Definition of the Satake category and convolution

Torsten Wedhorn

February 20, 2022

Introduction

The goal of this note is to introduce the Satake category in the "classical" setting of geometric Langlands theory using the ideas of Fargues and Scholze in [FS] for the p-adic geometric local Langlands correspondence, transferred to the classical setting, see Section 2.1.

The Satake category consists of flat (relative) perverse sheaves on the Hecke stack that are in addition universally locally acyclic. In this note we focus on the notion of (unbounded, not necessarily constructible) perverse sheaves as needed in this definition. They have been defined by Gabber in great generality for schemes and we recall that definition in Section 1.4. Thereafter we generalize this construction to Artin stacks and to ind schemes. This seems not to be available in the literature although it is rather straight forward.

Contents

1 Reminder on perverse sheaves

1.1 Derived categories of etale sheaves

Throughout we fix a ring Λ of coefficients. Later on we will have to make some restrictions on Λ bur for now let Λ be any (unital, commutative) ring.

For qcqs scheme X we denote by

$$\mathcal{D}(X) := \mathcal{D}_{\text{ét}}(X, \Lambda)$$

the left completion[1] of the derived category (viewed as a stable ∞-category with its natural t-structure) of the category of étale sheaves of Λ-modules. This left completion again is endowed with a natural t-structure.

For a map of qcqs schemes $f\colon X \to Y$ we have a t-exact colimit preserving functor $f^*\colon \mathcal{D}(Y) \to \mathcal{D}(X)$ of stable ∞-categories. Then $X \mapsto \mathcal{D}(X)$ is a sheaf for the topology of universal submersions [HS, 5.7 and the following remark].

We extend $\mathcal{D}(-)$ to all (derived) prestacks by right Kan extension, i.e., it is the unique extension such that if $\mathcal{X} = \operatorname{colim} \mathcal{X}_i$ is a colimit of prestacks, then

$$\mathcal{D}(\mathcal{X}) \xrightarrow{\sim} \lim \mathcal{D}(\mathcal{X}_i), \tag{1.1.1}$$

where lim is the limit in the ∞-category of stable ∞-category with colimit preserving functors (equivalently, the limit in the ∞-category of ∞-categories).

Moreover, if $\mathcal{X}$ is such a prestack and $s\colon \mathcal{X} \to \mathcal{X}_{\text{sub}}$ is its stackification for the submersive topology, then we have an equivalence

$$s^*\colon \mathcal{D}(\mathcal{X}_{\text{sub}}) \xrightarrow{\sim} \mathcal{D}(\mathcal{X}). \tag{1.1.2}$$

Example 1.1. Let S be a scheme, let G be a group pre-stack over S acting from the left on a prestack X over S, e.g. a presheaf of groups on (Aff/S)[2] acting on a sheaf of sets on (Aff/S). Then we have the bar resolution

$$\operatorname{Bar}(G, X) := \Big(\dots \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} G \times_S G \times_S X \substack{\rightarrow\\\rightarrow\\\rightarrow} G \times_S X \underset{p_2}{\overset{a}{\rightrightarrows}} X. \Big)$$

[1] See [Lu-HA, 1.2.1] for the notion of a left completion.
[2] Here (Aff/S) denotes the category of affine schemes equipped with a map to S.

The quotient of X be G is defined as the colimit

$$G\backslash_{\mathrm{pre}}X := \operatorname{colim}\operatorname{Bar}(G,X).$$

Here we take the (homotopy) colimit in the ∞-category of prestacks. If G and X are stacks with respect to some chosen topology τ, then the stackification of $G\backslash_{\mathrm{pre}}X$ with respect to this topology is denoted by $G\backslash_{\tau}X$ or simply by $G\backslash X$.

If G and X are presheaves of sets, then $G\backslash X$ is a presheaf with values in groupoids, i.e. with values in 1-truncated anima.

By (1.1.1) we obtain

$$\mathcal{D}(G\backslash_{\mathrm{pre}}X) = \lim\left(\cdots \substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow} \mathcal{D}(G\times_S G\times_S X) \substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow} \mathcal{D}(G\times_S X) \underset{p_2^*}{\overset{a^*}{\leftleftarrows}} \mathcal{D}(X). \right)$$

If $G\backslash X$ is the stackification for some topology that is coarser than the submersion topology, then (1.1.2) shows that $\mathcal{D}(G\backslash_{\mathrm{pre}}X) = \mathcal{D}(G\backslash X)$.

1.2 Reminder on bounded (perfect) constructible complexes

In this section, Λ will denote a coherent ring (e.g. if Λ is noetherian).

Let X be a qcqs scheme. In $\mathcal{D}(X)$ we have the full subcategory

$$\mathcal{D}^b_{\mathrm{cons}}(X) = \mathcal{D}^b_{\mathrm{cons}}(X_{\text{ét}},\Lambda)$$

spanned by complexes that are bounded with (classical) constructible[3] cohomology sheaves. Then $\mathcal{D}^b_{\mathrm{cons}}(X)$ is a stable ∞-category. It is not presentable as it only admits finite colimits but not arbitrary small colimits. The restriction of the standard t-structure induces a t-structure on $\mathcal{D}^b_{\mathrm{cons}}(X)$[4].

There is the full subcategory $\mathcal{D}_c(X) = \mathcal{D}_c(X_{\text{ét}},\Lambda)$ of $\mathcal{D}(X)$ consisting of *perfect constructible complexes*[5] which consists of those $\mathscr{F}$ in $\mathcal{D}(X)$ such that there exists a finite stratification $(X_i)_i$ of X into constructible locally closed subschemes such that $\mathscr{F}_{|X_i}$ is locally constant with perfect values. It is a subcategory of $\mathcal{D}^b_{\mathrm{cons}}(X)$.

The standard t-structure does in general not restrict to $\mathcal{D}_c(X)$ as truncations of perfect complexes in $D(\Lambda)$ are in general not perfect. If Λ is a regular noetherian ring, then the triangulated category of perfect complexes of Λ-modules is the same as the bounded derived category of finitely generated Λ-modules and in particular is stable under the standard t-structure. Hence $\mathcal{D}_c(X_{\text{ét}},\Lambda)$ carries an induced t-structure in this case.

We have the following result [BS, 6.4.8].

[3]Here we mean the classical definition of constructibility in the sense of [SGA4, IX, 2.3], i.e., an étale sheaf F on a qcqs scheme X is called *constructible* if there exists a finite decomposition of X into locally closed constructible subschemes Y_i such that $F_{|Y_i}$ is locally constant with values in a finitely presented Λ-module.

[4]For this we need that Λ is coherent: we have to ensure that finitely presented Λ-modules form an abelian category.

[5]The category $\mathcal{D}_c(X)$ can be defined for an arbitrary ring.

Proposition 1.2. *Suppose that X is locally of finite Λ-cohomological dimension. Then $\mathcal{D}(X_{et}, \Lambda)$ is already left complete (and hence equal to $\mathcal{D}_{\text{ét}}(X, \Lambda)$). $\mathcal{D}_{\text{ét}}(X, \Lambda)$ is compactly generated*[6] *and the full subcategory of compact objects is $\mathcal{D}_c(X_{\text{ét}}, \Lambda)$. In particular, the inclusion $\mathcal{D}_c(X) \to \mathcal{D}(X)$ induces an equivalence of stable ∞-categories $\operatorname{Ind}(\mathcal{D}_c(X)) \xrightarrow{\sim} \mathcal{D}(X)$.*

The hypothesis on X is for instance satisfied if X is of finite type over a separably closed field k and if Λ is a torsion ring as we have $H^i(U_{et}, \Lambda) = 0$ for $i > \dim(U)$ if U is an affine scheme of finite type over k by Artin's vanishing theorem.

If $f\colon X \to Y$ is a map of qcqs schemes, then the functor $f^*\colon \mathcal{D}(Y) \to \mathcal{D}(X)$ preserves colimits. In particular, it is exact[7]. It induces exact functors $f^*\colon \mathcal{D}^b_{\text{cons}}(Y) \to \mathcal{D}^b_{\text{cons}}(X)$ and $f^*\colon \mathcal{D}_c(Y) \to \mathcal{D}_c(X)$ and the first preserves the standard t-structure.

Moreover, there is the following result by Bhatt and Mathew [BM, 5.10 + 5.13] and Hansen and Scholze [HS, 2.2].

Theorem 1.3. *The functors $X \mapsto \mathcal{D}^b_{\text{cons}}(X)$ and $X \mapsto \mathcal{D}_c(X)$ are finitary*[8] *and are hypercomplete sheaves for the arc topology.*

Recall that a map $f\colon X \to Y$ of qcqs schemes is an *arc-cover* if for any rank ≤ 1 valuation ring V and any map $\operatorname{Spec} V \to Y$ there exists an injective local map $V \to W$ of valuation rings of rank ≤ 1 and a map $\operatorname{Spec} W \to X$ making

$$\begin{array}{ccc} \operatorname{Spec} W & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} V & \longrightarrow & Y \end{array}$$

commutative. Every universal submersion[9] is an arc cover by [BM, 2.19]. Every faithfully flat map between qcqs schemes is a universal submersion [**?**, 14.43] (note that any map between qcqs schemes is automatically quasi-compact).

Lemma 1.4. *If $f\colon X \to Y$ is surjective and locally of finite presentation and $\mathscr{G} \in \mathcal{D}(Y)$ such that $f^*\mathscr{G}$ is contained in $\mathcal{D}^b_{\text{cons}}(X)$ (resp. in $\mathcal{D}_c(X)$), then $\mathscr{G}$ is contained in $\mathcal{D}^b_{\text{cons}}(Y)$ (resp. in $\mathcal{D}_c(Y)$).*

There are similar results for surjective map of small v-stacks in [Sch, 20.5, 20.13] whose ideas allow to prove the lemma for more general f. Here we follow the classical proof in [SGA4, IX,2.8].

---

[6] in the sense of [Lu-HTT, 5.5.7.1]

[7] "exact" means, that f^* commutes with finite limits, finite colimits, and preserves fiber sequences. In fact each of these compatabilities implies the other two compatabilities for functors of stable ∞-categories by [Lu-HA, 1.1.4.1].

[8] "finitary" means that for every filtered projective system of qcqs schemes X_i with affine transition maps one has an equivalence $\operatorname{colim}_i \mathcal{D}^b_{\text{cons}}(X_i) \xrightarrow{\sim} \mathcal{D}^b_{\text{cons}}(\lim_i X_i)$.

[9] A map $f\colon X \to Y$ of schemes is called *submersion* if the underlying continuous map of topological spaces is a surjective quotient map. It is called a *universal submersion* if any base change of f is a submersion.

Proof. Replacing X by a scheme of the form $\coprod_i X_i$ for a finite constructible stratification $(X_i)_i$ of locally closed subschemes of finite presentation of X, we may assume that $f^*(\mathscr{G})$ has locally constant finitely presented cohomology sheaves (resp. is locally constant with perfect values). As these hypotheses are preserved under pullback, we may assume by [EGA, IV, 17.16.4] that f is surjective and étale (even finite étale). But then $\mathscr{G}$ has locally constant finitely presented cohomology sheaves (resp. is locally constant with perfect values). □

1.3 Perverse bounded constructible sheaves on schemes of finite type over a field

Let us first introduce the classical theory of perverse sheaves (with respect to the middle perversity) as defined in [BBDG, §4]. Let k be a field. In this section Λ will be a finite ring whose order is invertible on X.

Let X be a scheme[10] of finite type over k. If $i_{\bar{x}}\colon \bar{x} \to X$ is a geometric point, we denote by $x \in X$ its image and set $d(x) = \dim \overline{\{x\}} = \operatorname{trdeg}(\kappa(x)/k)$. Denoting by $a\colon X \to \operatorname{Spec} k$ the structure map, we denote by $K_X := a^!\Lambda$ the dualizing complex on X.

Recall [BBDG] that there is perverse t-structure on $\mathcal{D}^b_{\mathrm{cons}}(X)$ by defining

$$
\begin{aligned}
{}^p\mathcal{D}^{\leq 0}(X) &= \{\, \mathscr{F} \in \mathcal{D}^b_{\mathrm{cons}}(X) \;;\; i_{\bar{x}}^*\mathscr{F} \in \mathcal{D}^{\leq -d(\bar{x})}(\bar{x}) \,\} \\
&= \{\, \mathscr{F} \in \mathcal{D}^b_{\mathrm{cons}}(X) \;;\; \dim \operatorname{Supp} H^i\mathscr{F} \leq -i \,\}, \\
{}^p\mathcal{D}^{\geq 0}(X) &= \{\, \mathscr{F} \in \mathcal{D}^b_{\mathrm{cons}}(X) \;;\; i_{\bar{x}}^!\mathscr{F} \in \mathcal{D}^{\geq -d(\bar{x})}(\bar{x}) \,\} \\
&= \{\, \mathscr{F} \in \mathcal{D}^b_{\mathrm{cons}}(X) \;;\; D_X(\mathscr{F}) \in {}^p\mathcal{D}^{\leq 0}(X) \,\},
\end{aligned}
\tag{1.3.1}
$$

where $D_X(-) = R\mathscr{H}om(-, K_X)$ is the Verdier dual. The heart of this t-structure is (the nerve of) an abelian category. It is denoted by $\mathrm{Perv}^b_{\mathrm{cons}}(X)$. Elements in $\mathrm{Perv}^b_{\mathrm{cons}}(X)$ are called *bounded constructible perverse sheaves*.

Example 1.5 ([Ill, 1.4])**.** Let X be an lci scheme (e.g. if X is regular) that is equidimensional of dimension n. Then $\Lambda_X[n]$ is a bounded constructible perverse sheaf.

Recall the following fact by [BBDG, 4.1.3] about pushforward of perverse sheaves.

Proposition 1.6. *Let $f\colon X \to Y$ be a quasi-finite and affine map of schemes of finite type over k. Then f_* and $f_!$ are exact for the perverse t-structure. In particular, they induce functors $f_*, f_!\colon \mathrm{Perv}^b_{\mathrm{cons}}(X) \to \mathrm{Perv}^b_{\mathrm{cons}}(Y)$.*

By [BBDG, 4.2.5], the proof of [LO, 4.1], and ??? we have the following properties about pullback of perverse sheaves via smooth maps.

Proposition 1.7. *Let $f\colon X \to Y$ be a smooth map of schemes of finite type over k of relative dimension $d = \dim(f)$ (considered as locally constant function on X).*

[10] In [BBDG] perverse schemes are only defined for *separated* schemes of finite type over k, but separatedness is unnecessary, see [Ill].

(1) *Then* $f^*[d]$ *is exact for the perverse* t*-structure and in particular induces a functor* $f^*[d]\colon \mathrm{Perv}^b_{\mathrm{cons}}(Y) \to \mathrm{Perv}^b_{\mathrm{cons}}(X)$.
(2) *If* f *is in addition surjective, and* $\mathscr{G}$ *is in* $\mathcal{D}(Y)$ *such that* $f^*\mathscr{G}[d]$ *is in* ${}^p\mathcal{D}^{\leq 0}(X)$ *(resp. in* ${}^p\mathcal{D}^{\geq 0}(X)$*), then* $\mathscr{G}$ *is in* ${}^p\mathcal{D}^{\leq 0}(Y)$ *(resp.* ${}^p\mathcal{D}^{\geq 0}(Y)$*).*
(3) *Suppose that* f *has in addition geometrically connected fibers. Then the functor* $f^*[d]\colon \mathrm{Perv}^b_{\mathrm{cons}}(Y) \to \mathrm{Perv}^b_{\mathrm{cons}}(X)$ *is fully faithful and its essential image consists of those perverse sheaves* $\mathscr{F}$ *such that there exists an isomorphism* $p_1^*\mathscr{F} \cong p_2^*\mathscr{F}$ *in* $D^b_{\mathrm{cons}}(X \times_Y X)$*, where* $p_i\colon X \times_Y X \to X$ *are the projections.*

For smooth maps f we have $f^! = f^*[2\dim(f)](\dim f)$[11].

1.4 Unbounded perverse sheaves on schemes

The above results can be extended to define a perverse t-structure for arbitrary schemes and for unbounded complexes without any constructibility hypothesis. We essentially follow [Ga]. Although we work systematically with derived ∞-categories, the t-structures on them only depend on their homotopy categories which allows us to apply the results of [Ga]. In the end, for us it will be only important to define perverse sheaves for schemes of finite type over an algebraically closed field but without any constructibility or boundedness assumption on the underlying objects in $\mathcal{D}(X)$.

To define a perverse t-structure one has to fix a (weak) perversity function in the following sense.

Definition 1.8. *Let* X *be scheme with underlying topological space* $|X|$*. A function* $p\colon |X| \to \mathbb{Z} \cup \{\infty\}$ *is called a weak perversity function if for all* $n \in \mathbb{Z}$ *the set* $\{\, x \in X \;;\; p(x) \geq n \,\}$ *is ind-constructible*[12].

If $|X|$ is locally noetherian, then $p\colon |X| \to \mathbb{Z} \cup \{\infty\}$ is a weak perversity function if and only if for every for every $x \in X$ and every $n \in \mathbb{Z}$ one has $p(y) \geq \min(p(x), n)$ for y in some non-empty open subset of $\overline{x}$ [EGA, IV, 1.9.10]. This condition is for instance satisfied $y \in \overline{\{x\}}$ implies $p(y) \geq p(x)$.

Remark 1.9. Let $f\colon Y \to X$ be a map of qcqs schemes and let p be a weak perversity function on X. As pre-images of ind-constructible sets under f are again ind-constructible, $p \circ f$ is a weak perversity function on Y.

Example 1.10. Let X be a noetherian universally catenaire[13] scheme. A function $\delta\colon |X| \to \mathbb{Z}$ is called *dimension function*[14] if for every immediate specialization y of x

[11] Here $(-)$ denotes the tate twist which is of no consequence in this note since we ignore Galois actions.

[12] A subset E of a locally spectral space Z is called *ind-constructible* if there exists an open covering $(U_i)_i$ by spectral spaces such that $E \cap U_i$ is a union of constructible subsets of U_i for all i.

[13] See e.g. [GW, Section (14.25)] for a definition of a universally catenaire scheme. Most schemes "arising in pratice" are universally catenaire. For instance, every scheme locally of finite type over a Cohen Macaulay ring is universally catenaire [Mat, 17.9].

[14] There exist different definitions of a dimension function in the literature depending on whether one considers Zariski specializations or étale specializations. They agree under our hypothesis that X is noetherian and universally catenaire [ILO, XIV, 2.1.4].

for points $x, y \in X$ one has $\delta(y) = \delta(x) - 1$, see [ILO, XIV, 2.1.8]. Such a dimension function always exists Zariski locally on X by [ILO, XIV, 2.2.1] and any two dimension functions on X differ by a locally constant function [Stacks, 02IB].

If S is a noetherian universally catenaire scheme endowed with a dimension function δ_S and $f\colon X \to S$ is a map of schemes of finite type. Then the map $\delta_X\colon |X| \to \mathbb{Z}$ given by $\delta_X(x) = \delta_S(f(x)) + \operatorname{trdeg}_{\kappa(f(x))} \kappa(x)$ is a dimension function on X [Stacks, 02JW]. In particular, if X is a scheme of finite type over a field k, then $\delta(x) := \operatorname{trdeg}_k \kappa(x) = \dim \overline{\{x\}}$ is a dimension function.

If δ is a dimension function on X, then $-\delta$ is a weak perversity function.

Let X be a qcqs scheme endowed with the étale topology and let Λ be any sheaf of rings on $X_{\text{ét}}$, e.g. the locally constant sheaf $\mathbb{Z}/n\mathbb{Z}$ for some integer n[15]. Let $p\colon |X| \to \mathbb{Z} \cup \{\infty\}$ be a weak perversity function. As before, we denote geometric points over $x \in X$ by $i_{\bar{x}}\colon \bar{x} \to X$. Define full subcategories of $\mathcal{D}(X_{\text{ét}}, \Lambda)$ by

$$
\begin{aligned}
{}^p\mathcal{D}^{\leq 0} &:= \{\, \mathscr{F} \in \mathcal{D}(X_{\text{ét}}, \Lambda) \ ; \ i_{\bar{x}}^{*}\mathscr{F} \in \mathcal{D}^{\leq p(x)}(X_{\text{ét}}, \Lambda) \text{ for all } x \in X\},\\
{}^p\mathcal{D}^{\geq 0} &:= \{\, \mathscr{F} \in \mathcal{D}^{+}(X_{\text{ét}}, \Lambda) \ ; \ i_{\bar{x}}^{!}\mathscr{F} \in \mathcal{D}^{\geq p(x)}(X_{\text{ét}}, \Lambda) \text{ for all } x \in X\}.
\end{aligned}
\tag{1.4.1}
$$

Then Gabber has shown in [Ga, 6] that this defines a t-structure on $\mathcal{D}(X_{\text{ét}}, \Lambda)$, called the *perverse t-structure associated to the perversity function p*. It induces a t-structure on $\mathcal{D}^{+}(X_{\text{ét}}, \Lambda)$. If p is finite and bounded, then it also induces a t-structure on $\mathcal{D}^{b}(X_{\text{ét}}, \Lambda)$ and $\mathcal{D}^{-}(X_{\text{ét}}, \Lambda)$.

We denote the heart of the perverse t-structure on $\mathcal{D}(X_{\text{ét}}, \Lambda)$ by $\operatorname{Perv}_p(X, \Lambda)$ or simply by $\operatorname{Perv}(X)$. By [Ga, 7], $\operatorname{Perv}(X)$ has small colimits, a small set of generators, and filtered colimits are exact.

Let us wrap up everything in the case of interest to us.

Remark 1.11 (Essential Case Here)**.** Let X be of finite type over a separably closed field k. In this case we always choose $x \mapsto -\dim \overline{\{x\}}$ as perversity function (Example 1.10). Let Λ be a the constant ring sheaf $\mathbb{Z}/n\mathbb{Z}$ with n invertible in k. Then one has $\mathcal{D}_{\text{ét}}(X, \Lambda) = \mathcal{D}(X_{\text{ét}}, \Lambda)$ since $X_{\text{ét}}$ has finite Λ-cohomological dimension ([BS, 6.4], here we use that k is separably closed). Hence (1.4.1) defines a t-structure on $\mathcal{D}_{\text{ét}}(X, \Lambda)$ that we call *the* perverse t-structure on $\mathcal{D}_{\text{ét}}(X, \Lambda)$. Its heart is denoted by $\operatorname{Perv}(X)$.

The perverse truncation functors preserve $\mathcal{D}^{b}_{\text{cons}}(X)$ (see also [Ga, 8.2] for a very general result that is also applicable here) and the above perverse t-structure generalizes the t-structure defined in (1.3.1), i.e.

$$\operatorname{Perv}^{b}_{\text{cons}}(X) = \operatorname{Perv}(X) \cap \mathcal{D}^{b}_{\text{cons}}(X).$$

Moreover, by [Ga, 7.2], the inclusion $\operatorname{Perv}^{b}_{\text{cons}}(X) \to \operatorname{Perv}(X)$ induces an equivalence of abelian categories

$$\operatorname{Ind}(\operatorname{Perv}^{b}_{\text{cons}}(X)) \xrightarrow{\sim} \operatorname{Perv}(X), \tag{1.4.2}$$

[15] An other interesting example, which will not play any role in the sequel, is the structure sheaf $\mathscr{O}_X$.

where $\operatorname{Ind}(\mathcal{C})$ denotes the Ind-category[16] of a category $\mathcal{C}$, which is abelian if $\mathcal{C}$ is abelian. Finally, $\operatorname{Perv}^b_{\mathrm{cons}}(X)$ is a finite length abelian category by [Ga, 8.3].

Example 1.12. Let $X = \operatorname{Spec} k$ for a separably closed field k, let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n invertible in k. Then the perverse t-structure on $\mathcal{D}(\operatorname{Spec} k) = \mathcal{D}(\Lambda)$ is the standard t-structure and hence the abelian category of perverse sheaves on $\operatorname{Spec} k$ can be identified with the category of Λ-modules. Then the full subcategory $\operatorname{Perv}^b_c(\operatorname{Spec} k)$ is identified with the category of finitely generated Λ-modules.

The equivalence (1.4.2) allows to extend certain functors from $\operatorname{Perv}^b_{\mathrm{cons}}(-)$ to $\operatorname{Perv}(X)$ using the following general recipe.

Remark 1.13. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor of classical categories. Then there exists a unique functor $\operatorname{Ind}(F)\colon \operatorname{Ind}(\mathcal{C}) \to \operatorname{Ind}(\mathcal{C}')$ that extends F and commutes with filtered colimits. It is given by

$$\operatorname{Ind}(F)(A) = \operatorname*{colim}_{(U\to A)\in\mathcal{C}_{/A}} F(U) \qquad\qquad \text{for } A \in \operatorname{Ind}(\mathcal{C}).$$

If F is fully faithful, then $\operatorname{Ind}(F)$ is fully faithful.

We apply this as follows.

Remark 1.14. Let X and Y be schemes of finite type over a separably closed field k. Let $F\colon \mathcal{D}(X) \to \mathcal{D}(Y)$ be an exact functor of stable ∞-categories that commutes with filtered colimits and that sends $\operatorname{Perv}^b_{\mathrm{cons}}(X)$ to $\operatorname{Perv}^b_{\mathrm{cons}}(Y)$. Then F sends $\operatorname{Perv}(X)$ to $\operatorname{Perv}(Y)$. If $F\colon \operatorname{Perv}^b_{\mathrm{cons}}(X) \to \operatorname{Perv}^b_{\mathrm{cons}}(Y)$ is fully faithful, then $F\colon \operatorname{Perv}(X) \to \operatorname{Perv}(Y)$ is fully faithful.

We can apply this in particular to the functors $f_!$ and f^* as both have a right adjoint functor and in particular commute with colimits.

Corollary 1.15. *Let X and Y be schemes of finite type over a separably closed field k.*

(1) *Let $f\colon X \to Y$ be a quasi-finite and affine map. Then $f_!$ induces a functor* $\operatorname{Perv}(X) \to \operatorname{Perv}(Y)$.
(2) *Let $f\colon X \to Y$ be a smooth map of relative dimension d. Then $f^*[d] = f^![-d](-d)$ induces a functor* $\operatorname{Perv}(Y) \to \operatorname{Perv}(X)$. *If f has in addition geometrically connected fibers, then* $f^*[d]\colon \operatorname{Perv}(Y) \to \operatorname{Perv}(X)$ *is fully faithful with essential image consisting those $\mathscr{F}$ with $p_1^*\mathscr{F} \cong p_2^*\mathscr{F}$.*
(3) *If f is smooth and surjective and $\mathscr{G}$ is in $\mathcal{D}(Y)$ such that $f^*\mathscr{G}[d]$ is in* $\operatorname{Perv}(X)$, *then $\mathscr{G}$ is in* $\operatorname{Perv}(Y)$.

[16] It is the full subcategory of $\operatorname{Func}(\mathcal{C}^{\mathrm{opp}}, (\mathrm{Sets}))$ consisting of (set-valued) presheaves on $\mathcal{C}$ that are isomorphic to a filtered colimit of representable objects. Here the filtered colimit is taken in the category of presheaves. This is important since the Yonede embedding $\mathcal{C} \to \operatorname{Func}(\mathcal{C}^{\mathrm{opp}}, (\mathrm{Sets}))$ usually does not commute with colimits. See [KS, 6] for more details.

Proof. Assertion (1) (resp. (2)) follows from the principle explained in Remark 1.14 using Proposition 1.6 (resp. Proposition 1.7).

Let us show (3). By hypothesis, we have

$$f^{*p}H^i(\mathscr{G})[d] = {}^pH^i(f^*\mathscr{G}[d]) = 0 \qquad \text{for } i \neq 0.$$

As f is smooth and surjective, it has étale locally a section. In particular $f^*\colon \mathcal{D}(Y) \to \mathcal{D}(X)$ is faithful. Hence we see ${}^pH^i(\mathscr{G}) = 0$ for all $i \neq 0$, i.e. $\mathscr{G} \in \mathrm{Perv}(Y)$. □

Remark 1.16. Let $f\colon X \to Y$ be a universal homeomorphism. Then f^* induces equivalences $\mathcal{D}(Y) \xrightarrow{\sim} \mathcal{D}(X)$. Hence it induces an equivalence $\mathrm{Perv}(Y) \xrightarrow{\sim} \mathrm{Perv}(X)$.

1.5 Perverse sheaves on Artin stacks

Let k be a a separably closed field. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ for n invertible in k.

Corollary 1.15 implies that the following definitions of perverse sheaves on Artin stacks is reasonable.

Definition 1.17. Let $\mathcal{X}$ be an Artin stack of finite type over k. An element $\mathscr{F}$ of $\mathcal{D}(\mathcal{X}) = \mathcal{D}_{\text{ét}}(\mathcal{X}, \Lambda)$ is called perverse if there exists a smooth surjective map $f\colon X \to \mathcal{X}$ from a scheme X such that $f^*\mathscr{F}[\dim f]$[17] is in $\mathrm{Perv}(X)$.

Equivalently, $f^*\mathscr{F}[\dim f]$ is perverse for every smooth atlas f.

Remark 1.18. Let $\mathcal{X}$ be an Artin stack of finite type. Let $f\colon X \to \mathcal{X}$ be an atlas of relative dimension d and let $X^{\bullet/\mathcal{X}}$ be its Čech nerve. Then $\mathcal{X}$ is the colimit (in the ∞-category of stacks for the fppf topology) of $X^{\bullet/\mathcal{X}}$. As $X \to \mathcal{X}$ is a universal submersion we get $\mathcal{D}(\mathcal{X}) = \lim \mathcal{D}(X^{\bullet/\mathcal{X}})$.

All projections $X^{n+1/\mathcal{X}} \to X^{n/\mathcal{X}}$ are smooth of relative dimension d and all diagonal maps $X^{n/\mathcal{X}} \to X^{n+1/\mathcal{X}}$ are regular immersion of codimension d. We can now modify $\mathcal{D}(X^{\bullet/\mathcal{X}})$ by shifting all pullback maps via their relative dimensions. In other words, we shift pull back maps via projections $\mathcal{D}(X^{n/\mathcal{X}}) \to \mathcal{D}(X^{n+1/\mathcal{X}})$ by d and we shift pullback maps via diagonals $\mathcal{D}(X^{n+1/\mathcal{X}}) \to \mathcal{D}(X^{n/\mathcal{X}})$ by $-d$. We obtain a new simplicial system of stable ∞-categories that we denote by $\mathcal{D}'(X^{\bullet/\mathcal{X}})$. We still have $\lim \mathcal{D}'(X^{\bullet/\mathcal{X}}) = \mathcal{D}(\mathcal{X})$ and this equivalence induces an equivalence

$$\mathrm{Perv}(\mathcal{X}) = \lim \mathrm{Perv}(X^{\bullet/\mathcal{X}}).$$

Corollary 1.15 (2),(3) imply that one has:

Proposition 1.19. *Let $g\colon \mathcal{Y} \to \mathcal{X}$ be a smooth map of Artin stacks (not necessarily representable). Then $g^*[\dim g]$ induces a functor $\mathrm{Perv}(\mathcal{X}) \to \mathrm{Perv}(\mathcal{Y})$.*

This functor is fully faithful if g is in addition representable with geometrically connected fibers. In this case its essentially image consists of those perverse sheaves $\mathscr{F}$ on $\mathcal{Y}$ such that $p_2^*\mathscr{F} \cong p_1^*\mathscr{F}$, where $p_i\colon \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ are the projections.

[17] As before, we view the relative dimension of f as locally constant function on X.

Proof. Choosing an atlas $X \to \mathcal{X}$ and an atlas $Y \to X \times_{\mathcal{X}} \mathcal{Y}$ we obtain a commutative diagram

(*)
$$\begin{array}{ccc} Y & \xrightarrow{\tilde{g}} & X \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xrightarrow{g} & \mathcal{X}, \end{array}$$

where the vertical maps are smooth and surjective, X and Y are schemes and $\tilde{g}$ is smooth. Now the first assertion follows formally.

To see the second assertion, one can assume that (*) is cartesian since g is representable. Then $\tilde{g}$ is smooth with geometrically connected fibers and $\dim \tilde{g} = \dim g$. The map $\tilde{g}$ induces a map $\tilde{g}^{\bullet}\colon Y^{\bullet/\mathcal{Y}} \to X^{\bullet/\mathcal{X}}$ such that $\tilde{g}^{n}\colon Y^{n/\mathcal{X}} \to X^{n/\mathcal{X}}$ is a smooth map of relative dimension $\dim g$ between schemes and has geometrically connected fibers. Hence it is fully faithful on perverse sheaves. We conclude by Remark 1.18. □

Example 1.20. Let G be a smooth algebraic group over k and let BG be its classifying stack.

The quotient map $\pi\colon \operatorname{Spec} k \to BG$ is representable smooth of relative dimension $g := \dim G$. Hence $\pi^*[g]$ yields a functor $\mathrm{Perv}(BG) \to \mathrm{Perv}(k)$.

If G is connected, then $\pi^*[g]$ yields an equivalence $\mathrm{Perv}(BG) \to \mathrm{Perv}(k) = (\Lambda\text{-Mod})$. Indeed, we apply Proposition 1.19. It shows that $\pi^*[g]$ is fully faithful. We have $\operatorname{Spec} k \times_{BG} \operatorname{Spec} k = G$ and the projections to $\operatorname{Spec} k$ can be both identified with the structure map. Hence the functor $\pi^*[g]$ is also essentially surjective.

The structure map $\sigma\colon BG \to \operatorname{Spec} k$ is smooth of relative dimension $-\dim(G)$ and $\sigma^*[-g]$ defines a section of $\pi^*[g]$, which is an inverse if G is connected.

A special case of Proposition 1.19 easily implies the following description of perverse sheaves on quotient stacks.

Corollary 1.21. *Let X be a scheme of finite type over k, let H be a connected smooth algebraic group acting on X. Denote by $a, p_2\colon H \times X \to X$ the action and the projection, respectively. Let $\pi\colon X \to [H\backslash X]$ the canonical map. Then $\pi^*[\dim H]$ induces an equivalence of abelian categories*

$$\mathrm{Perv}([H\backslash X]) \cong \{\, \mathscr{F} \in \mathrm{Perv}(X)\ ;\ a^*\mathscr{F} \cong p_2^*\mathscr{F} \,\},$$

where $a, p_2\colon H \times_k X \to X$ are the action map and the second projection, respectively.

This is the classical definition of H-equivariant perverse sheaves on X (at least for bounded constructible perverse sheaves).

If H acts trivially on X in the situation of Corollary 1.21, then $a = p_2$ and one obtains an equivalence

$$\mathrm{Perv}(H\backslash X) \cong \mathrm{Perv}(X).$$

This is a special case of the following more general corollary.

Corollary 1.22. *Let X be a scheme of finite type over k, let G be smooth connected affine group scheme that acts on X. Let $H \subseteq G$ be a normal smooth connected subgroup scheme of G that acts trivially on X. Let*

$$\pi\colon G\backslash X \longrightarrow (G/H)\backslash X$$

be the canonical map of Artin stacks. The functor $\pi^*[-\dim H]$ yields an equivalence

$$\mathrm{Perv}(G\backslash X) \cong \mathrm{Perv}((G/H)\backslash X).$$

Proof. This follows from Corollary 1.21. □

Functoriality for certain pushforwards can also be transferred to Artin stacks.

Proposition 1.23. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-finite affine representable map of Artin stacks of finite type over k. Then $f_!$ induces a functor $\mathrm{Perv}(\mathcal{X}) \to \mathrm{Perv}(\mathcal{Y})$.*

If f is finite, then $f_! = f_*$ and hence f_* sends perverse sheaves to perverse sheaves.

Proof. Choose an atlas $g\colon Y \to \mathcal{Y}$ and let d be the relative dimension of g. Form the cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}. \end{array}$$

Then h is an atlas of relative dimension d. Hence if $\mathscr{F} \in \mathrm{Perv}(\mathcal{X})$, then $h^*\mathscr{F}[d] \in \mathrm{Perv}(X)$ and hence

$$\tilde{f}_! h^*\mathscr{F}[d] = g^* f_! \mathscr{F}[d] \in \mathrm{Perv}(Y),$$

where the equality holds by proper base change. But this means that $f_!\mathscr{F} \in \mathrm{Perv}(\mathcal{Y})$. □

1.6 Perverse sheaves on ind-schemes

We continue to denote by k a separably closed field and by $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n invertible in k. By Remark 1.16 perverse sheaves do not see nilpotent elements. This leads us to the following definition.

Definition 1.24. Let X be an ind-scheme and let $X_{\mathrm{red}} = \mathrm{colim}_i X_i$ be a representation, where X_i are reduced schemes of finite type over k. For $i \leq j$ one has closed immersions $X_i \to X_j$ which induce by push forward functors $\mathrm{Perv}(X_i) \to \mathrm{Perv}(X_j)$ (Proposition 1.6). We define

$$\mathrm{Perv}(X) := \operatorname*{colim}_i \mathrm{Perv}(X_i).$$

A standard argument shows that this is well defined, i.e., independent of the presentation $X_{\mathrm{red}} = \mathrm{colim}_i X_i$.

1.7 Flat perverse sheaves

If X is any geometric object for which we defined the notion of $\mathrm{Perv}(X) = \mathrm{Perv}(X, \Lambda) \subseteq \mathcal{D}_{\text{ét}}(X, \Lambda)$. Then there is usually no good functoriality of $\mathrm{Perv}(X, \Lambda)$ in Λ as the following trivial example shows.

Example 1.25. Let $X = \operatorname{Spec} k$ for a separably closed field k. Then $\mathrm{Perv}(X, \Lambda)$ is the category of Λ-modules. Hence if $\Lambda \to \Lambda'$ is a map of noetherian rings (e.g., $\mathbb{Z}/\ell^n \to \mathbb{Z}/\ell^m$ for $m \leq n$) and $\mathscr{F} \in \mathrm{Perv}(X, \Lambda)$, then $\mathscr{F} \otimes^{\mathbb{L}}{}_{\Lambda} \Lambda'$ will be in $\mathrm{Perv}(\Lambda')$ if and only if $\mathrm{Tor}_i^{\Lambda}(\Lambda', \mathscr{F}) = 0$ for $i > 0$, which is usually not the case. It is the case for all maps $\Lambda \to \Lambda'$ if and only if $\mathscr{F}$ is flat as a Λ-module.

Therefore we have the following definition that ensures preservation of perversity under change of coefficients.

Definition 1.26. We call $\mathscr{F} \in \mathrm{Perv}(X, \Lambda)$ *flat perverse* if $\mathscr{F} \otimes^{\mathbb{L}}{}_{\Lambda} M \in \mathrm{Perv}(X, \Lambda)$ for all Λ-modules M.

Then $\mathscr{F} \in \mathrm{Perv}(X, \Lambda)$ is flat perverse if and only if for every map $\Lambda \to \Lambda'$ of noetherian rings one has $\mathscr{F} \otimes^{\mathbb{L}}{}_{\Lambda} \Lambda' \in \mathrm{Perv}(X, \Lambda')$.

2 Perverse Sheaves on the Hecke stack

2.1 Reminder on notation

To define perverse sheaves on the Hecke stack and define the Satake category we use the following notation. We fix an algebraically closed field k. For any k-algebra R we set $\mathbb{D}_R := \operatorname{Spec} R[\![t]\!]$ and $\mathbb{D}_R^* := \operatorname{Spec} R(\!(t)\!)$.

We fix a (connected) reductive group G over k. We choose a Borel subgroup and a maximal torus $T \subseteq B \subseteq G$ and we denote by $X_*(T)_+$ the monoid of all B-dominant coharacters of T. If S is any k-scheme we denote the trivial G-bundle over S by $\mathscr{E}_S^{\mathrm{triv}}$.

Recall that there are defined (groupoid-valued) fpqc stacks

$$\mathrm{Hk}_G, \mathrm{Gr}_G \colon (k\text{-Alg}) \to (\mathrm{Grpd}).$$

Here Hk_G is the (local) Hecke stack and $\mathrm{Hk}_G(R)$ is the groupoid of triples $(\mathscr{E}_0, \mathscr{E}_1, \alpha)$, where $\mathscr{E}_0$ and $\mathscr{E}_1$ are G-bundles on $\mathbb{D}_R$ and where $\alpha \colon \mathscr{E}_{0|\mathbb{D}_R^*} \xrightarrow{\sim} \mathscr{E}_{1|\mathbb{D}_R^*}$ is an isomorphism of G-bundles on $\mathbb{D}_R^*$. There is the obvious notion of an isomorphism of such triples.

The (local) affine Grassmannian Gr_G is given by defining $\mathrm{Gr}_G(R)$ the groupoid of tuples $(\mathscr{E}_0, \mathscr{E}_1, \alpha, \beta)$ with $(\mathscr{E}_0, \mathscr{E}_1, \alpha) \in \mathrm{Hk}_G(R)$ and where $\beta \colon \mathscr{E}_1 \xrightarrow{\sim} \mathscr{E}_{\mathbb{D}_R}^{\mathrm{triv}}$ is a trivialization of $\mathscr{E}_1$ over $\mathbb{D}_R$. Here the isomorphisms are given by those isomorphism of objects in $\mathrm{Hk}_G(R)$ which induce on $\mathscr{E}^{\mathrm{triv}}$ the identity. Then $\mathrm{Gr}_G(R)$ is a static groupoid, i.e., it is equivalent to a groupoid defined by a set. Hence we can consider Gr_G as an fpqc-sheaf. There is an obvious forgetful map

$$\mathrm{Gr}_G \longrightarrow \mathrm{Hk}_G\,.$$

We have seen that there is a commutative diagram, where the horizontal maps are isomorphisms,

$$\begin{array}{ccc} LG/L^+G & \xrightarrow{\sim} & \mathrm{Gr}_G \\ \downarrow & & \downarrow \\ [L+G\backslash LG/L^+G] & \longrightarrow & \mathrm{Hk}_G, \end{array}$$

in particular, $\mathrm{Gr}_G \to \mathrm{Hk}_G$ is an L^+G-torsor.

We fix a prime ℓ which is invertible in k and a finite ring Λ which is annihilated by some power of ℓ.

2.2 Perverse sheaves on the affine Grassmannian

We have seen that we can write the underlying reduced ind-scheme of each connected component $\mathrm{Gr}_{G,\gamma}$ for $\gamma \in \pi_1(G)$ of the affine Grassmannian as Gr_G as filtered colimit of Schubert varieties

$$\mathrm{Gr}_{G,\gamma,\mathrm{red}} = \operatorname*{colim}_{\{\,\mu\in X_*(T)_+\ ;\ \mu^\#=\gamma\,\}} \mathrm{Gr}_{G,\leq\mu}\,.$$

Thus we obtain

$$\mathrm{Perv}(\mathrm{Gr}_{G,\gamma}) = \operatorname*{colim}_{\mu,\mu^\#=\gamma} \mathrm{Perv}(\mathrm{Gr}_{G,\leq\mu}).$$

Altogether we set[18]

$$\mathrm{Perv}(\mathrm{Gr}_G) := \bigoplus_{\gamma\in\pi_1(G)} \mathrm{Perv}(\mathrm{Gr}_{G,\gamma}).$$

Recall that the Schubert varieties $\mathrm{Gr}_{G,\leq\mu}$ are projective irreducible varieties of dimension

$$d(\mu) := \langle 2\rho, \mu\rangle,$$

where, as usual, ρ is the half sum of the positive roots of (G, B, T).

2.3 Approximation of the Hecke stack by Artin stacks

We recall same facts about Schubert varieties in the affine Grassmannian (e.g. [Zh]). The closed subschemes $\mathrm{Gr}_{G,\leq\mu}$ for $\mu \in X_*(T)_+$ of Gr_G are by definition $L^+(G)$-stable and the $L^+(G)$-action on $\mathrm{Gr}_{G,\leq\mu}$ factors through an algebraic quotient $L_\mu := L^+G/K_\mu$, where K_μ is a connected normal closed subgroup scheme of L^+G. We can arrange the K_μ in such a way that for $\mu \leq \lambda$ one has $K_\lambda \subseteq K_\mu$. Then one has by definition an isomorphism of smooth connected affine group schemes

$$L_\lambda/(K_\mu/K_\lambda) \cong L_\mu,$$

where K_μ/K_λ is a normal connected smooth algebraic subgroup of L_λ. We may also assume that $\lim_\mu L_\mu = L^+G$ and hence $\lim_\mu K_\mu = 1$.

[18]It would have been more conceptual to define ind-schemes as colimits of representable schemes indexed by a category whose every connected component is filtered.

We obtain maps of Artin stacks

$$(*) \qquad L_\mu \backslash \mathrm{Gr}_{G,\le\mu} \xleftarrow{\sigma} L_\lambda \backslash \mathrm{Gr}_{G,\le\mu} \xrightarrow{i} L_\lambda \backslash \mathrm{Gr}_{G,\le\lambda}\,.$$

The left map σ is induced by the projection $L_\lambda \to L_\mu$ and hence is smooth of relative dimension

$$d_{\mu,\lambda} := \dim(L_\mu) - \dim(L_\lambda) \le 0$$

Hence $\sigma^*[d_{\mu,\lambda}]$ induces by Corollary 1.22 an equivalence

$$\mathrm{Perv}(L_\mu \backslash \mathrm{Gr}_{G,\le\mu}) \xrightarrow{\sim} \mathrm{Perv}(L_\lambda \backslash \mathrm{Gr}_{G,\le\mu}).$$

The right map i in (*) is induced by the closed immersion $\mathrm{Gr}_{G,\le\mu} \to \mathrm{Gr}_{G,\le\lambda}$ and hence is itself representable and a closed immersion. By Proposition 1.23, i_* yields a pushforward functor

$$i_*\colon \mathrm{Perv}(L_\lambda \backslash \mathrm{Gr}_{G,\le\mu}) \longrightarrow \mathrm{Perv}(L_\lambda \backslash \mathrm{Gr}_{G,\le\lambda}).$$

Composing $\sigma^*[d_{\mu,\lambda}]$ and i_* yields for $\mu \le \lambda$ transition functors

$$\mathrm{Perv}(L_\mu \backslash \mathrm{Gr}_{G,\le\mu}) \longrightarrow \mathrm{Perv}(L_\lambda \backslash \mathrm{Gr}_{G,\le\lambda}).$$

We set

$$(2.3.1) \qquad \mathrm{Perv}(\mathrm{Hk}_G) := \operatorname*{colim}_{\mu \in X_*(T)_+} \mathrm{Perv}(L_\lambda \backslash \mathrm{Gr}_{G,\le\lambda}).$$

This definition and Proposition 2.1 below can be generalized to arbitrary ind-finite-type schemes endowed with an action by a pro-algebraic group such that the stabilizer of each geometric point has only finitely many connected components, see [Zh, A.1.4].

2.4 Description of perverse sheaves on the Hecke stack

We now motivate the definition of perverse sheaves on the Hecke stack given in (2.3.1) and show in particular that it is independent of the choice of the algebraic quotients L_μ.

Proposition 2.1. *Let $a, p_2\colon L^+G \times \mathrm{Gr}_G \to \mathrm{Gr}_G$ be the action and the second projection. Then we have an equivalence of abelian categories*

$$\mathrm{Perv}(\mathrm{Hk}_G) = \{\, \mathscr{F} \in \mathrm{Perv}(\mathrm{Gr}_G)\ ;\ a^*\mathscr{F} \cong p_2^*\mathscr{F} \,\}$$

Proof. Consider for $\mu \le \lambda$ the commutative diagram, where the square is cartesian,

$$\begin{array}{ccccc} & & \mathrm{Gr}_{G,\le\mu} & \xrightarrow{\tilde{\imath}} & \mathrm{Gr}_{G,\le\lambda} \\ & \swarrow^{\pi_\mu} & \downarrow{\scriptstyle \pi_{\mu,\lambda}} & & \downarrow{\scriptstyle \pi_\lambda} \\ L_\mu \backslash \mathrm{Gr}_{G,\le\mu} & \xleftarrow{\sigma} & L_\lambda \backslash \mathrm{Gr}_{G,\le\mu} & \xrightarrow{i} & L_\lambda \backslash \mathrm{Gr}_{G,\le\lambda}\,. \end{array}$$

We set $\ell_\mu := \dim L_\mu$. Then the above diagram induces a diagram

$$\begin{array}{ccccc}
 & & \mathrm{Perv}(\mathrm{Gr}_{G,\le\mu}) & \xrightarrow{\tilde{i}_*} & \mathrm{Perv}(\mathrm{Gr}_{G,\le\lambda}) \\
 & \nearrow^{\pi_\mu^*[\ell_\mu]} & \uparrow{\scriptstyle \pi^*_{\mu,\lambda}[\ell_\lambda]} & & \uparrow{\scriptstyle \pi^*_\lambda[\ell_\lambda]} \\
\mathrm{Perv}(L_\mu\backslash \mathrm{Gr}_{G,\le\mu}) & \xrightarrow{\sigma^*[\ell_\mu - \ell_\lambda]} & \mathrm{Perv}(L_\lambda\backslash \mathrm{Gr}_{G,\le\mu}) & \xrightarrow{i_*} & \mathrm{Perv}(L_\lambda\backslash \mathrm{Gr}_{G,\le\lambda})
\end{array}$$

which commutes by proper base change. Using Corollary 1.21 and passing to the limit implies the claim (details omitted). □

We have also the following description of $\mathrm{Perv}(\mathrm{Hk}_G)$ which is the analogue of the perversity definition on the Hecke stack in [FS].

Recall that the underlying topological space $|\,\mathrm{Hk}_G\,|$ is given by the partially ordered set $X_*(T)_+$. Let us describe for $\mu \in X_*(T)_+$ the residual gerbe in Hk_G. We set

$$L^+G_\mu := L^+G \cap t^\mu L^+(G) t^{-\mu},$$

which is the stabilizer in L^+G of the Schubert cell $\mathrm{Gr}_{G,\mu}$. As L^+G acts transitively on each Schubert cell, we have $\mathrm{Gr}_{G,\mu} = L^+G/L^+G_\mu$. Hence the residual gerbe is given by

$$L^+G\backslash \mathrm{Gr}_{G,\mu} = L^+G_\mu\backslash \operatorname{Spec} k.$$

Denote by i_μ the locally closed immersion of the residual gerbe at the geometric point μ of the Hecke stack and set $d_\mu = \dim \mathrm{Gr}_{G,\mu} = \langle 2\rho, \mu\rangle$. Then

$$\mathrm{Perv}(\mathrm{Hk}_G) = \{\, \mathscr{F} \in \mathcal{D}(\mathrm{Hk}_G)^{\mathrm{bd}} \ ; \ i_\mu^*\mathscr{F} \in \mathcal{D}^{\le -d(\mu)}, i_\mu^!\mathscr{F} \in \mathcal{D}^{\ge -d(\mu)} \,\},$$

where for some ind-scheme $X = \operatorname{colim} X_i$ one sets $\mathcal{D}(X)^{\mathrm{bd}} = \operatorname{colim} \mathcal{D}(X_i)$.

3 The Satake category

3.1 ULA-sheaves on ind-schemes

Definition and Remark 3.1. Let S be a scheme and let $X = \operatorname{colim}_i X_i$ be an ind-scheme over S such that $X_i \to S$ is separated and of finite presentation. Then we call $\mathscr{F} \in \mathcal{D}(X)^{\mathrm{bd}} = \operatorname{colim} \mathcal{D}(X_i)$ *universally locally acyclic over* S if there exists $\mathscr{F}_i \in \mathcal{D}(X_i)$ mapping to $\mathscr{F}$ such that $\mathscr{F}_i$ is universally locally acyclic over S.

For this to be a decent definition recall that the pushforward of relative ULA-sheaves under closed immersions (or, more generally, under proper maps of S-schemes) is again relative ULA.

Example 3.2. Let $S = \operatorname{Spec} k$ be a field and $X = \operatorname{colim} X_i$ an ind-scheme over k such that X_i are of finite type and separated over k. Then $\mathscr{F} \in \mathcal{D}(X_i)$ is ULA over $\operatorname{Spec} k$ if and only if $\mathscr{F}$ is perfect constructible.

Proof missing.

3.2 Definition of the Satake category

If $\mathscr{F}$ is in $\mathrm{Perv}(\mathrm{Hk}_G)$ we can view it as a perverse sheaf of Gr_G by Proposition 2.1. In particular, we have defined for $\mathscr{F}$ to be ULA over $\mathrm{Spec}\, k$. We now can come to the main definition.

Definition 3.3. We define the *Satake category*

$$\mathrm{Sat}_G(\Lambda) := \{\, \mathscr{F} \in \mathrm{Perv}(\mathrm{Hk}_G) \text{ flat perverse } ; \mathscr{F} \text{ is ULA over } \mathrm{Spec}\, k\}.$$

This is not an abelian category since cokernels of maps between flat perverse sheaves are not necessarily flat perverse.

This is also seen by the following trivial example.

Example 3.4. Let $G = 1$ be the trivial group. Then $\mathrm{Gr}_G = \mathrm{Hk}_G = \mathrm{Spec}\, k$ and $\mathrm{Perv}(\mathrm{Hk}_G)$ can be identified with the category of Λ-modules. A perverse sheaf $\mathscr{F} \in \mathrm{Perv}(\mathrm{Hk}_G)$ is flat perverse, if and only if it is flat as a Λ-module. It is ULA over $\mathrm{Spec}\, k$ if and only if $\mathscr{F}$ is perfect as a Λ-module. As perfect Λ-modules are in particular of finite presentation, we see that $\mathrm{Sat}_1(\Lambda)$ is the additive category of finite projective Λ-modules.

If $G = T$ is a torus, then $\mathrm{Gr}_{T,\mathrm{red}} = \coprod_{\mu \in X_*(T)} \mathrm{Spec}\, k$ and this trivial example will be the starting point for the proof of the geometric Satake equivalence.

3.3 The fiber functor

The Satake category carries a fiber functor

$$\begin{aligned} F\colon \mathrm{Sat}_G(\Lambda) &\to (\Lambda\text{-Mod}), \\ A &\mapsto \bigoplus_{k \in \mathbb{Z}} H^k(\mathrm{Gr}_G, A). \end{aligned} \tag{3.3.1}$$

4 Convolution

4.1 Convolution on smooth group schemes

Let S be a scheme and let $H \to S$ be a smooth group scheme. Then the multiplication $m\colon H \times_S H \to H$ is smooth as well (it is the composition of the isomorphism of S-schemes $H \times_S H \xrightarrow{\sim} H \times_S H$, $(h, h') \mapsto (hh', h')$ followed by the first projection $H \times_S H \to H$). We define a convolution product on $\mathcal{D}_{\text{ét}}(H, \Lambda)$ by

$$A * B := Rm_*(p_1^* A \otimes^{\mathbb{L}} p_2^* B). \tag{4.1.1}$$

This defines a monoidal structure on $\mathcal{D}_{\text{ét}}(H, \Lambda)$ (to see the associativity one uses smooth base change).

4.2 Convolutions on double quotients of smooth group schemes

More generally, let $K \subseteq H$ be a smooth subgroup scheme and set $X := K\backslash H/K$, where we mean the stack quotient. It is an algebraic stack with $H/K \to K\backslash H/K$ an atlas by an algebraic space. Moreover, we let K act from the left on the algebraic space $K\backslash H \times H/K$ by $k \cdot (h, h') = (hk^{-1}, kh')$ and denote the quotient stack as usual by

$$K\backslash H \times^K H/K.$$

Then the multiplication on H and the projections define a "convolution diagram"

$$X \times_S X \xleftarrow{(p_1,p_2)} K\backslash H \times^K H/K \xrightarrow{\quad m \quad} X. \tag{4.2.1}$$

Then (4.1.1) defines again a monoidal structure on $\mathcal{D}_{\text{ét}}(X, \Lambda)$.

4.3 Convolution on $\mathcal{D}(\mathrm{Hk}_G)$

Next we imitate the above definition for $\mathrm{Hk}_K = L^+G\backslash LG/L^+G$. We set

$$\mathrm{Hk}_G \tilde{\times} \mathrm{Hk}_G := L^+G\backslash LG \times^{L^+G} LG/L^+G.$$

It is the fpqc stack whose R-valued points for a k-algebra R is the groupoid of tuples $(\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3, \alpha_1, \alpha_2)$, where the $\mathscr{E}_i$ are G-bundles over $\mathbb{D}_R$ and where

$$\alpha_i \colon \mathscr{E}_{i|\mathbb{D}^*_R} \overset{\sim}{\to} \mathscr{E}_{i+1|\mathbb{D}^*_R}, \qquad\qquad i = 1, 2$$

are isomorphism of G-bundles over $\mathbb{D}^*_R$. Then the projections and the multiplication in the convolution diagram take the form

$$\begin{aligned} p_i \colon \mathrm{Hk}_G \tilde{\times} \mathrm{Hk}_G \to \mathrm{Hk}_G, \qquad &(\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3, \alpha_1, \alpha_2) \mapsto (\mathscr{E}_i, \mathscr{E}_{i+1}, \alpha_i), \\ m \colon \mathrm{Hk}_G \tilde{\times} \mathrm{Hk}_G \to \mathrm{Hk}_G, \qquad &(\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3, \alpha_1, \alpha_2) \mapsto (\mathscr{E}_1, \mathscr{E}_3, \alpha_2 \circ \alpha_1). \end{aligned}$$

Then we define again a monoidal structure on $\mathcal{D}(\mathrm{Hk}_G)^{\mathrm{bd}}$ by (4.1.1). One checks associativity by reducing to the finite-dimensional case (details omitted).

In the next talks it will be shown that the convolution preserves $\mathrm{Sat}_G(\Lambda)$ and that $(\mathrm{Sat}_G(\Lambda), *)$ can be endowed with a commutativity constraint making it into a symmetric monoidal exact category. Moreover, one shows that the fiber functor F (3.3.1) is compatible with monoidal structures.

If $G = 1$ (Example 3.4) and hence $\mathrm{Sat}_1(\Lambda)$ is the category of finite projective Λ-modules, the convolution is given by the tensor product of Λ-modules.

5 Variant for the global Hecke stack

For a smooth (geometrically) connected curve X over k and a finite set I one has global variants of the affine Grassmannian and the Hecke stack

$$\mathrm{Gr}_{G,I} \longrightarrow \mathrm{Hk}_{G,I} \longrightarrow X^I.$$

Then one defines the corresponding Satake category $\mathrm{Sat}^I_G(\Lambda)$, the convolution, and the fiber functor as above by replacing the words "perverse" by "relative perverse" and "ULA over k" by "ULA over X^I".

References

[BBDG] A. Beilinson, J. Bernstein, P. Deligne, O. Gabber, *Faisceaux pervers*, Astérisque 100

[BM] B. Bhatt, A. Mathew, *The arc-topology*, Duke Math. J. **170** (2021), 1899–1988

[BS] B. Bhatt, P. Scholze, *The pro-étale topology for schemes*, Astérisque **369** (2015), 99–201

[EGA] A. Grothendieck, J. Dieudonné: *Eléments de Géométrie Algébrique*, I Grundlehren der Mathematik **166** (1971) Springer, II-IV Publ. Math. IHES **8** (1961), **11** (1961), **17** (1963), **20** (1964), **24** (1965), **28** (1966), **32** (1967).

[FS] L. Fargues, P. Scholze, *Geometrization of the local Langlands Correspondence*, arXiv:2102.13459

[Ga] O. Gabber, *Notes on some t-structures*, In *Geometric aspects of Dwork theory*, Vol. I, II, 711–734.

[GW] U. Görtz, T. Wedhorn, *Algebraic Geometry I, Schemes*, 2nd edition, Springer 2020

[HS] D. Hansen, P. Scholze, *Relative Perversity*, arXiv:2109.06766

[KS] M. Kashiwara, P. Schapira, *Categories and Sheaves*, Springer 2006

[Ill] L. Illusie, *Perversité et variation*, Manuscripta Math. **112** (2003), 271–295

[ILO] L. Illusie, Y. Laszlo, F. Orgogozo, *Travaux de Gabber dur l'uniformisation locale et la cohomologie étale des schemas quasi-excellents*, Astérisque 363

[LO] Y. Laszlo, M. Olsson, *Pervers t-structure on Artin stacks*, Math. Zeitschrift **261** (2009), 737–748

[Lu-HA] J. Lurie, *Higher Algebra*

[Lu-HTT] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies **170**

[Mat] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics **8** (2009)

[Sch] P. Scholze, *Étale cohomology of diamonds*, arXiv:1709.07343

[SGA4] M. Artin, A. Grothendieck, J. L. Verdier, *Théorie des topos et cohomologie étale des schémas*, LNM 269,270, 305

[Stacks] The Stacks Project Authors, *Stacks Project.*

[Zh] X. Zhu, *An introduction to affine Grassmannians and the geometric Satake equivalence*, arXiv 1603.05593