Controllability of the heat equation submitted to unilateral constraint

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Abstract

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1 Introduction

Let $T > 0$ and $Q_T = (0,T) \times (0,1)$. We consider the following system

$$\begin{cases} y' - y_{xx} = 0, & (t,x) \in Q_T, \\ y(t,0) = u(t), & t \in (0,T), \\ y(t,1) \geq \psi(t),\ y_x(t,1) \geq 0,\ (y(t,1) - \psi(t)) y_x(t,1) = 0, & t \in (0,T), \\ y(0,x) = y^0(x), & x \in (0,1) \end{cases} \tag{1.1}$$

where the symbol $'$ denotes the derivative with respect to the variable t.

2 The Control Dirichlet-to-Neumann map of a linear system

Let $T > 0$, $Q_T = (0,T) \times (0,1)$ and consider the following system:

$$\begin{cases} \epsilon\phi'' + \phi' - \phi_{xx} = 0, & (t,x) \in Q_T, \\ \phi(t,0) = u(t), & t \in (0,T), \\ \phi(t,1) = f(t), & t \in (0,T), \\ \phi(0,x) = \phi^0(x), \quad \phi'(0,x) = \phi^1(x), & x \in (0,1) \end{cases} \tag{2.2}$$

where $u \in L^2(0,T)$ is a control function and $f \in L^2(0,T)$ is given.

Given $\left(\phi^0, \phi^1, f\right)$, our aim is to find a family of explicit controls u for which the solution ϕ of (2.2) satisfies $\phi(T) = \phi'(T) = 0$ in $(0,1)$. We take $T \in]2,3]\sqrt{\epsilon}$.

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Setting

$$p = \phi' - \frac{1}{\sqrt{\epsilon}}\phi_x,\ q = \phi' + \frac{1}{\sqrt{\epsilon}}\phi_x, \tag{2.3}$$

it follows that (2.2) is equivalent to

$$\begin{cases} p' + \frac{1}{\sqrt{\epsilon}}p_x + \frac{1}{2\epsilon}(p+q) = 0, & (t,x) \in Q_T, \\ q' - \frac{1}{\sqrt{\epsilon}}q_x + \frac{1}{2\epsilon}(p+q) = 0, & (t,x) \in Q_T, \\ (p+q)(\,\cdot\,,0) = 2u', & t \in (0,T), \\ (p+q)(\,\cdot\,,1) = 2f' & t \in (0,T), \\ p^0 = \phi^1 - \frac{1}{\sqrt{\epsilon}}\phi^0_x,\ q^0 = \phi^1 + \frac{1}{\sqrt{\epsilon}}\phi^0_x, & x \in (0,1). \end{cases} \tag{2.4}$$

We introduce $P = (p,q)$ the solution of

$$P' = \frac{1}{\sqrt{\epsilon}}MP_x - \frac{1}{2\epsilon}AP \tag{2.5}$$

with

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{2.6}$$

leading to

$$P(t,x(t)) = e^{-\frac{t}{2\epsilon}A}P(t_0,x(t_0)), \quad \forall t \geq t_0$$

where the characteristic $x(t)$ satisfies

$$x'(t) = \pm\frac{1}{\sqrt{\epsilon}}$$

Remark 2.1

$$e^{-\frac{t}{2\epsilon}A} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{1}{2}e^{-\frac{t}{\epsilon}}A \tag{2.7}$$

2.1 $p(x,T)$ in term of p^0 and q^0

Let

$$\overline{x} = \frac{T - 2\sqrt{\epsilon}}{\sqrt{\epsilon}} \in (0,1)$$

Let $x_0 \in (\overline{x},1)$ (two reflections at $x=0$ then $x=1$ before to reach $t=0$). We got

$$\begin{cases} p(x_0,T) = a(T)p(0,t_1) + b(T)q(0,t_1), \\ p(0,t_1) = 2u'(t_1) - q(0,t_1), \\ q(0,t_1) = b(t_1)p(1,t_2) + c(t_1)q(1,t_2), \\ q(1,t_2) = 2f'(t_2) - p(1,t_2), \\ p(1,t_2) = a(t_2)p(x_3,0) + b(t_2)q(x_3,0) = a(t_2)p^0(x_3) + b(t_2)q^0(x_3) \end{cases}$$

leading to

$$\begin{aligned} p(x_0,T) =& 2a(T)u'(t_1) \\ &+2c(t_1)[b(T)-a(T)]f'(t_2) \\ &+[b(T)-a(T)][b(t_1)-c(t_1)](a(t_2)p^0(x_3)+b(t_2)q^0(x_3)) \end{aligned} \tag{2.8}$$

with

$$\begin{aligned} t_1 &= T - \sqrt{\epsilon} x_0 \in (T - \sqrt{\epsilon}, 2\sqrt{\epsilon}) \\ t_2 &= t_1 - \sqrt{\epsilon} = T - \sqrt{\epsilon}(x_0 + 1) \in (T - 2\sqrt{\epsilon}, \sqrt{\epsilon}), \\ x_3 &= \frac{\sqrt{\epsilon} - t_2}{\sqrt{\epsilon}} \in (0, \frac{3\sqrt{\epsilon} - T}{\sqrt{\epsilon}}) \end{aligned} \tag{2.9}$$

and

$$\begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix} := e^{-\frac{t}{2\epsilon} A}. \tag{2.10}$$

Writing that $p(x_0, T) = 0$, we get that, for all $t_1 \in (T - \sqrt{\epsilon}, 2\sqrt{\epsilon})$, a first relation between u' and f':

$$\begin{aligned} 0 =& 2a(T)u'(t_1) \\ &+ 2c(t_1)[b(T) - a(T)]f'(t_1 - \sqrt{\epsilon}) \\ &+ [b(T) - a(T)][b(t_1) - c(t_1)]\Big(a(t_1 - \sqrt{\epsilon})p^0(\frac{2\sqrt{\epsilon} - t_1}{\sqrt{\epsilon}}) + b(t_1 - \sqrt{\epsilon})q^0(\frac{2\sqrt{\epsilon} - t_1}{\sqrt{\epsilon}})\Big) \end{aligned} \tag{2.11}$$

Let now $x_0 \in (1, \overline{x})$ (three reflections at $x = 0$ then $x = 1$ then $x = 0$ before to reach $t = 0$). We got .....

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