TALK 12: IDENTIFICATION OF THE DUAL GROUP

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The aim of this talk is to identify the group scheme $\check{G}$ constructed in Talk 11 as the Langlands dual group following the arguments in [FS21, VI.11]. I claim no originality.

1. Statement of the result

Let G be a reductive group over an algebraically closed field k. Fix a prime number $\ell \in \mathbb{N}$ invertible on k. Recall from Talk 11 that, for each $n \geqslant 1$, there exists a flat affine $\mathbb{Z}/\ell^n$-group scheme $\check{G}_{\mathbb{Z}/\ell^n}$ together with a Tannakian equivalence

$$\big(\mathrm{Sat}_{G,\mathbb{Z}/\ell^n}, \star, F_{G,\mathbb{Z}/\ell^n}\big) \cong \big(\mathrm{Rep}(\check{G}_{\mathbb{Z}/\ell^n}), \otimes, \mathrm{forget}\,\big), \tag{1.1}$$

where $F_{G,\mathbb{Z}/\ell^n} := \oplus_{m\in\mathbb{Z}} R^m\pi_{G,*}\colon \mathrm{Sat}_{G,\mathbb{Z}/\ell^n} \to \mathrm{Mod}^{\mathrm{fg,proj}}_{\mathbb{Z}/\ell^n}$ is the global cohomology functor.

Lemma 1.1. *For $m \geqslant n \geqslant 1$, the equivalence (1.1) fits in the commutative diagram*

$$\begin{array}{ccc} \mathrm{Sat}_{G,\mathbb{Z}/\ell^m} & \xrightarrow{\cong} & \mathrm{Rep}(\check{G}_{\mathbb{Z}/\ell^m}) \\ \Big\downarrow {\scriptstyle A\mapsto A\otimes^{\mathbb{L}}\mathbb{Z}/\ell^n} & & \Big\downarrow {\scriptstyle V\mapsto V\otimes\mathbb{Z}/\ell^n} \\ \mathrm{Sat}_{G,\mathbb{Z}/\ell^n} & \xrightarrow{\cong} & \mathrm{Rep}(\check{G}_{\mathbb{Z}/\ell^n}), \end{array} \tag{1.2}$$

compatibly with the Tannakian structures. Diagram (1.2) induces an equivalence of inverse systems $\{\mathrm{Sat}_{G,\mathbb{Z}/\ell^n}\}_{\geqslant 1} \cong \{\mathrm{Rep}(\check{G}_{\mathbb{Z}/\ell^n})\}_{n\geqslant 1}$. Passing to the 2-limit of categories gives the Tannakian equivalence

$$\mathrm{Sat}_{G,\mathbb{Z}_\ell} := \lim_{n\geqslant 1} \mathrm{Sat}_{G,\mathbb{Z}/\ell^n} \cong \mathrm{Rep}(\check{G}_{\mathbb{Z}_\ell}), \tag{1.3}$$

where $\check{G}_{\mathbb{Z}_\ell}$ is a flat affine $\mathbb{Z}_\ell$-group scheme equipped with compatible isomorphisms $\check{G}_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Z}/\ell^n \cong \check{G}_{\mathbb{Z}/\ell^n}$ of $\mathbb{Z}/\ell^n$-group schemes for all $n \geqslant 1$.

Proof. The operation $(\text{-}) \otimes^{\mathbb{L}} \mathbb{Z}/\ell^n$ preserves flat perversity, ULAness and convolution. So the functor $\mathrm{Sat}_{G,\mathbb{Z}/\ell^m} \to \mathrm{Sat}_{G,\mathbb{Z}/\ell^n}$, $A \mapsto A \otimes^{\mathbb{L}} \mathbb{Z}/\ell^n$ is well-defined and monoidal. To see that it is compatible with the global cohomology functor, we fix an auxiliary datum of a split maximal torus T contained in a Borel subgroup B in G. Using the $*$-pullback and !-push forward version of the shifted constant terms functor, the projection formula gives the isomorphism

$$\mathrm{CT}_B[\deg](A \otimes^{\mathbb{L}} \mathbb{Z}/\ell^n) \xrightarrow{\cong} \mathrm{CT}_B[\deg](A) \otimes^{\mathbb{L}} \mathbb{Z}/\ell^n, \tag{1.4}$$

functorially in $A \in \mathrm{Sat}_{G,\mathbb{Z}/\ell^m}$. As the objects (1.4) of $\mathrm{D}(\mathrm{Gr}_T, \Lambda)^{\mathrm{bd}}$ are in cohomological degree 0, the isomorphism of functors $F_G \cong \mathcal{H}^0(\mathrm{CT}_B[\deg])$ implies $F_G(\text{-} \otimes^{\mathbb{L}} \mathbb{Z}/\ell^n) \cong F_G(\text{-}) \otimes \mathbb{Z}/\ell^n$ as functors $\mathrm{Sat}_{G,\mathbb{Z}/\ell^m} \to \mathrm{Mod}^{\mathrm{fg,proj}}_{\mathbb{Z}/\ell^n}$. We conclude that (1.2) is well-defined, commutative and compatible with the Tannakian structures.

Recall from Talk 11 that $\check{G}_{\mathbb{Z}/\ell^n} = \mathrm{Spec}(\mathcal{A}_n)$ with $\mathcal{A}_n = \mathrm{colim}_W \mathcal{A}_{n,W}$ where $W \subset X_*^+$ runs through finite subsets and each $\mathcal{A}_{n,W}$ is a finite free $\mathbb{Z}/\ell^n$-module. One checks that $\mathrm{Sat}_{G,\mathbb{Z}_\ell}$ together with the induced convolution structure and the functor

$$F_{G,\mathbb{Z}_\ell}\colon \mathrm{Sat}_{G,\mathbb{Z}_\ell} \to \mathrm{Mod}^{\mathrm{fg,proj}}_{\mathbb{Z}_\ell},\ \{A_n\}_{n\geqslant 1} \mapsto \oplus_{m\in\mathbb{Z}} \lim_{n\geqslant 1} R^m\pi_{G,*}A_n \tag{1.5}$$

satisfies the Tannakian formalism of Talk 11. The associated co-algebra object in $\mathrm{Ind}(\mathrm{Mod}^{\mathrm{fg,proj}}_{\mathbb{Z}_\ell})$ is given by $\mathcal{A} := \mathrm{colim}_W \lim_{n\geqslant 1} \mathcal{A}_{n,W}$. By construction, it is equipped with compatible isomorphisms $\mathcal{A}\otimes\mathbb{Z}/\ell^n \cong \mathcal{A}_n$ of $\mathbb{Z}/\ell^n$-co-algebras for all $n \geqslant 1$. So $\check{G}_{\mathbb{Z}_\ell} := \mathrm{Spec}(\mathcal{A})$ is a flat affine $\mathbb{Z}_\ell$-group scheme equipped with compatible isomorphisms $\check{G}_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Z}/\ell^n \cong \check{G}_{\mathbb{Z}/\ell^n}$ for all $n \geqslant 1$. □

In the following, we abbreviate $\check{G} := \check{G}_{\mathbb{Z}_\ell}$ and likewise $\mathrm{Sat}_G := \mathrm{Sat}_{G,\mathbb{Z}_\ell}$, $F_{G,\mathbb{Z}_\ell} := F_G$. Let $\widehat{G}$ be the Langlands dual group formed over $\mathbb{Z}_\ell$. By the construction of Chevalley group schemes from based root data, $\widehat{G}$ is equipped with $\widehat{T} \subset \widehat{B}$ and isomorphisms $\mathrm{Lie}(U_a) \cong \mathbb{Z}_\ell$ for all simple coroots a of $\widehat{G}$. Recall that, by construction, the pinned group $\widehat{G}$ is uniquely determined up to pinning preserving automorphisms, which correspond to automorphisms of the based root datum, see [Co14, Theorem 6.1.17]. The aim of this talk is to explain the proof of the following theorem:

Theorem 1.2 ([FS21, Theorem VI.11.1])**.** *Fix a compatible system of ℓ^n-th roots of unity in k for all $n \geqslant 1$. Then there exists a pinned isomorphism $\check{G} \cong \widehat{G}$.*

Remark 1.3.

(1) In particular, the $\mathbb{Z}_\ell$-group scheme $\check{G}$ is reductive. Its pinning is constructed throughout the proof using the fixed system of ℓ^n-th roots of unity, which are needed in order to construct a canonical isomorphism $\mathrm{H}^2(\mathbb{P}^1_k, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell$, see (2.1) below. This choice can be circumvented by using Tate twists $\mathrm{Lie}(\widehat{U}_a) \cong \mathbb{Z}_\ell(1)$ in the construction of the dual group, see [FS21, Theorem VI.11.1]. Let us point out that even if $k = \mathbb{C}$ and we would work with sheaves in the analytic topology, then we would still need to make the choice of $\pm i$.

(2) By base change, one obtains pinned isomorphisms $\check{G}_{\mathbb{Q}_\ell} \cong \widehat{G}_{\mathbb{Q}_\ell}$ and $\check{G}_{\mathbb{Z}/\ell^n} \cong \widehat{G}_{\mathbb{Z}/\ell^n}$ for all $n \geqslant 1$.

(3) If k is any field and G any reductive k-group (not necessarily split), then $\check{G} \cong \widehat{G}$ Galois equivariantly up to the cyclotomic twist $\mathrm{Lie}(\widehat{U}_a) \cong \mathbb{Z}_\ell(1)$, see [FS21, Lemma VI.11.4 ff.].

2. Proof of the theorem in several steps

Fix an auxiliary pinning of G and denote by $T \subset B$ the maximal torus and the Borel subgroup. The proof of Theorem 1.2 proceeds in several steps as follows.

2.1. **The case $G = \{e\}$.** Then $\mathrm{Gr}_G = \mathrm{Spec}(k)$ and $F_G = \Gamma(\mathrm{Spec}(k), \text{-})$ induces an equivalence $\mathrm{Sat}_{\{e\}} \cong \lim_{n\geqslant 1} \mathrm{Mod}^{\mathrm{fg,proj}}_{\mathbb{Z}/\ell^n} \cong \mathrm{Mod}^{\mathrm{fg,proj}}_{\mathbb{Z}_\ell}$.

2.2. **The case $G = T$.** The map $\mu \mapsto \mu(t)$ induces an isomorphism $\underline{X_*(T)} \cong (\mathrm{Gr}_T)_{\mathrm{red}}$ of k-group schemes as we now argue. The source is the constant group scheme associated with the abelian group $X_*(T)$. The target denotes the underlying reduced ind-scheme of Gr_T which is a scheme in this case. As the map $\underline{X_*(T)} \to (\mathrm{Gr}_T)_{\mathrm{red}}$ is functorial in the torus T and compatible with products, we reduce to the case $T = \mathbb{G}_{m,k}$, for which see [Ri19, Section 2.3]. This shows $\underline{X_*(T)} \cong (\mathrm{Gr}_T)_{\mathrm{red}}$. Under this isomorphism, we have $F_T = \oplus_{\mu\in X_*(T)}\Gamma(\{\mu\}, \text{-})$ as functors $\mathrm{Sat}_T \to \mathrm{Mod}^{\mathrm{fg,proj}}_{\mathbb{Z}_\ell}$. Thus, F_T can naturally be upgraded to an equivalence between Sat_G with the category of finite free $\mathbb{Z}_\ell$-modules equipped with a $X_*(T)$-grading. This equivalence is symmetric monoidal: Indeed, for two objects $A_\mu, A_\lambda \in \mathrm{Sat}_G$ concentrated on $\{\mu\}$, respectively $\{\lambda\}$ for some $\mu, \lambda \in X_*(T)$, the convolution $A_\mu \star A_\lambda$ is concentrated on $\{\lambda + \mu\}$ and given by the (derived) tensor product of the underlying sheaves. We conclude that $\check{T}$ is the unique multiplicative group scheme over $\mathbb{Z}_\ell$ with character group $X^*(\check{T}) = X_*(T)$, that is, $\check{T} = \widehat{T}$.

2.3. **The closed immersion $\check{T} \hookrightarrow \check{G}$.** The constant term functor $\mathrm{CT}_B[\deg]\colon \mathrm{Sat}_G \to \mathrm{Sat}_T$ induces a map of $\mathbb{Z}_\ell$-group schemes $\check{T} \to \check{G}$. To check that the map is a closed immersion, we use the following result:

Theorem 2.1 ([DH18, Theorem 4.1.2 (ii)])**.** *Let $f\colon H \to H'$ be a map of flat affine $\mathbb{Z}_\ell$-group schemes. Then f is a closed immersion if and only if every object of* $\mathrm{Rep}(H)$ *is isomorphic to a strict subquotient of f^*V for some $V \in$* $\mathrm{Rep}(H')$.

Now, for any $\mu \in X_*(T)_+$, one has $\mathrm{H}^*_c(S_\mu \cap \mathrm{Gr}_{G,\leqslant\mu}, {}^p j_{\mu,*}\mathbb{Z}/\ell^n) \simeq \mathbb{Z}/\ell^n$ for every $n \geqslant 1$ as we now argue. Indeed, $S_\mu \cap \mathrm{Gr}_{G,\leqslant\mu} \subset \mathrm{Gr}_{G,\mu}$ and $S_\mu \cap \mathrm{Gr}_{G,\leqslant\mu} \cong \mathbb{A}^{d_\mu}_k$ for $d_\mu := \dim(\mathrm{Gr}_{G,\leqslant\mu})$ by [NP01, Lemme 5.2], so ${}^p j_{\mu,*}\mathbb{Z}/\ell^n|_{S_\mu\cap\mathrm{Gr}_{G,\leqslant\mu}} = \underline{\mathbb{Z}/\ell^n}_{\mathbb{A}^{d_\mu}_k}[d_\mu]$ and its compactly supported cohomology is concentrated in cohomological degree d_μ, where it is isomorphic to $\mathbb{Z}/\ell^n$.

We see that the μ-isotypical component of $\mathrm{CT}_B[\deg]({}^p j_{\mu,*}\mathbb{Z}_\ell)$ viewed as an object of Sat_T is isomorphic to $\mathbb{Z}_\ell$ concentrated in degree μ. For varying μ, these objects generate $\mathrm{Sat}_T = \mathrm{Rep}(\check{T})$ under finite direct sums as this category is identified with the category of $X_*(T)$-graded, finite free $\mathbb{Z}_\ell$-modules, see §2.2. So Theorem 2.1 implies that $\check{T} \to \check{G}$ is a closed immersion.

As explained by Torsten Wedhorn during my talk, an alternative argument uses the rigidity of tori as in [Co14, Corollary B.3.5] to reduce to show that $\check{T}_{\mathbb{F}_\ell} \to \check{G}_{\mathbb{F}_\ell}$ is a closed immersion. The analogue of Theorem 2.1 over fields is classical (see [DM82, Proposition 2.21 (b)]), and we conclude by the same argument as above.

2.4. **The reductive group $\check{G}_{\mathbb{Q}_\ell}$ and the pair $\check{T} \subset \check{B}$.** The group $\check{G}_{\mathbb{Q}_\ell}$ is reductive, the subgroup $\check{B} \subset \check{G}$ stabilizing the ascending filtration $F_{G,\leqslant i} := \oplus_{m\leqslant i} R^m\pi_{G,*}$, $i \in \mathbb{Z}$ contains $\check{T}$ and defines a Borel subgroup over $\mathbb{Q}_\ell$, and $\check{T}_{\mathbb{Q}_\ell} \subset \check{G}_{\mathbb{Q}_\ell}$ is a maximal torus, see [FS21, bottom of page 233].

2.5. **The case $G = \mathrm{PGL}_{2,k}$.** Assume that $G = \mathrm{PGL}_{2,k}$ equipped with the standard pinning. Fix $\mu \in X_*(T)_+ = \mathbb{Z}_{\geqslant 0}$ minuscule (corresponding to 1). Then $\mathrm{Gr}_{G,\leqslant\mu} = \mathrm{Gr}_{G,\mu} = \mathbb{P}^1_k$ and

$$F_G(\underline{\mathbb{Z}_\ell}_{\mathrm{Gr}_{G,\leqslant\mu}}[1]) = \mathrm{H}^0(\mathbb{P}^1_k, \mathbb{Z}_\ell) \oplus \mathrm{H}^2(\mathbb{P}^1_k, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell \oplus \mathbb{Z}_\ell(-1) \simeq \mathbb{Z}_\ell^2. \tag{2.1}$$

Note that the H^0-component is canonically isomorphic to $\mathbb{Z}_\ell$, but that the H^2-component is canonically isomorphic to $\mathbb{Z}_\ell(-1) = \mu_{\ell^\infty}^{\otimes -1}(k)$ by [De77, Corollaire 3.5], which we identify in (2.1) with $\mathbb{Z}_\ell$ using the fixed compatible system of ℓ^n-th roots of unity in k, compare with Remark 1.3 (1). By the Tannkian formalism, $\check{G}$ naturally acts on (2.1). We consider the induced morphism of $\mathbb{Z}_\ell$-group schemes

$$\check{G} \to \mathrm{GL}_{2,\mathbb{Z}_\ell}. \tag{2.2}$$

By construction, $\check{B}$ maps into the Borel subgroup of $\mathrm{GL}_{2,\mathbb{Z}_\ell}$ stabilizing the filtration $\mathbb{Z}_\ell \oplus 0 \subset \mathbb{Z}_\ell \oplus \mathbb{Z}_\ell$, that is, into the upper triangular matrices, and $\check{T}$ maps into the diagonal torus.

Claim 2.2. *The map (2.2) factors through $\mathrm{SL}_{2,\mathbb{Z}_\ell}$ and induces an isomorphism $\check{G} \cong \mathrm{SL}_{2,\mathbb{Z}_\ell}$.*

The torus $\check{T} \hookrightarrow \check{G}$ acts under (2.1) with weights ± 1 on $\mathbb{Z}_\ell^2$: Indeed, the decomposition into semi-infinite orbits is given by $\mathrm{Gr}_{G,\leqslant\mu} = S_\mu \cap \mathrm{Gr}_{G,\leqslant\mu} \sqcup S_{-\mu} \cap \mathrm{Gr}_{G,\leqslant\mu}$ and corresponds to the decomposition $\mathbb{P}^1_k = \mathbb{A}^1_k \sqcup \{*\}$. Using the $*$-pullback and !-pushforward version of $\mathrm{CT}_B[\deg]$, we see that the $\pm\mu$-component lies in weight ± 1 under $X_*(T) = \mathbb{Z}$. So $\check{T}$ acts with the prescribed weights, and hence maps under (2.2) isomorphically onto the diagonal torus $\mathbb{G}_{m,\mathbb{Z}_\ell} \subset \mathrm{SL}_{2,\mathbb{Z}_\ell}$.

Next, we prove Claim 2.2 over $\mathbb{Q}_\ell$. Since $\check{G}_{\mathbb{Q}_\ell}$ is a split reductive group of rank $\mathrm{rank}(\check{T}_{\mathbb{Q}_\ell}) = 1$ by §2.4 and the inclusion $\check{T}_{\mathbb{Q}_\ell} \subset \check{G}_{\mathbb{Q}_\ell}$ is strict, $\check{G}_{\mathbb{Q}_\ell}$ must be 3-dimensional by considering Lie algebras, hence also semisimple. As $\check{B}_{\mathbb{Q}_\ell}$ maps under (2.2) into the upper triangular matrices, the map (2.2) over $\mathbb{Q}_\ell$ induces an isogeny $\check{G}_{\mathbb{Q}_\ell} \to \mathrm{SL}_{2,\mathbb{Q}_\ell}$, which is necessarily central, so an isomorphism, compare with [Co14, Proof of Theorem 1.2.7, Proposition 4.3.1]. This proves Claim 2.2 over $\mathbb{Q}_\ell$.

As $\check{G}$ is flat over $\mathbb{Z}_\ell$ (so agrees with the scheme-theoretic closure of its generic fiber), the scheme-theoretic image of (2.2) is contained in $\mathrm{SL}_{2,\mathbb{Z}_\ell}$. Hence, (2.2) factors as a morphism of $\mathbb{Z}_\ell$-group schemes

$$\check{G} \to \mathrm{SL}_{2,\mathbb{Z}_\ell} \tag{2.3}$$

that is an isomorphism over $\mathbb{Q}_\ell$. Next, put $H := \mathrm{image}(\check{G}_{\mathbb{F}_\ell} \to \mathrm{SL}_{2,\mathbb{F}_\ell})$ which is a closed subgroup scheme of $\mathrm{SL}_{2,\mathbb{F}_\ell}$. The surjective map $\check{G}_{\mathbb{F}_\ell} \to H$ induces an injection on the set of isomorphism classes of its irreducible representations

$$\operatorname{Irrep}(H) \hookrightarrow \operatorname{Irrep}(\check{G}_{\mathbb{F}_\ell}) = \mathbb{Z}_{\geqslant 0}. \tag{2.4}$$

The last equality arises by taking highest weights for $\check{T}_{\mathbb{F}_\ell}$ and the notion of positivity induced by the group $\check{B}_{\mathbb{F}_\ell}$ stabilizing the filtration $\mathbb{F}_\ell \oplus 0 \subset \mathbb{F}_\ell \oplus \mathbb{F}_\ell$. The following lemma shows that one has $H = \mathrm{SL}_{2,\mathbb{F}_\ell}$, so (2.3) is surjective:

Lemma 2.3 ([FS21, Lemma VI.11.2])**.** *Let $H \subset \mathrm{SL}_{2,\mathbb{F}_\ell}$ be a closed subgroup containing the diagonal torus such that* $\operatorname{Irrep}(H) \hookrightarrow \operatorname{Irrep}(\mathrm{SL}_{2,\mathbb{F}_\ell}) = \mathbb{Z}_{\geqslant 0}$ *via consideration of highest weights. Then $H = \mathrm{SL}_{2,\mathbb{F}_\ell}$.*

Proof. We repeat the proof for convenience. The pullback of representations along the Frobenius endomorphism $\mathrm{Frob}_\ell : \mathrm{SL}_{2,\mathbb{F}_\ell} \to \mathrm{SL}_{2,\mathbb{F}_\ell}$ induces multiplication by ℓ on $\mathbb{Z}_{\geqslant 0}$ and, in particular, is injective. Passing to a sufficiently high power, we may assume without loss of generality that H is reduced and, further, that H is connected by [DM82, Corollary 2.22]. Since $\operatorname{Lie}(H) \subset \operatorname{Lie}(\mathrm{SL}_{2,\mathbb{F}_\ell})$ is stable under the action of the diagonal torus, there are only three possibilities according to the dimension of H: either H is the diagonal torus, or the upper triangular Borel subgroup, or all of $\mathrm{SL}_{2,\mathbb{F}_\ell}$. In the first two cases, H has too many representations so that $H = \mathrm{SL}_{2,\mathbb{F}_\ell}$. □

In conclusion, the map $\check{G} \to \mathrm{SL}_{2,\mathbb{Z}_\ell}$ in (2.3) is surjective, an isomorphism over $\mathbb{Q}_\ell$ and both schemes are affine and flat over $\mathbb{Z}_\ell$. Note that the induced map on the underlying rings is injective by surjectivity of (2.3) and reducedness of $\mathrm{SL}_{2,\mathbb{Z}_\ell}$. Therefore, the following lemma implies that (2.3) is an isomorphism, so proves the claim:

Lemma 2.4 ([FS21, Lemma VI.11.3])**.** *Let $f\colon M \to N$ be a morphism of flat $\mathbb{Z}_\ell$-modules such that $f \otimes \mathbb{Q}_\ell$ is an isomorphism and $f \otimes \mathbb{F}_\ell$ is injective. Then f is an isomorphism.*

Proof. We repeat the proof for convenience. The map f is injective because $f \otimes \mathbb{Q}_\ell$ is an isomorphism and ℓ is a non-zero divisor on M (by flatness). To see that f is surjective, pick any $n \in N$ and write $f(m) = \ell^k n$ for some $m \in M$ with $k \geqslant 0$ minimal (again, $f \otimes \mathbb{Q}_\ell$ is an isomorphism). If $k \geqslant 1$, then $m \mod \ell$ lies in $\operatorname{kernel}(f \otimes \mathbb{F}_\ell)$, hence vanishes. Writing $m = \ell m'$ and using that ℓ is a non-zero divisor on N, we get $f(m') = \ell^{k-1} n$, contradicting the minimality of k. Hence, $k = 0$, so $f(m) = n$. □

We point out that $\check{T} \subset \check{B}$ corresponds under $\check{G} \cong \mathrm{SL}_{2,\mathbb{Z}_\ell}$ to the diagonal torus contained in the upper triangular matrices. We equip $\check{G}$ with the standard pinning induced from $\mathrm{SL}_{2,\mathbb{Z}_\ell}$.

2.6. **The case G of semisimple rank 1.** The adjoint group G_{ad} is isomorphic to $\mathrm{PGL}_{2,k}$ by the classification of split reductive groups of rank 1. Note that the fixed pinning of G induces a pinning of G_{ad}. The isomorphism $G_{\mathrm{ad}} \cong \mathrm{PGL}_{2,k}$ is uniquely determined by requiring that the pinning of G_{ad} induces the standard pinning of $\mathrm{PGL}_{2,k}$: Indeed, the pinning preserving automorphisms of $\mathrm{PGL}_{2,k}$ correspond to automorphisms of the based roots datum. So any such automorphism must be the identity. In order to link the Satake categories Sat_G and $\mathrm{Sat}_{G_{\mathrm{ad}}}$, we study the map $\mathrm{Gr}_G \to \mathrm{Gr}_{G_{\mathrm{ad}}}$ of affine Grassmannians induced by the quotient $G \to G_{\mathrm{ad}}$. Recall that for the set of connected components $\pi_0(\mathrm{Gr}_G) = \pi_1(G)$, see Talk 6. This induces a locally constant morphism $\mathrm{Gr}_G \to \underline{\pi_1(G)}$ of k-ind-schemes that is functorial in G for morphisms of k-group schemes. Hence, the map $G \to G_{\mathrm{ad}}$ induces a canonical morphism of k-ind-schemes

$$f\colon \mathrm{Gr}_G \to \underline{\pi_1(G)} \times_{\underline{\pi_1(G_{\mathrm{ad}})}} \mathrm{Gr}_{G_{\mathrm{ad}}}. \tag{2.5}$$

Lemmas 2.5 and 2.7 hold for general reductive groups G:

Lemma 2.5. *The map (2.5) is a universal homeomorphism, compatibly with the stratification into Schubert varieties. Further, it is an isomorphism on the underlying reduced ind-schemes if* $\operatorname{char}(k)$ *does not divide $\#\pi_1(G_{\mathrm{ad}})$.*

Proof. By [HR19, Proposition 3.5], the induced map on Schubert varieties $\mathrm{Gr}_{G,\leqslant\mu} \to \mathrm{Gr}_{G_{\mathrm{ad}},\leqslant\mu_{\mathrm{ad}}}$ is a finite birational universal homeomorphism for all $\mu \in X_*(T)^+$, where μ_{ad} denotes the composition of μ with $T \subset G \to G_{\mathrm{ad}}$. If $k \nmid \#\pi_1(G_{\mathrm{ad}})$, then $\mathrm{Gr}_{G_{\mathrm{ad}},\leqslant\mu_{\mathrm{ad}}}$ is normal by [PR08, Theorem 0.3], in which case $\mathrm{Gr}_{G,\leqslant\mu} \to \mathrm{Gr}_{G_{\mathrm{ad}},\leqslant\mu_{\mathrm{ad}}}$ is an isomorphism (being a finite birational map of integral schemes with normal target). Now, passing to the colimit over μ and taking neutral components recovers the map (2.5) on the underlying reduced ind-schemes, which has therefore the desired properties. □

Remark 2.6. We note that the finer information, on whether (2.5) is an isomorphism, is not needed for §2.6. Also, we remark that (2.5) fails to be an isomorphism in the case $G = \mathrm{SL}_{2,k}$ and $\mathrm{char}(k) = 2$, see [HLR20]. This difficulty does not arise in the setting of [FS21, Section VI.11] and the analogue of (2.5) is an isomorphism.

Since universal homeomorphisms of schemes induce equivalences on the categories of étale sheaves [StP, 04DY], Lemma (2.5) gives an equivalence

$$f_* : \mathrm{D}(\mathrm{Gr}_G, \Lambda)^{\mathrm{bd}} \cong \mathrm{D}\big(\underline{\pi_1(G)} \times_{\underline{\pi_1(G_{\mathrm{ad}})}} \mathrm{Gr}_{G_{\mathrm{ad}}}, \Lambda\big)^{\mathrm{bd}} : f^* \tag{2.6}$$

on derived categories with bounded support.

Lemma 2.7. *The equivalence (2.6) induces a Tannakian equivalence between Sat_G and the category of objects $A \in \mathrm{Sat}_{G_{\mathrm{ad}}}$ together with a refinement of the $\pi_1(G_{\mathrm{ad}})$-grading to a $\pi_1(G)$-grading.*

Proof. The convolution of objects in Sat_G is compatible with the abelian group structure of $\pi_1(G)$ as follows. If $A, B \in \mathrm{Sat}_G$ is supported in the connected component of Gr_G corresponding to $\alpha, \beta \in \pi_1(G)$, then $A \star B$ is supported on $\alpha + \beta$. We leave it to the reader to check that (2.6) is compatible with the Tannakian structures. □

Lemma 2.7 implies that $\check{G} = \widecheck{G_{\mathrm{ad}}} \times^{\mu_2} \check{Z}$ where $\check{Z}$ is the multiplicative $\mathbb{Z}_\ell$-group scheme with $X^*(\check{Z}) = \pi_1(G)$. The scheme-theoretic center $\widehat{Z}$ of $\widehat{G}$ is split multiplicative [Co14, Corollary 3.3.6]. Following [Bo98], there is a natural isomorphism $X^*(\widehat{Z}) \cong \pi_1(G)$, so $\check{Z} \cong \widehat{Z}$. In particular, $\widehat{G} = \widehat{G}_{\mathrm{sc}} \times^{\mu_2} \widehat{Z}$ along with $\widecheck{G_{\mathrm{ad}}} \cong \mathrm{SL}_{2,\mathbb{Z}_\ell} \cong \widehat{G}_{\mathrm{sc}}$ from §2.5 induces the pinned isomorphism $\check{G} \cong \widehat{G}$.

2.7. **General case.** We return to the case of a general pinned reductive k-group G. For a simple coroot a, we get the Levi subgroup $M_a \subset G$ of semisimple rank 1 containing the torus T and the parabolic subgroup P_a containing M_a and the Borel subgroup B. The constant term functor $\mathrm{CT}_{P_a}[\deg_{P_a}] \colon \mathrm{Sat}_G \to \mathrm{Sat}_{P_a}$ is compatible with the constant term functors to Sat_T, see Talk 10. As $\mathrm{CT}_{P_a}[\deg_{P_a}]$ is equipped with a Tannakian structure, it induces a morphism of $\mathbb{Z}_\ell$-group schemes $\widecheck{M}_a \to \check{G}$ compatible with the closed subgroup scheme $\check{T}$. As both $\widecheck{M}_{a,\mathbb{Q}_\ell}, \check{G}_{\mathbb{Q}_\ell}$ are reductive, $a \in X_*(T) = X^*(\check{T})$ defines a root of $\check{G}_{\mathbb{Q}_\ell}$ and $a^\vee \in X^*(T) = X_*(\check{T})$ defines a coroot of $\check{G}_{\mathbb{Q}_\ell}$. In particular, the simple reflection s_a is contained in the Weyl group $\widecheck{W} := \widecheck{W}(\check{G}_{\mathbb{Q}_\ell}, \check{T}_{\mathbb{Q}_\ell})$. Varying a, this implies $W(G,T) \subset \widecheck{W}$ and

$$\Phi^\vee(G,T) \subset \Phi(\check{G}_{\mathbb{Q}_\ell}, \check{T}_{\mathbb{Q}_\ell}), \quad \Phi(G,T) \subset \Phi^\vee(\check{G}_{\mathbb{Q}_\ell}, \check{T}_{\mathbb{Q}_\ell}). \tag{2.7}$$

In fact, the inclusions (2.7) are equalities as both sets have the same cardinality. Thus, the pinned isomorphisms $\widecheck{M}_a \cong \widehat{M}_a$ over $\mathbb{Z}_\ell$ constructed in §2.6 extend, at least over $\mathbb{Q}_\ell$, to a pinned isomorphism

$$\check{G}_{\mathbb{Q}_\ell} \cong \widehat{G}_{\mathbb{Q}_\ell}. \tag{2.8}$$

Claim 2.8. *The map (2.8) extends to a pinned isomorphism $\check{G} \cong \widehat{G}$ over $\mathbb{Z}_\ell$.*

One argues as follows. Since $\widehat{G}(\breve{\mathbb{Z}}_\ell)$ is generated by the subgroups $\widehat{M}_a(\breve{\mathbb{Z}}_\ell)$ for varying a, the image of $\widehat{G}(\breve{\mathbb{Z}}_\ell) \subset \widehat{G}(\breve{\mathbb{Q}}_\ell) \cong \check{G}(\breve{\mathbb{Q}}_\ell)$ lies inside $\check{G}(\breve{\mathbb{Z}}_\ell)$. So, picking any $\check{G} \to \mathrm{GL}_{N,\mathbb{Z}_\ell}$ that is a closed immersion over $\mathbb{Q}_\ell$, the map $\widehat{G}_{\mathbb{Q}_\ell} \cong \check{G}_{\mathbb{Q}_\ell} \to \mathrm{GL}_{N,\mathbb{Q}_\ell}$ extends to a map

$$\widehat{G} \to \mathrm{GL}_{N,\mathbb{Z}_\ell} \tag{2.9}$$

by [BT84, Proposition 1.7.6]. Furthermore, (2.9) is a closed immersion:

Lemma 2.9 (special case of [PY06, Corollary 1.3])**.** *Let $f: H \to H'$ be a morphism of affine, finite type $\mathbb{Z}_\ell$-group schemes with H reductive. Assume either that $\ell \neq 2$ or that H is simply connected. Then f is a closed immersion.*

In order to apply the lemma in the case $\ell = 2$, we additionally use a reduction to the adjoint group as in §2.6. Next, the map (2.9) being a closed immersion together with the flatness of $\breve{G} \to \mathrm{Spec}(\mathbb{Z}_\ell)$ implies that (2.8) extends to a map

$$\breve{G} \to \widehat{G}. \tag{2.10}$$

Also, (2.10) is an isomorphism over $\mathbb{Q}_\ell$ and surjective on $\breve{\mathbb{Z}}_\ell$-valued points because the image of $\widehat{G}(\breve{\mathbb{Z}}_\ell) \subset \widehat{G}(\breve{\mathbb{Q}}_\ell) \cong \breve{G}(\breve{\mathbb{Q}}_\ell)$ lies inside $\breve{G}(\breve{\mathbb{Z}}_\ell)$. In particular, (2.10) is surjective: any element in $g \in \widehat{G}(\overline{\mathbb{F}}_\ell)$ lifts to an element in $\tilde{g} \in \widehat{G}(\breve{\mathbb{Z}}_\ell)$ by (formal) smoothness of $\widehat{G}$ over $\mathbb{Z}_\ell$. Finally, we conclude that (2.10) is an isomorphism by Lemma 2.4. As (2.10) is pinned by construction, Theorem 1.2 follows.

2.8. **Independence of auxiliary pinning.** It remains to show that the pinned isomorphism $\breve{G} \cong \widehat{G}$ is independent of the auxiliary pinning $T \subset B$ and $\mathrm{Lie}(U_a) \cong \mathbb{Z}_\ell$ chosen in the beginning of §2. This follows as in [FS21, top of page 236].

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