Optimal design of the support of the control for the wave equation

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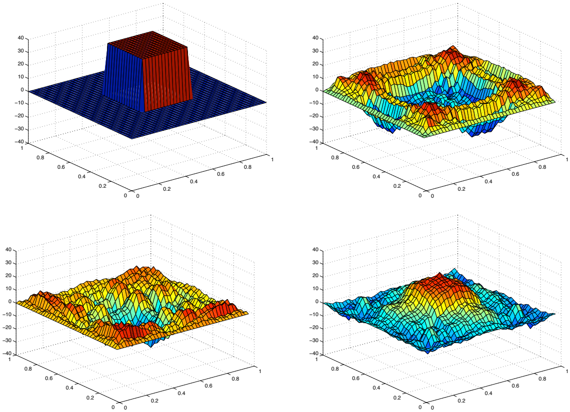

Figure: Boundary controllability of a discontinuous initial condition y^0 - Wave solution $y(x,t)$ for $t = 0, 3/7, 6/7, 9/7$

Exact (null) Controllability

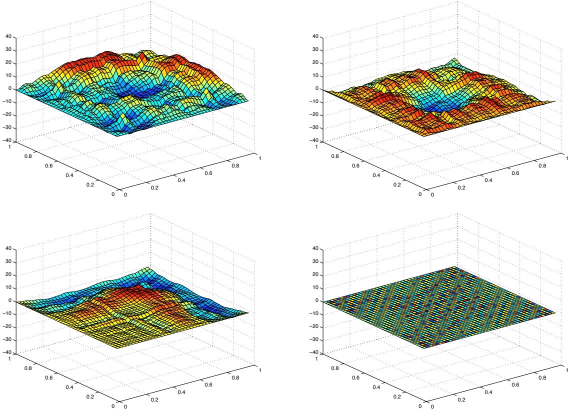

Figure: Boundary controllability of a discontinuous initial condition y^0 - Wave solution $y(x,t)$ for $t = 12/7, 15/7, 18/7$ and $t = T = 3$

Let Ω a Lipschitzian bounded domain in $\mathbb{R}^N$, $N = 1, 2$, two functions $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $T > 0$. Let

$$V(y^0, y^1, T) = \{\omega \subset \Omega \quad \text{such that} \quad (3) \quad \text{holds}\} : \tag{1}$$

There exists a control function [1] $v_\omega \in L^2(\omega \times (0, T))$ such that the unique solution $y \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ of

$$\begin{cases} y_{tt} - \Delta y = v_\omega \mathcal{X}_\omega, & \Omega \times (0, T), \\ y = 0, & \partial\Omega \times (0, T), \\ (y(\cdot, 0), y_t(\cdot, 0)) = (y^0, y^1), & \Omega, \end{cases} \tag{2}$$

satisfies

$$y(., T) = y_t(., T) = 0, \quad \text{in} \quad \Omega. \tag{3}$$

[1] J-L. Lions, Contrôlabilité exacte de systèmes distribués, RMA 8, 1988

$V(y^0, y^1, T)$

- $\forall T > 0, \qquad \Omega \subset V(y^0, y^1, T)$!
- For $N = 1$, any ω belongs to $V(y^0, y^1, T)$ provided that $T > diam(\Omega \backslash \omega)$.
- For $N = 2$, assuming $\Omega \in C^\infty$, any subset ω satisfying the Geometric Control Condition in Ω:

  "Every ray of geometric optics that propagates in Ω and is reflected on its boundary enters ω in time less than T"

  belongs to $V(y^0, y^1, T)$
- If Ω is a rectangular domain (convex ?), any ω belongs to $V(y^0, y^1, T)$ provided that $T > diam(\Omega \backslash \omega)$ [2]

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Let $L \in (0,1)$ and

$$V_L(y^0, y^1, T) = \{\omega \in V(y^0, y^1, T), \quad |\omega| = L|\Omega|\} \tag{4}$$

We consider the following NON LINEAR optimal design problem :

$$(\mathcal{P}_\omega) : \inf_{\omega \subset V_L(y^0, y^1, T)} J(\mathcal{X}_\omega), \quad \text{where} \quad J(\mathcal{X}_\omega) = \frac{1}{2} ||v_\omega||^2_{L^2((0,T)\times\omega)} \tag{5}$$

QUESTIONS -

1. Is the problem well-posed in the class of characteristic functions ?
   - Usually, the answer is no : the infimum is not reached and the optimal domain is composed of an infinite number of disjoint components
   - In this case, what is a well-posed relaxation of $(\mathcal{P}_\omega)$?
2. How to approximate an optimal domain ?

[3]

[3] M. Asch - G. Lebeau, Geometrical aspects of exact boundary controllability for the wave equation - A numerical study ; Esaim Cocv, 1998

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$(\mathcal{P}_\omega)$ is related to the following problem [4]

$$(\mathcal{D}_\omega): \quad \inf_{\omega\in\Omega, |\omega|=L|\Omega|} J_2(\mathcal{X}_\omega) \equiv \frac{1}{2}\int_0^T\int_\Omega (|y_t(x,t)|^2 + |\nabla y(x,t)|^2)dxdt = \int_0^T E(t)dt \tag{6}$$

where y is the unique solution of the damped wave equation ($a \in L^\infty(\Omega;\mathbb{R}^+)$)

$$\begin{cases} y_{tt} - \Delta y + \mathbf{a(x)\mathcal{X}_\omega y_t} = 0 & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y^0, \quad y_t(0,\cdot) = y^1 & \text{in } \Omega, \end{cases} \tag{7}$$

[4] AM, P. Pedregal, F. Periago, Optimal design of the damping set for the stabilization of the wave equation, J. Diff. Equations, 2006

Let us consider the homogeneous equation

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0, & \Omega \times (0,T), \\ \varphi = 0, & \partial\Omega \times (0,T), \\ (\varphi(\cdot,0), \varphi_t(\cdot,0)) = (\varphi^0, \varphi^1), & \Omega, \end{cases} \quad (8)$$

Lemma (Caracterizations of the controls v_ω)

$v_\omega \in L^2(\omega \times (0,T))$ is a control for $(y^0, y^1) \in H^1_0(\Omega) \times L^2(\Omega)$ iff

$$\int_0^T \int_\omega \varphi v_\omega \, dxdt = < \varphi_t(\cdot,0), y^0 >_{H^{-1}(\Omega),H^1_0(\Omega)} - \int_\Omega y^1 \varphi(\cdot,0)dx, \forall (\varphi^0,\varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega) \quad (9)$$

(9) is an optimality condition for the critical points of $\mathcal{J} : L^2(\Omega) \times H^{-1}(\Omega) \to \mathbb{R}$:

$$\mathcal{J}(\varphi^0,\varphi^1) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dtdx + < \varphi_t(\cdot,0), y^0 >_{H^{-1}(\Omega),H^1_0(\Omega)} - \int_\Omega y^1 \varphi(\cdot,0)dx, \quad (10)$$

Theorem

If $\mathcal{J}$ has a minimizer $(\varphi^0,\varphi^1) \in L^2 \times H^{-1}$, then $v_\omega = -\varphi \mathcal{X}_\omega$ is a control.

Observability inequality (Lions, Haraux):

$$|| (\varphi^0,\varphi^1) ||^2_{L^2(\Omega)\times H^{-1}(\Omega)} \le C(T,\omega) \int_0^T \int_\omega | \varphi |^2 dxdt, \quad \forall (\varphi^0,\varphi^1) \quad (11)$$

ω fixed - HUM control v_ω (Overview)

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IN PRACTICE, ω being fixed such a control is determined by introducing the isomorphism Λ from $L^2(\Omega) \times H^{-1}(\Omega)$ onto $H_0^1(\Omega) \times L^2(\Omega)$ defined by $\Lambda(\varphi^0, \varphi^1) := (\psi_t(0), -\psi(0))$ as follows :

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and then solve the linear problem

$$\Lambda(\varphi^0, \varphi^1) = (y^1, -y^0). \tag{13}$$

The HUM control is $v_\omega = -\varphi\mathcal{X}_\omega$ and $y = \psi$

The HUM control v_ω is of minimal $L^2(0, T)$ norm !!

$\Longrightarrow$ Problem $(\mathcal{P}_\omega)$ is then "reduced" to find the best HUM control !

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Let $\boldsymbol{\theta} \in W^{1,\infty}(\Omega, \mathbb{R}^2)$, $\eta > 0$ and the perturbation $\omega^\eta = (I + \eta\boldsymbol{\theta})(\omega)$. The Frechet derivative of J in the direction $\boldsymbol{\theta}$ is defined by

$$\frac{\partial J(\mathcal{X}_\omega)}{\partial \omega} \cdot \boldsymbol{\theta} \equiv \lim_{\eta \to 0} \frac{J(\mathcal{X}_{\omega^\eta}) - J(\mathcal{X}_\omega)}{\eta} \tag{14}$$

Let $(y^0, y^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ and $\omega \in C^1(\Omega)$. The derivative of J with respect to ω in the θ-direction exists and is given by the following expression:

$$\frac{\partial J(\mathcal{X}_\omega)}{\partial \omega} \cdot \theta = \frac{1}{2} \int_\omega \int_0^T (2 v_\omega V_\omega + v_\omega^2 div\theta) dt dx \tag{15}$$

where V_ω is the HUM control (of minimal L^2-norm) with support ω associated to the following system :

$$\begin{cases} Y_{tt} - \Delta Y - \nabla(div\theta).\nabla y + div((\nabla\theta + \nabla\theta^T).\nabla y) = V_\omega \mathcal{X}_\omega, & \Omega \times (0,T), \\ Y = 0, & \partial\Omega \times (0,T), \\ (Y(\cdot,0), Y_t(\cdot,0)) = (\nabla y^0.\theta, \nabla y^1.\theta), & \Omega. \end{cases} \tag{16}$$

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Theorem

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■

Idea of the Proof. Let assume that $\omega^\eta = (I + \eta\theta)(\omega) \in V(y^0, y^1, T)$ and let $(y^\eta(x), v^\eta(x)) = (y(x^\eta), v(x^\eta))$. Let $\mathcal{A}^\eta(\boldsymbol{\theta}) = \det(\nabla\mathcal{F}^\eta)(Id + \eta\nabla\boldsymbol{\theta})^{-1} \cdot (Id + \eta\nabla\boldsymbol{\theta})^{-T}$. (y^η, v^η) is solution of

$$\begin{cases} y^\eta_{tt} - det(\nabla\mathcal{F}^\eta)^{-1}\mathrm{div}(\mathcal{A}^\eta(\boldsymbol{\theta}) \cdot \nabla y^\eta) = v^\eta \mathcal{X}_\omega, & \Omega \times (0, T), \\ y^\eta = 0, & \partial\Omega \times (0, T), \\ (y^\eta(\cdot, 0), y^\eta_t(\cdot, 0)) = (y^0 + \eta\nabla y^0 \cdot \boldsymbol{\theta}, y^1 + \eta\nabla y^1 \cdot \boldsymbol{\theta}), & \Omega, \end{cases} \tag{17}$$

such that $y^\eta(\cdot, T) = 0$, $y^\eta_t(\cdot, T) = 0$ on Ω. This system is controllable thanks to

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Idea of the Proof. Let assume that $\omega^\eta = (I + \eta\theta)(\omega) \in V(y^0, y^1, T)$ and let $(y^\eta(x), v^\eta(x)) = (y(x^\eta), v(x^\eta))$. Let $\mathcal{A}^\eta(\boldsymbol{\theta}) = \det(\nabla \mathcal{F}^\eta)(Id + \eta\nabla\boldsymbol{\theta})^{-1} \cdot (Id + \eta\nabla\boldsymbol{\theta})^{-T}$. (y^η, v^η) is solution of

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Theorem

Let $\boldsymbol{\nu}$ be the outward normal derivative of ω. The derivative of J with respect to ω is given by the following expression:

$$\frac{\partial J(\mathcal{X}_\omega)}{\partial\omega}\cdot\boldsymbol{\theta} = -\frac{1}{2}\int_{\partial\omega}\int_0^T v_\omega^2(x,t)dt\,\boldsymbol{\theta}\cdot\boldsymbol{\nu}\,d\sigma \tag{20}$$

where v_ω is the HUM control (of minimal L^2-norm) with support on ω which drives to the rest at time $t = T$ the solution y of (46). ■

Remark

- *The derivative is independent of any adjoint problem !*
- $\omega_1 \subset \omega_2 \subset \Omega \Longrightarrow J(\mathcal{X}_{\omega_2}) \le J(\mathcal{X}_{\omega_1})$ *(because a descent direction is given by $\theta = c\nu$ with $c \ge 0$)*

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Proof. [Cea Method ,[5]]

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[5] J. Cea, Conception optimale ou identification de formes- calcul rapide de la dérivée directionnelle de la fonction cout, Math. Model. Num. Anal, 1986

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  $\Longrightarrow$ $q = -\varphi$ and $p = 0$ on $\omega \times (0,T)$ and

-

$$\frac{\partial}{\partial\omega}\mathcal{L}(\omega,\varphi,\psi,p,q)\cdot\theta = \frac{1}{2}\int_0^T\int_\omega \mathrm{div}(\varphi^2\theta)dtdx + \int_0^T\int_\omega \mathrm{div}(\varphi q\theta)dtdx = -\frac{1}{2}\int_0^T\int_\omega \mathrm{div}(\varphi^2\theta)dtdx \tag{27}$$

- The initial condition $(q^0, q^1) \in H_0^1(\Omega) \times L^2(\Omega)$ are such that the solution

$$\left\{\begin{array}{ll} q_{tt} - \Delta q = 0, & \Omega \times (0,T), \\ q = 0, & \partial\Omega \times (0,T), \\ (q(\cdot,0), q_t(\cdot,0)) = (q^0, q^1), & \Omega, \end{array}\right. \quad \left\{\begin{array}{ll} p_{tt} - \Delta p = -(\varphi + q)\mathcal{X}_\omega, & \Omega \times (0,T), \\ p = 0, & \partial\Omega \times (0,T), \\ (p(\cdot,T), p_t(\cdot,T)) = (0,0), & \Omega, \end{array}\right. \tag{25}$$

ensures $(p(\cdot,0), p_t(\cdot,0)) = (0,0)$ or equivalently, (defining $F = \varphi + q$)

$$\left\{\begin{array}{ll} F_{tt} - \Delta F = 0, & \Omega \times (0,T), \\ F = 0, & \partial\Omega \times (0,T), \\ (F(\cdot,0), F_t(\cdot,0)) = (\varphi^0 + q^0, \varphi^1 + q^1), & \Omega, \end{array}\right. \quad \left\{\begin{array}{ll} p_{tt} - \Delta p = -F\mathcal{X}_\omega, & \Omega \times (0,T), \\ p = 0, & \partial\Omega \times (0,T), \\ (p(\cdot,T), p_t(\cdot,T)) = (0,0), & \Omega, \end{array}\right. \tag{26}$$

- $-F\mathcal{X}_\omega$ is then the HUM-control which drives the solution p

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Let λ be a Lagrange multiplier and

$$J_\lambda(\mathcal{X}_\omega) = J(\mathcal{X}_\omega) + \lambda\left(\int_\omega dx - L\int_\Omega dx\right) \tag{28}$$

Theorem

The derivative of J_λ with respect to ω is given by the following expression:

$$\frac{\partial J_\lambda(\mathcal{X}_\omega)}{\partial \omega}\cdot \boldsymbol{\theta} = -\frac{1}{2}\int_{\partial\omega}\int_0^T v_\omega^2(x,t)dt\, \boldsymbol{\theta}\cdot\boldsymbol{\nu}\, d\sigma + \lambda\int_{\partial\omega}\boldsymbol{\theta}\cdot\boldsymbol{\nu}\, d\sigma \tag{29}$$

■

ALGORITHM - [6] $\omega^{(0)} \in V_L(y^0, y^1, T)$, $\omega^{(k+1)} = (I+\eta\theta^k)\omega^{(k)}$, $\quad k \geq 0$ with

$$\theta^{(k)} = \left(\frac{1}{2}\int_0^T v^2_{\omega^{(k)}}(\boldsymbol{x},t)dt - \lambda^{(k)}\right)\boldsymbol{\nu}^{(\boldsymbol{k})}, \forall x \in \Omega \tag{30}$$

and

$$\lambda^{(k)} = \frac{1}{2}\int_{\omega^{(k)}} div\left(\int_0^T v^2_{\omega^{(k)}}(\boldsymbol{x},t)dt\boldsymbol{\nu}^{(\boldsymbol{k})}\right)dx \Big/ \int_{\omega^{(k)}} div(\boldsymbol{\nu}^{(\boldsymbol{k})})dx \tag{31}$$

Remark (Fundamental)

if $\omega^{(k)} \in V(y^0, y^1, T)$, then $\omega^{(k+1)} \in V(y^0, y^1, T)$ because $J(\mathcal{X}_{\omega^{(k+1)}}) \leq J(\mathcal{X}_{\omega^{(k)}}) < \infty$

[6] M. Burger, A survey on level set methods for inverse problems and optimal design, Eur. J. Appl. Math., 2005

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[6] M. Burger, A survey on level set methods for inverse problems and optimal design, Eur. J. Appl. Math., 2005

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Theorem

For any $\mathbf{x_0} \in \Omega \subset \mathbb{R}^2$ and ρ such that $D(\mathbf{x_0}, \rho) \subset \Omega$, the functional associated to $\Omega \backslash D(\mathbf{x_0}, \rho)$ may be expressed as follows :

$$J_\lambda(\mathcal{X}_{\Omega \backslash D(\mathbf{x_0},\rho)}) = J_\lambda(\mathcal{X}_\Omega) + \pi\rho^2 \underbrace{\left(\frac{1}{2}\int_0^T v_\Omega^2(\mathbf{x_0}, t)dt - \lambda\right)}_{\equiv f(\mathbf{x_0})} + o(\rho^2) \tag{32}$$

in term only of the HUM control v_Ω associated to (46) with $\omega = \Omega$. ■

- The best support of the form $\Omega \backslash D(\mathbf{x_0}, \rho)$ is for $\mathbf{x_0}$ minimizing $f(\mathbf{x_0})$
- Consequently, the best support of the control is the points maximizing f ;
- INITIALIZATION OF THE ALGORITHM -

$$\omega^0 = \{\mathbf{x} \in \Omega, \tfrac{1}{2}\int_0^T v_\Omega^2(\mathbf{x}, t)dt > \lambda\} \text{ with } \lambda \text{ such that } |\omega^0| = L|\Omega|.$$

[7] Sokolowski J, Zochowski A, On the topological derivative in shape optimization, SIAM J. Control. Optim, 1999

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Find $(\varphi_h^0, \varphi_h^1) \in l_h^2 \times h_h^{-1}$ such that $\Lambda_h(\varphi_h^0, \varphi_h^1) = (\psi_h^1, -\psi_h^0) = (y_h^1, -y_h^0)$ where

$$\begin{cases} (\varphi_h)_{tt} - \Delta_h \varphi_h = 0, & \Omega \times (0,T), \\ \varphi_h = 0, & \partial\Omega \times (0,T), \\ (\varphi_h(\cdot,0), (\varphi_h)_t(\cdot,0) = (\varphi_h^0, \varphi_h^1), & \Omega, \end{cases} \qquad \begin{cases} (\psi_h)_{tt} - \Delta_h \psi_h = -\varphi_h \mathcal{X}_{\omega_h}, & \Omega \times (0,T), \\ \psi_h = 0, & \partial\Omega \times (0,T), \\ (\psi_h(\cdot,T), (\psi_h)_t(\cdot,T)) = (0,0), & \Omega, \end{cases} \tag{33}$$

leading to $v_h = -\varphi_h \mathcal{X}_\omega \in l_h^2((0,T)\times\omega)$.
Usual finite element method or finite difference method may leads to a divergence of v_h [8] :

$$||v - P(v_h)||_{L^2(\Omega)} > C \exp(1/h) \tag{34}$$

and to a very bad conditioning number:

$$cond(\Lambda_h) = O(\exp(1/h)) \tag{35}$$

$\Longrightarrow$ Problem comes from the spurious high frequencies components.

[8] Glowinski-Li-Lions, A numerical approach to exact boundary controllability of the wave equation, Int. J. Numer. Methods. Eng., 1989

The convergence of v_h is restored if we use, for instance, the following scheme to solve φ (and the same for ψ):

$$(I + \frac{h^2}{4}\partial^2_{xh})(I + \frac{h^2}{4}\partial^2_{yh})(\boldsymbol{\varphi_h})_{tt} - \boldsymbol{\Delta_h}\boldsymbol{\varphi_h} = 0 \tag{36}$$

We observe that $cond(\boldsymbol{\Lambda_h}) = O(h^{-2})$ and

Theorem

The semi-discrete scheme (36) is uniformly controllable with respect to h.

$$\| (\varphi^{\mathbf{0}}_{\boldsymbol{h}}, \varphi^{\mathbf{1}}_{\boldsymbol{h}}) \|^2_{L^2_h \times H^{-1}_h} \leq C_h \int_0^T \int_{\omega_h} | \boldsymbol{\varphi_h} |^2 \, dxdt \tag{37}$$

In addition, if $(P(\boldsymbol{y^0_h}), P(\boldsymbol{y^1_h}))$ converges strongly toward (y^0, y^1) in $H^1_0(\Omega) \times L^2(\Omega)$ as h goes to 0, then the corresponding control $\tilde{\boldsymbol{v}}_{\boldsymbol{h}}$ of minimal l^2-norm is such that $\lim_{h\to 0} \| P(\tilde{\boldsymbol{v}}_{\boldsymbol{h}}) - v \|_{L^2(((0,T)\times\omega)} = 0$. ■

(see [9] for the boundary case using mixed finite element).

[9] Castro C., AM, Micu S., Numerical approximations of the boundary control of the 2-D wave equation with mixed finite elements, IMA J. Numer. Analysis, (2007).

- Very efficient finite difference scheme in 1D: [10]:

$$\Delta_{\Delta t}\phi_{h,\Delta t} + \frac{1}{4}(h^2 - \Delta t^2)\Delta_h \Delta_{\Delta t}\phi_{h,\Delta t} - \Delta_h \phi_{h,\Delta t} = 0 \quad (38)$$

uniformly controllable (with respect to h and Δt) IIF $\Delta t \leq h\sqrt{T/2}$

- Very efficient finite difference scheme in 2D (same in 3D) [11]:

$$(I + \frac{h^2}{4}\partial^2_{xh})(I + \frac{h^2}{4}\partial^2_{yh})\Delta_{\Delta t}\varphi_{h,\Delta t} - (I + \frac{h^2}{4}\partial^2_{xh})\partial^2_{yh} - (I + \frac{h^2}{4}\partial^2_{yh})\partial^2_{xh} = 0 \quad (39)$$

+ Newmark strategy is uniformly (with respect to h and Δt) IIF $\Delta t \leq h/\sqrt{2}$.

[10] AM, A uniformly controllable and implicit scheme for the 1-D wave equation, M2AN (2005)

[11] AM, An implicit scheme uniformly controllable for the 2-D wave equation, J. Sci. Comp.

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$$y^0(\mathbf{x}) = \exp^{-100(x_1-0.3)^2-100(x_2-0.3)^2}; \qquad y^1(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega = (0,1)^2 \tag{40}$$

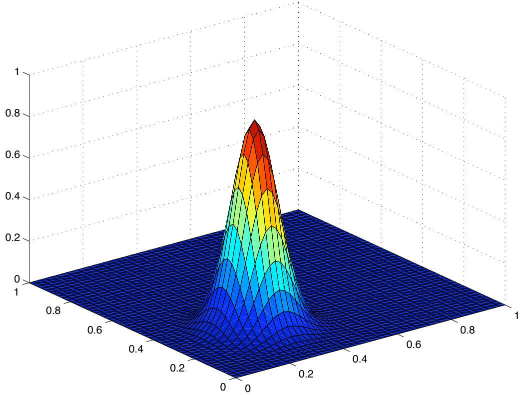


Figure: $y_0(\mathbf{x})$

[12] For numerical animations, see www-math.univ-fcomte.fr/amunch/gallery.htm.

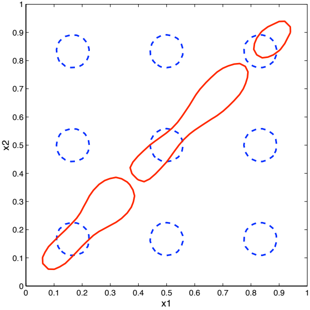


Figure: $T = 3$ - Zeros of the initial level set $\psi^0_{p=3}$ and of the limit one $\psi^{lim}_{p=3}$- $J(\mathcal{X}_{\omega^0_3}) \approx 12.50$, $J(\mathcal{X}_{\omega^{lim}_3}) \approx 6.42$

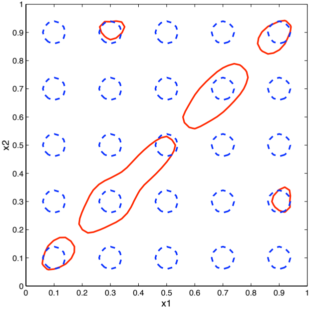


Figure: $T = 3$ - Zeros of the initial level set $\psi^0_{p=3}$ and of the limit one $\psi^{lim}_{p=5}$- $J(\mathcal{X}_{\omega^0_5}) \approx 8.47$, $J(\mathcal{X}_{\omega^{lim}_5}) \approx 5.07$

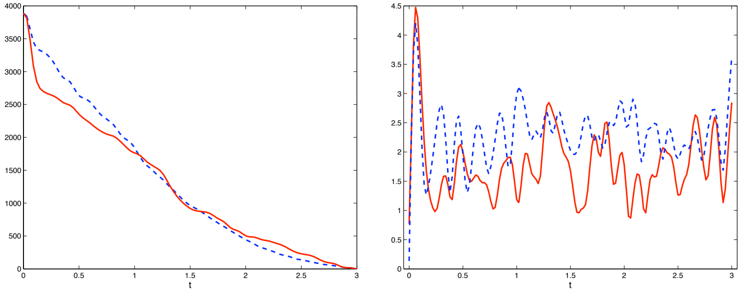
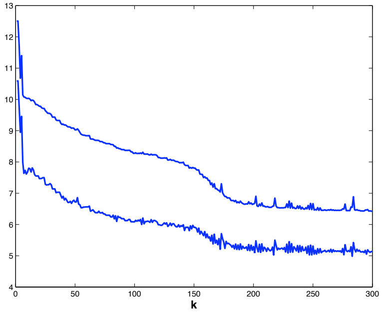


Figure: $T = 3$ - Energy $E(y,t)$ of the system (left) and $||v_\omega||_{L^2(\omega)}$ (right) vs. t- $E(y,T)/E(y,0) \approx 2.40 \times 10^{-6}$ corresponding to the initial level set function $\psi^0_{p=3}$ $(--)$ and to the limit one $\psi^{lim}_{p=3}$ $(-)$.



$$\frac{||\phi^0_{\omega^k}||^2_{L^2(\Omega)} + ||\phi^1_{\omega^k}||^2_{H^{-1}(\Omega)}}{\int_{\omega^k}\int_0^T (\phi_{\omega^k}(\boldsymbol{x},t))^2 dt\ dx} \quad (41)$$

Figure: $T = 3$- Evolution of $J_0(\mathcal{X}_{\omega^k})$ (top) and corresponding ratio (41) (bottom) vs. k.

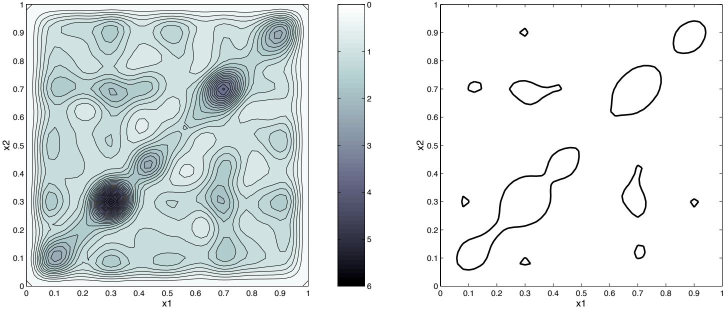
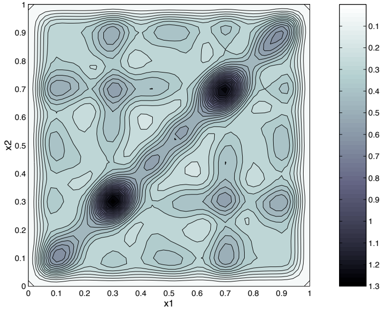


Figure: $T = 3$ - Left: Iso-values of $\int_0^T v_\Omega^2(\boldsymbol{x}, t)dt$ on Ω - Right : $\partial\omega^0 = \{\boldsymbol{x} \in \Omega, 1/2 \int_0^T (v_\Omega(\boldsymbol{x}, t))^2 dt - \lambda = 0\}$, $\lambda \approx 1.5$, $\omega^0 \in V(y^0, y^1, T)$.



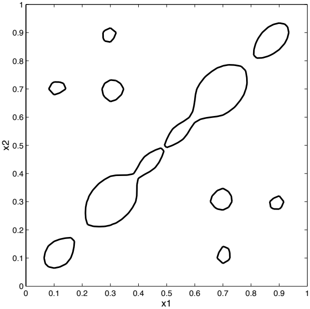


Figure: $T = 10$ - Left: Iso-values of $1/2 \int_0^T v_\Omega^2(\boldsymbol{x}, t)dt$ on Ω - Right : $\partial\omega^0 \equiv \{\boldsymbol{x} \in \Omega, 1/2 \int_0^T (v_\Omega(\boldsymbol{x}, t))^2 dt - \lambda = 0\}$, $\lambda \approx 0.82$, $\omega^0 \in V(y^0, y^1, T)$

Topological derivative - $T = 1$

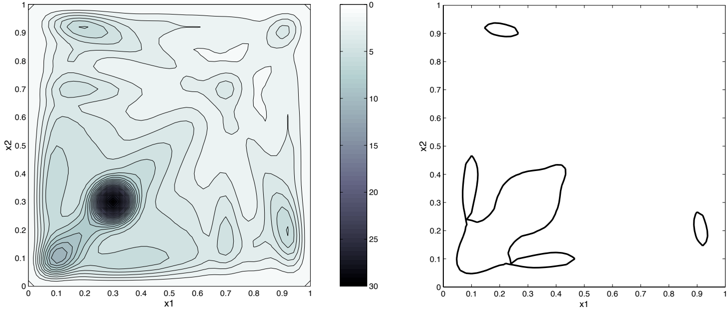


Figure: $T = 1$ - Left: Iso-values of $1/2 \int_0^T v_\Omega^2(\boldsymbol{x}, t) dt$ on Ω - Right : $\partial\omega^0 = \{\boldsymbol{x} \in \Omega, 1/2 \int_0^T (v_\Omega(\boldsymbol{x}, t))^2 dt - \lambda = 0\}$, $\lambda \approx 5.30$, $\omega^0 \notin V(T, y^0, y^1)$

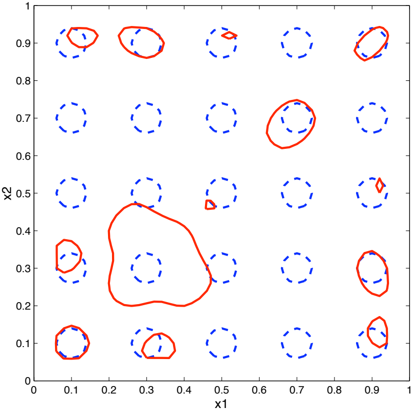


Figure: Limit of $\partial\omega_k = \{\boldsymbol{x} \in \Omega, \psi_k(\boldsymbol{x}) = 0\}$ vs. k for $T = 1$ - $J(\mathcal{X}_{\omega^0_{p=5}}) \approx 29.321$, $J(\mathcal{X}_{\omega^{lim}_{p=5}}) \approx 15.314$

$$(RP_\omega): \quad \inf_{s \in L^\infty(\Omega)} \frac{1}{2}\int_\Omega s(\boldsymbol{x}) \int_0^T v_s^2(\boldsymbol{x}, t) dt dx \tag{42}$$

where v_s (function of the density s) is such that sv_s if the HUM control associated to the unique solution of

$$\begin{cases} y_{tt} - \Delta y = s(x) v_s & \text{in } (0,T) \times \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega, \\ y(0,\cdot) = y^0, \quad y_t(0,\cdot) = y^1 & \text{in } \Omega, \\ 0 \le s(x) \le 1, \quad \int_\Omega s(x)\, dx = L\,|\Omega| & \text{in } \Omega. \end{cases} \tag{43}$$

$\Longrightarrow$ The set $\{\mathcal{X}_\omega \in L^\infty(\Omega, \{0,1\})\}$ is replaced by it convex envelopp $\{s \in L^\infty(\Omega, [0,1])\}$ for the weak-$\star$ topology.

Theorem

Problem (RP_ω) is a full relaxation of $(\mathcal{P}_\omega)$ in the sense that

- *there are optimal solutions for (RP_ω);*
- *the infimum of $(\mathcal{P}_\omega)$ equals the minimum of (RP_ω);*
- *if s is optimal for (RP_ω), then optimal sequences of damping subsets ω_j for $(\mathcal{P}_\omega)$ are exactly those for which the Young measure associated with the sequence of their characteristic functions $\mathcal{X}_{\omega_j}$ is precisely*

$$s(x)\delta_1 + (1 - s(x))\delta_0. \tag{44}$$

13

[13] AM., A shape optimal design problem related to the exact controllability of the 2-D wave equation. C.R.Acad. Sci. Paris Serie I (2006)

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where v_s (function of the density s) is such that sv_s if the HUM control associated to the unique solution of

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$\Longrightarrow$ The set $\{\mathcal{X}_\omega \in L^\infty(\Omega,\{0,1\})\}$ is replaced by it convex envelopp $\{s \in L^\infty(\Omega,[0,1])\}$ for the weak-$\star$ topology.

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Problem (RP_ω) is a full relaxation of $(\mathcal{P}_\omega)$ in the sense that

- *there are optimal solutions for (RP_ω);*
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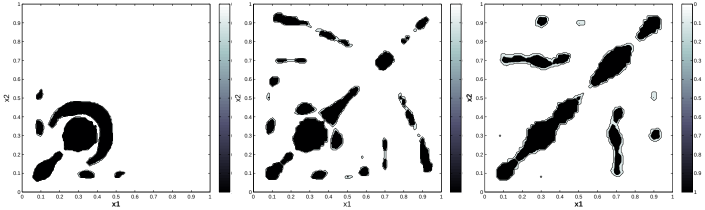
Let $\Omega = (0,1)^2$, and $(u^0, u^1) = (e^{-80(x_1-0.3)^2-80(x_2-0.3)^2}, 0)$ and $L = 1/10$



Iso-value of the optimal density s on Ω for $T = 0.5$, $T = 1$, $T = 3$

- $\{x \in \Omega, 0 < s(x) < 1\} = \emptyset$, $(P_\omega) = (RP_\omega)$ and is well-posed

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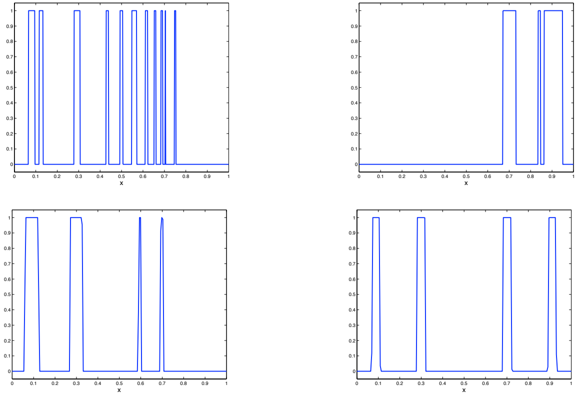


Figure: Limit density function s^{lim} for $T = 0.5$ (top left), $T = 1.5$ (top right), $T = 2.5$ (bottom left) and $T = 3$ (bottom right) initialized with $s^0 = L = 0.15$ on $\Omega = (0,1)$

Remark

T AND $|\omega|$ MAY BE ARBITRARILY SMALL !!!

For any $u^0, u_T \in L^2(\Omega)$, ϵ, ϵ_1, T (u_T is the target)

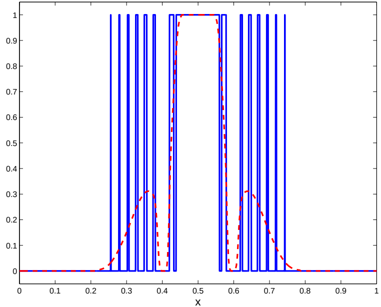
$$(\mathcal{P}_\omega): \inf_{\omega\subset\Omega, |\omega|=L|\Omega|} I(\mathcal{X}_\omega), \quad \text{where} \quad I(\mathcal{X}_\omega) = \frac{1}{2}||v_\omega||^2_{L^2((0,T)\times\omega)} + \frac{\epsilon^{-1}}{2}||u_T - u(T)||^2_{L^2(\Omega)} \tag{45}$$

where $v_\omega \in L^2(\omega \times (0,T))$ is the approximate control for

$$\begin{cases} u_t - \Delta u = v_\omega \mathcal{X}_\omega, & \Omega \times (0,T), \\ u = 0, & \partial\Omega \times (0,T), \\ u(\cdot, 0) = u^0, & \Omega, \end{cases} \tag{46}$$

so that $||u(.,T) - u_T||_{L^2(\Omega)} \leq \epsilon_1$.

$$u^0(x) = \sin(\pi x), \quad u_T(x) = e^{-100(x-0.5)^2}, \quad \epsilon^{-1} = 5 \times 10^{-4}, \quad L = 1/5, \quad T = 0.5 \tag{47}$$



Characteristic functions associated to the optimal density: $I(\mathcal{X}_\omega) \approx 22.4597$ $I(s^{opt}) \approx 22.4479$

For instance, for the HEAT equation, the problem is :

$$(\mathcal{C}_\omega) \quad \inf_{\omega\subset\Omega} \underbrace{\sup_{\varphi_T\in L^2(\Omega)} \frac{\int_\Omega \varphi^2(x,0)dx}{\int_\omega \int_0^T \varphi^2(x,t)dxdt}}_{C_{obs}(\omega,T)} \tag{48}$$

where φ is solution of (the backward heat equation)

$$\begin{cases} \varphi_t + \Delta\varphi = 0, & \Omega\times(0,T), \\ \varphi(x,t) = 0, & \partial\Omega\times(0,T), \\ \varphi(x,T) = \varphi_T(x) & \Omega, \end{cases} \tag{49}$$

Introducing the solution y of

$$\begin{cases} y_t - \Delta y = \mathcal{X}_\omega \varphi, & \Omega\times(0,T), \\ y = 0, & \partial\Omega\times(0,T), \\ y(\cdot,0) = 0, & \Omega, \end{cases} \tag{50}$$

we arrive at

$$\int_\omega \int_0^T \varphi^2(x,t)dxdt = \int_\Omega y(T)\varphi_T dx \tag{51}$$

If we note $\Lambda(\varphi_T) = y(T)$ and $A(\varphi_T) = \varphi(0)$, then

$$C_{obs}(\omega,T) = \sup_{\varphi_T\in L^2(\Omega)} \frac{\int_\Omega A(\varphi_T)A(\varphi_T)dx}{\int_\Omega \Lambda(\varphi_T)\varphi_T dx} \equiv \sup_{\varphi_T\in L^2(\Omega)} \frac{(A^2(\varphi_T),\varphi_T)}{(\Lambda(\varphi_T),\varphi_T)} \tag{52}$$

[14]

[14] Work in progress with F. Hubert and F. Boyer (Marseille, LATP, France)

$N = 1$ - $\Omega = (0, 1)$

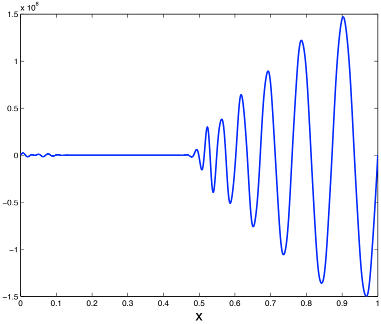


Figure: $T = 1$; Eigenfunction φ_T for $\omega = (0.2, 0.4)$

$C_{obs}(\omega, T) \approx 19.12$

Theorem (Shape derivative)

$$\frac{\partial C_{obs}(\omega)}{\partial \omega} \cdot \theta = -C(\phi, \omega) \int_{\omega} div\left(\int_0^T \phi^2(x,t) dt \theta \right) dx \tag{53}$$

with

$$C(\phi, \omega) = \frac{||\phi(\cdot, 0)||^2_{L^2(\Omega)}}{||\phi||^4_{L^2(\omega \times (0,T))}} \tag{54}$$

■

$\Longrightarrow$ Once again, a restriction on $|\omega| = L|\Omega|$ is necessary in order to avoid the optimal trivial solution $\omega = \Omega$

Theorem (Topological Derivative)

Let $\Omega \subset \mathbb{R}^N$. Let $\mathbf{x_0} \in \Omega$ and $D(\mathbf{x_0}, \rho)$ the ball of center $\mathbf{x_0}$ and radius ρ. Then,

$$C_{obs}(\Omega \backslash D(\mathbf{x_0}, \rho)) = C_{obs}(\Omega) + |D(\mathbf{x_0}, \rho)| C(\phi, \Omega) \int_0^T \phi^2(\mathbf{x_0}, t) dt + o(\rho^N) \tag{55}$$

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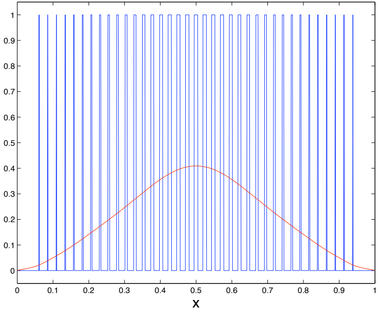

Figure: $|\omega| = 1/5$ - $h = 1/200$ - $T = 1$ - $c = 0.1$ - Optimal density and characteristic function - $C_{s^{opt},T} \approx 1.1792$

$\Longrightarrow$ The optimal position is approximatively uniformly distributed on Ω

M	1	2	3	4	5	7	9	$+\infty$
$C_{obs}(\omega_M)$	2.3160	3.1713	1.5812	1.4429	1.3430	1.2774	1.2385	**1.1792**

Table: $|\omega| = 1/5$ - $h = 1/200$ - $T = 1$ - $c = 0.1$ - Convergence of the observability constant toward the optimal one

- The ultimate (open but challenging !) goal is to consider TIME-DEPENDENT support of the form

$$\{\omega(t)\} \times (0, T), \qquad \text{with} \quad \omega(t) \subset \Omega, \forall t \in (0, T) \qquad (56)$$