)

- If $a \le a^\star(\Omega, L, u^0, u^1)$, $\{x \in \Omega, 0 < s(x) < 1\} = \emptyset$, $(P^1_\omega) = (RP^1_\omega)$ and is well-posed
- If $a > a^\star(\Omega, L, u^0, u^1)$, $\{x \in \Omega, 0 < s(x) < 1\} \neq \emptyset$, $(P^1_\omega) \neq (RP^1_\omega)$ and is NOT well-posed

(This property is related to the over-damping phenomena)

$$\Omega = (0,1), \quad (u^0(x), u^1(x)) = (\sin(\pi x), 0), \quad L = 1/5, \quad T = 1 \tag{65}$$

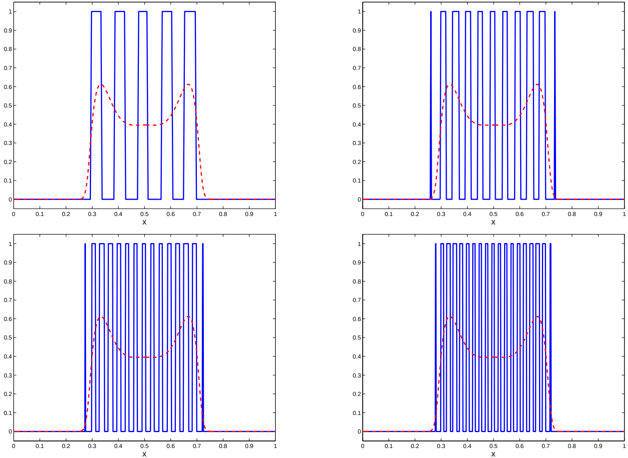



Optimal density for $a(x) = 1$ (Left) and $a(x) = 10$ (Right)

- If $a \leq a^\star(\Omega, L, u^0, u^1)$, $\{x \in \Omega, 0 < s(x) < 1\} = \emptyset$, $(P^1_\omega) = (RP^1_\omega)$ and is well-posed
- If $a > a^\star(\Omega, L, u^0, u^1)$, $\{x \in \Omega, 0 < s(x) < 1\} \neq \emptyset$, $(P^1_\omega) \neq (RP^1_\omega)$ and is NOT well-posed

(This property is related to the over-damping phenomena)

$$\Omega = (0,1), \quad (u^0(x), u^1(x)) = (\sin(\pi x), 0), \quad L = 1/5, \quad T = 1 \tag{66}$$



| $\sharp\omega_j$ | 10 | 20 | 30 | 40 |
|---|---|---|---|---|
| $I(\mathcal{X}_{\omega_j})$ | 4.1331 | 3.7216 | 3.5413 | 3.4313 |

$$lim_{\sharp\omega_j \to \infty} I(\mathcal{X}_{\omega_j}) = I(s_{opt}) = 3.4212$$

# The 2-D case (N=2)

- Similarly, the damped wave equation may be written as

$$u_{tt} - \Delta u + a(x_1, x_2)\mathcal{X}_\omega u_t = 0 \Longleftrightarrow \text{Div}\left(u_t + a\mathcal{X}_\omega u, -u_{x_1}, -u_{x_2}\right) = 0 \quad \text{in} \quad (0, T) \times \Omega \tag{67}$$

and so there exist two Clebsch's potentials [6] $v_1 = v_1(t, x_1, x_2)$ and $v_2 = v_2(t, x_1, x_2)$ such that

$$\left(u_t + a\mathcal{X}_\omega u, -u_{x_1}, -u_{x_2}\right) = \nabla v_1 \times \nabla v_2. \tag{68}$$

- Let the vector field $U = (u, v_1, v_2) \in (H^1((0, T) \times \Omega))^3$ and the two non-linear manifolds

$$\Lambda_0 = \left\{M \in \mathcal{M}^{3\times 3} : AM^{(1)} - M^{(2)} \times M^{(3)} = 0\right\},$$
$$\Lambda_{1,\lambda} = \left\{M \in \mathcal{M}^{3\times 3} : AM^{(1)} - M^{(2)} \times M^{(3)} = \lambda e_1\right\}, \tag{69}$$

where $\lambda \in \mathbb{R}$ and

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{70}$$

[6] Kotiuga, P.R, *Clebsch potentials and the vizualisation of three-dimensional solenoidal vectors fields*, 1991.

- Similarly, the damped wave equation may be written as

$$u_{tt} - \Delta u + a(x_1, x_2)\mathcal{X}_\omega u_t = 0 \Longleftrightarrow \text{Div}\left(u_t + a\mathcal{X}_\omega u, -u_{x_1}, -u_{x_2}\right) = 0 \quad \text{in} \quad (0,T)\times\Omega \tag{67}$$

and so there exist two Clebsch's potentials [6] $v_1 = v_1(t, x_1, x_2)$ and $v_2 = v_2(t, x_1, x_2)$ such that

$$\left(u_t + a\mathcal{X}_\omega u, -u_{x_1}, -u_{x_2}\right) = \nabla v_1 \times \nabla v_2. \tag{68}$$

- Let the vector field $U = (u, v_1, v_2) \in (H^1((0,T)\times\Omega))^3$ and the two non-linear manifolds

$$\begin{aligned} \Lambda_0 &= \left\{M \in \mathcal{M}^{3\times3} : AM^{(1)} - M^{(2)} \times M^{(3)} = 0\right\}, \\ \Lambda_{1,\lambda} &= \left\{M \in \mathcal{M}^{3\times3} : AM^{(1)} - M^{(2)} \times M^{(3)} = \lambda e_1\right\}, \end{aligned} \tag{69}$$

where $\lambda \in \mathbb{R}$ and

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{70}$$

[6] Kotiuga, P.R, *Clebsch potentials and the vizualisation of three-dimensional solenoidal vectors fields*, 1991.

# A numerical illustration in 2-D: $\Omega = (0,1)^2$

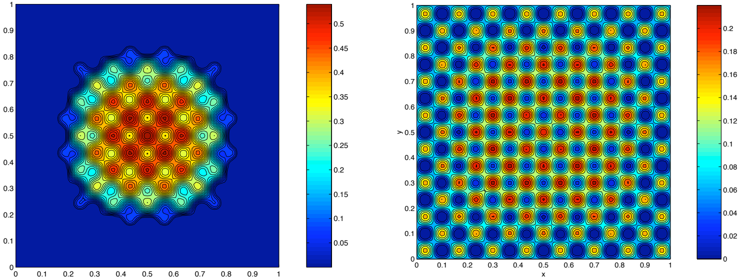



Figure: Iso-values of the optimal $s$ in $\Omega$ for $a(\boldsymbol{x}) = 25\mathcal{X}_\Omega(\boldsymbol{x})$ (**Left**) and $a(\boldsymbol{x}) = 50\mathcal{X}_\Omega(\boldsymbol{x})$ (**Right**) - $T = 1$

[Maestre-AM-Pedegral, IFB 08][7]

- Let $\Omega \subset \mathbb{R},\ 0 < \alpha < \beta < \infty,\ L \in (0,1),\ T > 0,\ (u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

$$(P_\omega^2): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + a(t,x,\mathcal{X}_\omega)|\nabla u|^2) dxdt \tag{71}$$

with

$$a(t,x,\mathcal{X}_\omega) = a_\alpha(t,x)\mathcal{X}_\omega + a_\beta(t,x)(1-\mathcal{X}_\omega) \tag{72}$$

subject to

$$\begin{cases} u_{tt} - div\Big([\alpha\mathcal{X}_\omega + \beta(1-\mathcal{X}_\omega)]\nabla u\Big) = 0 & (0,T)\times\Omega, \\ u = 0 & (0,T)\times\partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \Omega, \\ \mathcal{X}_\omega \in L^\infty((0,T)\times\Omega;\{0,1\}), & \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} \le L\|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0,T) \end{cases} \tag{73}$$

[7] F. Maestre, AM, P. Pedregal, *Optimal design under the one-dimensional wave equation*, Interfaces and Free Boundaries (2008)

- $h(t,x)=\beta a_\alpha(t,x)-\alpha a_\beta(t,x), \qquad a(t,x,\mathcal{X})=\mathcal{X}(t,x)a_\alpha(t,x)+(1-\mathcal{X}(t,x))a_\beta(t,x)$
- 

$$(RP^2_\omega): \qquad \min_{U,s}\int_0^T\int_\Omega CQW(t,\boldsymbol{x},\nabla U(t,\boldsymbol{x}),s(t,\boldsymbol{x}))dxdt$$

$$\begin{cases} U=(u,v)\in H^1([0,T]\times\Omega)^2, \quad tr(\nabla U(t,\boldsymbol{x}))=0,\\ U^{(1)}(0,\boldsymbol{x})=u_0(\boldsymbol{x}),\ U^{(1)}_t(0,\boldsymbol{x})=u_1(\boldsymbol{x}) \quad \text{in} \quad \Omega,\\ U^{(1)}(t,1)=U^{(1)}(t,0)=0 \quad \text{in} \quad [0,T],\\ 0\le s(t,\boldsymbol{x})\le 1,\ \int_\Omega s(t,\boldsymbol{x})\,dx\le V_\alpha|\Omega| \quad \forall t\in[0,T], \end{cases}$$

- $CQW(t,\boldsymbol{x},F,s)$ is defined by

$$\begin{cases} \frac{h}{s\beta(\beta-\alpha)^2}(\beta^2|F_{12}|^2+|F_{21}|^2+2\beta F_{12}F_{21})+|F_{11}|^2-\frac{a_\beta}{\beta}F_{12}F_{21} & \text{if } \boldsymbol{h(t,x)\ge 0,\ \psi(F,s)\le 0}\\ \frac{-h}{(1-s)\alpha(\beta-\alpha)^2}(\alpha^2|F_{12}|^2+|F_{21}|^2+2\alpha F_{12}F_{21})+|F_{11}|^2-\frac{a_\alpha}{\alpha}F_{12}F_{21}, & \text{if } \boldsymbol{h(t,x)\le 0,\ \psi(F,s)\le 0}\\ -\,detF+\frac{1}{s(1-s)(\beta-\alpha)^2}\Big(((1-s)\beta^2(\alpha+a_\alpha)+s\alpha^2(\beta+a_\beta))|F_{12}|^2 & \\ \qquad +((1-s)(\alpha+a_\alpha)+s(\beta+a_\beta))|F_{21}|^2+2((\alpha+a_\alpha)\beta-sh)F_{12}F_{21}\Big) & \text{if } \boldsymbol{\psi(F,s)\ge 0}.\\ +\infty & \text{if } \boldsymbol{Tr(F)\ne 0} \end{cases}$$

$$\psi(F,s)=\frac{(\alpha(1-s)+\beta s)}{(\beta-\alpha)}\Big(F_{21}+\lambda^-_{\alpha,\beta}(s)F_{12}\Big)\Big(F_{21}+\lambda^+_{\alpha,\beta}(s)F_{12}\Big)$$

# Problem $(P^2_\omega)$: Optimal $(\alpha,\beta)$ distribution - The result

- $h(t,x)=\beta a_\alpha(t,x)-\alpha a_\beta(t,x), \qquad a(t,x,\mathcal{X})=\mathcal{X}(t,x)a_\alpha(t,x)+(1-\mathcal{X}(t,x))a_\beta(t,x)$
-

$$(RP^2_\omega): \qquad \min_{U,s}\int_0^T\int_\Omega CQW(t,\boldsymbol{x},\nabla U(t,\boldsymbol{x}),s(t,\boldsymbol{x}))dxdt$$

$$\begin{cases} U=(u,v)\in H^1([0,T]\times\Omega)^2, \ \ tr(\nabla U(t,\boldsymbol{x}))=0, \\ U^{(1)}(0,\boldsymbol{x})=u_0(\boldsymbol{x}), \ U^{(1)}_t(0,\boldsymbol{x})=u_1(\boldsymbol{x}) \quad \text{in} \quad \Omega, \\ U^{(1)}(t,1)=U^{(1)}(t,0)=0 \quad \text{in} \quad [0,T], \\ 0\le s(t,\boldsymbol{x})\le 1, \displaystyle\int_\Omega s(t,\boldsymbol{x})\,dx\le V_\alpha|\Omega| \ \ \forall t\in[0,T], \end{cases}$$

- $CQW(t,\boldsymbol{x},F,s)$ is defined by

$$\begin{cases} \dfrac{h}{s\beta(\beta-\alpha)^2}(\beta^2|F_{12}|^2+|F_{21}|^2+2\beta F_{12}F_{21})+|F_{11}|^2-\dfrac{a_\beta}{\beta}F_{12}F_{21} & \text{if } \boldsymbol{h(t,x)\ge 0, \psi(F,s)\le 0} \\ \dfrac{-h}{(1-s)\alpha(\beta-\alpha)^2}(\alpha^2|F_{12}|^2+|F_{21}|^2+2\alpha F_{12}F_{21})+|F_{11}|^2-\dfrac{a_\alpha}{\alpha}F_{12}F_{21}, & \text{if } \boldsymbol{h(t,x)\le 0, \psi(F,s)\le 0} \\ -detF+\dfrac{1}{s(1-s)(\beta-\alpha)^2}\Big(((1-s)\beta^2(\alpha+a_\alpha)+s\alpha^2(\beta+a_\beta))|F_{12}|^2 & \\ \qquad +((1-s)(\alpha+a_\alpha)+s(\beta+a_\beta))|F_{21}|^2+2((\alpha+a_\alpha)\beta-sh)F_{12}F_{21}\Big) & \text{if } \boldsymbol{\psi(F,s)\ge 0}. \\ +\infty & \text{if } \boldsymbol{Tr(F)\ne 0} \end{cases}$$

$$\psi(F,s)=\frac{(\alpha(1-s)+\beta s)}{(\beta-\alpha)}\left(F_{21}+\lambda^-_{\alpha,\beta}(s)F_{12}\right)\left(F_{21}+\lambda^+_{\alpha,\beta}(s)F_{12}\right)$$

# Problem $(P^2_\omega)$: Optimal $(\alpha,\beta)$ distribution - The result

- $h(t,x) = \beta a_\alpha(t,x) - \alpha a_\beta(t,x), \qquad a(t,x,\mathcal{X}) = \mathcal{X}(t,x) a_\alpha(t,x) + (1-\mathcal{X}(t,x)) a_\beta(t,x)$
-
$$(RP^2_\omega): \qquad \min_{U,s} \int_0^T \int_\Omega CQW(t,\boldsymbol{x},\nabla U(t,\boldsymbol{x}), s(t,\boldsymbol{x}))dxdt$$
$$\begin{cases} U=(u,v)\in H^1([0,T]\times\Omega)^2, \;\; tr(\nabla U(t,\boldsymbol{x}))=0, \\ U^{(1)}(0,\boldsymbol{x}) = u_0(\boldsymbol{x}), \;\; U^{(1)}_t(0,\boldsymbol{x}) = u_1(\boldsymbol{x}) \quad \text{in} \quad \Omega, \\ U^{(1)}(t,1) = U^{(1)}(t,0) = 0 \quad \text{in} \quad [0,T], \\ 0 \le s(t,\boldsymbol{x}) \le 1, \; \int_\Omega s(t,\boldsymbol{x})\, dx \le V_\alpha |\Omega| \;\; \forall t \in [0,T], \end{cases}$$
- $CQW(t,\boldsymbol{x},F,s)$ is defined by

$$\begin{cases} \dfrac{h}{s\beta(\beta-\alpha)^2}(\beta^2|F_{12}|^2 + |F_{21}|^2 + 2\beta F_{12}F_{21}) + |F_{11}|^2 - \dfrac{a_\beta}{\beta}F_{12}F_{21} & \text{if } \boldsymbol{h(t,x) \ge 0, \psi(F,s) \le 0} \\ \dfrac{-h}{(1-s)\alpha(\beta-\alpha)^2}(\alpha^2|F_{12}|^2 + |F_{21}|^2 + 2\alpha F_{12}F_{21}) + |F_{11}|^2 - \dfrac{a_\alpha}{\alpha}F_{12}F_{21}, & \text{if } \boldsymbol{h(t,x) \le 0, \psi(F,s) \le 0} \\ -\, detF + \dfrac{1}{s(1-s)(\beta-\alpha)^2}\Big( \big((1-s)\beta^2(\alpha+a_\alpha) + s\alpha^2(\beta+a_\beta)\big)|F_{12}|^2 & \\ \qquad + \big((1-s)(\alpha+a_\alpha) + s(\beta+a_\beta)\big)|F_{21}|^2 + 2\big((\alpha+a_\alpha)\beta - sh\big)F_{12}F_{21}\Big) & \text{if } \boldsymbol{\psi(F,s) \ge 0}. \\ +\infty & \text{if } \boldsymbol{Tr(F) \ne 0} \end{cases}$$

$$\psi(F,s) = \tfrac{(\alpha(1-s)+\beta s)}{(\beta-\alpha)}\Big(F_{21} + \lambda^-_{\alpha,\beta}(s)F_{12}\Big)\Big(F_{21} + \lambda^+_{\alpha,\beta}(s)F_{12}\Big)$$

# The relaxation for $(P^3_\omega)$: First order laminate

- $$(RP^3_\omega) : \qquad \min_{U,s,r} \int_0^T \int_\Omega CQW(t, \boldsymbol{x}, \nabla U(t,x), s(t,\boldsymbol{x}), r(x))dxdt$$

soumis à

$$\begin{cases} U = (u,v) \in H^1([0,T]\times\Omega)^2, \quad \boldsymbol{\psi(t,x,\nabla U, s, r) = 0} \\ tr(\nabla U(t,\boldsymbol{x})) = u_t + v_x = a(x)r(x)u(t,x), \ \text{dans } (0,T)\times\Omega \\ U^{(1)}(0,\boldsymbol{x}) = u_0(\boldsymbol{x}), \ U_t^{(1)}(0,\boldsymbol{x}) = u_1(\boldsymbol{x}) \quad \text{dans} \quad \Omega, \\ U^{(1)}(t,1) = U^{(1)}(t,0) = 0 \quad \text{dans} \quad [0,T], \\ 0 \le s(t,x) \le 1, \ \int_\Omega s(t,x)\,dx \le L_\alpha|\Omega| \ \ \forall t \in [0,T], \\ 0 \le r(x) \le 1, \quad \int_\Omega r(x)dx \le L_d|\Omega| \end{cases}$$

- $CQW(t, \boldsymbol{x}, F, s, r)$ is given by

$$CQW(U,F,s,r) = |F_{11}|^2 + \frac{a_\alpha}{s(\beta-\alpha)^2}|\beta F_{12} + F_{21}|^2 + \frac{a_\beta}{(1-s)(\beta-\alpha)^2}|\alpha F_{12} + F_{21}|^2 \qquad (74)$$

$$\psi(F,s,r) = \frac{(\alpha(1-s)+\beta s)}{(\beta-\alpha)}\left(F_{21} + \lambda^-_{\alpha,\beta}(s)F_{12}\right)\left(F_{21} + \lambda^+_{\alpha,\beta}(s)F_{12}\right)$$

- First order laminate $\Longrightarrow$ (Regular effect on the optimal micro-structure) or (no second order laminates for $(RP^2_\omega)$).

- $$(RP^3_\omega): \qquad \min_{U,s,r} \int_0^T \int_\Omega CQW(t, \boldsymbol{x}, \nabla U(t,x), s(t,\boldsymbol{x}), r(x))dxdt$$

  soumis à

$$\begin{cases} U = (u,v) \in H^1([0,T]\times\Omega)^2, \quad \boldsymbol{\psi(t,x,\nabla U,s,r) = 0} \\ tr(\nabla U(t,\boldsymbol{x})) = u_t + v_x = a(x)r(x)u(t,x), \ \text{dans } (0,T)\times\Omega \\ U^{(1)}(0,\boldsymbol{x}) = u_0(\boldsymbol{x}), \ U_t^{(1)}(0,\boldsymbol{x}) = u_1(\boldsymbol{x}) \quad \text{dans} \quad \Omega, \\ U^{(1)}(t,1) = U^{(1)}(t,0) = 0 \quad \text{dans} \quad [0,T], \\ 0 \le s(t,x) \le 1, \ \int_\Omega s(t,x)\,dx \le L_\alpha|\Omega| \ \ \forall t\in[0,T], \\ 0 \le r(x) \le 1, \ \ \int_\Omega r(x)dx \le L_d|\Omega| \end{cases}$$

- $CQW(t, \boldsymbol{x}, F, s, r)$ is given by

$$CQW(U,F,s,r) = |F_{11}|^2 + \frac{a_\alpha}{s(\beta-\alpha)^2}|\beta F_{12}+F_{21}|^2 + \frac{a_\beta}{(1-s)(\beta-\alpha)^2}|\alpha F_{12}+F_{21}|^2 \qquad (74)$$

$$\psi(F,s,r) = \frac{(\alpha(1-s)+\beta s)}{(\beta-\alpha)}\left(F_{21}+\lambda^-_{\alpha,\beta}(s)F_{12}\right)\left(F_{21}+\lambda^+_{\alpha,\beta}(s)F_{12}\right)$$

- First order laminate $\Longrightarrow$ (Regular effect on the optimal micro-structure) or (no second order laminates for $(RP^2_\omega)$).

- $$(RP^3_\omega) : \qquad \min_{U,s,r} \int_0^T \int_\Omega CQW(t, \boldsymbol{x}, \nabla U(t,x), s(t,\boldsymbol{x}), r(x)) dxdt$$

  soumis à

  $$\begin{cases} U = (u,v) \in H^1([0,T]\times\Omega)^2, \quad \boldsymbol{\psi(t, x, \nabla U, s, r) = 0} \\ tr(\nabla U(t,\boldsymbol{x})) = u_t + v_x = a(x)r(x)u(t,x), \ \text{dans } (0,T)\times\Omega \\ U^{(1)}(0,\boldsymbol{x}) = u_0(\boldsymbol{x}), \ U_t^{(1)}(0,\boldsymbol{x}) = u_1(\boldsymbol{x}) \quad \text{dans} \quad \Omega, \\ U^{(1)}(t,1) = U^{(1)}(t,0) = 0 \quad \text{dans} \quad [0,T], \\ 0 \le s(t,x) \le 1, \ \int_\Omega s(t,x)\,dx \le L_\alpha|\Omega| \ \ \forall t \in [0,T], \\ 0 \le r(x) \le 1, \quad \int_\Omega r(x)dx \le L_d|\Omega| \end{cases}$$

- $CQW(t, \boldsymbol{x}, F, s, r)$ is given by

  $$CQW(U,F,s,r) = |F_{11}|^2 + \frac{a_\alpha}{s(\beta-\alpha)^2}|\beta F_{12} + F_{21}|^2 + \frac{a_\beta}{(1-s)(\beta-\alpha)^2}|\alpha F_{12} + F_{21}|^2 \qquad (74)$$

  $$\psi(F,s,r) = \frac{(\alpha(1-s)+\beta s)}{(\beta-\alpha)}\left(F_{21} + \lambda^-_{\alpha,\beta}(s)F_{12}\right)\left(F_{21} + \lambda^+_{\alpha,\beta}(s)F_{12}\right)$$

- First order laminate $\Longrightarrow$ (Regular effect on the optimal micro-structure) or (no second order laminates for $(RP^2_\omega)$).

$$\psi(F,s)=0 \Rightarrow \left(F_{21}+\frac{\lambda^+(s)+\lambda^-(s)}{2}F_{12}\right)^2=\frac{1}{4}(\lambda^+(s)-\lambda^-(s))^2|F_{12}|^2 \quad (75)$$

$$F_{21}=m(x,t)\left(\frac{\lambda^+(s)-\lambda^-(s)}{2}\right)|F_{12}|-\left(\frac{\lambda^+(s)+\lambda^-(s)}{2}\right)F_{12},\quad m(x,t)=\pm 1 \text{ in } (0,T)\times\Omega \quad (76)$$

*The relaxed formulation $(RP^2_\omega)$ is equivalent*

$$(RP^2_\omega):\qquad \min_{u,s,m}\int_0^T\int_\Omega CQW(x,t,u,s,m)dxdt$$

*subject to*

$$\begin{cases} u_{tt}-div\left(\frac{\lambda^+(s)+\lambda^-(s)}{2}\nabla u-m(x,t)\left(\frac{\lambda^+(s)-\lambda^-(s)}{2}\right)|\nabla u|\right)=0 & (0,T)\times\Omega,\\ u=0 & (0,T)\times\partial\Omega,\\ u(0,\cdot)=u^0,\quad u_t(0,\cdot)=u^1 & \Omega,\\ s\in L^\infty((0,T)\times\Omega;\{0,1\}),\quad |m(x,t)|=1 & \\ \|s\|_{L^1(\Omega)}\le L\|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0,T) \end{cases} \quad (77)$$

# Simplification of $(RP^2_\omega)$

$$\psi(F,s)=0 \Rightarrow \left(F_{21}+\frac{\lambda^+(s)+\lambda^-(s)}{2}F_{12}\right)^2=\frac{1}{4}(\lambda^+(s)-\lambda^-(s))^2|F_{12}|^2 \quad (75)$$

$$F_{21}=m(x,t)\left(\frac{\lambda^+(s)-\lambda^-(s)}{2}\right)|F_{12}|-\left(\frac{\lambda^+(s)+\lambda^-(s)}{2}\right)F_{12}, \quad m(x,t)=\pm 1 \text{ in } (0,T)\times\Omega \quad (76)$$

*The relaxed formulation $(RP^2_\omega)$ is equivalent*

$$(RP^2_\omega): \qquad \min_{u,s,m}\int_0^T\int_\Omega CQW(x,t,u,s,m)dxdt$$

*subject to*

$$\begin{cases} u_{tt}-div\left(\frac{\lambda^+(s)+\lambda^-(s)}{2}\nabla u - m(x,t)\left(\frac{\lambda^+(s)-\lambda^-(s)}{2}\right)|\nabla u|\right)=0 & (0,T)\times\Omega, \\ u=0 & (0,T)\times\partial\Omega, \\ u(0,\cdot)=u^0, \quad u_t(0,\cdot)=u^1 & \Omega, \\ s\in L^\infty((0,T)\times\Omega;\{0,1\}), \quad |m(x,t)|=1 & \\ \|s\|_{L^1(\Omega)}\le L\|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0,T) \end{cases} \quad (77)$$

# Simplification of $(RP^2_\omega)$

$$\psi(F,s)=0 \Rightarrow \left(F_{21}+\frac{\lambda^+(s)+\lambda^-(s)}{2}F_{12}\right)^2=\frac{1}{4}(\lambda^+(s)-\lambda^-(s))^2|F_{12}|^2 \tag{75}$$

$$F_{21}=m(x,t)\left(\frac{\lambda^+(s)-\lambda^-(s)}{2}\right)|F_{12}|-\left(\frac{\lambda^+(s)+\lambda^-(s)}{2}\right)F_{12}, \quad m(x,t)=\pm 1 \text{ in } (0,T)\times\Omega \tag{76}$$

**Theorem**

*The relaxed formulation $(RP^2_\omega)$ is equivalent*

$$(RP^2_\omega): \qquad \min_{u,s,m}\int_0^T\int_\Omega CQW(x,t,u,s,m)dxdt$$

*subject to*

$$\begin{cases} u_{tt}-div\left(\frac{\lambda^+(s)+\lambda^-(s)}{2}\nabla u-m(x,t)\left(\frac{\lambda^+(s)-\lambda^-(s)}{2}\right)|\nabla u|\right)=0 & (0,T)\times\Omega, \\ u=0 & (0,T)\times\partial\Omega, \\ u(0,\cdot)=u^0, \quad u_t(0,\cdot)=u^1 & \Omega, \\ s\in L^\infty((0,T)\times\Omega;\{0,1\}), \quad |m(x,t)|=1 & \\ \|s\|_{L^1(\Omega)}\le L\|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0,T) \end{cases} \tag{77}$$

# Problem $(P^2_\omega)$: Particular case : $(a_\alpha, a_\beta) = (\alpha, \beta)$

The relaxed formulation of

$$(P^2_\omega): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + [\alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega)]|\nabla u|^2) dxdt \tag{78}$$

subject to

$$\begin{cases} u_{tt} - div\Big([\alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega)]\nabla u\Big) = 0 & (0,T) \times \Omega, \\ u = 0 & (0,T) \times \partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \Omega, \\ \mathcal{X}_\omega \in L^\infty((0,T) \times \Omega; \{0,1\}), & \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} \leq L \|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0,T) \end{cases} \tag{79}$$

$$m = \min_{u,s} \int_0^T \int_\Omega \left( u_t(t,x)^2 + \frac{1}{(\alpha^{-1} s + \beta^{-1}(1-s))} u_x(t,x)^2 \right) dxdt \tag{80}$$

*subject to*

$$\begin{cases} u_{tt} - div(\frac{1}{\alpha^{-1} s(t,x) + \beta^{-1}(1 - s(t,x))} \nabla u) = 0 & in \quad (0,T) \times \Omega, \\ u = 0 & on \quad (0,T) \times \partial\Omega, \\ u(0,x) = u^0(x), \ u_t(0,x) = u^1(x) & in \quad \Omega, \\ 0 \leq s(t,x) \leq 1, \int_\Omega s(t,x)\, dx \leq L|\Omega| & in \quad [0,T] \end{cases} \tag{81}$$

*and the optimal measure is recovered with first order laminates with normal* $(0,1)$.

# Problem $(P^2_\omega)$: Particular case : $(a_\alpha, a_\beta) = (\alpha, \beta)$

The relaxed formulation of

$$(P^2_\omega): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + [\alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega)]|\nabla u|^2) dxdt \tag{78}$$

subject to

$$\begin{cases} u_{tt} - div\Big([\alpha \mathcal{X}_\omega + \beta(1-\mathcal{X}_\omega)]\nabla u\Big) = 0 & (0,T) \times \Omega, \\ u = 0 & (0,T) \times \partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \Omega, \\ \mathcal{X}_\omega \in L^\infty((0,T) \times \Omega; \{0,1\}), & \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} \leq L \|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0,T) \end{cases} \tag{79}$$

**Theorem**

$$m = \min_{u,s} \int_0^T \int_\Omega \left( u_t(t,x)^2 + \frac{1}{(\alpha^{-1} s + \beta^{-1}(1-s))} u_x(t,x)^2 \right) dxdt \tag{80}$$

*subject to*

$$\begin{cases} u_{tt} - div\big(\frac{1}{\alpha^{-1} s(t,x) + \beta^{-1}(1 - s(t,x))} \nabla u\big) = 0 & in \quad (0,T) \times \Omega, \\ u = 0 & on \quad (0,T) \times \partial\Omega, \\ u(0,x) = u^0(x), \ u_t(0,x) = u^1(x) & in \quad \Omega, \\ 0 \leq s(t,x) \leq 1, \ \int_\Omega s(t,x)\, dx \leq L|\Omega| & in \quad [0,T] \end{cases} \tag{81}$$

*and the optimal measure is recovered with first order laminates with normal* $(0,1)$.

# Problem $(P^2_\omega)$: Particular case : $(a_\alpha, a_\beta) = (1, 1)$

The relaxed formulation of

$$(P^2_\omega): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + |\nabla u|^2) dxdt \tag{82}$$

subject to

$$\begin{cases} u_{tt} - div\Big([\alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega)]\nabla u\Big) = 0 & (0, T) \times \Omega, \\ u = 0 & (0, T) \times \partial\Omega, \\ u(0, \cdot) = u^0, \quad u_t(0, \cdot) = u^1 & \Omega, \\ \mathcal{X}_\omega \in L^\infty((0, T) \times \Omega; \{0, 1\}), & \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} \le L\|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0, T) \end{cases} \tag{83}$$

$$m = \min_{u,s} \int_0^T \int_\Omega \Big( u_t(t, x)^2 + \Big[\alpha s(t, x) + \beta(1 - s(t, x))\Big] u_x(t, x)^2 \Big) dxdt \tag{84}$$

*subject to*

$$\begin{cases} u_{tt} - div([\alpha s(t, x) + \beta(1 - s(t, x))]\nabla u) = 0 & in \quad (0, T) \times \Omega, \\ u = 0 & on \quad (0, T) \times \partial\Omega, \\ u(0, x) = u^0(x), \ u_t(0, x) = u^1(x) & in \quad \Omega, \\ 0 \le s(t, x) \le 1, \int_\Omega s(t, x)\, dx \le L|\Omega| & in \quad [0, T] \end{cases} \tag{85}$$

*and the optimal measure is recovered with first order laminates with normal* $(1, 0)$.

The relaxed formulation of

$$(P^2_\omega): \quad \inf_{\mathcal{X}_\omega} I(\mathcal{X}_\omega) = \int_0^T \int_\Omega (|u_t|^2 + |\nabla u|^2) dxdt \tag{82}$$

subject to

$$\begin{cases} u_{tt} - div\Big([\alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega)]\nabla u\Big) = 0 & (0, T) \times \Omega, \\ u = 0 & (0, T) \times \partial\Omega, \\ u(0, \cdot) = u^0, \quad u_t(0, \cdot) = u^1 & \Omega, \\ \mathcal{X}_\omega \in L^\infty((0, T) \times \Omega; \{0, 1\}), & \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} \le L\|\mathcal{X}_\Omega\|_{L^1(\Omega)} & (0, T) \end{cases} \tag{83}$$

**Theorem**

$$m = \min_{u,s} \int_0^T \int_\Omega \Big( u_t(t, x)^2 + \Big[\alpha s(t, x) + \beta(1 - s(t, x))\Big] u_x(t, x)^2 \Big) dxdt \tag{84}$$

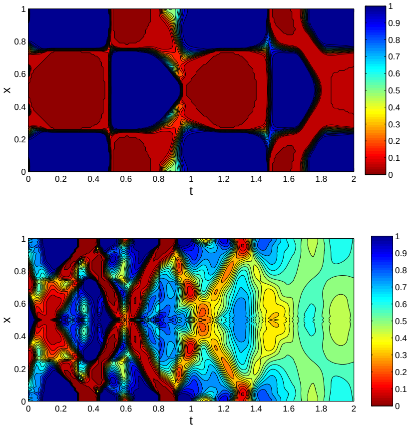
*subject to*

$$\begin{cases} u_{tt} - div([\alpha s(t, x) + \beta(1 - s(t, x))]\nabla u) = 0 & in \quad (0, T) \times \Omega, \\ u = 0 & on \quad (0, T) \times \partial\Omega, \\ u(0, x) = u^0(x), \; u_t(0, x) = u^1(x) & in \quad \Omega, \\ 0 \le s(t, x) \le 1, \int_\Omega s(t, x)\, dx \le L|\Omega| & in \quad [0, T] \end{cases} \tag{85}$$

*and the optimal measure is recovered with first order laminates with normal* $(1, 0)$.

# Some numerical results for $(RP^2_\omega)$

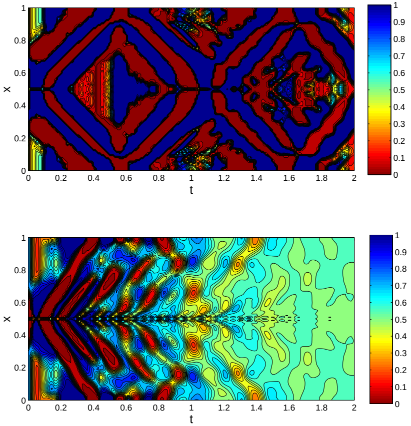
Let $\Omega = (0,1)$, $T = 2$ and $(u^0, u^1) = (\sin(\pi x), 0)$ and $L = 1/2$




Iso-values of the optimal density $s$ on $(0,T) \times \Omega$ Top: $(\alpha, \beta) = (1, 1.1)$ -Bottom:$(\alpha, \beta) = (1, 10)$

# Some numerical results for $(RP^2_\omega)$

Let $\Omega = (0,1)$, $T = 2$ and $(u^0, u^1) = (e^{-0.5(x-0.5)^2}, 0)$ and $L = 1/2$




Iso-values of the optimal density $s$ on $(0,T) \times \Omega$ Top: $(\alpha, \beta) = (1, 1.1)$ -Bottom:$(\alpha, \beta) = (1, 10)$

$$(\mathrm{P}_t) \quad \text{Minimize over } \mathcal{X} : \quad J_t(\mathcal{X}) = \frac{1}{2}\int_0^T\int_\Omega K(t,x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt$$

$$\begin{cases} (\beta(t,x)u(t,x))' - \operatorname{div}\,(K(t,x)\nabla u(t,x)) = f(t,x) & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{86}$$

with

$$\beta(t,x) = \mathcal{X}(t,x)\beta_1 + (1-\mathcal{X}(t,x))\beta_2, K(t,x) = \mathcal{X}(t,x)k_1 I_N + (1-\mathcal{X}(t,x))k_2 I_N,$$

**Theorem** (AM, Pedregal, Periago, JMPA 2008)

$$(RP_t) \quad \textit{Minimize over } \left(\theta, \overline{G}, u\right) : \quad \overline{J_t}(\theta,\overline{G},u) = \frac{1}{2}\int_0^T\int_\Omega \left[k_1\frac{\left|\overline{G}-k_2\nabla u\right|^2}{\theta(k_1-k_2)^2} + k_2\frac{\left|\overline{G}-k_1\nabla u\right|^2}{(1-\theta)(k_2-k_1)^2}\right] dxdt$$

$$\begin{cases} G \in L^2\left((0,T)\times\Omega;\mathbb{R}^{N+1}\right), & u\in H^1((0,T)\times\Omega;\mathbb{R}), \\ ((\theta\beta_1+(1-\theta)\beta_2)u)' - div\,\overline{G} = 0 & dans\ H^{-1}((0,T)\times\Omega), \\ u|_{\partial\Omega} = 0 \quad p.\,p.\ t\in[0,T], & u(0) = u_0 \quad dans\ \Omega, \\ \theta\in L^\infty((0,T)\times\Omega;[0,1]), \quad \int_\Omega \theta(t,x)\,dx = L|\Omega| \quad p.p.\ t\in(0,T). & \end{cases}$$

*is a relaxation of $(P_t)$ in the following sense :*

(i) *$(RP_t)$ is well-posed,*
(ii) *the infimum of $(VP_t)$ equals the minimum of $(RP_t)$, and*
(iii) *the Young measure associated with $(RP_t)$ (et donc la micro-structure optimale de $(VP_t)$) is expressed in term of an explicit first order laminate.*

# Optimization of the heat flux: Div-Rot Young Measure

$$(\mathrm{P}_t) \quad \text{Minimize over } \mathcal{X} : \quad J_t(\mathcal{X}) = \frac{1}{2}\int_0^T\int_\Omega K(t,x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt$$

$$\begin{cases} (\beta(t,x)u(t,x))' - \mathrm{div}\,(K(t,x)\nabla u(t,x)) = f(t,x) & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{86}$$

with

$$\beta(t,x) = \mathcal{X}(t,x)\beta_1 + (1-\mathcal{X}(t,x))\beta_2,\, K(t,x) = \mathcal{X}(t,x)k_1 I_N + (1-\mathcal{X}(t,x))k_2 I_N,$$

**Theorem** (AM, Pedregal, Periago, JMPA 2008)

$$(RP_t) \quad \textit{Minimize over } \left(\theta,\overline{G},u\right) : \quad \overline{J_t}(\theta,\overline{G},u) = \frac{1}{2}\int_0^T\int_\Omega \left[k_1\frac{\left|\overline{G}-k_2\nabla u\right|^2}{\theta(k_1-k_2)^2} + k_2\frac{\left|\overline{G}-k_1\nabla u\right|^2}{(1-\theta)(k_2-k_1)^2}\right] dxdt$$

$$\begin{cases} G \in L^2\left((0,T)\times\Omega;\mathbb{R}^{N+1}\right), & u\in H^1((0,T)\times\Omega;\mathbb{R}), \\ ((\theta\beta_1+(1-\theta)\beta_2)u)' - div\,\overline{G} = 0 & dans\ H^{-1}((0,T)\times\Omega), \\ u|_{\partial\Omega} = 0 \quad p.\,p.\ t\in[0,T], & u(0) = u_0 \quad dans\ \Omega, \\ \theta\in L^\infty((0,T)\times\Omega;[0,1]), \quad \int_\Omega\theta(t,x)\,dx = L|\Omega| \quad p.p.\ t\in(0,T). \end{cases}$$

*is a relaxation of ($P_t$) in the following sense :*

(i) *($RP_t$) is well-posed,*

(ii) *the infimum of ($VP_t$) equals the minimum of ($RP_t$), and*

(iii) *the Young measure associated with ($RP_t$) (et donc la micro-structure optimale de ($VP_t$)) is expressed in term of an explicit first order laminate.*

$$(P_t) \quad \text{Minimize over } \mathcal{X} : \quad J_t(\mathcal{X}) = \frac{1}{2}\int_0^T\int_\Omega K(t,x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt$$

$$\begin{cases} (\beta(t,x)u(t,x))' - \text{div}\,(K(t,x)\nabla u(t,x)) = f(t,x) & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{86}$$

with

$$\beta(t,x) = \mathcal{X}(t,x)\beta_1 + (1-\mathcal{X}(t,x))\beta_2, \; K(t,x) = \mathcal{X}(t,x)k_1 I_N + (1-\mathcal{X}(t,x))k_2 I_N,$$

**Theorem** (AM, Pedregal, Periago, JMPA 2008)

$$(RP_t) \quad \textit{Minimize over } \left(\theta,\overline{G},u\right) : \quad \overline{J_t}(\theta,\overline{G},u) = \frac{1}{2}\int_0^T\int_\Omega \left[k_1\frac{\left|\overline{G}-k_2\nabla u\right|^2}{\theta(k_1-k_2)^2} + k_2\frac{\left|\overline{G}-k_1\nabla u\right|^2}{(1-\theta)(k_2-k_1)^2}\right] dxdt$$

$$\begin{cases} G \in L^2\left((0,T)\times\Omega;\mathbb{R}^{N+1}\right), & u \in H^1((0,T)\times\Omega;\mathbb{R}), \\ ((\theta\beta_1+(1-\theta)\beta_2)u)' - div\,\overline{G} = 0 & dans\; H^{-1}((0,T)\times\Omega), \\ u|_{\partial\Omega} = 0 \quad p.\,p.\; t\in[0,T], & u(0) = u_0 \quad dans\;\Omega, \\ \theta \in L^\infty((0,T)\times\Omega;[0,1]), \quad \int_\Omega \theta(t,x)\,dx = L|\Omega| \quad p.p.\; t\in(0,T). & \end{cases}$$

*is a relaxation of $(P_t)$ in the following sense :*

(i) *$(RP_t)$ is well-posed,*

(ii) *the infimum of $(VP_t)$ equals the minimum of $(RP_t)$, and*

(iii) *the Young measure associated with $(RP_t)$ (et donc la micro-structure optimale de $(VP_t)$) is expressed in term of an explicit first order laminate.*

# Optimization of the heat flux: Div-Rot Young Measure

$$(\mathrm{P}_t)\quad \text{Minimize over } \mathcal{X}: \quad J_t(\mathcal{X}) = \frac{1}{2}\int_0^T\int_\Omega K(t,x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt$$

$$\begin{cases} (\beta(t,x)u(t,x))' - \text{div}\,(K(t,x)\nabla u(t,x)) = f(t,x) & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{86}$$

with

$$\beta(t,x) = \mathcal{X}(t,x)\beta_1 + (1-\mathcal{X}(t,x))\beta_2,\; K(t,x) = \mathcal{X}(t,x)k_1 I_N + (1-\mathcal{X}(t,x))k_2 I_N,$$

**Theorem** (AM, Pedregal, Periago, JMPA 2008)

$$(RP_t)\quad \textit{Minimize over } \left(\theta,\overline{G},u\right): \quad \overline{J_t}(\theta,\overline{G},u) = \frac{1}{2}\int_0^T\int_\Omega\left[k_1\frac{\left|\overline{G}-k_2\nabla u\right|^2}{\theta(k_1-k_2)^2} + k_2\frac{\left|\overline{G}-k_1\nabla u\right|^2}{(1-\theta)(k_2-k_1)^2}\right]dxdt$$

$$\begin{cases} G\in L^2\left((0,T)\times\Omega;\mathbb{R}^{N+1}\right), & u\in H^1((0,T)\times\Omega;\mathbb{R}), \\ ((\theta\beta_1+(1-\theta)\beta_2)u)' - div\,\overline{G} = 0 & dans\; H^{-1}((0,T)\times\Omega), \\ u|_{\partial\Omega} = 0 \quad p.\,p.\; t\in[0,T], & u(0) = u_0 \quad dans\;\Omega, \\ \theta\in L^\infty((0,T)\times\Omega;[0,1]), \quad \int_\Omega\theta(t,x)\,dx = L|\Omega| \quad p.p.\; t\in(0,T). & \end{cases}$$

*is a relaxation of $(P_t)$ in the following sense :*

(i) *$(RP_t)$ is well-posed,*

(ii) *the infimum of $(VP_t)$ equals the minimum of $(RP_t)$, and*

(iii) *the Young measure associated with $(RP_t)$ (et donc la micro-structure optimale de $(VP_t)$) is expressed in term of an explicit first order laminate.*

# Problem 4: Optimal design and exact controllability

[AM 06,07,08] [Asch-Lebeau 99], [Chambolle-Santosa 03], [Periago 09]

- Let $\Omega \subset \mathbb{R}^N$, $N = 1, 2$, $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, $L \in (0, 1)$, $T > 0$ [8]

$$(P_\omega^4): \quad \inf_{\mathcal{X}_\omega} \|v_\omega\|^2_{L^2(\omega\times(0,T))} \tag{87}$$

where $v_\omega$ is an exact control, supported on $\omega \times (0, T)$ for

$$\begin{cases} u_{tt} - \Delta u = v_\omega \mathcal{X}_\omega & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u^0, \quad u_t(0,\cdot) = u^1 & \text{in } \Omega \end{cases} \tag{88}$$

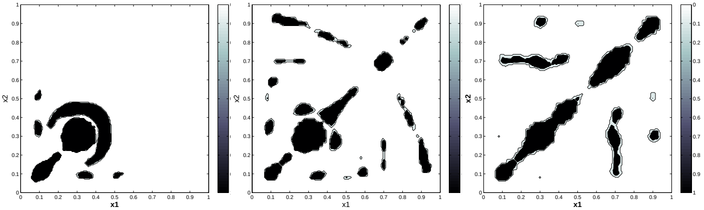
and subject to

$$\begin{cases} \text{The system (88) may be observed from } \quad \omega \times (0,T), \\ \|\mathcal{X}_\omega\|_{L^1(\Omega)} \le L\|\mathcal{X}_\Omega\|_{L^1(\Omega)} \end{cases} \tag{89}$$

[8] AM, *Optimal design of the support of the control for the 2-D wave equation*, C.R.Acad Sci., Paris Serie I (2006)

# Some numerical results for $(RP^4_\omega)$

Let $\Omega = (0,1)^2$, and $(u^0, u^1) = (e^{-80(x_1-0.3)^2-80(x_2-0.3)^2}, 0)$ and $L = 1/10$




Iso-value of the optimal density $s$ on $\Omega$ for $T = 0.5$, $T = 1$, $T = 3$

- $\{x \in \Omega, 0 < s(x) < 1\} = \emptyset$, $(P^4_\omega) = (RP^4_\omega)$ and is well-posed

[Ph. Destuynder 87,88,89]

- Let $\omega \subset \Omega \in \mathbb{R}^2$, $0 < \alpha \leq \beta$, $u_0 \in H^{1/2}(\Gamma_0)$, $g \in L^2(\Gamma_g)$ et $u$ solution de

$$\begin{cases} -\,div(a_{\mathcal{X}_\omega} \nabla u) = 0, \quad a_{\mathcal{X}_\omega} = \alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega) & \Omega, \\ u = u_0 & \Gamma_0 \subset \partial\Omega, \\ \beta\, \nabla u \cdot \boldsymbol{\nu} = g & \Gamma_g \subset \partial\Omega. \end{cases} \tag{90}$$

- For any $L \in (0,1)$, the problem is

$$(P): \quad \inf_{\mathcal{X}_\omega \in \mathcal{X}_L} g_\psi(u, \mathcal{X}_\omega) = \int_\Omega a_{\mathcal{X}_\omega}(\boldsymbol{x})(A_\psi(\boldsymbol{x})\nabla u, \nabla u)dx, \quad A_\psi = \frac{1}{2}\begin{pmatrix} \psi_{1,1} & 2\psi_{1,2} \\ 0 & -\psi_{1,1} \end{pmatrix},$$

$$\mathcal{X}_L = \{\mathcal{X} \in L^\infty(\Omega, \{0,1\}), \mathcal{X} = 0 \text{ on } \mathcal{D} \cup \partial\Omega, \|\mathcal{X}\|_{L^1(\Omega)} = L|\Omega|\} \tag{91}$$

$g_\psi$ - Energy release rate

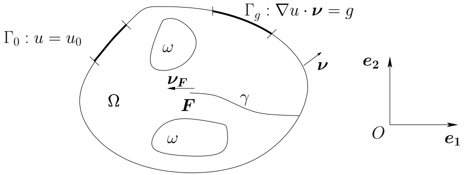



Figure: Domain $\Omega$ with a cut $\Gamma_0$ - Optimization of the distribution $(\alpha, \beta)$.

[Ph. Destuynder 87,88,89]

- Let $\omega \subset \Omega \in \mathbb{R}^2$, $0 < \alpha \leq \beta$, $u_0 \in H^{1/2}(\Gamma_0)$, $g \in L^2(\Gamma_g)$ et $u$ solution de

$$\begin{cases} -\,div(a_{\mathcal{X}_\omega} \nabla u) = 0, \quad a_{\mathcal{X}_\omega} = \alpha \mathcal{X}_\omega + \beta(1 - \mathcal{X}_\omega) & \Omega, \\ u = u_0 & \Gamma_0 \subset \partial\Omega, \\ \beta\, \nabla u \cdot \boldsymbol{\nu} = g & \Gamma_g \subset \partial\Omega. \end{cases} \quad (90)$$

- For any $L \in (0,1)$, the problem is

$$(P): \quad \inf_{\mathcal{X}_\omega \in \mathcal{X}_L} g_\psi(u, \mathcal{X}_\omega) = \int_\Omega a_{\mathcal{X}_\omega}(\boldsymbol{x})(A_\psi(\boldsymbol{x})\nabla u, \nabla u)dx, \quad A_\psi = \frac{1}{2}\begin{pmatrix} \psi_{1,1} & 2\psi_{1,2} \\ 0 & -\psi_{1,1} \end{pmatrix},$$

$$\mathcal{X}_L = \{\mathcal{X} \in L^\infty(\Omega, \{0,1\}), \mathcal{X} = 0 \text{ on } \mathcal{D} \cup \partial\Omega, \|\mathcal{X}\|_{L^1(\Omega)} = L|\Omega|\} \quad (91)$$

$g_\psi$ - Energy release rate

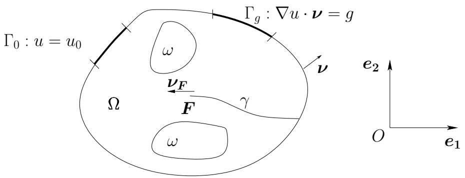



Figure: Domain $\Omega$ with a cut $\Gamma_0$ - Optimization of the distribution $(\alpha, \beta)$.

**Theorem** (AM, Pedregal (COCV 09) )

*The problem*

$$(RP): \qquad \min_{s,t} I(s,t) = \int_\Omega g_\psi(\overline{u}, s)dx \tag{92}$$

*subject to*

$$\begin{cases} s \in L^\infty(\Omega, [0,1]), s = 0 \ in \ \mathcal{D} \cup \partial\Omega, \|s\|_{L^1(\Omega)} = L|\Omega|, \\ t \in L^\infty(\Omega, \mathbb{R}^2), \quad |t| = 1, \end{cases} \tag{93}$$

*where* $\overline{u} = \overline{u}(s,t)$ *is solution of the nonlinear problem*

$$\begin{cases} div\Big(A(s)\nabla\overline{u} + B(s)|\nabla\overline{u}|t\Big) = 0, & in \ \Omega, \\ \overline{u} = u_0, & on \ \Gamma_0, \\ \beta\nabla\overline{u}\cdot\boldsymbol{\nu} = g, & on \ \Gamma_g. \end{cases} \tag{94}$$

$$A(s) = \frac{\lambda^+(s) + \lambda^-(s)}{2} = \frac{2\alpha\beta + s(1-s)(\beta-\alpha)^2}{2(\alpha(1-s)+\beta s)}, \quad B(s) = \frac{\lambda^+(s) - \lambda^-(s)}{2} = \frac{s(1-s)(\beta-\alpha)^2}{2(\alpha(1-s)+\beta s)}. \tag{95}$$

*is a relaxion of the initial problem* $(P)$.

**Remark**

*If* $s \in \{0,1\}$*, then* $A(s) = \alpha s + \beta(1-s) = a_{\mathcal{X}_\omega}$*,* $B(s) = 0$ *and* $u \equiv \overline{u}$.

**Theorem** (AM, Pedregal (COCV 09) )

*The problem*

$$(RP): \qquad \min_{s,t} I(s,t) = \int_\Omega g_\psi(\overline{u}, s)dx \tag{92}$$

*subject to*

$$\begin{cases} s \in L^\infty(\Omega,[0,1]), s = 0 \ in \ \mathcal{D} \cup \partial\Omega, \|s\|_{L^1(\Omega)} = L|\Omega|, \\ t \in L^\infty(\Omega,\mathbb{R}^2), \quad |t| = 1, \end{cases} \tag{93}$$

*where* $\overline{u} = \overline{u}(s,t)$ *is solution of the nonlinear problem*

$$\begin{cases} div\Big(A(s)\nabla\overline{u} + B(s)|\nabla\overline{u}|t\Big) = 0, & in \ \Omega, \\ \overline{u} = u_0, & on \ \Gamma_0, \\ \beta\nabla\overline{u}\cdot\boldsymbol{\nu} = g, & on \ \Gamma_g. \end{cases} \tag{94}$$

$$A(s) = \frac{\lambda^+(s) + \lambda^-(s)}{2} = \frac{2\alpha\beta + s(1-s)(\beta-\alpha)^2}{2(\alpha(1-s)+\beta s)}, \quad B(s) = \frac{\lambda^+(s) - \lambda^-(s)}{2} = \frac{s(1-s)(\beta-\alpha)^2}{2(\alpha(1-s)+\beta s)}. \tag{95}$$

*is a relaxion of the initial problem* $(P)$.

**Remark**

*If* $s \in \{0,1\}$, *then* $A(s) = \alpha s + \beta(1-s) = a_{\mathcal{X}_\omega}$, $B(s) = 0$ *and* $u \equiv \overline{u}$.

$$\Omega = (0,1),\ \gamma = [0.5,1] \times \{1/2\},\ L = 2/5,\ u = 0 \text{ on } \{0\} \times [0,1],\ u = 1/2 \text{ on } [0.5,0.8] \times \{1\} \qquad (96)$$

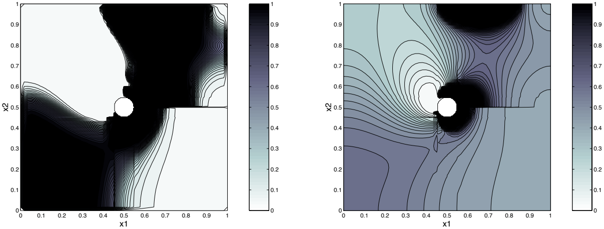



Figure: Iso-values of the optimal density : $(\alpha,\beta) = (1,2)$ and $(\alpha,\beta) = (1,10)$.

# Two related works in progress

- Non linear heat equation [Fernandez-Cara, Zuazua (00,01)]

$$\Omega \subset \mathbb{R}^N, \omega \subset \Omega, \tag{97}$$

$$\begin{cases} u_t - \Delta u + f(u) = v\mathcal{X}_\omega, & (0,T) \times \Omega, \\ u = 0 & (0,T) \times \partial\Omega, \\ u = u^0 \in L^2(\Omega) & \{0\} \times \Omega \end{cases} \tag{98}$$

$\Rightarrow$ Optimal position of the support of the control $v$ in order to prevent the blow up of $u$ : $\inf_{\mathcal{X}_\omega} \|v\|_{L^2((0,T)\times\omega)}$

- Null controllability of shell - $\Omega \subset \mathbb{R}^2$, $\omega \subset \Omega$

$$\begin{cases} y_\epsilon'' + A_M y_\epsilon + \epsilon^2 A_F y_\epsilon = 0 & (0,T) \times \Omega \\ (y_\epsilon^0, y_\epsilon^1) & \{0\} \times \Omega \\ y_\epsilon = v_\epsilon & (0,T) \times \partial\Omega \end{cases} \tag{99}$$

$$(\lambda(\xi), \mu(\xi)) = (\lambda_\alpha, \mu_\alpha)\mathcal{X}_\omega(\xi) + (\lambda_\beta, \mu_\beta)(1 - \mathcal{X}_\omega(\xi)), \quad \xi \in \omega, \quad \omega \subset \Omega$$

$$\inf_{\omega \subset \Omega} \sup_{\phi^0, \phi^1} \frac{\|\phi^0, \phi^1\|^2_{V \times H}}{\int_0^T \int_{\partial\Omega} b_M(\phi, \phi) d\sigma dt} \tag{100}$$

# Two related works in progress

- Non linear heat equation [Fernandez-Cara, Zuazua (00,01)]

$$\Omega \subset \mathbb{R}^N, \omega \subset \Omega, \quad (97)$$

$$\begin{cases} u_t - \Delta u + f(u) = v\mathcal{X}_\omega, & (0,T)\times\Omega, \\ u = 0 & (0,T)\times\partial\Omega, \\ u = u^0 \in L^2(\Omega) & \{0\}\times\Omega \end{cases} \quad (98)$$

$\Rightarrow$ Optimal position of the support of the control $v$ in order to prevent the blow up of $u$ : $\inf_{\mathcal{X}_\omega} \|v\|_{L^2((0,T)\times\omega)}$

- Null controllability of shell - $\Omega \subset \mathbb{R}^2$, $\omega \subset \Omega$

$$\begin{cases} \boldsymbol{y}_\epsilon'' + \boldsymbol{A_M y_\epsilon} + \epsilon^2 \boldsymbol{A_F y_\epsilon} = 0 & (0,T)\times\Omega \\ (\boldsymbol{y}_\epsilon^0, \boldsymbol{y}_\epsilon^1) & \{0\}\times\Omega \\ \boldsymbol{y}_\epsilon = \boldsymbol{v}_\epsilon & (0,T)\times\partial\Omega \end{cases} \quad (99)$$

$$(\lambda(\boldsymbol{\xi}), \mu(\boldsymbol{\xi})) = (\lambda_\alpha, \mu_\alpha)\mathcal{X}_\omega(\boldsymbol{\xi}) + (\lambda_\beta, \mu_\beta)(1 - \mathcal{X}_\omega(\boldsymbol{\xi})), \quad \boldsymbol{\xi} \in \omega, \quad \omega \subset \Omega$$

$$\inf_{\omega\subset\Omega} \sup_{\boldsymbol{\phi^0},\boldsymbol{\phi^1}} \frac{\|\boldsymbol{\phi^0}, \boldsymbol{\phi^1}\|^2_{\boldsymbol{V}\times\boldsymbol{H}}}{\int_0^T\int_{\partial\Omega} b_M(\phi,\phi)d\sigma dt} \quad (100)$$