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Optimal design of the damping set for the stabilization of the wave equation

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Abstract

We consider the problem of optimizing the shape and position of the damping set for the internal stabilization of the linear wave equation in $\mathbb{R}^N$, $N = 1, 2$. In a first theoretical part, we reformulate the problem into an equivalent non-convex vector variational one using a characterization of free divergence vector fields. Then, by means of gradient Young measures, we obtain a relaxed formulation of the problem in which the original cost density is replaced by its constrained quasi-convexification. This implies that the new relaxed problem is well-posed in the sense that there exists a minimizer and, in addition, the infimum of the original problem coincides with the minimum of the relaxed one. In a second numerical part, we address the resolution of the relaxed problem using a first order gradient descent method. We present some numerical experiments which highlight the influence of the over-damping phenomena and show that for large values of the damping potential the original problem has no a minimizer. We then propose a penalization technique to recover the minimizing sequences of the original problem from the optimal solution of the relaxed one.

1 Introduction

Let us consider the following damped wave equation

$$\left\{\begin{array}{ll} u_{tt} - \Delta u + a\,(x)\,\mathcal{X}_\omega\,(x)\,u_t = 0 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1 & \text{in } \Omega, \end{array}\right. \tag{1}$$

where $\Omega \subset C^2(\mathbb{R}^N)$, $N = 1, 2$, is a bounded domain. $\mathcal{X}_\omega$ designates the characteristic function of ω, a subset of Ω of positive Lebesgue measure and independent of the time $t \in (0, T)$. Moreover, the damping potential $a \in L^\infty(\Omega; \mathbb{R}^+)$ is such that

$$a\,(x) \geq \alpha > 0 \quad \text{a. e. } x \in \omega. \tag{2}$$

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We assume that the initial data (u_0, u_1) are in $H_0^1(\Omega) \times L^2(\Omega)$ and are independent of both ω and a. System (1) is then well-posed and there exists a unique weak solution [10] such that

$$u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right). \tag{3}$$

It is also well-known that system (1) models, for instance, the stabilization of an elastic string (resp. membrane) when $N = 1$ (or resp. $N = 2$) by an internal dissipative mechanism located on the subset ω. The unknown $u(t,x)$ represents the transversal displacement of the string (resp. membrane) at the point x and at time t, while u_0 and u_1 designate the initial position and velocity, respectively.

The energy associated with system (1) is given by

$$E(t) = \frac{1}{2}\int_\Omega \left(|u_t(t,x)|^2 + |\nabla u(t,x)|^2\right) dx, \quad \forall t > 0 \tag{4}$$

and fulfills the following dissipation law

$$\frac{dE(t)}{dt} = -\int_\omega a(x)\,|u_t(t,x)|^2 dx < 0, \quad \forall t > 0. \tag{5}$$

Subsequently, the energy E is a decreasing function of the time variable t.

In this work, we address the very important in practice question of determining the best location and shape of the domain ω in order to minimize some cost related to the energy E. More precisely, we consider the following nonlinear optimal shape design problem :

$$(P): \quad \inf_{\omega \in \Omega_L} J(\mathcal{X}_\omega) \quad \text{where} \quad J(\mathcal{X}_\omega) = \frac{1}{2}\int_0^T \int_\Omega \left(|u_t|^2 + |\nabla u|^2\right) dxdt \tag{6}$$

with

$$\Omega_L = \{\omega \subset \Omega : \quad |\omega| = L\,|\Omega|, \quad 0 < L < 1\}, \tag{7}$$

$|\omega|$ and $|\Omega|$ being the Lebesgue measures of ω and Ω, respectively. Problem (P) consists in finding the location and shape of ω which minimizes the integral of the energy over the time interval $(0,T)$.

In the literature, the problem of optimal stabilization for the wave equation has been extensively from different perspectives, but mainly for $N = 1$ and using spectral analysis (see for instance [2, 3, 4, 6, 7]). The mathematical attraction (and difficulty!) on this problem lies in the fact that, with respect to damping,"more is not better". This is the so-called over-damping phenomena. For instance, for a constant in $\Omega = \omega = (0,1)$, the exponential rate (and similarly the energy) is not a strictly decreasing function of a and reaches its minimum for $a = \pi$. Thus, returning to problem (P), this non-monotony may lead, for a large, to non intuitive optimal position of the subset ω (loss of symmetry of ω with respect to the symmetry of (u_0, u_1) and Ω) [8]. In the same spirit, it is known since the work [6] that a locally negative damping function may lead to a better dissipation (we refer to [12] for some numerical illustrations of this fact).

A second source of difficulty is the possible non well-posedeness character (non existence of minimizer in the class of characteristic functions) of the optimal shape

design problem (P), if no regularity is assumed on ω. For small and constant value of a and $N = 1$, it is proved in [8] that when ω is composed of a number of subintervals which is not fixed a priori, then an optimal domain does not exist. Similarly, by minimizing the total energy over a large time interval, it is observed numerically in [5] that the damping designs do not converge for increasing number of damping subintervals.

Our approach to tackle from a theoretical point of view the problem (P) does not rely on spectral analysis. It consists in a suitable variational reformulation of the initial problem inspired in an strategy that has been successfully implemented in several optimal design problems with steady-state equations (see [15] for a nice account on this), and recently used for some dynamical cases [11]. Two important features to take into account in our problem are, on one hand, the nonlinear dependence of the integrand in the cost function with respect to the gradient ∇u and, on another hand, the possible non existence of minimizer for (P). In this context, the use of gradient Young measures has revealed as a very important tool not only because it leads to an appropriate relaxation of the original problem, but also because it gives the information contained in some minimizing sequences.

We emphasize that we only make on ω the above two constraints, namely, ω is independent of time and its Lebesgue measure is strictly less than the measure of Ω. In particular, ω may be composed of an infinite number of subsets. It is also important to note that in the problem we are dealing with, the damping potential a is fixed. That is, we minimize only in ω for a given a. Other problems would be to optimize in a for a fixed ω or to minimize in (a, ω) jointly, but these issues will not be addressed here.

The rest of the paper is organized as follows. In Section 2, we deal with the theoretical part of the problem. Our main result in this section concerns a full relaxation of (P). Precisely, consider the optimization problem

$$(RP): \quad \inf_{s\in L^\infty(\Omega)} \frac{1}{2}\int_0^T\int_\Omega (u_t^2 + |\nabla u|^2)\,dx\,dt \tag{8}$$

where u (function of s) is the unique solution of

$$\begin{cases} u_{tt} - \Delta u + a\,(x)\,s(x)u_t = 0 & \text{in } (0,T)\times\Omega,\\ u = 0 & \text{on } (0,T)\times\partial\Omega,\\ u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1 & \text{in } \Omega,\\ 0 \le s(x) \le 1, \quad \int_\Omega s(x)\,dx = L\,|\Omega| & \text{in } \Omega. \end{cases} \tag{9}$$

Then we have the following main result.

THEOREM 1.1 *Problem (RP) is a full relaxation of (P) in the sense that*

- *there are optimal solutions for (RP);*
- *the infimum of (P) equals the minimum of (RP);*
- *if s is optimal for (RP), then*
  - *optimal sequences of damping subsets ω_j for (P) in the 1-d case are exactly those for which the Young measure associated with the sequence of their characteristic functions $\mathcal{X}_{\omega_j}$ is precisely*

$$s(x)\delta_1 + (1 - s(x))\delta_0; \tag{10}$$

– *for the 2-d case, optimal sequences of damping sets are those first-order laminates with any normal (in space) which may depend on the space variable but independent of time.*

Notice that in both situations there is a huge non-uniqueness of the distribution of the optimal damping sets, not only because of the arbitrariness of the normals in the 2-d case, but also because first-order laminates can be generated in various length scales. This non-uniqueness may be the explanation why the numerical results in Section 3 are dependent on initial densities.

As an immediate consequence of Theorem 1.1, we have that if the optimal solution s for (RP) is such that the subset $\{x \in \Omega : 0 < s(x) < 1\}$ has positive measure, then no finite admissible collection of subsets can be optimal for (P). The physical meaning of this is that if we want to damp a string (or a membrane), then the best way is to split the actuators into smaller parts.

Notice also that problem (RP) basically consists in replacing the space $L^\infty(\Omega; \{0,1\})$ of admissible designs for problem (P) by its weak * closure $L^\infty(\Omega; [0,1])$.

In Section 3, we address the numerical resolution of the relaxed problem (RP). In this respect, we first compute analytically the first derivative of J with respect to s, in terms of the solution u and of the solution of an appropriate adjoint problem (see eq. 80). We present several experiments which highlight the influence of the over-damping phenomena on the optimal density. Precisely, for small values of the damping potential a we obtain that the original problem (P) is well-posed in the sense that there is a minimizer in the class of characteristic functions. However, when a increases we find that the optimal solution is no longer in $L^\infty(\Omega; \{0,1\})$ but in $L^\infty(\Omega; [0,1])$. We then analyze two penalization techniques to obtain some elements of a minimizing sequence for (P) from the relaxed optimal density of (RP).

2 Variational Reformulation and Relaxation

Although the method of proving Theorem 1.1 is based on the same ideas for $N = 1$ and $N = 2$, we treat separately both cases because of the different characterization of the free-divergence vector fields, which is a key fact in our method. We explain in detail the 1-d case and in 2-d we just point out the main differences with respect to 1-d.

2.1 The one-dimensional case

Throughout this section, we assume that $\Omega = (0,1)$.

2.1.1 Variational Reformulation of problem (P)

First, ω being time independent, we rewrite the main equation in (1) as

$$\text{div } (u_t + a\mathcal{X}_\omega u, -u_x) = 0, \tag{11}$$

where the div operator is considered with respect to the variables t and x. Then, from the characterization of the 2-d free-divergence vector fields, there exists an scalar potential (or stream function) $v = v(t,x) \in H^1((0,T) \times \Omega)$ such that

$$A\nabla u + B\nabla v = -a\mathcal{X}_\omega \overline{u} \tag{12}$$

where $\nabla u = \begin{pmatrix} u_t \\ u_x \end{pmatrix}$, $\nabla v = \begin{pmatrix} v_t \\ v_x \end{pmatrix}$, $\overline{u} = \begin{pmatrix} u \\ 0 \end{pmatrix}$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

We introduce the vector field $U = (u, v) \in \left(H^1\left((0,T)\times\Omega\right)\right)^2$ and the two sets of matrices

$$\begin{aligned} \Lambda_0 &= \left\{ M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = 0 \right\}, \\ \Lambda_{1,\lambda} &= \left\{ M \in \mathcal{M}^{2\times 2} : AM^{(1)} + BM^{(2)} = \lambda e_1 \right\}, \end{aligned} \tag{14}$$

where $M^{(i)}$, $i = 1, 2$ stands for the i-th row of the matrix M, $\lambda \in \mathbb{R}$ and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then considering the following functions,

$$W(x,U,M) = \begin{cases} \frac{1}{2}\left|M^{(1)}\right|^2, & M \in \Lambda_0 \cup \Lambda_{1,-a(x)U^{(1)}} \\ +\infty, & \text{else} \end{cases} \tag{15}$$

and

$$V(x,U,M) = \begin{cases} 1, & M \in \Lambda_{1,-a(x)U^{(1)}} \\ 0, & M \in \Lambda_0 \setminus \Lambda_{1,-a(x)U^{(1)}} \\ +\infty, & \text{else} \end{cases} \tag{16}$$

the optimization problem (P) is equivalent to the following vector variational problem

$$(VP) \quad m \equiv \inf_U \int_0^T \int_0^1 W\left(x, U(t,x), \nabla U\left(t,x\right)\right) dx\, dt \tag{17}$$

subject to

$$\begin{cases} U \in \left(H^1\left((0,T)\times\Omega\right)\right)^2 & \\ U^{(1)} = 0 & (0,T)\times\partial\Omega \\ U^{(1)}\left(0,\cdot\right) = u_0\left(\cdot\right), \quad U_t^{(1)}\left(0,\cdot\right) = u_1\left(\cdot\right), & \Omega \\ \int_0^1 V\left(x, U(t,x), \nabla U\left(t,x\right)\right) dx = L, & (0,T). \end{cases} \tag{18}$$

Therefore, this procedure transforms the scalar optimization problem (P), with differentiable, integrable and pointwise constraints, into a non-convex, vector variational problem (VP) with only pointwise and integral constraints.

2.1.2 Relaxation. Proof of Theorem 1.1

In this section we use the methodology of gradient Young measures (see [14] for the basic properties of these measures) in order to obtain a relaxed form for problem (VP). Our main goal in this section is to prove Theorem 1.1. To this end we follow the same lines as in the case of optimal design problems for elliptic equations. We refer the reader to [15] for a more detailed analysis of this procedure, but in order to

make the paper easier to read and to have a guide for the rest of this section, we now briefly describe the basic underlying ideas.

The main problem is that since the functional W is not convex, we can not ensure the existence of solutions for (VP). A natural thing to do to overcome this difficulty is to consider a larger class of admissible solutions which includes all the minimizing sequences for (VP) and in a way that the resulting functional to be weakly lower semi-continuous. To this end, we consider

$$\overline{m} = \inf_{U} \left\{ \int_0^T \int_\Omega CQW\left(t, x, \nabla U\left(t, x\right), s\left(x\right)\right) dxdt \right\} \tag{19}$$

where the infimum is taken over the fields $U \in \left(H^1\left((0,T)\times\Omega\right)\right)^2$ which satisfy the initial and boundary conditions and the function s satisfies the constraints

$$0 \leq s\left(x\right) \leq 1 \quad \forall x \in \Omega, \quad \text{and} \quad \int_\Omega s\left(x\right) dx = L\left|\Omega\right|. \tag{20}$$

Moreover, the expression $CQW\left(t, x, \nabla U\left(t, x\right), s\left(x\right)\right)$ stands for the constrained quasi-convexification of the density W and, for a fixed $(F, s) \in \mathcal{M}^{2\times 2} \times \mathbb{R}$, is defined as

$$CQW\left(t, x, F, s\right) = \inf_{\nu} \left\{ \int_{\mathcal{M}^{2\times 2}} W\left(t, x, M\right) d\nu\left(M\right) :\ \nu \in \mathcal{A}\left(F, s\right) \right\}, \tag{21}$$

where

$$\begin{aligned} \mathcal{A}\left(F, s\right) \quad = \quad & \big\{ \nu : \nu \text{ is a homogeneous } H^1 - \text{Young measure}, \\ & F = \int_{\mathcal{M}^{2\times 2}} M d\nu\left(M\right) \quad \text{and} \quad \int_{\mathcal{M}^{2\times 2}} V\left(M\right) d\nu\left(M\right) = s \Big\}. \end{aligned}$$

Then, it can be proved (see [15] and the references therein) that the infimum $\overline{m}$ is attained and, in addition, that $m = \overline{m}$, that is, the minimum of the relaxed problem (19) is equal to the infimum of the original one (VP).

So, our main task is to compute the constrained quasi-convexification associated with the density W. The difficulty now is that we do not know explicitly the measures entering in the class $\mathcal{A}\left(F, s\right)$ that we need to compute $CQW\left(t, x, F, s\right)$. We then follow the same strategy as in [15]. First, we consider a larger class of measures $\mathcal{A}^*$, the class of polyconvex measures, and minimize over this class. Second, we will prove that the infimum over the class of polyconvex measures is actually attained at a measure which belongs to a class of measures, say $\mathcal{A}_*$, (the laminates) which is included in $\mathcal{A}\left(F, s\right)$, that is, $\mathcal{A}_* \subset \mathcal{A} \subset \mathcal{A}^*$. As a result, we obtain the exact value of the constrained quasi-convex density $CQW\left(t, x, F, s\right)$.

To sum up, all we have to do in the rest of this section is: (1) to compute the constrained polyconvexification of the density W, (2) to prove that this new relaxed density is obtained from a first-order laminate, and (3) to reinterpret all of this information in the form of the relaxed problem (RP). We proceed in different steps.

Step 1: computation of the constrained polyconvexification. The constrained polyconvexification, CPW, associated with the density W is given by

$$CPW\left(x, U, F, s\right) = \min_{\nu} \int_{\mathcal{M}^{2\times 2}} W\left(x, U, M\right) d\nu\left(M\right) \tag{22}$$

where the measure ν satisfies the constraints

$$\begin{cases} \nu \text{ commutes with } \det, \\ F = \displaystyle\int_{\mathcal{M}^{2\times 2}} M d\nu\left(M\right), \\ s = \displaystyle\int_{\mathcal{M}^{2\times 2}} V\left(x,U,M\right) d\nu\left(M\right). \end{cases} \tag{23}$$

Note that in computing $CPW\left(x,U,F,s\right)$, the variables (x,U) just play the role of parameters so that they are regarded as constants. Hence, if we take λ as given, put Λ_1 for $\Lambda_{1,\lambda}$ for short, and consider the integrands

$$W(M) = \begin{cases} \frac{1}{2}\left|M^{(1)}\right|^2, & M \in \Lambda_0 \cup \Lambda_1 \\ +\infty, & \text{else} \end{cases} \tag{24}$$

and

$$V\left(M\right) = \begin{cases} 1, & M \in \Lambda_1 \\ 0, & M \in \Lambda_0 \setminus \Lambda_1 \\ +\infty, & \text{else} \end{cases} \tag{25}$$

then we are concerned with the computation of

$$CPW\left(F,s\right) = \min_{\nu} \int_{\mathcal{M}^{2\times 2}} W\left(M\right) d\nu\left(M\right) \tag{26}$$

where the measure ν satisfies (23). From the volume constraint we deduce that this measure has the form

$$\nu = s\nu_1 + \left(1-s\right)\nu_0, \quad \text{with } \operatorname{supp}\left(\nu_j\right) \subset \Lambda_j,\ j = 0,1, \tag{27}$$

and hence for each pair $\left(F,s\right)$, the constrained polyconvexification $CPW\left(F,s\right)$ is computed by solving

$$\min_{\nu} \quad \frac{s}{2}\int_{\Lambda_1}\left|M^{(1)}\right|^2 d\nu_1\left(M\right) + \frac{1-s}{2}\int_{\Lambda_0}\left|M^{(1)}\right|^2 d\nu_0\left(M\right) \tag{28}$$

subject to

$$\begin{cases} \nu = s\nu_1 + \left(1-s\right)\nu_0 \text{ commutes with } \det, \\ \operatorname{supp}\left(\nu_j\right) \subset \Lambda_j,\ j = 0,1, \\ F = s\displaystyle\int_{\Lambda_1} M d\nu_1\left(M\right) + \left(1-s\right)\int_{\Lambda_0} M d\nu_0\left(M\right). \end{cases} \tag{29}$$

Let us introduce the following variables

$$S_i = \int_{\mathbb{R}} \left(M_{1i}\right)^2 d\nu^{(1i)}, \quad i = 1,2, \tag{30}$$

where $\nu^{(1i)}$ stands for the projection of ν onto the $(1i)-$th component, and

$$F^j = \int_{\Lambda_j} M d\nu_j\left(M\right), \quad j = 0,1. \tag{31}$$

Since $F^j \in \Lambda_j$, we have

$$\left\{ \begin{array}{l} F_{11}^0 = F_{22}^0 \\ F_{12}^0 = F_{21}^0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} F_{11}^1 = F_{22}^1 + \lambda \\ F_{12}^1 = F_{21}^1 \end{array} \right. \tag{32}$$

On the other hand, from the third condition in (29) it follows that

$$\left\{ \begin{array}{l} F_{11} = sF_{11}^1 + (1-s)\, F_{11}^0 \\ F_{12} = sF_{12}^1 + (1-s)\, F_{12}^0 \\ F_{21} = sF_{21}^1 + (1-s)\, F_{21}^0 \\ F_{22} = sF_{22}^1 + (1-s)\, F_{22}^0 \end{array} \right. \tag{33}$$

Substituting (32) into (33) we obtain the system

$$\left\{ \begin{array}{l} F_{11} = sF_{11}^1 + (1-s)\, F_{11}^0 \\ F_{12} = sF_{12}^1 + (1-s)\, F_{12}^0 \\ F_{21} = sF_{12}^1 + (1-s)\, F_{12}^0 \\ F_{22} + s\lambda = sF_{11}^1 + (1-s)\, F_{11}^0 \end{array} \right. \tag{34}$$

which has a solution if and only if the compatibility condition

$$F_{12} = F_{21}, \quad F_{11} = F_{22} + s\lambda \tag{35}$$

holds. In this case, the solution is given by

$$\left\{ \begin{array}{l} F_{11}^0 = \alpha, \quad F_{12}^0 = \beta \\ F_{11}^1 = \frac{1}{s}\left(F_{11} - (1-s)\,\alpha\right) \\ F_{12}^1 = \frac{1}{s}\left(F_{12} - (1-s)\,\beta\right) \end{array} \right. \tag{36}$$

where $(\alpha, \beta) \in \mathbb{R}^2$ are two parameters. Notice then that there is no restriction on F_{11}^1, as it can take on any value. Moreover, the constraint on the commutation with det yields to

$$\begin{array}{rcl} \det F & = & s\displaystyle\int_{\Lambda_1} \det M d\nu_1\,(M) + (1-s)\int_{\Lambda_0} \det M d\nu_0\,(M) \\ & = & S_1 - S_2 - s\lambda F_{11}^1 \end{array}$$

since

$$\det M = \left\{ \begin{array}{ll} (M_{11})^2 - (M_{12})^2 & \text{if} \quad M \in \Lambda_0 \\ (M_{11})^2 - \lambda M_{11} - (M_{12})^2 & \text{if} \quad M \in \Lambda_1 \end{array} \right. \tag{37}$$

Finally, from Jensen's inequality we obtain the conditions

$$S_i \geq |F_{1i}|^2, \quad i = 1, 2. \tag{38}$$

To sum up, we have to solve the mathematical programming problem

$$\text{Minimize in } \left(S_j, F_{11}^1\right): \quad \frac{1}{2}\left(S_1 + S_2\right) \tag{39}$$

subject to

$$\left\{ \begin{array}{l} S_1 - S_2 - s\lambda F_{11}^1 = \det F \\ S_i \geq |F_{1i}|^2, \quad i = 1, 2. \end{array} \right. \tag{40}$$

We obtain easily that the solution is

$$S_i = |F_{1i}|^2, \quad i = 1, 2. \tag{41}$$

This implies that

$$CPW(F,s) = \begin{cases} \frac{1}{2}|F^{(1)}|^2 & \text{if (35) holds} \\ +\infty & \text{else.} \end{cases} \tag{42}$$

Step 2: first-order laminates. Notice that the two equalities

$$S_i = |F_{1i}|^2, \quad i = 1, 2, \tag{43}$$

imply, by the strict convexity of the 2-norm, that

$$\nu^{(1i)} = \delta_{F_{1i}}, \quad i = 1, 2. \tag{44}$$

It is now straightforward to check that the unique, optimal ν furnishing the value of $CPW(F,s)$ is

$$\nu = (1-s)\delta_{G^0} + s\delta_{G^1},$$

where

$$G^0 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11} \end{pmatrix} \quad \text{and} \quad G^1 = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11} + \lambda \end{pmatrix}. \tag{45}$$

Note that $G^j \in \Lambda_j$, $j = 0, 1$. Moreover, since $G^1 - G^0 = b \otimes n$, with

$$b = (0, \lambda) \quad \text{and} \quad n = (0, 1),$$

the optimal measure ν is a first-order laminate with normal n. As a matter of fact, this implies that the constrained quasi-convexification of the density W is also given by (42).

Remark 1 *It is important to realize that sequences of gradients associated with the measure ν have the form*

$$\begin{aligned} \nabla U^j(t,y) &= G^0 + \mathcal{X}_s(j\ (t,y)\cdot n)\, b \otimes n \\ &= \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{11} + \mathcal{X}_s(jy)\,\lambda \end{pmatrix} \end{aligned}$$

with $\mathcal{X}_s$ the characteristic function of $(0,s)$ in $(0,1)$. This means that the sequence of gradients ∇U^j oscillates between the two manifolds Λ_0 and Λ_1 with a frequency s and normal n to the layers.

Step 3: reinterpretation in terms of the initial variables. We now return to the variables of the original problem (P). So, we put

$$\lambda = -a(x)\, U^{(1)}(t,x) = -a(x)\, u(t,x), \tag{46}$$

which, in particular implies that the laminate we have computed depends on (t,x). However, since the direction of lamination is independent of time, we avoid to mention explicitly this variable and therefore we write $\nu = \{\nu_x\}_{x\in\Omega}$. We also have to take into account the compatibility condition on $(0,T)\times\Omega$ of the first moment of the measure ν, namely, that the vector field $U \in \left(H^1((0,T)\times(0,1))\right)^2$ satisfies

$$\nabla U(t,x) = \int_{\mathcal{M}^{2\times 2}} M d\nu_x(M) \quad \text{a.e. } (t,x)\in(0,T)\times\Omega \tag{47}$$

and the volume constraint

$$\int_{\mathcal{M}^{2\times 2}} V(x, U(t,x), \nabla U(t,x))\, d\nu_x = s(x) \tag{48}$$

with

$$0 \le s(x) \le 1, \quad \int_\Omega s(x)dx = L|\Omega|. \tag{49}$$

Hence, the compatibility condition (35) reads as

$$u_x(t,x) = v_t(t,x), \quad u_t(t,x) = v_x(t,x) - a(x)\, s(x)\, u(t,x), \tag{50}$$

or equivalently

$$u_{tt}(t,x) - u_{xx}(t,x) + a(x)\, s(x)\, u_t(t,x) = 0, \tag{51}$$

and the relaxed integrand of the cost function (42) as

$$CPW(x, U(t,x), \nabla U(t,x)) = \begin{cases} \frac{1}{2}\left(u_t^2(t,x) + u_x^2(t,x)\right) & \text{if (51) holds} \\ +\infty & \text{else} \end{cases}. \tag{52}$$

Finally, taking into account that the hole minimizing process is complete if we now minimize in $U(t,x)$ the expression

$$\frac{1}{2}\int_0^T\int_0^1 CPW(x, U(t,x), \nabla U(t,x))\, dxdt \tag{53}$$

and noticing that this is equivalent to minimize in all possible functions $s \in L^\infty(0,1)$ satisfying (49) we arrive to the relaxed problem (RP).

The existence of optimal solutions for (RP) is a consequence of the fact that the new density $CPW(x, U(t,x), \nabla U(t,x))$ is quasi-convex. Moreover, the minima of (RP) coincide with the infima of (VP) because minimizers for (RP) are obtained in the form of a first-order laminate associated with a sequence of gradients of admissible vector fields $U^j(t,x)$ for (VP). This is the way in which the information concerning minimizing sequences for (VP) is codified in the relaxed problem (RP). That is, since the normal to the layers of all admissible laminates is $n = (0,1)$, in the $(t,x)-$plane a minimizing sequence for (VP) looks like horizontal layers, limiting the regions of damping and in which for each time t the damping region is limited to have a total length of L.

2.2 The two-dimensional case

In this section we aim to extend Theorem 1 to the case $N = 2$. So, from now on in this section Ω will be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and we note $x = (x_1, x_2)$. The main novelty with respect to the 1-d case concerns the characterization of the free-divergence vector fields. These vector fields may be characterized by using Clebsch's potentials (see [9, 13, 16] for more information on these potentials). Precisely, if $V \in \mathbb{R}^3$ is such that $\mathrm{div}(V) = 0$, then there exist two potentials v_1, v_2 such that $V = \nabla v_1 \times \nabla v_2$. We will apply this result in order to obtain a variational reformulation of our system.

2.2.1 Variational Reformulation

Similarly to the 1-d case, the wave equation in system (1) may be written as

$$\mathrm{div}\left(u_t + a\mathcal{X}_\omega u, -u_{x_1}, -u_{x_2}\right) = 0 \quad \text{in} \quad (0,T)\times\Omega \tag{54}$$

and so there exist two potentials $v_1 = v_1(t, x_1, x_2)$ and $v_2 = v_2(t, x_1, x_2)$ such that

$$\left(u_t + a\mathcal{X}_\omega u, -u_{x_1}, -u_{x_2}\right) = \nabla v_1 \times \nabla v_2. \tag{55}$$

Let us now introduce the vector field $U = (u, v_1, v_2) \in (H^1((0,T)\times\Omega))^3$ and the two manifolds

$$\begin{aligned} \Lambda_0 &= \left\{ M \in \mathcal{M}^{3\times 3} : AM^{(1)} - M^{(2)} \times M^{(3)} = 0 \right\}, \\ \Lambda_{1,\lambda} &= \left\{ M \in \mathcal{M}^{3\times 3} : AM^{(1)} - M^{(2)} \times M^{(3)} = \lambda e_1 \right\}, \end{aligned} \tag{56}$$

where $\lambda \in \mathbb{R}$ and

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{57}$$

Notice that the introduction of the Clebsch's potentials leads to the fact that the two manifolds Λ_0 and $\Lambda_{1,\lambda}$ are nonlinear contrary to the 1-d case where they are linear. Fortunately, this will add no additional difficulty in the process.

If we define the density function W and the function V appearing in the volume constraint in the same way as in 1-d, then our optimization problem is equivalent to the following vector variational problem

$$\min_U \int_0^T \int_\Omega W\left(x, U(t,x), \nabla U(t,x)\right) dxdt \tag{58}$$

subject to

$$\begin{cases} U \in \left(H^1((0,T)\times\Omega)\right)^3 & \\ U^{(1)} = 0, & (0,T)\times\partial\Omega \\ U^{(1)}(0,\cdot) = u_0(\cdot), \quad U_t^{(1)}(0,\cdot) = u_1(\cdot), & \Omega \\ \displaystyle\int_\Omega V\left(x, U(t,x), \nabla U(t,x)\right) dx = L|\Omega|, & (0,T). \end{cases} \tag{59}$$

2.2.2 Relaxation. Proof of Theorem 1.1

Next, we prove Theorem 1.1 for the 2-d case. We proceed as in 1-d by computing the constrained polyconvexification, that is, for a fixed (F, s) we have to minimize in ν the integral

$$\int_{\mathcal{M}^{3\times 3}} W(M)\, d\nu(M) \tag{60}$$

where the measure ν satisfies

$$\begin{cases} \nu \text{ commutes with all minors,} \\ F = \displaystyle\int_{\mathcal{M}^{3\times 3}} M d\nu(M), \\ s = \displaystyle\int_{\mathcal{M}^{3\times 3}} V(M)\, d\nu(M), \end{cases} \tag{61}$$

¿From the volume constraint we have

$$\nu = s\nu_1 + (1-s)\,\nu_0, \quad \text{with } \operatorname{supp}(\nu_j) \subset \Lambda_j,\ j = 0, 1, \tag{62}$$

and hence the optimization problem to be solved is

$$\text{Minimize in } \nu: \quad \frac{s}{2}\int_{\Lambda_1} \left|M^{(1)}\right|^2 d\nu_1(M) + \frac{1-s}{2}\int_{\Lambda_0} \left|M^{(1)}\right|^2 d\nu_0(M) \tag{63}$$

subject to

$$\begin{cases} \nu = s\nu_1 + (1-s)\,\nu_0 \text{ commutes with all minors,} \\ \operatorname{supp}(\nu_j) \subset \Lambda_j,\ j = 0, 1, \\ F = s\displaystyle\int_{\Lambda_1} M d\nu_1(M) + (1-s)\int_{\Lambda_0} M d\nu_0(M). \end{cases} \tag{64}$$

Let us introduce the following variables

$$S_1 = \int_{\mathbb{R}} (M_{11})^2\, d\nu^{(11)}, \quad S_2 = \int_{\mathbb{R}} (M_{12})^2\, d\nu^{(12)} + \int_{\mathbb{R}} (M_{13})^2\, d\nu^{(13)}, \tag{65}$$

where $\nu^{(1i)}$ stands for the projection of ν onto the $(1i)-$th component, and

$$F^j = \int_{\Lambda_j} M d\nu_j(M), \quad j = 0, 1. \tag{66}$$

Since the components of $F^{(2)} \times F^{(3)}$ are the second order minors obtained from the second and third row of F and taking into account the commutation of the measure ν with these minors we obtain

$$\begin{aligned} F^{(2)} \times F^{(3)} &= s\int_{\Lambda_1} \left(M^{(2)} \times M^{(3)}\right) d\nu_1(M) + (1-s)\int_{\Lambda_0} \left(M^{(2)} \times M^{(3)}\right) d\nu_0(M) \\ &= s\int_{\Lambda_1} \left(AM^{(1)} - \lambda e_1\right) d\nu_1(M) + (1-s)\int_{\Lambda_0} \left(AM^{(1)}\right) d\nu_0(M) \\ &= s\left(AF^{1(1)} - \lambda e_1\right) + (1-s)\, AF^{0(1)}. \end{aligned}$$

This, together with the condition on the first moment of the measure ν, that is,

$$F^{(1)} = sF^{1(1)} + (1-s)\,F^{0(1)} \tag{67}$$

leads to the system

$$\begin{cases} sF^{1(1)} + (1-s)\,F^{0(1)} = A^{-1}\left(F^{(2)} \times F^{(3)} + \lambda s e_1\right) \\ sF^{1(1)} + (1-s)\,F^{0(1)} = F^{(1)} \end{cases} \tag{68}$$

which has a solution if and only if the compatibility condition

$$AF^{(1)} - F^{(2)} \times F^{(3)} = \lambda s e_1 \tag{69}$$

holds. Again the commutation of ν with det implies

$$\begin{aligned} \det F &= s\int_{\Lambda_1} \det M d\nu_1(M) + (1-s)\int_{\Lambda_0} \det M d\nu_0(M) \\ &= S_1 - S_2 - \lambda s F^1_{11} \end{aligned}$$

since

$$\det M = \begin{cases} (M_{11})^2 - (M_{12})^2 - (M_{13})^2, & M \in \Lambda_0 \\ (M_{11})^2 - \lambda M_{11} - (M_{12})^2 - (M_{13})^2, & M \in \Lambda_1 \end{cases} \tag{70}$$

Finally, from Jensen's inequality it follows that

$$S_1 \geq |F_{11}|^2, \quad S_2 \geq |F_{12}|^2 + |F_{13}|^2. \tag{71}$$

Similarly to the 1-d case, the problem (63) is then reduced to the resolution of the mathematical programming problem

$$\text{Minimize in } \left(S_j, F^1_{11}\right): \quad \frac{1}{2}(S_1 + S_2) \tag{72}$$

subject to

$$\begin{cases} S_1 - S_2 - s\lambda F^1_{11} = \det F \\ S_1 \geq |F_{11}|^2, \quad S_2 \geq |F_{12}|^2 + |F_{13}|^2. \end{cases} \tag{73}$$

The solution corresponds to taking

$$S_1 = |F_{11}|^2, \quad S_2 = |F_{12}|^2 + |F_{13}|^2. \tag{74}$$

and implies that

$$CPW(F,s) = \begin{cases} \frac{1}{2}|F^{(1)}|^2 & \text{if (69) holds} \\ \\ +\infty & \text{else} \end{cases} \tag{75}$$

Next, we prove that these extreme points of $CPW(F,s)$ can be attained as the second moments of some measures ν_j, $j = 0, 1$, such that the linear combination $\nu = (1-s)\,\nu_0 + s\nu_1$ is a laminate, actually a first-order laminate. To this end, we first note that from the equalities (74) it follows that

$$\nu^{(1i)} = \delta_{F_{1i}}, \quad i = 1, 2, 3.$$

So, all we have to do is to look for two matrices G^j, $j = 0, 1$, such that (i) $G^j_{i1} = F_{1i}$, with $i = 1, 2, 3$ and $j = 0, 1$, (ii) $F = sG^1 + (1 - s)\, G^0$, (iii) $G^j \in \Lambda_j$, $j = 0, 1$, and (iv) $G^1 - G^0$ has rank one. This would imply that the measure that we are looking for is

$$\nu = (1 - s)\, \delta_{G^0} + s\delta_{G^1}.$$

Condition (iv) above may be also written as

$$G^1 = G^0 + b \otimes n$$

for some appropriate vectors b and n. Hence,

$$G^1 = F + (1 - s)b \otimes n, \quad G^0 = F - sb \otimes n.$$

In order to have condition (i) we necessarily must take $n = (0, n_2, n_3)$. Moreover, after some simple algebra, condition $G^j \in \Lambda_j$, $j = 0, 1$, leads to

$$\lambda e_1 = (b_3 F^{(2)} - b_2 F^{(3)}) \times n. \tag{76}$$

This means that any normal of the form $(0, n_2, n_3)$ is optimal, and optimal first-order laminates with such a normal are then obtained by solving the relaxed problem, taking any normal in space, and building these first-order laminates with volume fraction given by the optimal s (depending on space).

As a consequence of the fact that the constrained polyconvex density $CPW\,(F, s)$ is attained at a first-order laminate, we obtain, as in the 1-d case, the exact value of the quasi-convex density $CQW\,(F, s)$.

3 Numerical resolution of the relaxed problem

In this section, we address the problem of computing numerically the optimal density for (RP). To this end, we first describe the algorithm of minimization and then present some numerical experiments.

3.1 Algorithm of minimization

Next, we briefly discuss the resolution of the relaxed problem (RP) using a gradient descent method. In this respect, we compute the first variation of the cost function

$$J(s) = \frac{1}{2} \int_0^T \int_\Omega (|u_t|^2 + |\nabla u|^2)\, dxdt. \tag{77}$$

where $u = u(s)$ is a solution of (9). For any $\eta \in \mathbb{R}^+$, $\eta << 1$, and any $s_1 \in L^\infty(\Omega)$, we associate to the perturbation $s^\eta = s + \eta s_1$ of s the derivative of J with respect to s in the direction s_1 as follows :

$$\frac{\partial J(s)}{\partial s} \cdot s_1 = \lim_{\eta \to 0} \frac{J(s + \eta s_1) - J(s)}{\eta}. \tag{78}$$

We obtain the following result.

THEOREM 3.1 *If $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, then the derivative of J with respect to s in any direction s_1 exists and takes the form*

$$\frac{\partial J(s)}{\partial s} \cdot s_1 = \int_\Omega a(x) s_1(x) \int_0^T u_t(t,x) p(t,x)\, dtdx \tag{79}$$

where u is the solution of (9) and p is the solution in $C^1([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ of the adjoint problem

$$\begin{cases} p_{tt} - \Delta p - a(x)s(x)p_t = u_{tt} + \Delta u, & \text{in} \quad (0,T) \times \Omega, \\ p = 0, & \text{on} \quad (0,T) \times \partial\Omega, \\ p(T,\cdot) = 0, \quad p_t(T,\cdot) = u_t(T,\cdot) & \text{in} \quad \Omega. \end{cases} \tag{80}$$

Sketch of the proof. Let us explain briefly how we obtain the expression (79). We introduce the lagrangian functional

(81)
$$\mathcal{L}(s,\phi,\psi) = \frac{1}{2}\int_\Omega \int_0^T (\mid \phi_t \mid^2 + \mid \nabla\phi \mid^2)\, dtdx + \int_\Omega \int_0^T (\phi_{tt} - \Delta\phi + a(x)s(x)\phi_t)\psi\, dtdx$$

for any $s \in L^\infty(\Omega)$, $\phi \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; H_0^1(\Omega))$ and $\psi \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ and then write formally that

(82)
$$\frac{d\mathcal{L}}{ds}(s_1) = \frac{\partial}{\partial s}\mathcal{L}(s,\phi,\psi) \cdot s_1 + < \frac{\partial}{\partial \phi}\mathcal{L}(s,\phi,\psi), \frac{\partial \phi}{\partial s} \cdot s_1 > + < \frac{\partial}{\partial \psi}\mathcal{L}(s,\phi,\psi), \frac{\partial \psi}{\partial s} \cdot s_1 >$$

The first term is

$$\frac{\partial}{\partial s}\mathcal{L}(s,\phi,\psi) \cdot s_1 = \int_\Omega a(x) s_1(x) \int_0^T \phi_t(t,x)\psi(t,x)\, dtdx \tag{83}$$

for any s, ϕ, ψ whereas the third term is equal to zero if $\phi = u$ solution of (9). We then determine the solution p so that, for all $\phi \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; H_0^1(\Omega))$, we have

$$< \frac{\partial}{\partial \phi}\mathcal{L}(s,\phi,p), \frac{\partial \phi}{\partial s} \cdot s_1 >= 0, \tag{84}$$

which leads to the formulation of the adjoint problem (80). Next, writing that $J(s) = \mathcal{L}(s,u,p)$, we obtain (79) from (83). Moreover, notice that the integral (79) is well defined, i.e. $u_t p \in C([0,T], L^1(\Omega))$ since from the regularity assumed on (u_0, u_1), we have $u_{tt} + \Delta u \in C([0,T]; L^2(\Omega))$ and hence $p \in C([0,T]; L^2(\Omega))$. ■

In order to take into account the volume constraint on s, we introduce the Lagrange multiplier $\gamma \in \mathbb{R}$ and the functional

$$J_\gamma(s) = J(s) + \gamma ||s||_{L^1(\Omega)}. \tag{85}$$

Using Theorem 3.1, we obtain that the derivative of J_γ is

$$\frac{\partial J_\gamma(s)}{\partial s} \cdot s_1 = \int_\Omega s_1(x) \left(a(x) \int_0^T u_t(t,x) p(t,x) dt + \gamma \right) dx \tag{86}$$

which permits to define the following descent direction :

$$s_1(x) = -\left(a(x)\int_0^T u_t(t,x)p(t,x)dt + \gamma\right), \quad \forall x \in \Omega. \tag{87}$$

Consequently, for any function $\eta \in L^\infty(\Omega, \mathbb{R}^+)$ with $||\eta||_{L^\infty(\Omega)}$ small enough, we have $J_\gamma(s + \eta s_1) \leq J_\gamma(s)$. The multiplier γ is then determined in order that, for any function $\eta \in L^\infty(\Omega, \mathbb{R}^+)$ and $\eta \neq 0$, $||s + \eta s_1||_{L^1(\Omega)} = L|\Omega|$ leading to

$$\gamma = \frac{(\int_\Omega s(x)dx - L|\Omega|) - \int_\Omega \eta(x)a(x)\int_0^T u_t(t,x)p(t,x)\,dtdx}{\int_\Omega \eta(x)dx}. \tag{88}$$

At last, the function η is chosen so that $s(x) + \eta(x)s_1(x) \in [0,1]$, for all $x \in \Omega$. A simple and efficient choice consists in taking $\eta(x) = \varepsilon s(x)(1 - s(x))$ for all $x \in \Omega$ with ε a small real positive.

Consequently, the descent algorithm to solve numerically the relaxed problem (RP) may be structured as follows : let $\Omega \subset \mathbb{R}^N$, $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, $L \in (0,1)$, $T > 0$, and $\varepsilon < 1$, $\varepsilon_1 << 1$ be given :

- Initialization of the density function $s^0 \in L^\infty(\Omega;]0,1[)$;
- For $k \geq 0$, iteration until convergence (i.e. $|J(s^{k+1}) - J(s^k)| \leq \varepsilon_1 |J(s^0)|$) as follows :
  - Computation of the solution u_{s^k} of (9) and then the solution p_{s^k} of (80), both corresponding to $s = s^k$.
  - Computation of the descent direction s_1^k defined by (87) where the multiplier γ^k is defined by (88).
  - Update the density function in Ω:

    $$s^{k+1} = s^k + \varepsilon s^k(1 - s^k)s_1^k \tag{89}$$

    with $\varepsilon \in \mathbb{R}^+$ small enough in order to ensure the decrease of the cost function and $s^{k+1} \in L^\infty(\Omega, [0,1])$.

3.2 Numerical experiments

In this section, we present some numerical simulations for $\Omega = (0,1)$ when $N = 1$ and for $\Omega = (0,1)^2$ when $N = 2$. We use a finite centered difference scheme of order two in space and time to solve wave systems (9) and (80). We consider uniform meshes and note h the parameter of discretization in space. Moreover, without loss of generality, we consider a constant function a in Ω since the dependence in x is contained in the density s. So, we introduce $a > 0$ such that $a(x) = a\mathcal{X}_\Omega(x)$. We take $\varepsilon = 10^{-1}$ and $\varepsilon_1 = 10^{-5}$.

3.2.1 Example 1

Let us first consider in $\Omega = (0,1)$ the following simple initial condition :

$$u_0(x) = \sin(\pi x); \quad u_1(x) = 0. \tag{90}$$

and the data $T = 1$, $L = 1/5$. The numerical simulation highlights the important influence of the value of a on the optimal position of the damping set ω. For small values of a - for instance here $a = 1$ - we observe that the limit of the sequence $\{s^k\}_{k\geq 0}$ converges to the characteristic function $\mathcal{X}_{[1/2-L/2,1/2+L/2]}$ whatever the initialization s^0. The optimal position is therefore $\omega = [0.4, 0.6]$ (which is an expected result according to the symmetry of u_0) and is the global minimum for the cost function. Thus, for this value, the original problem (P) is well-posed and the infimum is reached.

The situation is completely different when a is large (for instance here $a = 10$). The limit of the sequence depends now on the initialization and implies the existence of several local minima for J. Moreover, the centered position is not the optimal one. Let us consider the following initialization

$$s_n^0(x) = L\frac{n\pi(1+\sin(n\pi x))}{n\pi + (1-\cos(n\pi))}, \quad n \in \mathbb{N}, x \in \Omega \tag{91}$$

which satisfies $||s_n^0||_{L^1(\Omega)} = L$ for all n. Figure 1 presents results for $a = 10$ and $n = 5, 15, 25, 45$. For each value of n, the function s_n^0 is plotted in dash dot $(--)$ while the corresponding limit density s_n^{lim} of the s_n^k sequence is plotted in full line $(-)$. Table 1 summarizes the value of the cost function for the different cases. We observe that the limit density s_n^{lim} takes its values in $[0, 1[$ and is strictly positive in an interval included in $I_a = [0.2, 0.8]$ which depends on a. We also observe that $n \to J(s_n^{lim})$ is a decreasing function of n.

n	5	15	25	45	60
$J(s_n^0)$	1.3595	1.3065	1.2986	1.2953	1.2921
$J(s_n^{lim})$	1.1370	1.1357	1.1354	1.1353	1.1352
$J(s_n^{pen})$	1.1671	1.1433	1.1413	1.1395	1.1371
$J(s_{n,20}^{pen})$	1.1371	1.1358	1.1355	1.1353	1.1352

Table 1: Example 1 - $T = 1$ - $a(x) = 10\mathcal{X}_{(0,1)}(x)$ - Values of the cost function J - $h = 1/500$

Over-damping phenomena - These computations exhibit a value of bifurcation $a^\star(T, L, \Omega, u_0, u_1)$ depending on the data of the problem. We have obtained numerically $a^\star(T, L, \Omega, u_0, u_1) \approx 1.725$. This sensitivity with respect to a is related to the following over-damping phenomena : the function $a \to J(s, a)$ first decreases, reaches a minimum $J(s, \overline{a})$ at $\overline{a} \in]0, +\infty[$, and then increases to $J(s, \infty) = J(s, 0)$ corresponding to the conservative case. From (79), one may write that

$$J(s + \eta s_1, a) = J(s, a) + \eta \int_\Omega a(x)s_1(x) \int_0^T u_{t(s)}p_{(s)}dtdx + \eta^2 aO(u_{t(s)}, p_{(s)}) \tag{92}$$

such that for the conservative case $s = 0$ and $a(x) = a\mathcal{X}_\Omega(x)$,

$$\begin{aligned} J(\eta s_1, a) =& J(0, a) + \eta a \int_\Omega s_1(x) \int_0^T u_{t(0)}p_{(0)}dtdx + \eta^2 aO(u_{t(0)}, p_{(0)}) \\ =& J(s_1, 0) + a \int_\Omega (\eta s_1(x)) \int_0^T u_{t(0)}p_{(0)}dtdx + \eta^2 aO(u_{t(0)}, p_{t(0)}) \end{aligned} \tag{93}$$

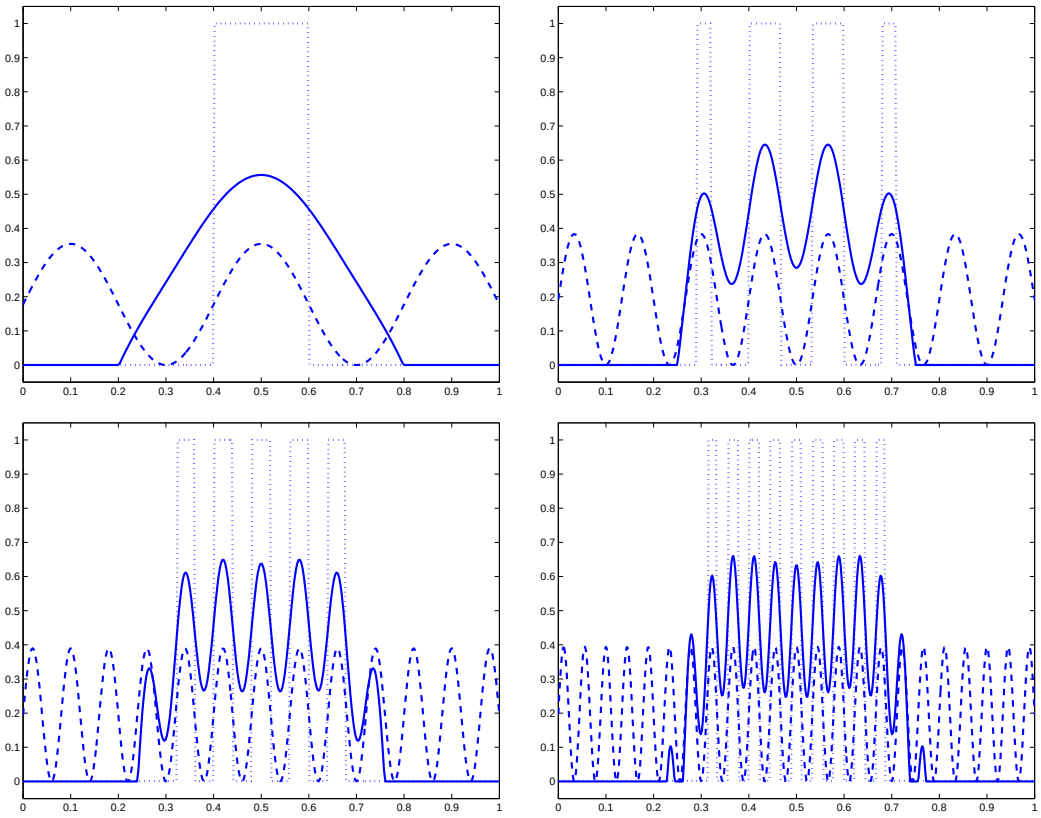

Figure 1: Example 1 - Initialized density s_n^0 $(--)$, limit density s_n^{lim} $(-)$ and penalized density $s_n^{pen}((\cdots))$ for $n = 5$ (top left), $n = 15$ (top right), $n = 25$ (bottom left) and $n = 45$ (bottom right) - $a = 10$ - $T = 1$.

where $u_{t(0)}, p_{t(0)}$ are the solutions of (9) and (80) in the conservative case. Then, writing that $J(\eta s_1, a) = J(s_1, \eta a)$, one get

$$J(s_1, \eta a) = J(s_1, 0) + \eta a \int_\Omega s_1(x) \int_0^T u_{t(0)} p_{(0)} dt dx + \eta^2 a O(u_{t(0)}, p_{(0)}) \tag{94}$$

For η small, the last term may be neglected. Therefore, the optimal density associated with the damping coefficient ηa (small) is related to the minima in Ω of $\int_0^T u_{t(0)} p_{(0)} dt$. With the initial condition (90), we obtain explicitly

$$u_{(0)}(t,x) = \cos(\pi t) u_0(x), \quad p_{(0)}(t,x) = \pi(T-t)\sin(\pi t) u_0(x) \tag{95}$$

and then

$$\int_0^T u_{t(0)}(t,x) p_{(0)}(t,x) dt = -\frac{1}{4}\Big((\pi T)^2 - \sin(\pi T)^2\Big)(u_0(x))^2 \leq 0 \tag{96}$$

We conclude that the second right term of (94) is minimal for $s_1 = \mathcal{X}_{[1/2-L/2, 1/2+L/2]}$.

For large values of the damping coefficient ηa (i.e. when η is not small), the last term in (94) can not be neglected. The optimal position is not necessarily related to the minima of the function $\int_0^T u_{t(0)}(t,x) p_{(0)}(t,x) dt$.

Remark 2 *For a small damping coefficient, note that the equality (94) may be used to efficiently initiate the descent algorithm: it suffices to take*

$$s^0(x) = L\frac{f(x)}{||f||_{L^1(\Omega)}} \quad \textit{with} \quad f(x) = -\int_0^T u_{t(0)}(t,x) p_{(0)}(t,x) dt \quad \textit{in} \quad \Omega \tag{97}$$

which is positive since $J(s_1, \eta a) < J(s_1, 0)$ for all $s_1 \in L^\infty(\Omega;]0,1])$ (the same argument implies that the multiplier γ defined by (88) is positive).

Characteristic function associated with the density s^{lim} - Associated with the density s^{lim}, local solution of the relaxed problem (RP), it is very interesting to determine a bi-valued function characterizing the subset ω, local solution of problem (P).

A first approach, used in homogenization theory, is to penalize s^{lim} (we refer to [1], page 381). After the convergence of the algorithm leading to s^{lim}, the idea is to perform a few more iterations with the instruction

$$s^{pen} = \frac{1 - \cos(\pi s^{lim})}{2} \quad \text{in} \quad \Omega. \tag{98}$$

Results are presents in Figure 1 and Table 1. s_n^{pen} associated with s_n^{lim}, $n = 5, 15, 25, 45$ are plotted in dot line $(\cdots)$. For each n, the penalized density s_n^{pen} is concentrated on an interval included in $[0.3, 0.7]$ and we observe that $n \to J(s_n^{pen})$ is a decreasing function with $J(s_n^{pen}) > J(s_n^{lim})$ for all n. By taking larger values for n, we have checked that the function $n \to J_0(s_n^{pen}) - J_0(s_n^{lim})$ decreases toward zero. This seems to indicate that the infimum for the original problem (P) is equal to the minimum of the relaxed problem (RP). This is an illustration of Theorem 1.1. While for $a = 1$, the optimal subset is $\omega = [0.4, 0.6]$, the optimal one for $a = 10$ is composed of an infinity of subintervals concentrated in the region $[0.3, 0.7]$.

This simple penalization technique which does not strictly preserve the volume constraint and fails if $s^{lim}(x) < 1/2$ for all x in Ω may be replaced by the following one. Let us decompose the interval Ω into $M > 0$ non-empty subintervals such that $\Omega = \cup_{j=1,M}[x_j, x_{j+1}]$, . Then, we associate with each interval $[x_j, x_{j+1}]$ the mean value $m_j \in [0, 1]$ defined by

$$m_j = \frac{1}{x_{j+1} - x_j} \int_{x_j}^{x_{j+1}} s^{lim}(x)dx \tag{99}$$

and the division into two parts

$$[x_j, (1 - m_j)x_j + m_j x_{j+1}] \cup [(1 - m_j)x_j + m_j x_{j+1}, x_{j+1}]. \tag{100}$$

At last, we introduce the function s_M^{pen} in $L^\infty(\Omega, \{0, 1\})$ by

$$s_M^{pen}(x) = \sum_{j=1}^{M} \mathcal{X}_{[x_j,(1-m_j)x_j+m_j x_{j+1}]}(x) \tag{101}$$

We easily check that $||s_M^{pen}||_{L^1(\Omega)} = ||s^{lim}||_{L^1(\Omega)}$, for all $M > 0$. The bi-valued function s_M^{pen} takes more advantage of the information codified in the density s^{lim}. Figure 2 represents the functions s_M^{pen} obtained with $M = 21$ from the density s^{lim} of Figure 1. We observe that the corresponding values of the cost functions $J(s_{21}^{pen})$ and $J(s^{lim})$ are very close (see Table 1). A similar approach may be used for the case $N = 2$.

3.2.2 Example 2

We now consider the 2-d case with the following initial condition :

$$u_0(x) = \sin(\pi x_1)\sin(\pi x_2); \quad u_1(x) = 0, \quad x = (x_1, x_2) \in (0, 1)^2. \tag{102}$$

We take $T = 1$ and $L = 1/10$.

The same behavior with respect to the value of a appears. When a is small enough (here $a = 5$ for instance), then the centered disc

$$D = \left\{ x = (x_1, x_2) \in \Omega, (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leq \frac{L}{\pi} \right\} \tag{103}$$

corresponds to the optimal global position of ω. We obtain $J(\mathcal{X}_D) \approx 1.3619$. The original problem is then well-posed. For larger values of a, the system is not well-posed, and the numerical simulations exhibit many local minima for the density s, depending on the initialization. Similarly to (91), we initialize the density function with the following periodic functions :

$$s_n^0(x) = LK\sin(n\pi x_1)\sin(n\pi x_2), \quad K = 1 + \left(\frac{1 - \cos(n\pi)}{n\pi}\right)^2, n \in \mathbb{N}, x \in \Omega. \tag{104}$$

Figure 3 depicts in Ω the isovalues of s_n^{lim} obtained with $n = 15$, $a = 15$ and the corresponding bi-value function $s_n^{pen} \in \{0, 1\}$. We obtain $J(s_p^{lim}) \approx 0.8881$ and $J(s_n^{pen}) \approx 0.9411$ respectively. We check that the centered disc D such that $|D| = L$ is not optimal : $J(\mathcal{X}_D) \approx 0.9743$. Similarly, figure 4 displays the results for $a = 50$. We obtain $J(s_n^{lim}) \approx 0.7839$, $J(s_n^{pen}) \approx 0.8543$ and $J(\mathcal{X}_D) \approx 1.3109$. In both cases, the results are in agreement with [12] where the level set method is used.

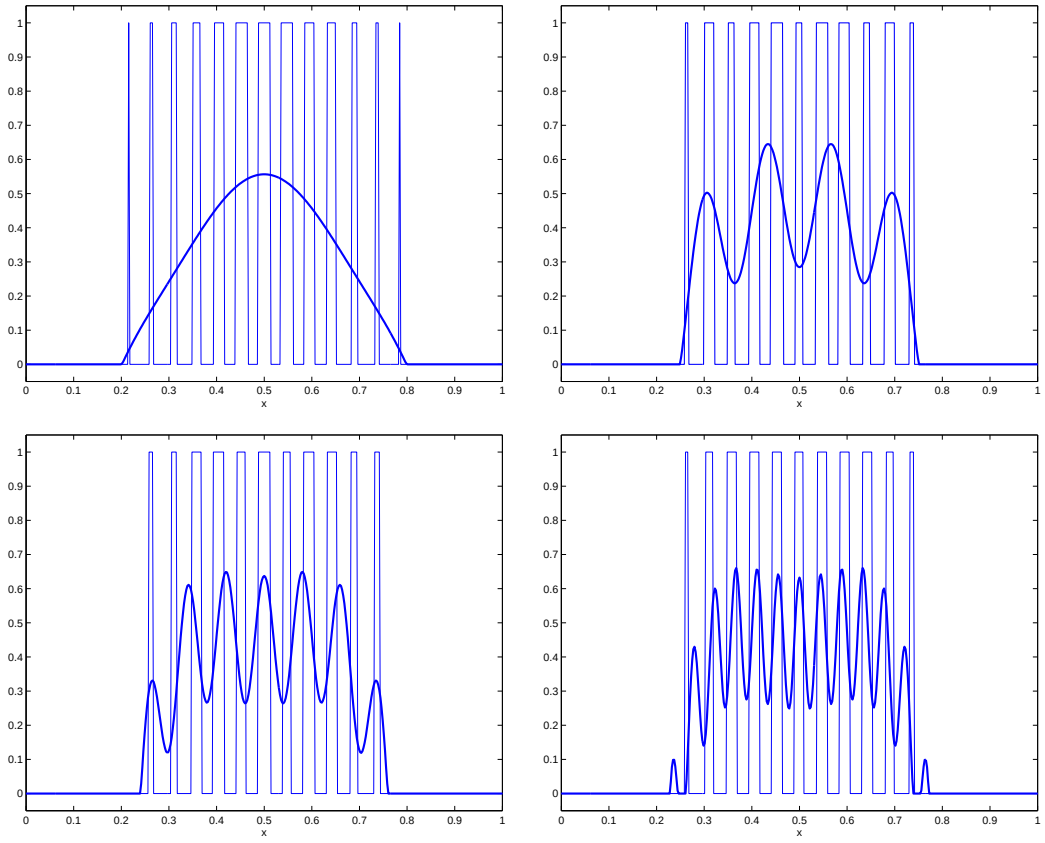

Figure 2: Example 1 - Limit density s^{lim} (from Figure 1) and corresponding bi-valued density s_{21}^{pen} - $a(x) = 10\mathcal{X}_\Omega(x)$ - $T = 1$.

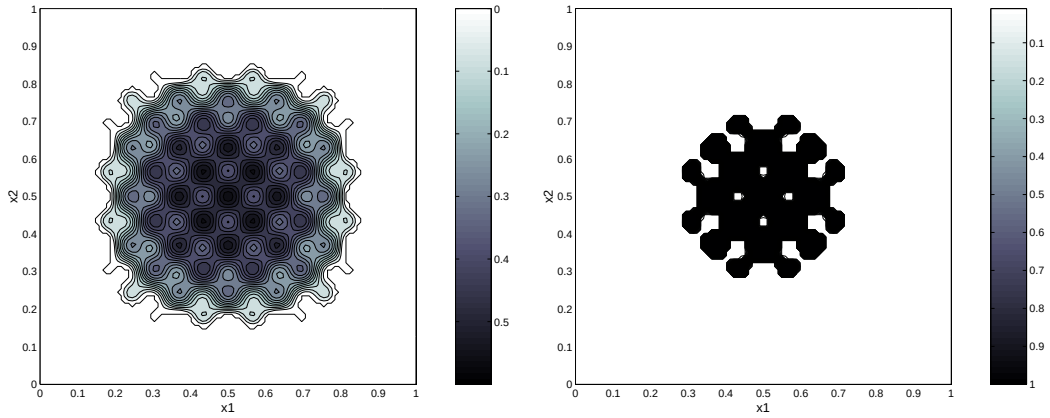

Figure 3: Example 2 - $T = 1$ - $n = 15$ - $a(x) = 15\mathcal{X}_\Omega(x)$ - density function $s^{lim} \in L^\infty(\Omega; [0,1])$ (left) and penalized density function $s_n^{pen} \in L^\infty(\Omega; \{0,1\})$ (right) - $J(s_n^{lim}) \approx 0.8881$ and $J(s_n^{pen}) \approx 0.9411$

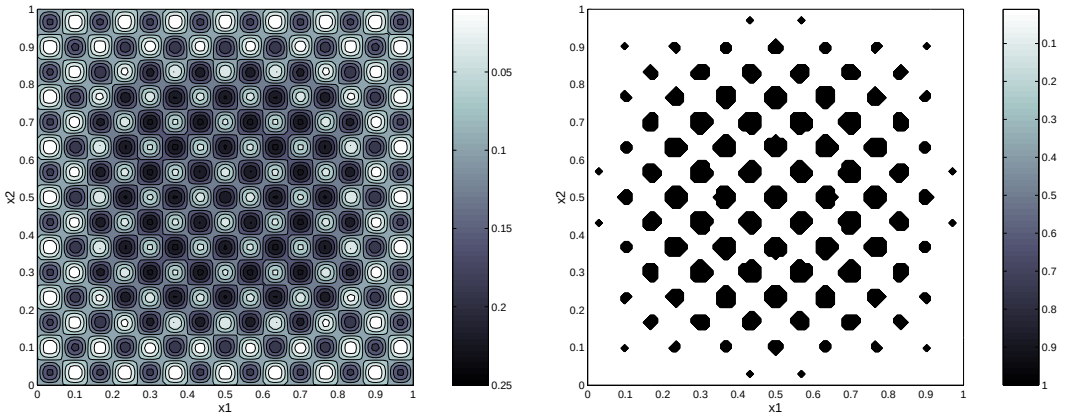

Figure 4: Example 2 - $T = 1$ - $n = 15$ - $a(x) = 50\mathcal{X}_\Omega(x)$ - density function $s_n^{lim} \in L^\infty(\Omega; [0,1])$ (left) and penalized density function $s^{pen} \in L^\infty(\Omega; \{0,1\})$ (right) - $J(s_n^{lim}) \approx 0.7839$ and $J(s_n^{pen}) \approx 0.8543$ - $h = 1/102$

3.2.3 Example 3

Let us now consider the following simple initial condition
(105)

$$u_0(x) = \exp^{-100(x_1-0.3)^2-100(x_2-0.3)^2}\,\mathcal{X}_\Omega; \quad u_1(x) = 0, \quad x = (x_1, x_2) \in (0,1)^2,$$

and $L = 0.2$, $a(x) = 5\mathcal{X}_\Omega(x)$. The initial position is concentrated around the point $(0.3, 0.3)$. Therefore, according to the finite speed of propagation, the optimal position is expected to be concentrated around the point $(0.3, 0.3)$, for small values of T. Figures 5-left depicts the density limit $s_0^{lim,T}$ (initialized with $s^0(x) = L\mathcal{X}_\Omega(x)$) for $T = 1$. For $L = 0.2$, we obtain a laminate. Moreover, the zone of pure damping material $\overline{\omega} = \{x \in \Omega, s^{lim}(x) = 1\}$ is such that $|\overline{\omega}| \approx 0.094$. Thus, for $L \leq 0.094$ and $T = 1$, we check that the optimal density is a characteristic function and the original problem (P) is well-posed. Figures highlight also the reflection of the waves on the corner $(0, 0)$ of the unit square. Figures 5-right depicts the corresponding penalized density which provides a slightly higher value of the cost function.

Figures 6 and 7 displays the results obtained for $T = 2$ and $T = 4$ respectively. For these larger values of T, the optimal density is less concentrated and take into account the propagation of the wave during the whole interval of time. Observe that for symmetry reason, the first diagonal of Ω plays an important role. Moreover, for values $T > 4$, the corresponding limit density $s^{lim,T}$ have a weak variation with respect to T. This is due to the fact that almost all the energy is dissipated at time $T = 4$. Finally, we give in figure 8 the evolution of the energy E with respect to time for $s^{lim,1}, s^{lim,2}$ and $s^{lim,4}$. The decay is exponential. Moreover, we check (Table 2) that for instance, on the interval of time $t \in [0, 1]$, $J(s^{lim,1}, T = 1) \leq J(s^{lim,2}, T = 1) \leq J(s^{lim,4}, T = 1)$. In what concern the energy, we observe however that $E(s^{lim,2}, T = 1) \leq E(s^{lim,1}, T = 1)$.

This example highlights the influence of the value T on the optimal position. Moreover, although we have assumed the time independence of the density s and of the subset ω, it would be easy to obtain numerically a time dependent density. It

suffices to discretize the interval of time $[0,T] = \cup_i [t_i, t_{i+1}]$ and then solve on each interval $[t_i, t_{i+1}]$ the problem (RP) with initial data $(u(t_i,\cdot), u_t(t_i,\cdot))$ associated to the optimal density $s^{lim,i-1}$ for $t \in [t_{i-1}, t_i]$. This would permit to obtain a discretized time sequence of density $(s^{lim}(t))_{\{t\in[0,T]\}}$ and then increases the dissipation of the system.

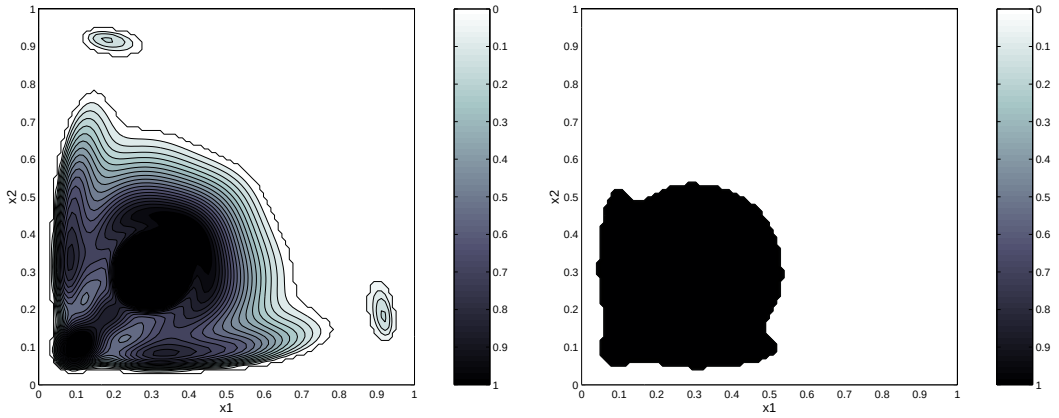

Figure 5: Example 3 - $T = 1$ - $n = 0$ - $a(x) = 5\mathcal{X}_\Omega(x)$ - density function $s_n^{lim} \in L^\infty(\Omega;[0,1])$ (left) and penalized density function $s^{pen} \in L^\infty(\Omega;\{0,1\})$ (right) - $J(s_n^{lim,T}) \approx 0.5334$ and $J(s_n^{pen,T}) \approx 0.5501$

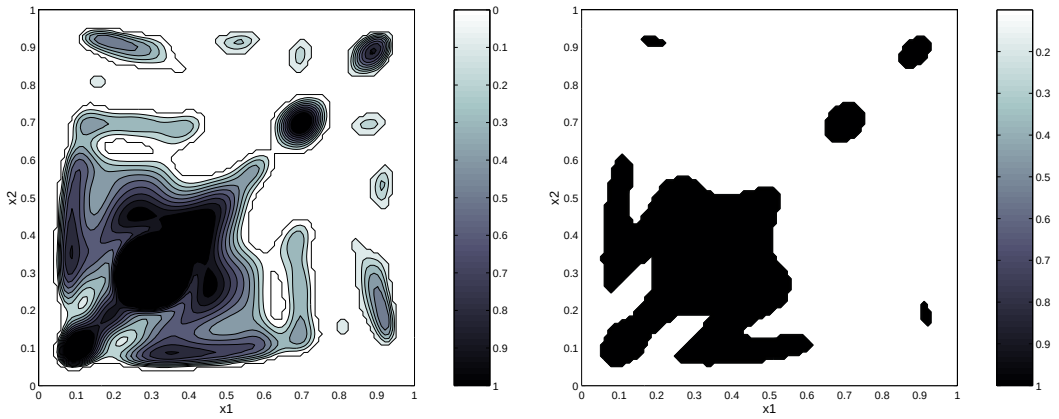

Figure 6: Example 3 - $T = 2$ - $n = 0$ - $a(x) = 5\mathcal{X}_\Omega(x)$ - density function $s_n^{lim,T} \in L^\infty(\Omega;[0,1])$ (left) and penalized density function $s^{pen,T} \in L^\infty(\Omega;\{0,1\})$ (right) - $J(s_n^{lim,T}) \approx 0.7296$ and $J(s_n^{pen,T}) \approx 0.7415$

4 Concluding remarks

The nonlinear optimal design problem which consists in finding the optimal shape and position of the internal damping set for the stabilization of the linear wave equation in

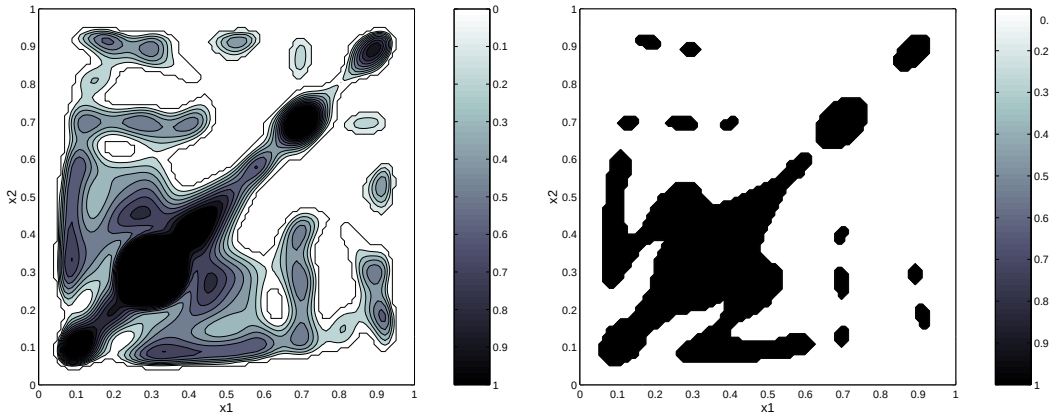

Figure 7: Example 3 - $T = 4$ - $n = 0$ - $a(x) = 5\mathcal{X}_\Omega(x)$ - density function $s_n^{lim} \in L^\infty(\Omega; [0,1])$ (left) and penalized density function $s^{pen,T} \in L^\infty(\Omega; \{0,1\})$ (right) - $J(s_n^{lim,T}) \approx 0.8053$ and $J(s_n^{pen,T}) \approx 0.8257$.

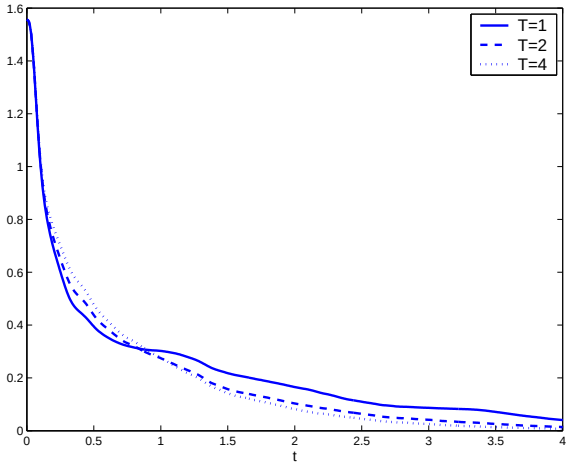

Figure 8: Example 3 - Evolution of the energy E vs. $t \in [0,4]$ of the dissipative wave system associated to the density $S^{lim,T}$, $T = 1, 2, 4$

	$J(s^{lim,1},T)$	$E(s^{lim,1},T)$	$J(s^{lim,2},T)$	$E(s^{lim,2},T)$	$J(s^{lim,4},T)$	$E(s^{lim,4},T)$
$T=1$	0.5323	0.3020	0.5570	0.2740	0.5853	0.2767
$T=2$	0.7624	0.1649	0.7294	0.1028	0.7434	0.0812
$T=4$	0.9457	0.0401	0.8220	0.0142	0.8053	0.0073

Table 2: Example 3 - Value of the cost function $J(s^{lim,T})$ and of the energy $E(s^{lim,T},T)$ for $T = 1, 2, 4$.

1-d and 2-d has been solved both theoretically and numerically. Only independence with respect to time and a volume constraint is assumed on the damping set.

For the theoretical part, our approach is based on a characterization of the free-divergence vector fields. This methodology, which may be only used when the state equation has a particular free-divergence form, has, for the contrary, two important advantages:

- On one hand, no additional regularity hypotheses on the initial data are needed to carry out the method. In fact, we only assume the regularity which is needed for the state equation to be well-posed and for the cost function to make sense.
- On the other hand, the computation of the relaxed optimal solutions in the form of Young measures which in fact are first-order laminates leads to a numerical penalization technique to recover the quasi-optimal classical designs from the relaxed ones.

It would be very interesting to check this approach in some other more complicated systems such as the elasticity system.

For the numerical part, it is worthwhile to emphasize the influence of the value of the damping potential a on the well/ill-posedeness character of the optimization problem. The numerical simulations seem to indicate that when a is small the problem is well-posed, that is, there is a minimizer in the class of characteristic functions, but if a is large, then the problem is ill-posed and it is necessary to relax it. The computation of this bifurcation value of the potential is an interesting open problem.

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