Lectures on modular representation theory of reductive algebraic groups

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Introduction

0.1. Presentation. The goal of these notes is to present a new approach to the study of the representation theory of reductive algebraic groups over algebraically closed fields of positive characteristic, in particular to the problem of character computation, that has emerged in the last 10 years. This approach was found as part as a joint work with Geordie Williamson, and was implemented in the following years involving joint work with various collaborators, in particular Geordie Williamson and Pramod Achar.

Since the 1980's, the study of representation theory of reductive algebraic groups in positive characteristic has been driven by a conjectural character formula for simple modules proposed by Lusztig in 1980, in terms of the Kazhdan–Lusztig polynomials associated with the corresponding affine Weyl group. This formula has been proved in the mid-1990's by the combination of deep works of Kazhdan–Lusztig, Kashiwara–Tanisaki and Andersen–Jantzen–Soergel (following a strategy outlined by Lusztig and involving the study of a similar problem for quantum groups at a root of unity), but only under the assumption that the characteristic of the base field is large i.e. larger that a nonexplicit bound depending on the root datum of the group). Later work by Fiebig allowed to give an explicit lower bound for how large the characteristic should be, but this lower bound is huge.

At the surprise of many expert, in 2013 Geordie Williamson found a family of examples in case of the group GL_n, showing that Lusztig's formula cannot be true under any assumption of the form $p \geq P(n)$ where P is a fixed polynomial. In other words, Lusztig's formula is only an *asymptotic* answer to the question of computing simple characters for a reductive group, but a general answer has to be more subtle.

0.2. Characters and the p-canonical basis. After the discovery of these counterexamples, in joint with with Williamson [**RW1**] we expressed the idea that the combinatorics that should be used to express characters of a reductive groups G over a field of characteristic p (for general p) is not the Kazhdan–Lusztig combinatorics as predicted by Lusztig, but rather its "p-canonical" version that started to emerge from joint work of Williamson with Mautner and Juteau on parity complexes on the one hand, and with Elias on a presentation of Soergel bimodules by generators and relations on the other hand. (The equivalence between the two approaches was morally clear, and was proved explicitly in Part III of [**RW1**].) To make this idea concrete one should switch perspective a bit; instead of giving an explicit character formula for simple modules as done by Lusztig, it is more convenient to give a character formula for another family of modules, namely the indecomposable tilting modules, from which the characters of simples can be obtained. (The observation that characters of tilting modules determine characters of simple modules is due to Andersen.) Explicitly, in [**RW1**] we proposed a conjectural character formula for indecomposable tilting modules in terms of (antispherical) p-Kazhdan–Lusztig polynomials. This formula was later proved for regular blocks in joint work with Achar, Makisumi and Williamson [**AMRW**], and then for all blocks (in all characteristics) in joint work with Williamson [**RW3**].

0.3. What this book might be good for. Our hope in writing this book is that it can serve as a guide for the reader interested in this topic to go from the classical approach on this subject on which Lusztig's formula is based, which is summarized in a marvelous way in Jantzen's classical book [**J3**], to the recent literature on this subject, in particular Williamson's construction of counterexamples to Lusztig's formula [**W3**] and the proofs of the tilting character formula in [**AMRW**] and [**RW3**]. We will not be able to give detailed proofs of any deep result in this direction, but what we have tried to do is to explain the main constructions involved, state the most important results in the largest reasonable generality that is available in the literature, and clarify some results that are usually considered "well-known" but whose explicit proofs are difficult to find. We have also tried to give precise references for all the results we require, sometimes pointing some gaps in the original literature that have been filled by later work.

0.4. Contents. Will be completed later.

0.5. Prerequisites. We will assume that the reader is familiar with the structure theory of connected reductive algebraic groups over algebraically closed fields, as explained e.g. in the classical books of Borel [**Bo**], Humphreys [**H2**] and Springer [**Sp2**]. All the results from representation theory of algebraic groups that we will need will be recalled, usually with the appropriate reference to [**J3**]. We will also rely on the basic theory of Coxeter groups, for which we refer to [**H3**] or [**Mi**].

0.6. Some notation and conventions.

0.6.1. *Grothendieck groups.* If A is an essential small additive, resp. abelian, resp. triangulated, category, we denote by

$$[\mathsf{A}]_\oplus, \quad \text{resp. } [\mathsf{A}], \quad \text{resp. } [\mathsf{A}]_\Delta,$$

its split Grothendieck group, resp. its Grothendieck group, resp. its Grothendieck group. In each case, the class of an object $M \in \mathsf{A}$ will be denoted $[M]$.

0.6.2. *Modules and bimodules.* If A is a ring, we will denote by

$$A\text{-}\mathsf{Mod}, \quad \text{resp. } \mathsf{Mod}\text{-}A,$$

the category of left A-modules, resp. of right A-modules, and by

$$A\text{-}\mathsf{Mod}_{\mathrm{fg}}, \quad \text{resp. } \mathsf{Mod}_{\mathrm{fg}}\text{-}A,$$

the subcategory of finitely generated modules.

If A is a k-algebra for some commutative ring k, by an A-bimodule we will mean a left A-module endowed with a commuting right action of A such that the left and right actions of k coincide; in other words, an A-bimodule is an $A \otimes_k A^{\mathrm{op}}$-module. The choice of k is of course not unique, but it will always be the obvious one (in general the base field). The category of A-bimodules will be denoted

$$A\text{-}\mathsf{Mod}\text{-}A.$$

0.6.3. *Coxeter groups.* For all Coxeter systems $(\mathcal{W}, \mathcal{S})$ considered in this book, the set $\mathcal{S}$ is assumed to be finite.

Given a Coxeter system $(\mathcal{W}, \mathcal{S})$, we will denote by $\ell : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ the length function, such that $\ell(w)$ is the minimal possible number of terms occurring when writing w as a product of elements in $\mathcal{S}$. We will denote by $\leq$ the *Bruhat order* on $\mathcal{W}$, i.e. the order generated by

$$w \leq wt \qquad \text{if } t \in \{xsx^{-1} : x \in \mathcal{W},\, s \in \mathcal{S}\} \text{ and } \ell(wt) > \ell(w).$$

Recall that for any subset $I \subset \mathcal{S}$, if we denote by $\mathcal{W}_I$ the subgroup of $\mathcal{W}$ generated by I, then $(\mathcal{W}_I, I)$ is a Coxeter system. A subgroup of the form $\mathcal{W}_I$ (or sometimes the corresponding pair $(\mathcal{W}_I, I)$) will be called a parabolic subgroup. (Note that the term "parabolic subgroup" is sometimes used for something more general in the theory of Coxeter groups.)

We will call *expression* a word in $\mathcal{S}$, or in other words an r-tuple $(s_1, \cdots, s_r)$ of elements in $\mathcal{S}$. (The case $r = 0$ is allowed, corresponding to the empty word.) The *length* $\ell(\underline{w})$ of an expression $\underline{w}$ is its length as a word, i.e. the number of letters appearing in it (counting repetitions). The expression $(s_1, \cdots, s_r)$ is called reduced if $\ell(s_1 \cdots s_r) = r$ (where $s_1 \cdots s_r$ is the product of these elements in $\mathcal{W}$). We will denote by $\mathcal{S}^2_\circ \subset \mathcal{S}^2$ the subset consisting of pairs (s,t) with $s \neq t$ generating a finite subgroup of $\mathcal{W}$. If $(s,t) \in \mathcal{S}^2$, we will denote by $\langle s, t \rangle$ the subgroup generated by s and t, and if $(s,t) \in \mathcal{S}^2_\circ$ we will denote by $m_{s,t}$ the order of the product st.

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0.8. Warning. This book is in preliminary form. Use at your own risks! It will grow and be corrected whenever I find time for that. Any constructive comments are welcome.

0.9. Latest edits. December 11, 2023

0.10. Status of the chapters. Chapter 1: essentially in final form.
Chapter 2: essentially in final form.
Chapter 3: incomplete.
Chpater 4: complete, but needs to be proofread.
Chapter 5: empty.

Chapter 6: complete, but needs to be proofread.
Appendix A: complete, but needs to be proofread.
Appendix B: incomplete.

CHAPTER 1

Modular representation theory of reductive groups

The goal of this chapter is twofold. First, we aim at recalling the main classical results from the representation theory of connected reductive algebraic groups. Most of the proofs will be omitted; they can e.g. be found in [J3]. Second, we will try to explain how to reinterpret, or sometimes restate, some of these results in a way that will be more convenient for the point of view we want to emphasize in the later chapters. This new point of view will often involve the affine Weyl group and its Coxeter group[1] structure.

Historically, the importance of the affine Weyl group for the description of the representation theory of connected reductive algebraic groups was first suggested by Verma [Ve]. A more concrete incarnation of this idea, which greatly influenced the later study of these questions, was given by Lusztig [L1]. In a sense, the main idea of the approach to character formulas presented in these lectures is that this idea should be taken at the level of *categories* rather than combinatorics. This will be made more concrete in Chapter 6.

1. Representations of reductive algebraic groups

1.1. Definitions. We will denote by $\Bbbk$ an algebraically closed field of characteristic p. (For us the most interesting case is when $p > 0$, but for now the case $p = 0$ is also allowed.) Let $\mathbf{G}$ be a connected reductive algebraic group over $\Bbbk$, and let us choose a Borel subgroup $\mathbf{B} \subset \mathbf{G}$ and a maximal torus $\mathbf{T} \subset \mathbf{B}$. We will denote by $\mathbb{X} = X^*(\mathbf{T})$ the lattice of characters of $\mathbf{T}$, i.e. morphisms of algebraic groups from $\mathbf{T}$ to $\mathbb{G}_{\mathrm{m},\Bbbk} = \Bbbk^\times$. Elements of $\mathbb{X}$ will usually be called *weights*. If $\mathbf{U} \subset \mathbf{B}$ is the unipotent radical of $\mathbf{B}$, then multiplication induces an isomorphism of algebraic groups

$$\mathbf{T} \ltimes \mathbf{U} \xrightarrow{\sim} \mathbf{B}.$$

In particular, it follows that any $\lambda \in \mathbb{X}$ extends in a unique way to a morphism of algebraic groups from $\mathbf{B}$ to $\Bbbk^\times$, which will again be denoted λ. We will also denote by $\mathbf{B}^+$ the Borel subgroup opposite to $\mathbf{B}$ with respect to $\mathbf{T}$, and by $\mathbf{U}^+$ its unipotent radical.

We will denote by $\mathfrak{R} \subset \mathbb{X}$ the root system of $(\mathbf{G}, \mathbf{T})$, i.e. the set of nonzero $\mathbf{T}$-weights in $\mathrm{Lie}(\mathbf{G})$. The subset of positive roots consisting of the $\mathbf{T}$-weights in $\mathrm{Lie}(\mathbf{U}^+)$ will be denoted $\mathfrak{R}^+$, and the associated system of simple roots will be denoted $\mathfrak{R}^{\mathrm{s}} \subset \mathfrak{R}^+$. We will also denote by $\mathbb{X}^\vee := X_*(\mathbf{T})$ the cocharacter lattice of $\mathbf{T}$, and by $\mathfrak{R}^\vee \subset \mathbb{X}^\vee$ the coroots of $(\mathbf{G}, \mathbf{T})$. There is a canonical bijection $\mathfrak{R} \xrightarrow{\sim} \mathfrak{R}^\vee$, which we denote as usual by $\alpha \mapsto \alpha^\vee$. Our choice of basis of $\mathfrak{R}$ determines a subset of dominant weights in $\mathbb{X}$, defined by

$$\mathbb{X}^+ = \{\lambda \in \mathbb{X} \mid \forall \alpha \in \mathfrak{R}^+,\ \langle \lambda, \alpha^\vee \rangle \geq 0\},$$

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[1]

and an order $\preceq$ on $\mathbb{X}$ defined by

$$\lambda \preceq \mu \quad \Leftrightarrow \quad \mu - \lambda \in \mathbb{Z}_{\geq 0}\mathfrak{R}^+.$$

We will denote by $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ the Weyl group of $(\mathbf{G}, \mathbf{T})$, and by $S \subset W$ the set of simple reflections, so that $S = \{s_\alpha : \alpha \in \mathfrak{R}^{\mathrm{s}}\}$. It is well known that the pair (W, S) is a Coxeter system. The longest element with respect to that structure will be denoted w_0. Since $\mathbf{T}$ is its own centralizer in $\mathbf{G}$, W identifies with a subgroup of the group of automorphisms of $\mathbf{T}$ (as a $\Bbbk$-algebraic group), or equivalently of $\mathbb{X}$ (as an abelian group).

EXAMPLE 1.1. The main example the reader should keep in mind is $\mathbf{G} = \mathrm{SL}_n(\Bbbk)$. In this case one can choose

$$\mathbf{B} = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} \right\} \subset \mathbf{G}$$

as the subgroup of lower triangular matrices and

$$\mathbf{T} = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subset \mathbf{B}$$

as the subgroup of diagonal matrices.

In this case we have a canonical identification $\mathbb{X} = \mathbb{Z}^n/\mathbb{Z}(1, \dots, 1)$, where the class $[\lambda_1, \dots, \lambda_n]$ of an n-tuple $(\lambda_1, \dots, \lambda_n)$ corresponds to the morphism

$$\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \mapsto \prod_{i=1}^{n} x_i^{\lambda_i}.$$

If we denote (for $i \in \{1, \cdots, n\}$) by $\epsilon_i \in \mathbb{Z}^n/\mathbb{Z}(1, \dots, 1)$ the class of the vector whose only nonzero entry is 1 in the ith position, then we have

$$\begin{aligned}
\mathfrak{R} &= \{\epsilon_i - \epsilon_j : 1 \leq i \neq j \leq n\}, \\
\mathfrak{R}^+ &= \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq n\}, \\
\mathfrak{R}^{\mathrm{s}} &= \{\epsilon_i - \epsilon_{i+1} : i \in \{1, \cdots, n-1\}\}, \\
\mathbb{X}^+ &= \{[\lambda_1, \dots, \lambda_n] \in \mathbb{Z}^n/\mathbb{Z}(1, \dots, 1) \mid \lambda_1 \geq \cdots \geq \lambda_n\}.
\end{aligned}$$

If we set, for any $i \in \{1, \cdots, n-1\}$, $\varpi_i := \epsilon_1 + \cdots + \epsilon_i$, then we have

$$\mathbb{X}^+ = \{a_1\varpi_1 + \cdots + a_{n-1}\varpi_{n-1} : a_1, \cdots, a_{n-1} \in \mathbb{Z}_{\geq 0}\}.$$

(The weights $\varpi_1, \cdots, \varpi_n$ are the *fundamental weights* of $(\mathbf{G}, \mathbf{B}, \mathbf{T})$.) We also have a natural identification $W \xrightarrow{\sim} \mathfrak{S}_n$, where $\mathfrak{S}_n$ acts on $\mathbf{T}$ by permuting the entries, and in this way S identifies with $\{(i, i+1) : i \in \{1, \cdots, n-1\}\}$. Moreover, we have

$$w_0 = \left(\begin{smallmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{smallmatrix}\right).$$

(See Exercise 1.2 for the closely related example of $\mathbf{G} = \mathrm{GL}_n(\Bbbk)$.)

For some aspects of the theory it is important to notice that one can consider "the same group" for various values of the field $\Bbbk$. One way of making sense of this idea is by using the classification of connected reductive algebraic groups over algebraically closed fields. Namely, such a group is uniquely determined (up to isomorphism) by its root datum, i.e. the quadruple $(\mathbb{X}, \mathbb{X}^\vee, \mathfrak{R}, \mathfrak{R}^\vee)$ (implicitly, together with the pairing between $\mathbb{X}$ and $\mathbb{X}^\vee$ and the bijection between roots and coroots). Root data do not involve $\Bbbk$ in any way. Fixing a root datum, we therefore obtain an attached connected reductive algebraic group over any algebraically closed field, which we can consider as "the same group" over different fields. By construction, the character lattice $X^*(\mathbf{T})$ is the same for all these groups. Soon we will construct some families of $\mathbf{G}$-modules parametrized by (dominant) weights; using this point of view we will be able to consider "the same" module over different fields $\Bbbk$: by this we will mean the modules attached to the same weight for each $\Bbbk$. Most of the questions we will consider will not really depend on $\Bbbk$ itself, but only on its characteristic p.

REMARK 1.2. A more subtle way of expressing the idea of "the same group over different fields" is by using the notion of reductive group schemes over rings. Namely, any connected reductive algebraic group $\mathbf{G}$ over $\Bbbk$ can be obtained by base change from a split reductive group scheme $\mathbf{G}_{\mathbb{Z}}$ over $\mathbb{Z}$. After fixing such $\mathbf{G}_{\mathbb{Z}}$ (which is unique up to isomorphism), one obtains for any field $\Bbbk'$ a "version" of $\mathbf{G}$ over $\Bbbk'$, namely $\mathrm{Spec}(\Bbbk') \times_{\mathrm{Spec}(\mathbb{Z})} \mathbf{G}_{\mathbb{Z}}$. Here the subgroups $\mathbf{B}$ and $\mathbf{T}$ can also be obtained by base change from subgroups of $\mathbf{G}_{\mathbb{Z}}$, which give rise to subgroups of $\mathrm{Spec}(\Bbbk') \times_{\mathrm{Spec}(\mathbb{Z})} \mathbf{G}_{\mathbb{Z}}$.

REMARK 1.3. For simplicity we have chosen a Borel subgroup and a maximal torus in $\mathbf{G}$. In order to avoid these noncanonical choices, one can instead work with the "universal maximal torus." Namely, given any two Borel subgroups $\mathbf{B}_1, \mathbf{B}_2 \subset \mathbf{G}$, whose unipotent radicals will be denoted $\mathbf{U}_1$ and $\mathbf{U}_2$, there exists a canonical isomorphism of $\Bbbk$-tori

$$\mathbf{B}_1/\mathbf{U}_1 \cong \mathbf{B}_2/\mathbf{U}_2.$$

In fact all Borel subgroups are conjugate; hence there exists $g \in \mathbf{G}$ such that $\mathbf{B}_2 = g\mathbf{B}_1 g^{-1}$. Any two choices for this element g differ by right multiplication by an element in $\mathbf{B}_1$; since the $\mathbf{B}_1$-action on $\mathbf{B}_1/\mathbf{U}_1$ by conjugation is trivial, it follows that the isomorphism $\mathbf{B}_1/\mathbf{U}_1 \xrightarrow{\sim} \mathbf{B}_2/\mathbf{U}_2$ induced by conjugation by such a g does not depend on the choice of element, which provides the desired canonical isomorphism. The universal maximal torus is then defined as the torus $\mathbf{A} = \mathbf{B}/\mathbf{U}$, for any choice of Borel subgroup $\mathbf{B} \subset \mathbf{G}$, whose unipotent radical is denoted $\mathbf{U}$. (As explained above, this torus is canonically independent of the choice of $\mathbf{B}$.) Note that $\mathbf{A}$ is *not* a subgroup of $\mathbf{G}$. Given an arbitrary maximal torus $\mathbf{T} \subset \mathbf{G}$, for any choice of Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$ we have a canonical identification $\mathbf{T} \xrightarrow{\sim} \mathbf{A}$, provided by the composition $\mathbf{T} \hookrightarrow \mathbf{B} \twoheadrightarrow \mathbf{A}$. (This identification *does* depend on the choice of $\mathbf{B}$.)

To continue in this vein, we can then define $\mathbb{X}$ as the lattice of characters of $\mathbf{A}$; in this way, for any Borel subgroup $\mathbf{B} \subset \mathbf{G}$ we have a canonical identification of $\mathbb{X}$ with the lattice of algebraic group morphisms $\mathbf{B} \to \Bbbk^\times$, sending a character $\mathbf{A} \to \Bbbk^\times$ to its composition with the projection $\mathbf{B} \to \mathbf{A}$. The root system $\mathfrak{R}$, and its positive system $\mathfrak{R}^+$, can also be defined universally, as the image in $\mathbb{X}$ of the $\mathbf{T}$-weights in $\mathrm{Lie}(\mathbf{G})$ and in $\mathrm{Lie}(\mathbf{G})/\mathrm{Lie}(\mathbf{B})$ respectively, for any choice of a Borel subgroup $\mathbf{B} \subset \mathbf{G}$ and a maximal torus $\mathbf{T} \subset \mathbf{B}$, where we identify $\mathbb{X}$ with the lattice of characters of $\mathbf{T}$ using the canonical isomorphism $\mathbf{T} \xrightarrow{\sim} \mathbf{A}$ considered above.

1.2. Categories of representations and induction functor. For any $\Bbbk$-algebraic group[2] $\mathbf{H}$, we will denote by $\mathsf{Rep}(\mathbf{H})$ the category of finite dimensional

[2]By a $\Bbbk$-algebraic group we mean a smooth affine group scheme of finite type over $\Bbbk$. This terminology is that of [**Bo, H2, Sp2**].

algebraic $\mathbf{H}$-modules, and by $\mathsf{Rep}^\infty(\mathbf{H})$ the category of all (not necessarily finite-dimensional) algebraic $\mathbf{H}$-modules. In other words, the algebra $\mathscr{O}(\mathbf{H})$ has a canonical structure of Hopf algebra over $\Bbbk$ (with comultiplication defined by the multiplication morphism $\mathbf{H} \times \mathbf{H} \to \mathbf{H}$), and an algebraic $\mathbf{H}$-module is nothing but an $\mathscr{O}(\mathbf{H})$-comodule.

Recall that for any $V \in \mathsf{Rep}(\mathbf{G})$, the dual vector space V^* has a canonical structure of object in $\mathsf{Rep}(\mathbf{G})$ with action defined by

$$(h \cdot f)(v) = f(h^{-1} \cdot v)$$

for $h \in \mathbf{H}$, $f \in V^*$ and $v \in V$. Recall also that for any (closed) subgroup $\mathbf{K} \subset \mathbf{H}$, we have an "induction functor"

$$\mathrm{Ind}_{\mathbf{K}}^{\mathbf{H}} : \mathsf{Rep}^\infty(\mathbf{K}) \to \mathsf{Rep}^\infty(\mathbf{H}),$$

see [J3, Chap. I.3]. This functor sends a representation (M, ϱ) (where $\varrho : \mathbf{K} \to \mathrm{GL}(M)$ is the morphism defining the $\mathbf{K}$-action) to the space of algebraic functions $f : \mathbf{H} \to M$ (i.e. elements of $\mathscr{O}(\mathbf{H}) \otimes M$) which satisfy

$$f(hk) = \varrho(k^{-1})(f(h))$$

for any $h \in \mathbf{H}$ and $k \in \mathbf{K}$, the action of $\mathbf{H}$ being induced by left multiplication on itself. This functor is left exact, and is right adjoint to the restriction functor

$$\mathsf{For}_{\mathbf{K}}^{\mathbf{H}} : \mathsf{Rep}^\infty(\mathbf{H}) \to \mathsf{Rep}^\infty(\mathbf{K}).$$

(This property is usually called "Frobenius reciprocity" see [J3, Proposition I.3.4].) In general, it is not true that $\mathrm{Ind}_{\mathbf{K}}^{\mathbf{H}}$ restricts to a functor from $\mathsf{Rep}(\mathbf{K})$ to $\mathsf{Rep}(\mathbf{H})$.

Remark 1.4. The definition of the functor $\mathrm{Ind}_{\mathbf{K}}^{\mathbf{H}}$ can be "localized" in the following way. (For details on all of this, see [J3, §§I.5-8–9].) Consider the quotient $\mathbf{H}/\mathbf{K}$ (a separated $\Bbbk$-scheme of finite type, whose construction is explained e.g. [Mil, Chap. 7]) and the projection morphism $p : \mathbf{H} \to \mathbf{H}/\mathbf{K}$. Given (M, ϱ) as above, for any open subvariety $V \subset \mathbf{H}/\mathbf{K}$ one can consider the vector space consisting of the functions $f \in \mathscr{O}(p^{-1}(V)) \otimes M$, seen as maps $V \to M$, which satisfy

$$f(hk) = \varrho(k^{-1})(f(h))$$

for any $h \in p^{-1}(V)$ and $k \in \mathbf{K}$. (Note that in this setting hk belongs to $p^{-1}(V)$, so that this equality makes sense.) This space admits a natural action of $\mathscr{O}(V)$ (by composition with p and multiplication), and can easily be seen to define a quasi-coherent $\mathscr{O}_{\mathbf{H}/\mathbf{K}}$-module denoted $\mathscr{L}_{\mathbf{H}/\mathbf{K}}(M)$. By construction we then have

$$\mathrm{Ind}_{\mathbf{K}}^{\mathbf{H}}(M) = \Gamma(\mathbf{H}/\mathbf{K}, \mathscr{L}_{\mathbf{H}/\mathbf{K}}(M)).$$

In fact, $\mathscr{L}_{\mathbf{H}/\mathbf{K}}(M)$ has a canonical structure of $\mathbf{H}$-equivariant quasi-coherent sheaf on $\mathbf{H}/\mathbf{K}$ (in the sense that its pullbacks under the projection and action morphisms $\mathbf{H} \times \mathbf{H}/\mathbf{K} \to \mathbf{H}/\mathbf{K}$ are canonically isomorphic, with this isomorphism satisfying a certain "cocycle" condition), and this construction induces an equivalence of categories between $\mathsf{Rep}(\mathbf{K})$ and the category of $\mathbf{H}$-equivariant quasi-coherent sheaves on $\mathbf{H}/\mathbf{K}$ (see e.g. [Bri, §2]).

1.3. Induced and Weyl G-modules. The main player of this book will be the category $\mathsf{Rep}(\mathbf{G})$. It is a general scheme in Representation Theory that in order to construct interesting representations of a group (or module over an algebra) one should start with some "simple enough" representations of a "large" subgroup (or module over a "large" subalgebra) and then induce to the whole group (or algebra). In the setting of representations of reductive groups, such a "large" subgroup can be chosen as the Borel subgroup $\mathbf{B}$, and the "simple enough" representations can be chosen to the 1-dimensional representations $\Bbbk_{\mathbf{B}}(\lambda)$ associated with the weights $\lambda \in \mathbb{X}$, considered as morphisms from $\mathbf{B}$ to $\Bbbk$.[3]

Definition 1.5 (Induced or co-Weyl modules). For $\lambda \in \mathbb{X}$, the *induced module* $\mathsf{N}(\lambda)$ associated with λ is defined as

$$\mathsf{N}(\lambda) := \mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(\lambda)) = \{f \in \mathscr{O}(\mathbf{G}) \mid \forall b \in \mathbf{B}, \forall g \in \mathbf{G}, \ f(gb) = \lambda(b)^{-1} f(g)\},$$

with the $\mathbf{G}$-action given by $(g \cdot f)(h) = f(g^{-1}h)$ for $g, h \in \mathbf{G}$ and $f \in \mathsf{N}(\lambda)$.

Example 1.6. In case $\lambda = 0$, one finds that

$$\mathsf{N}(0) = \mathscr{O}(\mathbf{G}/\mathbf{B}) = \Bbbk$$

since $\mathbf{G}/\mathbf{B}$ is an irreducible projective variety (see [H2, §21.3]), so that any morphism $\mathbf{G}/\mathbf{B} \to \Bbbk$ must be constant (see e.g. [H2, §6.1]). More generally, if $\lambda \in \mathbb{X}$ satisfies $\langle \lambda, \alpha^\vee \rangle = 0$ for any $\alpha \in \mathfrak{R}$, then λ extends uniquely to a character $\mathbf{G} \to \Bbbk^\times$, hence defines a 1-dimensional $\mathbf{G}$-module $\Bbbk_{\mathbf{G}}(\lambda)$. (In fact, it is a classical fact that restriction to $\mathbf{T}$ induces an isomorphism between the lattice of algebraic group morphisms $\mathbf{G} \to \Bbbk^\times$ and the subset of $\mathbb{X}$ consisting of weights orthogonal to all coroots.) By the tensor identity (see [J3, Proposition I.3.6]) we deduce that

$$\mathsf{N}(\lambda) = \mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(\lambda)) \cong \mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(0)) \otimes \Bbbk_{\mathbf{G}}(\lambda) = \Bbbk_{\mathbf{G}}(\lambda).$$

Remark 1.7. In the case considered in Definition 1.5, using the fact that the projection morphism $\mathbf{G} \to \mathbf{G}/\mathbf{B}$ locally has sections (in fact it has a natural section on the "big cell" $\mathbf{U}^+\mathbf{B}/\mathbf{B} \cong \mathbf{U}^+$, and then one obtains further local sections by translation) one sees that the sheaf $\mathscr{L}_\lambda := \mathscr{L}_{\mathbf{G}/\mathbf{B}}(\Bbbk_{\mathbf{B}}(\lambda))$ considered in Remark 1.4 is a line bundle (i.e. a locally free sheaf of rank 1) on $\mathbf{G}/\mathbf{B}$.

In the special case $\mathbf{G} = \mathrm{SL}_2(\Bbbk)$, the flag variety $\mathbf{G}/\mathbf{B}$ identifies with the projective space $\mathbb{P}^1$. Under this identification, for any $a \in \mathbb{Z}$ the line bundle $\mathscr{L}_{a\varpi_1}$ identifies with the line bundle $\mathscr{O}_{\mathbb{P}^1}(a)$.

It turns out that the module $\mathsf{N}(\lambda)$ is finite-dimensional and algebraic, for any $\lambda \in \mathbb{X}$. (One possible way of proving this fact is to use Remark 1.7 and classical facts on sections of coherent sheaves on proper schemes, see [J3, Proposition I.5.12(c)].) In particular, the dual $\mathbf{G}$-modules are also finite-dimensional and algebraic, and will be called Weyl modules.

Definition 1.8 (Weyl modules). For $\lambda \in \mathbb{X}$, the *Weyl module* $\mathsf{M}(\lambda)$ is defined as

$$\mathsf{M}(\lambda) = (\mathsf{N}(-w_0\lambda))^* .$$

[3] In fact, every irreducible representation of $\mathbf{B}$ is 1-dimensional, hence of the form $\Bbbk_{\mathbf{B}}(\lambda)$ for some $\lambda \in \mathbb{X}$.

REMARK 1.9. The definition of induced and Weyl modules we have given above seems to depend on the choice of Borel subgroup $\mathbf{B}$ and maximal torus $\mathbf{T}$. In fact these modules are uniquely defined up to isomorphism, if one uses the "universal" constructions considered in Remark 1.3. Namely, with $\mathbb{X}$ defined as in this remark, if $\lambda \in \mathbb{X}$ and if $\mathbf{B}' \subset \mathbf{G}$ is any Borel subgroup, the weight λ defines a morphism $\lambda_{\mathbf{B}'} : \mathbf{B}' \to \Bbbk^\times$, and one can consider the module $\mathrm{Ind}_{\mathbf{B}'}^{\mathbf{G}}(\Bbbk_{\mathbf{B}'}(\lambda_{\mathbf{B}'}))$. Now if $\mathbf{B}'' \subset \mathbf{G}$ is any other Borel subgroup, then we also have a character $\lambda_{\mathbf{B}''} : \mathbf{B}'' \to \Bbbk^\times$, and if $g \in \mathbf{G}$ is any element such that $g\mathbf{B}'g^{-1} = \mathbf{B}''$, we have $\lambda_{\mathbf{B}''}(gbg^{-1}) = \lambda_{\mathbf{B}'}(b)$ for any $b \in \mathbf{B}'$. We therefore obtain an isomorphism of $\mathbf{G}$-modules

$$\mathrm{Ind}_{\mathbf{B}'}^{\mathbf{G}}(\Bbbk_{\mathbf{B}'}(\lambda_{\mathbf{B}'})) \xrightarrow{\sim} \mathrm{Ind}_{\mathbf{B}''}^{\mathbf{G}}(\Bbbk_{\mathbf{B}''}(\lambda_{\mathbf{B}''}))$$

which sends a function $f : \mathbf{G} \to \Bbbk$ in the left-hand side to the function $h \mapsto f(hg)$. (This isomorphism does depend on the choice of g.)

1.4. Examples in classical groups.

1.4.1. *Special linear groups.* Let us consider the case $\mathbf{G} = \mathrm{SL}_n(\Bbbk)$, with the conventions and notations of Example 1.1, and consider the natural action of $\mathbf{G}$ on $V = \Bbbk^n$. It is not difficult (see Exercise 1.6) to show that for any $i \in \{1, \cdots, n-1\}$ we have

$$\bigwedge^i V \cong \mathsf{N}(\omega_i).$$

It is known also that for any $r \geq 0$ we have

$$\mathsf{N}(r\omega_{n-1}) \cong \mathrm{Sym}^r(V^*) \quad \text{and} \quad \mathsf{N}(r\omega_1) \cong \mathrm{Sym}^r(V)$$

see [**J3**, §II.2.16]. (For $r = 1$, this is equivalent to the description above since $\bigwedge^n V \cong \Bbbk$, so that $\bigwedge^{n-1} V \cong V^*$.)

In particular, when $n = 2$, we deduce that $\mathsf{N}(r\omega_1)$ identifies with the space $\Bbbk[X, Y]_r$ of homogeneous polynomials in two variables X and Y of degree r, with the natural action of $\mathrm{SL}_2(\Bbbk)$ obtained by viewing a polynomial in X, Y as a function on $\mathbb{A}^2$.

1.4.2. *Symplectic groups.* Now, let us assume that $\mathbf{G} = \mathrm{Sp}_{2n}(\Bbbk)$, with the conventions and notations of Exercise 1.3. We consider the natural action of $\mathbf{G}$ on $V = \Bbbk^{2n}$, which we equip with the standard basis $(e_1, \cdots, e_{2n})$. In this basis, the associated alternating form is given by

$$\omega(e_i, e_j) = \begin{cases} 1 & \text{if } i \in \{1, \cdots, n\} \text{ and } j = i + n; \\ -1 & \text{if } j \in \{1, \cdots, n\} \text{ and } i = j + n; \\ 0 & \text{otherwise.} \end{cases}$$

For $m \in \{1, \cdots, n\}$, we denote by M_m the $\mathbf{G}$-submodule of $\bigwedge^m V$ generated by the vector $e_1 \wedge \ldots \wedge e_m$. (By Witt's theorem, M_m is spanned as a vector space by the vectors $v_1 \wedge \ldots \wedge v_m$ such that $\mathrm{span}(v_1, \ldots, v_m)$ is an isotropic subspace of V.) Then it is proved in [**PS**] that we have

$$M_m \cong \mathsf{M}(\omega_m).$$

(In [**PS**] it is assumed that $p \neq 2$, but this restriction is not necessary for this particular claim; see [**AdRy**, p. 20].) In particular this shows that there exists an embedding $\mathsf{M}(\omega_m) \hookrightarrow \bigwedge^m V$. Dualizing, and using the canonical isomorphisms

$$\left(\bigwedge^m V\right)^* \xrightarrow{\sim} \bigwedge^m V^* \xrightarrow{\sim} \bigwedge^m V \tag{1.1}$$

(here the first isomorphism is induced by the pairing $(\bigwedge^m V)\times(\bigwedge^m V^*) \to \Bbbk$ defined by $(\varphi_1 \wedge \cdots \wedge \varphi_m, v_1 \wedge \cdots \wedge v_m) \mapsto \det(\varphi_i(v_j))_{i,j}$, and the second one is induced by the isomorphism $V \xrightarrow{\sim} V^*$ defined by $v \mapsto \omega(v,-)$ where ω is as above) and the fact that w_0 acts on $\mathbb{X}$ as $-\mathrm{id}$ we deduce a surjection

$$\bigwedge^m V \twoheadrightarrow \mathsf{N}(\omega_m). \tag{1.2}$$

If $m = 1$, this surjection has to be an isomorphism since $\dim(\mathsf{N}(\omega_1)) = 2n$ (which can e.g. be derived from Theorem 1.20 below and [**FH**, Exercise 24.21]). On the other hand, if $m \geq 2$ there exists a $\mathbf{G}$-equivariant embedding

$$\bigwedge^{m-2} V \hookrightarrow \bigwedge^m V$$

defined by $x \mapsto (\sum_{i=1}^n e_i \wedge e_{n+i}) \wedge x$. (Here, the element $\sum_{i=1}^n e_i \wedge e_{n+i}$ is the image under the isomorphism (1.1)—in case $m = 2$—of the aternating form ω.) By Frobenius reciprocity, the composition of this embedding with (1.2) vanishes. Since

$$\dim(\mathsf{N}(\omega_m)) = \binom{2n}{m} - \binom{2n}{m-2}$$

(again by Theorem 1.20 below and [**FH**, Exercise 24.21]) we deduce an isomorphism

$$\mathsf{N}(\omega_m) \cong \bigwedge^m V \Big/ \bigwedge^{m-2} V.$$

1.4.3. *Even orthogonal groups.* Now we turn to the case $\mathbf{G} = \mathrm{SO}_{2n}(\Bbbk)$ (with $p \neq 2$), with the conventions and notations of Exercise 1.4. We consider the natural action of $\mathbf{G}$ on $V = \Bbbk^{2n}$, which we equip with the standard basis $(e_1, \cdots, e_{2n})$. In this basis, the associated symmetric bilinear form is given by

$$\omega(e_i, e_j) = \begin{cases} 1 & \text{if } i \in \{1, \cdots, n\} \text{ and } j = i+n \text{ or } j \in \{1, \cdots, n\} \text{ and } i = j+n; \\ 0 & \text{otherwise.} \end{cases}$$

If $m \leq n-2$, then by Frobenius reciprocity there exists a $\mathbf{G}$-equivariant morphism

$$\bigwedge^m V \to \mathsf{N}(\omega_m),$$

or dually a $\mathbf{G}$-equivariant morphism

$$\mathsf{M}(\omega_m) \to \bigwedge^m V.$$

(Here again we use the identification of $\bigwedge^m V$ with its dual, and the fact that $w_0(\omega_m) = -\omega_m$.) Now, as noticed e.g. in [**AdRy**, p. 20], the $\mathbf{G}$-module $\bigwedge^m V$ is generated by its weight-subspace of weight ω_m; this morphism must therefore be surjective. Since $\dim(\mathsf{N}(\omega_m)) = \binom{2n}{m}$ (e.g. by Theorem 1.20 below and [**FH**, Exercise 24.43]), this surjection must be an isomorphism, and we deduce that

$$\mathsf{N}(\omega_m) \cong \bigwedge^m V.$$

Similar considerations show that we also have

$$\mathsf{N}(\varepsilon_1 + \cdots + \varepsilon_{n-1}) \cong \bigwedge^{n-1} V.$$

In fact, all of these modules are simple.

1.4.4. *Odd orthogonal groups.* Finally we consider the case $\mathbf{G} = \mathrm{SO}_{2n+1}(\Bbbk)$ (with $p \neq 2$), with the conventions and notations of Exercise 1.5. We consider the natural action of $\mathbf{G}$ on $V = \Bbbk^{2n+1}$, which we equip with the standard basis $(e_1, \cdots, e_{2n+1})$. In this basis, the associated symmetric bilinear form is given by

$$\omega(e_i, e_j) = \begin{cases} 1 & \text{if } i \in \{1, \cdots, n\} \text{ and } j = i+n \text{ or } j \in \{1, \cdots, n\} \text{ and } i = j+n; \\ 1 & \text{if } i = j = 2n+1; \\ 0 & \text{otherwise.} \end{cases}$$

If $m \leq n-1$, then the same considerations as in §1.4.3 (using [FH, Exercise 24.31]) show that we have

$$\mathsf{N}(\omega_m) \cong \bigwedge^m V,$$

and that similarly we have

$$\mathsf{N}(\varepsilon_1 + \cdots + \varepsilon_n) \cong \bigwedge^n V.$$

Once again, all of these modules are simple.

1.5. Weights and characters of G-modules. The representation theory of tori is particularly simple, in that each algebraic representation is a direct sum of 1-dimensional representations. In more concrete terms, in our situation this means that for any algebraic $\mathbf{T}$-module M we have

$$M = \bigoplus_{\lambda \in \mathbb{X}} M_\lambda,$$

where for $\lambda \in \mathbb{X}$ we set

$$M_\lambda := \{m \in M \mid \forall t \in \mathbf{T},\, t \cdot m = \lambda(t)m\}.$$

The set of *weights* of M is the subset

$$\mathrm{wt}(M) = \{\lambda \in \mathbb{X} \mid M_\lambda \neq 0\} \subset \mathbb{X}.$$

In case $\dim(M_\lambda) < \infty$ for any $\lambda \in \mathbb{X}$, a more interesting invariant is given by the character $\mathrm{ch}(M)$ of M, defined by

$$\mathrm{ch}(M) = \sum_{\lambda \in \mathbb{X}} \dim(M_\lambda) \cdot e^\lambda \quad \in \mathbb{Z}[\mathbb{X}].$$

We will mainly consider weights and characters in case $M = \mathsf{For}_{\mathbf{T}}^{\mathbf{G}}(M')$ for some $M' \in \mathsf{Rep}^\infty(\mathbf{G})$. (In this case we will write $\mathrm{wt}(M')$ and $\mathrm{ch}(M')$ for $\mathrm{wt}(M)$ and $\mathrm{ch}(M)$ respectively.)

Lemma 1.10. *In case $M \in \mathsf{Rep}^\infty(\mathbf{G})$, the subset $\mathrm{wt}(M) \subset \mathbb{X}$ is invariant under the action of W. In fact, if moreover $\dim(M_\lambda) < \infty$ for any $\lambda \in \mathbb{X}$, the element $\mathrm{ch}(M)$ is invariant under the action of W.*

Proof. Both claims follow from the observation that if $w \in W$ and $n \in N_{\mathbf{G}}(\mathbf{T})$ is a lift of w, then for any $v \in M$ and $\lambda \in \mathbb{X}$ we have $v \in M_\lambda$ iff $n \cdot v \in M_{w(\lambda)}$. □

The weights of the modules $\mathsf{N}(\lambda)$ considered in §1.3 have the following property.

Lemma 1.11. *For any $\lambda \in \mathbb{X}$ we have*

$$\mu \in \mathrm{wt}(\mathsf{N}(\lambda)) \quad \Rightarrow \quad \mu \preceq \lambda.$$

Moreover, if $\mathsf{N}(\lambda) \neq 0$ we have

$$\dim(\mathsf{N}(\lambda)_\lambda) = 1.$$

Proof. It is a standard consequence of the Bruhat decomposition that multiplication induces an open (dense) embedding

$$\mathbf{U}^+ \times \mathbf{B} \hookrightarrow \mathbf{G}.$$

We deduce an embedding of $\mathbf{U}^+$-modules

$$\mathsf{N}(\lambda) \hookrightarrow \mathscr{O}(\mathbf{U}^+),$$

where the right-hand side is the algebra of (algebraic) functions on $\mathbf{U}^+$, with the action of $\mathbf{U}^+$ induced by left multiplication on itself. In fact this embedding can be also seen as an embedding of $\mathbf{T}$-modules

$$\mathsf{N}(\lambda) \hookrightarrow \mathscr{O}(\mathbf{U}^+) \otimes \Bbbk_{\mathbf{T}}(\lambda),$$

where the action on $\mathscr{O}(\mathbf{U}^+)$ is induced by the action on $\mathbf{U}^+$ by conjugation. Now we have

$$\mathrm{wt}(\mathscr{O}(\mathbf{U}^+)) = \mathbb{Z}_{\geq 0}(-\mathfrak{R}^+),$$

which implies that $\mathrm{wt}(\mathsf{N}(\lambda)) \subset \{\mu \in \mathbb{X} \mid \mu \preceq \lambda\}$.

Now, let us assume that $\mathsf{N}(\lambda) \neq 0$. Since $\mathscr{O}(\mathbf{U}^+)_0$ has dimension 1, the considerations above imply that

$$\dim(\mathsf{N}(\lambda)_\lambda) \leq 1.$$

On the other hand, by Frobenius reciprocity we have

$$\mathrm{Hom}_{\mathbf{G}}(\mathsf{N}(\lambda), \mathsf{N}(\lambda)) \cong \mathrm{Hom}_{\mathbf{B}}(\mathsf{N}(\lambda), \Bbbk_{\mathbf{B}}(\lambda)).$$

Our assumption implies that this space is nonzero; hence there exists a nonzero morphism of $\mathbf{B}$-modules (in particular, of $\mathbf{T}$-modules) from $\mathsf{N}(\lambda)$ to $\Bbbk_{\mathbf{B}}(\lambda)$, which implies that $\mathsf{N}(\lambda)_\lambda \neq 0$. □

Remark 1.12. As noted in the proof of Lemma 1.11 we have

$$\mathrm{Hom}_{\mathbf{G}}(\mathsf{N}(\lambda), \mathsf{N}(\lambda)) \cong \mathrm{Hom}_{\mathbf{B}}(\mathsf{N}(\lambda), \Bbbk_{\mathbf{B}}(\lambda)) \subset \mathrm{Hom}_{\mathbf{T}}(\mathsf{N}(\lambda), \Bbbk_{\mathbf{T}}(\lambda)) \cong (\mathsf{N}(\lambda)_\lambda)^*.$$

Since the right-hand side is 1-dimensional, we deduce that $\mathrm{Hom}_{\mathbf{G}}(\mathsf{N}(\lambda), \mathsf{N}(\lambda)) = \Bbbk \cdot \mathrm{id}$.

Note that Lemma 1.10 and Lemma 1.11 also imply that

$$\mu \in \mathrm{wt}(\mathsf{N}(\lambda)) \quad \Rightarrow \quad \mu \succeq w_0\lambda. \tag{1.3}$$

These two simple observations already have the following interesting consequence.

Corollary 1.13. *For $\lambda \in \mathbb{X}$, if $\mathsf{N}(\lambda) \neq 0$ then $\lambda \in \mathbb{X}^+$.*

Proof. Assume that $\lambda \in \mathbb{X} \smallsetminus \mathbb{X}^+$. Then there exists $\alpha \in \mathfrak{R}^{\mathrm{s}}$ such that $\langle \lambda, \alpha^\vee \rangle < 0$, i.e. such that $s_\alpha(\lambda) \succeq \lambda$. By Lemma 1.11, if $\mathsf{N}(\lambda)$ were nonzero we would have $\lambda \in \mathrm{wt}(\mathsf{N}(\lambda))$, hence (by Lemma 1.10) $s_\alpha(\lambda) \in \mathrm{wt}(\mathsf{N}(\lambda))$. This would contradict the fact that $\mathrm{wt}(\mathsf{N}(\lambda)) \subset \{\mu \in \mathbb{X} \mid \mu \preceq \lambda\}$ (see Lemma 1.11). □

It turns out that the converse of the implication of Corollary 1.13 is also true, see [**J3**, Proposition II.2.6]. The proof is more subtle, and will not be reviewed here. Of course, this also implies that $\mathsf{M}(\lambda) \neq 0$ iff $\lambda \in \mathbb{X}^+$.

1.6. Classification of simple modules. The next task we consider is the classification of the simple objects of the category $\mathsf{Rep}(\mathbf{G})$. (These will also be the simple objects in $\mathsf{Rep}^\infty(\mathbf{G})$.) The answer is given in the following statement, whose first proof is due to Chevalley.

THEOREM 1.14. *For any $\lambda \in \mathbb{X}^+$, the $\mathbf{G}$-module $\mathsf{N}(\lambda)$ admits a unique simple submodule, which we will denote $\mathsf{L}(\lambda)$. Moreover, the assignment $\lambda \mapsto \mathsf{L}(\lambda)$ induces a bijection between $\mathbb{X}^+$ and the set of isomorphism classes of simple algebraic $\mathbf{G}$-modules.*

This theorem says in particular that if M is a simple $\mathbf{G}$-module, then there exists a unique $\lambda \in \mathbb{X}^+$ such that $M \cong \mathsf{L}(\lambda)$. This dominant weight is called the *highest weight* of M. (See below for a justification of this terminology.)

The proof of Theorem 1.14 turns out to be quite simple. Namely, we start with the following observation.

LEMMA 1.15. *For any $\lambda \in \mathbb{X}^+$, we have $(\mathsf{N}(\lambda))^{\mathbf{U}^+} = \mathsf{N}(\lambda)_\lambda$.*

PROOF. Recall the embedding of $\mathbf{U}^+$-modules $\mathsf{N}(\lambda) \subset \mathscr{O}(\mathbf{U}^+)$ considered in the proof of Lemma 1.11. Since $\mathscr{O}(\mathbf{U}^+)^{\mathbf{U}^+} = \Bbbk$, this embedding shows that $(\mathsf{N}(\lambda))^{\mathbf{U}^+} \subset \mathsf{N}(\lambda)_\lambda$. On the other hand $\mathsf{N}(\lambda)_\lambda$ is 1-dimensional (see Lemma 1.11), and by the group version of Engel's theorem (see [H2, Theorem 17.5]) we have $(\mathsf{N}(\lambda))^{\mathbf{U}^+} \neq 0$ since $\mathsf{N}(\lambda) \neq 0$ and $\mathbf{U}^+$ is unipotent. The equality follows. □

PROOF OF THEOREM 1.14. Lemma 1.15 is already enough to show that the socle of $\mathsf{N}(\lambda)$ is simple. In fact, if $V \subset \mathsf{N}(\lambda)$ is any submodule then as in the proof of the lemma we must have $V^{\mathbf{U}^+} \neq 0$, so that $V \supset \mathsf{N}(\lambda)_\lambda$. Hence $\mathsf{N}(\lambda)$ cannot have two distinct simple submodules.

If we denote by $\mathsf{L}(\lambda)$ this simple socle, then $\mathbf{T}$ acts on $\mathsf{L}(\lambda)^{\mathbf{U}^+}$ with weight λ, which implies that $\mathsf{L}(\lambda)$ is not isomorphic to $\mathsf{L}(\mu)$ if $\lambda \neq \mu$. Finally, let V be a simple $\mathbf{G}$-module. Then the (nonzero) subspace $(V^*)^{\mathbf{U}} \subset V^*$ is a direct sum of modules of the form $\Bbbk_{\mathbf{B}}(\lambda)$ with $\lambda \in \mathbb{X}$. In particular there exists $\lambda \in \mathbb{X}$ and a nonzero morphism $\Bbbk_{\mathbf{B}}(-\lambda) \to V^*$, hence a nonzero morphism $V \to \Bbbk_{\mathbf{B}}(\lambda)$. By Frobenius reciprocity there exists a nonzero morphism $V \to \mathsf{N}(\lambda)$, which must then be injective by simplicity of V, and therefore identify V with $\mathsf{L}(\lambda)$. (Here we necessarily have $\lambda \in \mathbb{X}^+$ since $\mathsf{N}(\lambda) \neq 0$.) □

REMARK 1.16. By Schur's lemma we have $\mathrm{Hom}_{\mathbf{G}}(\mathsf{L}(\lambda), \mathsf{L}(\lambda)) = \Bbbk \cdot \mathrm{id}$.

Lemma 1.11 implies that $\mathrm{wt}(\mathsf{L}(\lambda)) \subset \{\mu \in \mathbb{X} \mid \mu \preceq \lambda\}$, and as seen in the proof of Theorem 1.14 we have $\mathsf{L}(\lambda)_\lambda \neq 0$. Hence λ is the unique maximal element in $\mathrm{wt}(\mathsf{L}(\lambda))$ with respect to $\preceq$, which justifies the terminology of "highest weight." Using Lemma 1.10 we deduce that $w_0(\lambda)$ is the unique minimal element in $\mathrm{wt}(\mathsf{L}(\lambda))$, and then that for any $\lambda \in \mathbb{X}^+$ we have

$$\mathsf{L}(\lambda)^* \cong \mathsf{L}(-w_0(\lambda)). \tag{1.4}$$

This shows that $\mathsf{L}(\lambda)$ is also isomorphic to the unique simple quotient of $\mathsf{M}(\lambda)$.

EXAMPLE 1.17. Consider the case $\mathbf{G} = \mathrm{SL}_2(\Bbbk)$, with $p > 0$. As explained in §1.4.1, $\mathsf{N}([p, 0])$ identifies with the space $\Bbbk[X, Y]_p$ of homogeneous polynomials of degree p in two variables X and Y, with the obvious action of $\mathrm{SL}_2(\Bbbk)$. Using the fact that the map $x \mapsto x^p$ is additive in $\Bbbk[X, Y]$, it is not difficult to see that the subspace

$$\Bbbk \cdot X^p \oplus \Bbbk \cdot Y^p \subset \Bbbk[X, Y]_p$$

is stable under the action of $\mathrm{SL}_2(\Bbbk)$. In fact, under the identification $\mathsf{N}([p, 0]) = \Bbbk[X, Y]_p$ this subspace is exactly $\mathsf{L}([p, 0])$.

Theorem 1.14 provides a classification of simple algebraic $\mathbf{G}$-modules. However, the construction of these simple modules is far from explicit; even though the induced modules $\mathsf{N}(\lambda)$ are relatively well understood (see, in particular, §1.9 below), this theorem does not explain how "big" the submodule $\mathsf{L}(\lambda)$ is. A very important problem in this area (which is one of the main topics of this book, and is still not solved in any satisfactory way in general) is therefore to understand what these simple modules "look like." To make this problem more precise, one can e.g. ask for the description of the characters $\mathrm{ch}(\mathsf{L}(\lambda))$. Given $M \in \mathsf{Rep}(\mathbf{G})$ and $\lambda \in \mathbb{X}^+$, we will denote by

$$[M : \mathsf{L}(\lambda)]$$

the multiplicity of $\mathsf{L}(\lambda)$ as a composition factor of M.

To finish this subsection we note the following result for later use.

Lemma 1.18. *For $\lambda \in \mathbb{X}^+$, we have $\dim(\mathsf{L}(\lambda)) = 1$ if and only if $\langle \lambda, \alpha^\vee \rangle = 0$ for any $\alpha \in \mathfrak{R}$.*

Proof. As noted in Example 1.6, if λ satisfies $\langle \lambda, \alpha^\vee \rangle = 0$ for any $\alpha \in \mathfrak{R}$, then $\dim(\mathsf{N}(\lambda)) = 1$, so that we must have $\mathsf{L}(\lambda) = \mathsf{N}(\lambda)$, hence $\dim(\mathsf{L}(\lambda)) = 1$. On the other hand, if $\dim(\mathsf{L}(\lambda)) = 1$ then λ must be the restriction of a group morphism $\mathbf{G} \to \Bbbk^\times$, so that $\langle \lambda, \alpha^\vee \rangle = 0$ for any $\alpha \in \mathfrak{R}$. □

1.7. Central characters. Let $Z(\mathbf{G})$ be the scheme-theoretic center of $\mathbf{G}$, as defined in [**J3**, §I.2.6]. This group scheme can be described very explicitly: we have $Z(\mathbf{G}) \subset \mathbf{T}$, and $Z(\mathbf{G})$ identifies with the diagonalizable group scheme (in the sense of [**J3**, §I.2.5]) associated with the quotient $\mathbb{X}/\mathbb{Z}\mathfrak{R}$ of $\mathbb{X}$. (Here, $\mathbb{Z}\mathfrak{R}$ is the sublattice in $\mathbb{X}$ generated by $\mathfrak{R}$, or equivalently by $\mathfrak{R}^{\mathrm{s}}$.) In particular, this group scheme might not be smooth if $p > 0$, but its representations are still very easy to describe: the category of representations of $Z(\mathbf{G})$ is semi-simple, with simple objects (up to isomorphism) in bijection with $\mathbb{X}/\mathbb{Z}\mathfrak{R}$, and all of them are 1-dimensional. In other words, the datum of a representation of $Z(\mathbf{G})$ is equivalent to that of a $\mathbb{X}/\mathbb{Z}\mathfrak{R}$-graded $\Bbbk$-vector space. (These facts are special cases of general results about representations of diagonalizable group schemes, see [**J3**, §I.2.11].)

Every $V \in \mathsf{Rep}^\infty(\mathbf{G})$ can be seen as a representation of $Z(\mathbf{G})$ by restriction. For any $x \in \mathbb{X}/\mathbb{Z}\mathfrak{R}$ we will denote by $V_{Z=x}$ the subspace consisting of vectors on which $Z(\mathbf{G})$ acts via the character x. Then we have

$$V = \bigoplus_{x \in \mathbb{X}/\mathbb{Z}\mathfrak{R}} V_{Z=x},$$

and each $V_{Z=x}$ is a $\mathbf{G}$-stable subspace of V. If V' is another object of $\mathsf{Rep}^\infty(\mathbf{G})$, then any morphism $f \in \mathrm{Hom}_{\mathbf{G}}(V, V')$ must send $V_{Z=x}$ to $V'_{Z=x}$ for any $x \in \mathbb{X}/\mathbb{Z}\mathfrak{R}$. Hence we have a decomposition of the category $\mathsf{Rep}^\infty(\mathbf{G})$ as

$$\mathsf{Rep}^\infty(\mathbf{G}) = \bigoplus_{x \in \mathbb{X}/\mathbb{Z}\mathfrak{R}} \mathsf{Rep}^\infty(\mathbf{G})_{Z=x} \tag{1.5}$$

where $\mathsf{Rep}(\mathbf{G})^{\infty}_{Z=x}$ is the full subcategory of $\mathsf{Rep}^{\infty}(\mathbf{G})$ whose objects are the representations V such that $V = V_{Z=x}$.

Each indecomposable object in $\mathsf{Rep}^{\infty}(\mathbf{G})$ (in particular, each object $\mathsf{L}(\lambda)$, $\mathsf{N}(\lambda)$ or $\mathsf{M}(\lambda)$ for $\lambda \in \mathbb{X}^+$) must belong to one of the summands $\mathsf{Rep}^{\infty}(\mathbf{G})_{Z=x}$. In fact, considering the action of $Z(\mathbf{G})$ on the highest-weight line one sees that for any $\lambda \in \mathbb{X}^+$, $\mathsf{L}(\lambda)$, $\mathsf{N}(\lambda)$ and $\mathsf{M}(\lambda)$ belong to the summand corresponding to the image of λ in $\mathbb{X}/\mathbb{Z}\mathfrak{R}$.

1.8. Characters and the Grothendieck group. Consider now the Grothendieck group $[\mathsf{Rep}(\mathbf{G})]$ of the abelian category $\mathsf{Rep}(\mathbf{G})$. This abelian group admits as a basis the classes $([\mathsf{L}(\lambda)] : \lambda \in \mathbb{X}^+)$ of the simple modules. For $M \in \mathsf{Rep}(\mathbf{G})$, the expansion of the class $[M]$ in this basis is given by

$$[M] = \sum_{\lambda \in \mathbb{X}^+} [M : \mathsf{L}(\lambda)] \cdot [\mathsf{L}(\lambda)].$$

Any short exact sequence $V' \hookrightarrow V \twoheadrightarrow V''$ in $\mathsf{Rep}(\mathbf{G})$ induces, for any $\mu \in \mathbb{X}$, an exact sequence of vector spaces

$$(V')_{\mu} \hookrightarrow V_{\mu} \twoheadrightarrow (V'')_{\mu}.$$

It follows that the map $V \mapsto \mathrm{ch}(V)$ induces a group morphism

$$[\mathsf{Rep}(\mathbf{G})] \to \mathbb{Z}[\mathbb{X}]$$

which we will also denote ch, Moreover, it follows from Lemma 1.10 that this morphism takes values in $\mathbb{Z}[\mathbb{X}]^W$.

PROPOSITION 1.19. *The morphism*

$$\mathrm{ch} : [\mathsf{Rep}(\mathbf{G})] \to \mathbb{Z}[\mathbb{X}]^W$$

is an isomorphism.

PROOF. As seen above the classes $([\mathsf{L}(\lambda)] : \lambda \in \mathbb{X}^+)$ form a basis of the $\mathbb{Z}$-module $[\mathsf{Rep}(\mathbf{G})]$. On the other hand, since $\mathbb{X}^+$ is a system of representatives for the W-orbits on $\mathbb{X}$, the $\mathbb{Z}$-module $\mathbb{Z}[\mathbb{X}]^W$ is free, with a basis consisting of the elements $o_{\lambda} := \sum_{\mu \in W(\lambda)} e^{\mu}$ where λ runs over $\mathbb{X}^+$. Since $\mathsf{L}(\lambda)$ is a submodule of $\mathsf{N}(\lambda)$ containing $\mathsf{N}(\lambda)_{\lambda}$, we deduce from Lemma 1.11 that

$$\mathrm{ch}(\mathsf{L}(\lambda)) \in o_{\lambda} + \sum_{\substack{\mu \in \mathbb{X}^+ \\ \mu \prec \lambda}} \mathbb{Z} \cdot o_{\mu}.$$

This observation shows that $(\mathrm{ch}(\mathsf{L}(\lambda)) : \lambda \in \mathbb{X}^+)$ forms a basis of $\mathbb{Z}[\mathbb{X}]^W$, which implies that our morphism is an isomorphism. $\square$

This proposition and its proof show that the composition factors of a finite-dimensional algebraic $\mathbf{G}$-module are determined by its character. Moreover, if one knows the characters of the modules $\mathsf{L}(\lambda)$ for $\lambda \in \mathbb{X}^+$, then the determination of these composition factors is equivalent to the determination of the coefficients of this character in the basis $(\mathrm{ch}(\mathsf{L}(\lambda)) : \lambda \in \mathbb{X}^+)$ of $\mathbb{Z}[\mathbb{X}]^W$.

In fact, as mentioned in §1.6, the determination of the characters of the simple $\mathbf{G}$-modules is a very delicate question. We will therefore also consider other bases of $[\mathsf{Rep}(\mathbf{G})]$. Namely, with the notation in the proof of Proposition 1.19, by Lemma 1.11 we also have

$$\mathrm{ch}(\mathsf{N}(\lambda)) \in o_\lambda + \sum_{\substack{\mu \in \mathbb{X}^+ \\ \mu \prec \lambda}} \mathbb{Z} \cdot o_\mu.$$

Therefore the classes $[\mathsf{N}(\lambda)]$ for $\lambda \in \mathbb{X}^+$ also constitute a basis of $[\mathsf{Rep}(\mathbf{G})]$. (The main difference with the basis $([\mathsf{L}(\lambda)] : \lambda \in \mathbb{X}^+)$ is that in this basis the coefficients of the class of an object of $\mathsf{Rep}(\mathbf{G})$ are not necessarily nonnegative.)

The tensor product of $\mathbf{G}$-modules endows the category $\mathsf{Rep}(\mathbf{G})$ with a structure of monoidal category, which in turns induces a ring structure on the Grothendieck group $[\mathsf{Rep}(\mathbf{G})]$. For V, V' in $\mathsf{Rep}(\mathbf{G})$ and $\lambda \in \mathbb{X}$ we have

$$(V \otimes V')_\lambda = \bigoplus_{\substack{\mu,\nu \in \mathbb{X} \\ \mu+\nu=\lambda}} V_\mu \otimes (V')_\nu;$$

it follows that the morphism of Proposition 1.19 is a ring isomorphism.

1.9. Weyl's character formula. The next statement we will consider is an analogue in the setting of (algebraic) representations of algebraic groups of Weyl's character formula, originally discovered in the setting of compact Lie groups. It is usually also referred to as Weyl's character formula. Here we denote by

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}^+} \alpha \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}$$

the half sum of the positive roots. Given $w \in W$ and $\lambda \in \mathbb{X}$ we set

$$w \bullet \lambda = w(\lambda + \rho) - \rho.$$

It is a standard fact that the right-hand side always belongs to $\mathbb{X}$, and that this formula defines an action of W on $\mathbb{X}$.

For the proof of the following theorem, we refer to [**J3**, Proposition II.5.10]. (This statement involves a quotient of two elements in $\mathbb{Z}[\mathbb{X}]$. It is a classical fact that this fraction belongs to $\mathbb{Z}[\mathbb{X}]$, i.e. that its numerator is divisible by its denominator in the domain $\mathbb{Z}[\mathbb{X}]$.)

THEOREM 1.20. *For every $\lambda \in \mathbb{X}^+$ we have*

$$\mathrm{ch}(\mathsf{N}(\lambda)) = \frac{\sum_{w \in W} (-1)^{\ell(w)} \cdot e^{w \bullet \lambda}}{\sum_{w \in W} (-1)^{\ell(w)} \cdot e^{w \bullet 0}}.$$

Using this formula one can check that

$$\mathrm{ch}(\mathsf{M}(\lambda)) = \mathrm{ch}(\mathsf{N}(\lambda))$$

for any $\lambda \in \mathbb{X}^+$. (This equality can also be seen more directly, see [**J3**, §II.2.13].) In view of Proposition 1.19, this implies that we also have

$$[\mathsf{M}(\lambda)] = [\mathsf{N}(\lambda)] \tag{1.6}$$

in $[\mathsf{Rep}(\mathbf{G})]$, or in other words that

$$[\mathsf{M}(\lambda) : \mathsf{L}(\mu)] = [\mathsf{N}(\lambda) : \mathsf{L}(\mu)] \tag{1.7}$$

for any $\mu \in \mathbb{X}^+$.

The fraction appearing in Weyl's character formula can be difficult to compute, but it appears in several other contexts (in particular, compact Lie groups and complex semisimple Lie algebras), and has been extensively studied. (For explicit examples of how to compute this fraction for classical groups, see e.g. [**FH**, §24.2].) For us, we will hence considered that the characters $(\mathrm{ch}(\mathsf{N}(\lambda)) : \lambda \in \mathbb{X}^+)$ are understood. In view of Proposition 1.19, we will therefore consider that computing the character of a $\mathbf{G}$-module M is equivalent to expressing $[M]$ in the basis $([\mathsf{N}(\lambda)] : \lambda \in \mathbb{X}^+)$ of $[\mathsf{Rep}(\mathbf{G})]$.[4] From this point of view, the problem evoked in §1.6 asks for the description, for each $\lambda \in \mathbb{X}^+$, of the expansion of the element $[\mathsf{L}(\lambda)] \in [\mathsf{Rep}(\mathbf{G})]$ in the basis $([\mathsf{N}(\mu)] : \mu \in \mathbb{X}^+)$.

The formula in Theorem 1.20 does not involve the field $\Bbbk$ (or its characteristic p) in any way. From the point of view described in §1.1, we will therefore consider that induced and Weyl modules are independent of $\Bbbk$ (or of p).

Remark 1.21. Once $\mathrm{ch}(\mathsf{N}(\lambda))$ is known, one can in particular compute the dimension $\dim \mathsf{N}(\lambda)$ by evaluating each e^μ to 1. The result one gets in this way is well known from the representation theory of complex semisimple Lie algebras (or of compact Lie groups): we obtain that

$$\dim \mathsf{N}(\lambda) = \frac{\prod_{\alpha \in \mathfrak{R}^+} \langle \lambda + \rho, \alpha^\vee \rangle}{\prod_{\alpha \in \mathfrak{R}^+} \langle \rho, \alpha^\vee \rangle};$$

see [**H1**, Corollary in §24.3].

2. Structure of the category $\mathsf{Rep}(\mathbf{G})$

In this section we recall (mostly without proof) some important structural results on the category $\mathsf{Rep}(\mathbf{G})$ that will allow us to explain the way in which one can try to answer the problem considered in §1.9.

2.1. Kempf's vanishing theorem. Another useful statement that we will require below is the following theorem due to Kempf, and called *Kempf's vanishing theorem*. This statement makes use, for a subgroup $\mathbf{K}$ of an algebraic group $\mathbf{H}$, of the derived functors

$$R^i\mathrm{Ind}_{\mathbf{K}}^{\mathbf{H}} : \mathsf{Rep}^\infty(\mathbf{K}) \to \mathsf{Rep}^\infty(\mathbf{H}) \qquad (i \geq 0)$$

of the induction functor $\mathrm{Ind}_{\mathbf{K}}^{\mathbf{H}}$. (Note that the category of algebraic representations of a $\Bbbk$-algebraic group always has enough injectives, see [**J3**, Proposition I.3.9], so that these functors are well defined.)

Theorem 2.1. *For any $\lambda \in \mathbb{X}^+$ and any $i \in \mathbb{Z}_{>0}$, we have*

$$R^i\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(\lambda)) = 0.$$

For the proof, see [**J3**, Theorem II.4.5].

Remark 2.2. Recall the notation of Remark 1.4. For any $\mathbf{H}$-equivariant quasi-coherent sheaf $\mathscr{F}$ on $\mathbf{H}/\mathbf{K}$, and any $i \geq 0$, the space $\mathsf{H}^i(\mathbf{H}/\mathbf{K}, \mathscr{F})$ has a canonical structure of $\mathbf{H}$-module. For and $i \geq 0$ and $M \in \mathsf{Rep}^\infty(\mathbf{K})$, there exists a canonical isomorphism

$$R^i\mathrm{Ind}_{\mathbf{K}}^{\mathbf{H}}(M) \cong \mathsf{H}^i(\mathbf{H}/\mathbf{K}, \mathscr{L}_{\mathbf{H}/\mathbf{K}}(M))$$

in $\mathsf{Rep}^\infty(\mathbf{H})$, see [**J3**, §I.5.12].

[4]We are not saying that there does not exist any other interesting way of computing characters; but this point of view is the one which is adopted in most works on the subject, and which will be considered in these notes.

From this point of view, in the setting of Remark 1.7 and in the special case $\mathbf{G} = \mathrm{SL}_2(\Bbbk)$, Theorem 2.1 specializes to the standard fact that the higher cohomology spaces of the line bundles $\mathscr{L}_a$ are trivial when $a \geq 0$.

There exists another general vanishing result on the spaces $R^i\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(\lambda))$: it states that
$$R^i\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(\lambda)) = 0 \quad \text{for any } i \geq 0$$
in case $\lambda \in \mathbb{X}$ satisfies $\langle \lambda, \alpha^\vee \rangle = -1$ for some $\alpha \in \mathfrak{R}^{\mathrm{s}}$, see [J3, Proposition II.5.4(a)]. (This statement is much easier to prove than Theorem 2.1: by using a standard transitivity result for derived induction functors it suffices to prove that $R^i\mathrm{Ind}_{\mathbf{B}}^{\mathbf{P}_\alpha}(\Bbbk_{\mathbf{B}}(\lambda)) = 0$ for any $i \geq 0$, where $\mathbf{P}_\alpha$ is the parabolic subgroup containing $\mathbf{B}$ attached to α. The latter statement comes down to the standard fact that $\mathsf{H}^i(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-1)) = 0$ for any i.)

2.2. Highest weight structure. In these notes we will make extensive use of a certain structure on $\mathsf{Rep}(\mathbf{G})$ called a structure of *highest weight category*. The general theory of such structures is reviewed in Appendix A.

THEOREM 2.3. *The category* $\mathsf{Rep}(\mathbf{G})$*, together with the poset* $(\mathbb{X}^+, \preceq)$*, the collection of "standard objects"* $(\mathsf{M}(\lambda) : \lambda \in \mathbb{X}^+)$*, and the collection of "costandard objects"* $(\mathsf{N}(\lambda) : \lambda \in \mathbb{X}^+)$*, is a highest weight category.*

In view of Remark 1.9, this structure of highest weight category is intrinsic, i.e. it does not depend on the choice of Borel subgroup and maximal torus.

The proof of Theorem 2.3 will make use of the following lemma, for which we refer to [J3, Proposition II.4.10]. (The proof of this lemma uses the description of the injective hulls of the simple $\mathbf{B}$-modules $\Bbbk_{\mathbf{B}}(\lambda)$.) Here we denote by
$$H^i(\mathbf{B}, -) : \mathsf{Rep}^\infty(\mathbf{B}) \to \mathrm{Vect}_\Bbbk$$
the i-th derived functor of the functor of $\mathbf{B}$-fixed points (where $\mathrm{Vect}_\Bbbk$ is the category of $\Bbbk$-vector spaces), i.e.
$$H^i(\mathbf{B}, M) = \mathrm{Ext}^i_{\mathsf{Rep}^\infty(\mathbf{B})}(\Bbbk, M),$$
and for $\lambda \in \mathbb{Z}_{\geq 0}\mathfrak{R}^+$, written as $\lambda = \sum_{\alpha \in \mathfrak{R}^{\mathrm{s}}} n_\alpha \cdot \alpha$, we set $\mathrm{ht}(\lambda) = \sum_{\alpha \in \mathfrak{R}^{\mathrm{s}}} n_\alpha$.

LEMMA 2.4. *If* $H^i(\mathbf{B}, M) \neq 0$*, then there exists* $\lambda \in \mathrm{wt}(M)$ *such that*
$$-\lambda \in \mathbb{Z}_{\geq 0}\mathfrak{R}^+ \quad \textit{and} \quad \mathrm{ht}(-\lambda) \geq i.$$

The main step of the proof of Theorem 2.3 is the following proposition, due to Cline–Parshall–Scott–van der Kallen.

PROPOSITION 2.5. *For* $\lambda \in \mathbb{X}^+$ *and* $i \in \mathbb{Z}$ *we have*
$$\mathrm{Ext}^i_{\mathsf{Rep}^\infty(\mathbf{G})}(\mathsf{M}(\lambda), \mathsf{N}(\mu)) = 0$$
unless $i = 0$ *and* $\lambda = \mu$.

PROOF. We reproduce the proof given in [J3, Proposition II.4.13]. If we denote by
$$R\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} : D^+\mathsf{Rep}^\infty(\mathbf{B}) \to D^+\mathsf{Rep}^\infty(\mathbf{G})$$
the derived functor of the left exact functor $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} : \mathsf{Rep}^\infty(\mathbf{B}) \to \mathsf{Rep}^\infty(\mathbf{G})$, then Theorem 2.1 implies that for any $\mu \in \mathbb{X}^+$ we have an isomorphism
$$R\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(\mu)) \cong \mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(\mu)),$$

where the right-hand side denotes the module $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(\mu))$ seen as a complex concentrated in degree 0. The derived version of Frobenius reciprocity (stated in the form of a spectral sequence in [J3, Proposition I.4.5]) can also be stated as saying the functor $R\,\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}$ is right adjoint to the forgetful functor

$$\mathsf{For}_{\mathbf{B}}^{\mathbf{G}} : D^+\mathsf{Rep}^\infty(\mathbf{G}) \to D^+\mathsf{Rep}^\infty(\mathbf{B}).$$

Using this we deduce that

$$\begin{aligned}\mathrm{Ext}^i_{\mathsf{Rep}^\infty(\mathbf{G})}(\mathsf{M}(\lambda), \mathsf{N}(\mu)) &\cong \mathrm{Ext}^i_{\mathsf{Rep}^\infty(\mathbf{B})}\big(\mathsf{For}_{\mathbf{B}}^{\mathbf{G}}(\mathsf{M}(\lambda)), \Bbbk_{\mathbf{B}}(\mu)\big)\\ &\cong \mathrm{Ext}^i_{\mathsf{Rep}^\infty(\mathbf{B})}\big(\Bbbk, \mathsf{For}_{\mathbf{B}}^{\mathbf{G}}(\mathsf{N}(-w_0\lambda)) \otimes \Bbbk_{\mathbf{B}}(\mu)\big)\\ &\cong H^i(\mathbf{B}, \mathsf{For}_{\mathbf{B}}^{\mathbf{G}}(\mathsf{N}(-w_0\lambda)) \otimes \Bbbk_{\mathbf{B}}(\mu)).\end{aligned}$$

Similar arguments show that we also have

$$\mathrm{Ext}^i_{\mathsf{Rep}^\infty(\mathbf{G})}(\mathsf{M}(\lambda), \mathsf{N}(\mu)) \cong H^i(\mathbf{B}, \Bbbk_{\mathbf{B}}(-w_0\lambda) \otimes \mathsf{For}_{\mathbf{B}}^{\mathbf{G}}(\mathsf{N}(\mu))).$$

From these equalities and Lemma 2.4 we see that if $\mathrm{Ext}^i_{\mathsf{Rep}^\infty(\mathbf{G})}(\mathsf{M}(\lambda), \mathsf{N}(\mu)) \neq 0$, then there exist $\nu \in \mathrm{wt}(\mathsf{N}(-w_0\lambda))$ and $\nu' \in \mathrm{wt}(\mathsf{N}(\mu))$ such that

$$\begin{aligned}-\mu - \nu \in \mathbb{Z}_{\geq 0}\mathfrak{R}^+ \quad &\text{and} \quad \mathrm{ht}(-\mu - \nu) \geq i,\\ w_0\lambda - \nu' \in \mathbb{Z}_{\geq 0}\mathfrak{R}^+ \quad &\text{and} \quad \mathrm{ht}(w_0\lambda - \nu') \geq i.\end{aligned}$$

Then we have $\nu \succeq -\lambda$ and $\nu' \succeq w_0(\mu)$ by (1.3), so that

$$\begin{aligned}-\mu + \lambda &= (-\mu - \nu) + (\nu + \lambda) \in \mathbb{Z}_{\geq 0}\mathfrak{R}^+,\\ w_0(-\mu + \lambda) &= (w_0\lambda - \nu') + (\nu' - w_0\mu) \in \mathbb{Z}_{\geq 0}\mathfrak{R}^+.\end{aligned}$$

We deduce that $\mu - \lambda \in (\mathbb{Z}_{\geq 0}\mathfrak{R}^+) \cap (-\mathbb{Z}_{\geq 0}\mathfrak{R}^+) = \{0\}$, so that $\mu = \lambda$. We must also have $\nu = -\mu$ so that $i \leq 0$, and finally $i = 0$. □

We can now complete the proof of Theorem 2.3.

Proof of Theorem 2.3. We need to check the various conditions in Definition 1.1 from Appendix A. Here Condition (1) is a standard fact from the theory of root systems, see e.g. [H1, Lemma B in §13.2]. Condition (2) follows from the fact that

$$\mathrm{Hom}_{\mathsf{Rep}(\mathbf{G})}(\mathsf{L}(\lambda), \mathsf{N}(\lambda)) \cong \mathrm{Hom}_{\mathsf{Rep}(\mathbf{B})}(\mathsf{L}(\lambda), \Bbbk_{\mathbf{B}}(\lambda))$$

(by Frobenius reciprocity), and that this space is at most 1-dimensional, since

$$\mathrm{Hom}_{\mathsf{Rep}(\mathbf{T})}(\mathsf{L}(\lambda), \Bbbk_{\mathbf{T}}(\lambda)) \cong (\mathsf{L}(\lambda)_\lambda)^*$$

is 1-dimensional.

For condition (3), we consider an ideal $\Lambda \subset \mathbb{X}^+$ and a maximal element $\lambda \in \Lambda$. For any V in $\mathsf{Rep}(\mathbf{G})$ we have

$$\mathrm{Ext}^1_{\mathsf{Rep}(\mathbf{G})}(V, \mathsf{N}(\lambda)) \cong \mathrm{Ext}^1_{\mathsf{Rep}^\infty(\mathbf{G})}(V, \mathsf{N}(\lambda))$$

because the subcategory $\mathsf{Rep}(\mathbf{G}) \subset \mathsf{Rep}^\infty(\mathbf{G})$ is closed under extensions. The same arguments as in the proof of Proposition 2.5 then show that if $\mathrm{Ext}^1_{\mathsf{Rep}(\mathbf{G})}(V, \mathsf{N}(\lambda)) \neq 0$, then exists $\nu \in \mathrm{wt}(V)$ such that $\nu - \lambda \in \mathbb{Z}_{\geq 0}\mathfrak{R}^+$ and $\mathrm{ht}(\nu - \lambda) \geq 1$. Then we have $\nu \succ \lambda$, so that V must admit a composition factor of the form $\mathsf{L}(\eta)$ with $\eta \succ \lambda$, hence with $\eta \notin \Lambda$. This implies that $\mathsf{N}(\lambda)$ is injective in the Serre subcategory of $\mathsf{Rep}(\mathbf{G})$ generated by the simple objects $\mathsf{L}(\mu)$ with $\mu \in \Lambda$. Since its socle is $\mathsf{L}(\lambda)$, it must be the injective hull of $\mathsf{L}(\lambda)$ in this subcategory. By duality, we deduce that $\mathsf{M}(\lambda)$ is the projective cover of $\mathsf{L}(\lambda)$ in this Serre subcategory.

Condition (4) follows from the fact that the weights μ of the cokernel of the embedding $\mathsf{L}(\lambda) \hookrightarrow \mathsf{N}(\lambda)$ satisfy $\mu \prec \lambda$, so that the composition factors of this cokernel must be of the form $\mathsf{L}(\nu)$ with $\mu \prec \lambda$, and a similar observation for the kernel of the surjection $\mathsf{M}(\lambda) \twoheadrightarrow \mathsf{L}(\lambda)$.

Finally, to prove Condition (5), we remark that the natural functor

$$D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G}) \to D^{\mathrm{b}}\mathsf{Rep}^{\infty}(\mathbf{G}) \tag{2.1}$$

induces isomorphisms

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G})}(M, N[i]) \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Rep}^{\infty}(\mathbf{G})}(M, N[i])$$

for any $M, N \in \mathsf{Rep}(\mathbf{G})$ and $i \in \{0,1\}$, since $\mathsf{Rep}(\mathbf{G})$ is a full subcategory of $\mathsf{Rep}^{\infty}(\mathbf{G})$ closed under extensions. It follows that the similar morphism

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G})}(M, N[2]) \to \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Rep}^{\infty}(\mathbf{G})}(M, N[2])$$

is injective for any $M, N \in \mathsf{Rep}(\mathbf{G})$, see e.g. [**BBD**, Remarque 3.1.17(i)] or [**BGS**, Lemma 3.2.3]. In particular if $M = \mathsf{M}(\lambda)$ and $N = \mathsf{N}(\mu)$ for some $\lambda, \mu \in \mathbb{X}^+$, the natural morphism

$$\mathrm{Ext}^2_{\mathsf{Rep}(\mathbf{G})}(\mathsf{M}(\lambda), \mathsf{N}(\mu)) \to \mathrm{Ext}^2_{\mathsf{Rep}^{\infty}(\mathbf{G})}(\mathsf{M}(\lambda), \mathsf{N}(\mu))$$

is injective. Since the right-hand side vanishes by Proposition 2.5, it follows that the left-hand side also vanishes, which finishes the proof. □

REMARK 2.6. (1) By Corollary 2.3 from Appendix A, Theorem 2.3 implies that we have

$$\mathrm{Ext}^i_{\mathsf{Rep}(\mathbf{G})}(\mathsf{M}(\lambda), \mathsf{N}(\mu)) = 0$$

unless $\lambda = \mu$ and $i = 0$. Comparing with Proposition 2.5, and using the fact that the category $D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G})$ is generated (as a triangulated category) both by the objects $(\mathsf{N}(\lambda) : \lambda \in \mathbb{X}^+)$ and by the objects $(\mathsf{M}(\lambda) : \lambda \in \mathbb{X}^+)$, it is not difficult to deduce that the functor (2.1) is fully faithful. This property is in fact a general fact on categories of representations of affine group schemes over fields, see [**Co**, Theorem 2.3.1].

(2) By Lemma 1.4 in Appendix A we have

$$\dim \mathrm{Hom}_{\mathbf{G}}(\mathsf{M}(\lambda), \mathsf{N}(\lambda)) = 1.$$

Now $\mathsf{M}(\lambda)$ has head isomorphic to $\mathsf{L}(\lambda)$, which by definition is the socle of $\mathsf{L}(\lambda)$. It follows that any nonzero morphism in $\mathrm{Hom}_{\mathbf{G}}(\mathsf{M}(\lambda), \mathsf{N}(\lambda))$ factors as a composition

$$\mathsf{M}(\lambda) \twoheadrightarrow \mathsf{L}(\lambda) \hookrightarrow \mathsf{N}(\lambda).$$

2.3. The case $p = 0$. The results we have discussed so far are uniform across all characteristics. Starting from §2.4 below we will restrict to the case $p > 0$, which is the main topic of this book. Before that, for completeness (and comparison) we state two important results which are specific to the case $p = 0$.

The first one is the *Borel–Weil–Bott theorem* (or Borel–Bott–Weil theorem, or Bott–Borel–Weil theorem). This theorem is due to Bott, and is based on an earlier result of Borel–Weil describing irreducible representations of compact Lie groups as sections of line bundles on flag varieties. To state this theorem, we note that a fundamental domain for the restriction of W on $\mathbb{X}$ via $\bullet$ is given by

$$\{\lambda \in \mathbb{X} \mid \forall \alpha \in \mathfrak{R}^{\mathrm{s}},\ \langle \lambda, \alpha^\vee \rangle \geq -1\}.$$

In other words, any $\mu \in \mathbb{X}$ can be written as $\mu = w \bullet \lambda$ where $w \in W$ and $\lambda \in \mathbb{X}$ satisfies $\langle \lambda, \alpha^\vee \rangle \geq -1$ for any $\alpha \in \mathfrak{R}^{\mathrm{s}}$. Here λ is uniquely determined, but w is determined only up to multiplication on the right by an element of the stabilizer of λ (which is the parabolic subgroup of W generated by the simple reflections in S which stabilize λ). In case $\lambda \in \mathbb{X}^+$, this stabilizer is trivial, so that w is uniquely determined in this case.

THEOREM 2.7. *Assume that $p = 0$. Let $\lambda \in \mathbb{X}$ such that $\langle \lambda, \alpha^\vee \rangle \geq -1$ for any $\alpha \in \mathfrak{R}^{\mathrm{s}}$, and let $w \in W$.*

(1) If $\lambda \notin \mathbb{X}^+$, then $R^i\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(w \bullet \lambda)) = 0$ for any $i \geq 0$.
(2) If $\lambda \in \mathbb{X}^+$, then for $i \geq 0$ we have

$$R^i\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(w \bullet \lambda)) \cong \begin{cases} \mathsf{N}(\lambda) & \text{if } i = \ell(w) \\ 0 & \text{otherwise.} \end{cases}$$

For the proof of this theorem, see [**J3**, Corollary II.5.5]. This statement also has a variant in the case $p > 0$, which is useful but more restricted. Namely, in case $p > 0$, the same statement holds if $\lambda \in \mathbb{X}$ satisfies $0 \leq \langle \lambda + \rho, \beta^\vee \rangle \leq p$ for all $\beta \in \mathfrak{R}^+$.

The second fundamental result we want to mention is the following.

THEOREM 2.8. *Assume that $p = 0$. Then the category $\mathsf{Rep}^\infty(\mathbf{G})$ is semisimple, and for each $\lambda \in \mathbb{X}^+$ the module $\mathsf{N}(\lambda)$ is simple.*

For the proof, see [**J3**, §II.5.6]. These statements are definitely false in case $p > 0$, as seen already in Example 1.17.

2.4. The Frobenius morphism and Steinberg's tensor product formula. From now on we will assume that $p > 0$.

For any $\Bbbk$-scheme X, the *Frobenius twist* of X is the fiber product

$$X^{(1)} := \mathrm{Spec}(\Bbbk) \times_{\mathrm{Spec}(\Bbbk)} X,$$

where the morphism $\mathrm{Spec}(\Bbbk) \to \mathrm{Spec}(\Bbbk)$ is induced by the ring morphism $x \mapsto x^p$. In fact, the projection morphism $X^{(1)} \to X$ is an isomorphism of abstract schemes, but not of $\Bbbk$-schemes: if $X = \mathrm{Spec}(A)$ for some $\Bbbk$-algebra A, then $X^{(1)}$ is the spectrum of A, seen as a $\Bbbk$-algebra with the same multiplication map, but with the structure of $\Bbbk$-vector space given by $\lambda \cdot a = \lambda^{1/p} a$, where $(-)^{1/p}$ is the inverse to $x \mapsto x^p$. In this setting the *Frobenius morphism*

$$\mathrm{Fr}_X : X \to X^{(1)}$$

is the morphism of $\Bbbk$-schemes corresponding to the algebra morphism $A \to A$ defined by $a \mapsto a^p$. (In general, the Frobenius morphism can be obtained by gluing these morphisms on an affine open cover.)

In particular, we can consider the connected reductive group $\mathbf{G}^{(1)}$, with its Borel subgroup $\mathbf{B}^{(1)}$, and its maximal torus $\mathbf{T}^{(1)}$. The Frobenius morphism

$$\mathrm{Fr}_{\mathbf{G}} : \mathbf{G} \to \mathbf{G}^{(1)}$$

is a group morphism, which sends $\mathbf{B}$ into $\mathbf{B}^{(1)}$ and $\mathbf{T}$ into $\mathbf{T}^{(1)}$. In particular, given V in $\mathsf{Rep}(\mathbf{G}^{(1)})$ we can consider the $\mathbf{G}$-module $\mathrm{Fr}_{\mathbf{G}}^*(V)$ obtained by pullback. We will also denote by

$$\mathrm{Fr}_{\mathbf{T}}^* : X^*(\mathbf{T}^{(1)}) \to \mathbb{X} \tag{2.2}$$

the morphism sending a morphism to its composition with $\mathrm{Fr}_{\mathbf{T}}$. It is easily checked that this morphism is injective, with image $p \cdot \mathbb{X}$.

The classification of simple modules from §1.6 holds also for $\mathbf{G}^{(1)}$ with the subgroups $\mathbf{B}^{(1)}$ and $\mathbf{T}^{(1)}$. If the corresponding subset of dominant weights is denoted $X^*(\mathbf{T}^{(1)})^+$, then for $\lambda \in X^*(\mathbf{T}^{(1)})$ we will denote by $\mathsf{L}^{(1)}(\lambda)$ the corresponding simple $\mathbf{G}^{(1)}$-module. Note that the image of $X^*(\mathbf{T}^{(1)})^+$ under (2.2) is $p\mathbb{X}^+$.

We set

$$\mathbb{X}^+_{\mathrm{res}} = \{\lambda \in \mathbb{X} \mid \forall\, \alpha \in \mathfrak{R}^{\mathrm{s}},\, 0 \leq \langle \lambda, \alpha^\vee \rangle < p\}.$$

(The weights in this subset are called *restricted* dominant weights.) The following theorem is due to Steinberg, and is called *Steinberg's tensor product theorem*. For the proof, we refer to [J3, Proposition II.3.16].

THEOREM 2.9. *For any* $\lambda \in \mathbb{X}^+_{\mathrm{res}}$ *and* $\mu \in X^*(\mathbf{T}^{(1)})^+$ *we have*

$$\mathsf{L}(\lambda + \mathrm{Fr}^*_{\mathbf{T}}(\mu)) \cong \mathsf{L}(\lambda) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{L}^{(1)}(\mu)).$$

Usually we will fix an isomorphism of $\Bbbk$-algebraic groups $\mathbf{G}^{(1)} \cong \mathbf{G}$ identifying $\mathbf{B}^{(1)}$ with $\mathbf{B}$ and $\mathbf{T}^{(1)}$ with $\mathbf{T}$, such that the morphism $\mathrm{Fr}^*_{\mathbf{T}}$ of (2.2) identifies with multiplication by p (which is always possible, see [J3, §II.3.1]); if we still denote by $\mathrm{Fr}_{\mathbf{G}} : \mathbf{G} \to \mathbf{G}$ the morphism obtained using this identification, then the isomorphism of Theorem 2.9 then reads

$$\mathsf{L}(\lambda + p\mu) \cong \mathsf{L}(\lambda) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{L}(\mu)) \tag{2.3}$$

for $\lambda \in \mathbb{X}^+_{\mathrm{res}}$ and $\mu \in \mathbb{X}^+$.

Consider now the derived subgroup $\mathscr{D}(\mathbf{G})$ of $\mathbf{G}$ (a semisimple group), and its maximal torus $\mathbf{T} \cap \mathscr{D}(\mathbf{G})$. The restriction to $\mathfrak{R}$ of the (surjective) morphism

$$\mathbb{X} \to X^*(\mathbf{T} \cap \mathscr{D}(\mathbf{G})) \tag{2.4}$$

induced by restriction to $\mathbf{T} \cap \mathscr{D}(\mathbf{G})$ is injective, and its image is the root system of $(\mathscr{D}(\mathbf{G}), \mathbf{T} \cap \mathscr{D}(\mathbf{G}))$. The roots $\mathfrak{R}^{\mathrm{s}}$ therefore also provide a basis of this root system. Any coroot in $\mathfrak{R}^\vee$ factors through $\mathscr{D}(\mathbf{G})$, hence can be considered as a coroot if this group. If $\mathscr{D}(\mathbf{G})$ is simply connected,[5] for any $\alpha \in \mathfrak{R}^{\mathrm{s}}$ there exists a weight $\varpi_\alpha \in X^*(\mathbf{T} \cap \mathscr{D}(\mathbf{G}))$ which satisfies

$$\langle \varpi_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$$

for all $\beta \in \mathfrak{R}^{\mathrm{s}}$. If $\varpi'_\alpha \in \mathbb{X}$ is any element whose image under (2.4) is ϖ_α, then we also have

$$\langle \varpi'_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$$

for all $\beta \in \mathfrak{R}^{\mathrm{s}}$. Using these weights one sees that any $\nu \in \mathbb{X}^+$ can be written (possibly non uniquely) as a sum $\nu = \lambda + p\mu$ with $\lambda \in \mathbb{X}^+_{\mathrm{res}}$ and $\mu \in \mathbb{X}^+$. Therefore, in this case, applying Theorem 2.9 repeatedly reduces the description of all simple $\mathbf{G}$-modules to the description of those associated with restricted dominant weights. In particular, if $\mathbf{G}$ is semisimple (and simply connected) there exists a finite number of restricted dominant weights, so that only finitely many simple modules have to be considered.

[5] Let us recall that this condition is not really a restriction, since for any $\mathbf{G}$ there exists a finite central isogeny $\mathbf{G}' \to \mathbf{G}$ where $\mathbf{G}$ has simply connected derived subgroup; see e.g. [J3, §II.1.17].

EXAMPLE 2.10. In case $\mathbf{G} = \mathrm{SL}_n(\mathbb{k})$, there exists a canonical isomorphism of $\mathbb{k}$-algebraic groups

$$\mathbf{G}^{(1)} \cong \mathrm{SL}_n(\mathbb{k})$$

under which the Frobenius morphism $\mathrm{Fr}_{\mathbf{G}}$ identifies with the morphism sending a matrix $(a_{i,j})_{1 \leq i,j \leq n}$ to the matrix $(a_{i,j}^p)_{1 \leq i,j \leq n}$. With the notation of Example 1.1, we have

$$\mathbb{X}^+_{\mathrm{res}} = \{a_1\varpi_1 + \cdots + a_{n-1}\varpi_{n-1} : a_1, \cdots, a_{n-1} \in \{0, \cdots, p-1\}\}.$$

In case $n = 2$, it is easily seen that for any $a \in \{0, \cdots, p-1\}$ we have $\mathsf{L}(a\varpi_1) = \mathsf{N}(a\varpi_1)$, and this module is described in §1.4.1. For a general $a \in \mathbb{Z}_{\geq 0}$, writing

$$a = a^{(0)} + a^{(1)}p + \cdots + a^{(r)}p^r$$

with each $a^{(i)}$ in $\{0, \cdots, p-1\}$, we therefore have

$$\mathsf{L}(a\varpi_1) = \mathsf{N}(a^{(0)}\varpi_1) \otimes \mathsf{N}(a^{(1)}\varpi_1)^{(1)} \otimes \cdots \otimes \mathsf{N}(a^{(r)}\varpi_1)^{(r)}$$

where $(-)^{(s)}$ means the pullback under the morphism $(a_{i,j}) \mapsto \big((a_{i,j})^{p^s}\big)$. See Exercise 1.1 for more details.

Theorem 2.9 is the first statement so far were the characteristic p of $\mathbb{k}$ appears. This statement shows that simple $\mathbf{G}$-modules *do* depend on p in an essential way if we use the parametrization by dominant weights. "Independence of p" phenomena for simple modules (as discussed in §1.1) can be expected, but they will be more subtle than what we have encountered so far, and will require a different parametrization of these modules, which will be introduced in the next subsections.

2.5. The affine Weyl group and the linkage principle. Recall the sublattice $\mathbb{Z}\mathfrak{R} \subset \mathbb{X}$ generated by $\mathfrak{R}$. The *affine Weyl group* is the semi-direct product

$$W_{\mathrm{aff}} = W \ltimes \mathbb{Z}\mathfrak{R}.$$

For $\lambda \in \mathbb{Z}\mathfrak{R}$, we will denote by t_λ the corresponding element in W_{aff}. We will consider the *dot action* of W_{aff} on $\mathbb{X}$, defined by

$$(wt_\lambda) \cdot_p \mu = w(\mu + p\lambda + \rho) - \rho$$

for $\lambda \in \mathbb{Z}\mathfrak{R}$, $\mu \in \mathbb{X}$ and $w \in W$. (As noted in §1.9, the right-hand side indeed belongs to $\mathbb{X}$. Note also that the restriction of this action to W coincides with the action $\bullet$ considered above.)

A fundamental result is the following statement, called the *linkage principle* and due to Jantzen (under certain assumptions) and Andersen (in full generality).

THEOREM 2.11. *For $\lambda, \mu \in \mathbb{X}^+$, if $\mathrm{Ext}^1_{\mathsf{Rep}(\mathbf{G})}(\mathsf{L}(\lambda), \mathsf{L}(\mu)) \neq 0$ then $W_{\mathrm{aff}} \cdot_p \lambda = W_{\mathrm{aff}} \cdot_p \mu$.*

For the proof of this theorem (and in fact, of a more precise version known as the *strong* linkage principle, discussed in §2.6 below), we refer to [**A1**], see [**J3**, Corollary II.6.17]. The proof in full generality is subtle, but one can give a simple proof under the following assumptions:

- the derived subgroup $\mathscr{D}(\mathbf{G})$ is simply connected;
- the quotient $\mathbb{X}/\mathbb{Z}\mathfrak{R}$ has no p-torsion.

In fact, under the first assumption one can describe an important subalgebra of the center of the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$, as follows. Consider the adjoint action of $\mathbf{G}$ on $\mathcal{U}\mathfrak{g}$, and denote by $(\mathcal{U}\mathfrak{g})^{\mathbf{G}}$ the fixed points for this action. It is clear that $(\mathcal{U}\mathfrak{g})^{\mathbf{G}}$ is a subalgebra of $\mathcal{U}\mathfrak{g}$, and since the differential of the $\mathbf{G}$-action is the action of $\mathfrak{g}$ given by $x \cdot y = xy - yx$ for $x \in \mathfrak{g}$ and $y \in \mathcal{U}\mathfrak{g}$ this subalgebra is contained in the center of $\mathcal{U}\mathfrak{g}$. Let us denote by $\mathfrak{t}$, resp. $\mathfrak{u}$, resp. $\mathfrak{u}^+$, the Lie algebra of $\mathbf{T}$, resp. $\mathbf{U}$, resp. $\mathbf{U}^+$. Then we have a triangular decomposition

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^+,$$

so that multiplication induces an isomorphism of $\Bbbk$-vector spaces

$$\mathcal{U}\mathfrak{u} \otimes \mathcal{U}\mathfrak{t} \otimes \mathcal{U}\mathfrak{u}^+ \xrightarrow{\sim} \mathcal{U}\mathfrak{g}.$$

Consider the natural algebra morphism $\mathcal{U}\mathfrak{u} \to \Bbbk$, resp. $\mathcal{U}\mathfrak{u}^+ \to \Bbbk$, sending each element of $\mathfrak{u}$, resp. $\mathfrak{u}^+$, to 0, and the induced map

$$\mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{t} = \mathrm{S}(\mathfrak{t}). \tag{2.5}$$

This map is *not* an algebra morphism, but an adaptation of classical results of Harish-Chandra in the analogous characteristic-0 setting (see [J2, §9.3]) shows that its restriction to $(\mathcal{U}\mathfrak{g})^{\mathbf{G}}$ is an injective algebra morphism, whose image can be described as follows. There exists a unique action $\bullet$ of the group W on $\mathfrak{t}^*$ such that

$$s_\alpha \bullet v = s_\alpha(v) - \alpha$$

for any $\alpha \in \mathfrak{R}^{\mathrm{s}}$, where in the right-hand side we consider the obvious action on $\mathfrak{t}^*$. This action is compatible with the action on $\mathbb{X}$ denoted with the same symbol in the sense that if $\lambda \in \mathbb{X}$ has differential $\xi \in \mathfrak{t}^*$, then for any $w \in W$ the differential of $w \bullet \lambda$ is $w \bullet \xi$. Consider the induced action on $\mathscr{O}(\mathfrak{t}^*) = \mathrm{S}(\mathfrak{t})$, and denote by $\mathrm{S}(\mathfrak{t})^{(W,\bullet)}$ its fixed points. Then the map (2.5) restricts to an algebra isomorphism

$$(\mathcal{U}\mathfrak{g})^{\mathbf{G}} \xrightarrow{\sim} \mathrm{S}(\mathfrak{t})^{(W,\bullet)}.$$

By definition the right-hand side is the algebra of functions on the quotient scheme $\mathfrak{t}^*/(W,\bullet)$ (an affine scheme); the datum of a $\Bbbk$-point in this scheme (i.e. of an element of the quotient set $\mathfrak{t}^*/(W,\bullet)$) is therefore equivalent to the datum of a character of $\mathrm{S}(\mathfrak{t})^{(W,\bullet)}$, hence of $(\mathcal{U}\mathfrak{g})^{\mathbf{G}}$.

If $V \in \mathsf{Rep}(\mathbf{G})$, one can consider the action of $\mathcal{U}\mathfrak{g}$ on V obtained by differentiation, and its restriction to $(\mathcal{U}\mathfrak{g})^{\mathbf{G}}$ (which is an action by morphisms of $\mathbf{G}$-modules). If $V = \mathsf{L}(\lambda)$ for some $\lambda \in \mathbb{X}^+$, since $\mathrm{End}_{\mathbf{G}}(V) = \Bbbk \cdot \mathrm{id}$ this action must be given by a character of $(\mathcal{U}\mathfrak{g})^{\mathbf{G}}$. Considering the action on the highest-weight line one sees that this character corresponds to the image of the differential of λ in $\mathfrak{t}^*/(W,\bullet)$.

Fix now $\lambda, \mu \in \mathbb{X}^+$ such that $\mathrm{Ext}^1_{\mathsf{Rep}(\mathbf{G})}(\mathsf{L}(\lambda), \mathsf{L}(\mu)) \neq 0$. Then $(\mathcal{U}\mathfrak{g})^{\mathbf{G}}$ must act on $\mathsf{L}(\lambda)$ and $\mathsf{L}(\mu)$ by the same character; hence there exists $w \in W$ such that $\lambda - w \cdot_p \mu$ has vanishing differential, i.e. belongs to $p\mathbb{X}$. On the other hand, recall the (scheme-theoretic) center $Z(\mathbf{G}) \subset \mathbf{G}$ considered in §1.7. In view of the decomposition (1.5), since $\mathrm{Ext}^1_{\mathsf{Rep}(\mathbf{G})}(\mathsf{L}(\lambda), \mathsf{L}(\mu)) \neq 0$ we must have $\lambda - \mu \in \mathbb{Z}\mathfrak{R}$. We therefore obtain that

$$\lambda - w \cdot_p \mu \in (p\mathbb{X}) \cap \mathbb{Z}\mathfrak{R}.$$

Now, under our second assumption above we have $(p\mathbb{X}) \cap \mathbb{Z}\mathfrak{R} = p\mathbb{Z}\mathfrak{R}$ (because multiplication by p is injective on $\mathbb{X}/\mathbb{Z}\mathfrak{R}$), hence $W_{\mathrm{aff}} \cdot_p \lambda = W_{\mathrm{aff}} \cdot_p \mu$.

REMARK 2.12. Recently, a new general proof of Theorem 2.11, based on the geometric Satake equivalence, has been obtained by G. Williamson and the author in [**RW3**].

Theorem 2.11 has strong consequences for the structure of $\mathsf{Rep}^\infty(\mathbf{G})$. Namely, if $c \in \mathbb{X}/(W_{\mathrm{aff}}, \cdot_p)$, we will denote by $\mathsf{Rep}^\infty(\mathbf{G})_c$ the Serre subcategory of $\mathsf{Rep}^\infty(\mathbf{G})$ consisting of modules all of whose simple subquotients have the form $\mathsf{L}(\lambda)$ with $\lambda \in c \cap \mathbb{X}^+$.

COROLLARY 2.13. *The assignment* $(M_c)_{c\in\mathbb{X}/(W_{\mathrm{aff}},\cdot_p)} \mapsto \bigoplus_c M_c$ *induces an equivalence of categories*

$$\prod_{c\in\mathbb{X}/(W_{\mathrm{aff}},\cdot_p)} \mathsf{Rep}^\infty(\mathbf{G})_c \xrightarrow{\sim} \mathsf{Rep}^\infty(\mathbf{G}).$$

In more concrete terms, this corollary says that any object in $\mathsf{Rep}^\infty(\mathbf{G})$ decomposes in a canonical way as a direct sum of objects in the subcategories $\mathsf{Rep}^\infty(\mathbf{G})_c$, and that any morphism between such modules is a direct sum of morphisms between the components in these subcategories. This statement is an essentially immediate consequence of Theorem 2.11; for details, see [**J3**, §II.7.1]. Below we will also consider the restriction of this decomposition to $\mathsf{Rep}(\mathbf{G})$. For $c \in \mathbb{X}/(W_{\mathrm{aff}}, \cdot_p)$ we will denote by $\mathsf{Rep}(\mathbf{G})_c$ the Serre subcategory of $\mathsf{Rep}(\mathbf{G})$ generated by the simple objects $\mathsf{L}(\lambda)$ where $\lambda \in c \cap \mathbb{X}^+$; then we have

$$\mathsf{Rep}(\mathbf{G}) = \bigoplus_{c\in\mathbb{X}/(W_{\mathrm{aff}},\cdot_p)} \mathsf{Rep}(\mathbf{G})_c. \tag{2.6}$$

REMARKS 2.14.
(1) If $\lambda \in \mathbb{X}^+$, then $\mathsf{M}(\lambda)$ and $\mathsf{N}(\lambda)$ are indecomposable; they therefore belong to $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}}\cdot_p\lambda}$.
(2) The subcategory $\mathsf{Rep}(\mathbf{G})_c$ is often called "the block of c," even though this is not a block in the strict sense in general (that is, sometimes it can be decomposed as a direct sum in a nontrivial way). For more details on this question, see [**J3**, §II.7.2].
(3) The decomposition in Corollary 2.13 refines the decomposition (1.5) in the sense that any W_{aff}-orbit in $\mathbb{X}$ is included in a (unique) coset in $\mathbb{X}/\mathbb{Z}\mathfrak{R}$, and that for any $x \in \mathbb{X}/\mathbb{Z}\mathfrak{R}$ we have

$$\mathsf{Rep}(\mathbf{G})_{Z=x} = \bigoplus_{\substack{c\in\mathbb{X}/(W_{\mathrm{aff}},\cdot_p)\\ c\subset x}} \mathsf{Rep}(\mathbf{G})_c.$$

In particular, Corollary 2.13 shows that at the level of Grothendieck groups we have

$$[\mathsf{Rep}(\mathbf{G})] = \bigoplus_{c\in\mathbb{X}/(W_{\mathrm{aff}},\cdot_p)} [\mathsf{Rep}(\mathbf{G})_c].$$

In terms of the bases considered in §1.8, the subfamilies $([\mathsf{L}(\lambda)] : \lambda \in c \cap \mathbb{X}^+)$ and $([\mathsf{N}(\lambda)] : \lambda \in c \cap \mathbb{X}^+)$ both form bases of the summand $[\mathsf{Rep}(\mathbf{G})_c]$, for any $c \in \mathbb{X}/(W_{\mathrm{aff}}, \cdot_p)$.

2.6. Highest weight structure on blocks and the strong linkage principle. Recall from Theorem 2.3 that the category $\mathsf{Rep}(\mathbf{G})$ has a natural structure of highest weight category, with weight poset $(\mathbb{X}^+, \preceq)$. As explained in Remark 2.14(1), the objects $\mathsf{M}(\lambda)$ and $\mathsf{N}(\lambda)$ belong to the block $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}}\cdot_p\lambda}$, for any $\lambda \in \mathbb{X}^+$. From this it is easily seen that for any $c \in \mathbb{X}/(W_{\mathrm{aff}}, \cdot_p)$, the category $\mathsf{Rep}(\mathbf{G})_c$ also has a structure of highest weight category, with weight poset $c \cap \mathbb{X}^+$ (for the order obtained by restricting $\preceq$), standard objects $(\mathsf{M}(\lambda) : \lambda \in c \cap \mathbb{X}^+)$, and costandard objects $(\mathsf{N}(\lambda) : \lambda \in c \cap \mathbb{X}^+)$. (The proof of this claim is identical to that of Lemma 1.3 in Appendix A.)

The *strong* linkage principle provides a refinement of this claim. Here we will mainly consider a special case of this result which we first state; for the full statement, see Remark 2.16 below. The *reflections* in W_{aff} are the elements of the form $t_{r\beta} s_\beta$ with $\beta \in \mathfrak{R}$ and $r \in \mathbb{Z}$. We define a new order $\uparrow$ on $\mathbb{X}$ by declaring that $\lambda \uparrow \mu$ if there exist reflections $s_1, \cdots, s_n$ such that

$$\lambda \preceq s_1 \cdot_p \lambda \preceq s_2 s_1 \cdot_p \lambda \preceq \cdots \preceq (s_{n-1} \cdots s_1) \cdot_p \lambda \preceq (s_n \cdots s_1) \cdot_p \lambda = \mu.$$

Of course, when two elements are comparable for this order they belong to the same $(W_{\mathrm{aff}}, \cdot_p)$-orbit.

The following statement was conjectured by Verma, and first proved in full generality by Andersen; see [**J3**, Proposition II.6.13]. (See [**J3**, Chap. II.6] for historical remarks and references.)

Proposition 2.15 (The strong linkage principle). *If $\lambda, \mu \in \mathbb{X}^+$ and $\mathsf{L}(\lambda)$ is a composition factor of $\mathsf{N}(\mu)$, then $\lambda \uparrow \mu$.*

Using (1.6), we see that the statement of Proposition 1.6 also holds with $\mathsf{N}(\mu)$ replaced by $\mathsf{M}(\mu)$. As a consequence, using Remark 2.4 in Appendix A one sees that for any orbit $c \subset \mathbb{X}^+$ the category $\mathsf{Rep}(\mathbf{G})_c$ has a highest weight structure for the order $\uparrow$ on $c \cap \mathbb{X}^+$ (with the same standard and costandard objects as above). We will explain a different (and, in a sense, more explicit) description of the intersections $c \cap \mathbb{X}^+$ and the order $\uparrow$ on it in §2.8 below.

Remark 2.16. The strong linkage principle as proved by Andersen in [**A1**] and presented in [**J3**, §§II.6.13–16] is in fact a more general statement, which gives information on all modules $R^i\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(\lambda))$ with $\lambda \in \mathbb{X}$ and $i \in \mathbb{Z}$. Namely, as in §2.3, each weight λ can be written in the form $w \cdot_p \mu$ with $w \in W$ and $\mu \in \mathbb{X}$ which satisfies $\langle \mu, \alpha^\vee \rangle \geq -1$ for all $\alpha \in \mathfrak{R}^{\mathrm{s}}$. Here μ is uniquely determined, but w is determined only up to multiplication on the right by an element in the stabilizer of μ for the action of W via $\cdot_p$ (a parabolic subgroup of W). Any composition factor of $R^i\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(\lambda))$ is of the form $\mathsf{L}(\nu)$ with ν satisfying $\nu \uparrow \mu$. In case $\mu \notin \mathbb{X}^+$, of course we must have $\nu \neq \mu$; if $\mu \in \mathbb{X}^+$ then we have

$$[R^i\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(\lambda)) : \mathsf{L}(\mu)] = \begin{cases} 1 & \text{if } i = \ell(w); \\ 0 & \text{otherwise.} \end{cases}$$

See [**J3**, Propositions II.6.15–16] for details.

We do not know any alternative proof of this statement, nor do we understand its categorical meaning. See Exercise 1.20 for a proof of a weaker statement regarding the higher induced modules $R^i\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(\lambda))$ based on the same considerations as in the proof of the linkage principle in §2.5.

2.7. Translation functors.

2.7.1. *Definition.* For $\lambda \in \mathbb{X}$, we will denote by

$$\mathrm{pr}_\lambda : \mathsf{Rep}(\mathbf{G}) \to \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}$$

the functor of projection on the summand $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}$ in the decomposition (2.6).

DEFINITION 2.17. Let $\lambda, \mu \in \mathbb{X}$, and let ν be the unique dominant weight in $W(\mu - \lambda)$. The translation functor from $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}$ to $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \mu}$ is the functor

$$T_\lambda^\mu := \mathrm{pr}_\mu\big(\mathsf{L}(\nu) \otimes (-)\big) : \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda} \to \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \mu}.$$

REMARK 2.18. In the definition of T_λ^μ, if one replaces the module $\mathsf{L}(\nu)$ by any $M \in \mathsf{Rep}(\mathbf{G})$ such that $\dim(M_\nu) = 1$ and $\mathrm{wt}(M) \subset \{\eta \in \mathbb{X} \mid \eta \preceq \nu\}$, then one obtains an isomorphic functor; see [**J3**, Remark II.7.6]. For this reason, we find it useful to consider that translation functors are only defined "up to isomorphism," and that there is no prefered choice of these functors among their isomorphism classes.

The following claim gathers easy (though important) properties of the translation functors, whose proofs are easy. For details, see [**J3**, §II.7.6].

PROPOSITION 2.19. *Let $\lambda, \mu \in \mathbb{X}$.*

(1) For any $w \in W_{\mathrm{aff}}$ we have $T_\lambda^\mu \cong T_{w \cdot_p \lambda}^{w \cdot_p \mu}$.
(2) The functor T_λ^μ is exact.
(3) The functor T_λ^μ is both left and right adjoint to T_μ^λ.

REMARK 2.20. Even if one wants to ignore the comments in Remark 2.18, and consider that T_λ^μ is canonically defined using the simple module $\mathsf{L}(\nu)$, the adjointness in Proposition 2.19(3) is not canonical: it depends on a choice of isomorphism $\mathsf{L}(\nu)^* \cong \mathsf{L}(-w_0(\nu))$ (where ν is the only dominant W-translate of $\mu - \lambda$). Such an isomorphism exists (see (1.4)), and is unique up to scalar, but there does not exist any canonical choice for it in general.

2.7.2. *Alcove geometry.* In order to state more subtle properties of the translation functors, we will need to introduce the system of *facets* in the real vector space

$$\mathbf{V} := \mathbb{X} \otimes_{\mathbb{Z}} \mathbb{R}.$$

The same formula as for the dot-action $\cdot_p$ on $\mathbb{X}$ defines an action of W_{aff} on $\mathbf{V}$ which stabilizes $\mathbb{X}$, and which will be denoted similarly. A facet is any nonempty subset of $\mathbf{V}$ of the form

$$F = \{\lambda \in \mathbf{V} \mid \forall \alpha \in \mathfrak{R}_0^+,\ \langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha p \\ \text{and } \forall \alpha \in \mathfrak{R}_1^+,\ (n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle < n_\alpha p\}$$

for some partition $\mathfrak{R}^+ = \mathfrak{R}_0^+ \sqcup \mathfrak{R}_1^+$ and some integers $n_\alpha \in \mathbb{Z}$. (We insist that a subset defined by such conditions may well be empty; we only consider *nonempty* subsets of this form.) Equivalently, a facet is a connected component of the complement in an intersection of hyperplanes

$$H_{\alpha,n} = \{\lambda \in \mathbf{V} \mid \langle \lambda + \rho, \alpha^\vee \rangle = np\}$$

for $(\alpha, n) \in \mathfrak{R}^+ \times \mathbb{Z}$ of all strictly smaller intersections of such hyperplanes. A facet determined by a partition $\mathfrak{R}^+ = \mathfrak{R}_0^+ \sqcup \mathfrak{R}_1^+$ is called an *alcove* if $\mathfrak{R}_0^+ = \varnothing$, and a *wall* if $\#\mathfrak{R}_0^+ = 1$.

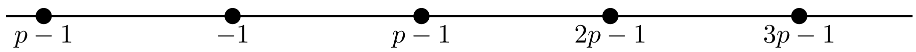


FIGURE 2.1. Weights and facets for SL_2

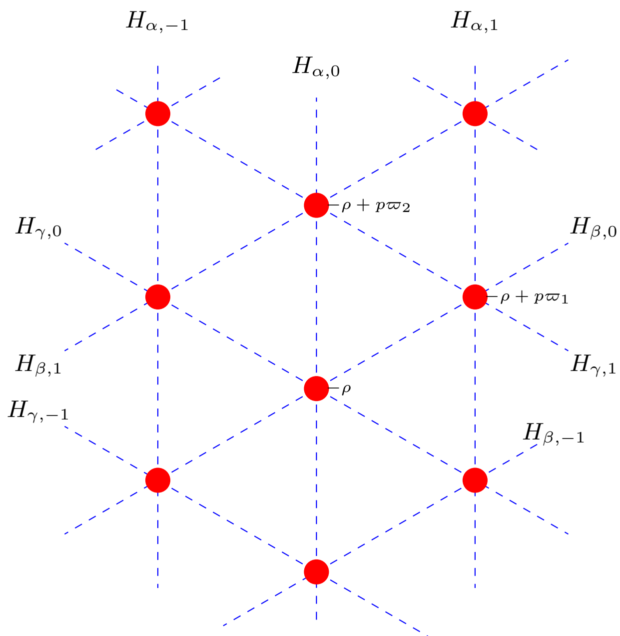


FIGURE 2.2. Weights and facets for SL_3

The space $\mathbf{V}$ is the disjoint union of all facets, and the alcoves are the connected components of

$$\mathbf{V} \smallsetminus \left(\bigcup_{\substack{\alpha \in \mathfrak{R}^+ \\ n \in \mathbb{Z}}} H_{\alpha,n} \right).$$

If F is a wall, defined by a partition $\mathfrak{R}^+ = \mathfrak{R}_0^+ \sqcup \mathfrak{R}_1^+$ and integers $(n_\alpha : \alpha \in \mathfrak{R}^+)$, then we can associate to F a reflection $s_F \in W_{\mathrm{aff}}$ as follows. Let β be the unique element in $\mathfrak{R}_0^+$, and let $n = n_\beta$ be the corresponding integer. Then

$$s_F = t_{n\beta} s_\beta.$$

With this definition, the fixed points of the action of s_F on $\mathbf{V}$ consist of the unique affine hyperplane containing F.

EXAMPLE 2.21. In case $\mathbf{G} = \mathrm{SL}_2(\Bbbk)$, recall that we have a canonical identification $\mathbb{X} = \mathbb{Z}$. The decomposition of $\mathbf{V} = \mathbb{R}$ is given in Figure 2.1. Namely, each facet is either an alcove or a wall. Alcoves are intervals of the form $(np-1, (n+1)p-1)$ with $n \in \mathbb{Z}$, and walls are singletons $\{np-1\}$ with $n \in \mathbb{Z}$.

EXAMPLE 2.22. In case $\mathbf{G} = \mathrm{SL}_3(\Bbbk)$, the decomposition of the plane $\mathbf{V}$ into facets is illustrated in Figure 2.2. (Here we follow the notation of Example 1.1, with $\alpha = \epsilon_1 - \epsilon_2$, $\beta = \epsilon_2 - \epsilon_3$ and $\gamma = \alpha + \beta = \epsilon_1 - \epsilon_3$.) Each facet is either an alcove, a wall or a singleton. The singleton facets are the red points. The walls are the blue intervals between red dots. The alcoves are the triangles delimited by the walls.

The following properties are standard, but very important. They follow from the general theory of discrete groups generated by affine reflections; see [**J3**, §6.2–6.3] for references.

LEMMA 2.23. *(1) The action of W_{aff} on $\mathbf{V}$ via $\cdot_p$ induces a simply transitive action on the set of alcoves.*

(2) For any alcove A, the closure $\overline{A}$ of A for the standard metric topology is a fundamental domain for the action of W_{aff} on $\mathbf{V}$.

(3) If A is an alcove, and if we denote by $\Sigma(A) \subset W_{\mathrm{aff}}$ the subset consisting of the reflections s_F where F is a wall contained in $\overline{A}$, then $(W_{\mathrm{aff}}, \Sigma(A))$ is a Coxeter system.

(4) If A is an alcove and $x \in \overline{A}$, then the stabilizer $\mathrm{Stab}_{(W_{\mathrm{aff}},\cdot_p)}(x)$ of x in W_{aff} (for the action $\cdot_p$) is generated by the subset $S_x \subset \Sigma(A)$ of reflections s_F where F is a wall contained in $\overline{A}$ and containing x in its closure. Moreover, the pair $(\mathrm{Stab}_{(W_{\mathrm{aff}},\cdot_p)}(x), S_x)$ is a Coxeter system.

A particularly important example of an alcove is the "fundamental alcove," defined as

$$C = \{v \in \mathbf{V} \mid \forall\, \alpha \in \mathfrak{R}^+,\ 0 < \langle v + \rho, \alpha^\vee \rangle < p\}.$$

The corresponding subset of Coxeter generators of W_{aff} will be denoted

$$S_{\mathrm{aff}} := \Sigma(C).$$

(This is in fact the only set of Coxeter generators of W_{aff} which will be considered below.) It can be checked that S_{aff} does not depend on p: in fact it is the union of S and the set of elements $t_\beta s_\beta$ where $\beta \in \mathfrak{R}^+$ is a maximal short root. We have $S \subset S_{\mathrm{aff}}$, so that W identifies with a parabolic subgroup in $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$. The choice of this set of Coxeter generators determines a Bruhat order and a length function on W_{aff}. This function has an explicit description which builds on work of Iwahori–Matsumoto [**IM**]: for $w \in W$ and $\lambda \in \mathbb{Z}\mathfrak{R}$ we have

$$\ell(w \cdot t_\lambda) = \sum_{\substack{\alpha \in \mathfrak{R}^+ \\ w(\alpha) \in \mathfrak{R}^+}} |\langle \lambda, \alpha^\vee \rangle| + \sum_{\substack{\alpha \in \mathfrak{R}^+ \\ w(\alpha) \in -\mathfrak{R}^+}} |1 + \langle \lambda, \alpha^\vee \rangle|. \tag{2.7}$$

What is particularly important for the study of $\mathsf{Rep}(\mathbf{G})$ is not really $\mathbf{V}$, but rather its subset $\mathbb{X}$. Since $\overline{C}$ is a fundamental domain for the action of W_{aff} on $\mathbf{V}$ (see Lemma 2.23(2)), the intersection $\overline{C} \cap \mathbb{X}$ is a fundamental domain for the action of W_{aff} on $\mathbb{X}$. In particular, the decomposition in Corollary 2.13 can be now written as

$$\mathsf{Rep}(\mathbf{G}) = \bigoplus_{\lambda \in \overline{C} \cap \mathbb{X}} \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}. \tag{2.8}$$

The closure $\overline{C}$ decomposes as a disjoint union of facets, but it is *not* the case that every facet contained in $\overline{C}$ intersects $\mathbb{X}$. This question already occurs in the case of the facet C; in this case, it is a standard fact that the following conditions are equivalent:

$\mathbf{A}_n$	$\mathbf{B}_n$, $\mathbf{C}_n$	$\mathbf{D}_n$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$	$\mathbf{F}_4$	$\mathbf{G}_2$
$n+1$	$2n$	$2n-2$	12	18	30	9	4

FIGURE 2.3. Coxeter numbers of irreducible root systems

(1) $C \cap \mathbb{X} \neq \varnothing$;
(2) $0 \in C$;
(3) some alcove contains an element in $\mathbb{X}$;
(4) for any alcove A we have $A \cap \mathbb{X} \neq \varnothing$;
(5) $p \geq h$, where
$$h := \max\{\langle \rho, \beta^\vee \rangle + 1 : \beta \in \mathfrak{R}^+\}$$
is the Coxeter number of $\mathfrak{R}$.

See [**J3**, §II.6.2] for more details on this question.

Explicitly, the Coxeter numbers of the indecomposable root systems are given by the table in Figure 2.3. In general, the Coxeter number of a root system is the maximum of the Coxeter numbers of its indecomposable factors.

REMARK 2.24. The facets contained in $\overline{C}$ are in bijection with the subsets of S_{aff} which generate a finite subgroup of W_{aff}, via the operation sending a facet to the set of elements of S_{aff} which fix it pointwise. (Such subsets of S_{aff} are sometimes called *finitary*.)

If $\mu \in \mathbb{X}$, we will say that μ is *regular* if μ belongs to an alcove, or in other words if its stabilizer in W_{aff} is trivial, or in other words if $p \nmid \langle \lambda + \rho, \alpha^\vee \rangle$ for any $\alpha \in \mathfrak{R}$. As explained above, such weights exist iff $p \geq h$. A weight which is not regular will be called *singular*.

REMARK 2.25. In addition to the affine Weyl group W_{aff}, it is sometimes useful to consider the *extended* affine Weyl group
$$W_{\mathrm{ext}} := W \ltimes \mathbb{X},$$
in which W_{aff} is a normal subgroup. The formula for the dot-action of W_{aff} on $\mathbb{X}$ makes sense more generally for elements in W_{ext}, and defines an action of this group on $\mathbb{X}$, and also on $\mathbf{V}$. It is easily seen that this action sends each facet to a facet, and in particular each alcove to an alcove. If we set
$$\Omega := \{w \in W_{\mathrm{ext}} \mid w \cdot_p C = C\},$$
then conjugation by Ω preserves S_{aff}, hence acts on W_{aff} by Coxeter group automorphisms, and multiplication induces an isomorphism
$$W_{\mathrm{aff}} \rtimes \Omega \xrightarrow{\sim} W_{\mathrm{ext}}.$$
This can be used to extend the length function ℓ and the Bruhat order $\leq$ on W_{aff} (see §4.1 below for details) to W_{ext}, by defining $\ell(w\omega) = \ell(w)$ for $w \in W_{\mathrm{aff}}$ and $\omega \in \Omega$, and $w\omega \leq w'\omega'$ iff $\omega = \omega'$ and $w \leq w'$ for $w, w' \in W_{\mathrm{aff}}$ and $\omega, \omega' \in \Omega$. (The same formulas with the order of terms inverted then also hold.) With this extension, formula (2.7) holds for any $\lambda \in \mathbb{X}$.

2.7.3. *Image of standard, costandard, and simple modules.* If F is a facet, determined by a partition $\mathfrak{R}^+ = \mathfrak{R}_0^+ \sqcup \mathfrak{R}_1^+$ and some integers $n_\alpha \in \mathbb{Z}$, then the closure of F is

$$\overline{F} = \{\lambda \in \mathbf{V} \mid \forall \alpha \in \mathfrak{R}_0^+,\ \langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha p \\ \text{and } \forall \alpha \in \mathfrak{R}_1^+,\ (n_\alpha - 1)p \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq n_\alpha p\}.$$

The *upper closure* of F is the union of facets defined by

$$\widehat{F} = \{\lambda \in \mathbf{V} \mid \forall \alpha \in \mathfrak{R}_0^+, \langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha p \\ \text{and } \forall \alpha \in \mathfrak{R}_1^+,\ (n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle \leq n_\alpha p\}.$$

This notion is crucial for the following statement.

PROPOSITION 2.26. *Let $\lambda, \mu \in \overline{C}$.*

(1) If λ and μ belong to the same facet, then T_λ^μ and T_μ^λ induce quasi-inverse equivalences of categories between $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}$ and $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \mu}$.

(2) Assume that μ belongs to the closure of the facet containing λ. Let $w \in W_{\mathrm{aff}}$ be such that $w \cdot_p \lambda \in \mathbb{X}^+$, and let F be the facet of $w \cdot_p \lambda$. We have

$$T_\lambda^\mu(\mathsf{M}(w \cdot_p \lambda)) \cong \begin{cases} \mathsf{M}(w \cdot_p \mu) & \text{if } w \cdot_p \mu \in \mathbb{X}^+, \\ 0 & \text{otherwise}; \end{cases}$$

$$T_\lambda^\mu(\mathsf{N}(w \cdot_p \lambda)) \cong \begin{cases} \mathsf{N}(w \cdot_p \mu) & \text{if } w \cdot_p \mu \in \mathbb{X}^+, \\ 0 & \text{otherwise}; \end{cases}$$

$$T_\lambda^\mu(\mathsf{L}(w \cdot_p \lambda)) \simeq \begin{cases} \mathsf{L}(w \cdot_p \mu) & \text{if } w \cdot_p \mu \in \widehat{F}, \\ 0 & \text{otherwise}. \end{cases}$$

(3) Assume that $\lambda \in C$, and that μ belongs to a wall contained in $\overline{C}$, with associated reflection $s \in S_{\mathrm{aff}}$. Let $w \in W_{\mathrm{aff}}$ be such that $w \cdot_p \lambda \in \mathbb{X}^+$ and $w \cdot_p \lambda \prec ws \cdot_p \lambda$. Then $w \cdot_p \mu$ and $ws \cdot_p \lambda$ are dominant, and there exist short exact sequences

$$\mathsf{N}(w \cdot_p \lambda) \hookrightarrow T_\mu^\lambda(\mathsf{N}(w \cdot_p \mu)) \twoheadrightarrow \mathsf{N}(ws \cdot_p \lambda),$$
$$\mathsf{M}(ws \cdot_p \lambda) \hookrightarrow T_\mu^\lambda(\mathsf{M}(w \cdot_p \mu)) \twoheadrightarrow \mathsf{M}(w \cdot_p \lambda).$$

(4) More generally, let $w \in W_{\mathrm{aff}}$ such that $w \cdot_p \lambda \in \mathbb{X}^+$. Then $T_\lambda^\mu \mathsf{N}(w \cdot_p \lambda)$, resp. $T_\lambda^\mu \mathsf{M}(w \cdot_p \lambda)$, admits a filtration whose subquotients are the modules $\mathsf{N}(wx \cdot_p \mu)$, resp. $\mathsf{M}(wx \cdot_p \mu)$, where x runs over the elements of $\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\lambda)$ such that $wx \cdot_p \mu$ belongs to $\mathbb{X}^+$, each occurring once.

For (1), we refer to [**J3**, Proposition II.7.9]. For (2), see [**J3**, Proposition II.7.11] for the second isomorphism, and [**J3**, Proposition II.7.15] for the third one. The first isomorphism follows from the second one by duality (or can be proved by similar arguments). For (3), the first exact sequence is constructed in [**J3**, Proposition II.7.19]; the second one follows by duality (or, again, can be proved by the same considerations). For (4), see [**J3**, Proposition II.7.13].

REMARK 2.27. We have explained above that $C \cap \mathbb{X} \neq \varnothing$ iff $p \geq h$. Proposition 2.26(3) shows the importance of also having weights which belong to the walls contained in $\overline{C}$. As explained in [**J3**, §II.6.3], such weights always exist when $p \geq h$ and $\mathscr{D}(\mathbf{G})$ is simply connected.

Recall the decomposition (2.8). Proposition 2.26(1) shows that, in this decomposition, all factors corresponding to weights in a given facet give rise to equivalent categories. This shows that in order to understand the structure of $\mathsf{Rep}(\mathbf{G})$ it suffices, for any facet F contained in $\overline{C}$ and such that $F \cap \mathbb{X} \neq \varnothing$, to understand the category $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}$ for some choice of weight $\lambda \in F \cap \mathbb{X}$.

These statements also interact nicely with our strategy to describe characters of simple $\mathbf{G}$-modules (see §1.9). In fact, assume that for some $\lambda \in \mathbb{X}^+$ we can express $[\mathsf{L}(\lambda)]$ in the basis $([\mathsf{N}(\nu)] : \nu \in (W_{\mathrm{aff}} \cdot_p \lambda) \cap \mathbb{X}^+)$ of $[\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}]$, see §2.5. Then Proposition 2.26(2) allows to deduce the expansion of $[\mathsf{L}(\mu)]$ in the basis $([\mathsf{N}(\nu)] : \nu \in (W_{\mathrm{aff}} \cdot_p \mu) \cap \mathbb{X}^+)$ of $[\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \mu}]$, for any μ in the upper closure of the facet of λ. In particular, assuming that $p \geq h$ (so that $0 \in C$), if we can express $[\mathsf{L}(w \cdot_p 0)]$ in the basis $([\mathsf{N}(\nu)] : \nu \in (W_{\mathrm{aff}} \cdot_p 0) \cap \mathbb{X}^+)$ of $[\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p 0}]$ for any $w \in W_{\mathrm{aff}}$ such that $w \cdot_p 0 \in \mathbb{X}^+$, then we can deduce the similar expansions of all simple modules corresponding to a dominant weight which belongs to the upper closure of an alcove containing a point $w \cdot_p 0$ with $w \in W_{\mathrm{aff}}$ such that $w \cdot_p 0 \in \mathbb{X}^+$. In fact any dominant weight belongs to such an upper closure, hence we can compute the characters of *all* simple modules.

Recall (see §2.4) that (assuming that $\mathscr{D}(\mathbf{G})$ is simply connected) Steinberg's tensor product formula reduces the problem of computing characters of all simple $\mathbf{G}$-modules to computing the characters of simple modules corresponding to restricted dominant weights, i.e. dominant weights which belong to the region

$$\{\lambda \in \mathbf{V} \mid \forall \alpha \in \mathfrak{R}^{\mathrm{s}},\, 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq p\}. \tag{2.9}$$

This region is a union of facets. The considerations above show that if $p \geq h$, to compute (in theory) the characters of all simple $\mathbf{G}$-modules it therefore suffices to compute the characters of the finitely many simple $\mathbf{G}$-modules $\mathsf{L}(w \cdot_p 0)$ where $w \in W_{\mathrm{aff}}$ is such that $w \cdot_p 0$ is restricted dominant.

REMARK 2.28. There is a (sometimes important) subtlety in the restriction to restricted dominant weights: if $w \in W_{\mathrm{aff}}$ and $w \cdot_p 0$ is restricted dominant, it is *not* the case that in the expansion of $[\mathsf{L}(w \cdot_p 0)]$ in the basis $([\mathsf{N}(y \cdot_p 0)] : y \in W_{\mathrm{aff}})$ only elements y such that $y \cdot_p 0$ is restricted dominant can appear with nonzero coefficients.

2.8. Coxeter-theoretic parametrization of simple objects in blocks.

2.8.1. *Dominant weights in orbits.* If $c \subset \mathbb{X}$ is a W_{aff}-orbit, the simple objects in $\mathsf{Rep}(\mathbf{G})_c$ are naturally parametrized by $c \cap \mathbb{X}^+$. On the other hand, the behaviour of translation functors as described in Proposition 2.26, as well as many subsequent statements (as e.g. Lusztig's conjecture, see Conjecture 4.6), involve the group W_{aff} and its Coxeter generators S_{aff} (see §2.7). It is therefore important to understand the relation between these two parametrizations.

Recall (see e.g. [**H3**] or [**Mi**]) that if $(\mathcal{W}, \mathcal{S})$ is a Coxeter system, given any subset $I \subset \mathcal{S}$ the *standard parabolic subgroup* of $\mathcal{W}$ associated with I is the subgroup $\mathcal{W}_I$ generated by I; then the pair $(\mathcal{W}_I, I)$ is again a Coxeter system. Moreover, for any $w \in \mathcal{W}$, the left coset $w\mathcal{W}_I$ admits a unique element x of minimal length; it is characterized by the property that $\ell(xy) = \ell(x) + \ell(y)$ for any $y \in \mathcal{W}_I$; see [**Mi**, Lemma-Definition 5.12]. (In particular, x is also minimal in $w\mathcal{W}_I$ for the restriction of the Bruhat order.) We will say simply that x is minimal in $w\mathcal{W}_I$. In case $\mathcal{W}_I$ is finite, each coset $w\mathcal{W}_I$ also contains a unique element of maximal length, characterized by the property that $\ell(xy) = \ell(x) - \ell(y)$ for any $y \in \mathcal{W}_I$; we will say that this element is maximal in $w\mathcal{W}_I$. In fact, if x is the minimal element in $w\mathcal{W}_I$, and if w_I is the longest element in $\mathcal{W}_I$, then the maximal element in $w\mathcal{W}_I$ is xw_I. Similar comments apply to right cosets $\mathcal{W}_I w$ ($w \in \mathcal{W}$).

As an example of this setting, the finite Weyl group $W \subset W_{\mathrm{aff}}$ is a standard parabolic subgroup (associated with the subset $S \subset S_{\mathrm{aff}}$). The elements $w \in W_{\mathrm{aff}}$ which are minimal in Ww can be described explicitly (see e.g. [**AHR**, Lemma 6.1]): if $w = t_\lambda v$ with $\lambda \in \mathbb{Z}\mathfrak{R}$ and $v \in W$, then w is minimal in Ww if and only if

$$\lambda \in \mathbb{X}^+ \text{ and for any } \alpha \in \mathfrak{R}^+ \text{ s. t. } v^{-1}(\alpha) \in -\mathfrak{R}^+ \text{ we have } \langle \lambda, \alpha^\vee \rangle \geq 1. \tag{2.10}$$

We will denote by

$$^{\mathrm{f}}W_{\mathrm{aff}} \subset W_{\mathrm{aff}}$$

the subset of elements which satisfy this condition.

Let us now return to the question we wanted to consider. By Lemma 2.23(2), the intersection $\mathbb{X} \cap \overline{C}$ is a fundamental domain for the action of W_{aff} on $\mathbb{X}$; in other words, each orbit can be written as $W_{\mathrm{aff}} \cdot_p \mu$ for some unique $\mu \in \mathbb{X} \cap \overline{C}$. Fix some $\mu \in \mathbb{X} \cap \overline{C}$; by Lemma 2.23(4), the stabilizer $\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$ of μ is then the parabolic subgroup of W_{aff} associated with the subset of S_{aff} consisting of reflections fixing μ. As a consequence, if we denote by $W_{\mathrm{aff}}^{(\mu)} \subset W_{\mathrm{aff}}$ the subset of elements w which are maximal in $w\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$, we obtain a bijection

$$W_{\mathrm{aff}}^{(\mu)} \xrightarrow{\sim} W_{\mathrm{aff}} \cdot_p \mu$$

defined by $w \mapsto w \cdot_p \mu$.

The following statement explains, in terms of this parametrization, which elements of $W_{\mathrm{aff}} \cdot_p \mu$ are dominant weights.

Proposition 2.29. *If $w \in W_{\mathrm{aff}}^{(\mu)}$, the weight $w \cdot_p \mu$ is dominant if and only if $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$.*

In order to prove this proposition, we will need two lemmas. The first one only involves Coxeter combinatorics.

Lemma 2.30. *Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system. If $x \in \mathcal{W}$ and $s, r \in \mathcal{S}$ satisfy $rx < x$ and $rxs > xs$, then $rxs = x$.*

Proof. By [**H3**, Proposition in §5.9] we have either $rxs \leq x$ or $rxs \leq xs$. By assumption the second alternative is impossible; we must therefore have $rxs \leq x$. Now $\ell(rxs) = \ell(xs) + 1 \geq \ell(x)$, hence $rxs = x$, as desired. □

Now we set

$$D := \{x \in \mathbf{V} \mid \forall \alpha \in \mathfrak{R}^+, \ \langle x + \rho, \alpha^\vee \rangle > 0\}. \tag{2.11}$$

Then D is a union of facets; in particular, an alcove meets D if and only if it is contained in D.

Lemma 2.31. *For $w \in W_{\mathrm{aff}}$, we have $w \cdot_p C \subset D$ if and only if $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$.*

Proof. Let us fix $x \in C$, and write $w = t_\lambda v$ with $\lambda \in \mathbb{Z}\mathfrak{R}$ and $v \in W$. In view of the comments above and the characterization (2.10) of the property that w is minimal in Ww, what we have to prove is that $w \cdot_p x \in D$ if and only if $\lambda \in \mathbb{X}^+$ and for any $\alpha \in \mathfrak{R}^+$ such that $\langle \lambda, \alpha^\vee \rangle = 0$ we have $v^{-1}(\alpha) \in \mathfrak{R}^+$.

For any $\beta \in \mathfrak{R}^+$ we have

$$\langle w \cdot_p x + \rho, \beta^\vee \rangle = \langle v(x + \rho) + p\lambda, \beta^\vee \rangle = \langle x + \rho, v^{-1}(\beta)^\vee \rangle + p\langle \lambda, \beta^\vee \rangle.$$

If $\lambda \notin \mathbb{X}^+$, then there exists $\beta \in \mathfrak{R}^+$ such that $\langle \lambda, \beta^\vee \rangle \leq -1$. We have $\langle x + \rho, v^{-1}(\beta)^\vee \rangle < p$, so that $\langle w \cdot_p x + \rho, \beta^\vee \rangle < 0$, and therefore $w \cdot_p x \notin D$. On the other hand, if there exists $\beta \in \mathfrak{R}^+$ such that $\langle \lambda, \beta^\vee \rangle = 0$ and $v^{-1}(\beta) \in -\mathfrak{R}^+$, then since $\langle x + \rho, v^{-1}(\beta)^\vee \rangle < 0$, the formula above shows that $\langle w \cdot_p x + \rho, \beta^\vee \rangle < 0$, and therefore $w \cdot_p x \notin D$.

Now we assume that $\lambda \in \mathbb{X}^+$ and $v^{-1}(\alpha) \in \mathfrak{R}^+$ for any $\alpha \in \mathfrak{R}^+$ such that $\langle \lambda, \alpha^\vee \rangle = 0$. Then if $\beta \in \mathfrak{R}^+$ we have

$$\langle w \cdot_p x + \rho, \beta^\vee \rangle = \langle x + \rho, v^{-1}(\beta)^\vee \rangle + p\langle \lambda, \beta^\vee \rangle.$$

If $v^{-1}(\beta) \in \mathfrak{R}^+$ then we have $\langle x + \rho, v^{-1}(\beta)^\vee \rangle > 0$, so that $\langle w \cdot_p x + \rho, \beta^\vee \rangle > 0$. And if $v^{-1}(\beta) \in -\mathfrak{R}^+$ then $\langle \lambda, \beta^\vee \rangle \geq 1$; since $\langle x + \rho, v^{-1}(\beta)^\vee \rangle > -p$ we deduce that again $\langle w \cdot_p x + \rho, \beta^\vee \rangle > 0$. This implies that $w \cdot_p x \in D$, and finishes the proof. □

We can now prove Proposition 2.29.

Proof of Proposition 2.29. The proof is based on the observation that $w \cdot_p \mu \in \mathbb{X}^+$ if and only if $x(C) \subset D$ for any $x \in w\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$. By Lemma 2.31, this condition is equivalent to requiring that $w\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu) \subset {}^{\mathrm{f}}W_{\mathrm{aff}}$. Of course this implies that $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$. On the other hand, assume that $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ and that $wx \notin {}^{\mathrm{f}}W_{\mathrm{aff}}$ for some $x \in \mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$. Choose x of minimal length with this property; then there exists $s \in S_{\mathrm{aff}} \cap \mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$ such that $xs < x$. Since $wx \notin {}^{\mathrm{f}}W_{\mathrm{aff}}$, there exists $r \in S$ such that $rwx < wx$. On the other hand, by minimality we have $wxs \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, hence $rwxs > wxs$. By Lemma 2.30 these conditions imply that $rwxs = wx$, hence $wxs = rwx < wx$. But $\ell(wxs) = \ell(w) - \ell(xs) = \ell(w) - \ell(x) + 1 = \ell(wx) + 1$ by maximality; we have therefore reached a contradiction. □

Remark 2.32. In the course of the proof of Proposition 2.29 we have seen that if $w \in W_{\mathrm{aff}}^{(\mu)}$, we have $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ if and only if $w\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu) \subset {}^{\mathrm{f}}W_{\mathrm{aff}}$. This property is a special case of a general fact about coset representatives in Coxeter groups. For other characterizations of the elements satisfying these properties, see [**AR5**, Lemma 2.4]. Let us note that it is *not* the case that any double coset $Ww\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$ contains an element in ${}^{\mathrm{f}}W_{\mathrm{aff}} \cap W_{\mathrm{aff}}^{(\mu)}$.

We will set

$${}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)} := {}^{\mathrm{f}}W_{\mathrm{aff}} \cap W_{\mathrm{aff}}^{(\mu)}.$$

Proposition 2.29 then says that the assignment $w \mapsto w \cdot_p \mu$ defines a bijection

$$(2.12) \qquad {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)} \xrightarrow{\sim} (W_{\mathrm{aff}} \cdot_p \mu) \cap \mathbb{X}^+.$$

In this way, the simple objects in $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \mu}$ can be parametrized by ${}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)}$.

2.8.2. *Orders.* Fix again $\mu \in \overline{C} \cap \mathbb{X}$. As explained in §2.6, the category $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \mu}$ has a natural highest weight structure with underlying weight poset $((W_{\mathrm{aff}} \cdot_p \mu) \cap \mathbb{X}^+, \uparrow)$. It is therefore interesting to describe the transport along (2.12) of the restriction of the order $\uparrow$ to $(W_{\mathrm{aff}} \cdot_p \mu) \cap \mathbb{X}^+$. In fact we will now explain that this order is nothing but the restriction of the Bruhat order on W_{aff} to the subset ${}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)}$.

To check this, and in particular to compare this construction with others appearing in the literature, it is useful to recall another, related but different, notion of alcoves. (To distinguish the two cases, we will call these new objects “alcoves” with quotation marks.) Namely set $\mathbf{V}' := \mathbb{X} \otimes_{\mathbb{Z}} \mathbb{R}$, which we endow with the action of W_{aff} determined by $(t_\lambda v) \cdot x = v(x) + \lambda$ for $\lambda \in \mathbb{Z}\mathfrak{R}$ and $v \in W$, where in the right-hand side we consider the obvious action of W on $\mathbb{X} \otimes_{\mathbb{Z}} \mathbb{R}$. (The vector space $\mathbf{V}'$ therefore coincides with $\mathbf{V}$, but the actions of W_{aff} differ.) We will call "alcoves" the connected components of

$$\mathbf{V}' \smallsetminus \left(\bigcup_{\substack{\alpha \in \mathfrak{R}^+ \\ n \in \mathbb{Z}}} \{v \in \mathbf{V}' \mid \langle v, \alpha^\vee \rangle = n\} \right).$$

We have a fundamental "alcove" defined by

$$A := \{v \in \mathbf{V}' \mid \forall\, \alpha \in \mathfrak{R}^+,\, 0 < \langle v, \alpha^\vee \rangle < 1\},$$

and obvious analogues of the statements in Lemma 2.23 hold; in fact the map $v \mapsto -\rho + pv$ defines a W_{aff}-equivariant bijection $\mathbf{V}' \xrightarrow{\sim} \mathbf{V}$ which matches A with C and "alcoves" with alcoves. In particular, the action morphism $w \mapsto w \cdot A$ defines a bijection between W_{aff} and the set of "alcoves", and restricts (as in Lemma 2.31) to a bijection between ${}^{\mathrm{f}}W_{\mathrm{aff}}$ and the set of "alcoves" contained in the dominant Weyl chamber

$$\{x \in \mathbf{V}' \mid \forall \alpha \in \mathfrak{R}^+,\, \langle x, \alpha^\vee \rangle > 0\}.$$

Recall that if $(\mathcal{W}, \mathcal{S})$ is a Coxeter system, the *reflections* in $\mathcal{W}$ are the conjugates of the elements in $\mathcal{S}$. In the case when $(\mathcal{W}, \mathcal{S}) = (W_{\mathrm{aff}}, S_{\mathrm{aff}})$, the reflections are the elements of the form $t_{n\alpha} s_\alpha$ with $\alpha \in \mathfrak{R}^+$ and $n \in \mathbb{Z}$.

We have an order $\uparrow$ on the set of alcoves defined in [**J3**, §II.6.5] as follows: given $\alpha \in \mathfrak{R}^+$ and $n \in \mathbb{Z}$, if C_1 is an alcove then we have either $\langle x + \rho, \alpha^\vee \rangle < np$ for all $x \in C_1$, or $\langle x + \rho, \alpha^\vee \rangle > np$ for all $x \in C_1$. In the first case we set $C_1 \uparrow t_{n\alpha} s_\alpha \cdot_p C_1$, and in the second case we set $t_{n\alpha} s_\alpha \cdot_p C_1 \uparrow C_1$. Then if C_1, C_2 are alcoves we set $C_1 \uparrow C_2$ if and only if there exist reflections $s_1, \cdots, s_r$ such that

$$C_1 \uparrow s_1 \cdot_p C_1 \uparrow s_2 s_1 \cdot_p C_1 \uparrow \cdots \uparrow (s_r \cdots s_1) \cdot_p C_1 = C_2.$$

Comparing this definition with that given in [**S3**, p. 95], we see that the bijection considered above between "alcoves" and alcoves matches the order $\preceq$ (on "alcoves") from [**S3**] with $\uparrow$.

On the other hand, one can consider the "periodic order" $\preceq$ on W_{aff} considered in [**AR6**, §2.5].[6] Comparing the definition with [**S3**, Claim 4.14] we see that the bijection between W_{aff} and the set of "alcoves" defined by $w \mapsto w^{-1} \cdot A$ matches this periodic order with $\preceq$. Next, as explained in [**AR6**, Lemma 2.5(3)], the restriction of the periodic order to the subset of W_{aff} consisting of elements w which are minimal in wW coincides with the restriction of the Bruhat order. Given into account the fact that the bijection $w \mapsto w^{-1}$ matches the Bruhat order with itself, we have finally proved the following lemma.

LEMMA 2.33. *The assignment $w \mapsto w \cdot_p C$ identifies the restriction of the Bruhat order of W_{aff} to ${}^{\mathrm{f}}W_{\mathrm{aff}}$ with the restriction of the order $\uparrow$ on the set of alcoves to the subset of alcoves contained in D (see (2.11)).*

We deduce the following claim, that was announced above.

[6] In [**AR6**, §2.5] the order is defined on W_{ext}; more explicitly, what we consider here is the restriction of this order to W_{aff}. The definition of this order (with a different normalization) is due to Lusztig in [**L2**].

PROPOSITION 2.34. *For any $\mu \in \overline{C} \cap \mathbb{X}$, the bijection (2.12) matches the restriction of the Bruhat order to ${}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)}$ with the restriction of the order $\uparrow$ to $(W_{\mathrm{aff}} \cdot_p \mu) \cap \mathbb{X}^+$.*

PROOF. Let $x, y \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)}$. If $x \leq y$ for the Bruhat order, then by Lemma 2.33 we have $x \cdot_p C \uparrow y \cdot_p C$, which implies that $x \cdot_p \mu \uparrow y \cdot_p \mu$ by [**J3**, Equation (2) in §II.6.5]. On the other hand, assume that $x \cdot_p \mu \uparrow y \cdot_p \mu$. The alcove "$C^-$" associated with the facet containing $x \cdot_p \mu$ in [**J3**, §II.6.11] is $xw_{(\mu)} \cdot_p C$, where $w_{(\mu)}$ is the longest element in the parabolic subgroup $\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$, and similarly for $y \cdot_p \mu$. (This follows e.g. by comparing [**J3**, Equation (3) in §II.6.11] with Lemma 2.33.) In view of [**J3**, Equation (4) in §II.6.11], we therefore have $xw_{(\mu)} \cdot_p C \uparrow yw_{(\mu)} \cdot_p C$, hence $xw_{(\mu)} \leq yw_{(\mu)}$ in the Bruhat order. By Exercise 1.9, this implies that $x \leq y$, which finishes the proof. $\square$

REMARK 2.35. (1) In the special case when $p \geq h$ and $\mu = 0$, Proposition 2.34 already appeared in the literature, see [**AR3**, Lemma 10.1] for references. We do not know any reference for the general case.

(2) Assume that there exists a weight $\varsigma \in \mathbb{X}$ such that $\langle \varsigma, \alpha^\vee \rangle = 1$ for each $\alpha \in \mathfrak{R}^{\mathrm{s}}$. (Such a weight exists at least under the assumption that the derived subgroup $\mathscr{D}(\mathbf{G})$ is simply connected. In case $\mathbf{G}$ is semisimple such a weight is unique if it exists, and equal to ρ. For a general reductive group, there might exist several choices.) Then the weight $-\varsigma$ belongs to $\overline{C}$, and its stabilizer is W; we therefore have

$$W_{\mathrm{aff}} \cdot_p (-\varsigma) = -\varsigma + p\mathbb{Z}\mathfrak{R}.$$

In fact, in this case we have

$$ {}^{\mathrm{f}}W_{\mathrm{aff}}^{(-\varsigma)} = \{t_\lambda w_0 : \lambda \in \mathbb{Z}\mathfrak{R} \cap (\varsigma + \mathbb{X}^+)\}, \tag{2.13}$$

see [**AR5**, Lemma 2.5].

2.8.3. *Translation of standard, costandard, and simple modules (new version).* We can now translate Proposition 2.26 in Coxeter-theoretic terms. (Here, as in the proof of Proposition 2.34, we denote by $w_{(\mu)}$ the longest element in $\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$.)

PROPOSITION 2.36. *Let $\lambda, \mu \in \overline{C}$.*

(1) Assume that μ belongs to the closure of the facet containing λ, and let $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\lambda)}$. We have

$$T_\lambda^\mu(\mathsf{M}(w \cdot_p \lambda)) \cong \begin{cases} \mathsf{M}(w \cdot_p \mu) & \text{if } w\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu) \cap {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)} \neq \varnothing, \\ 0 & \text{otherwise}; \end{cases}$$

$$T_\lambda^\mu(\mathsf{N}(w \cdot_p \lambda)) \cong \begin{cases} \mathsf{N}(w \cdot_p \mu) & \text{if } w\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu) \cap {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)} \neq \varnothing, \\ 0 & \text{otherwise}; \end{cases}$$

$$T_\lambda^\mu(\mathsf{L}(w \cdot_p \lambda)) \simeq \begin{cases} \mathsf{L}(w \cdot_p \mu) & \text{if } ww_{(\mu)} \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)}, \\ 0 & \text{otherwise.} \end{cases}$$

(2) Assume that $\lambda \in C$, and that μ belongs to a wall contained in $\overline{C}$, with associated reflection $s \in S_{\mathrm{aff}}$. Let $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)}$. Then $w \cdot_p \lambda$, $w \cdot_p \mu$ and

$ws \cdot_p \lambda$ are dominant, and there exist short exact sequences

$$\mathsf{N}(ws \cdot_p \lambda) \hookrightarrow T_\mu^\lambda(\mathsf{N}(w \cdot_p \mu)) \twoheadrightarrow \mathsf{N}(w \cdot_p \lambda),$$
$$\mathsf{M}(w \cdot_p \lambda) \hookrightarrow T_\mu^\lambda(\mathsf{M}(w \cdot_p \mu)) \twoheadrightarrow \mathsf{M}(ws \cdot_p \lambda).$$

2.8.4. *Consequence for simple characters.* Using the Coxeter-theoretic parametrization of simple modules in a given block, one can also make the procedure of computing characters of simple modules in "singular" blocks from those of modules in a "regular" block (see §2.7) more explicit.

Namely, assume that $p \geq h$, so that $0 \in C \cap \mathbb{X}$. As explained above the simple, induced, and Weyl modules in $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p 0}$ can (and will) be parametrized by ${}^{\mathrm{f}}W_{\mathrm{aff}}^{(0)} = {}^{\mathrm{f}}W_{\mathrm{aff}}$. Consider the matrix $(a_{y,w} : y, w \in {}^{\mathrm{f}}W_{\mathrm{aff}})$ such that

$$[\mathsf{L}(w \cdot_p 0)] = \sum_{y \in {}^{\mathrm{f}}W_{\mathrm{aff}}} a_{y,w} \cdot [\mathsf{N}(y \cdot_p 0)] \tag{2.14}$$

in $[\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p 0}]$ for any $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$. Fix now $\mu \in \overline{C} \cap \mathbb{X}$. Each weight in $(W_{\mathrm{aff}} \cdot_p \mu) \cap \mathbb{X}^+$ belongs to the upper closure of exactly one alcove, which is moreover contained in the domain D of (2.11): explicitly, for $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)}$ the weight $w \cdot_p \mu$ is in the upper closure of $ww_{(\mu)} \cdot_p C$. (Here $ww_{(\mu)}$ is the *minimal* element in $w\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$). Applying the translation functor T_0^μ to the formula (2.14) and using Proposition 2.26 we obtain that for any $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)}$ we have

$$[\mathsf{L}(w \cdot_p \mu)] = \sum_{y \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)}} \left(\sum_{x \in \mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)} a_{yx, ww_{(\mu)}} \right) \cdot [\mathsf{N}(y \cdot_p \mu)].$$

REMARK 2.37. In case $p < h$, one can apply similar considerations to compute characters of simple modules in blocks corresponding to weights in the closure of a given facet contained in $\overline{C}$, if one knows the characters in the block of a weight in this facet. However, it is not clear how to determine the "most regular" weights in $\overline{C} \cap \mathbb{X}$ in general, and in any case these weights might belong to several different facets.

2.9. Some simple cases. In this subsection we explain how the characters of some simple modules can be easily computed.

First, let assume that $\mu \in \mathbb{X}^+$ is minimal (for the order $\uparrow$) in $(W_{\mathrm{aff}} \cdot_p \mu) \cap \mathbb{X}^+$. Then the linkage principle implies that the canonical morphisms

$$\mathsf{M}(\mu) \to \mathsf{L}(\mu) \to \mathsf{N}(\mu)$$

are isomorphisms. This happens for instance if $\mu \in \overline{C} \cap \mathbb{X}^+$. (See [J3, Corollary II.5.6] for a different proof of the simplicity of $\mathsf{N}(\mu)$ in this case, which does not use the linkage principle.) If there exists a weight $\varsigma \in \mathbb{X}$ such that $\langle \varsigma, \alpha^\vee \rangle = 1$ for each $\alpha \in \mathfrak{R}^{\mathrm{s}}$ (see Remark 2.35(2)) then this also applies to the weight $(p-1)\varsigma$, since we have

$$W_{\mathrm{ext}} \cdot_p (p-1)\varsigma = (p-1)\varsigma + p\mathbb{X},$$

hence

$$(W_{\mathrm{ext}} \cdot_p (p-1)\varsigma) \cap \mathbb{X}^+ = (p-1)\varsigma + p\mathbb{X}^+.$$

The modules $\mathsf{L}((p-1)\varsigma)$ are called the *Steinberg modules*. For some of their properties, see [**J3**, §§II.3.18–19].[7]

Next, let us assume that $p \geq h$ and that $\mathbf{G}$ is quasi-simple. In this case $S_{\mathrm{aff}} \smallsetminus S$ contains a unique element, which we will denote $s_\circ$. Consider the induced module $\mathsf{N}(s_\circ \cdot_p 0)$. We know that its socle is $\mathsf{L}(s_\circ \cdot_p 0)$, that this simple module appears only once as a composition factor of $\mathsf{N}(s_\circ \cdot_p 0)$, and that the only other possible composition factor is $\mathsf{L}(0)$. Since $\mathrm{Ext}^1_{\mathsf{Rep}(\mathbf{G})}(\mathsf{L}(0), \mathsf{L}(0)) = 0$, we deduce that there exists an exact sequence

$$\mathsf{L}(s_\circ \cdot_p 0) \hookrightarrow \mathsf{N}(s_\circ \cdot_p 0) \twoheadrightarrow \mathsf{L}(0)^{\oplus r}$$

for some $r \in \mathbb{Z}_{\geq 0}$. We then have

$$[\mathsf{N}(s_\circ \cdot_p 0)] = [\mathsf{L}(s_\circ \cdot_p 0)] + r \cdot [\mathsf{L}(0)].$$

If μ is a weight on the wall contained in $\overline{C}$ fixed by $s_\circ$, then applying the functor T_0^μ and using Proposition 2.26 we deduce that

$$[\mathsf{N}(\mu)] = r \cdot [\mathsf{L}(\mu)].$$

On the other hand we have $\mathsf{N}(\mu) = \mathsf{L}(\mu)$ since $\mu \in \overline{C}$, hence $r = 1$. This shows that $\mathsf{N}(s_\circ \cdot_p 0)$ sits in an exact sequence

$$\mathsf{L}(s_\circ \cdot_p 0) \hookrightarrow \mathsf{N}(s_\circ \cdot_p 0) \twoheadrightarrow \mathsf{L}(0).$$

EXAMPLE 2.38. In case $\mathbf{G} = \mathrm{SL}_3(\Bbbk)$, assuming that $p \geq 3$ the region (2.9) is the union of the closures of C and $s_\circ \cdot_p C$. In view of the considerations above, this shows that the problem of computing characters of simple modules can be considered solved in this case also.

2.10. The Steinberg (extended) block. In this subsection we assume (as in Remark 2.35(2)) that there exists a weight $\varsigma \in \mathbb{X}$ such that $\langle \varsigma, \alpha^\vee \rangle = 1$ for each $\alpha \in \mathfrak{R}^{\mathrm{s}}$. We consider the "extended block of $-\varsigma$," i.e. the Serre subcategory $\mathsf{Rep}_{\mathrm{Stein}}(\mathbf{G})$ generated by the simple modules whose highest weight belongs to

$$(W_{\mathrm{ext}} \cdot_p (-\varsigma)) \cap \mathbb{X}^+ = (-\varsigma + p\mathbb{X}) \cap \mathbb{X}^+ = (p-1)\varsigma + p\mathbb{X}^+.$$

(Here, the subscript "Stein" refers to Steinberg.) This subcategory is a direct summand in $\mathsf{Rep}(\mathbf{G})$, in fact it is a direct sum of some blocks in the decomposition (2.6). Note also that, as explained in §2.9, the canonical morphisms

$$\mathsf{M}((p-1)\varsigma) \to \mathsf{L}((p-1)\varsigma) \to \mathsf{N}((p-1)\varsigma)$$

are isomorphisms.

The following result is due to Andersen; for a proof, see [**J3**, Proposition II.3.19]. Here we consider the Frobenius twist $\mathbf{G}^{(1)}$ as in §2.4. Given $\lambda \in X^*(\mathbf{T}^{(1)})^+$, we will denote by $\mathsf{N}^{(1)}(\lambda)$, resp. $\mathsf{M}^{(1)}(\lambda)$, the associated induced, resp. Weyl, $\mathbf{G}^{(1)}$-module (defined with respect to the Borel subgroup $\mathbf{B}^{(1)}$).

PROPOSITION 2.39. *For any $\lambda \in X^*(\mathbf{T}^{(1)})^+$, there exist isomorphisms of $\mathbf{G}$-modules*

$$\mathsf{N}\big((p-1)\varsigma + \mathrm{Fr}^*_{\mathbf{T}}(\lambda)\big) \cong \mathsf{L}\big((p-1)\varsigma\big) \otimes \mathrm{Fr}^*_{\mathbf{G}}\big(\mathsf{N}^{(1)}(\lambda)\big),$$
$$\mathsf{M}\big((p-1)\varsigma + \mathrm{Fr}^*_{\mathbf{T}}(\lambda)\big) \cong \mathsf{L}\big((p-1)\varsigma\big) \otimes \mathrm{Fr}^*_{\mathbf{G}}\big(\mathsf{M}^{(1)}(\lambda)\big).$$

[7]Jantzen only considers the case $\varsigma = \rho$, assuming that $(p-1)\rho \in \mathbb{X}$. However, all the properties of the Steinberg module proved in [**J3**] also hold for the modules we consider here.

REMARK 2.40. Assume that $\mathbf{G} = \mathrm{SL}_2(\Bbbk)$, and recall the notation of Example 2.10. In this case, the first isomorphism in Proposition 2.39 takes the form

$$\mathsf{N}((p-1+pn)\varpi_1) \cong \mathsf{N}((p-1)\varpi_1) \otimes \mathsf{N}(n\varpi_1)^{(1)}$$

for $n \in \mathbb{Z}_{\geq 0}$. Explicitly, with the identifications of §1.4.1, this isomorphism is induced by the isomorphism

$$\Bbbk[X,Y]_{p-1} \otimes \Bbbk[X,Y]_n \xrightarrow{\sim} \Bbbk[X,Y]_{p-1+pn}$$

given by $P \otimes Q \mapsto P \cdot Q^p$. (To check that this indeed is an isomorphism, one uses the observation that if $a, b \in \mathbb{Z}_{\geq 0}$ have respective remainders r, s modulo p, and if $a + b \equiv p-1 \mod p$, then $r + s = p - 1$.)

This result has the following consequence.

COROLLARY 2.41. *The functor*

$$\mathsf{Rep}(\mathbf{G}^{(1)}) \to \mathsf{Rep}(\mathbf{G})$$

defined by $V \mapsto \mathsf{L}\big((p-1)\varsigma\big) \otimes \mathrm{Fr}_{\mathbf{G}}^*(V)$ *induces an equivalence of categories*

$$\mathsf{Rep}(\mathbf{G}^{(1)}) \xrightarrow{\sim} \mathsf{Rep}_{\mathrm{Stein}}(\mathbf{G}).$$

Moreover, this functor sends $\mathsf{N}^{(1)}(\lambda)$*, resp.* $\mathsf{M}^{(1)}(\lambda)$*, to* $\mathsf{N}\big((p-1)\varsigma + \mathrm{Fr}_{\mathbf{T}}^*(\lambda)\big)$*, resp.* $\mathsf{M}\big((p-1)\varsigma + \mathrm{Fr}_{\mathbf{T}}^*(\lambda)\big)$.

PROOF. Let us consider the induced functor on derived categories

$$\varphi : D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G}^{(1)}) \to D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G}).$$

By Proposition 2.39, this functor sends $\mathsf{N}^{(1)}(\lambda)$, resp. $\mathsf{M}^{(1)}(\lambda)$, to $\mathsf{N}\big((p-1)\varsigma + \mathrm{Fr}_{\mathbf{T}}^*(\lambda)\big)$, resp. $\mathsf{M}\big((p-1)\varsigma + \mathrm{Fr}_{\mathbf{T}}^*(\lambda)\big)$. Now by Theorem 2.3 and Corollary 2.3 from Appendix A, we have

$$\mathrm{Ext}^n_{\mathsf{Rep}(\mathbf{G}^{(1)})}(\mathsf{M}^{(1)}(\lambda), \mathsf{N}^{(1)}(\mu)) \cong \begin{cases} \Bbbk & \text{if } \lambda = \mu \text{ and } n = 0; \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Ext}^n_{\mathsf{Rep}(\mathbf{G})}\big(\mathsf{M}((p-1)\varsigma + \mathrm{Fr}_{\mathbf{T}}^*(\lambda)), \mathsf{N}((p-1)\varsigma + \mathrm{Fr}_{\mathbf{T}}^*(\mu))\big) \cong \begin{cases} \Bbbk & \text{if } \lambda = \mu \text{ and } n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for fixed $\mu \in X^*(\mathbf{T}^{(1)})^+$, our functor induces an isomorphism

$$\mathrm{Ext}^n_{\mathsf{Rep}(\mathbf{G}^{(1)})}(\mathsf{M}^{(1)}(\lambda), \mathsf{N}^{(1)}(\mu)) \xrightarrow{\sim} \mathrm{Ext}^n_{\mathsf{Rep}(\mathbf{G})}\big(\varphi(\mathsf{M}^{(1)}(\lambda)), \mathsf{N}((p-1)\varsigma + \mathrm{Fr}_{\mathbf{T}}^*(\mu))\big)$$

for any $n \in \mathbb{Z}$ and $\lambda \in X^*(\mathbf{T}^{(1)})^+$. Since the objects $(\mathsf{M}^{(1)}(\lambda) : \lambda \in X^*(\mathbf{T}^{(1)})^+)$ generate $D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G}^{(1)})$ as a triangulated category, we deduce that for any M in $D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G}^{(1)})$ our functor induces an isomorphism

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G}^{(1)})}(M, \mathsf{N}^{(1)}(\mu)) \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G})}\big(\varphi(M), \mathsf{N}((p-1)\varsigma + \mathrm{Fr}_{\mathbf{T}}^*(\mu))\big).$$

For fixed M in $D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G}^{(1)})$, using the fact that the objects $(\mathsf{M}^{(1)}(\mu) : \mu \in X^*(\mathbf{T}^{(1)})^+)$ generate $D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G}^{(1)})$ as a triangulated category, we deduce that for any N in $D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G}^{(1)})$ our functor induces an isomorphism

$$\mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G}^{(1)})}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G})}(\varphi(M), \varphi(N)),$$

i.e. that this functor is fully faithful. Since $D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G}^{(1)})$ is generated as a triangulated category by the objects $(\mathsf{L}^{(1)}(\mu) : \mu \in X^*(\mathbf{T}^{(1)})^+)$, the essential image of φ is the triangulated subcategory of $D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G})$ generated by the objects $(\varphi(\mathsf{L}^{(1)}(\mu)) : \mu \in X^*(\mathbf{T}^{(1)})^+)$, i.e. by the objects $\mathsf{L}((p-1)\varsigma + \mathrm{Fr}_{\mathbf{T}}^*(\lambda))$ (see Theorem 2.9), i.e. the full subcategory $D^{\mathrm{b}}\mathsf{Rep}_{\mathrm{Stein}}(\mathbf{G})$. Restricting φ to the full subcategory $\mathsf{Rep}(\mathbf{G}^{(1)})$, we obtain the desired claim. □

The comments in §2.7 suggest that the singular blocks (those associated with weights in $\overline{C} \smallsetminus C$) are "simpler" than the regular blocks (those associated with weights in C), in that their structure can in theory be derived if we understand the regular blocks. However, this point of view is a bit contradicted by Corollary 2.41: since $\mathbf{G}^{(1)}$ is isomorphic to $\mathbf{G}$, the block associated to the highly singular weight $(p-1)\varsigma$ (associated with a weight in a facet of maximal codimension) is equivalent to the block of the regular weight 0. In this way, the category $\mathsf{Rep}(\mathbf{G})$ exhibits some kind of "fractal" behaviour.

Remark 2.42. See [**A3**] for some applications of Corollary 2.41.

3. Soergel's modular category $\mathcal{O}$

In this section we explain a construction due to Soergel [**S5**], which allows to produce an analogue in the setting of representations of $\mathbf{G}$ of the celebrated "category $\mathcal{O}$" of Bernstein–Gelfand–Gelfand for complex semisimple Lie algebras (see [**H4**]). This construction is the basis for the construction of Williamson's counterexamples which will be explained in Chapter 5. The definition uses the notion of Serre quotient of an abelian category, whose construction is recalled in §3.1 in Chapter A.

3.1. Motivation. As illustred in Theorem 2.11, and as will be made clearer below, the structure of the category $\mathsf{Rep}(\mathbf{G})$ is closely related with the combinatorics of the Coxeter group $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$. The way the problem of computing characters of simple modules will be tackled is inspired by the Kazhdan–Lusztig conjecture in the study of highest weight simple modules for complex semisimple Lie algebras, which is closely related to the simpler combinatorics of (W, S).[8] In an effort to continue the parallel between these two problems, and to allow the use of some of the techniques used in the latter problem for the study of the former one, Soergel introduced in [**S5**] a category defined in terms of representations of $\mathbf{G}$, but whose combinatorics is governed by (W, S). This category is now called *Soergel's modular category* $\mathcal{O}$, and can serve as a "toy model" for $\mathsf{Rep}(\mathbf{G})$. (This toy model turns to be quite complicated already, as we will later see!)

3.2. Definition. In this section we assume that $p > h$ and that $\mathbf{G}$ is semisimple and simply connected. We will denote by A the Serre subcategory of $\mathsf{Rep}(\mathbf{G})$ generated by the simple objects $\mathsf{L}(\lambda)$ with $\lambda \in \mathbb{X}^+$ which satisfies $\lambda \uparrow p\rho$, and by B the Serre subcategory of $\mathsf{Rep}(\mathbf{G})$ (or equivalently, of A) generated by the simple objects $\mathsf{L}(\lambda)$ with $\lambda \in \mathbb{X}^+$ which satisfies $\lambda \uparrow p\rho$ but $\lambda \notin \{(p-1)\rho + W\rho\}$. Then Soergel's modular category $\mathcal{O}$ is defined as

$$\mathcal{O}_{\Bbbk} = \mathsf{A}/\mathsf{B}.$$

[8]More specifically, regular integral blocks of category $\mathcal{O}$ of a complex semisimple Lie algebra are highest-weight categories with underlying poset the associated Weyl group endowed with the Bruhat order (or its inverse, depending on the choice of parametrization of simple objects).

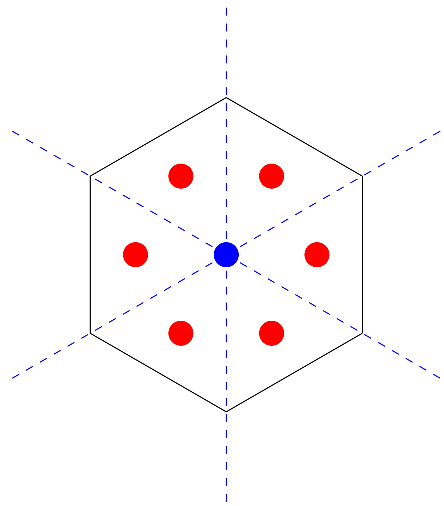


FIGURE 3.1. Weights for Soergel's modular category $\mathcal{O}$ for SL_3

EXAMPLE 3.1. For $\mathbf{G} = \mathrm{SL}_3$, the picture the reader can keep in mind is illustrated in Figure 3.1. Here the blue dot is $(p-1)\rho$, and the six dominant weights one has to consider belong to the six alcoves containing red dots.

3.3. Highest weight structure. Let μ be the unique element in $C \cap W_{\mathrm{aff}} \cdot_p (p\rho)$. Then the results of §2.8 show that the assignment $w \mapsto w \cdot_p \mu$ induces a bijection ${}^{\mathrm{f}}W_{\mathrm{aff}} \xrightarrow{\sim} \mathbb{X}^+ \cap W_{\mathrm{aff}} \cdot_p (p\rho)$ which identifies the Bruhat order on ${}^{\mathrm{f}}W_{\mathrm{aff}}$ with the order $\uparrow$ on $\mathbb{X}^+ \cap W_{\mathrm{aff}} \cdot_p (p\rho)$. If we denote by $w \in W_{\mathrm{aff}}$ the unique element such that $w \cdot_p \mu = p\rho$ (or, equivalently, such that $w \cdot_p C = C + p\rho$), then the simple objects in A are in a canonical bijection with $\{y \in {}^{\mathrm{f}}W_{\mathrm{aff}} \mid y \leq w\}$. Since this subset is an ideal in ${}^{\mathrm{f}}W_{\mathrm{aff}}$, Lemma 1.3 in Appendix A implies that this category has a structure of highest weight category with underlying poset $\{y \in {}^{\mathrm{f}}W_{\mathrm{aff}} \mid y \leq w\}$ (for the restriction of the Bruhat order).

We have $w \cdot_p C = t_\rho \cdot_p C$; hence $\omega := w^{-1}t_\rho \in W_{\mathrm{ext}}$ belongs to the subgroup Ω of Remark 2.25. (With this notation we have $\mu = \omega \cdot_p 0$.) Set $S^\omega := \omega S \omega^{-1}$ and $W^\omega := \omega W \omega^{-1}$. Then S^ω is a finitary subset of S_{aff}, with associated parabolic subgroup W^ω.

LEMMA 3.2. *The element w defined above is maximal in the coset wW^ω. As a consequence:*

(1) we have $wW^\omega \subset \{y \in {}^{\mathrm{f}}W_{\mathrm{aff}} \mid y \leq w\}$, and $\{y \in {}^{\mathrm{f}}W_{\mathrm{aff}} \mid y \leq w\} \smallsetminus wW^\omega$ is an ideal in ${}^{\mathrm{f}}W_{\mathrm{aff}}$;
(2) if $y \in W_{\mathrm{aff}}$ satisfies $y \leq w$ and if $s \in S^\omega$, then $ys \leq w$;
(3) the bijection

$$W^\omega \xrightarrow{\sim} \{y \in {}^{\mathrm{f}}W_{\mathrm{aff}} \mid y \leq w\} \smallsetminus wW^\omega$$

given by $x \mapsto wx$ identifies the inverse of the Bruhat order on W^ω with the restriction of the Bruhat order on the right-hand side.

PROOF. To prove that w is maximal in wW^ω it suffices to prove that t_ρ has maximal length in $t_\rho W$, i.e. that for any $x \in W$ we have

$$\ell(t_\rho x) = \ell(t_\rho) - \ell(x).$$

Now we have

$$\ell(t_\rho x) = \ell(x^{-1}t_{-\rho}),$$

and applying the formula (2.7) (see Remark 2.25) we deduce the desired claim.

Now that this property is established, if $x \in W^\omega$ then we have $wx \leq w$. Since $\ell(wx) = \ell(w) - \ell(x)$, by Exercise 1.10 we also have $wx \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, which shows the first assertion in (1). To prove that the complement in an ideal, we choose $u \in \{y \in {}^{\mathrm{f}}W_{\mathrm{aff}} \mid y \leq w\} \smallsetminus wW^\omega$ and $z \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ such that $z \leq u$. Then $z \leq w$. If we assume for a contradiction that $z \in wW^\omega$, and denote by u' the maximal element in uW^ω then by [Dou, Lemma 2.2] we have $w \leq u'$. On the other hand, the same claim (applied to the inequality $u \leq w$) shows that $u' < w$, which provides a contradiction.

In (2), denoting by y' the maximal element in yW^ω, then again by [Dou, Lemma 2.2] we have $y' \leq w$. Then $ys \leq y' \leq w$, proving the desired inequality.

For (3), we note that our map is a bijection by (1). Set $w_0^\omega := \omega w_0 \omega^{-1}$; then w_0^ω is the longest element in W^ω. Since the element ww_0^ω is mimimal in wW^ω, by Exercise 1.9 the assignment $x \mapsto ww_0^\omega x$ identifies the Bruhat order on W^ω with the restriction of the Bruhat order to wW^ω. The claim follows, since $x \mapsto w_0^\omega x$ intertwines the Bruhat order and its inverse on W^ω. □

Lemma 3.2(1)–(3) and Lemma 3.1 in Appendix A guarantee that the category $\mathcal{O}_\Bbbk$ has a natural structure of highest weight category with underlying poset W^ω endowed with the inverse of the Bruhat order, such that the standard object associated with x is the image of the Weyl module $\mathsf{M}(wx \cdot_p \mu)$. In fact, it will be more convenient to identify this poset with W via $x \mapsto \omega x \omega^{-1}$. Observing that for $x \in W$ we have

$$w\omega x \omega^{-1} \cdot_p \mu = t_\rho x \cdot_p 0 = (p-1)\rho + x(\rho),$$

we see that if for $x \in W$ we denote by N_x, M_x and L_x the images of the modules $\mathsf{N}((p-1)\rho+x(\rho))$, $\mathsf{M}((p-1)\rho+x(\rho))$ and $\mathsf{L}((p-1)\rho+x(\rho))$ respectively, then $\mathcal{O}_\Bbbk$ has a structure of highest weight category with underlying poset W (with the inverse of the Bruhat order) and parametrization of standard objects given by $x \mapsto \mathsf{M}_x$. For any $x, y \in W$ the multiplicity $[\mathsf{N}_y : \mathsf{L}_x]$ of the simple object L_x as a composition factor of the object N_y is given by

$$[\mathsf{N}_y : \mathsf{L}_x] = [\mathsf{N}((p-1)\rho + y(\rho)) : \mathsf{L}((p-1)\rho + x(\rho))].$$

Similarly we have

$$[\mathsf{M}_y : \mathsf{L}_x] = [\mathsf{M}((p-1)\rho + y(\rho)) : \mathsf{L}((p-1)\rho + x(\rho))]$$

where the left-hand side denotes the multiplicity of L_x as a composition factor of M_y. In view of (1.7), we therefore have

$$[\mathsf{N}_y : \mathsf{L}_x] = [\mathsf{M}_y : \mathsf{L}_x] \tag{3.1}$$

for any $x, y \in W$.

Consider the Grothendieck group $[\mathcal{O}_\Bbbk]$. It admits as a basis the classes of the simple objects $([\mathsf{L}_x] : x \in W)$. On the other hand, as for any highest weight category, this Grothendieck admits another basis consisting of classes of standard objects. We will therefore identify it with the group algebra $\mathbb{Z}[W]$ in such a way

that $w \in W$ corresponds to $[\mathsf{M}_w]$. In fact, the comments above show that for any $w \in W$ we have
$$[\mathsf{M}_w] = [\mathsf{N}_w].$$

3.4. Wall-crossing functors. For later use, we explain now how to define a collection of endofunctors of $\mathcal{O}_0$ parametrized by S. For any $s \in S$ we fix a cocharacter $\mu_s \in \mathbb{X}$ which belongs to the wall contained in $\overline{C}$ corresponding to s. (For instance, if $(\varpi_\alpha : \alpha \in \mathfrak{R}^{\mathrm{s}})$ is the collection of fundamental weights, see §2.4, one can choose $\mu_s = \rho - \varpi_{\alpha_s}$ for any $s \in S$.) Then we consider the self-adjoint exact endofunctor
$$\vartheta_s := T^{p\rho}_{p\rho+\mu_s} \circ T^{p\rho+\mu_s}_{p\rho}$$
of $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p (p\rho)}$.

Lemma 3.3. *For any $s \in S$ the functor ϑ_s stabilizes A and B.*

Proof. By exactness, proving the lemma amounts to proving that if $\lambda \uparrow p\rho$ (resp. if $\lambda \uparrow p\rho$ and $\lambda \notin \{(p-1)\rho + x(\rho) : x \in W\}$) then $\vartheta_s(\mathsf{L}(\lambda))$ belongs to A (resp. to B). As explained in §3.3 we have $\lambda = y \cdot_p \mu$ with $y \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ such that $y \leq w$, resp. with y which satisfies these conditions and does not belong to wW^ω. Moreover, since $\mathsf{L}(\lambda)$ is a submodule of $\mathsf{N}(\lambda)$, by exactness again it suffices to prove that $\vartheta_s(\mathsf{N}(\lambda))$ belongs to A, resp. to B.

Write $\mu'_s := w^{-1} \cdot_p (p\rho + \mu_s)$. Then μ'_s belongs to the wall of $\overline{C}$ corresponding to the simple reflection $s' = \omega s \omega^{-1}$, and by Proposition 2.19(1) we have
$$\vartheta_s = T^{p\rho}_{p\rho+\mu_s} \circ T^{p\rho+\mu_s}_{p\rho} = T^{\mu}_{\mu'_s} \circ T^{\mu'_s}_{\mu}.$$
By Proposition 2.36, if $ys' \notin {}^{\mathrm{f}}W_{\mathrm{aff}}$ we have $\vartheta_s(\mathsf{N}(\lambda)) = 0$, and otherwise $\vartheta_s(\mathsf{N}(\lambda))$ admits a 2-step filtration with associated graded
$$\mathsf{N}(y \cdot_p \mu) \oplus \mathsf{N}(ys' \cdot_p \mu).$$
Here $s' \in S^\omega$, hence by Lemma 3.2(2) we have $ys' \leq w$, which implies that $ys' \cdot_p \mu \uparrow p\rho$ by Proposition 2.34, and of course $ys' \notin wW^\omega$ if $y \notin wW^\omega$. This implies the desired claim. □

Let us denote by $\pi : \mathsf{A} \to \mathcal{O}_\Bbbk$ the quotient functor. In view of the universal property of the Serre quotient (see §3.1 in Chapter A), Lemma 3.3 implies that there exists a unique endofunctor of $\mathcal{O}_\Bbbk$ whose composition with π is $\pi \circ \vartheta_s$; it is again self-adjoint. This functor will also be denoted ϑ_s. The proof of Lemma 3.3 shows that for any $w \in W$, in $[\mathcal{O}_\Bbbk]$ we have
$$[\vartheta_s(\mathsf{M}_w)] = [\mathsf{M}_w] + [\mathsf{M}_{ws}].$$
In other words, under the identification $[\mathcal{O}_\Bbbk] = \mathbb{Z}[W]$ considered in §3.3, the morphism induced by ϑ_s is given by right multiplication by $e + s$.

3.5. Projective objects. Let us denote by $\mathsf{Proj}(\mathcal{O}_\Bbbk)$ the full subcategory of $\mathcal{O}_\Bbbk$ whose objects are the projective objects, and consider its split Grothendieck group $[\mathsf{Proj}(\mathcal{O}_\Bbbk)]_\oplus$. For $w \in W$ we will denote by P_w the projective cover of L_w. The obvious morphism
$$[\mathsf{Proj}(\mathcal{O}_\Bbbk)]_\oplus \to [\mathcal{O}_\Bbbk]$$

is an isomorphism; under the identification of the right-hand side with $\mathbb{Z}[W]$, this isomorphism is given by

$$[P] \mapsto \sum_w (P : \mathsf{M}_w) \cdot w$$

for $P \in \mathcal{O}_\Bbbk$ projective. The identification $[\mathsf{Proj}(\mathcal{O}_\Bbbk)]_\oplus \xrightarrow{\sim} \mathbb{Z}[W]$ will be denoted $\varkappa$.

One can compute the dimensions of morphism spaces between projective objects in terms of this identification, as follows (see [S5, Lemma 2.11.2]). Let us denote by b the bilinear form on $\mathbb{Z}[W]$ which satisfies $b(w, y) = \delta_{w,y}$.

LEMMA 3.4. *For any $P, Q \in \mathsf{Proj}(\mathcal{O}_\Bbbk)$ we have*

$$\dim_\Bbbk \operatorname{Hom}_{\mathcal{O}_\Bbbk}(P, Q) = b(\varkappa(P), \varkappa(Q)).$$

PROOF. It suffices to prove the formula when P is indecomposable, i.e. $P = \mathsf{P}_x$ for some $x \in W$. Then we have

$$\dim_\Bbbk \operatorname{Hom}_{\mathcal{O}_\Bbbk}(\mathsf{P}_x, Q) = [Q : \mathsf{L}_x] = \sum_y (Q : \mathsf{M}_y) \cdot [\mathsf{M}_y : \mathsf{L}_x],$$

where we use the fact that Q has a standard filtration (see Theorem 2.1 in Appendix A). Using reciprocity (see (2.1) in Appendix A) and (3.1), we deduce that

$$\dim_\Bbbk \operatorname{Hom}_{\mathcal{O}_\Bbbk}(\mathsf{P}_x, Q) = \sum_y (Q : \mathsf{M}_y) \cdot [\mathsf{P}_x : \mathsf{M}_y] = b(\varkappa(\mathsf{P}_x), \varkappa(Q)),$$

as desired. □

REMARK 3.5. Lemma 3.4 implies that the problem of computing the multiplicities $((\mathsf{P}_x : \mathsf{M}_y) : x, y \in W)$ (or, equivalently by reciprocity, the multiplicities $([\mathsf{M}_y : \mathsf{L}_x] : x, y \in W)$) is equivalent to the problem of computing the dimensions $(\dim_\Bbbk \operatorname{Hom}_{\mathcal{O}_\Bbbk}(\mathsf{P}_x, \mathsf{P}_y) : x, y \in W)$. In fact, the formula in the lemma shows that if one knows the multiplicities $([\mathsf{M}_y : \mathsf{L}_x] : x, y \in W)$ one can compute the dimensions $(\dim_\Bbbk \operatorname{Hom}_{\mathcal{O}_\Bbbk}(\mathsf{P}_x, \mathsf{P}_y) : x, y \in W)$. Reciprocally, if one knows the dimensions $(\dim_\Bbbk \operatorname{Hom}_{\mathcal{O}_\Bbbk}(\mathsf{P}_x, \mathsf{P}_y) : x, y \in W)$ one can compute the multiplicities $((\mathsf{P}_x : \mathsf{M}_y) : x, y \in W)$ by induction on x as follows. In fact, for $x = e$ we have $\mathsf{P}_e = \mathsf{M}_e$ by maximality. Then, if $x \in W$ and if the multiplicities are known for indecomposable projective objects with labels $< x$, one computes the multiplicities $((\mathsf{P}_x : \mathsf{M}_y) : y \in W)$ by induction on y as follows. For $y = e$ we have

$$(\mathsf{P}_x : \mathsf{M}_e) = \dim_\Bbbk \operatorname{Hom}_{\mathcal{O}_\Bbbk}(\mathsf{P}_x, \mathsf{P}_e).$$

Then if $y < x$ and if the multiplicities $(\mathsf{P}_x : \mathsf{M}_z)$ are known for all the elements $z < y$, we use the fact that $(\mathsf{P}_y : \mathsf{M}_y) = 1$ to see that

$$(\mathsf{P}_x : \mathsf{M}_y) = \dim_\Bbbk \operatorname{Hom}_{\mathcal{O}_\Bbbk}(\mathsf{P}_x, \mathsf{P}_y) - \sum_{z<y} (\mathsf{P}_x : \mathsf{M}_z) \cdot (\mathsf{P}_y : \mathsf{M}_z).$$

Finally if $y = x$ we have $(\mathsf{P}_x : \mathsf{M}_x) = 1$, and if $y \not\leq x$ we have $(\mathsf{P}_x : \mathsf{M}_y) = 0$, which completes the procedure.

The projective objects in $\mathcal{O}_\Bbbk$ admit an inductive construction as follows. First, note that since each functor ϑ_s is self-adjoint and exact, it sends projective objects to projective objects. As seen already in Remark 3.5, by maximality the object $[\mathsf{M}_e]$ is projective; we therefore have

$$\mathsf{P}_e = \mathsf{M}_e.$$

Now if $n \geq 0$ is such that all the objects P_y with $y \in W$ of length $\leq n$ are know, and if $w \in W$ has length $n+1$, then we can choose $s \in S$ such that $ws < w$. By consideration of standard multiplicities (see Remark 2.2 in Appendix A) one sees that P_w is then a direct summand of $\vartheta_s(\mathsf{P}_{ws})$ with multiplicity 1, and that all other direct summands have a label of length $\leq n$. In particular, this implies that the subcategory $\mathsf{Proj}(\mathcal{O}_\Bbbk)$ of $\mathcal{O}_\Bbbk$ is the smallest full subcategory which contains the object M_e and is stable under the functors ϑ_s and under taking direct summands.

In another formulation, given a word $\underline{w} = (s_1, \cdots, s_r)$ in S we set

$$\vartheta_{\underline{w}} = \vartheta_{s_r} \circ \cdots \circ \vartheta_{s_1}.$$

Then if $\underline{w}$ is a reduced expression for some $w \in W$ we have

$$\vartheta_{\underline{w}}(\mathsf{M}_e) \cong \mathsf{P}_w \oplus \bigoplus_{\substack{y \in W \\ y < w}} \mathsf{P}_y^{\oplus b_{y,\underline{w}}} \tag{3.2}$$

for some nonnegative integers $b_{y,\underline{w}}$.

3.6. The object P_{w_0}. There is one nontrivial projective object that can be described explicitly, namely P_{w_0}. In fact, consider the object

$$T_{(p-1)\rho}^{p\rho}(\mathsf{M}((p-1)\rho)).$$

By Proposition 2.26(4), this objects admits a filtration with subquotients $\mathsf{M}((p-1)\rho + x(\rho))$ where x runs over W; in particular, it belongs to A. If $\lambda \in \mathbb{X}^+$ satisfies $\lambda \uparrow p\rho$, we see using Proposition 2.26(2) that

$$T_{p\rho}^{(p-1)\rho}(\mathsf{L}(\lambda)) \cong \begin{cases} \mathsf{L}((p-1)\rho) & \text{if } \lambda = (p-2)\rho; \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{Ext}^1_{\mathsf{Rep}(\mathbf{G})}(\mathsf{L}((p-1)\rho), \mathsf{L}((p-1)\rho)) = 0$ (e.g. because $\mathsf{L}((p-1)\rho)$ is both standard and costandard, see §2.9), we deduce using adjunction that $T_{(p-1)\rho}^{p\rho}(\mathsf{M}((p-1)\rho))$ is projective in A; in fact it is the projective cover of $\mathsf{L}((p-2)\rho)$. In view of Remark 3.3 in Appendix A, we deduce that its image in $\mathcal{O}_\Bbbk$ is P_{w_0}, and that the natural morphism

$$\mathrm{End}_{\mathsf{Rep}(\mathbf{G})}(T_{(p-1)\rho}^{p\rho}(\mathsf{M}((p-1)\rho))) \to \mathrm{End}_{\mathcal{O}_\Bbbk}(\mathsf{P}_{w_0})$$

is an isomorphism. This analysis also shows that

$$[\mathsf{P}_{w_0}] = \sum_{y \in W} [\mathsf{M}_y].$$

3.7. The functor $\mathbb{V}$. The starting point of the work in [S5] is a description of the algebra $\mathrm{End}_{\mathcal{O}_\Bbbk}(\mathsf{P}_{w_0})$ which is reminiscent of a statement for complex Lie algebras also due to Soergel [S1]; see §1.10 in Chapter 2. Namely, recall the notation of §2.5, and denote by $\langle \mathrm{S}(\mathfrak{t})_+^W \rangle$ the ideal of $\mathrm{S}(\mathfrak{t})$ generated by homogeneous W-invariant elements of positive degree (for the natural grading on $\mathrm{S}(\mathfrak{t})$). We set

$$C := \mathrm{S}(\mathfrak{t}) / \langle \mathrm{S}(\mathfrak{t})_+^W \rangle.$$

(This algebra is sometimes called the "coinvariant algebra," but this terminology might be misleading since C is different from the coinvariants for the action of W on $\mathrm{S}(\mathfrak{t})$.) This algebra admits a canonical action of W, and for $s \in S$ we will denote by C^s the subalgebra of s-invariant elements.

In [**AJS**, §19.8] the authors construct a canonical algebra isomorphism

$$\mathrm{S}(\mathfrak{t})/\langle \mathrm{S}(\mathfrak{t})^W_+ \rangle \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_\Bbbk}(\mathsf{P}_{w_0}).$$

In particular, using this isomorphism we obtain that the functor

$$\mathbb{V} := \mathrm{Hom}_{\mathcal{O}_\Bbbk}(\mathsf{P}_{w_0}, -) : \mathcal{O}_\Bbbk \to \mathsf{Vect}_\Bbbk$$

factors through a functor (still denoted $\mathbb{V}$) taking values in the category C-Mod of C-modules.

The following statement gathers some of the main results of the "algebraic part" of [**S5**].

Theorem 3.6. *(1) The restriction of the functor*

$$\mathbb{V} : \mathcal{O}_\Bbbk \to C\text{-}\mathsf{Mod}$$

to the subcategory $\mathsf{Proj}(\mathcal{O}_\Bbbk)$ *is fully faithful.*

(2) The image of M_e *under* $\mathbb{V}$ *is the trivial* $\mathrm{S}(\mathfrak{t})$*-module, seen as a* C*-module.*

(3) For any $s \in S$ *there exists a canonical isomorphism of functors*

$$\mathbb{V} \circ \vartheta_s(-) \cong C \otimes_{C^s} \mathbb{V}(-).$$

For (1), see [**S5**, Theorem 2.6.1]. For (2) and (3), see [**S5**, Theorem 2.6.2].

Transferring the results of §3.5 through the fully faithful functor $\mathbb{V}$ we deduce the following results:

(1) the category $\mathsf{Proj}(\mathcal{O}_\Bbbk)$ is equivalent to the smallest full subcategory of C-Mod which contains the trivial module $\Bbbk$ and is stable under the functors $C \otimes_{C^s} (-)$;

(2) for any $w \in W$ there exists a unique indecomposable C-module D_w which is a direct summand of the module

$$C \otimes_{C^{s_r}} C \otimes_{C^{s_{r-1}}} \cdots \otimes_{C^{s_2}} C \otimes_{C^{s_1}} \Bbbk$$

for any reduced expression $(s_1, \cdots, s_r)$ of w, but not a direct summand of a module

$$C \otimes_{C^{s_k}} C \otimes_{C^{s_{k-1}}} \cdots \otimes_{C^{s_2}} C \otimes_{C^{s_1}} \Bbbk$$

for any word $(s_1, \cdots, s_k)$ in S with $k < \ell(w)$; moreover we have

$$D_w = \mathbb{V}(\mathsf{P}_w).$$

The category in (1) is an example of a category of Soergel modules, which will be studied systematically in Chapter 2. Using these results and Remark 3.5, one sees that the problem of computing the multiplicities (3.1) can be rephrased completely in terms of these modules. We will come back to this question repeatedly in the following chapters, culminating in Chapter 5 where we will explain how these considerations are the basis for Williamson's construction of counterexamples to Lusztig conjectures.

4. Lusztig's character formula

We are now ready to explain Lusztig's conjecture, which provides an answer to the question of computing the characters of the simple algebraic $\mathbf{G}$-modules under appropriate assumptions.

4.1. (Iwahori–)Hecke algebras. In this subsection we consider a Coxeter system $(\mathcal{W}, \mathcal{S})$. (See §0.6 for our conventions on Coxeter systems.) Thus $\mathcal{W}$ admits a presentation with generators $\mathcal{S}$, and with the following relations:

- for any $s \in \mathcal{S}$, $s^2 = e$;
- for any $(s,t) \in \mathcal{S}^2_\circ$, $(st)^{m_{s,t}} = e$.

It is a classical observation that, given the first set of relations, the second one can be rephrased as saying that for any $s, t \in \mathcal{S}^2_\circ$,

$$\underbrace{st\cdots}_{m_{s,t}\text{ terms}} = \underbrace{ts\cdots}_{m_{s,t}\text{ terms}}.$$

(These relations are called the *braid relations*.) We will consider an indeterminate v, and the ring $\mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in v with coefficients in $\mathbb{Z}$.

Recall the definition of the *Hecke algebra* (sometimes called the *Iwahori–Hecke algebra*) associated with $(\mathcal{W}, \mathcal{S})$.

Definition 4.1. The Hecke algebra associated with $(\mathcal{W}, \mathcal{S})$ is the $\mathbb{Z}[v, v^{-1}]$-algebra $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ with a basis $(H_w : w \in \mathcal{W})$ and with multiplication determined by the following rules:

(1) $(H_s + vH_e) \cdot (H_s - v^{-1}H_e) = 0$ if $s \in \mathcal{S}$;
(2) $H_x \cdot H_y = H_{xy}$ if $x, y \in \mathcal{W}$ and $\ell(xy) = \ell(x) + \ell(y)$.

Note in particular that H_e is the unit in $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$; this element will therefore sometimes be denoted 1. The relations (1) are called the *quadratic relations* (because they say that a certain quadratic polynomial in H_s vanishes). The relations (2) imply in particular that the elements $(H_s : s \in \mathcal{S})$ satisfy the braid relations in the sense that for any $s, t \in \mathcal{S}^2_\circ$ we have

$$\underbrace{H_s H_t \cdots}_{m_{st}\text{ terms}} = \underbrace{H_t H_s \cdots}_{m_{st}\text{ terms}}. \tag{4.1}$$

Remark 4.2. If $w \in \mathcal{W}$ and $w = s_1 \cdots s_r$ is a reduced expression (i.e. each s_i belongs to $\mathcal{S}$, and $r = \ell(w)$), then we have

$$H_w = H_{s_1} \cdots H_{s_r}. \tag{4.2}$$

In particular, the elements $(H_s : s \in \mathcal{S})$ generate $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ as a $\mathbb{Z}[v, v^{-1}]$-algebra. In fact, it is standard that this algebra admits a presentation with generators $(H_s : s \in \mathcal{S})$ and relations the quadratic relations (for any $s \in \mathcal{S}$) and the braid relations (4.1) for any $s, t \in \mathcal{S}^2_\circ$.

The existence of the algebra $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ is standard but not completely obvious; for details, see [**H3**, Chap. 7]. Here we follow the notation and conventions of [**S3**]. Another popular convention involves a basis $(T_w : w \in \mathcal{W})$, where for $s \in \mathcal{S}$ we have

$$T_s^2 = v^{-2}T_1 + (v^{-2} - 1)T_s.$$

The relation between these bases is such that $H_x = v^{\ell(x)} \cdot T_x$. Some authors also use an indeterminate q rather than v; these conventions are related by the relation $q = v^{-2}$.

The second relation in Definition 4.1 implies that each H_s ($s \in \mathcal{S}$) is invertible in $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$, with

$$H_s^{-1} = H_s + (v - v^{-1}). \tag{4.3}$$

In view of (4.2), it follows that each element H_w ($w \in \mathcal{W}$) is invertible.

Note that if we view $\mathbb{Z}$ as a $\mathbb{Z}[v, v^{-1}]$-module with v acting as the identity, then we have a canonical algebra isomorphism

$$\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H}_{(\mathcal{W},\mathcal{S})} \cong \mathbb{Z}[\mathcal{W}] \tag{4.4}$$

(where the right-hand side is the group algebra of $\mathcal{W}$) where $1 \otimes H_w$ corresponds to the element $w \in \mathbb{Z}[\mathcal{W}]$ for any $w \in \mathcal{W}$.

If $I \subset \mathcal{S}$ is a subset, recall that we have the standard parabolic subgroup $\mathcal{W}_I \subset \mathcal{W}$ associated with I, such that $(\mathcal{W}_I, I)$ is a Coxeter system; see §2.8. Since the restriction of the length function of $\mathcal{W}$ to $\mathcal{W}_I$ is the length function of $\mathcal{W}_I$, it is clear that we have a canonical $\mathbb{Z}[v, v^{-1}]$-algebra embedding

$$\mathcal{H}_{(\mathcal{W}_I,I)} \hookrightarrow \mathcal{H}_{(\mathcal{W},\mathcal{S})} \tag{4.5}$$

sending the basis element H_w in the left-hand side to the basis element H_w in the right-hand side, for any $w \in \mathcal{W}_I$.

4.2. The Kazhdan–Lusztig basis. The basis $(H_w : w \in \mathcal{W})$ is called the *standard basis* of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$. This algebra has another basis with a very rich combinatorics, whose definition is due to Kazhdan–Lusztig [**KL1**], and which we now introduce.

The *Kazhdan-Lusztig involution* is the unique ring involution ι of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ which satisfies

$$\iota(v) = v^{-1}, \qquad \iota(H_x) = (H_{x^{-1}})^{-1}.$$

The following theorem is due to Kazhdan–Lusztig [**KL1**]. For a simple proof, we refer to [**S3**, Theorem 2.1].

THEOREM 4.3. *For all $w \in \mathcal{W}$, there exists a unique element $\underline{H}_w \in \mathcal{H}_{(\mathcal{W},\mathcal{S})}$ such that*

$$\iota(\underline{H}_w) = \underline{H}_w, \qquad \underline{H}_w \in H_w + \sum_{y \in \mathcal{W}} v\mathbb{Z}[v] H_y.$$

The elements $(\underline{H}_w : w \in \mathcal{W})$ form a basis of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$, called the Kazhdan–Lusztig basis *(or sometimes the* canonical basis*).*

Again we are following the notational conventions of [**S3**]. In [**KL1**] the authors denote the element $\underline{H}_w$ by C'_w. They also consider another basis $(C_w : w \in \mathcal{W})$; it is related to the basis $(\underline{H}_w : w \in \mathcal{W})$ by $C_w = (-1)^{\ell(w)} \tau(\underline{H}_w)$, where τ is the ring involution of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ defined by $\tau(H_x) = (-1)^{\ell(x)} \cdot H_x$ and $\tau(v) = v^{-1}$.

The condition that

$$\underline{H}_w \in H_w + \sum_{y \in \mathcal{W}} v\mathbb{Z}[v] H_y$$

is sufficient to characterize the element $\underline{H}_w$; but it turns out that a more precise statement holds; we in fact have

$$\underline{H}_w \in H_w + \sum_{\substack{y \in \mathcal{W} \\ y < w}} v\mathbb{Z}[v] H_y.$$

If one writes

$$\underline{H}_x = \sum_{y \in \mathcal{W}} h_{y,x} \cdot H_y,$$

then the polynomials $(h_{y,x} : y, x \in \mathcal{W})$ are called the *Kazhdan–Lusztig polynomials*. These polynomials satisfy $h_{w,w} = 1$ for any $w \in \mathcal{W}$, and $h_{y,w} = 0$ unless $y < w$.

Remark 4.4. Using the formula (4.3), it is easy to see that $\underline{H}_s = H_s + v$ for any $s \in \mathcal{S}$. More generally, if $I \subset \mathcal{S}$ is such that $\mathcal{W}_I$ is finite, and if w_I is the unique element of maximal length in $\mathcal{W}_I$, then we have

$$\underline{H}_{w_I} = \sum_{y \in \mathcal{W}_I} v^{\ell(w_I)-\ell(y)} H_y;$$

see Exercise 1.16.

The proof of Theorem 4.3 provides some kind of algorithm to compute the Kazhdan–Lusztig basis inductively. Namely, let $w \in \mathcal{W}$, $s \in \mathcal{S}$ be such that $sw > w$, and assume that the elements $(\underline{H}_y : y \in \mathcal{W}, y < sw)$ are known. Then it is easily seen that the element $\underline{H}_s \cdot \underline{H}_w$ can be written as

$$\underline{H}_s \cdot \underline{H}_w = \sum_{\substack{y \in \mathcal{W} \\ y \leq sw}} p_y \cdot H_y \tag{4.6}$$

for some polynomials $p_y \in \mathbb{Z}[v]$. Then one has

$$\underline{H}_{sw} = \underline{H}_s \cdot \underline{H}_w - \sum_{\substack{y \in \mathcal{W} \\ y < sw}} p_y(0) \cdot \underline{H}_y.$$

It is clear that if $I \subset \mathcal{S}$ is a subset, the embedding (4.5) sends the Kazhdan–Lusztig element $\underline{H}_w$ in the left-hand side to the Kazhdan–Lusztig element $\underline{H}_w$ in the right-hand side, for any $w \in \mathcal{W}_I$.

4.3. Lusztig's conjecture. We now specialize the considerations above to the special case $(\mathcal{W}, \mathcal{S}) = (W_{\mathrm{aff}}, S_{\mathrm{aff}})$. (As explained above W is a standard parabolic subgroup in W_{aff}; hence the Kazhdan–Lusztig combinatorics of $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ in particular contains that of (W, S).) We will write $\mathcal{H}_{\mathrm{aff}}$ for $\mathcal{H}_{(W_{\mathrm{aff}}, S_{\mathrm{aff}})}$.

Remark 4.5. Recall the group W_{ext} introduced in Remark 2.25. Even through this group has no natural Coxeter group structure, it admits a "length function" ℓ, and it is not difficult to check that there exists a $\mathbb{Z}[v, v^{-1}]$-algebra structure on the free $\mathbb{Z}[v, v^{-1}]$-module $\mathcal{H}_{\mathrm{ext}}$ with a basis $(H_w : w \in W_{\mathrm{ext}})$ where multiplication is defined by the same rule as in §4.1. The submodule spanned by $(H_w : w \in W_{\mathrm{aff}})$ identifies with the Hecke algebra $\mathcal{H}_{\mathrm{aff}}$ of $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$, the submodule $\mathcal{H}_\Omega$ spanned by $(H_\omega : \omega \in \Omega)$ identifies with the group algebra of Ω over $\mathbb{Z}[v, v^{-1}]$, and multiplication induces an isomorphism

$$\mathcal{H}_{\mathrm{aff}} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H}_\Omega \xrightarrow{\sim} \mathcal{H}_{\mathrm{ext}}.$$

Moreover, for any $\omega \in \Omega$, conjugation by H_ω stabilizes $\mathcal{H}_{\mathrm{aff}}$, and acts on this subalgebra by the automorphism induced by the automorphism of W_{aff} given by conjugation by ω in W_{ext}. Hence $\mathcal{H}_{\mathrm{ext}}$ is some kind of semi-direct product of $\mathcal{H}_{\mathrm{aff}}$ with Ω.

One can define a Kazhdan–Lusztig basis $(\underline{H}_w : w \in W_{\mathrm{ext}})$ in $\mathcal{H}_{\mathrm{ext}}$ in the same way as for Hecke algebras of Coxeter groups; in fact, for any $w \in W_{\mathrm{aff}}$ and $\omega \in \Omega$ we have

$$\underline{H}_{\omega w} = H_\omega \underline{H}_w, \qquad \underline{H}_{w\omega} = \underline{H}_w H_\omega.$$

By expanding the Kazhdan–Lusztig basis in the standard basis we obtain Kazhdan–Lusztig polynomials $(h_{y,w} : y, w \in W_{\mathrm{ext}})$. In fact these polynomials are determined by those attached to $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$; more precisely we have

$$h_{\omega y,\omega' w} = \begin{cases} h_{y,w} & \text{if } \omega = \omega'; \\ 0 & \text{otherwise} \end{cases}$$

for $w, y \in W_{\mathrm{aff}}$ and $\omega, \omega' \in \Omega$.

From now on we assume that $p \geq h$, so that $C \cap \mathbb{X} \neq \varnothing$. Recall that for $\lambda \in C \cap \mathbb{X}$ the simple objects in $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}$ are parametrized by ${}^{\mathrm{f}}W_{\mathrm{aff}}$. The following (extremely important) conjecture is due to Lusztig [L1], and is usually called *Lusztig's conjecture*.

Conjecture 4.6. *Assume that $p \geq h$, and fix $\lambda \in C \cap \mathbb{X}$. For any $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ such that*

$$\langle w \cdot_p \lambda + \rho, \alpha^\vee \rangle \leq p(p-h+2) \quad \text{for all } \alpha \in \mathfrak{R}^+, \tag{4.7}$$

we have

$$[\mathsf{L}(w \cdot_p \lambda)] = \sum_{y \in {}^{\mathrm{f}}W_{\mathrm{aff}}} (-1)^{\ell(w)+\ell(y)} h_{w_0 y, w_0 w}(1) \cdot [\mathsf{N}(y \cdot_p \lambda)] \tag{4.8}$$

in $[\mathsf{Rep}(\mathbf{G})]$.

Remark 4.7. A number of remarks on this conjecture are in order.

(1) By Proposition 2.26(1)–(2), the choice of λ in Conjecture 4.6 does not matter. Namely, the conjecture holds for one specific choice of λ iff it holds for all λ's. For simplicity, we will usually assume that $\lambda = 0$.
(2) Conjecture 4.6 is stated in terms of the group W_{aff}. In Remark 4.5 we have explained how to define Kazhdan–Lusztig polynomials for the group W_{ext}. Let us denote by ${}^{\mathrm{f}}W_{\mathrm{ext}} \subset W_{\mathrm{ext}}$ the subset of elements y which have minimal length in Wy; then ${}^{\mathrm{f}}W_{\mathrm{ext}} = \sqcup_{\omega \in \Omega} {}^{\mathrm{f}}W_{\mathrm{aff}} \cdot \omega$. It is easy to see that if Conjecture 4.6 holds then the formula (4.8) will also hold for $w \in {}^{\mathrm{f}}W_{\mathrm{ext}}$ (if one replace the condition $y \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ by the condition $y \in {}^{\mathrm{f}}W_{\mathrm{ext}}$).
(3) A very important aspect of the formula (4.8) is that the coefficients appearing there do not depend on p. The conjecture therefore expresses in particular the idea that with the correct parametrization of simple modules (based on the dot-action of W_{aff}) and if one restricts the problem to a suitable region, then the characters of simple $\mathbf{G}$-modules "do not depend on p" in the sense that the coefficients in the expansion of $[\mathsf{L}(w \cdot_p \lambda)]$ in the basis $([\mathsf{N}(y \cdot_p \lambda)] : y \in {}^{\mathrm{f}}W_{\mathrm{aff}})$ do not depend on p.
(4) The condition (4.7) is called "Jantzen's condition" since it appeared earlier in work of Jantzen. To explain the meaning of this condition, write a dominant weight μ as $\mu_0 + p\mu_1$ with $\mu_0 \in \mathbb{X}^+_{\mathrm{res}}$ and $\mu_1 \in \mathbb{X}^+$ (assuming this is possible; see §2.4). Identifying $\mathbf{G}^{(1)}$ with $\mathbf{G}$ in such a way that $\mathrm{Fr}^*_{\mathbf{T}}$ identifies with $\mu \mapsto p\mu$, by Theorem 2.9 we then have
$$\mathsf{L}(\mu) \cong \mathsf{L}(\mu_0) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{L}(\mu_1)).$$
On the other hand, if μ satisfies the condition that
$$\langle \mu + \rho, \alpha^\vee \rangle \leq p(p-h+2)$$
for any $\beta \in \mathfrak{R}^+$ then we have
$$\langle \mu_1, \beta^\vee \rangle < p - h + 2$$

for any $\beta \in \mathfrak{R}^+$, and finally

$$\langle \mu_1 + \rho, \beta^\vee \rangle \leq p$$

for any $\beta \in \mathfrak{R}^+$. Hence μ_1 belongs to $\overline{C}$, so that $\mathsf{L}(\mu_1) \cong \mathsf{N}(\mu_1)$, see §2.9. Jantzen's condition can therefore be seen as a simple condition that ensures that when we apply Steinberg's decomposition theorem, the simple module which is pulled back under the Frobenius morphism is in fact a simple induced (and Weyl) module. Since characters of induced modules do not depend on p (see Theorem 1.20), this condition seems favorable if one expects characters to enjoy some "independence of p." (For an explicit example where the formula 4.8 does no hold when one leaves the region determined by (4.7), see Exercise 1.23.)

(5) If $\mu \in \mathbb{X}^+_{\mathrm{res}}$, then for any $\alpha \in \mathfrak{R}^+$ we have

$$\langle \mu + \rho, \alpha^\vee \rangle \leq \langle (p-1)\rho + \rho, \alpha^\vee \rangle = p(h-1).$$

Hence if $p \geq 2h-3$ all the elements $w \in W_{\mathrm{aff}}$ such that $w \cdot_p \lambda$ is dominant restricted satisfy (4.7). In view of the comments in §2.7 it follows that, under this assumption, from Conjecture 4.6 one can deduce (in theory) the characters of *all* simple $\mathbf{G}$-modules. For a more explicit description of the character formula one obtains in this way, see [**L7**].

Example 4.8. The first nontrivial example in which Conjecture 4.6 can be checked is when $\mathbf{G}$ is quasi-simple and $w = s_\circ$, where we use the notation of §2.9. In this case we have seen that

$$[\mathsf{L}(s_\circ \cdot_p 0)] = [\mathsf{N}(s_\circ \cdot 0)] - [\mathsf{L}(0)].$$

On the other hand, using Remark 4.4 we see that

$$\underline{H}_{w_0 s_\circ} = \underline{H}_{w_0}\underline{H}_{s_\circ} = \sum_{x \in W} v^{\ell(w_0)-\ell(x)} \cdot (H_{xs_\circ} + vH_x).$$

In particular, $h_{w_0, w_0 s_\circ} = v$.

4.4. Some history. Let us explain some important steps in the history of Lusztig's conjecture. This conjecture has guided and motivated most of the later works on this subject. For more details on some aspects of this history, we refer to [**J4**].

The conjecture was stated in 1980, and presented as an analogue of the Kazhdan–Lusztig conjecture [**KL1**] for characters of simple highest weight modules for complex semisimple Lie algebras. (See §1.10 in Chapter 2 for a discussion of the latter conjecture. It involves Kazhdan–Lusztig polynomials for the group (W, S) rather than $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$.) Lusztig writes the following in [**L7**]: "*The evidence for the conjecture is very strong. I have verified it in the cases where $\mathbf{G}$ is of type $\mathbf{A}_2$, $\mathbf{B}_2$ or $\mathbf{G}_2$. (In these cases, [the characters have] been computed by Jantzen.)*[9]" In fact, at that time some characters for the group of type $\mathbf{A}_3$ had also been computed by Jantzen (see Exercise 1.21); the conjecture also holds in these cases. Shortly thereafter, as further evidence for his conjecture, Lusztig proved (independently of the conjecture) in [**L3**] a formula for characters of induced modules which follows from Conjecture 4.6; see §4.5 below for details.

[9] See Exercise 1.22 for this computation.

A few years later, in [Ka] Kato proved some formulas for Kazhdan–Lusztig polynomials for the Coxeter system $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$, and used them to show that Conjecture 4.6 holds iff the formula (4.8) holds for any $w \in W_{\mathrm{aff}}$ which satisfies (4.7) and such that $w \cdot_p \lambda$ is dominant restricted. This suggests to modify the conjecture slightly and say that the formula (4.8) should hold for any $w \in W_{\mathrm{aff}}$ such that $w \cdot_p \lambda$ is dominant restricted, for any $p \geq h$. An important aspect of this result is that it reduces the proof of Lusztig's conjecture to proving a collection of formulas, the cardinality of this collection being independent of p. (More precisely, this cardinality is the number of alcoves contained in (2.9), which can be shown to be equal to the quotient of $\#W$ by the cardinality of the fundamental group of $\mathfrak{R}$.)

In the early 1990's, Lusztig proposed a program for solving his conjecture, see e.g. [L4]. This program involved the versions of the quantized enveloping algebras at roots of unity that he had introduced a few years before, and proposed three main steps:

(1) show that the characters of simples $\mathbf{G}$-modules attached to restricted dominant weights in the W_{aff}-orbit of 0 are equal to similar characters for the quantum groups at a root of unity;
(2) build a bridge relating quantum groups at a root of unity and some category of representations of affine Lie algebras (over the complex numbers);
(3) build a "localization theory" for affine Lie algebras, relating their representations to some category of $\mathcal{D}$-modules on an affine flag variety, analogous to the constructions for complex semisimple Lie algebras due to Beĭlinson–Bernstein and Brylinski–Kashiwara (which led to the first proof of the Kazhdan–Lusztig conjecture).

With these three steps completed, one would obtain a proof of the conjecture by passing from $\mathcal{D}$-modules to perverse sheaves via the Riemann–Hilbert correspondence, and then using the computation of dimensions of fibers of intersection cohomology complexes on affine flag varieties in terms of Kazhdan–Lusztig polynomials due to Kazhdan–Lusztig [KL2].

This program was tackled in the following years. A solution for step (2) was obtained by Kazhdan–Lusztig [KL3] and Lusztig [L6], and a solution for step (3) was obtained by Kashiwara–Tanisaki [KT]. Step (1) however revealed more subtle than expected. Namely, in [AJS] the desired equality was proved, but under the assumption than p is bigger than a non explicit bound depending on $\mathfrak{R}$. Combining all these works, one therefore obtains that given a root datum Δ, there exists a bound $N(\Delta)$ such that, for any algebraically closed field $\Bbbk$ with $\mathrm{char}(\Bbbk) > N(\Delta)$, Conjecture 4.6 holds for the connected reductive algebraic group over $\Bbbk$ with root datum Δ. But no estimate of $N(\Delta)$ can be obtained from the techniques used for the proof in [AJS]. This situation is described by Soergel in [S5] in the following terms: "*It is proven up to now that this conjecture is valid for every given root system in sufficiently high characteristic. If however the root system is none of* $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{B}_2$, $\mathbf{G}_2$*, one does not know for a single characteristic whether it is sufficiently high.*"

In the late 2000's, Fiebig found a new way to relate the "combinatorial category" constructed by Andersen–Jantzen–Soergel, which played a crucial role in the proofs in [AJS], to perverse sheaves on affine flag varieties. Using this tool, he was able to give a new proof of Lusztig's conjecture in large characteristics (in the same sense as above) in [F2], and then to provide an explicit bound over which the conjecture holds in [**F3**]. These results make the status of Lusztig's conjecture a bit more satisfactory; however the bound obtained in [**F3**] is difficult to compute in practice, and in any case several orders of magnitude bigger than the expected bound, namely the Coxeter number h.

The next important contribution is due to Williamson. In [**W3**] he provided a family of examples, for the special case of the group $\mathbf{G} = \mathrm{GL}_n(\Bbbk)$, which show (using some results from number theory due to Kontorovich, McNamara and Williamson, see the appendix to [**W3**]) that there cannot exist any polynomial $P \in \mathbb{Z}[X]$ such that Conjecture 4.6 holds for the group $\mathrm{GL}_n(\Bbbk)$ provided that $\mathrm{char}(\Bbbk) > P(n)$. In particular, the expected bound for the validity of Conjecture 4.6 (namely h, which equals n in this case) is not sufficient, and in fact no polynomial in h can be sufficient. The construction of these counterexamples is based on a relation between Soergel's modular category $\mathcal{O}$ (see Section 3) and the Soergel bimodules associated with (W, S) and its action on $\Bbbk \otimes_{\mathbb{Z}} \mathbb{X}$, proved by Soergel in [**S5**], and a description of the category of Soergel bimodules by generators and relations due to Elias–Williamson [**EW2**]; see Chapter 5 for details. So, in fact, what these examples contradict is not directly Lusztig's conjecture, but rather a consequence of this conjecture which can be seen in the combinatorics of the category $\mathcal{O}_\Bbbk$; see Proposition 4.9 below. Williamson published later a different proof of these counterexamples in [**W4**], where the arguments involving [**EW2**] are replaced by geometric considerations involving the singularities of Schubert varieties.

4.5. Relation with characters of induced modules. In this subsection we explain how one can deduce from Conjecture 4.6 a formula for dimensions of weight spaces of induced modules in terms of Kazhdan–Lusztig polynomials. This formula was in fact proved independently of the conjecture by Lusztig in [**L3**], which provided further evidence for the truth of Conjecture 4.6. This result was also the starting point of a very fruitful subject, namely the geometric Satake equivalence; see [**BR**].

Fix $\lambda \in \mathbb{X}^+$. We are interested in computing the dimension of $\mathsf{N}(\lambda)_\mu$ for any $\mu \in \mathbb{X}$. In fact, by Lemma 1.10 we can (and will) assume that $\mu \in \mathbb{X}^+$. As explained in §1.9, this dimension does not depend on p; we will therefore assume that $p \gg 0$, and more precisely that $p \geq h$ and that

$$\langle \lambda, \alpha^\vee \rangle \leq p - h + 2$$

for any $\alpha \in \mathfrak{R}^+$. As explained in Remark 4.7(4), this condition implies in particular that $\lambda \in \overline{C}$, so that that $\mathsf{L}(\lambda) = \mathsf{N}(\lambda)$. By (2.3) we then have $\mathsf{L}(p\lambda) \cong \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{N}(\lambda))$, which implies that

$$\mathrm{ch}(\mathsf{L}(p\lambda)) = \sum_{\mu \in \mathbb{X}^+} \left(\dim(\mathsf{N}(\lambda)_\mu) \cdot \sum_{\nu \in W(\mu)} e^{p\nu} \right). \tag{4.9}$$

On the other hand, for $\eta \in \mathbb{X}$ we set

$$\chi(\eta) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w \bullet \eta}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w \bullet 0}}.$$

It isclear from definition that for any $w \in W$ and $\eta \in \mathbb{X}$ we have

$$\chi(w \bullet \eta) = (-1)^{\ell(w)} \chi(\eta). \tag{4.10}$$

We also have

$$\begin{aligned}
\sum_{w\in W} \chi(w\eta) &= \frac{\sum_{y,w\in W}(-1)^{\ell(y)} e^{y(w(\eta)+\rho)-\rho}}{\sum_{z\in W}(-1)^{\ell(z)} e^{z(\rho)-\rho}} \\
&= \sum_{x\in W} e^{x(\eta)} \cdot \frac{\sum_{y\in W}(-1)^{\ell(y)} e^{y(\rho)-\rho}}{\sum_{z\in W}(-1)^{\ell(z)} e^{z(\rho)-\rho}} \\
&= \sum_{x\in W} e^{x(\eta)}
\end{aligned}$$

where in the second line we set $x = yw$. Applying this to $\eta = p\mu$ and dividing by the order of the stabilizer W_μ of μ (a standard parabolic subgroup of W) we deduce that

$$\begin{aligned}
\sum_{\nu\in W(\mu)} e^{p\nu} &= \sum_{\nu\in W(\mu)} \chi(p\nu) \\
&= \sum_{w\in W^\mu} e^{w(p\mu)} \\
&= \sum_{w\in W^\mu} (-1)^{\ell(w)} \chi(p\mu - \rho + w^{-1}(\rho)),
\end{aligned}$$

where $W^\mu \subset W$ is the subset of elements w which are minimal in wW^μ, and where the third equality uses (4.10).

Using this equality in (4.9), we obtain that

$$\mathrm{ch}(\mathsf{L}(p\lambda)) = \sum_{\mu\in\mathbb{X}^+} \left(\dim(\mathsf{N}(\lambda)_\mu) \cdot \sum_{w\in W^\mu} (-1)^{\ell(w)} \chi(p\mu - \rho + w^{-1}(\rho)) \right).$$

Here each $p\mu - \rho + w^{-1}(\rho)$ is dominant. Indeed, for $\alpha \in \mathfrak{R}^{\mathrm{s}}$, if $\langle \mu, \alpha^\vee \rangle > 0$ then

$$\langle p\mu - \rho + w^{-1}(\rho), \alpha^\vee \rangle = \langle p\mu + w^{-1}(\rho), \alpha^\vee \rangle - 1 \geq p - 1 + \langle \rho, w(\alpha)^\vee \rangle,$$

and the right-hand side is nonnegative because $p \geq h$. On the other hand, if $\langle \mu, \alpha^\vee \rangle = 0$ then $s_\alpha \in W_\mu$, so that $ws_\alpha > w$, which implies that $w(\alpha) \in \mathfrak{R}^+$, and finally that

$$\langle p\mu - \rho + w^{-1}(\rho), \alpha^\vee \rangle = \langle \rho, w(\alpha)^\vee \rangle - 1 \geq 0.$$

Using this fact and Weyl's character formula (see §1.9), the formula above can be written as

$$\mathrm{ch}(\mathsf{L}(t_\lambda \cdot_p 0)) = \sum_{\mu\in\mathbb{X}^+} \left(\dim(\mathsf{N}(\lambda)_\mu) \cdot \sum_{w\in W^\mu} (-1)^{\ell(w)} \, \mathrm{ch}\big(\mathsf{N}((t_\mu w^{-1}) \cdot_p 0)\big) \right).$$

Our assumption on p implies that the element $t_\lambda \in W_{\mathrm{ext}}$ satisfies the condition in (4.7); comparing the formula above with that in (4.8) (see also Remark 4.7(2)), we deduce that for any $\mu \in \mathbb{X}^+ \cap (\lambda + \mathbb{Z}\mathfrak{R})$ we have

$$h_{w_0 t_\mu, w_0 t_\lambda}(1) = \dim(\mathsf{N}(\lambda)_\mu).$$

4.6. Consequence for Soergel's modular category $\mathcal{O}$. In this subsection we assume that $\mathbf{G}$ is semisimple and simply connected, and that $p \geq h$. Recall the category $\mathcal{O}_{\Bbbk}$ from Section 3. Our goal is to explain that, if Lusztig's conjecture holds, then one can express the multiplicities of the simple objects $(\mathsf{L}_y : y \in W)$ in $\mathcal{O}_{\Bbbk}$ in the costandard objects $(\mathsf{N}_y : y \in W)$ in terms of the Kazhdan–Lusztig combinatorics of the group W.

Here we find it convenient to work with the extended affine Weyl group W_{ext} (see Remark 2.25 and Remark 4.5). In fact, for any $y \in W$ we have

$$(p-1)\rho + y(\rho) = (t_\rho y) \cdot_p 0,$$

so that we will consider the elements $(t_\rho y : y \in W)$ of W_{ext}. We will assume that (4.7) holds for these elements and $\lambda = 0$, i.e. that

$$[\mathsf{L}((p-1)\rho + y(\rho))] = \sum_{\substack{x \in W_{\mathrm{ext}} \\ x \cdot_p 0 \in \mathbb{X}^+}} (-1)^{\ell(t_\rho y)+\ell(x)} h_{w_0 x, w_0 t_\rho y}(1) \cdot [\mathsf{N}(x \cdot_p 0)] \tag{4.11}$$

in $[\mathsf{Rep}(\mathbf{G})]$ for any $y \in W$.

The following statement is implicit in [S5].

Proposition 4.9. *Under the assumptions above, for any $x, y \in W$ we have*

$$[\mathsf{N}_y : \mathsf{L}_x] = h_{y,x}(1).$$

The proof of Proposition 4.9 will require the following preliminary lemma.

Lemma 4.10. *For any $x, y \in W$ we have $h_{w_0 t_\rho x, w_0 t_\rho y} = h_{w_0 x, w_0 y}$.*

Sketch of proof. It would probably be possible to give a combinatorial proof of this lemma; but we will rather use arguments from geometry. Namely, thanks to results of Kazhdan–Lusztig [KL2], it is known that the polynomials $h_{w_0 x, w_0 x'}$ with $x, x' \in W_{\mathrm{ext}}$ of minimal length in their cosets Wx, Wx' compute the local intersection cohomology groups of Iwahori orbits on the affine Grassmannian attached to the complex reductive group $G^\vee$ which is Langlands dual to G (see REF below for details), while the polynomials $(h_{z,z'} : z, z' \in W)$ compute the local intersection cohomology groups of the Bruhat orbits in the flag variety of $G^\vee$. Since the elements $t_\rho x$ $(x \in W)$ are minimal in their coset since they satisfy the condition 2.10, the lemma is thus a consequence of a geometric relation between the corresponding orbits.

More specifically, consider the loop group $LG^\vee$ associated with $G^\vee$, and the corresponding arc group $L^+G^\vee$ (see §4.1 in Chapter 3 for details). Let also $B^\vee$ be the negative Borel subgroup of $G^\vee$, and let $I^\vee$ be its inverse image of $B^\vee$ under the morphism $L^+G^\vee \to G^\vee$ sending the indeterminate z to 0. Then one can consider the "opposite" affine Grassmannian $\mathrm{Gr}' := L^+G^\vee \backslash LG^\vee$, and the action of $I^\vee$ induced by multiplication on the right. The orbits for this action are naturally parametrized by ${}^{\mathrm{f}}W_{\mathrm{ext}}$, and as explained in Section 4 of Chapter 3 the graded dimensions of the stalks of the intersection cohomology complexes (with rational coefficients) associated with these orbits are computed by the polynomials $(h_{w_0 x, w_0 x'} : x, x' \in {}^{\mathrm{f}}W_{\mathrm{ext}})$.

Consider also the "opposite" flag variety $B^\vee \backslash G^\vee$ and the action of $B^\vee$ induced by multiplication on the right. By the Bruhat decomposition, the orbits for this action are naturally parametrized by W, and as explained in Theorem 1.3 in Chapter 3 the graded dimensions of the stalks of the intersection cohomology complexes (with rational coefficients) associated with these orbits are computed by the polynomials $(h_{y,y'} : y, y' \in W)$.

We now consider the $L^+G^\vee$-orbit

$$\mathrm{Gr}'_\rho := L^+G^\vee \backslash L^+G^\vee z^\rho L^+G^\vee$$

in Gr', which is the union of the $I^\vee$-orbits associated with the elements $(t_\rho x : x \in W)$. It is well known that there exists a canonical $L^+G^\vee$-equivariant morphism $\mathrm{Gr}'_\rho \to B^\vee \backslash G^\vee$ (where $L^+G^\vee$ acts on $B^\vee\backslash G^\vee$ via the natural morphism $LG^\vee \to G^\vee$ and the action of $G^\vee$ induced by multiplication on the right) which is Zariski locally trivial with fiber an affine space; in particular this morphism is smooth. This morphism sends the point $L^+G^\vee \backslash L^+G^\vee z^\rho$ to $B^\vee \backslash B^\vee w_0$, hence for an $x \in W$ it sends the $I^\vee$-orbit associated with $t_\rho x$ to the $B^\vee$-orbit associated with $w_0 x$.

Let us now fix $y \in W$. The pullback under the open embedding

$$\mathrm{Gr}'_\rho \hookrightarrow \overline{\mathrm{Gr}'_\rho}$$

of the intersection cohomology complex associated with the $I^\vee$-orbit of $t_\rho y$ in Gr' is the intersection cohomology complex associated with this same orbit, now seen in the variety Gr'_ρ. Hence the graded dimensions of the stalks of the latter complex are computed by the polynomials $(h_{w_0 t_\rho x, w_0 t_\rho y} : x \in W)$. On the other hand, this intersection cohomology complex also identifies with the shifted pullback under the smooth morphism $\mathrm{Gr}'_\rho \to B^\vee\backslash G^\vee$ by [**BBD**, §4.2.6] (see also [**Ac**, Corollary 3.6.9]). Since the graded dimension of the stalks of the latter complex are computed by the polynomials $(h_{x,w_0 y} : x \in W)$, we deduce the desired equality. □

Now we can give the proof of the proposition.

Proof of Proposition 4.9. One can easily check using (2.7) that for any $x \in W$ we have $\ell(t_\rho x) = \ell(t_\rho) - \ell(x)$; therefore the formula (4.11) implies that for $x \in W$ we have

$$[\mathsf{L}_x] = \sum_{y \in W} (-1)^{\ell(x)+\ell(y)} h_{w_0 t_\rho y, w_0 t_\rho x}(1) \cdot [\mathsf{N}_y]$$

in $[\mathcal{O}_\Bbbk]$. In view of Lemma 4.10, this implies that

$$[\mathsf{L}_x] = \sum_{y \in W} (-1)^{\ell(x)+\ell(y)} h_{w_0 y, w_0 x}(1) \cdot [\mathsf{N}_y],$$

again for any $x \in W$. Now recall that the Kazhdan–Lusztig inversion formula (see [**S3**, Remark 3.10]) states that for $z, z' \in W$ we have

$$\sum_{u \in W} (-1)^{\ell(z)+\ell(u)} h_{u,z} h_{uw_0, z'w_0} = \delta_{z,z'}.$$

Hence the formula above can be inverted to obtain that for any $y \in W$ we have

$$[\mathsf{N}_y] = \sum_{x \in W} h_{w_0 y w_0, w_0 x w_0}(1) \cdot [\mathsf{L}_x].$$

We conclude using the fact that the map $z \mapsto w_0 z w_0$ induces an automorphism of the Coxeter system (W, S), so that for $y, w \in W$ we have $h_{w_0 y w_0, w_0 w w_0} = h_{y,w}$. □

CHAPTER 2

Soergel bimodules in their various algebraic incarnations

Soergel bimodules are certain graded bimodules over a polynomial algebra, attached to a choice of a Coxeter system $(\mathcal{W}, \mathcal{S})$ and a representation of $\mathcal{W}$, which can often be used to relate categories of different origins. They were initially introduced by Soergel [S7] (under the name "special bimodules"), as an abstraction of some objects that appeared in his study of category $\mathcal{O}$ of a complex semisimple Lie algebra (see [S1]) and of Harish-Chandra bimodules (see [S2]). Since then they have proved to be invaluable tools in Geometric Representation Theory, in particular because of their great flexibility of use. In this chapter we explain three incarnations of these objects, which make sense (and behave in the expected way) in various levels of generality: the original definition of Soergel (see §1), a "diagrammatic" variant introduced by Elias–Williamson (see §2), and finally a more recent incarnation due to Abe (see §3). The latter two play important roles in the geometric approach to representations of reductive groups (see Chapter 6), while the former one is important in the construction of Williamson's counterexamples (see Chapter 5) and for historical reasons.

Soergel bimodules are also closely related to the parity complexes that will be studied in Chapter 3 so that these objects can also be considered a "topological" incarnation of Soergel bimodules. There are other interesting incarnations that we will not discuss here, like sheaves on moment graphs. For a thorough study of this subject, we refer to [EMTW]; for a brief introduction to Soergel bimodules and a presentation of one of the most exciting recent developments in this subject, we refer to [Ri].

1. "Classical" Soergel bimodules

1.1. Origin: total cohomology of semisimple complexes on flag varieties. Let us start by explaining how one can construct interesting families of bimodules out of semisimple complexes on flag varieties. Consider a complex connected reductive algebraic group $\mathscr{G}$ with a choice of Borel subgroup $\mathscr{B}$ and maximal torus $\mathscr{T}$ contained in $\mathscr{B}$. Let W be the Weyl group of $(\mathscr{G}, \mathscr{T})$, and $S \subset W$ be the system of Coxeter generators determined by $\mathscr{B}$. Let $\mathscr{X} := \mathscr{G}/\mathscr{B}$ be the flag variety of $\mathscr{G}$, and consider the $\mathscr{B}$-equivariant derived category

$$D^{\mathrm{b}}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q})$$

of sheaves of $\mathbb{Q}$-vector spaces on $\mathscr{X}$ (with respect to the obvious action by left multiplication). (The reader not familiar with equivariant derived categories is referred to [BL] or to [Ac, Chap. 6].) A standard construction provides a monoidal product $\star_{\mathscr{B}}$ on this category; see [Ac, §7.2] for details. Recall that the Bruhat decomposition provides a stratification

$$\mathscr{X} = \bigsqcup_{w \in W} \mathscr{X}_w \tag{1.1}$$

of $\mathscr{X}$, see (1.1) in Chapter 3. On the category $D^{\mathrm{b}}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q})$ we have the perverse t-structure, and the general theory of perverse sheaves tells us that the simple objects in the heart of this t-structure are parametrized by W, via the assignment to $w \in W$ of the intersection cohomology complex

$$\mathcal{IC}_w := \mathrm{IC}(\mathscr{X}_w, \underline{\mathbb{Q}})$$

associated with the constant local system on the stratum $\mathscr{X}_w$.

We will denote by

$$\mathsf{IC}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q}) \subset D^{\mathrm{b}}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q})$$

the full subcategory whose objects are the *semisimple complexes*, i.e. the direct sums of cohomological shifts of objects $\mathcal{IC}_w$ ($w \in W$). It follows from the *decomposition theorem* (a deep result in the theory of perverse sheaves) that this subcategory is stable under the convolution product $\star_{\mathscr{B}}$; see [**Ac**, Proposition 7.2.6] for details. (This crucially relies on the fact that our coefficient field, here $\mathbb{Q}$, has characteristic 0.) This category is a Krull–Schmidt category,[1] and its isomorphism classes of indecomposable objects are in bijection with $W \times \mathbb{Z}$ via the map

$$(w, n) \mapsto \mathcal{IC}_w[n].$$

REMARK 1.1. The category $\mathsf{IC}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q})$ is in fact the category of parity complexes in $D^{\mathrm{b}}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q})$; see REF in Chapter 3.

Consider the character lattice $X^*(\mathscr{T})$ of $\mathscr{T}$, and the $\mathbb{Q}$-algebra

$$R := \mathrm{S}(\mathbb{Q} \otimes_{\mathbb{Z}} X^*(\mathscr{T})),$$

which we endow with the grading such that $\mathbb{Q} \otimes_{\mathbb{Z}} X^*(\mathscr{T})$ is in degree 2. We will denote by

$$R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R$$

the abelian category of $\mathbb{Z}$-graded R-bimodules,[2] and consider the functor

$$\mathbb{H} : D^{\mathrm{b}}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q}) \to R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R$$

defined as follows. Given $\mathcal{F}$ in $D^{\mathrm{b}}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q})$, the underlying graded $\mathbb{Q}$-vector space of $\mathbb{H}(\mathcal{F})$ is

$$\mathsf{H}^{\bullet}_{\mathscr{B}}(\mathscr{X}, \mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \mathsf{H}^{n}_{\mathscr{B}}(\mathscr{X}, \mathcal{F})$$

with the obvious grading. (Here, $\mathsf{H}_{\mathscr{B}}$ denotes equivariant cohomology; see [**Ac**, §6.7].) To explain the R-bimodule structure, or in other words the action of $R \otimes_{\mathbb{Q}} R$, on this object, recall that since the unipotent radical of $\mathscr{B}$ is unipotent the natural morphism

$$\mathsf{H}^{\bullet}_{\mathscr{B}}(\mathrm{pt}; \mathbb{Q}) \to \mathsf{H}^{\bullet}_{\mathscr{T}}(\mathrm{pt}; \mathbb{Q})$$

[1] An additive category is called *Krull–Schmidt* if any object has a decomposition as a direct sum of indecomposable objects with local endomorphism rings. In this case, any object has a *unique* decomposition as a direct sum of indecomposable objects up to permutation and isomorphisms, and an object is indecomposable iff its endomorphism algebra is a local ring. For basic properties of this notion, and references, see e.g. [**CYZ**, Appendix A].

[2] See §0.6 for our conventions on bimodules.

is an isomorphism (this follows e.g. from [Ac, Theorem 6.6.16]), and that the right-hand side identifies canonically with R (as a graded ring), see [Ac, Theorem 6.7.7]. Similarly, we have a canonical identification

$$\mathsf{H}^\bullet_{\mathscr{B}\times\mathscr{B}}(\mathrm{pt};\mathbb{Q}) \xrightarrow{\sim} R \otimes_{\mathbb{Q}} R.$$

Now it follows e.g. from [Ac, Theorem 6.5.9] that we have a canonical isomorphism

$$\mathsf{H}^\bullet_{\mathscr{B}}(\mathscr{X};\mathbb{Q}) \xrightarrow{\sim} \mathsf{H}^\bullet_{\mathscr{B}\times\mathscr{B}}(\mathscr{G};\mathbb{Q})$$

where $\mathscr{B}\times\mathscr{B}$ acts on $\mathscr{G}$ via $(b,c)\cdot g = bgc^{-1}$ for $b,c\in\mathscr{B}$ and $g\in\mathscr{G}$. We therefore have a natural morphism of graded algebras

$$R\otimes_{\mathbb{Q}} R \to \mathsf{H}^\bullet_{\mathscr{B}}(\mathscr{X};\mathbb{Q}).$$

By construction $\mathbb{H}(\mathcal{F})$ has a canonical action of $\mathsf{H}^\bullet_{\mathscr{B}}(\mathscr{X};\mathbb{Q})$; using this morphism it therefore acquires a natural action of $R\otimes_{\mathbb{Q}} R$, which finishes the construction of the functor $\mathbb{H}$.

For any $r\in\mathbb{Z}$ we will denote by

$$(r) : R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R \to R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R \tag{1.2}$$

the "shift of grading" autoequivalence which sends an object M to the graded bimodule whose n-th graded piece is

$$(M(r))^n = M^{n+r}$$

for any $n\in\mathbb{Z}$ (with the same R-actions as M). Then it is clear that the functor $\mathbb{H}$ satisfies

$$\mathbb{H}\circ[1] = (1)\circ\mathbb{H}.$$

The following result is proved in [S6, Proposition 2].

PROPOSITION 1.2. *The functor*

$$\mathbb{H} : \mathsf{IC}_{\mathscr{B}}(\mathscr{X},\mathbb{Q}) \to R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R$$

is fully faithful. In other words, for any $w,y\in W$ and $n\in\mathbb{Z}$ this functor induces an isomorphism

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathscr{B}}(\mathscr{X},\mathbb{Q})}(\mathcal{IC}_w, \mathcal{IC}_y[n]) \xrightarrow{\sim} \mathrm{Hom}_{R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R}(\mathbb{H}(\mathcal{IC}_w), \mathbb{H}(\mathcal{IC}_y)(n)). \tag{1.3}$$

REMARK 1.3.
(1) Basic commutative algebra shows that if M,N are graded R-bimodules with M finitely generated, the canonical functor

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R}(M, N(n)) \to \mathrm{Hom}_{R\otimes_{\mathbb{Q}}R}(M,N)$$

is an isomorphism. In this way one sees that Proposition 1.2 is indeed equivalent to [S6, Proposition 2].

(2) Proposition 1.2 is an "equivariant" version of an earlier result for nonequivariant cohomology also due to Soergel, see [S1, Erweiterungssatz 5]. These statements are connected via the fact that for any $\mathcal{F}$ in $\mathsf{IC}_{\mathscr{B}}(\mathscr{X},\mathbb{Q})$ we have a canonical isomorphism

$$\mathbb{Q}\otimes_R \mathsf{H}^\bullet_{\mathscr{B}}(\mathscr{X},\mathcal{F}) \xrightarrow{\sim} \mathsf{H}^\bullet(\mathscr{X},\mathcal{F}) \tag{1.4}$$

where $\mathbb{Q}$ is seen as the trivial R-module. (This fact follows from standard considerations involving an appropriate spectral sequence and parity vanishing, once one knows that, for any $w\in W$, $\mathsf{H}^\bullet(\mathscr{X},\mathcal{IC}_w)$ is concentrated in degrees of the same parity as $\ell(w)$.) For a generalization of the

nonequivariant version to a larger geometric setting, see [Gi]. For yet another proof, see [BGS, Proposition 3.4.2].
(3) This result has variants for flag varieties of Kac–Moody groups; see [Hä] and (in an étale setting) [BY, Proposition 3.1.6].
(4) Proposition 1.2 also has versions for parity complexes (with arbitrary coefficients), where on the right-hand side one considers morphisms as modules over the equivariant cohomology ring of the flag variety; see [MR2, Proposition 3.13 and Remark 3.19]. For an earlier "non equivariant" variant, see [ARi1, Theorem 4.1].

Proposition 1.2 implies that the structure of the category $\mathsf{IC}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q})$ is reflected in a certain full subcategory of $R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R$. To proceed further one has to identify the essential image of $\mathbb{H}$. This is equivalent to describing the image of the objects $\mathcal{IC}_w$; but in fact it turns out to be much easier to describe the image of another family of objects. Namely, for any expression $\underline{w} = (s_1, \cdots, s_r)$, we set

$$\mathcal{IC}_{\underline{w}} := \mathcal{IC}_{s_1} \star_{\mathscr{B}} \cdots \star_{\mathscr{B}} \mathcal{IC}_{s_r}.$$

These objects "generate" the category $\mathsf{IC}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q})$ in the following sense. If $w \in W$, and if $\underline{w}$ is a reduced expression for w, then $\mathcal{IC}_w$ is a direct summand of $\mathcal{IC}_{\underline{w}}$ (see the proof of Theorem 1.3 in Chapter 3). As a consequence, $\mathsf{IC}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q})$ is the full subcategory of $D^{\mathrm{b}}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q})$ whose objects are the direct sums of cohomological shifts of direct summands of objects $\mathcal{IC}_{\underline{w}}$ for expressions $\underline{w}$.

On the other hand, for any $s \in S$ we consider the subalgebra $R^s \subset R$ of s-invariant elements, and set

$$\mathsf{B}_s^{\mathrm{bim}} := R \otimes_{R^s} R(1) \quad \in R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R.$$

Next, given an expression $\underline{w} = (s_1, \cdots, s_r)$, we set

$$\mathsf{B}_{\underline{w}}^{\mathrm{bim}} := \mathsf{B}_{s_1}^{\mathrm{bim}} \otimes_R \cdots \otimes_R \mathsf{B}_{s_r}^{\mathrm{bim}} = R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_r}} R(r).$$

The following statement is the main step of the proof of [S6, Lemma 5].

Proposition 1.4. *For any expression $\underline{w}$ there exists a canonical isomorphism*

$$\mathbb{H}(\mathcal{IC}_{\underline{w}}) \cong \mathsf{B}_{\underline{w}}^{\mathrm{bim}}.$$

The proof of Proposition 1.4 in [S6] proceeds by induction on the length of $\underline{w}$ to reduce to the case of words of length 1, which is standard. In fact one can endow $\mathbb{H}$ with the structure of a monoidal functor to streamline this argument; see [BY, Proposition 3.2.1] or [Ac, Proposition 7.6.9].

Remark 1.5. (1) Once again Proposition 1.4 has an earlier "nonequivariant" version in [S1, §3].
(2) For extensions to Kac–Moody flag varieties, see [Hä] and [BY, §3.2]. For versions for parity complexes, see [MR2, Proposition 3.11] or [Ac, Theorem 7.6.11].

Let us now denote by $X_*(\mathscr{T})$ the cocharacter lattice of $\mathscr{T}$ (which identifies to the dual of $X^*(\mathscr{T})$), and consider the full subcategory $\mathsf{SBim}(W, \mathbb{Q} \otimes_{\mathbb{Z}} X_*(\mathscr{T}))$ of $R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R$ whose objects are the direct sums of grading shifts of direct summands of objects $\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ (for $\underline{w}$ an expression). It is clear from this definition that the tensor product of graded R-bimodules endows this category with a monoidal structure, and Propositions 1.2 and 1.4 (and the comments above) imply that this category is equivalent to $\mathsf{IC}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q})$ as a monoidal category; this is exactly the category of Soergel bimodules associated with the Coxeter system (W, S) and the representation $\mathbb{Q} \otimes_{\mathbb{Z}} X_*(\mathscr{T})$ of W.

The description of indecomposable objects in $\mathsf{IC}_{\mathscr{B}}(\mathscr{X}, \mathbb{Q})$ in terms of the "Bott–Samelson objects" $\mathcal{IC}_{\underline{w}}$ can be transferred to the category $\mathsf{SBim}(W, \mathbb{Q} \otimes_{\mathbb{Z}} X_*(\mathscr{T}))$ via the equivalence $\mathbb{H}$. Namely, for any $w \in W$ we set

$$\mathsf{B}_w^{\mathrm{bim}} := \mathbb{H}(\mathcal{IC}_w). \tag{1.5}$$

Then $\mathsf{B}_w^{\mathrm{bim}}$ is an indecomposable object in $\mathsf{SBim}(W, \mathbb{Q} \otimes_{\mathbb{Z}} X_*(\mathscr{T}))$, and the assignment $(w, n) \mapsto \mathsf{B}_w^{\mathrm{bim}}(n)$ induces a bijection between $W \times \mathbb{Z}$ and the set of isomorphism classes of indecomposable objects in $\mathsf{SBim}(W, \mathbb{Q} \otimes_{\mathbb{Z}} X_*(\mathscr{T}))$. The object $\mathsf{B}_w^{\mathrm{bim}}$ can be characterized intrinsequely as follows: for any reduced expression $\underline{w}$ for w, $\mathsf{B}_w^{\mathrm{bim}}$ is the unique indecomposable direct summand of $\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ which is not isomorphic to a direct summand of $\mathsf{B}_{\underline{y}}^{\mathrm{bim}}(n)$ for some expression $\underline{y}$ of length strictly smaller than $\ell(w)$ and $n \in \mathbb{Z}$. (This follows from the similar characterization of $\mathcal{IC}_w$ in terms of the objects $\mathcal{IC}_{\underline{w}}$.) These statements are prototypes for the main results in the theory of Soergel bimodules.

REMARK 1.6. As explained in Remarks 1.3 and 1.5, what Soergel initially introduced are not the bimodules considered above, but the associated *Soergel modules*, i.e. the objects one obtains by tensoring on the right with the trivial R-module $\mathbb{Q}$. (See §1.9 below for more on Soergel modules.) The *bi*modules for (W, S) and $\mathbb{Q} \otimes_{\mathbb{Z}} X_*(\mathscr{T})$ as above were introduced in [**S2**], considered again in [**S6**], and finally studied algebraically and in a more general context (as explained in §1.4 below) in [**S7**].

1.2. Reflection faithful representations.

1.2.1. *Definition.* As explained above the initial data for the definition of Soergel bimodules are a Coxeter system $(\mathcal{W}, \mathcal{S})$ and a finite-dimensional representation V of $\mathcal{W}$ over some field $\Bbbk$. The definition makes sense for any representation, but for these objects to behave in a reasonable way one needs to impose a technical condition on V that Soergel called *reflection faithful*, and that we now explain.

Denote by $\mathcal{T} \subset \mathcal{W}$ the set of reflections in $\mathcal{W}$, i.e. of conjugates of elements of $\mathcal{S}$. A finite-dimensional representation V of $\mathcal{W}$ over a field $\Bbbk$ with $\mathrm{char}(\Bbbk) \neq 2$ is called *reflection faithful* if it is faithful and if for $x \in \mathcal{W}$ we have

$$\dim(V^x) = \dim(V) - 1 \quad \Leftrightarrow \quad x \in \mathcal{T}. \tag{1.6}$$

Let us note some easy "stability" properties of this notion.

LEMMA 1.7. *Let V be a finite-dimensional representation of $\mathcal{W}$ over the field $\Bbbk$.*

1. *If $I \subset \mathcal{S}$ is a subset and if V is a reflection faithful representation of $(\mathcal{W}, \mathcal{S})$, then the restriction of V to $\mathcal{W}_I$ is a reflection faithful representation of $(\mathcal{W}_I, I)$.*
2. *If $\Bbbk'$ is an extension of $\Bbbk$, then $\Bbbk' \otimes_{\Bbbk} V$ is a reflection faithful representation of $(\mathcal{W}, \mathcal{S})$ (as a representation over $\Bbbk'$) if and only if V is a reflection faithful representation of $(\mathcal{W}, \mathcal{S})$.*
3. *V is a reflection faithful representation of $(\mathcal{W}, \mathcal{S})$ iff the contragredient representation V^* is a reflection faithful representation of $(\mathcal{W}, \mathcal{S})$.*

PROOF. (1) This follows from the definition and the fact that any element in $\mathcal{W}_I \cap \mathcal{T}$ is a reflection for $(\mathcal{W}_I, I)$, i.e. is $\mathcal{W}_I$-conjugate to an element of I, see [**Da**, Lemma 4.2.3].

(2) This follows from the fact that the dimension of the kernel of a matrix with coefficients in $\Bbbk$ is the same as the dimension of the kernel of that matrix regarded as a matrix with coefficients in $\Bbbk'$.

(3) This follows from the fact that the kernel of a matrix and of its transpose have the same dimension. □

There are 2 natural families of representations of Coxeter systems which are known to satisfy this definition, which we now explain.

1.2.2. *Soergel's representation.* Given any Coxeter system $(\mathcal{W}, \mathcal{S})$, one can consider a $\mathbb{R}$-vector space V endowed with a linearly independent family $(e_s : s \in \mathcal{S}) \subset V$ and a linearly independent family $(e_s^* : s \in \mathcal{S}) \subset V^*$ which satisfy, for any $s, t \in \mathcal{S}$,

$$\langle e_t, e_s^* \rangle = \begin{cases} -2\cos(\pi/m_{s,t}) & \text{if } (s,t) \in \mathcal{S}_\circ^2; \\ 2 & \text{if } s = t; \\ -2 & \text{if } s \neq t \text{ and } \langle s,t \rangle \text{ is infinite.} \end{cases}$$

Then the formula

$$s \cdot v = v - \langle v, e_s^* \rangle e_s$$

defines a reflection faithful representation of $(\mathcal{W}, \mathcal{S})$ on V. For a proof of this fact, we refer to [**S7**, §2]. (In this reference it is assumed that V has minimal dimension among $\mathbb{R}$-vector spaces admitting such data, but this condition is not used.)

REMARK 1.8. Recall the *geometric representation* of Coxeter groups, see [**Mi**]. Namely, let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system, and set $V = \mathbb{R}^S$, with canonical basis $(e_s : s \in \mathcal{S})$. We define a symmetric bilinear form $\langle -,- \rangle$ on V by setting for $s, t \in \mathcal{S}$

$$\langle e_s, e_t \rangle = \begin{cases} -\cos(\pi/m_{s,t}) & \text{if } (s,t) \in \mathcal{S}_\circ^2; \\ 1 & \text{if } s = t; \\ -1 & \text{if } s \neq t \text{ and } \langle s,t \rangle \text{ is infinite.} \end{cases}$$

By [**Mi**, Lemma 5.10], the assignment

$$s \mapsto \big(x \mapsto x - 2\langle x, e_s \rangle e_s\big)$$

extends to a representation of $\mathcal{W}$ on V. In case $\mathcal{W}$ is finite, the bilinear form $\langle -,- \rangle$ is positive definite (in particular, non-degenerate), see [**Mi**, Proposition 5.14]. One can therefore choose for V as above this vector space, with the family $(e_s : s \in \mathcal{S})$, and the family $(e_s^* : s \in \mathcal{S})$ defined by $e_s^* = 2\langle e_s, - \rangle$. In particular, the geometric representation is reflection faithful if $\mathcal{W}$ is finite.

For general $\mathcal{W}$, the geometric representation is faithful (see [**Mi**, Lemma 5.11]), but it is not always reflection faithful (see [**EMTW**, Example 5.34]). Note that the main results of the theory of Soergel bimodules still apply for this representation thanks to the results of [**Li2**].

1.2.3. *Representations arising from Kac–Moody algebras.* The other example arises in the theory of Kac–Moody groups and algebras. Namely, let $A = (a_{i,j})_{i,j \in I}$ be a generalized Cartan matrix, with rows and columns parametrized by a finite set I, and let

$$\big(\mathfrak{h}, (\alpha_i : i \in I), (\alpha_i^\vee : i \in I)\big)$$

be a realization of A over $\mathbb{Q}$ in the sense of [**Kac**]. Concretely, this means that $\mathfrak{h}$ is a finite-dimensional $\mathbb{Q}$-vector space, $(\alpha_i : i \in I)$ is a collection of elements in $\mathfrak{h}^*$ parametrized by I, $(\alpha_i^\vee : i \in I)$ is a collection of elements in $\mathfrak{h}$ parametrized by I, and we assume that:

(1) the sets $(\alpha_i : i \in I)$ and $(\alpha_i^\vee : i \in I)$ consist of linearly independent vectors;
(2) for any $i, j \in I$ we have $\langle \alpha_i^\vee, \alpha_j \rangle = a_{i,j}$;
(3) $\dim(\mathfrak{h}) = \#I + \mathrm{cork}(A)$.

(For more about this construction, see [**Ca**, §14.1], replacing the field $\mathbb{C}$ by $\mathbb{Q}$. Note in particular that these data are unique up to isomorphism, see [**Ca**, Proposition 14.3].)

To each generalized Cartan matrix A one can associate a Coxeter system $(\mathcal{W}, \mathcal{S})$ using the following recipe. The set of simple reflections $\mathcal{S}$ is equipped with a fixed bijection $\mathcal{S} \xrightarrow{\sim} I$ (denoted by $s \mapsto i_s$), and for distinct $s, t \in S$, the order of st is determined by the following rule:

$a_{i_s i_t} a_{i_t i_s}$	$m_{s,t}$
0	2
1	3
2	4
3	6
≥ 4	∞

In particular we have $m_{s,t} \in \{2, 3, 4, 6\}$ for any $(s,t) \in \mathcal{S}^2_\circ$; the Coxeter systems which satisfy this condition are called *crystallographic*.

It is a basic fact in the theory of Kac–Moody algebras that the assignment

$$s \mapsto (\lambda \mapsto \lambda - \langle \alpha_{i_s}^\vee, \lambda \rangle \alpha_{i_s})$$

defines an action of $\mathcal{W}$ on $\mathbb{C} \otimes_{\mathbb{Q}} \mathfrak{h}^*$, see [**Kac**, Proposition 3.13] or [**Ku**, Definition 1.3.1 and Proposition 1.3.11]; since these automorphisms are induced by automorphisms of $\mathfrak{h}^*$, it follows that the same recipe defines an action of $\mathcal{W}$ on $\mathfrak{h}^*$. This representation turns out to be reflection faithful. In fact, by Lemma 1.7(2) it suffices to prove that the representation on $\mathbb{C} \otimes_{\mathbb{Q}} \mathfrak{h}^*$ is reflection faithful. Faithfulness follows from the fact that $\mathcal{W}$ can be defined as a subgroup of $\mathrm{GL}(\mathbb{C} \otimes_{\mathbb{Q}} \mathfrak{h}^*)$, see [**Ku**, Definition 1.3.1]. (See [**Ku**, Proposition 1.3.21] for the identification with the group defined above.) The condition (1.6) is checked in [**Ku**, Lemma 11.2.2]. By Lemma 1.7(3), the representation $\mathfrak{h}$ is also a reflection faithful representation of $(\mathcal{W}, \mathcal{S})$.

REMARK 1.9. See [**Ri**, Proposition 1.1(2)] for a different proof of the fact that this representation is reflection faithful, based on the arguments in [**S7**, §2], under the additional assumption that A is symmetrizable.

1.2.4. *More representations arising from Kac–Moody theory.* Let again A be a generalized Cartan matrix, whose rows and columns are parametrized by some finite set I. Recall that a *Kac–Moody root datum* associated with A is a triple

$$(\mathbf{X}, (\alpha_i : i \in I), (\alpha_i^\vee : i \in I))$$

where $\mathbf{X}$ is a finite free $\mathbb{Z}$-module, $(\alpha_i : i \in I)$ is a family of elements of $\mathbf{X}$ parametrized by I, and $(\alpha_i^\vee : i \in I)$ is a family of elements of $\mathbf{X}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}, \mathbb{Z})$ which satisfy

$$\langle \alpha_i^\vee, \alpha_j \rangle = a_{i,j},$$

see [Ti]. As in §1.2.3, to A we associate a Coxeter system $(\mathcal{W}, \mathcal{S})$; once again, it is a basic fact that the assignment

$$s \mapsto (\lambda \mapsto \lambda - \langle \alpha_{i_s}^\vee, \lambda \rangle \alpha_{i_s})$$

defines an action of $\mathcal{W}$ on $\mathbf{X}$, see [Ti, §3.1].

EXAMPLE 1.10. Let $\mathscr{G}$ be a connected reductive algebraic group (over some algebraically closed field) with a choice of Borel subgroup $\mathscr{B}$ and maximal torus $\mathscr{T} \subset \mathscr{B}$, with associated Cartan matrix A and Weyl group $(\mathcal{W}, \mathcal{S})$. Then A is a generalized Cartan matrix (called *of finite type*), the associated Coxeter system is $(\mathcal{W}, \mathcal{S})$, and one can take for $\mathbf{X}$ the character lattice $X^*(\mathscr{T})$, for $(\alpha_i : i \in I)$ the collection of simple roots, and for $(\alpha_i^\vee : i \in I)$ the collection of simple coroots. The associated Coxeter system is the pair (W, S) where W is the Weyl group of $(\mathscr{G}, \mathscr{T})$ and S is the system of Coxeter generators determined by $\mathscr{B}$, and the associated representation over $\mathbb{Q}$ is that considered in §1.1.

Given any field $\Bbbk$, one deduces a representation of $\mathcal{W}$ with underlying vector space $\Bbbk \otimes_{\mathbb{Z}} \mathbf{X}$. In general this representation is not faithful, hence a fortiori not reflection faithful. (For instance, if $p = \mathrm{char}(\Bbbk) > 0$ and $\mathcal{W}$ is infinite this representation cannot be faithful because it factors through an action on $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbf{X}$ and $\mathrm{GL}(\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbf{X})$ is finite.)

We claim that the representation $\mathfrak{h}$ considered in §1.2.3 is a special case of this construction, with $\Bbbk = \mathbb{Q}$. In fact, consider a triple $(\mathfrak{h}, (\alpha_i : i \in I), (\alpha_i^\vee : i \in I))$ as in §1.2.3. Let $(v_j : j \in J)$ be a set of vectors in $\mathfrak{h}$ such that

- $(\alpha_i^\vee : i \in I) \cup (v_j : j \in J)$ is a $\mathbb{Q}$-basis of $\mathfrak{h}$;
- for any $i \in I$ and $j \in J$, $\langle \alpha_i, v_j \rangle \in \mathbb{Z}$.

(Such a family of vectors exists: it suffices to start with any family that completes $(\alpha_i^\vee : i \in I)$ to a basis, and then to multiply these vectors by appropriate integers to ensure that the second conditions is satisfied.) Set

$$\mathbf{Y} := \left(\bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee \right) \oplus \left(\bigoplus_{j \in J} \mathbb{Z} v_j \right) \quad \subset \mathfrak{h}$$

and $\mathbf{X} := \mathrm{Hom}_{\mathbb{Z}}(\mathbf{Y}, \mathbb{Z})$. Then we have an identification

$$\mathbf{X} = \{\lambda \in \mathfrak{h}^* \mid \forall i \in I,\ \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ and } \forall j \in J,\ \langle \lambda, v_j \rangle \in \mathbb{Z}\}, \tag{1.7}$$

and natural isomorphisms

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{Y} \xrightarrow{\sim} \mathfrak{h}, \quad \mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{X} \xrightarrow{\sim} \mathfrak{h}^*.$$

Each $\alpha_i^\vee$ belongs to $\mathbf{Y}$, and under the identification (1.7) each α_i belongs to $\mathbf{X}$. Hence the triple $(\mathbf{X}, (\alpha_i : i \in I), (\alpha_i^\vee : i \in I))$ is a Kac–Moody root datum for A, and the associated representation over $\mathbb{Q}$ is $\mathfrak{h}$.

There is another special case of this construction which *does* produce reflection faithful representations. Namely, let $\mathscr{G}$ be a simply connected semisimple algebraic group (over some algebraically closed field $\mathbb{F}$), with a Borel subgroup $\mathscr{B}$ and a maximal torus $\mathscr{T} \subset \mathscr{B}$. If we denote by $\mathbf{X} = X^*(\mathscr{T})$ the character lattice of $\mathscr{T}$, by $(\alpha_i : i \in I)$ the basis of the root system of $(\mathscr{G}, \mathscr{T})$ determined by $\mathscr{B}$, and by $(\alpha_i^\vee : i \in I)$ the associated coroots, then $(\mathbf{X}, (\alpha_i : i \in I), (\alpha_i^\vee : i \in I))$ is a Kac–Moody root datum associated with the (generalized) Cartan matrix $A = (\langle \alpha_i^\vee, \alpha_j \rangle)_{i,j \in I}$, see Example 1.10. In this setting, it is proved in [**Li3**, Appendix A] that the representation of W on $\Bbbk \otimes_{\mathbb{Z}} \mathbf{X}$ is a reflection faithful representation of (W, S) if $\mathrm{char}(\Bbbk) \notin \{2, 3\}$.

The categories of Soergel bimodules and Soergel modules in this case are closely related to the categories of parity complexes on $\mathscr{G}/\mathscr{B}$, as explained in case $\Bbbk = \mathbb{Q}$ in §1.1. This observation can be used to study them without reference to the general theory, even in the case when $\mathrm{char}(\Bbbk) > 0$; see [**S5**] and [**AR1**] for this approach.

1.3. Soergel bimodules. From now on we fix a Coxeter system $(\mathcal{W}, \mathcal{S})$ and a finite-dimensional representation V of $\mathcal{W}$ over a field $\Bbbk$ whose characteristic is not 2. We set

$$R := \mathrm{S}(V^*).$$

(Here R can be interpreted geometrically as the algebra of functions—in the sense of algebraic geometry—on V seen as an affine space over k.) We endow this $\Bbbk$-algebra with the grading such that V^* is concentrated in degree 2, and then consider the category

$$R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R$$

of $\mathbb{Z}$-graded R-bimodules. This category admits a monoidal structure, with product the tensor product $\otimes_R$ over R (for the right-action on the left factor, and the left action on the right factor). We define the “shift of grading” equivalences (r) as in (1.2) (for $r \in \mathbb{Z}$).

The algebra R also admits an action of $\mathcal{W}$ by graded algebra automorphisms, induced by the action on V. For any $s \in \mathcal{S}$ we denote by $R^s \subset R$ the subalgebra of s-invariant elements, and set

$$\mathsf{B}_s^{\mathrm{bim}} := R \otimes_{R^s} R(1) \quad \in R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R.$$

Recall that a *reflection* of a finite-dimensional vector space (over a field whose characteristic is not 2) is an endomorphism whose square is id and which acts as the identity on a hyperplane. Note that our assumption on $\mathrm{char}(\Bbbk)$ implies that such an endomorphism is diagonalizable, with eigenvalues 1 and -1; the associated eigenspaces have dimension $\dim(V) - 1$ and 1 respectively.

LEMMA 1.11. *Assume that s acts on V as a reflection, and let $\alpha \in V^*$ be an element such that $s(\alpha) = -\alpha$. Then, as a graded R^s-module, R is graded free with basis $(1, \alpha)$. As a consequence, as graded left R-modules (or as graded right R-modules) we have*

$$\mathsf{B}_s^{\mathrm{bim}} \cong R(1) \oplus R(-1).$$

PROOF. If s acts as a reflection on V, then it also acts as a reflection on V^*. Hence there exists a hyperplane $H \subset V^*$ on which s acts as the identity and such that

$$V^* = H \oplus \Bbbk \cdot \alpha.$$

Then we have
$$R = \bigoplus_{n \geq 0} \mathrm{S}(H) \cdot \alpha^n,$$
and
$$R^s = \bigoplus_{\substack{n \geq 0 \\ n \text{ even}}} \mathrm{S}(H) \cdot \alpha^n.$$
The first claim is then clear. This claim implies that as graded left R^s-modules (or as graded right R^s-modules) we have $R \cong R^s \oplus R^s(-2)$. The claim about $\mathsf{B}_s^{\mathrm{bim}}$ follows. □

Next, for any expression $\underline{w} = (s_1, \cdots, s_r)$ we set
$$\mathsf{B}_{\underline{w}}^{\mathrm{bim}} := \mathsf{B}_{s_1}^{\mathrm{bim}} \otimes_R \cdots \otimes_R \mathsf{B}_{s_r}^{\mathrm{bim}} = R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_r}} R(r).$$
(By convention, this tensor product is interpreted as R with its canonical bimodule structure in case $r = 0$, i.e. $\underline{w}$ is the empty word.) These bimodules are sometimes called *Bott–Samelson bimodules*, because of their relation with Bott–Samelson resolutions of Schubert varieties (see REF).

It is clear that if $\underline{w}$ and $\underline{y}$ are expressions we have
$$\mathsf{B}_{\underline{w}}^{\mathrm{bim}} \otimes_R \mathsf{B}_{\underline{y}}^{\mathrm{bim}} = \mathsf{B}_{\underline{wy}}^{\mathrm{bim}}$$
where $\underline{wy}$ is the word obtained by concatenation of $\underline{w}$ and $\underline{y}$. Lemma 1.11 implies that if each element in $\mathcal{S}$ acts on V as a reflection, then $\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ is graded free as a graded left R-module and as a graded right R-module (with a graded rank which can easily be computed).

The category $\mathsf{SBim}(\mathcal{W}, V)$ of Soergel bimodules associated with $(\mathcal{W}, \mathcal{S})$ and the representation V is the full subcategory of R-$\mathsf{Mod}^{\mathbb{Z}}$-$R$ whose objects are the direct sums of grading shifts of direct summands of objects $\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ (for $\underline{w}$ a word in $\mathcal{S}$). There is no reason to expect that this category is well behaved for an arbitrary choice of V. But the magic of this theory is that the properties observed in §1.1 (for a very specific choice closely related to geometry) continue to hold when V is reflection faithful.

REMARK 1.12. The category R-$\mathsf{Mod}^{\mathbb{Z}}$-$R$ admits an autoequivalence φ that switches the left and right actions of R. This autoequivalence is "anti-monoidal" in the sense that for M, N in R-$\mathsf{Mod}^{\mathbb{Z}}$-$R$ there exists a canonical isomorphism
$$\varphi(M \otimes_R N) \cong \varphi(N) \otimes_R \varphi(M).$$
It is clear that $\varphi(\mathsf{B}_s^{\mathrm{bim}}) = \mathsf{B}_s^{\mathrm{bim}}$ for any $s \in \mathcal{S}$, hence for any expression $\underline{w}$ the bimodule $\varphi(\mathsf{B}_{\underline{w}}^{\mathrm{bim}})$ is the bimodule associated with the expression obtained from $\underline{w}$ by reversing the order of the simple reflections. In particular, φ stabilizes $\mathsf{SBim}(\mathcal{W}, V)$.

A simple property which is true under a much weaker assumption is that $\mathsf{SBim}(\mathcal{W}, V)$ is a Krull-Schmidt category if each element in $\mathcal{S}$ acts on V as a reflection. Indeed, consider first the abelian full subcategory of R-$\mathsf{Mod}^{\mathbb{Z}}$-$R$ whose objects are the graded bimodules which are finitely generated (as bimodules). For any M, N in this subcategory we have $\dim \mathrm{Hom}_{R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R}(M, N) < \infty$. In view of [**CYZ**, Remark A.2] it follows that this subcategory is Krull–Schmidt. It follows that the subcategory of R-$\mathsf{Mod}^{\mathbb{Z}}$-$R$ whose objects are the graded bimodules which are finitely generated as left R-modules is Krull–Schmidt too. This subcategory is easily seen to be stable under the tensor product $\otimes_R$, direct sums, and direct summands. Lemma 1.11 shows that it contains each $\mathsf{B}_s^{\mathrm{bim}}$; it therefore contains $\mathsf{SBim}(\mathcal{W}, V)$, which is therefore Krull–Schmidt.

Under the same assumption, Lemma 1.11 implies that any object of $\mathsf{SBim}(\mathcal{W}, V)$ is finitely generated projective as a graded left R-module (or right R-module), hence that it is in fact graded free (both as a left R-module and as a right R-module, but of course not as an R-bimodule).

REMARK 1.13. By Lemma 1.7(1), for any $I \subset \mathcal{S}$ the restriction of V to $\mathcal{W}_I$ is a reflection faithful representation of $(\mathcal{W}_I, I)$. If $\underline{w}$ is an expression for $(\mathcal{W}_I, I)$, then it can be considered as an expression of $(\mathcal{W}, \mathcal{S})$, and the bimodules $\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ are the same for the two possible interpretations of $\underline{w}$. It follows that $\mathsf{SBim}(\mathcal{W}_I, V)$ is contained in $\mathsf{SBim}(\mathcal{W}, V)$ (as full subcategories of $R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R$).

1.4. Structure for reflection-faithful representations. We now assume that V is reflection faithful, and that $\Bbbk$ is infinite. The first main result on Soergel bimodules is the following theorem; see [**S7**, Theorem 1.10].

THEOREM 1.14. *There exists a unique ring homomorphism*

$$\varepsilon : \mathcal{H}_{(\mathcal{W},\mathcal{S})} \to [\mathsf{SBim}(\mathcal{W}, V)]_{\oplus}$$

such that $\varepsilon(v) = [R(1)]$ *and* $\varepsilon(\underline{H}_s) = [\mathsf{B}_s^{\mathrm{bim}}]$.

In view of Remark 4.2 in Chapter 1, there is an obvious strategy for proving Theorem 1.14. The $\mathbb{Z}$-module $[\mathsf{SBim}(\mathcal{W}, V)]_{\oplus}$ has a ring structure induced by the monoidal product on the category $\mathsf{SBim}(\mathcal{W}, V)$. One turns this ring into a $\mathbb{Z}[v, v^{-1}]$-algebra by defining the action of v as multiplication (on the left or on the right) by $[R(1)]$. Then one should prove that the elements $[\mathsf{B}_s^{\mathrm{bim}}] - v$ satisfy the quadratic relations and the braid relations. The quadratic relations are immediate consequences of Lemma 1.11; in fact we have

$$\begin{aligned}(1.8)\quad \mathsf{B}_s^{\mathrm{bim}} \otimes_R \mathsf{B}_s^{\mathrm{bim}} &= R \otimes_{R^s} R \otimes_{R^s} R(2) \\ &= R \otimes_{R^s} R(1) \oplus R \otimes_{R^s} R(-1) = \mathsf{B}_s^{\mathrm{bim}}(1) \oplus \mathsf{B}_s^{\mathrm{bim}}(-1)\end{aligned}$$

since $R \cong R^s \oplus R^s(-2)$ as a graded R^s-bimodule, see Lemma 1.11. This implies that

$$[\mathsf{B}_s^{\mathrm{bim}}] \cdot [\mathsf{B}_s^{\mathrm{bim}}] = (v + v^{-1}) \cdot [\mathsf{B}_s^{\mathrm{bim}}],$$

which is equivalent to the quadratic relation for $[\mathsf{B}_s^{\mathrm{bim}}] - v$.

Checking the braid relations turns out to be a bit more difficult. The verification in [**S7**] relies on a fine study of Soergel bimodules in case $\mathcal{W}$ is a finite dihedral group (for the obvious choice of Coxeter generators). This is proved in fact under slightly weaker assumptions, namely that

(1) $\Bbbk$ is infinite;
(2) each $t \in \mathcal{T}$ acts on V as a reflection;
(3) distinct elements in $\mathcal{T}$ act with distinct -1-eigenspaces.

(These condition is satisfied for reflection faithful representations by Exercise 2.1.) Theorem 1.14 is therefore valid under this weaker assumption.

An earlier proof, based on different arguments, appears in [**S2**] for the special case when V is the complexification of the geometric representation. (As explained in Remark 1.8, this representation is not always reflection faithful.)

REMARK 1.15. (1) In [S7], Soergel denotes by $\mathcal{R}$ the full subcategory of $R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R$ whose objects are the bimodules which are finitely generated both as left and as right R-modules. He defines ε as taking values in $[\mathcal{R}]_\oplus$, and then defines the "special bimodules" as the objects in $\mathcal{R}$ whose class in $[\mathcal{R}]_\oplus$ belongs to the image of ε. By [S7, Lemma 5.13], any special bimodule is a direct summand of a direct sum of objects of the form $\mathsf{B}^{\mathrm{bim}}_{\underline{w}}(n)$, hence belongs to our category $\mathsf{SBim}(\mathcal{W}, V)$. Conversely, it is clear that each $\mathsf{B}^{\mathrm{bim}}_{\underline{w}}(n)$ is special. Since the category of special bimodules is stable under direct sums, and also under direct summands (by [S7, Satz 6.14(4)]), evey object of $\mathsf{SBim}(\mathcal{W}, V)$ is special. In the end, our definition is thus consistent with that in [S7].

(2) One might ask for a description of the "standard basis" $(H_w : w \in \mathcal{W})$ of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ under the morphism of Theorem 1.14. One thing which is clear is that the image of such elements is not in general the class of a bimodule; for instance for $s \in \mathcal{S}$ we have $H_s = \underline{H}_s - v$, hence $\varepsilon(H_s) = [\mathsf{B}^{\mathrm{bim}}_s] - [R(1)]$. If $\varepsilon(H_s)$ was the class of an object M of $\mathsf{SBim}(\mathcal{W}, V)$, then decomposing M as a sum of indecomposable objects and using Theorem 1.16 below would provide a contradiction. However, by the main result of [Ros] the obvious morphism

$$[\mathsf{SBim}(\mathcal{W}, V)]_\oplus \to [K^{\mathrm{b}}\mathsf{SBim}(\mathcal{W}, V)]_\Delta$$

is an isomorphism. One can thus ask a different question: does there exist a "natural" object in $K^{\mathrm{b}}\mathsf{SBim}(\mathcal{W}, V)$ whose class in $[K^{\mathrm{b}}\mathsf{SBim}(\mathcal{W}, V)]_\Delta$ is H_w? This question turns out to have a very interesting answer, provided by Rouquier [Ro1]. As a warm-up, consider the case $w = s \in \mathcal{S}$. Then there exists an obvious "multiplication" morphism $R \otimes_{R^s} R \to R$, which provides a morphism $m_s : \mathsf{B}^{\mathrm{bim}}_s \to R(1)$ in $\mathsf{SBim}(\mathcal{W}, V)$. One can therefore consider the complex

$$F_s := (\cdots \to 0 \to \mathsf{B}^{\mathrm{bim}}_s \xrightarrow{m_s} R(1) \to 0 \to \cdots)$$

where $\mathsf{B}^{\mathrm{bim}}_s$ is placed in degree 0. The class of this complex is clearly H_s. Rouquier proves[3] in [Ro1] that for any $(s,t) \in \mathcal{S}^2_\circ$ we have

$$\underbrace{F_s \otimes_R F_t \otimes_R \cdots}_{m_{s,t}\text{ terms}} \cong \underbrace{F_t \otimes_R F_s \otimes_R \cdots}_{m_{s,t}\text{ terms}}$$

in $K^{\mathrm{b}}\mathsf{SBim}(\mathcal{W}, V)$, where we still denote by $\otimes_R$ denotes the obvious extension of this bifunctor to $K^{\mathrm{b}}\mathsf{SBim}(\mathcal{W}, V)$. As a consequence, using Matsumoto's lemma (see [Mi, Theorem 4.2(iv)]), if $w \in \mathcal{W}$ and $w = s_1 \cdots s_r$ is a reduced expression the complex

$$F_w := F_{s_1} \otimes_R \cdots \otimes_R F_{s_r}$$

does not depend on the choice of reduced expression, i.e. only depends on w. It is clear that its class in $[K^{\mathrm{b}}\mathsf{SBim}(\mathcal{W}, V)]_\Delta$ is H_w.

The complexes $(F_w : w \in \mathcal{W})$ are called *Rouquier complexes*; they have found important applications to link invariants; for a discussion,

[3] In this paper Rouquier works with V being the geometric representation. His proof however applies similarly for reflection faithful representations.

see [EMTW, Chap. 21]. For more on Rouquier complexes, and a discussion of their role in the proof of the Elias–Williamson theorem discussed in §1.8 below, see [EMTW, §19.3].

The next important result is the classification of indecomposable objects in $\mathsf{SBim}(\mathcal{W}, V)$.

THEOREM 1.16. *For any $w \in \mathcal{W}$ there exists a unique indecomposable object $\mathsf{B}_w^{\mathrm{bim}} \in \mathsf{SBim}(\mathcal{W}, V)$ which satisfies the property that for any reduced expression $\underline{w}$ for w, $\mathsf{B}_w^{\mathrm{bim}}$ is the unique indecomposable summand of $\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ which is not a direct summand of an object $\mathsf{B}_{\underline{y}}^{\mathrm{bim}}(n)$ with $\underline{y}$ a reduced expression of an element $y < w$ and $n \in \mathbb{Z}$. Moreover, the assignment*

$$(w, n) \mapsto \mathsf{B}_w^{\mathrm{bim}}(n)$$

induces a bijection between $\mathcal{W} \times \mathbb{Z}$ and the set of isomorphism classes of indecomposable objects in $\mathsf{SBim}(\mathcal{W}, V)$.

As a consequence, the family $([\mathsf{B}_w^{\mathrm{bim}}] : w \in \mathcal{W})$ forms a basis of $[\mathsf{SBim}(\mathcal{W}, V)]_{\oplus}$ over $\mathbb{Z}[v, v^{-1}]$.

Given an object $M \in \mathsf{SBim}(\mathcal{W}, V)$, the integers $(a_{w,n} : w \in \mathcal{W},\ n \in \mathbb{Z})$ in the decomposition

$$M \cong \bigoplus_{\substack{w \in \mathcal{W} \\ n \in \mathbb{Z}}} (\mathsf{B}_w^{\mathrm{bim}}(n))^{\oplus a_{w,n}}$$

(which are well defined thanks to the Krull–Schmidt property) are determined by the coefficients of the expansion of $[M]$ in the basis $([\mathsf{B}_w^{\mathrm{bim}}] : w \in \mathcal{W})$: we have

$$[M] = \sum_{w \in \mathcal{W}} \left(\sum_{n \in \mathbb{Z}} a_{w,n} v^n \right) \cdot [\mathsf{B}_w^{\mathrm{bim}}].$$

REMARK 1.17. (1) The characterization of the object $\mathsf{B}_w^{\mathrm{bim}}$ given in [S7, Satz 6.14(1)] is different from that given in Theorem 1.16. However, Soergel explains in [S7, Bemerkung 6.16] that for any $w \in \mathcal{W}$ and any reduced expression $\underline{w}$ for w we have

$$\mathsf{B}_{\underline{w}}^{\mathrm{bim}} \cong \mathsf{B}_w^{\mathrm{bim}} \oplus \bigoplus_{\substack{y \in \mathcal{W}, y < w \\ n \in \mathbb{Z}}} (\mathsf{B}_y^{\mathrm{bim}}(n))^{a_{y,n}^{\underline{w}}} \tag{1.9}$$

for some nonnegative integers $a_{y,n}^{\underline{w}}$ (which moreover satisfy $a_{y,n}^{\underline{w}} = a_{y,-n}^{\underline{w}}$). It follows that $\mathsf{B}_w^{\mathrm{bim}}$ is also characterized by the condition stated in Theorem 1.16.

(2) Recall the autoequivalence φ from Remark 1.12. Then for any $w \in \mathcal{W}$ we have $\varphi(\mathsf{B}_w^{\mathrm{bim}}) \cong \mathsf{B}_{w^{-1}}^{\mathrm{bim}}$.

(3) In case (W, S) is as in §1.1 and $V = \mathbb{Q} \otimes_{\mathbb{Z}} X_*(\mathscr{T})$, the object $\mathsf{B}_w^{\mathrm{bim}}$ coincides with the graded bimodule denoted in the same way in (1.5).

(4) The characterization of $\mathsf{B}_w^{\mathrm{bim}}$ given in Theorem 1.16 provides an inductive procedure for constructing it: if $\underline{w}$ is a reduced expression for w, and if one knows the objects $\mathsf{B}_y^{\mathrm{bim}}$ for any $y \in \mathcal{W}$ such that $y < w$, the integer $a_{y,n}^{\underline{w}}$ is the largest integer a such that there exist morphisms $f_i : \mathsf{B}_y^{\mathrm{bim}}(n) \to \mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ and $g_i : \mathsf{B}_{\underline{w}}^{\mathrm{bim}} \to \mathsf{B}_y^{\mathrm{bim}}(n)$ $(i \in \{1, \cdots, a\})$ such that $g_j \circ f_i = \delta_{i,j}\,\mathrm{id}$; using these maps one determines a subbimodule of $\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$

isomorphic to $\bigoplus_{\substack{y<w \\ n\in\mathbb{Z}}} (\mathsf{B}^{\mathrm{bim}}_y(n))^{a^w_{y,n}}$, and $\mathsf{B}^{\mathrm{bim}}_w$ is a complement. Of course, this procedure is very difficult to run in practice, and there are very few examples for which the object $\mathsf{B}^{\mathrm{bim}}_w$ admits an explicit description. Among those, one can cite the elements belonging to the subgroup generated by a pair of elements in $\mathcal{S}$ (see [S7, §4], to which one can reduce by Lemma 1.7(1)). Another case is when w is the longest element in a finite parabolic subgroup $\mathcal{W}_I$ of $\mathcal{W}$ ($I \subset \mathcal{S}$); in this case we have $\mathsf{B}^{\mathrm{bim}}_w = R \otimes_{R^{\mathcal{W}_I}} R(\ell(w))$. A quick way to check this uses the theory of *singular Soergel bimodules* from [W1], see Remark 1.19 below. In fact, R^{W_I} is (up to shift) the indecomposable singular Soergel bimodule with "singularity" (I, I) associated with the trivial coset in $\mathcal{W}_I\backslash\mathcal{W}/\mathcal{W}_I$. Then the claim follows from [W1, Proposition 7.11(1)].

(5) As explained in Remark 1.13, if $I \subset \mathcal{S}$ the category $\mathsf{SBim}(\mathcal{W}_I, V)$ is contained in $\mathsf{SBim}(\mathcal{W}, V)$. If $w \in \mathcal{W}_I$, then the two possible meanings of $\mathsf{B}^{\mathrm{bim}}_w$ define isomorphic bimodules.

(6) In the course of the proof of Theorem 1.16 Soergel proves another useful theorem, which allows to compute the dimension of the space of morphisms of graded bimodules between any pair of Soergel bimodules; for the precise formula, see [S7, Theorem 5.15] or [EMTW, Theorem 5.27]. This statement is often called "Soergel's Hom formula." We will encounter a formula with similar flavor in the geometric setting of Chapter 3: see Proposition 2.8 in that chapter.

Once Theorem 1.16 is proved, one obtains the following.

Corollary 1.18. *The morphism ε of Theorem 1.14 is an isomorphism. Moreover, for any $w \in \mathcal{W}$ we have*

$$\varepsilon^{-1}([\mathsf{B}^{\mathrm{bim}}_w]) \in H_w + \sum_{y<w} \mathbb{Z}[v, v^{-1}] \cdot H_y. \tag{1.10}$$

Proof. Choose, for any $w \in \mathcal{W}$, a reduced expression $\underline{w}$ for w, and denote by $\mathcal{W}'$ the set of expressions obtained in this way. The Krull–Schmidt property implies that the classes $([\mathsf{B}^{\mathrm{bim}}_w] : w \in \mathcal{W})$ form a basis of $[\mathsf{SBim}(\mathcal{W}, V)]$ over $\mathbb{Z}[v, v^{-1}]$. In view of the decompositions (1.9), the same is true for the classes $([\mathsf{B}^{\mathrm{bim}}_{\underline{w}}] : \underline{w} \in \mathcal{W}')$. On the other side, for an expression $\underline{w} = (s_1, \cdots, s_r)$ we set

$$\underline{H}_{\underline{w}} = \underline{H}_{s_1} \cdots \underline{H}_{s_r}. \tag{1.11}$$

Then if $\underline{w}$ is a reduced expression for w it is easily seen that

$$\underline{H}_{\underline{w}} \in H_w + \sum_{y<w} \mathbb{Z}[v, v^{-1}] \cdot H_y.$$

As a consequence, the set $(\underline{H}_{\underline{w}} : \underline{w} \in \mathcal{W}')$ forms a basis of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$. It is clear from definitions that for any expression $\underline{w}$ we have

$$\varepsilon(\underline{H}_{\underline{w}}) = [\mathsf{B}^{\mathrm{bim}}_{\underline{w}}].$$

Hence ε sends a basis to a basis, and is therefore an isomorphism.

To prove (1.10), one proceeds by induction on the Bruhat order using (1.9). $\square$

In the course of the proof of Theorem 1.16, Soergel constructs a map $h_\Delta : [\mathsf{SBim}(\mathcal{W}, V)]_\oplus \to \mathcal{H}_{(\mathcal{W},\mathcal{S})}$ which is a left inverse (hence an inverse) to ε (see [S7, Proposition 5.7(3) and Bemerkung 5.14], or [**EMTW**, Definition 5.11 and Theorem 5.24(3)]). Manifestly, for any object M in $\mathsf{SBim}(\mathcal{W}, V)$, all the coefficients of the expansion of $h_\Delta([M])$ in the basis $(H_w : w \in \mathcal{W})$ have nonnegative coefficients. In particular, (1.10) can be refined to the statement that

$$\varepsilon^{-1}([\mathsf{B}_w^{\mathrm{bim}}]) \in H_w + \sum_{y<w} \mathbb{Z}_{\geq 0}[v, v^{-1}] \cdot H_y. \tag{1.12}$$

For any expression $\underline{w}$, from the definition one sees that the lowest degree in which $\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ is nonzero is $-\ell(\underline{w})$, and that the component in this degree has dimension 1. If $w \in \mathcal{W}$ and $\underline{w}$ is a reduced expression for w, since $\mathsf{B}_w^{\mathrm{bim}}$ is a direct summand in $\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$, we deduce that $\mathsf{B}_w^{\mathrm{bim}}$ vanishes in degrees $< -\ell(w)$, and that its component of degree $-\ell(w)$ has dimension at most 1. In fact, using the explicit description of ε^{-1} and (1.10) one can check that

$$\dim((\mathsf{B}_w^{\mathrm{bim}})^{-\ell(w)}) = 1; \tag{1.13}$$

see [**Ri**, Remarque 1.10(1)] for details.

Remark 1.19. In [**W1**], G. Williamson developed a "singular" version of Soergel bimodules, in the form of categories of graded $(R^{\mathcal{W}_I}, R^{\mathcal{W}_J})$-bimodules for subsets $I, J \subset \mathcal{S}$. In the case of representations as in §1.2.3–1.2.4 these objects are connected[4] to semisimple complexes (or parity complexes) on *parabolic* flag varieties in the same way that "usual" Soergel bimodules are related to semisimple complexes on flag varities; see §1.1. As we have already used in Remark 1.17(4), the description of indecomposable singular Soergel bimodules reduces to the description of the indecomposable objects in $\mathsf{SBim}(\mathcal{W}, V)$, see [**W1**, Proposition 7.11(1)].

1.5. Extension of scalars. Let V be a reflection faithful representation of $(\mathcal{W}, \mathcal{S})$ over a field $\Bbbk$, and let $\Bbbk'$ be an extension of $\Bbbk$. By Lemma 1.7(2) the tensor product $V' := \Bbbk' \otimes_\Bbbk V$ is a reflection faithful representation of $(\mathcal{W}, \mathcal{S})$ over $\Bbbk'$, hence we can consider both the categories $\mathsf{SBim}(\mathcal{W}, V)$ (a $\Bbbk$-linear category) and $\mathsf{SBim}(\mathcal{W}, V')$ (a $\Bbbk'$-linear category). In $\mathsf{SBim}(\mathcal{W}, V)$ we have the objects $\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ attached to expressions $\underline{w}$ and the indecomposable objects $(\mathsf{B}_w^{\mathrm{bim}} : w \in \mathcal{W})$, and in $\mathsf{SBim}(\mathcal{W}, V')$ we have the similar objects, which we denote by ${}'\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ and ${}'\mathsf{B}_w^{\mathrm{bim}}$ respectively.

If we set $R = \mathrm{S}_\Bbbk(V^*)$ and $R' := \mathrm{S}_{\Bbbk'}((V')^*)$, considered as graded algebras as in §1.3, then the functor $\Bbbk' \otimes (-)$ induces a functor

$$R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R \to R'\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R'$$

For any M, N in $R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R$ with M finitely generated (as a bimodule) we have

$$\mathrm{Hom}_{R'\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R'}(\Bbbk' \otimes_\Bbbk M, \Bbbk' \otimes_\Bbbk N) = \Bbbk' \otimes_\Bbbk \mathrm{Hom}_{R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R}(M, N), \tag{1.14}$$

and for any expression $\underline{w}$ this functor sends $\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ to ${}'\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$. In particular, this functor restricts to a functor

$$\mathsf{SBim}(\mathcal{W}, V) \to \mathsf{SBim}(\mathcal{W}, V'). \tag{1.15}$$

Lemma 1.20. *For any $w \in \mathcal{W}$ we have*

$$\Bbbk' \otimes_\Bbbk \mathsf{B}_w^{\mathrm{bim}} \cong {}'\mathsf{B}_w^{\mathrm{bim}}.$$

[4]This connection is morally clear, but it does not appear in the literature as far as we know.

PROOF. First we prove that $\Bbbk' \otimes_\Bbbk \mathsf{B}_w^{\mathrm{bim}}$ is indecomposable for any $w \in \mathcal{W}$. For that, we remark that in view of (1.13), restriction to the components of degree $-\ell(w)$ defines an algebra morphism

$$\mathrm{End}_{\mathsf{SBim}(\mathcal{W},V)}(\mathsf{B}_w^{\mathrm{bim}}) \to \mathrm{End}_\Bbbk((\mathsf{B}_w^{\mathrm{bim}})^{-\ell(w)}) = \Bbbk.$$

The kernel K of this morphism is a maximal ideal of $\mathrm{End}_{\mathsf{SBim}(\mathcal{W},V)}(\mathsf{B}_w^{\mathrm{bim}})$, which is a local ring, hence it coincides with its radical. Since this algebra is finite-dimensional, its radical is nilpotent. Using (1.14) we see that

$$\mathrm{End}_{\mathsf{SBim}(\mathcal{W},V)}(\Bbbk' \otimes_\Bbbk \mathsf{B}_w^{\mathrm{bim}}) = (\Bbbk' \otimes_\Bbbk K) \oplus (\Bbbk' \cdot \mathrm{id})$$

with $(\Bbbk' \otimes_\Bbbk K)$ a nilpotent ideal; this algebra is therefore local, which implies that $\Bbbk' \otimes_\Bbbk \mathsf{B}_w^{\mathrm{bim}}$ is indecomposable, as desired.

Now we prove the claim. Let $w \in \mathcal{W}$, and let $\underline{w}$ be a reduced expression for w. Then $\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ is a direct sum of $\mathsf{B}_w^{\mathrm{bim}}$ and some objects which are direct summands of objects $\mathsf{B}_{\underline{y}}^{\mathrm{bim}}(n)$ for some reduced expressions $\underline{y}$ with $\ell(\underline{y}) < \ell(w)$. Hence $'\mathsf{B}_{\underline{w}}^{\mathrm{bim}} = \Bbbk' \otimes_\Bbbk \mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ is a direct sum of $\Bbbk' \otimes_\Bbbk \mathsf{B}_w^{\mathrm{bim}}$ and some objects which are direct summands of objects $'\mathsf{B}_{\underline{y}}^{\mathrm{bim}}(n)$ for some reduced expressions $\underline{y}$ with $\ell(\underline{y}) < \ell(w)$. Hence $\Bbbk' \otimes_\Bbbk \mathsf{B}_w^{\mathrm{bim}}$ is the only possible indecomposable summand of $'\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$ which is not isomorphic to a direct summand of an object $'\mathsf{B}_{\underline{y}}^{\mathrm{bim}}(n)$ with $\underline{y}$ a reduced expression with $\ell(\underline{y}) < \ell(\underline{w})$, so that it must be isomorphic to $'\mathsf{B}_w^{\mathrm{bim}}$. □

It is clear that the morphism

$$[\mathsf{SBim}(\mathcal{W}, V)]_\oplus \to [\mathsf{SBim}(\mathcal{W}, V')]_\oplus$$

induced by (1.15) intertwines the isomorphisms ϵ of Theorem 1.14 for V and for V'. Lemma 1.20 shows that this morphism sends the basis $([\mathsf{B}_w^{\mathrm{bim}}] : w \in \mathcal{W})$ of $[\mathsf{SBim}(\mathcal{W}, V)]_\oplus$ to the basis $([{}'\mathsf{B}_w^{\mathrm{bim}}] : w \in \mathcal{W})$ of $[\mathsf{SBim}(\mathcal{W}, V')]_\oplus$.

1.6. Multiplication by a simple reflection. In this subsection we explain what happens to indecomposable Soergel bimodules when one tensors them with a bimodule associated with a simple reflection.

LEMMA 1.21. *Let $w \in \mathcal{W}$ and $s \in \mathcal{S}$.*

(1) If $sw > w$, then there exist nonnegative integers $d_{w,s}^{y,n}$ for $y \in \mathcal{W}$ such that $y < sw$ and $n \in \mathbb{Z}$ such that

$$\mathsf{B}_s^{\mathrm{bim}} \otimes_R \mathsf{B}_w^{\mathrm{bim}} \cong \mathsf{B}_{sw}^{\mathrm{bim}} \oplus \bigoplus_{\substack{y \in \mathcal{W}, y < sw \\ n \in \mathbb{Z}}} (\mathsf{B}_y^{\mathrm{bim}})^{\oplus d_{w,s}^{y,n}}.$$

Moreover, for any y and n we have $d_{w,s}^{y,n} = d_{w,s}^{y,-n}$.

(2) If $sw < w$ we have

$$\mathsf{B}_s^{\mathrm{bim}} \otimes_R \mathsf{B}_w^{\mathrm{bim}} \cong \mathsf{B}_w^{\mathrm{bim}}(1) \oplus \mathsf{B}_w^{\mathrm{bim}}(-1).$$

PROOF. (1) By (1.9) we have integers $a_{y,n}^w$ satisfying

$$\mathsf{B}_{\underline{w}}^{\mathrm{bim}} \cong \mathsf{B}_w^{\mathrm{bim}} \oplus \bigoplus_{\substack{y \in \mathcal{W}, y < w \\ n \in \mathbb{Z}}} (\mathsf{B}_y^{\mathrm{bim}}(n))^{a_{y,n}^w},$$

and integers $a^{s\underline{w}}_{y,n}$ satisfying

$$\mathsf{B}^{\mathrm{bim}}_{s\underline{w}} \cong \mathsf{B}^{\mathrm{bim}}_{sw} \oplus \bigoplus_{\substack{y\in\mathcal{W}, y<sw \\ n\in\mathbb{Z}}} (\mathsf{B}^{\mathrm{bim}}_{y}(n))^{a^{s\underline{w}}_{y,n}}.$$

(Here, $s\underline{w}$ is the concatenation of the words (s) and $\underline{w}$.) By the Krull–Schmidt property, we deduce that there exist $\delta \in \{0,1\}$ and integers $d^{y,n}_{w,s}$ such that

$$\mathsf{B}^{\mathrm{bim}}_{s} \otimes_R \mathsf{B}^{\mathrm{bim}}_{w} \cong (\mathsf{B}^{\mathrm{bim}}_{sw})^{\delta} \oplus \bigoplus_{\substack{y\in\mathcal{W}, y<sw \\ n\in\mathbb{Z}}} (\mathsf{B}^{\mathrm{bim}}_{y})^{\oplus d^{y,n}_{w,s}}.$$

The object $\mathsf{B}^{\mathrm{bim}}_{sw}$ cannot appear as a direct summand of any $\mathsf{B}^{\mathrm{bim}}_{s} \otimes_R \mathsf{B}^{\mathrm{bim}}_{y}(n)$ with $y < w$, which shows that $\delta = 1$. In fact, assume the contrary. If $sy > y$ then for any reduced expression $\underline{y}$ for y the word $s\underline{y}$ is a reduced expression for sy. Then $\mathsf{B}^{\mathrm{bim}}_{y}$ is direct summand in $\mathsf{B}^{\mathrm{bim}}_{\underline{y}}$, hence $\mathsf{B}^{\mathrm{bim}}_{sw}$ is a direct summand in $\mathsf{B}^{\mathrm{bim}}_{s\underline{y}}(n)$, which is excluded by the characterization of $\mathsf{B}^{\mathrm{bim}}_{sw}$. If $sy < y$, one can choose a reduced expression $\underline{y}$ for y starting with s. Then $\mathsf{B}^{\mathrm{bim}}_{y}$ is direct summand in $\mathsf{B}^{\mathrm{bim}}_{\underline{y}}$, hence $\mathsf{B}^{\mathrm{bim}}_{s} \otimes_R \mathsf{B}^{\mathrm{bim}}_{y}(n)$ is direct summand in

$$\mathsf{B}^{\mathrm{bim}}_{s} \otimes_R \mathsf{B}^{\mathrm{bim}}_{\underline{y}}(n) \cong \mathsf{B}^{\mathrm{bim}}_{\underline{y}}(n+1) \oplus \mathsf{B}^{\mathrm{bim}}_{\underline{y}}(n-1),$$

where the isomorphism follows from (1.8). It follows that $\mathsf{B}^{\mathrm{bim}}_{sw}$ is a direct summand of $\mathsf{B}^{\mathrm{bim}}_{\underline{y}}(n+1)$ or of $\mathsf{B}^{\mathrm{bim}}_{\underline{y}}(n-1)$, which provides a contradiction as before.

Now, consider the autoequivalence D introduced in [S7, Proof of Proposition 5.9]. We have $D \circ (n) \cong (-n) \circ D$ for any n and, by [S7, Satz 6.14(3)], for any $y \in \mathcal{W}$ we have $D(\mathsf{B}^{\mathrm{bim}}_{y}) \cong \mathsf{B}^{\mathrm{bim}}_{y}$. From the considerations in [S7, Proof of Proposition 5.10] one sees that we also have

$$D(\mathsf{B}^{\mathrm{bim}}_{s} \otimes_R \mathsf{B}^{\mathrm{bim}}_{w}) \cong \mathsf{B}^{\mathrm{bim}}_{s} \otimes_R \mathsf{B}^{\mathrm{bim}}_{w}.$$

Using the Krull–Schmidt property, these properties imply that $d^{y,n}_{w,s} = d^{y,-n}_{w,s}$ for any y and n.

(2) First we remark that for any $w \in \mathcal{W}$ and $s \in \mathcal{S}$ such that $sw < w$, using (1.10) we have

$$\varepsilon^{-1}([\mathsf{B}^{\mathrm{bim}}_{s} \otimes_R \mathsf{B}^{\mathrm{bim}}_{w}]) = \underline{H}_s \cdot \varepsilon^{-1}([\mathsf{B}^{\mathrm{bim}}_{w}]) \in (v+v^{-1}) \cdot H_w + \sum_{y<w} \mathbb{Z}[v,v^{-1}] \cdot H_y,$$

which implies that $\mathsf{B}^{\mathrm{bim}}_{w}(1) \oplus \mathsf{B}^{\mathrm{bim}}_{w}(-1)$ is a direct summand of $\mathsf{B}^{\mathrm{bim}}_{s} \otimes_R \mathsf{B}^{\mathrm{bim}}_{w}$.

Next, we prove that for any reduced expression $\underline{w}$ starting with s, in the decomposition (1.9), any $y \in \mathcal{W}$ such that $a^{\underline{w}}_{y,n} \neq 0$ for some n satisfies $sy < y$. In fact, by (1.8) we have

$$\mathsf{B}^{\mathrm{bim}}_{s} \otimes_R \mathsf{B}^{\mathrm{bim}}_{\underline{w}} \cong \mathsf{B}^{\mathrm{bim}}_{\underline{w}}(1) \oplus \mathsf{B}^{\mathrm{bim}}_{\underline{w}}(-1). \tag{1.16}$$

If $y \in \mathcal{W}$ satisfies $sy > y$, then as explained above $\mathsf{B}^{\mathrm{bim}}_{sy}(1) \oplus \mathsf{B}^{\mathrm{bim}}_{sy}(-1)$ is a direct summand in $\mathsf{B}^{\mathrm{bim}}_{s} \otimes_R \mathsf{B}^{\mathrm{bim}}_{sy}$. Using also (1), we obtain that the sum of the multiplicities of the objects $\mathsf{B}^{\mathrm{bim}}_{sy}(n)$ in $\mathsf{B}^{\mathrm{bim}}_{s} \otimes_R \mathsf{B}^{\mathrm{bim}}_{\underline{w}}$ is at least

$$\sum_n (2a^{\underline{w}}_{sy,n} + a^{\underline{w}}_{y,n}).$$

On the other hand, the sum of the multiplicities of the objects $\mathsf{B}^{\mathrm{bim}}_{sy}(n)$ in $\mathsf{B}^{\mathrm{bim}}_{\underline{w}}(1) \oplus \mathsf{B}^{\mathrm{bim}}_{\underline{w}}(-1)$ is

$$\sum_n 2a^{\underline{w}}_{sy,n}.$$

In view of (1.16), we deduce that

$$\sum_n 2a^{\underline{w}}_{sy,n} \geq \sum_n (2a^{\underline{w}}_{sy,n} + a^{\underline{w}}_{y,n}),$$

hence $\sum_n a^{\underline{w}}_{y,n} \leq 0$, which implies that $a^{\underline{w}}_{y,n} = 0$ for any n.

Once this property is established, one proves the desired claim by induction on w, based on (1.16) and the decomposition (1.9). □

Remark 1.22. (1) Using e.g. the autoequivalence φ from Remark 1.12, one deduces from Lemma 1.21 a similar result for the tensor product on the *right* with $\mathsf{B}^{\mathrm{bim}}_s$.

(2) Let w, s be as in Lemma 1.21(1). If $\underline{w}$ is a reduced expression for w then $s\underline{w}$ is a reduced expression for w starting with s. Since $\mathsf{B}^{\mathrm{bim}}_s \otimes_R \mathsf{B}^{\mathrm{bim}}_w$ is a direct summand in $\mathsf{B}^{\mathrm{bim}}_{s\underline{w}}$, the property proved in the course of the proof of Lemma 1.21(2) shows that any $y \in \mathcal{W}$ such that $d^{y;n}_{w,s} \neq 0$ for some n satisfies $sy < y$.

1.7. Decomposition of Bott–Samelson bimodules. Recall that $\mathcal{W}$ admits a unique associated product $*$ (sometimes called the *Hecke product*) such that for $w \in \mathcal{W}$ and $s \in \mathcal{S}$ we have

$$w * s = \begin{cases} ws & \text{if } ws > w; \\ w & \text{if } ws < w, \end{cases}$$

see e.g. [BM, §3]. For any $w \in \mathcal{W}$ and $s \in \mathcal{S}$ we then also have

$$s * w = \begin{cases} sw & \text{if } sw > w; \\ w & \text{if } sw < w. \end{cases}$$

Moreover, for $w \in \mathcal{W}$ the maps $w * (-)$ and $(-) * w$ are increasing with respect to the Bruhat order, see [BM, Lemma 3.1(3)].

For an expression $\underline{w} = (s_1, \cdots, s_r)$ we set

$$*\underline{w} = s_1 * \cdots * s_r.$$

With this notation we can generalize part of the decomposition (1.9) as follows.

Proposition 1.23. *For any expression $\underline{w}$, the bimodule $\mathsf{B}^{\mathrm{bim}}_{\underline{w}}$ is a direct sum of modules of the form $\mathsf{B}^{\mathrm{bim}}_y(n)$ with $y \leq *\underline{w}$.*

Proof. We proceed by induction on $\ell(\underline{w})$, the case $\ell(\underline{w}) = 0$ being obvious. Let $\underline{w}$ be an expression of positive length, and write $\underline{w} = s\underline{y}$ with $s \in \mathcal{S}$ and $\underline{y}$ an expression. By induction we can assume that

$$\mathsf{B}^{\mathrm{bim}}_{\underline{y}} = \sum_{\substack{z \leq *\underline{y} \\ n \in \mathbb{Z}}} (\mathsf{B}^{\mathrm{bim}}_z(n))^{\oplus a_{z,n}}$$

for some nonnegative integers $a_{z,n}$. Then

$$\mathsf{B}^{\mathrm{bim}}_{\underline{w}} = \sum_{\substack{z \leq *\underline{y} \\ n \in \mathbb{Z}}} (\mathsf{B}^{\mathrm{bim}}_s \otimes_R \mathsf{B}^{\mathrm{bim}}_z(n))^{\oplus a_{z,n}}.$$

By Lemma 1.21, each $\mathsf{B}^{\mathrm{bim}}_s \otimes_R \mathsf{B}^{\mathrm{bim}}_z$ is a direct sum of shifts of modules $\mathsf{B}^{\mathrm{bim}}_x$ with $x \leq s * z$. Here, since $z \leq *\underline{y}$ we have $s * z \leq s * (*\underline{y}) = *\underline{w}$, so that indeed $\mathsf{B}^{\mathrm{bim}}_{\underline{w}}$ is a direct sum of modules of the form $\mathsf{B}^{\mathrm{bim}}_y(n)$ with $y \leq *\underline{w}$. □

Remark 1.24. One can easily check by induction that the multiplicity of $\mathsf{B}^{\mathrm{bim}}_{*\underline{w}}(n)$ as a direct summand of $\mathsf{B}^{\mathrm{bim}}_{\underline{w}}$ is at least the coefficient of v^n in $(v + v^{-1})^{\ell(\underline{w})-\ell(*\underline{w})}$.

1.8. Soergel's conjecture. In [S7, Vermutung 1.13] Soergel conjectures that “at least if $\Bbbk = \mathbb{C}$,” for any $w \in \mathcal{W}$ we have

$$\varepsilon(\underline{H}_w) = [\mathsf{B}^{\mathrm{bim}}_w], \tag{1.17}$$

where $\underline{H}_w$ is as in Theorem 4.3 of Chapter 1. Before explaining its status, let us explain the point of this conjecture. In [S7, Proposition 5.7(3)], Soergel defines a map $h_\Delta : [\mathsf{SBim}(\mathcal{W}, V)]_\oplus \to \mathcal{H}_{(\mathcal{W},\mathcal{S})}$ which is a left inverse to ε (see also [S7, Bemerkung 5.14]). Manifestly, for any $M \in \mathsf{SBim}(\mathcal{W}, V)$, all the coefficients of the expansion of $h_\Delta([M])$ in the basis $(H_w : w \in \mathcal{W})$ have nonnegative coefficients. In particular, (1.10) can be refined to the statement that

$$\varepsilon^{-1}([\mathsf{B}^{\mathrm{bim}}_w]) \in H_w + \sum_{y<w} \mathbb{Z}_{\geq 0}[v, v^{-1}] \cdot H_y.$$

Hence if (1.17) holds, it follows that $h_{y,w}$ has nonnegative coefficients for any $y \in \mathcal{W}$; this property (for all $w \in \mathcal{W}$) was conjectured by Kazhdan–Lusztig (see [KL1, Sentence above Definition 1.2]), and has since then become a major question in Coxeter groups combinatorics (known as the *Kazhdan–Lusztig positivity conjecture*).

Remark 1.25.
(1) The results of §1.5 show that Soergel's conjecture is “stable under field extensions” in the sense that if it is known for a reflection faithful representation V over $\Bbbk$ and if $\Bbbk'$ is an extension of $\Bbbk$, then it is true for the representation $\Bbbk' \otimes_\Bbbk V$.
(2) As explained in Remark 1.17(4), the indecomposable Soergel bimodule $\mathsf{B}^{\mathrm{bim}}_w$ is known in case w belongs to a subgroup of $\mathcal{W}$ generated by two simple reflections. In this case, the equality (1.17) holds.

In the setting considered in §1.1, Soergel's conjecture can be deduced from Theorem 1.3 in Chapter 3 and Remark 1.17(3). (This was first observed in [S2].) This proof can be generalized to the setting of flag varieties of Kac–Moody groups.

In [EW1] it was shown by Elias–Williamson that Soergel's conjecture holds in case V is a reflection faithful representation of $(\mathcal{W}, \mathcal{S})$ over $\mathbb{R}$ which satisfies the following condition. For any $s \in \mathcal{S}$, since s acts on V as a reflection there exist $\alpha_s \in V^*$ and $\alpha_s^\vee \in V$ such that $s \cdot v = v - \langle \alpha_s, v \rangle \alpha_s^\vee$ for any $v \in V$. (These vectors are unique up to scalar, in the sense that any two choices differ by the replacement of $(\alpha_s, \alpha_s^\vee)$ by $(\lambda \cdot \alpha_s, \lambda^{-1} \cdot \alpha_s^\vee)$ for some $\lambda \in \mathbb{R}^\times$.) We assume that these elements can be chosen in such a way that there exists $\rho \in V^*$ such that for any $s \in \mathcal{S}$ and $w \in \mathcal{W}$ we have

$$\langle w(\rho), \alpha_s^\vee \rangle > 0 \quad \Leftrightarrow \quad sw > w.$$

This condition is satisfied in the following cases (see [**Ri**, §2.1] for details):

- for $(\mathcal{W}, \mathcal{S})$ any Coxeter system, if V is as in §1.2.2 we can choose $\alpha_s^\vee = e_s$ and $\alpha_s = e_s^*$ for any s, and then take for ρ any element such that $\langle \rho, \alpha_s^\vee \rangle > 0$ for any $s \in \mathcal{S}$;
- if A is a generalized Cartan matrix and if we take $V = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{h}$ where $\mathfrak{h}$ is as §1.2.3, with α_s and $\alpha_s^\vee$ the vectors denoted in this way in §1.2.3, and again for ρ any element such that $\langle \rho, \alpha_s^\vee \rangle > 0$ for any $s \in \mathcal{S}$.

In particular, since the construction of §1.2.2 provides a reflection faithful representation for each Coxeter system, this is sufficient to imply the Kazhdan–Lusztig positivity conjecture in all cases.

The proof of this result (which we will not discuss in detail here; see [**Ri**] for an overview) is inspired by the special case considered in §1.1 (or, more precisely, its variant for $\mathbb{R}$ instead of $\mathbb{Q}$). Namely, if $\mathscr{G}$, $\mathscr{B}$, $\mathscr{T}$ are as in §1.1, by (a variant over $\mathbb{R}$ of) (1.4) and (1.5) the vector space $\mathbb{R} \otimes_R \mathsf{B}_w^{\mathrm{bim}}$ identifies with the intersection cohomology of the closure in $\mathscr{X}$ of the Bruhat cell attached to w. As such, this space satisfies some "Hodge theoretic" properties such as the hard Lefschetz theorem and the Hodge–Riemann bilinear relations. The spectacular idea at the heart of the proof of [**EW1**] is that these properties admit completely algebraic formulations, which can be shown (by a complicated inductive argument based on ideas of de Cataldo–Migliorini in the geometric context) to hold for any reflection faithful representation satisfying the condition considered above, independently of any geometry.

1.9. Soergel modules. As explained in §1.6, historically, what Soergel introduced first are not the bimodules in $\mathsf{SBim}(\mathcal{W}, V)$, but rather the graded R-modules one obtains by tensoring (either on the right or on the left; one has to make a choice but this does not affect the theory in any serious way) these bimodules with the trivial module $\Bbbk$. This theory behaves well only under the assumption that $\mathcal{W}$ is finite, which we therefore assume here. Given any expression $\underline{w} = (s_1, \cdots, s_r)$ one can consider the graded right R-module

$$\mathsf{B}_{\underline{w}}^{\mathrm{mod}} := \Bbbk \otimes_R \mathsf{B}_{\underline{w}}^{\mathrm{bim}} = \Bbbk \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_r}} R(r).$$

(Here (r) is the shift-of-grading by r for graded modules, which is defined similarly as for bimodules. The action of R is induced by the right action on $\mathsf{B}_{\underline{w}}^{\mathrm{bim}}$.) Clearly, the action of R on this module factors through an action of the finite-dimensional graded algebra $R/\langle R_+^{\mathcal{W}} \rangle$, where $\langle R_+^{\mathcal{W}} \rangle$ is the ideal in R generated by homogenous $\mathcal{W}$-invariant elements of positive degree. (The fact that this algebra is finite-dimensional follows from the fact that R is finite over $R^{\mathcal{W}}$, see e.g. [**Bou**, Chap. V, §1, 9, Théorème 2].) We will denote by $\mathsf{SMod}(\mathcal{W}, V)$ the full subcategory of the category of $\mathbb{Z}$-graded right R-modules whose objects are the direct sums of direct summands of objects of the form $\mathsf{B}_{\underline{w}}^{\mathrm{mod}}(n)$ with $\underline{w}$ an expression and $n \in \mathbb{Z}$. It is clear from this definition that the functor $\Bbbk \otimes_R (-)$ induces a functor

$$\mathsf{SBim}(\mathcal{W}, V) \to \mathsf{SMod}(\mathcal{W}, V). \tag{1.18}$$

Considerations similar to those used for $\mathsf{SBim}(\mathcal{W}, V)$ (see §1.3) show that the category $\mathsf{SMod}(\mathcal{W}, V)$ is Krull–Schmidt. The tensor product $\otimes_R$ defines a right action of the monoidal category $\mathsf{SBim}(\mathcal{W}, V)$ on $\mathsf{SMod}(\mathcal{W}, V)$, such that the functor (1.18) is a functor of module categories.

For any $M, N \in R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R$, the vector space

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R}(M, N(n))$$

has a natural structure of graded R-bimodule. The result that allows to connect more precisely the categories $\mathsf{SBim}(\mathcal{W}, V)$ and $\mathsf{SMod}(\mathcal{W}, V)$ is the following. For a proof (following unpublished work of Soergel), we refer to [**Ri**, Proposition 1.13].

Proposition 1.26. *For any M, N in $\mathsf{SBim}(\mathcal{W}, V)$, the natural morphism*

$$\Bbbk \otimes_R \left(\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{SBim}(\mathcal{W},V)}(M, N(n)) \right) \to \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{SMod}(\mathcal{W},V)}(\Bbbk \otimes_R M, \Bbbk \otimes_R N(n))$$

is an isomorphism.

Note that in settings where Soergel (bi)modules can be related to parity complexes on flag varieties, Proposition 1.26 can often be deduced from general properties of equivariant cohomology; see e.g. [**AR1**, Footnote 3 on p. 339].

Let us note the following consequences.

Corollary 1.27. *(1) For any $w \in \mathcal{W}$, the graded right R-module*

$$\mathsf{B}_w^{\mathrm{mod}} := \Bbbk \otimes_R \mathsf{B}_w^{\mathrm{bim}}$$

is indecomposable.

(2) For any $w \in \mathcal{W}$, $\mathsf{B}_w^{\mathrm{mod}}$ is indecomposable as an ungraded *right R-module.*

(3) For any $w \in \mathcal{W}$, $\mathsf{B}_w^{\mathrm{bim}}$ is indecomposable as an ungraded *R-bimodule.*

Proof. (1) To prove the claim it suffices to prove that $\mathrm{End}_{\mathsf{SMod}(\mathcal{W},V)}(\Bbbk \otimes_R \mathsf{B}_w^{\mathrm{bim}})$ is a local ring. Now by Proposition 1.26 this ring is a quotient of the ring $\mathrm{End}_{\mathsf{SBim}(\mathcal{W},V)}(\mathsf{B}_w^{\mathrm{bim}})$, which is local since $\mathsf{SBim}(\mathcal{W}, V)$ is Krull–Schmidt and $\mathsf{B}_w^{\mathrm{bim}}$ is indecomposable; it is therefore local too.

(2) The claim follows from (1) and the general result that a graded module over a finite-dimensional $\mathbb{Z}$-graded algebra is indecomposable as a graded module if and only if it is indecomposable as an ungraded module; see [**GG**, Theorem 3.2].

(3) Let M, N be R-bimodules such that $\mathsf{B}_w^{\mathrm{bim}} \cong M \oplus N$ as ungraded R-bimodules. Here $\mathsf{B}_w^{\mathrm{bim}}$ is free as a left R-module (see §1.3), hence so are M and N by the Quillen–Suslin theorem. On the other hand, as right R-modules we have

$$\Bbbk \otimes_R \mathsf{B}_w^{\mathrm{bim}} \cong (\Bbbk \otimes_R M) \oplus (\Bbbk \otimes_R N).$$

Using (2) we deduce that either $\Bbbk \otimes_R M = 0$ or $\Bbbk \otimes_R N = 0$. By freeness we deduce that $M = 0$ or $N = 0$, which shows indecomposability. □

From Corollary 1.27(1) we deduce the following.

Corollary 1.28. *The assignment*

$$(w, n) \mapsto \mathsf{B}_w^{\mathrm{mod}}(n)$$

induces a bijection between $\mathcal{W} \times \mathbb{Z}$ and the set of isomorphism classes of indecomposable objects in $\mathsf{SMod}(\mathcal{W}, V)$.

Proof. By Corollary 1.27(1) each object $\mathsf{B}_w^{\mathrm{mod}}(n)$ is indecomposable. It is also clear that any object in $\mathsf{SMod}(\mathcal{W}, V)$ is a direct sum of such objects. What remains to be proved is that these objects remain nonisomorphic. Let $w, w' \in \mathcal{W}$ and $n, n' \in \mathbb{Z}$, and assume that there exists an isomorphism

$$\mathsf{B}_w^{\mathrm{mod}}(n) \cong \mathsf{B}_{w'}^{\mathrm{mod}}(n').$$

Then, using Proposition 1.26, there exist morphisms

$$\phi : \mathsf{B}_w^{\mathrm{bim}}(n) \to \mathsf{B}_{w'}^{\mathrm{bim}}(n'), \quad \psi : \mathsf{B}_{w'}^{\mathrm{bim}}(n') \to \mathsf{B}_w^{\mathrm{bim}}(n)$$

such that

$$\psi \circ \phi \in \mathrm{id} + R_+ \cdot \left(\bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}(\mathsf{B}_w^{\mathrm{bim}}(n), \mathsf{B}_w^{\mathrm{bim}}(n+m)) \right)$$

and similarly for $\phi \circ \psi$. (Here, $R_+ \subset R$ is the ideal consisting of sums of elements of positive degrees.) Since the graded left R-module

$$\bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}(\mathsf{B}_w^{\mathrm{bim}}(n), \mathsf{B}_w^{\mathrm{bim}}(n+m))$$

is finitely generated, its grading is bounded below. Hence, for degree reasons, $\psi \circ \phi - \mathrm{id}$ is nilpotent, hence $\psi \circ \phi$ is invertible. Similarly, $\phi \circ \psi$ is invertible. Hence $\mathsf{B}_w^{\mathrm{bim}}(n)$ and $\mathsf{B}_{w'}^{\mathrm{bim}}(n')$ are isomorphic, which implies that $w = w'$ and $n = n'$. □

In particular, it follows from Corollary 1.28 that the family $(\mathsf{B}_w^{\mathrm{mod}} : w \in \mathcal{W})$ is a $\mathbb{Z}[v, v^{-1}]$-basis of $[\mathsf{SMod}(\mathcal{W}, V)]_\oplus$, which implies that the functor (1.18) induces an isomorphism

$$[\mathsf{SBim}(\mathcal{W}, V)]_\oplus \xrightarrow{\sim} [\mathsf{SMod}(\mathcal{W}, V)]_\oplus.$$

This map is in fact an isomorphism of right $[\mathsf{SBim}(\mathcal{W}, V)]_\oplus$-modules for the actions induced by the right actions of the category $\mathsf{SBim}(\mathcal{W}, V)$ on itself and on $\mathsf{SMod}(\mathcal{W}, V)$. In view of Corollary 1.18, we deduce an isomorphism

$$[\mathsf{SMod}(\mathcal{W}, V)]_\oplus \cong \mathcal{H}_{(\mathcal{W}, \mathcal{S})}.$$

In case Soergel's conjecture holds (see §1.8), the image of $[\mathsf{B}_w^{\mathrm{mod}}]$ under this identification is $\underline{H}_w$, for any $w \in \mathcal{W}$.

Let us now denote by $\overline{\mathsf{SMod}}(\mathcal{W}, V)$ the full subcategory of the category of (ungraded) R-modules whose objects are the direct sums of direct summands of objects $\mathsf{B}_{\underline{w}}^{\mathrm{mod}}$ for $\underline{w}$ an expression (seen as ungraded modules). We have a natural functor

$$\mathsf{For} : \mathsf{SMod}(\mathcal{W}, V) \to \overline{\mathsf{SMod}}(\mathcal{W}, V) \tag{1.19}$$

of forgetting the grading. Using once again [CYZ, Remark A.2] one checks that $\overline{\mathsf{SMod}}(\mathcal{W}, V)$ is Krull–Schmidt.

Corollary 1.29. *The assignment*

$$w \mapsto \mathsf{For}(\mathsf{B}_w^{\mathrm{mod}})$$

induces a bijection between $\mathcal{W}$ and the set of isomorphism classes of indecomposable objects in $\overline{\mathsf{SMod}}(\mathcal{W}, V)$.

Proof. By Corollary 1.27(2), and each object $\mathsf{For}(\mathsf{B}_w^{\mathrm{mod}})$ is indecomposable, and it is clear that any object in $\overline{\mathsf{SMod}}(\mathcal{W}, V)$ is a direct sum of such objects. To conclude, it remains to prove that these objects are pairwise nonisomorphic. This follows from [GG, Theorem 4.1] and Corollary 1.28. □

As above, Corollary 1.29 implies that $([\mathsf{For}(\mathsf{B}_w^{\mathrm{mod}})] : w \in \mathcal{W})$ is a $\mathbb{Z}$-basis of $[\overline{\mathsf{SMod}}(\mathcal{W}, V)]_\oplus$, and that the functor (1.19) induces an isomorphism

$$\mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} [\mathsf{SMod}(\mathcal{W}, V)]_\oplus \xrightarrow{\sim} [\overline{\mathsf{SMod}}(\mathcal{W}, V)]_\oplus,$$

where in the left-hand side $\mathbb{Z}$ is considered as a $\mathbb{Z}[v, v^{-1}]$-algebra where v acts as the identity. We deduce an identification

$$[\overline{\mathsf{SMod}}(\mathcal{W}, V)]_\oplus \cong \mathbb{Z}[\mathcal{W}]. \tag{1.20}$$

In case Soergel's conjecture holds, the image of $[\mathsf{For}(\mathsf{B}_w^{\mathrm{mod}})]$ under this identification is $\underline{H}_{w|v=1}$, where $h_{|v=1}$ is the image of $h \in \mathcal{H}_{(\mathcal{W},\mathcal{S})}$ in $\mathcal{H}_{(\mathcal{W},\mathcal{S})}/v \cdot \mathcal{H}_{(\mathcal{W},\mathcal{S})} \cong \mathbb{Z}[\mathcal{W}]$.

1.10. Application to the Kazhdan–Lusztig conjecture. The first important application of the theory of Soergel bimodules (developed before this subject was really introduced) was to the proof of the Kazhdan–Lusztig conjecture. This conjecture was formulated by Kazhdan–Lusztig in [**KL1**], and proved shortly thereafter by Brylinski–Kashiwara and Beĭlinson–Bernstein independently using geometry; see [**Ac**, Remark 7.3.10] for a brief overview of these proofs, and [**HTT**] for more details. Soergel proposed in [**S1**] a new approach to this question which, combined with the later algebraic proof of Soergel's conjecture in [**EW1**] (see §1.8) can be used to provide a completely algebraic solution to this problem. Here we provide a brief overview of this approach; for more details see [**EMTW**, Part III].

Consider the setting of §1.1, with $\mathscr{G}$ semisimple and simply connected, and also the categories of Soergel (bi)modules for (W, S) and the reflection faithful representation $V := \mathbb{C} \otimes_{\mathbb{Z}} X_*(\mathscr{T})$ (with the standard action). Let also $\mathfrak{g}$ be the semisimple complex Lie algebra whose root system is dual to $\mathfrak{R}$; thus we are given a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and an identification $\mathfrak{t} = V^*$ such that the roots of $(\mathfrak{g}, \mathfrak{t})$ are the coroots $(\alpha^\vee : \alpha \in \mathfrak{R})$. Let also $\mathfrak{b} \subset \mathfrak{g}$ be the Borel subalgebra whose roots are the positive coroots of $\mathscr{G}$. Bernstein–Gelfand–Gelfand have defined and studied a very nice category of modules over the enveloping algebra $\mathcal{U}\mathfrak{g}$ of $\mathfrak{g}$, called *category* $\mathcal{O}$, whose simple objects are the simple highest representations. (These include in particular the finite-dimensional representations, whose structure is understood via Weyl's character formula.) For this theory, see [**H4**].

The category $\mathcal{O}$ breaks into direct summands according to the action of the center of $\mathcal{U}\mathfrak{g}$; in particular we have the *principal block* $\mathcal{O}_0$, whose simple objects are in bijection with W. More precisely, for $w \in W$ we denote by Δ_w the Verma module of highest weight $w(\rho) - \rho$ (where ρ is the half sum of the positive roots), and by L_w its head; then the assignment $w \mapsto L_w$ induces a bijection between W and the set of isomorphism classes of simple objects in $\mathcal{O}_0$. The category $\mathcal{O}_0$ has a structure of highest weight category with underlying poset W endowed with the opposite of the Bruhat order, and standard objects $(\Delta_w : w \in W)$. (The costandard objects are the *dual Verma modules*.) The structure of $\mathcal{O}_0$ is very similar to that of the category $\mathcal{O}_{\Bbbk}$ studied in Section 3 of Chapter 1 (which is one reason which justifies the name *modular category* $\mathcal{O}$).

As for any highest weight category, the Grothendieck group $[\mathcal{O}_0]$ admits a basis consisting of classes of standard objects. We can therefore identify

$$[\mathcal{O}_0] = \mathbb{Z}[W]$$

where $w \in W$ corresponds to $[\Delta_w]$. We will be interested in particular in the full subcategory $\mathsf{Proj}(\mathcal{O}_0)$ of $\mathcal{O}_0$ whose objects are the projective objects. For any $w \in W$ we denote by P_w the projective cover of L_w; then the assignment $w \mapsto P_w$

induces a bijection between W and the set of isomorphism classes of indecomposable objects in $\mathsf{Proj}(\mathcal{O}_0)$. The natural morphism $[\mathsf{Proj}(\mathcal{O}_0)]_\oplus \to [\mathcal{O}_0]$ is an isomorphism, which provides an identification

$$[\mathsf{Proj}(\mathcal{O}_0)]_\oplus \xrightarrow{\sim} \mathbb{Z}[W] \tag{1.21}$$

sending $[P]$ to $\sum_y (P : \Delta_y) \cdot y$. The Kazhdan–Lusztig conjecture predicts a formula for computing the multiplicities $(\Delta_y : L_w)_{y,w\in W}$; by reciprocity (see (2.1) in Appendix A) and since $\mathcal{O}_0$ admits a "duality" operation which fixes each simple object (in the sense of Exercise 7.9), it is equivalent to determine the multiplicities $(P_w : \Delta_y)_{y,w\in W}$ or, in other words, the images of the classes $([P_w] : w \in W)$ under (1.21); in these terms, the Kazhdan–Lusztig formula amounts to the equality

$$[P_w] = \underline{H}_{w|v=1} \tag{1.22}$$

(where we use the notation introduced at the end of §1.9.)

The projective objects in $\mathcal{O}_0$ admit an "inductive" construction similar to that considered in §3.5 in Chapter 1, as follows. For any $s \in S$ we can consider the endofunctor

$$\vartheta_s : \mathcal{O}_0 \to \mathcal{O}_0$$

given by wall crossing along the s-wall of the dominant Weyl chamber. Standard formulas for translation of Verma modules (see e.g. [**H4**, §7.6 and §7.12]) show that, under the identification (1.21), the induced endomorphism of $\mathbb{Z}[W]$ is given by right multiplication by $e + s = \underline{H}_{s|v=1}$. Given an expression $\underline{w} = (s_1, \cdots, s_r)$ we set

$$\vartheta_{\underline{w}} := \vartheta_{s_r} \circ \cdots \circ \vartheta_{s_1}.$$

The Verma module Δ_e is projective by maximality; hence $P_e = \Delta_e$. Now if $w \in W$ and if $\underline{w}$ is a reduced expression for w, the comments above on standard filtrations imply that

$$\vartheta_{\underline{w}}(P_e) \cong P_w \oplus \bigoplus_{y<w} P_y^{b_{y,\underline{w}}} \tag{1.23}$$

for some nonnegative integers $b_{y,\underline{w}}$.

The first main result of the "representation theoretic" part of [**S1**] is the construction of an algebra isomorphism

$$R/\langle R_+^W \rangle \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_0}(P_{w_0}).$$

This isomorphism is somewhat explicit; the morphism from the left-hand side to the right-hand side is induced by the action of the center of $\mathcal{U}\mathfrak{g}$. (This isomorphism has a different proof due to Bernstein, see [**Be**].) Via this morphism, for any $P \in \mathsf{Proj}(\mathcal{O}_0)$ the finite-dimensional vector space

$$\mathbb{V}(P) = \mathrm{Hom}_{\mathcal{O}_0}(P_{w_0}, P)$$

acquires a right R-module structure, which allows to define a functor

$$\mathbb{V} : \mathsf{Proj}(\mathcal{O}_0) \to \mathsf{Mod}_{\mathrm{fg}}\text{-}R.$$

One can also show that for any $s \in S$ there is an isomorphism of functors

$$\mathbb{V} \circ \vartheta_s(-) \cong \mathbb{V}(-) \otimes_R \mathsf{B}_s^{\mathrm{bim}}. \tag{1.24}$$

Since $\mathbb{V}(P_e) \cong \Bbbk$, this implies that the functor $\mathbb{V}$ takes values in $\overline{\mathsf{SMod}}(W, V)$. The second main result of the "representation theoretic" part of [S1] is that this functor induces an equivalence of categories

$$\mathsf{Proj}(\mathcal{O}_0) \xrightarrow{\sim} \overline{\mathsf{SMod}}(V, W). \tag{1.25}$$

Comparing the formulas (1.9) and (1.23) and using (1.24), one easily checks by induction on the Bruhat order that for any $w \in W$ we have

$$\mathbb{V}(P_w) \cong \mathsf{For}(\mathsf{B}_w^{\mathrm{mod}}).$$

Now, using (1.24) one checks that, under the identifications (1.20) and (1.21), the automorphism of $\mathbb{Z}[W]$ induced by the equivalence (1.25) is the identity. Since Soergel's conjecture is known for our choice of V (at the time of [S1] the proof used geometry; now this can be replaced by the proof in [EW1]), as explained in §1.9 the class of $\mathsf{For}(\mathsf{B}_w^{\mathrm{mod}})$ is $\underline{H}_{w|v=1}$; we deduce the formula (1.22), as desired.

Remark 1.30. As in §1.1, for the choice of V considered here, the categories $\mathsf{SBim}(W, V)$ and $\mathsf{SMod}(W, V)$ admit descriptions in terms of semisimple complexes on the flag variety $\mathscr{X}$. More explicitly, consider the $\mathscr{B}$-equivariant derived category $D^{\mathrm{b}}_{\mathscr{B}}(\mathscr{X}, \mathbb{C})$ of $\mathbb{C}$-sheaves on $\mathscr{X}$, and the constructible derived category $D^{\mathrm{b}}_{(\mathscr{B})}(\mathscr{X}, \mathbb{C})$ with respect to the Bruhat stratification (1.1). Let also $\mathsf{IC}_{\mathscr{B}}(\mathscr{X}, \mathbb{C})$ and $\mathsf{IC}_{(\mathscr{B})}(\mathscr{X}, \mathbb{C})$ be the subcategories of semisimple complexes (or, in other words, of parity complexes). Then the functors

$$\mathbb{H} := \mathsf{H}^{\bullet}_{\mathscr{B}}(\mathscr{X}, -) \quad \text{and} \quad \mathbb{H}' := \mathsf{H}^{\bullet}(\mathscr{X}, -)$$

induce equivalences of additive categories

$$\mathbb{H} : \mathsf{IC}_{\mathscr{B}}(\mathscr{X}, \mathbb{C}) \xrightarrow{\sim} \mathsf{SBim}(W, V), \quad \mathbb{H}' : \mathsf{IC}_{(\mathscr{B})}(\mathscr{X}, \mathbb{C}) \xrightarrow{\sim} \mathsf{SMod}(W, V).$$

Here, $\mathbb{H}$ is an equivalence of monoidal categories with respect to convolution on $\mathsf{IC}_{\mathscr{B}}(\mathscr{X}, \mathbb{C})$, and $\mathbb{H}'$ intertwines the actions of $\mathsf{IC}_{\mathscr{B}}(\mathscr{X}, \mathbb{C})$ and $\mathsf{SBim}(W, V)$ via $\mathbb{H}$.

The situation is thus summarized in the diagram

$$\begin{tikzcd} \mathsf{IC}_{(\mathscr{B})}(\mathscr{X}, \mathbb{C}) \arrow[r, "\mathbb{H}'", "\sim"'] & \mathsf{SMod}(V, W) \arrow[d, "\mathsf{For}"] & \\ & \overline{\mathsf{SMod}}(\mathcal{W}, V) & \mathsf{Proj}(\mathcal{O}_0). \arrow[l, "\mathbb{V}"', "\sim"] \end{tikzcd} \tag{1.26}$$

From this point of view, Soergel (bi)modules appear as a "bridge" relating the two categories we want to compare (one of topological nature, and one of representation-theoretic nature). This point of view has become a model for most applications of these techniques; for some examples, see [BY, AR1, MR2].

Note that there exists another way to relate $\mathcal{O}_0$ to a category of perverse sheaves on a flag variety. Namely, if we denote by $\mathscr{X}^\vee$ the flag variety of the complex simply-connected semisimple algebraic group $\mathscr{G}^\vee$ whose Lie algebra is $\mathfrak{g}$, and by $\mathsf{Perv}_{(\mathscr{B}^\vee)}(\mathscr{X}^\vee, \mathbb{C})$ the category of $\mathbb{C}$-perverse sheaves on $\mathscr{X}^\vee$ constructible with respect to the Bruhat stratification, then combining the Beĭlinson–Bernstein localization theorem and a result of Soergel one obtains an equivalence of abelian categories

$$\mathcal{O}_0 \cong \mathsf{Perv}_{(\mathscr{B}^\vee)}(\mathscr{X}^\vee, \mathbb{C}); \tag{1.27}$$

see [**BGS**, Proposition 3.5.2]. Here the Weyl group of $\mathscr{G}^\vee$ identifies canonically with W, so that $\mathsf{Perv}_{(\mathscr{B}^\vee)}(\mathscr{X}^\vee, \mathbb{C})$ has a canonical structure of highest weight category with weight poset W for the Bruhat order. The equivalence (1.27) sends L_w to the simple perverse sheaf $\mathcal{IC}_{w_0 w^{-1}}$.

Note that the equivalence (1.27) is *different* from the relation provided by (1.26). In fact, understanding the relation between these two approaches was one of the motivations for the construction of Koszul duality for constructible sheaves on flag varieties; see [**BGS**].

1.11. Application to Soergel's modular category $\mathcal{O}$. Recall now the setting of Section 3 in Chapter 1. The main reason why Soergel called the category $\mathcal{O}_\Bbbk$ the "modular category $\mathcal{O}$" is that a large part of the theory of §1.10 can be adapted to this setting, as explained in §§3.5–3.7 of Chapter 1.

Namely, let $\mathbf{G}$, $\mathbf{B}$, $\mathbf{T}$ be as in Section 3 in Chapter 1. Let $\mathscr{G}$ be a complex semisimple algebraic group which is Langlands dual to $\mathbf{G}$; hence $\mathscr{G}$ is of adjoint type, and its root system is the coroot system of $\mathbf{G}$. Fix also a maximal torus $\mathscr{T} \subset \mathscr{G}$ and an identification $\mathbb{X} = X^*(\mathbf{T}) = X_*(\mathscr{T})$, and denote by $\mathscr{B} \subset \mathscr{G}$ the Borel subgroup containing $\mathscr{T}$ and whose set of roots is the set of coroots of $(\mathbf{G}, \mathbf{T})$ corresponding to $\mathbf{B}$. Now we consider the representation of W given by

$$V := \Bbbk \otimes_{\mathbb{Z}} X_*(\mathscr{T}) = \Bbbk \otimes_{\mathbb{Z}} \mathbb{X}.$$

REMARK 1.31. Consider the root lattice $\mathbb{Z}\mathfrak{R} \subset \mathbb{X}$; our assumption that $p > h$ implies in particular that the natural morphism $\Bbbk \otimes_{\mathbb{Z}} \mathbb{Z}\mathfrak{R} \to V$ is an isomorphism. In view of the comments in §1.2.4, it follows that if $p \notin \{2, 3\}$ (which follows from the assumption $p > h$ unless $\mathbf{G}$ is a product of copies of SL_2) V is a reflection faithful representation of (W, S). In any case, Soergel proves in [**S5**, Theorem 2.8.1] that the categories $\overline{\mathsf{SMod}}(W, V)$ and $\mathsf{SMod}(W, V)$ satisfy the properties of §1.9 without reference to the general theory of Soergel (bi)modules.

The results of §3.7 in Chapter 1 can now be restated as saying that $\mathbb{V}$ restricts to an equivalence of categories

$$\mathsf{Proj}(\mathcal{O}_\Bbbk) \xrightarrow{\sim} \overline{\mathsf{SMod}}(W, V).$$

Comparing (3.2) in Chapter 1 with (1.9) one checks by induction that

$$\mathbb{V}(\mathsf{P}_w) \cong \mathsf{For}(\mathsf{B}_w^{\mathrm{mod}})$$

for any $w \in W$.

As in §1.10, these results show that for any $w \in W$ the sum

$$\sum_{y \in W} (\mathsf{P}_w : \mathsf{M}_y) \cdot y$$

is the image of $[\mathsf{For}(\mathsf{B}_w^{\mathrm{mod}})]$ under the identification (1.20), or in other words is obtained from the image of $[\mathsf{B}_w^{\mathrm{bim}}]$ under ε^{-1} (see Corollary 1.18) by setting $v = 1$. In case Soergel's conjecture holds for this choice of V, this implies that the coefficients $(\mathsf{P}_w : \mathsf{M}_y) = (\mathsf{M}_y, \mathsf{L}_w)$ are given by the same formula as in §1.10, in accordance with what is predicted by Lusztig's conjecture (see §4.6 in Chapter 1). Unfortunately, Soergel's conjecture is not known in general in this case, and in fact we will see in Chapter 5 that it fails in many cases.

REMARK 1.32. As in Remark 1.30, the category $\mathsf{SMod}(W, V)$ admits a description in terms of constructible sheaves on flag varieties. Namely, if $\mathscr{X} = \mathscr{G}/\mathscr{B}$, and if $\mathsf{Parity}_{(\mathscr{B})}(\mathscr{X}, \Bbbk)$ is the category of parity complexes with coefficients in $\Bbbk$ on $\mathscr{X}$, then Soergel proves in [S5, Theorem 4.2.1] that the functor

$$\mathbb{H}' := \mathsf{H}^\bullet(\mathscr{X}, -)$$

induces an equivalence of categories

$$\mathsf{Parity}_{(\mathscr{B})}(\mathscr{X}, \Bbbk) \xrightarrow{\sim} \mathsf{SMod}(W, V).$$

(In fact the theory of parity complexes did not exist when [S5] was written, so his definition of $\mathsf{Parity}_{(\mathscr{B})}(\mathscr{X}, \Bbbk)$ is different. Understanding the meaning of Soergel's construction was one of the motivations for the study that led to [JMW2].) We therefore have an analgue of (1.26) in this setting, in the form of a diagram

$$\begin{tikzcd} \mathsf{Parity}_{(\mathscr{B})}(\mathscr{X}, \mathbb{C}) \arrow[r, "\mathbb{H}'", "\sim"'] & \mathsf{SMod}(V, W) \arrow[d, "\mathsf{For}"] \\ & \overline{\mathsf{SMod}}(\mathcal{W}, V) & \mathsf{Proj}(\mathcal{O}_0). \arrow[l, "\mathbb{V}"', "\sim"] \end{tikzcd} \tag{1.28}$$

There is also an analogue of (1.27) in this setting: if $\mathscr{X}^\vee$ is the flag variety of the complex simply-connected semisimple algebraic group whose root system is $\mathfrak{R}$, then by [AR1, Theorem 2.4] there exists an equivalence of abelian categories

$$\mathcal{O}_\Bbbk \cong \mathsf{Perv}_{(\mathscr{B}^\vee)}(\mathscr{X}^\vee, \Bbbk)$$

sending L_w to the simple perverse sheaf $\mathcal{IC}_{w_0 w^{-1}}$ for any $w \in W$.[5] The construction of this equivalence is quite different from that of (1.27): in fact it is obtain as a *consequence* of a Koszul duality formalism.

2. The Elias–Williamson category

In this section we explain the definition of the "diagrammatic" category associated with a Coxeter system $(\mathcal{W}, \mathcal{S})$ and a "realization" (see §2.2), following Elias–Williamson [EW2]. We will also explain the relation between this construction and the category of Soergel bimodules as considered in Section 1 (which, historically, was the main motivation behind its definition), see §2.12. The definition itself is given in §2.5. Before we can explain it we need to discuss a number of technicalities, which are important but can be ignored at first reading.

2.1. Quantum numbers. The definition of the Elias–Williamson category will involve a two-colored version of quantum numbers, which we now explain. These quantum numbers will live in the ring $\mathbb{Z}[x, y]$, where x, y are indeterminates. They are defined by induction, starting with

$$[0]_x = [0]_y = 0, \quad [1]_x = [1]_y = 1, \quad [2]_x = x, \ [2]_y = y,$$

and the relations

$$[n+1]_y = [2]_y [n]_x - [n-1]_y, \quad [n+1]_x = [2]_x [n]_y - [n-1]_x. \tag{2.1}$$

One can e.g. compute that

- $[3]_x = [3]_y = xy - 1$;
- $[4]_x = x^2 y - 2x$, $[4]_y = xy^2 - 2y$;
- $[5]_x = [5]_y = x^2 y^2 - 3xy + 1$;

[5]The formula in [AR1] looks a bit different. For the comparison between the two versions, see Exercise 3.1.

- $[6]_x = x^3y^2 - 4x^2y + 3x$; $[6]_y = x^2y^3 - 4xy^2 + 3y$.

These numbers are not symmetric in x and y (in the sense that $[n]_x \neq [n]_y$ for some n's), but in a very simple way explained in the following lemma.

LEMMA 2.1. *(1) For any $n \in \mathbb{Z}_{\geq 0}$ odd we have $[n]_x = [n]_y$;*
(2) For any $n \in \mathbb{Z}_{\geq 0}$ even we have $[2]_y[n]_x = [2]_x[n]_y$.

PROOF. Both formulas are proved in parallel by induction. □

It is clear also that the polynomial obtained from $[n]_x$ by switching x and y is $[n]_y$ for any $n \in \mathbb{Z}_{\geq 0}$, and vice versa. In view of Lemma 2.1(1), if n is odd we will sometimes write $[n]$ for $[n]_x = [n]_y$.

REMARK 2.2. Two-colored quantum numbers are generalization of "usual" quantum numbers, in a sense explained in [El, §3.1]. The (one-colored) quantum number $\langle n \rangle_x$ is a polynomial in x, which can be obtained from $[n]_x$ or $[n]_y$ by setting $y = x$.

There are also 2-colored quantum binomial coefficients, which can be defined as follows: for $n, m \geq 0$ with $n \leq m$ we set

$$\begin{bmatrix} m \\ n \end{bmatrix}_x = \frac{[m]_x[m-1]_x \cdots [m-n+1]_x}{[n]_x[n-1]_x \cdots [1]_x}, \quad \begin{bmatrix} m \\ n \end{bmatrix}_y = \frac{[m]_y[m-1]_y \cdots [m-n+1]_y}{[n]_y[n-1]_y \cdots [1]_y}.$$

It is not difficult to check that these fractions actually belong to $\mathbb{Z}[x,y]$, see [Ab3, Comments before Lemma 2.6].

2.2. Realizations.

2.2.1. *Definition.* Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system, and let $\Bbbk$ be a commutative domain. We consider a free $\Bbbk$-module V of finite rank, together with collections $(\alpha_s : s \in \mathcal{S})$ of vectors in $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ and $(\alpha_s^\vee : s \in \mathcal{S})$ of vectors of V. For any $s, t \in \mathcal{S}$ and $n \in \mathbb{Z}_{\geq 1}$, we denote[6] by $[n]_{s,t}$ the value of $[n]_x$ at

$$x = -\langle \alpha_s^\vee, \alpha_t \rangle \quad \text{and} \quad y = -\langle \alpha_t^\vee, \alpha_s \rangle.$$

Note that the corresponding evaluation of $[n]_y$ is $[n]_{t,s}$. In particular, if n is odd we have $[n]_{s,t} = [n]_{t,s}$; to emphasize the independence on the order between s and t, this element will sometimes be denoted $[n]_{\{s,t\}}$. Similarly, if $0 \leq n \leq m$ we will denote by $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_{s,t}$ the value of $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_x$ at $x = -\langle \alpha_s^\vee, \alpha_t \rangle$ and $y = -\langle \alpha_t^\vee, \alpha_s \rangle$.

Following [EW2, Definition 3.1], the triple

$$\big(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S})\big)$$

is called a *realization* of $(\mathcal{W}, \mathcal{S})$ over $\Bbbk$ if it satisfies the following conditions:

(1) for any $s \in \mathcal{S}$ we have $\langle \alpha_s^\vee, \alpha_s \rangle = 2$;
(2) the assignment $s \mapsto (v \mapsto v - \langle v, \alpha_s \rangle \alpha_s^\vee)$ defines a representation of $\mathcal{W}$ on V;
(3) we have

$$[m_{s,t}]_{s,t} = 0 \quad \text{for any } (s,t) \in \mathcal{S}_\circ^2. \tag{2.2}$$

[6] The element we denote by $[n]_{s,t}$ is often denoted $[n]_s$. We find this notation misleading since it hides the dependency on the other simple reflection, and hence follow a heavier but more explicit convention inspired by [ELi].

For an explanation of the origin of condition (3), see [**EW2**, §3.1] and [**El**].

There are further technical conditions on realizations that we will consider. First we will say that our realization *satisfies Demazure surjectivity* if for any $s \in \mathcal{S}$ the morphisms

$$\alpha_s : V \to \Bbbk \quad \text{and} \quad \alpha_s^\vee : V^* \to \Bbbk$$

are surjective. Note that this condition is automatic if $2 \in \Bbbk^\times$, due to the condition (1).

Next, we consider the numbers $[m_{s,t} - 1]_{s,t}$ for $(s,t) \in \mathcal{S}_\circ^2$. As explained in [**EW3**, (6.11), (6.12)], the condition that $[m_{s,t}]_{s,t} = [m_{s,t}]_{t,s} = 0$ implies that $[m_{s,t} - 1]_{s,t} \cdot [m_{s,t} - 1]_{t,s} = 1$. (In case $m_{s,t}$ is even, this condition simplifies to $([m_{s,t} - 1]_{\{s,t\}})^2 = 1$.) But the combinatorics involved simplified greatly when each of these numbers is actually equal to 1. We will therefore say that the realization is *balanced* if

$$[m_{s,t} - 1]_{s,t} = 1 \quad \text{for any } (s,t) \in \mathcal{S}_\circ^2. \tag{2.3}$$

Below all of our realizations will be assumed to be balanced and to satisfy Demazure surjectivity. The latter assumption is necessary for the results discussed in §2.8 to hold. The former assumption can be relaxed a little bit at the cost of some complications (see [**EW3**, §7]), but we will not consider this variant here.

A further condition that one needs to impose on realizations to obtain a complete theory is that

$$\begin{bmatrix} m_{s,t} \\ k \end{bmatrix}_{s,t} = 0 \quad \text{for all } s,t \in \mathcal{S}_\circ^2 \text{ and all integers } k \in \{1, \cdots, m_{s,t} - 1\}. \tag{2.4}$$

(Here the case $k = 1$ recovers (2.2).) This condition was overlooked in [**EW2**], but it was later considered in [**EW3**], [**Ab3**] and finally in [**Haz**] (as we will explain below).

REMARK 2.3. (1) Given a realization $(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S}))$ of $(\mathcal{W}, \mathcal{S})$ over $\Bbbk$ and a ring morphism $\Bbbk \to \Bbbk'$ (where $\Bbbk'$ is again a commutative domain), there exists a natural realization of $(\mathcal{W}, \mathcal{S})$ over $\Bbbk'$ with underlying $\Bbbk'$-module $\Bbbk' \otimes_{\Bbbk} V$. If the original realization is balanced, resp. satisfies Demazure surjectivity, resp. satisfies (2.4), then so does this new realization.

(2) See [**Ab3**, Proposition 3.4] for some reformulations of this assumption. By [**Ab3**, Proposition 3.6], it is satisfied if, for any $(s,t) \in \mathcal{S}_\circ^2$, the action of $\langle s, t \rangle$ on $\Bbbk\alpha_s + \Bbbk\alpha_t$ is faithful. (See also Lemma 2.5 below for a variant of this result.)

We will say that a realization is *symmetric* if for any distinct $s, t \in \mathcal{S}$ we have

$$\langle \alpha_s^\vee, \alpha_t \rangle = \langle \alpha_t^\vee, \alpha_s \rangle.$$

This condition is really useful, because when it is satisfied we have $[n]_{s,t} = [n]_{t,s}$ for any $n \in \mathbb{Z}_{\geq 1}$ and $s, t \in \mathcal{S}$, so that one can use a "one-colored" combinatorics rather than a "two-colored" one. Unfortunately, it is not satisfied for some important examples we want to consider (see §2.2.2 below), so we will generally not assume it is satisfied.

The following lemma can help checking that some data form realizations.

LEMMA 2.4. *Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system and $\Bbbk$ be a commutative ring. Assume we are given a free $\Bbbk$-module V of finite rank together with collections $(\alpha_s : s \in \mathcal{S})$ of vectors in V^* and $(\alpha_s^\vee : s \in \mathcal{S})$ of vectors in V which satisfy conditions (1)–(2) above. Let also $s, t \in \mathcal{S}$ be distinct reflections, and assume that $\alpha_s^\vee$ and $\alpha_t^\vee$ are linearly independent. Then the rank-2 free submodule $\Bbbk\alpha_s^\vee \oplus \Bbbk\alpha_t^\vee \subset V$ is stable under the actions of s and t, and for any $k \geq 0$ the matrix of $(st)^k$, resp. $(st)^k s$, in the basis $(\alpha_s^\vee, \alpha_t^\vee)$ of this module is*

$$\begin{pmatrix} [2k+1]_{\{s,t\}} & -[2k]_{t,s} \\ [2k]_{s,t} & -[2k-1]_{\{s,t\}} \end{pmatrix}, \quad \textit{resp.} \quad \begin{pmatrix} -[2k+1]_{\{s,t\}} & -[2k+2]_{t,s} \\ -[2k]_{s,t} & [2k+1]_{\{s,t\}} \end{pmatrix}.$$

PROOF. It is clear from definitions that $\Bbbk\alpha_s^\vee \oplus \Bbbk\alpha_t^\vee \subset V$ is stable under the actions of s and t, and that the matrix of s, resp. t, in the basis $(\alpha_s^\vee, \alpha_t^\vee)$ is

$$\begin{pmatrix} -1 & [2]_{t,s} \\ 0 & 1 \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} 1 & 0 \\ [2]_{t,s} & -1 \end{pmatrix}.$$

The claims can be checked together using these formulas and (2.1) by induction on k. □

The following lemma (explained to us by N. Abe) can also help checking condition (2.4).

LEMMA 2.5. *Consider a balanced realization*

$$(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S}))$$

of $(\mathcal{W}, \mathcal{S})$ over $\Bbbk$ in the sense of §2.2.1. Assume that

- *$\Bbbk$ is a field with $\operatorname{char}(\Bbbk) \neq 2$;*
- *for any $(s,t) \in \mathcal{S}_\circ^2$ the action of $\langle s, t \rangle$ on V^* is faithful, and we have $\Bbbk\alpha_s \neq \Bbbk\alpha_t$.*

Then (2.4) is satisfied.

PROOF. Fix $(s,t) \in \mathcal{S}_\circ^2$. By assumption the sum $\Bbbk\alpha_s + \Bbbk\alpha_t$ is direct. If $4 - [2]_{s,t}[2]_{t,s} \neq \{0\}$ we have

$$V = \big(\Bbbk\alpha_s \oplus \Bbbk\alpha_t\big) \oplus \{\lambda \in V^* \mid \langle \lambda, \alpha_s^\vee \rangle = \langle \lambda, \alpha_t^\vee \rangle = 0\}$$

because the matrix

$$\begin{pmatrix} \langle \alpha_s, \alpha_s^\vee \rangle & \langle \alpha_t, \alpha_s^\vee \rangle \\ \langle \alpha_s, \alpha_t^\vee \rangle & \langle \alpha_t, \alpha_t^\vee \rangle \end{pmatrix}$$

is invertible. Since $\langle s, t \rangle$ acts trivially on the rightmost summand and faithfully on V^*, it must act faithfully on $\Bbbk\alpha_s \oplus \Bbbk\alpha_t$. This implies our claim by Remark 2.3(2).

Now, assume that $[2]_{s,t}[2]_{t,s} = 4$. By Exercise 2.7, for any $n \geq 0$ we then have

$$[2n]_{s,t} = [2]_{s,t} n, \quad [2n]_{t,s} = [2]_{t,s} n, \quad [2n+1]_{\{s,t\}} = 2n+1. \tag{2.5}$$

We have

$$(st)(\alpha_t) = -\alpha_t - [2]_{s,t}\alpha_s, \quad (st)(\alpha_s) = 3\alpha_s + [2]_{t,s}\alpha_t,$$

and both s and t act trivially on $V^*/(\Bbbk\alpha_s \oplus \Bbbk\alpha_t)$. Hence, in a suitable basis of V^* extending $([2]_{t,s}\alpha_t, -2\alpha_s)$, the matrix of st has the form

$$\begin{pmatrix} -1 & -2 & x \\ 2 & 3 & y \\ 0 & 0 & \mathrm{id} \end{pmatrix}$$

for some vectors x and y. By induction, one then checks that for any $n \geq 0$ the matrix of $(st)^n$ in this basis is

$$\begin{pmatrix} -2n+1 & -2n & -n(n-2)x - n(n-1)y \\ 2n & 2n+1 & n(n-1)x + n^2 y \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

By the faithfulness assumption, $m_{s,t}$ is the smallest positive integer such that this matrix is the identity; we deduce that $\Bbbk$ has positive characteristic, equal to $m_{s,t}$. On the other hand, by (2.5), if $k \in \{1, \cdots, m_{s,t}-1\}$ there exists $a \in \Bbbk^\times$ such that

$$\begin{bmatrix} m_{s,t} \\ k \end{bmatrix}_{s,t} = a \cdot \binom{m_{s,t}}{k}.$$

The right-hand side vanishes, hence so does the left-hand side. □

2.2.2. *Cartan realizations of crystallographic Coxeter systems.* For the purposes of this book, the main example of a realization of a Coxeter system the reader should have in mind is the following. Let A be a generalized Cartan matrix, and let $(\mathbf{X}, (\alpha_i : i \in I), (\alpha_i^\vee : i \in I))$ be an associated Kac–Moody root datum; see §1.2.4.

EXAMPLE 2.6. Following [Ti, bottom of p. 8], there are three "natural" Kac–Moody root data one can associate to an arbitrary generalized Cartan matrix:

- the *adjoint* datum, given by $\mathbf{X} = \mathbb{Z}^I$ with canonical basis denoted $(\alpha_i : i \in I)$ and the vectors $\alpha_i^\vee \in \mathbf{X}^\vee$ defined by the equality $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ for $i, j \in I$;
- the *simply connected* datum, given by $\mathbf{X}^\vee = \mathbb{Z}^I$ with canonical basis denoted $(\alpha_i^\vee : i \in I)$ and the vectors $\alpha_i \in \mathbf{X} = (\mathbf{X}^\vee)^\vee$ defined by the equality $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ for $i, j \in I$;
- the *universal* datum, given by $\mathbf{X} = \mathbb{Z}^{I \sqcup I}$ with canonical basis $(\alpha_i)_{i \in I} \cup (\beta_i)_{i \in I}$, $\mathbf{X}^\vee = \mathbb{Z}^{I \sqcup I}$ with canonical basis $(\beta_i^\vee)_{i \in I} \cup (\alpha_i^\vee)_{i \in I}$ and the pairing between $\mathbf{X}$ and $\mathbf{X}^\vee$ defined by

$$\langle \alpha_i, \beta_j^\vee \rangle = \delta_{i,j}, \quad \langle \alpha_i, \alpha_j^\vee \rangle = a_{j,i}, \quad \langle \beta_i, \beta_j^\vee \rangle = 0, \quad \langle \beta_i, \alpha_j^\vee \rangle = \delta_{i,j}.$$

We have recalled in §1.2.3 how to associate to A a (crystallographic) Coxeter system $(\mathcal{W}, \mathcal{S})$. Let $\Bbbk$ be an integral domain. Using the Kac–Moody root datum, we can construct a realization of $(\mathcal{W}, \mathcal{S})$ over $\Bbbk$ as follows: we set $V := \Bbbk \otimes_{\mathbb{Z}} \mathbf{X}^\vee$ (so that V^* is identified with $\Bbbk \otimes_{\mathbb{Z}} \mathbf{X}$), and for $s \in \mathcal{S}$, we define α_s, resp. $\alpha_s^\vee$, to be the image of α_{i_s}, resp. $\alpha_{i_s}^\vee$, in V^*, resp. in V. To justify this assertion we need to explain why conditions (1)–(3) above are satisfied. Condition (1) is obvious, and (2) is part of the theory of Kac–Moody groups; see [Ti, §3.1]. Condition (3) can be checked by explicit computation, depending on the value of $a_{i_s i_t} a_{i_t i_s}$. (Only the values 0, 1, 2, 3 need to be considered.) For instance, if $a_{i_s i_t} a_{i_t i_s} = 2$, then by definition of a generalized Cartan matrix we have either $a_{i_s i_t} = -1$ and $a_{i_t i_s} = -2$, or $a_{i_s i_t} = -2$ and $a_{i_t i_s} = -1$. In both cases, using the formulas for $[4]_x$ and $[4]_y$ given in §2.1 one sees that $[4]_{s,t} = 0$ when $\Bbbk = \mathbb{Z}$, hence in general.

A realization of $(\mathcal{W}, \mathcal{S})$ obtained in this way is called a *Cartan realization*. Such a realization is always balanced, it always satisfies (2.4) (this can be checked explicitly by the same considerations as above), but it might not satisfy Demazure surjectivity. More precisely, let us define $\mathbb{Z}'$ to be $\mathbb{Z}$ if the maps $\alpha_i : \mathbf{X}^\vee \to \mathbb{Z}$ and

$\alpha_i^\vee : \mathbf{X} \to \mathbb{Z}$ are surjective for all $i \in I$, and as $\mathbb{Z}[\frac{1}{2}]$ otherwise. Then Demazure surjectivity holds as soon as there exists a ring morphism $\mathbb{Z}' \to \Bbbk$.

EXAMPLE 2.7. For the cases given in Example 2.6:

- for the adjoint datum we have $\mathbb{Z}' = \mathbb{Z}[\frac{1}{2}]$ if A has a line consisting only of even numbers, and $\mathbb{Z}' = \mathbb{Z}$ otherwise;
- for the simply connected datum we have $\mathbb{Z}' = \mathbb{Z}[\frac{1}{2}]$ if A has a column consisting only of even numbers, and $\mathbb{Z}' = \mathbb{Z}$ otherwise;
- for the universal datum, we have $\mathbb{Z}' = \mathbb{Z}$ in all cases.

2.2.3. *The geometric realization.* There exists another systematic construction of realizations, which will not play any role in the present book, but which has the advantage of providing realizations for *all* Coxeter systems. (Recall that for us, a Coxeter system always has a finite number of simple reflections!) Namely, if $(\mathcal{W}, \mathcal{S})$ is a Coxeter system, let V be the representation considered in Remark 1.8. For $s \in \mathcal{S}$ we set

$$\alpha_s^\vee := e_s \in V, \quad \alpha_s = 2\langle e_s, -\rangle \in V^*.$$

These data satisfy conditions (1)–(2) in the definition above. In order to check (2.2) (and, at the same time, (2.3) and (2.4)), we fix $(s,t) \in \mathcal{S}^2_\circ$. We identify $\mathbb{R}\alpha_s^\vee \oplus \mathbb{R}\alpha_t^\vee$ with the plane $\mathbb{R}^2$ in such a way that $\alpha_s^\vee$, resp. $\alpha_t^\vee$, corresponds to the vector $(1,0)$, resp. $(-\cos(\pi/m_{s,t}), \sin(\pi/m_{s,t}))$. With this identification, the restriction of s, resp. t, is the orthogonal reflection with respect to the line orthogonal to $\alpha_s^\vee$, resp. $\alpha_t^\vee$. As a consequence, the restriction of st identifies with rotation of angle $2\pi/m_{s,t}$. If $m_{s,t} = 2k$ is even, then these remarks show that $(st)^k = -\mathrm{id}$. Comparing with the information provided by Lemma 2.4, we deduce that

$$[m_{s,t}]_{s,t} = [m_{s,t}]_{t,s} = 0 \quad \text{and} \quad [m_{s,t} - 1]_{\{s,t\}} = 1.$$

If $m_{s,t} = 2k+1$ is odd, we use the fact that

$$(st)^k(\alpha_s^\vee) = \alpha_t^\vee \quad \text{and} \quad (st)^k(\alpha_t^\vee) = -\alpha_s^\vee - 2\cos(\pi/m_{s,t})\alpha_t^\vee$$

and again Lemma 2.4 to show that

$$[m_{s,t}]_{\{s,t\}} = 0 \quad \text{and} \quad [m_{s,t} - 1]_{s,t} = [m_{s,t} - 1]_{t,s} = 1.$$

In summary, in both cases (2.2) is satisfied, so that these data define a realization (called the *geometric realization*), and this realization is balanced. It satisfies Demazure surjectivity (because $2 \in \Bbbk^\times$), and it is clear that it is symmetric.

Finally, we note that if $k \in \{1, \cdots, m_{s,t} - 1\}$ we have

$$[k]_{s,t} = [k]_{t,s} \neq 0. \tag{2.6}$$

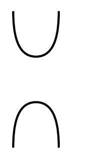
In fact, the equality of quantum numbers follows from symmetry. To prove that these numbers are nonzero we have to distinguish the cases when $m_{s,t}$ and k are even or odd. If $m_{s,t} = 2j$ is even and $k = 2l$ is also even, if $[k]_{s,t} = 0$ then by Lemma 2.4 we have $(st)^k(\alpha_t^\vee) \in \mathbb{R} \cdot \alpha_t^\vee$. This is absurd since the restriction of st to $\mathbb{R} \cdot \alpha_s^\vee \oplus \mathbb{R} \cdot \alpha_t \cong \mathbb{R}^2$ identifies with rotation of angle $\frac{l\pi}{j}$, which belongs to $(0,\pi)$. Similarly, if $m_{s,t} = 2j$ is even and $k = 2l+1$ is odd, if $[k]_{s,t} = 0$ then $(st)^k(\alpha_s^\vee) \in \mathbb{R} \cdot \alpha_t^\vee$, which implies (since $(st)^k$ identifies with rotation of an angle in $(0,\pi)$) that $(st)^k(\alpha_s^\vee) = \alpha_t^\vee$, hence that $\frac{l\pi}{j} = \pi - \frac{\pi}{2j} = \frac{(2j-1)\pi}{2j}$, which again is absurd. The cases when $m_{s,t}$ is odd can be checked similarly.

These conditions together with condition (3) imply that (2.4) is satisfied.

REMARK 2.8. Recall the representation of $(\mathcal{W}, \mathcal{S})$ considered in §1.2.2. This representation can be upgraded to a balanced realization by setting $\alpha_s^\vee = e_s$ and $\alpha_s = e_s^*$. In fact the matrix $(\langle \alpha_t^\vee, \alpha_s \rangle)_{s,t \in \mathcal{S}}$ for these data is the same as for the geometric realization, hence it also satisfies (2.2), (2.3) and (2.4).

2.3. Jones–Wenzl projectors.

2.3.1. *The two-colored Temperley–Lieb category.* Given two colors[7] s and t and a $\mathbb{Z}[x, y]$-algebra A, we can define the two-colored Temperley–Lieb category $2\mathscr{TL}_A$ over A as follows (see [**El**] or [**EW1**, §6.4]; see also [**ELi**, §2.6] for a multicolored extension of this definition). The objects in this category are alternating words in the alphabet $\{s, t\}$. There exists no nonzero morphism between two words unless they start and finish with the same letter. If they do, then the space of morphisms between them has an A-basis consisting of two-colored crossingless matchings between them. (Here a two-colored crossingless matching is a crossingless matching where the regions are colored either by s or by t, and adjacent regions have different colors.) Diagrams should always be read from bottom to top. For instance,

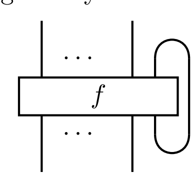


is a crossingless matching. If we color the leftmost region by s and alternate the colors, it defines a morphism from (s, t, s, t) to itself in $2\mathscr{TL}_A$. This morphism factors through (s, t).

Composition in this category consists of the A-bilinear maps induced by vertical concatenation of two-colored crossingless matchings and evaluation of circles as follows: a circle whose interior is labeled by t inside a region labeled by s evaluates to $-x$, and a circle whose interior is labeled by s inside a region labeled by t evaluates to $-y$. Note that by forgetting the coloring, a morphism from $(s, t, \cdots)$ (with n alternating letters) to $(s, t, \cdots)$ (with m alternating letters) provides a crossingless matching with $n - 1$ points at the bottom and $m - 1$ points on top.

The category $2\mathscr{TL}_A$ admits an anti-autoequivalence ι which fixes every object, and acts on morphisms by reflecting the two-colored crossingless matchings along an horizontal axis.

Later we will use the notion of *partial trace* of an endomorphism of an object of $2\mathscr{TL}_A$. If $\underline{w} = (u_1, \cdots, u_n)$ and $f \in \mathrm{End}_{2\mathscr{TL}_A}(\underline{w})$, the partial trace $\mathrm{pTr}(f)$ is the endomorphism of $(u_1, \cdots, u_{n-1})$ given by

$\mathrm{pTr}(f) =$ 

REMARK 2.9. The two-colored Temperley–Lieb category is a generalization of the "usual" Temperley–Lieb category $\mathscr{TL}_{A'}$, which is defined as follows. (Here, A' is a $\mathbb{Z}[x]$-algebra.) Objects are $\mathbb{Z}_{\geq 0}$, and morphisms from n to m are spanned

[7]In practice, below s and t will be two distinct simple reflections in a Coxeter system. But this interpretation plays no role in the present subsection, and s and t will just be considered as some colors.

by crossingless matchings with n points at the bottom and m points at the top. Composition is induced by concatenation of diagrams, where circles are evaluated to $-x$. The combinatorics of this category involves the "one-colored" quantum numbers mentioned in Remark 2.2.

There are cases where the two-colored Temperley–Lieb category in fact "reduces to the usual version." Namely, assume that A is a $\mathbb{Z}[x, y]$-algebra in which the images of x and y coincide. (We will refer to this setting as the "symmetric case.") Then A can also be considered as a $\mathbb{Z}[x]$-algebra, hence we can consider the category $\mathscr{TL}_A$. There are two fully faithful functors

$$\mathscr{TL}_A \to 2\mathscr{TL}_A$$

one can consider. The first option is to send $n \in \mathbb{Z}_{\geq 0}$ to the unique alternating word in $\{s, t\}$ of length $n + 1$ starting with s, and any crossingless matching to its unique two-colored version whose leftmost part is colored by s. The second option is to follow the same recipe with t in place of s. These two functors "capture" the combinatorics of $2\mathscr{TL}_A$ in this case, in a sense that should be obvious.

2.3.2. *Jones–Wenzl projectors.* Let $\underline{w}$ be an alternating word in $\{s, t\}$. The two-colored crossingless matching which consists only of vertical lines and has colors given by $\underline{w}$ is called the *trivial matching* associated with $\underline{w}$. The following lemma is taken from [**ELi**, Claim 2.14].

LEMMA 2.10. *Let $\underline{w}$ be an alternating word in $\{s, t\}$. Assume that $\mathrm{End}_{2\mathscr{TL}_A}(\underline{w})$ contains an element f whose expansion in the basis of two-colored crossingless matchings has coefficients 1 on the trivial matching and whose pre-composition with any morphism of the form*

$$\Big| \cdots \Big| \quad \cup \quad \Big| \cdots \Big| \tag{2.7}$$

(with appropriate coloring) vanishes. Then f is the unique element satisfying such properties, it is an idempotent, it satisfies $f = \iota(f)$, and it is killed by post-composition with any morphism of the form

$$\Big| \cdots \Big| \quad \cap \quad \Big| \cdots \Big| . \tag{2.8}$$

PROOF. Let us denote by $I \subset \mathrm{End}_{2\mathscr{TL}_A}(\underline{w})$ the submodule spanned by all the nontrivial two-colored crossingless matchings. Then by assumption we have $f = \mathrm{id} + g$ for some $g \in I$. Any nontrivial two-colored crossingless matching involves a cup on top, which implies that

$$f \circ h = 0 \quad \text{for any } h \in I,$$

hence that f is an idempotent. Similarly, since any nontrivial two-colored crossingless matching involves a cap on bottom, we have

$$h \circ \iota(f) = 0 \quad \text{for any } h \in I.$$

Since I is stable under ι, we have $\iota(f) \in \mathrm{id} + I$, which implies that

$$f = f \circ \iota(f) = \iota(f).$$

Hence f is killed by post-composition with any morphism of the form (2.8).

Finally, if $g \in \mathrm{End}_{2\mathscr{TL}_A}(\underline{w})$ is another element satisfying the properties of the lemma, then we have $g \in \mathrm{id} + I$, hence $f = g \circ f = g$, proving unicity. $\square$

A morphism satisfying the conditions in Lemma 2.10 is called a *Jones–Wenzl projector* associated with $\underline{w}$, and is denoted $\mathcal{JW}_{\underline{w}}$. Note that if $\mathcal{JW}_{\underline{w}}$ exists, then the subspace of $\mathrm{Hom}_{2\mathscr{TL}_A}(\underline{w})$ consisting of morphisms whose whose pre-composition with any morphism of the form (2.7) (with appropriate coloring) vanishes is exactly the span of $\mathcal{JW}_{\underline{w}}$. In fact, let $\lambda \in \Bbbk$ be the coefficient of the trivial matching in the expansion of f on the basis of two-colored crossingless matchings. Then $f-\lambda\cdot\mathcal{JW}_{\underline{w}}$ has a trivial coefficient on the trivial matching, so that $\mathcal{JW}_{\underline{w}} + (f - \lambda \cdot \mathcal{JW}_{\underline{w}})$ satisfies the properties of Lemma 2.10. By unicity we deduce that $f = \lambda \cdot \mathcal{JW}_{\underline{w}}$, which finishes the proof of our claim. Of course, a similar property holds for post-composition with morphisms of the form (2.8).

Remark 2.11. Recall the setting of Remark 2.9. The considerations above have obvious analogues in the category $\mathscr{TL}_{A'}$. This is in fact the setting where these morphisms were introduced by Jones and Wenzl independently; see [**El**, §4.1] for details. In this case there are no colors to consider, so Jones–Wenzl projectors are attached to nonnegative integers.

2.3.3. *Existence.* Given a $\mathbb{Z}[x,y]$-algebra A and an alternating word $\underline{w}$ in s,t, it is a priori a difficult question to determine whether a Jones–Wenzl projector associated with $\underline{w}$ exists. A solution to this question was asserted in [**EW2**], but it turned out to be wrong, as explained in [**EW3**]. The correct solution was finally found by Hazi in [**Haz**], following an earlier result in the symmetric case due to Webster (see the appendix to [**ELi**]): if $\underline{w}$ is a word of length n starting with s, then $\mathcal{JW}_{\underline{w}}$ exists if and only if the image of $\begin{bmatrix} n-1 \\ k \end{bmatrix}_x$ in $\Bbbk$ is invertible for any integer $k \in \{1, \cdots, n-2\}$.

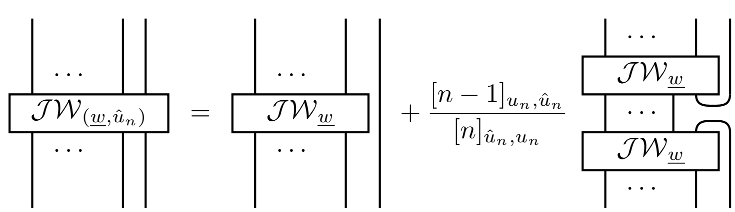
The following lemma, taken from [**EW3**, §6.6], can sometimes be used to compute $\mathcal{JW}_{\underline{w}}$ explicitly (see below for details). Here, if $u \in \{s,t\}$ we denote by $\hat{u}$ the unique element in $\{s,t\} \smallsetminus \{u\}$.

Lemma 2.12. *Let A be a $\mathbb{Z}[x,y]$-algebra with structure morphism $\varphi : \mathbb{Z}[x,y] \to A$, and for $n \geq 0$ denote by $[n]_{s,t}$, resp. $[n]_{t,s}$, the image of $[n]_x$, resp. $[n]_y$, in A. Let $m \in \mathbb{Z}_{>0}$, and assume that $[k]_{s,t}$ and $[k]_{t,s}$ are invertible for any $k < m$. Then $\mathcal{JW}_{\underline{w}}$ exists for any alternating word $\underline{w}$ in $\{s,t\}$ of length $\leq m$, and these morphisms satisfy*

$$\mathrm{pTr}(\mathcal{JW}_{\underline{w}}) = -\frac{[n]_{\hat{v},v}}{[n-1]_{v,\hat{v}}} \tag{2.9}$$

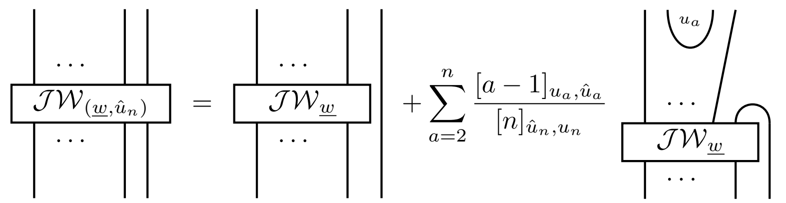
where n is the length of $\underline{w}$ and v is the last letter in $\underline{w}$. Moreover the following recursion formulas hold if $\underline{w} = (u_1, \cdots, u_n)$:

(1)



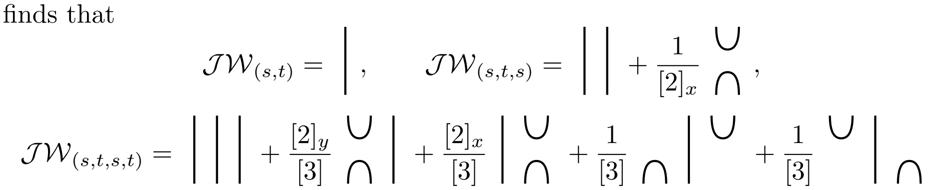
;

(2)



One can sometimes use this lemma to prove existence of (and compute) Jones–Wenzl projectors even when some quantum numbers vanish in A. Namely, first consider the case when $A = \mathbb{Q}(x, y)$. In this case, Lemma 2.12 implies the existence of all Jones–Wenzl projectors. Assume that, for a given $\underline{w}$, one has an explicit expression of $\mathcal{JW}_{\underline{w}}$ (e.g. obtained by using one of the recursion formulas in Lemma 2.12) and that the coefficients in the expansion of these morphisms in the basis of two-colored crossingless matchings all belong to $\mathbb{Z}[x, y][1/f]$ for some $f \in \mathbb{Z}[x, y]$. Then if the given morphism $\mathbb{Z}[x, y] \to A$ extends to a morphism $\mathbb{Z}[x, y][1/f] \to A$ (in other words, if the image of f in A is invertible), one obtains morphisms in $2\mathscr{T}\mathscr{L}_A$ by evaluating all coefficients in A using such an extension. It is clear from definitions that this morphism is a Jones–Wenzl projector for $\underline{w}$. (Note that Jones–Wenzl projectors associated with the other words of shorter length might not exist, constrary to the situation considered in Lemma 2.12.)

For our purposes, the most important cases will be when the length of $\underline{w}$ belongs to $\{2, 3, 4, 6\}$. In these cases, in $2\mathscr{T}\mathscr{L}_{\mathbb{Q}(x,y)}$ the Jones–Wenzl projectors are as follows. (We will only write projectors for words starting with s, and will not indicate the colors of the regions since they can be easily determined. The projectors for the words starting with t can be obtained by switching $s \leftrightarrow t$ and $x \leftrightarrow y$.) One finds that



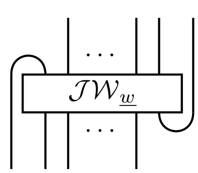
The next relevant case is $\mathcal{JW}_{(s,t,s,t,s,t)}$, whose expression is shown on Figure 2.1.

2.3.4. *Rotatability.* Consider a $\mathbb{Z}[x, y]$-algebra A, with structure morphism $\varphi : \mathbb{Z}[x, y] \to A$, and the associated category $2\mathscr{T}\mathscr{L}_A$. Fix $m \in \mathbb{Z}_{\geq 1}$, and denote by $\underline{w}$, $\underline{w}'$ the two alternating words in $\{s, t\}$ of length n. In the rest of this subsection we assume that $\mathcal{JW}_{\underline{w}}$ and $\mathcal{JW}_{\underline{w}'}$ exist. We will say that these morphisms are *rotatable* if we have

$$\mathrm{pTr}(\mathcal{JW}_{\underline{w}}) = 0 \quad \text{and} \quad \mathrm{pTr}(\mathcal{JW}_{\underline{w}'}) = 0.$$

The reason for this terminology is explained by the following lemma, which is copied from [EW3, Lemma 6.15].

Lemma 2.13. *The morphism*



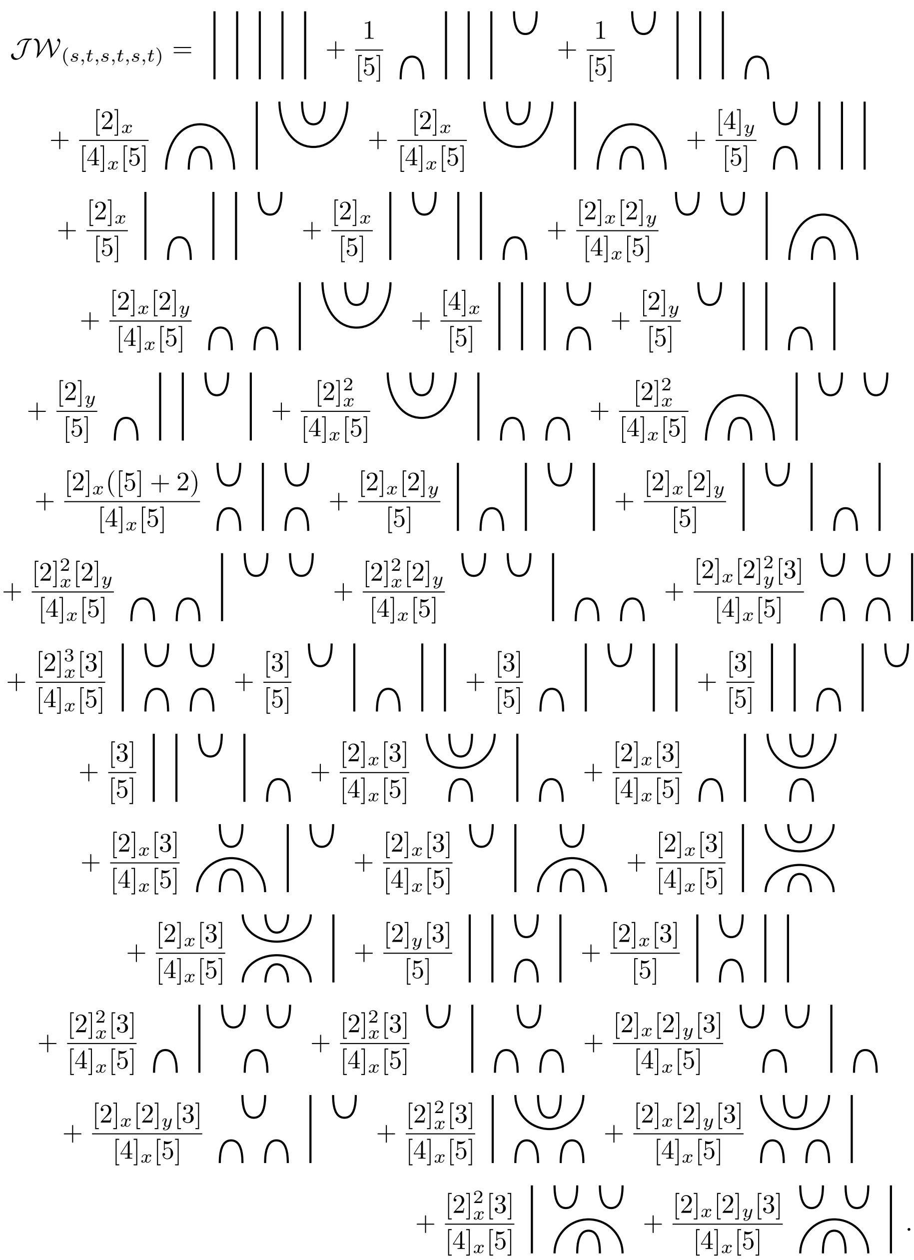


FIGURE 2.1. Jones–Wenzel projector for (s,t,s,t,s,t)

belongs to $A \cdot \mathcal{JW}_{\underline{w}'}$ *if and only if* $\mathrm{pTr}(\mathcal{JW}_{\underline{w}}) = 0$.

PROOF. As explained in §2.3.2, an endomorphism of $\underline{w}'$ belongs to $A \cdot \mathcal{JW}_{\underline{w}'}$ iff it is killed by post-composition with any morphism of the form (2.8). Our given morphism is killed by composition with such a morphism if the cap is not on the rightmost strands. It is killed by composition with the cap on the rightmost strand iff $\mathrm{pTr}(\mathcal{JW}_{\underline{w}}) = 0$. ☐

REMARK 2.14. If the morphisms $\mathcal{JW}_{\underline{w}}$ and $\mathcal{JW}_{\underline{w}'}$ exist and are rotatable, one can determine the coefficient appearing in Lemma 2.13 explicitly, see [**EW3**, Lemma 6.21]. In the cases we will consider in the setting of the Elias–Williamson category (see §2.4 below), the condition that the realization is balanced will in fact imply that this coefficient is 1.

As for existence, it is a priori a delicate question to determine when this condition is satisfied. One case when it is easy to conclude is the setting of Lemma 2.12.

LEMMA 2.15. *Consider the setting of Lemma 2.12, and assume that $[k]_{s,t}$ and $[k]_{t,s}$ are invertible for any $k < m$. Then $\mathcal{JW}_{\underline{w}}$ and $\mathcal{JW}_{\underline{w}'}$ are rotatable iff $[m]_{s,t} = [m]_{t,s} = 0$.*

PROOF. The claim is a direct consequence of (2.9). ☐

The rotatability for general realizations was also considered by Hazi in [**Haz**], where he proved the following result.

THEOREM 2.16. *Let $n \geq 1$. The Jones–Wenzl projectors associated with the two alternating words in $\{s,t\}$ of length n exist and are rotatable if and only if the images of $\begin{bmatrix} n \\ k \end{bmatrix}_x$ and $\begin{bmatrix} n \\ k \end{bmatrix}_y$ in $\Bbbk$ vanish for any integer $k \in \{1, \cdots, n-1\}$.*

In cases where one has an explicit formula for the projectors $\mathcal{JW}_{\underline{w}}$ and $\mathcal{JW}_{\underline{w}'}$, checking the rotatability condition is just a matter of computation. Using the formulas given in §2.3.3 one can check explicitly (if one is patient enough) that the condition in Theorem 2.16 is indeed sufficient in these cases.

2.4. Some consequences of the technical conditions. From now on we fix a Coxeter system $(\mathcal{W}, \mathcal{S})$, an integral domain $\Bbbk$, and a balanced realization

$$\big(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S})\big)$$

of $(\mathcal{W}, \mathcal{S})$ over $\Bbbk$ which satisfies (2.4). We will consider the symmetric algebra

$$R := \mathrm{S}_\Bbbk(V^*)$$

as a graded ring with V^* in degree 2. This algebra admits a natural action of $\mathcal{W}$ (induced by the action on V), and for $s \in \mathcal{S}$ we will denote by $R^s \subset R$ the subalgebra of s-invariants. The following lemma (which generalizes some of the computations in the proof of Lemma 1.11) is one of the justifications for the assumption of Demazure surjectivity.

LEMMA 2.17. *Assume that $\alpha_s^\vee : V^* \to \Bbbk$ is surjective and that $\alpha_s \neq 0$. If $\delta_s \in V^*$ satisfies $\langle \delta_s, \alpha_s^\vee \rangle = 1$, then we have*

$$R = R^s \oplus \delta_s \cdot R^s.$$

PROOF. Since $s(\delta_s) \neq \delta_s$ (because $\alpha_s \neq 0$ by assumption), it is clear that $R^s \cap (\delta_s R^s) = \{0\}$. On the other hand, using the fact that

$$V^* = (V^*)^s \oplus \Bbbk \cdot \delta_s \tag{2.10}$$

(where $(V^*)^s = \ker(\alpha_s^\vee)$) and the formula

$$(\delta_s)^2 = (\delta_s + s(\delta_s)) \cdot \delta_s - s(\delta_s)\delta_s \tag{2.11}$$

one checks by induction that for any $n \in \mathbb{Z}_{\geq 0}$, any element of R^{2n} belongs to $R^s + \delta_s \cdot R^s$; it follows that $R = R^s \oplus \delta_s \cdot R^s$, as desired. □

From now on we assume in addition that our realization satisfies Demazure surjectivity. In particular, Lemma 2.17 holds for any s. For $s \in \mathcal{S}$ we will denote by

$$\partial_s : R \to R^s$$

the Demazure operator associated with s, i.e. the $\Bbbk$-linear map sending $a = a_1 + \delta_s a_2$ (with $a_1, a_2 \in R^s$, and where δ_s is as in Lemma 2.17) to a_2. This map does not depend on the choice of δ_s: in fact we have

$$\partial_s(f) = \frac{f - s(f)}{\alpha_s}$$

in the fraction field of R.

In [**EW2**], the authors associate to such data a $\Bbbk$-linear graded (strict) monoidal category[8] $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$. (In [**EW2**] it is not assumed that (2.4) is satisfied, but the teatment of Jones–Wenzl projectors has a gap. This gap was identified and partially solved in [**EW3**], and later completely solved in [**Haz**].) The definition of this category is given in §2.5; in the rest of this subsection we discuss some technical details required in this definition.

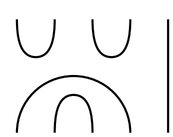
First, for any $(s,t) \in \mathcal{S}_\circ^2$, considering $\Bbbk$ as a $\mathbb{Z}[x,y]$-algebra via

$$x \mapsto -\langle \alpha_s^\vee, \alpha_t \rangle, \quad y \mapsto -\langle \alpha_t^\vee, \alpha_s \rangle,$$

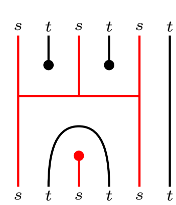
the assumption (2.4) and Theorem 2.16 ensure that in the category $2\mathscr{TL}_\Bbbk$ the Jones–Wenzl projectors

$$\mathcal{JW}_{(s,t,\cdots)} \quad \text{and} \quad \mathcal{JW}_{(t,s,\cdots)}$$

(with $m_{s,t}$ letters in each case) exist and are rotatable. Using a deformation retract (see [**EW2**, §5.2]), from a two-colored crossingless matching one obtains a diagram of the form used below in the definition of $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ (see §2.5); for instance, from the matching



with the leftmost region colored by s we obtain the diagram

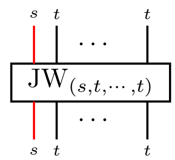


.

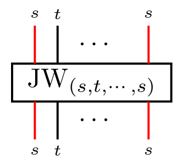
[8]Of course this category also depends on $\mathcal{S}$ and the collections $(\alpha_s^\vee : s \in \mathcal{S})$ and $(\alpha_s : s \in \mathcal{S})$. These data are not indicated to lighten the notation.

The diagram obtained from $\mathcal{JW}_{(s,t,\cdots,t)}$ (if $m_{s,t}$ is even) or $\mathcal{JW}_{(s,t,\cdots,s)}$ (if $m_{s,t}$ is odd) will be denoted

(2.12)



or



.

This morphism is again called the Jones–Wenzl projector associated with the pair (s,t)

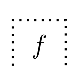
REMARK 2.18. As we will see below the Jones–Wenzl projectors, or rather their images (2.12), appear in the relations defining the Elias–Williamson category; it is therefore clear that we need to assume their existence for the definition to make sense. The necessity of rotatability is less immediate. It should be seen as some kind of compatibility of the cyclicity of the $2m_{s,t}$-valent vertex (relation (4) in §2.5) with the relation involving Jones–Wenzl projectors (relation (12) in §2.5) which prevents the category from collapsing. For a more formal discussion, see [**EW3**, §3.3].

There is an extra technical condition that has to be considered in case $\mathcal{W}$ admits a parabolic subgroup of type $\mathbf{H}_3$. In this case, the "Zamolodchikov" relation one needs to impose (see [**EW2**, (5.12)]) is not known explicitly. One therefore needs to assume that there exists a linear combination of this form that is sent to 0 by the operation described in [**EW3**, §2]. Such a linear combination is then fixed, and its vanishing is imposed in the definition of the category $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ (see (13) in §2.5 below). There does not seem to be any understanding of when this condition holds at this stage; we will therefore not discuss it any further. (This condition is empty for Cartan realizations of crystallographic Coxeter groups, since such groups do not have any parabolic subgroup of type H_3.)

2.5. Definition. We continue with the realization fixed in §2.4.

The category $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ is graded, in the sense that its morphism spaces are *graded* $\Bbbk$-modules. Its objects are parametrized by expressions; the object attached to $\underline{w}$ will be denoted by $\mathsf{B}_{\underline{w}}$. The morphisms are generated (under horizontal and vertical concatenation, and $\Bbbk$-linear combinations) by four kinds of morphisms depicted by diagrams (to be read from bottom to top):

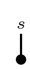
(1) for any homogeneous $f \in R$, a "box" morphism



from $\mathsf{B}_\varnothing$ to itself, of degree $\deg(f)$;

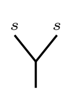
(2) for any $s \in \mathcal{S}$, "dot" morphisms

and

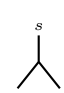


from B_s to $\mathsf{B}_\varnothing$ and from $\mathsf{B}_\varnothing$ to B_s, respectively, of degree 1;

(3) for any $s \in \mathcal{S}$, trivalent morphisms

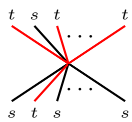


and

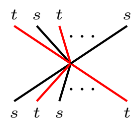


from B_s to $\mathsf{B}_{(s,s)}$ and from $\mathsf{B}_{(s,s)}$ to B_s, respectively, of degree -1;

(4) for any $(s,t) \in \mathcal{S}^2_\circ$, a morphism



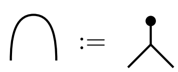
if m_{st} is odd or

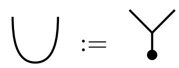


if m_{st} is even

from $\mathsf{B}_{(s,t,\cdots)}$ to $\mathsf{B}_{(t,s,\cdots)}$ (where each expression has length m_{st}, and colors alternate), of degree 0.

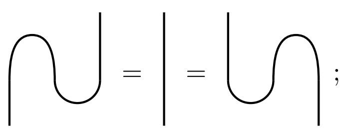
(Below we will sometimes omit the labels "s" or "t" when they do not play any role.) We set:

 ,

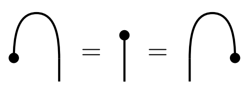
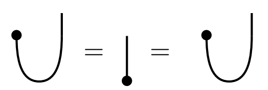
 .

These morphisms are subject to a number of relations that we now explain. First, there are the "isotopopy relations:"

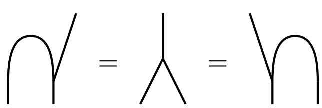
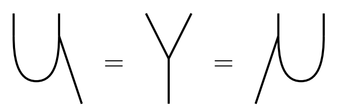
(1) biadjunction:

 ;

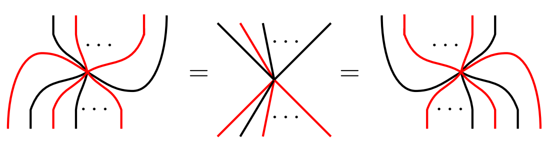
(2) rotation of univalent vertices :

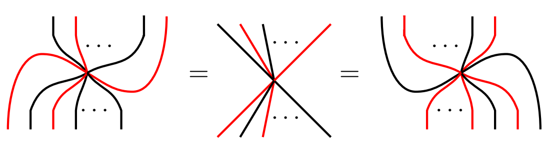
 and  ;

(3) rotation of trivalent vertices:

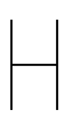
 and  ;

(4) cyclicity of the $2m_{st}$-valent vertex:

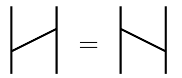
 if m_{st} is odd;

 if m_{st} is even.

Once these relations are known, as explained in [EMTW, Proposition 7.18], an isotopy class of diagrams unambiguously represents a morphism in our category. This also allows us to use some pictures that are not in the strict sense obtained by concatenating our diagrams above: for instance, we will write

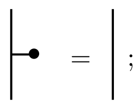


for

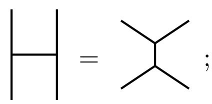


After this remark we can state the remaining relations:

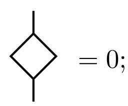
(5) the boxes add and multiply in the obvious way;
(6) Frobenius unit:



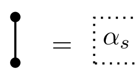
(7) Frobenius associativity:



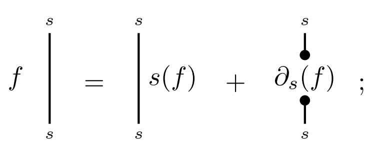
(8) needle relation:



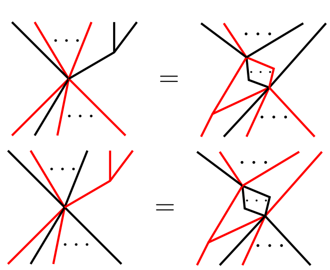
(9) barbell relation:



(where s is the color of the diagram on the left-hand side);
(10) nil-Hecke relation:



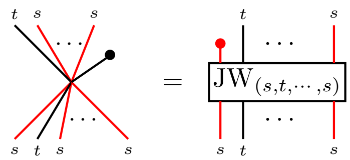
(11) 2-color associativity:



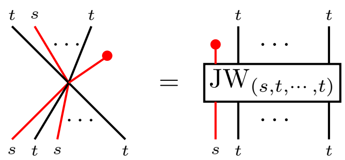
if m_{st} is odd;

if m_{st} is even;

(12) Jones–Wenzl relations (or two-color dot contraction):



if $m_{s,t}$ is odd;



if $m_{s,t}$ is even;

(13) Zamolodchikov relations: see [**EW2**, §5.1].

These relations can be gathered in four groups:

- the polynomial relation (5), which does not involve any simple reflection;
- the 1-color relations, which involve only 1 simple reflection in each case, namely (1)–(3); and (6)–(10);
- the 2-color relations, which involve pairs of simple reflections generating a finite subgroup of $\mathcal{W}$, namely (4) and (11)–(12);
- the 3-color relation, which involves triples of simple reflections generating a finite subgroup of $\mathcal{W}$, namely (13).

The composition of morphisms is induced by vertical concatenation. The monoidal product in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ is induced by the assignment $\mathsf{B}_{\underline{v}} \star \mathsf{B}_{\underline{w}} := \mathsf{B}_{\underline{vw}}$, and horizontal concatenation of diagrams.

Remark 2.19. (1) The letters "*BS*" in the notation $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ again refer to Bott–Samelson, because the objects $\mathsf{B}_{\underline{w}}$ play the role of equivariant cohomology of Bott–Samelson resolutions of Schubert varieties.

(2) As checked in [**EMTW**, Exercise 9.39], from the relations (11)–(12) above one deduces that the composition of the $2m_{s,t}$-valent morphism from $(s, t, \cdots)$ to $(t, s, \cdots)$ with the $2m_{s,t}$-valent morphism from $(t, s, \cdots)$ to $(s, t, \cdots)$ is $\mathrm{JW}_{(s,t,\cdots)}$.

(3) In some sources (e.g. [**EMTW**]) the needle relation (see (8) above) is presented in a different form; it is explained in Exercise 2.9 that this gives rise to the same category.

(4) As explained in Remarks 1.30 and 1.32, Soergel bimodules are often used as a bridge between two categories of representation-theoretic or geometric interest, by constructing functors $\mathbb{H}$ and $\mathbb{V}$ with values in Soergel (bi)modules. By design a category defined by generators and relations makes it easy to define a functor *from* it. In the case where one wants to use the Elias–Williamson category as a replacement for Soergel bimodules, one therefore usually constructs functors *from* this category to categories of representation-theoretic or geometric interest. For illustrations of this procedure, see [**AMRW**] and Conjecture 1.3 in Chapter 6.

When we consider $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ as a graded category as above, the graded $\Bbbk$-module of morphisms from $\mathsf{B}_{\underline{w}}$ to $\mathsf{B}_{\underline{v}}$ will be denoted

$$\mathrm{Hom}^{\bullet}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}_{\underline{w}}, \mathsf{B}_{\underline{v}}).$$

This $\Bbbk$-module has a canonical structure of graded R-bimodule, given by putting boxes to the left and to the right of a given morphism. But sometimes it will be more convenient to consider $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ as a usual category endowed with a "shift of grading" autoequivalence (1), whose n-th power will be denoted by (n). From this perspective the objects of $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ are the $\mathsf{B}_{\underline{w}}(n)$ where $\underline{w}$ is an expression and $n \in \mathbb{Z}$. The morphism space from $\mathsf{B}_{\underline{w}}(n)$ to $\mathsf{B}_{\underline{v}}(m)$, denoted

$$\mathrm{Hom}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}_{\underline{w}}(n), \mathsf{B}_{\underline{v}}(m)),$$

is the $\Bbbk$-submodule of $\mathrm{Hom}^{\bullet}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}_{\underline{w}}, \mathsf{B}_{\underline{v}})$ consisting of elements of degree $m-n$.

The category $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ admits a nice symmetry, which is explained in the following lemma. For applications, see Exercise 2.10.

LEMMA 2.20. *Assume that $(\mathcal{W}, \mathcal{S})$ has no parabolic subgroup of type H_3. There admits a canonical monoidal anti-autoequivalence*

$$\iota : \mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V) \xrightarrow{\sim} \mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$$

which acts on objects by the formula

$$\iota(\mathsf{B}_{\underline{w}}(n)) = \mathsf{B}_{\underline{w}}(-n)$$

for any expression $\underline{w}$ and any $n \in \mathbb{Z}$, and on morphisms by reflecting diagrams along a horizontal axis.

PROOF. Since our category is defined by generators and relations, and since we know the behaviour of our functor on objects, we consider the assignment sending each generating morphism to its reflection along an horizontal axis; what we need to check is that this assignment satisfies the relations defining $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$. This is easily seen for all relations except 2-color associativity and the Jones–Wenzl relations. For 2-color associativity, this follows from these relations together with cyclicity of the $2m_{s,t}$-valent vertex and of trivalent vertices. Finally, instead of checking explicitly the Jones–Wenzl relations, we remark that by [**EMTW**, Exercise 9.39] these relations are equivalent (modulo cyclicity and 2-color associativity) to the relation stating that the composition of two $2m_{s,t}$-valent vertices associated with s, t equals the corresponding Jones–Wenzl projector; see [**EMTW**, (9.27b)]. This relation is visibly invariant under horizontal reflection thanks to the corresponding property of Jones–Wenzl projectors (see Lemma 2.10), which allows to check this relation. □

REMARK 2.21. Similar considerations allow to construct an analogue of the autoequivalence φ of Remark 1.12, i.e. an autoequivalence of $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ that sends $\mathsf{B}_{\underline{w}}$ to the object associated with the word obtained from $\underline{w}$ by reversing the order of the letters, and acts on morphisms by reflection along a vertical axis. This autoequivalence respects degrees of morphisms, and reverses the order of the factors in a monoidal product.

2.6. Additive and Karoubian versions. The category $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ is "only" a preadditive (in fact, $\Bbbk$-linear) category. We will denote by $\mathsf{D}^{\oplus}_{\mathrm{BS}}(\mathcal{W}, V)$ the additive hull of $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ (considered as an ordinary, non graded, category). The objects of this category are the formal direct sums

$$\bigoplus_{i=1}^{r} \mathsf{B}_{\underline{w}_i}(n_i)$$

where each $\underline{w}_i$ is an expression and each n_i is an integer. The morphisms are defined in the obvious way, as matrices of morphisms in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$. This category admits an obvious (additive) monoidal product extending the product $\star$, and denoted by the same symbol.

In case $\Bbbk$ is a field or a complete local ring,[9] we will denote by $\mathsf{D}(\mathcal{W}, V)$ the Karoubian envelope of $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$. Once again this category admits a natural monoidal product extending $\star$, and which will also be denoted $\star$. It follows from Theorem 2.30 below that morphisms spaces in $\mathsf{D}(\mathcal{W}, V)$ are finitely generated over $\Bbbk$. Since a $\Bbbk$-algebra which is finitely generated as a $\Bbbk$-module is semi-perfect (see [**La**, Example 23.3]), it then follows from [**CYZ**, Theorem A.1] that $\mathsf{D}(\mathcal{W}, V)$ is a Krull–Schmidt category.

2.7. The quadratic relations. Below we will explain that (under suitable assumptions) the split Grothendieck group of the category $\mathsf{D}^{\oplus}_{\mathrm{BS}}(\mathcal{W}, V)$ identifies with the Hecke algebra $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$. The following lemma expresses in categorical terms that the quadratic relations in $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ are satisfied in $\mathsf{D}^{\oplus}_{\mathrm{BS}}(V, W)$ (without any further assumption).

LEMMA 2.22. *For any $s \in S$ there exists an isomorphism*

$$\mathsf{B}_s \star \mathsf{B}_s \cong \mathsf{B}_s(1) \oplus \mathsf{B}_s(-1)$$

in $\mathsf{D}^{\oplus}_{\mathrm{BS}}(V, W)$.

PROOF. To prove the lemma we need to construct morphisms

$$f_1 : \mathsf{B}_{(s,s)} \to \mathsf{B}_s(1), \quad f_2 : \mathsf{B}_{(s,s)} \to \mathsf{B}_s(-1),$$
$$f_3 : \mathsf{B}_s(1) \to \mathsf{B}_{(s,s)}, \quad f_4 : \mathsf{B}_s(-1) \to \mathsf{B}_{(s,s)}$$

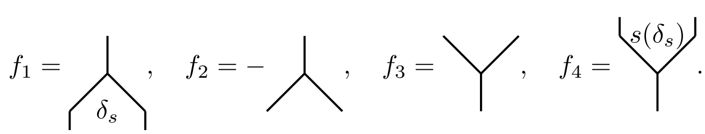
which satisfy

$$f_1 \circ f_3 = \mathrm{id}, \quad f_1 \circ f_4 = 0, \quad f_2 \circ f_3 = 0, \quad f_2 \circ f_4 = \mathrm{id}$$

and

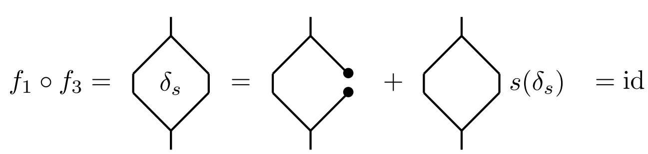
$$f_4 \circ f_2 + f_3 \circ f_1 = \mathrm{id}\,.$$

These morphisms are defined as follows (where all lines are labelled s, and δ_s is as in Lemma 2.17):



We have

(2.13)



where the second equality uses the nil-Hecke relation (and the fact that $\partial_s(\delta_s) = 1$) and the third one the Frobenius unit relation and the needle relation. A very similar computation shows that $f_2 \circ f_4 = \mathrm{id}$ and that $f_1 \circ f_4 = 0$. (In the former case we

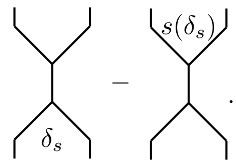
---

[9]This restriction is not necessary for the definition to make sense, but we will only consider it in this generality.

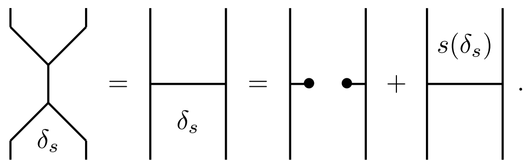
use that $\partial_s(s(\delta_s)) = -1$; in the latter case we use that $\partial_s(\delta_s s(\delta_s)) = 0$.) The fact that $f_2 \circ f_3 = 0$ follows directly from the needle relation.

Finally, we have

$$f_4 \circ f_2 + f_3 \circ f_1 =$$



Using Frobenius associativity and then the nil-Hecke relation we see that



It follows that $f_4 \circ f_2 + f_3 \circ f_1 = \mathrm{id}$, in view of the Frobenius associativity relation and the Frobenius unit relation. $\square$

2.8. The categorification theorem and indecomposable objects. Elias and Williamson prove in [**EW2**] that, under appropriate assumptions, the categories $\mathsf{D}_{\mathrm{BS}}^{\oplus}(\mathcal{W}, V)$ and $\mathsf{D}(\mathcal{W}, V)$ have properties very similar to those of the category of Soergel bimodules, see §1.4.

First, they explain in [**EW2**, §6.5] that there exists a morphism

$$\mathrm{ch}_{\mathsf{D}} : [\mathsf{D}_{\mathrm{BS}}^{\oplus}(\mathcal{W}, V)]_{\oplus} \to \mathcal{H}_{(\mathcal{W},\mathcal{S})}$$

which, for any expression $\underline{w}$ and any $n \in \mathbb{Z}$, satisfies

$$\mathrm{ch}_{\mathsf{D}}([\mathsf{B}_{\underline{w}}(n)]) = v^n \cdot \underline{H}_{\underline{w}}$$

(where $\underline{H}_{\underline{w}}$ is defined in (1.11)). (This construction relies on the construction of the light leaves basis presented in §2.10 below.) Since the classes $[\mathsf{B}_{\underline{w}}(n)]$ generate the $\mathbb{Z}$-module $[\mathsf{D}_{\mathrm{BS}}^{\oplus}(\mathcal{W}, V)]_{\oplus}$, the morphism η is therefore an algebra morphism.

For the next results, we assume that $\Bbbk$ is a complete local domain. In [**EW2**, Theorem 6.26], Elias and Williamson prove the following analogue of Theorem 1.16.

THEOREM 2.23. *For any $w \in \mathcal{W}$ there exists a unique indecomposable object $\mathsf{B}_w \in \mathsf{D}(\mathcal{W}, V)$ which satisfies the property that for any reduced expression $\underline{w}$ for w, B_w is the unique indecomposable summand of $\mathsf{B}_{\underline{w}}$ which is not a direct summand of an object $\mathsf{B}_{\underline{y}}(n)$ with $\underline{y}$ a reduced expression for an element $y < w$ and $n \in \mathbb{Z}$. Moreover, the assignment*

$$(w, n) \mapsto \mathsf{B}_w(n)$$

induces a bijection between $\mathcal{W} \times \mathbb{Z}$ and the set of isomorphism classes of indecomposable objects in $\mathsf{D}(\mathcal{W}, V)$.

Using the characterization of B_w in Theorem 2.23, one easily checks by induction on the length of $\underline{w}$ that for any reduced expression $\underline{w}$ for an element $w \in \mathcal{W}$ there exist nonnegative integers $b_{y,n}^{\underline{w}}$ such that

$$\mathsf{B}_{\underline{w}} \cong \mathsf{B}_w \oplus \bigoplus_{\substack{y \in \mathcal{W}, y < w \\ n \in \mathbb{Z}}} (\mathsf{B}_y(n))^{b_{y,n}^{\underline{w}}}. \tag{2.14}$$

Using the same considerations as for Corollary 1.18, one deduces the following result. (Here again, the positivity statement follows from the explicit description of ch_{D}, which was not explained in detail above.)

COROLLARY 2.24. *The morphism* ch_{D} *is an isomorphism. Moreover, for any* $w \in \mathcal{W}$ *we have*

$$\mathrm{ch}_{\mathsf{D}}([\mathsf{B}_w^{\mathrm{bim}}]) \in H_w + \sum_{y<w} \mathbb{Z}_{\geq 0}[v, v^{-1}] \cdot H_y.$$

REMARK 2.25. (1) It follows from the first sentence in Corollary 2.24 that, when $\Bbbk$ is a complete local domain, the assignment

$$\underline{H}_s \mapsto [\mathsf{B}_s]$$

extends to an algebra morphism $\mathcal{H}_{(\mathcal{W},\mathcal{S})} \to [\mathsf{D}(\mathcal{W}, V)]_{\oplus}$, which provides an analogue of Theorem 1.14. The proof of this fact is however quite different from that of the latter theorem; in particular the fact that the elements $([\mathsf{B}_s] - v : s \in \mathcal{S})$ satisfy the braid relations is not checked explicitly. For an interpretation of this relation in the category $\mathsf{D}(\mathcal{W}, V)$, see [**EW2**, Remark 6.29].

(2) In fact one can prove that there exists an algebra isomorphism

$$\mathcal{H}_{(\mathcal{W},\mathcal{S})} \cong [\mathsf{D}_{\mathrm{BS}}^{\oplus}(\mathcal{W}, V)]_{\oplus}$$

without any assumption on $\Bbbk$, see [**ARV**, Theorem 6.13].[10] The proof in this setting does not use the classification of indecomposable objects (because no classification is known); instead it is based on the construction of analogues of the Rouquier complexes (see Remark 1.15(2)).

For $w \in \mathcal{W}$ we set

$$\underline{H}_w(V) := \mathrm{ch}_{\mathsf{D}}([\mathsf{B}_w^{\mathrm{bim}}]).$$

Corollary 2.24 implies that the family

$$\big(\underline{H}_w(V) : w \in \mathcal{W}\big) \tag{2.15}$$

is a basis of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$. This basis "encodes" the combinatorics of the category $\mathsf{D}(\mathcal{W}, V)$, in the sense that computing it is equivalent (in theory) to computing the integers $b_{y,n}^{w}$ appearing in (2.14).

The same proof as for Lemma 1.21 (using Exercise 2.10 as a replacement for the arguments involving the duality D) gives the following result.

LEMMA 2.26. *Let* $w \in \mathcal{W}$ *and* $s \in \mathcal{S}$.

(1) *If* $sw > w$*, then there exist nonnegative integers* $d_{w,s}^{y,n}$ *for* $y \in \mathcal{W}$ *such that* $y < sw$ *and* $n \in \mathbb{Z}$ *such that*

$$\mathsf{B}_s \star \mathsf{B}_w \cong \mathsf{B}_{sw} \oplus \bigoplus_{\substack{y \in \mathcal{W}, y<sw \\ n \in \mathbb{Z}}} (\mathsf{B}_y)^{\oplus d_{w,s}^{y,n}}.$$

Moreover, for any y *and* n *we have* $d_{w,s}^{y,n} = d_{w,s}^{y,-n}$.

(2) *If* $sw < w$ *we have*

$$\mathsf{B}_s \star \mathsf{B}_w \cong \mathsf{B}_w(1) \oplus \mathsf{B}_w(-1).$$

[10] In [**ARV**] the technical conditions of §2.4 are not mentioned. They should be imposed however, since the proof involves the standard properties of the category $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ (in particular, the double leaves basis of §2.10 below.).

In particular, as a consequence of Lemma 2.26(2), for any $w \in \mathcal{W}$ and $s \in \mathcal{S}$ such that $sw < w$ we have

$$\underline{H}_s \cdot \underline{H}_w(V) = (v + v^{-1}) \cdot \underline{H}_w(V). \tag{2.16}$$

REMARK 2.27. (1) The comments in Remark 1.22, as well as Proposition 1.23, also apply in this context, with identical proofs.

(2) As explained in Remark 1.19, usual Soergel bimodules have "singular" variants. In the setting of the Elias–Williamson diagrammatic category, such a theory is not available in full generality as of now. In the case of dihedral groups, it was developed (under appropriate assumptions) in [**El**]. A solution to this problem has been announced by Elias–Williamson in (finite and affine) type $\mathbf{A}$, but no detailed treatment appears in the literature at present. For some details, and an important application, see [**ELo**].

2.9. Rex moves. We now come back to the general setting of §2.5 (i.e. we omit the condition that $\Bbbk$ is a complete local domain.)

To any $w \in \mathcal{W}$ we associate its "rex graph" Γ_w constructed as follows. The vertices of this graph are the reduced expressions for w, and an edge connects two vertices if they differ by the application of a braid relation, i.e. by the replacement of a subexpression $(s, t, \cdots)$ by $(t, s, \cdots)$, where $(s, t) \in \mathcal{S}^2_\circ$, and each sequence alternates the letters s and t and has length $m_{s,t}$. In these terms, Matsumoto's lemma in the theory of Coxeter groups states that the graph Γ_w is connected, for any $w \in \mathcal{W}$.

If $\underline{w}$ and $\underline{w}'$ are two vertices in Γ_w connected by an edge (associated with a pair (s, t) of simple reflections as above), then we have canonical morphisms

$$\mathsf{B}_{\underline{w}} \to \mathsf{B}_{\underline{w}'} \quad \text{and} \quad \mathsf{B}_{\underline{w}'} \to \mathsf{B}_{\underline{w}} \tag{2.17}$$

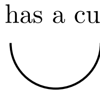
in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ obtained by adding vertical lines to the morphisms $\mathsf{B}_{(s,t,\cdots)} \to \mathsf{B}_{(t,s,\cdots)}$ and $\mathsf{B}_{(t,s,\cdots)} \to \mathsf{B}_{(s,t,\cdots)}$ appearing in the generators of $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$.

LEMMA 2.28. *Let $w \in \mathcal{W}$, and let $\underline{w}$ and $\underline{w}'$ be two vertices in Γ_w connected by an edge. Then there exist words $\underline{x}_1, \cdots, \underline{x}_r$ of length at most $\ell(w)-2$ and morphisms $f_1, \cdots, f_r : \mathsf{B}_{\underline{w}} \to \mathsf{B}_{\underline{w}}$ where each f_i factors through a shift of $\mathsf{B}_{\underline{x}_i}$ such that the composition*

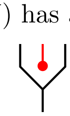
$$\mathsf{B}_{\underline{w}} \to \mathsf{B}_{\underline{w}'} \to \mathsf{B}_{\underline{w}}$$

(where both morphisms are as in (2.17)) equals $\mathrm{id} + \sum_{i=1}^r f_i$.

PROOF. Let s, t be the simple reflections associated with the edge under consideration. By Remark 2.19(2), the morphism we consider is obtained from $\mathrm{JW}_{(s,t,\cdots)}$ by adding appropriate vertical lines on both sides. Hence it is sufficient to prove the similar claim for the morphism $\mathrm{JW}_{(s,t,\cdots)}$. Now, by construction, this morphism is obtained from $\mathcal{JW}_{(s,t,\cdots)}$ by the deformation-retract process explained in §2.4. The morphism $\mathcal{JW}_{(s,t,\cdots)}$ is a linear combination of the identity morphism (giving rise to the identity morphism in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$) and nontrivial two-colored crossingless matchings. Each of these matchings has a cup



on top. Hence its image in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ has a diagram of the form



on top. This image therefore factors through an object associated with a word of length at most $m_{s,t}-2$, which implies our claim. □

We continue with our element $w \in \mathcal{W}$ and the rex graph Γ_w. We will call "rex move" a directed path in Γ_w. To each (directed) edge in this path we have associated above a morphism in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$. By composing these morphisms we therefore obtain a morphism

$$\mathsf{B}_{\underline{w}} \to \mathsf{B}_{\underline{w}'}$$

in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$, where $\underline{w}$, resp. $\underline{w}'$, is the starting point, resp. the end point, of our path. Given a rex move from $\underline{w}$ to $\underline{w}'$, we can also consider the "reversed" rex move, a path from $\underline{w}'$ to $\underline{w}$. The following statement is an immediate consequence of Lemma 2.28, which will be used in Chapter 6.

PROPOSITION 2.29. *Let $w \in \mathcal{W}$, and consider a rex move from a vertex $\underline{w}$ to a vertex $\underline{w}'$. Then there exist words $\underline{x}_1, \cdots, \underline{x}_r$ of length at most $\ell(w)-2$ and morphisms $f_1, \cdots, f_r : \mathsf{B}_{\underline{w}} \to \mathsf{B}_{\underline{w}}$ where each f_i factors through a shift of $\mathsf{B}_{\underline{x}_i}$ such that the composition*

$$\mathsf{B}_{\underline{w}} \to \mathsf{B}_{\underline{w}'} \to \mathsf{B}_{\underline{w}}$$

(where the first morphism is the morphism associated with our given rex move, and the second one is the morphism associated with the reversed rex move) equals $\mathrm{id} + \sum_{i=1}^r f_i$.

2.10. Light leaves and double leaves. One of the main technical tools used in [**EW2**] is the construction of bases of morphism spaces in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ inspired by a construction in the setting of "usual" Soergel bimodules due to Libedinsky [**Li1, Li3**], that we review here.

Given an expression $\underline{w} = (s_1, \cdots, s_r)$, we call *subexpression* of $\underline{w}$ a sequence $\underline{e} = (e_1, \cdots, e_r)$ where $e_i \in \{0, 1\}$ for any i. We will say that $\underline{e}$ expresses the element $(s_1)^{e_1} \cdots (s_r)^{e_r} \in \mathcal{W}$. To such a subexpression we assign its *Bruhat stroll*, the sequence $x_0 = e, x_1, \cdots, x_r$ with

$$x_i = (s_1)^{e_1} \cdots (s_i)^{e_i}$$

for any i, and a sequence $(X_1, \cdots, X_r)$ of labels in $\{U0, U1, D0, D1\}$ with

$$X_i = \begin{cases} U1 & \text{if } e_i = 1 \text{ and } x_{i-1}s_i > x_{i-1}; \\ U0 & \text{if } e_i = 0 \text{ and } x_{i-1}s_i > x_{i-1}; \\ D1 & \text{if } e_i = 1 \text{ and } x_{i-1}s_i < x_{i-1}; \\ D0 & \text{if } e_i = 0 \text{ and } x_{i-1}s_i < x_{i-1}. \end{cases}$$

(Here "D" stands for "down", and "U" for "up".) We define the *defect* $d(\underline{e})$ of $\underline{e}$ by

$$d(\underline{e}) = \#\{i \in \{1, \cdots, r\} \mid X_i = U0\} - \#\{i \in \{1, \cdots, r\} \mid X_i = D0\}.$$

To each expression $\underline{w}$ and each subexpression $\underline{e}$, with associated Bruhat stroll $(x_0, \cdots, x_r)$ and sequence of labels $(X_1, \cdots, X_r)$ we will assign a "light leaf" morphism

$$\mathrm{LL}_{\underline{w},\underline{e}} : \mathsf{B}_{\underline{w}} \to \mathsf{B}_{\underline{x}}(d(\underline{e}))$$

for some reduced expression $\underline{x}$ for x_r. (This construction will depend on some choices; in particular we do not specify the choice of $\underline{x}$.) The construction proceeds by induction on the length on $\underline{w}$. If $\underline{w} = \varnothing$ is the empty expression, then there is only one choice for $\underline{e}$, namely $\underline{e} = \varnothing$, and the corresponding morphism $\mathrm{LL}_{\varnothing,\varnothing}$ is the identity morphism of $\mathsf{B}_\varnothing$. Now consider a nonempty expression $\underline{w} = (s_1, \cdots, s_r)$ and a subexpression $\underline{e}$. Denote by $\underline{w}_{<r}$ the expression $(s_1, \cdots, s_{r-1})$ and by $\underline{e}_{<r} = (e_1, \cdots, e_{r-1})$ the subexpression of $\underline{w}_{<r}$ induced by $\underline{e}$, and assume that the morphism

$$\mathrm{LL}_{\underline{w}_{<r},\underline{e}_{<r}} : \mathsf{B}_{\underline{w}_{<r}} \to \mathsf{B}_{\underline{x}'}(d(\underline{e}_{<r}))$$

has been defined. (Here $\underline{x}'$ is a certain reduced expression for x_{r-1}.) Then we will set

$$\mathrm{LL}_{\underline{w},\underline{e}} = \phi_r \circ (\mathrm{LL}_{\underline{w}_{<r},\underline{e}_{<r}} \star \mathrm{id}_{\mathsf{B}_{s_r}})$$

for a certain morphism

$$\phi_r : \mathsf{B}_{\underline{x}'} \star \mathsf{B}_{s_r}(d(\underline{e}_{<r})) \to \mathsf{B}_{\underline{x}}(d(\underline{e}))$$

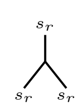
where $\underline{x}$ is a reduced expression of x_r. This morphism is determined by the following rules. (Here, to lighten notation, ϕ_r is described as an element in the graded $\Bbbk$-module $\mathrm{Hom}^\bullet_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}_{\underline{x}'} \star \mathsf{B}_{s_r}, \mathsf{B}_{\underline{x}})$.)

- If $X_r = U1$, then $(\underline{x}', s_r)$ is a reduced expression for x_r. In this case, we choose a reduced expression $\underline{x}$ for x_r and a rex move from $(\underline{x}', s_r)$ to $\underline{x}$ and define ϕ_r to be the associated morphism. (Here we can choose $\underline{x} = (\underline{x}', s_r)$ and the rex move staying at this reduced expression, but we do not impose this.)
- If $X_r = U0$, then $\underline{x}'$ is a reduced expression for x_r. We choose a reduced expression $\underline{x}$ for x_r and a rex move from $\underline{x}'$ to $\underline{x}$, denote by f the associated morphism, and set

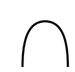
$$\phi_r = f \star$$

.

- If $X_r = D1$, then we choose a reduced expression $\underline{y}$ for x_{r-1} which has s_r in position $r-1$, and a rex move from $\underline{x}'$ to $\underline{y}$; we denote by $f : \mathsf{B}_{\underline{x}'} \to \mathsf{B}_{\underline{y}}$ the associated morphism. We denote by $\underline{z}$ the reduced expression for x_r obtained by deleting the rightmost s_r in $\underline{y}$, and choose a reduced expression $\underline{x}$ for x_r and a rex move from $\underline{z}$ to $\underline{x}$; we denote by $g : \mathsf{B}_{\underline{z}} \to \mathsf{B}_{\underline{x}}$ the associated morphism. Then we set

$$\phi_r = g \circ \Big(\mathrm{id}_{\mathsf{B}_{\underline{z}}} \star$$

$$\Big) \circ (f \star \mathrm{id}_{\mathsf{B}_{s_r}}).$$

- If $X_r = D0$, then we choose a reduced expression $\underline{y}$ for x_{r-1} which has s_r in position $r-1$, and a rex move from $\underline{x}'$ to $\underline{y}$; we denote by $f : \mathsf{B}_{\underline{x}'} \to \mathsf{B}_{\underline{y}}$ the associated morphism. Next we choose a reduced expression $\underline{x}$ for x_r and a rex move from $\underline{z}$ to $\underline{x}$; we denote by $g : \mathsf{B}_{\underline{y}} \to \mathsf{B}_{\underline{x}}$ the associated morphism. Then we set

$$\phi_r = g \circ \Big(\mathrm{id}_{\mathsf{B}_{\underline{z}}} \star$$

$$\Big) \circ (f \star \mathrm{id}_{\mathsf{B}_{s_r}}).$$

Now that light leaves morphisms have been defined, we can define the *double leaves* morphisms. These are associated to a pair of expressions $(\underline{x}, \underline{y})$ and a pair of subexpressions $(\underline{e}, \underline{f})$ of $\underline{x}$ and $\underline{y}$ respectively which express the same element $w \in \mathcal{W}$. Thanks to the construction above we have morphisms

$$\mathrm{LL}_{\underline{x},\underline{e}} : \mathsf{B}_{\underline{x}} \to \mathsf{B}_{\underline{w}}(d(\underline{e})), \quad \mathrm{LL}_{\underline{y},\underline{f}} : \mathsf{B}_{\underline{y}} \to \mathsf{B}_{\underline{w}'}(d(\underline{f}))$$

where $\underline{w}$ and $\underline{w}'$ are reduced expressions for w. We choose a rex move from $\underline{w}$ to $\underline{w}'$, denote by $f : \mathsf{B}_{\underline{w}} \to \mathsf{B}_{\underline{w}'}$ the associated morphism, and set

$$\mathbb{LL}_{\underline{x},\underline{e}}^{\underline{y},\underline{f}} := \big(\iota(\mathrm{LL}_{\underline{f},\underline{f}})(d(\underline{e}))\big) \circ \big(f(d(\underline{e}))\big) \circ \mathrm{LL}_{\underline{x},\underline{e}} : \mathsf{B}_{\underline{x}} \to \mathsf{B}_{\underline{y}}(d(\underline{e}) + d(\underline{e}')).$$

(Here, ι is the functor of Lemma 2.20.)

The following statement is proved in [EW2, Theorem 6.12], and is the main step for the proof of Theorem 2.23.

THEOREM 2.30. *Let $\underline{x}, \underline{y}$ be expression, and choose for any subexpressions $\underline{e}, \underline{f}$ of $\underline{x}$ and $\underline{y}$ respectively expressing the same element of $\mathcal{W}$ a double leaf morphism $\mathbb{LL}_{\underline{x},\underline{e}}^{\underline{y},\underline{f}}$. Then the family of such morphisms is a (homogeneous) basis of $\mathrm{Hom}^\bullet(\mathsf{B}_{\underline{x}}, \mathsf{B}_{\underline{y}})$ both as a left R-module and as a right R-module. In particular, this space is graded free as a left R-module and as a right R-module.*

2.11. Some applications. As explained above the main application of Theorem 2.30 is to the proof of [EW2, Theorem 6.12]. But this theorem has other very interesting implications, that we explore here. A general idea one can keep in mind is that "the category $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ does not really depend on the choice of realization." This should not be taken in the strict sense, but in this subsection we explain a few statements that go in this direction.

2.11.1. *Extension of scalars, I.* Recall from Remark 2.3 that given a realization $(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in S))$ of a Coxeter system $(\mathcal{W}, \mathcal{S})$ over a commutative domain $\Bbbk$ and a ring morphism $\Bbbk \to \Bbbk'$ (where again $\Bbbk'$ is a commutative domain) we obtain naturally a realization of $(\mathcal{W}, \mathcal{S})$ over $\Bbbk'$ with underlying $\Bbbk'$-module $\Bbbk' \otimes_\Bbbk V$. We will assume that the technical conditions considered in §2.4 are satisfied by V. Then these conditions are also satisfied for our new realization over $\Bbbk'$, so that we can also consider the category $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, \Bbbk' \otimes_\Bbbk V)$. To distinguish the two cases, we will add subscripts $\Bbbk$ or $\Bbbk'$ to all the notations considered above.

REMARK 2.31. We have to be a bit careful in case $(\mathcal{W}, \mathcal{S})$ admits a parabolic subgroup of type $\mathbf{H}_3$. Namely, in this case we have explained in §2.4 that we need to fix a corresponding "Zamolodchikov relation" in the definition of $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$. The image in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, \Bbbk' \otimes_\Bbbk V)$ of this relation will be taken as the corresponding Zamolodchikov relation in this category. Note that we have a natural morphism from the algebra $R_\Bbbk$ involved in the definition of $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ to the algebra $R_{\Bbbk'}$ involved in the definition of $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, \Bbbk' \otimes_\Bbbk V)$, which induces a morphism between localizations at $\mathcal{W}$-conjugates of the simple roots. Since the coefficients in [EW3, §2] only involve elements in these localizations, the image considered above is indeed suitable to be taken as a Zamolodchikov relation.

It is clear from definitions that there exists a canonical monoidal functor

$$\Bbbk' : \mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V) \to \mathsf{D}_{\mathrm{BS}}(\mathcal{W}, \Bbbk' \otimes_\Bbbk V)$$

which is defined on objects by

$$\Bbbk'(\mathsf{B}_{\underline{w}}^{\Bbbk}(n)) = \mathsf{B}_{\underline{w}}^{\Bbbk'}(n)$$

for any expression $\underline{w}$ and any $n \in \mathbb{Z}$. From Theorem 2.30 we deduce that for any expressions $\underline{w}, \underline{w}'$ and any $n, n' \in \mathbb{Z}$ this functor induces an isomorphism

$$\Bbbk' \otimes_{\Bbbk} \mathrm{Hom}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}^{\Bbbk}_{\underline{w}}(n), \mathsf{B}^{\Bbbk}_{\underline{w}'}(n')) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},\Bbbk' \otimes_{\Bbbk} V)}(\mathsf{B}^{\Bbbk'}_{\underline{w}}(n), \mathsf{B}^{\Bbbk'}_{\underline{w}'}(n')).$$

In case $\Bbbk$ and $\Bbbk'$ are complete local domains, this functor induces a functor

$$\mathsf{D}(\mathcal{W}, V) \to \mathsf{D}(\mathcal{W}, \Bbbk' \otimes_{\Bbbk} V)$$

which will again be denoted $\Bbbk'$, and which has the same effect on morphism spaces as above. The induced algebra morphism

$$[\mathsf{D}(\mathcal{W}, V)]_{\oplus} \to [\mathsf{D}(\mathcal{W}, \Bbbk' \otimes_{\Bbbk} V)]_{\oplus}$$

is an isomorphism; in fact, under the isomorphisms ch_{D} (used on both sides) it identifies with the identity morphism of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$. What we will consider more closely below is the effect of this functor on indecomposable objects.

We start with an easy case.

Lemma 2.32. *Assume that $\Bbbk$ and $\Bbbk'$ are complete local domains, and that the morphism $\Bbbk \to \Bbbk'$ is surjective. Then for any $w \in \mathcal{W}$ there exists an isomorphism*

$$\Bbbk'(\mathsf{B}^{\Bbbk}_w) \cong \mathsf{B}^{\Bbbk'}_w.$$

Proof. From the characterizations of the objects $\mathsf{B}^{\Bbbk}_w$ and $\mathsf{B}^{\Bbbk'}_w$ we see that it is enough to prove that $\Bbbk'(\mathsf{B}^{\Bbbk}_w)$ is indecomposable. Now, as explained above we have a canonical isomorphism

$$\Bbbk' \otimes_{\Bbbk} \mathrm{End}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}^{\Bbbk}_w) \xrightarrow{\sim} \mathrm{End}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},\Bbbk' \otimes_{\Bbbk} V)}(\Bbbk'(\mathsf{B}^{\Bbbk}_w)).$$

We deduce that this ring is a quotient of the local ring $\mathrm{End}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}^{\Bbbk}_w)$, hence is local, which finishes the proof. □

Lemma 2.32 implies that in this setting, for any $w \in \mathcal{W}$ we have

$$\underline{H}_w(V) = \underline{H}_w(\Bbbk' \otimes_{\Bbbk} V). \tag{2.18}$$

Remark 2.33. Lemma 2.32 applies in particular in the case when $\Bbbk = \mathbb{O}$ is a complete local domain and $\Bbbk' = \mathbb{F}$ is its residue field. In this case there is another natural morphism one can consider, namely the embedding $\mathbb{O} \to \mathbb{K}$ where $\mathbb{K}$ is the fraction field of $\mathbb{O}$. For this morphism it is *not* true that $\mathbb{K}(\mathsf{B}^{\mathbb{O}}_w)$ is indecomposable in general. What follows from the characterization of indecomposable objects in $\mathsf{D}(\mathcal{W}, V)$ and $\mathsf{D}(\mathcal{W}, \mathbb{K} \otimes_{\mathbb{O}} V)$ is that there exist nonnegative integers $(a_{y,w,n} : y < w \in \mathcal{W}, n \in \mathbb{Z})$ such that

$$\mathbb{K}(\mathsf{B}^{\mathbb{O}}_w) \cong \mathsf{B}^{\mathbb{K}}_w \oplus \bigoplus_{\substack{y<w \\ n \in \mathbb{Z}}} (\mathsf{B}^{\mathbb{K}}_y(n))^{\oplus a_{y,w,n}}.$$

It is also not difficult to check that $a_{y,w,n} = a_{y,w,-n}$ for any $n \in \mathbb{Z}$; see Exercise 2.10. If the basis

$$(\underline{H}_w(\mathbb{K} \otimes_{\mathbb{O}} V) : w \in \mathcal{W})$$

is known, the problem of computing the basis

$$(\underline{H}_w(V) : w \in \mathcal{W})$$

or, equivalently (see (2.18)), of the basis

$$(\underline{H}_w(\mathbb{F} \otimes_{\mathbb{O}} V) : w \in \mathcal{W}),$$

is equivalent to the problem of computing the integers $a_{y,w,n}$.

2.11.2. *Extension of scalars, II.* Now we assume that $\Bbbk$ and $\Bbbk'$ are fields. We fix $w \in \mathcal{W}$, and denote by

$$\mathrm{End}^+_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}^{\Bbbk}_w) \subset \mathrm{End}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}^{\Bbbk}_w)$$

the ideal consisting of morphisms which factor through a sum of objects of the form $\mathsf{B}_y(n)$ with $y < w$.

LEMMA 2.34. *Assume that $\Bbbk$ is a field. Then for any $w \in \mathcal{W}$ we have*

$$\mathrm{End}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}^{\Bbbk}_w) = \mathrm{End}^+_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}^{\Bbbk}_w) \oplus \Bbbk \cdot \mathrm{id}\,.$$

PROOF. First we remark that

$$\mathrm{End}^+_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}^{\Bbbk}_w) \cap (\Bbbk \cdot \mathrm{id}) = \{0\}.$$

In fact, this property is equivalent to saying that id does not belong to the ideal $\mathrm{End}^+_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}^{\Bbbk}_w)$, which follows from the fact that B_w is not a direct summand of a sum of objects $\mathsf{B}_y(n)$ with $y < w$ (by the Krull–Schmidt property). To conclude it therefore suffices to show that

$$\mathrm{End}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}^{\Bbbk}_w) = \mathrm{End}^+_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}^{\Bbbk}_w) + \Bbbk \cdot \mathrm{id}\,.$$

For this, choose a reduced expression $\underline{w}$ for w and morphisms

$$\mathsf{B}^{\Bbbk}_w \xrightarrow{i} \mathsf{B}^{\Bbbk}_{\underline{w}} \xrightarrow{p} \mathsf{B}^{\Bbbk}_w$$

such that $p \circ i = \mathrm{id}$. If $f \in \mathrm{End}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}^{\Bbbk}_w)$, then the morphism $i \circ f \circ p$ can be written in the double leaves basis of Theorem 2.30. Since $\underline{w}$ is a reduced expression, and for degree reasons, we deduce that there exist $\lambda \in \Bbbk$ and a morphism g which factors through a sum of objects $\mathsf{B}_y(n)$ with $y < w$ such that

$$i \circ f \circ p = \lambda \cdot \mathrm{id} + g.$$

Then we have

$$f = (p \circ i) \circ f \circ (p \circ i) = \lambda \cdot \mathrm{id} + p \circ g \circ i,$$

which proves the claim and finishes the proof of the lemma. □

Once this lemma is established, using the same considerations as for Lemma 1.20 we deduce the following property.

PROPOSITION 2.35. *Assume that $\Bbbk$ and $\Bbbk'$ are fields. Then for any $w \in \mathcal{W}$ we have*

$$\Bbbk'(\mathsf{B}^{\Bbbk}_w) \cong \mathsf{B}^{\Bbbk'}_w.$$

Proposition 2.35 shows that, in this setting, for any $w \in \mathcal{W}$ we have

$$\underline{H}_w(V) = \underline{H}_w(\Bbbk' \otimes_{\Bbbk} V). \tag{2.19}$$

2.11.3. *Diagrammatic Soergel modules.* We explained in §1.9 that, in the setting of reflection faithful representations, the category of Soergel bimodules has a variant where the left (or right) action of R is “killed,” giving rise to the theory of Soergel modules. Such a procedure has no obvious analogue in the setting of the present section, but we can copy Proposition 1.26 to define a category which plays the same role as Soergel modules. Namely, consider a balanced realization $(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S}))$ which satisfies the technical assumptions of §2.4. Then we define the category $\overline{\mathsf{D}}_{\mathrm{BS}}(\mathcal{W}, V)$ with

- objects the symbols $\overline{\mathsf{B}}_{\underline{w}}(n)$ where $\underline{w}$ is an expression and $n \in \mathbb{Z}$;

- morphisms from $\overline{\mathsf{B}}_{\underline{w}}(n)$ to $\overline{\mathsf{B}}_{\underline{w}'}(n')$ the elements of degree $n'-n$ in the graded $\Bbbk$-module
$$\Bbbk \otimes_R \mathrm{Hom}^\bullet_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}_{\underline{w}}, \mathsf{B}_{\underline{w}'})$$
(where $\Bbbk$ is the trivial R-module concentrated in degree 0);
- composition induced in the obvious way by composition in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)$.

Given expressions $\underline{w}$ and $\underline{w}'$, we set
$$\begin{aligned}\mathrm{Hom}^\bullet_{\overline{\mathsf{D}}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}_{\underline{w}}, \mathsf{B}_{\underline{w}'}) &= \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\overline{\mathsf{D}}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}_{\underline{w}}, \mathsf{B}_{\underline{w}'}(n)) \\ &= \Bbbk \otimes_R \mathrm{Hom}^\bullet_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}_{\underline{w}}, \mathsf{B}_{\underline{w}'}).\end{aligned}$$

Of course, From Theorem 2.30 we deduce that these spaces are graded free over $\Bbbk$, with bases consisting of images of double leaves morphisms. We will also denote by $\overline{\mathsf{D}}^{\oplus}_{\mathrm{BS}}(\mathcal{W},V)$ the additive hull of $\overline{\mathsf{D}}_{\mathrm{BS}}(\mathcal{W},V)$. In case $\Bbbk$ is a complete local domain, we denote by $\overline{\mathsf{D}}(\mathcal{W},V)$ the Karoubian envelope of $\overline{\mathsf{D}}^{\oplus}_{\mathrm{BS}}(\mathcal{W},V)$. It is easily seen that this category is Krull–Schmidt.

There exists a canonical bifunctor
$$\star : \overline{\mathsf{D}}_{\mathrm{BS}}(\mathcal{W},V) \times \mathsf{D}_{\mathrm{BS}}(\mathcal{W},V) \to \overline{\mathsf{D}}_{\mathrm{BS}}(\mathcal{W},V)$$
which defines a right action of the monoidal category $\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)$ on $\overline{\mathsf{D}}_{\mathrm{BS}}(\mathcal{W},V)$. We have obvious functors
$$\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V) \to \overline{\mathsf{D}}_{\mathrm{BS}}(\mathcal{W},V), \quad \mathsf{D}^{\oplus}_{\mathrm{BS}}(\mathcal{W},V) \to \overline{\mathsf{D}}^{\oplus}_{\mathrm{BS}}(\mathcal{W},V).$$
In case $\Bbbk$ is a complete local domain, the second functor induces a functor
$$\mathsf{D}(\mathcal{W},V) \to \overline{\mathsf{D}}(\mathcal{W},V). \tag{2.20}$$

Lemma 2.36. *Assume that $\Bbbk$ is a complete local domain. The functor (2.20) sends indecomposable objects to indecomposable objects. As a consequence, denoting for $w \in \mathcal{W}$ by $\overline{\mathsf{B}}_w$ the image of B_w in $\overline{\mathsf{D}}(\mathcal{W},V)$, the assignment*
$$(w,n) \mapsto \overline{\mathsf{B}}_w(n)$$
induces a bijection between $\mathcal{W} \times \mathbb{Z}$ and the set of isomorphism classes of indecomposable objects in $\overline{\mathsf{D}}(\mathcal{W},V)$.

Proof. From the definition we see that for any $M,N \in \mathsf{D}(\mathcal{W},V)$, with images $\overline{M}$ and $\overline{N}$ respectively, our functor induces an isomorphism
$$\Bbbk \otimes_R \left(\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{D}(\mathcal{W},V)}(M, N(n)) \right) \xrightarrow{\sim} \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\overline{\mathsf{D}}(\mathcal{W},V)}(\overline{M}, \overline{N}(n)).$$
In particular, if M is indecomposable then $\mathrm{End}_{\overline{\mathsf{D}}(\mathcal{W},V)}(\overline{M})$ is a quotient of the local ring $\mathrm{End}_{\mathsf{D}(\mathcal{W},V)}(M)$, hence is local. It follows that $\overline{M}$ is indecomposable. The rest of the proof is similar to that of Corollary 1.28. □

Lemma 2.36 implies that, if $\Bbbk$ is a complete local domain, the functor (2.20) induces an isomorphism
$$[\mathsf{D}(\mathcal{W},V)]_\oplus \xrightarrow{\sim} [\overline{\mathsf{D}}(\mathcal{W},V)]_\oplus$$
sending $[\mathsf{B}_w]$ to $[\overline{\mathsf{B}}_w]$ for any $w \in \mathcal{W}$. Combining this with Corollary 2.24 we deduce an isomorphism
$$\mathcal{H}_{(\mathcal{W},\mathcal{S})} \xrightarrow{\sim} [\overline{\mathsf{D}}(\mathcal{W},V)]_\oplus.$$

These properties can be translated roughly as saying that that "the categories $\mathsf{D}(\mathcal{W}, V)$ and $\overline{\mathsf{D}}(\mathcal{W}, V)$ contain the same combinatorial information." Another incarnation of this idea is that if the integers $b^{\underline{w}}_{y,n}$ are as in (2.14) we have

$$\overline{\mathsf{B}}_{\underline{w}} \cong \overline{\mathsf{B}}_w \oplus \bigoplus_{\substack{y \in \mathcal{W}, y < w \\ n \in \mathbb{Z}}} (\overline{\mathsf{B}}_y(n))^{b^{\underline{w}}_{y,n}}. \tag{2.21}$$

2.11.4. *Functoriality.* To avoid subtleties related to the Zamolodchikov relation, from now in this subsection we assume that $(\mathcal{W}, \mathcal{S})$ does not admit a parabolic subgroup of type $\mathbf{H}_3$.

There exists a notion of morphism of realizations, defined as follows. Given realizations

$$(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S})) \quad \text{and} \quad (\tilde{V}, (\tilde{\alpha}_s : s \in \mathcal{S}), (\tilde{\alpha}_s^\vee : s \in \mathcal{S}))$$

of a Coxeter system $(\mathcal{W}, \mathcal{S})$ over the same ring $\Bbbk$, a *morphism of realizations* from $(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S}))$ to $(\tilde{V}, (\tilde{\alpha}_s : s \in \mathcal{S}), (\tilde{\alpha}_s^\vee : s \in \mathcal{S}))$ is a $\Bbbk$-linear morphism $f : V \to \tilde{V}$ which satisfies

$$\tilde{\alpha}_s \circ f = \alpha_s \quad \text{and} \quad f(\alpha_s^\vee) = \tilde{\alpha}_s^\vee$$

for any $s \in \mathcal{S}$. Note that in this situation we have

$$\langle \alpha_s, \alpha_t^\vee \rangle = \langle \tilde{\alpha}_s, \tilde{\alpha}_t^\vee \rangle$$

for any $s, t \in \mathcal{S}$. In particular, all the technical conditions involving the quantum numbers are satisfied for $(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S}))$ if and only if they are satisfied for $(\tilde{V}, (\tilde{\alpha}_s : s \in \mathcal{S}), (\tilde{\alpha}_s^\vee : s \in \mathcal{S}))$.

Fix a morphism

$$f : (V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S})) \to (\tilde{V}, (\tilde{\alpha}_s : s \in \mathcal{S}), (\tilde{\alpha}_s^\vee : s \in \mathcal{S})),$$

and assume that the technical conditions of §2.4 are satisfied for these realizations. If we denote by R, resp. $\tilde{R}$, the symmetric algebra of V^*, resp. of $\tilde{V}^*$, then f induces a morphism of graded $\Bbbk$-algebras

$$f^* : \tilde{R} \to R.$$

In this setting we can consider the categories $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, \tilde{V})$ and $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$. The objects in both categories are in a canonical bijection with pairs $(\underline{w}, n)$ where $\underline{w}$ is an expression and $n \in \mathbb{Z}$; to distinguish them we will denote by $\tilde{\mathsf{B}}_{\underline{w}}(n)$ the object attached to $(\underline{w}, n)$ in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, \tilde{V})$, and by $\mathsf{B}_{\underline{w}}(n)$ the corresponding object in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$. We have a monoidal functor

$$f^* : \mathsf{D}_{\mathrm{BS}}(\mathcal{W}, \tilde{V}) \to \mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$$

which is defined on objects by

$$f^*(\tilde{\mathsf{B}}_{\underline{w}}(n)) = \mathsf{B}_{\underline{w}}(n)$$

for any expression $\underline{w}$ and any $n \in \mathbb{Z}$, and which sends a box labeled by $r \in \tilde{R}$ to the box labeled by $f^*(r)$, and each other generating morphism of $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, \tilde{V})$ to the corresponding morphism in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$. In case $\Bbbk$ is a complete local domain, f^* induces a functor $\mathsf{D}(\mathcal{W}, \tilde{V}) \to \mathsf{D}(\mathcal{W}, V)$ such that the induced morphism

$$[\mathsf{D}(\mathcal{W}, \tilde{V})]_\oplus \to [\mathsf{D}(\mathcal{W}, V)]_\oplus$$

is an isomorphism. (Under the isomorphisms ch_{D}, this morphism corresponds to the identity morphism of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$.)

The following lemma is a consequence of Theorem 2.30, once one remarks that double leaves morphisms do not involve "box" morphisms.

LEMMA 2.37. *For any expressions $\underline{w}, \underline{w}'$, the functor f^* induces an isomorphism of graded left R-modules, resp. graded right R-modules*

$$R \otimes_{\tilde{R}} \mathrm{Hom}^{\bullet}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},\tilde{V})}(\tilde{\mathsf{B}}_{\underline{w}}, \tilde{\mathsf{B}}_{\underline{w}'}) \xrightarrow{\sim} \mathrm{Hom}^{\bullet}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}_{\underline{w}}, \mathsf{B}_{\underline{w}'}),$$
$$\textit{resp.} \quad \mathrm{Hom}^{\bullet}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},\tilde{V})}(\tilde{\mathsf{B}}_{\underline{w}}, \tilde{\mathsf{B}}_{\underline{w}'}) \otimes_{\tilde{R}} R \xrightarrow{\sim} \mathrm{Hom}^{\bullet}_{\mathsf{D}_{\mathrm{BS}}(\mathcal{W},V)}(\mathsf{B}_{\underline{w}}, \mathsf{B}_{\underline{w}'}).$$

It is clear that the composition of f^* with the functor (2.20) (for V) factors through a functor

$$\overline{f}^* : \overline{\mathsf{D}}_{\mathrm{BS}}(\mathcal{W}, \tilde{V}) \to \overline{\mathsf{D}}_{\mathrm{BS}}(\mathcal{W}, V).$$

The following statement is a direct consequence of Lemma 2.37.

LEMMA 2.38. *The functor $\overline{f}^*$ is a equivalence of categories.*

From now on we assume that $\Bbbk$ is a complete local domain. Under this assumption we can consider the "normalized" indecomposable objects $(\mathsf{B}_w : w \in \mathcal{W})$ in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$, and the corresponding "normalized" indecomposable objects $(\tilde{\mathsf{B}}_w : w \in \mathcal{W})$ in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, \tilde{V})$.

PROPOSITION 2.39. *For any $w \in \mathcal{W}$ we have*

$$f^*(\tilde{\mathsf{B}}_w) \cong \mathsf{B}_w.$$

PROOF. We proceed by induction on w (for the Bruhat order). The claim is clear if $w = e$. Now let $w \in \mathcal{W}$, and assume the claim is known for smaller elements. Let $\underline{w}$ be a reduced expression for w, and consider the decompositions

$$\tilde{\mathsf{B}}_{\underline{w}} \cong \tilde{\mathsf{B}}_w \oplus \bigoplus_{\substack{y \in \mathcal{W}, y<w \\ n \in \mathbb{Z}}} (\tilde{\mathsf{B}}_y(n))^{\tilde{b}^{\underline{w}}_{y,n}}, \quad \mathsf{B}_{\underline{w}} \cong \mathsf{B}_w \oplus \bigoplus_{\substack{y \in \mathcal{W}, y<w \\ n \in \mathbb{Z}}} (\mathsf{B}_y(n))^{b^{\underline{w}}_{y,n}},$$

see (2.14). The comments above (2.21) and Lemma 2.38 imply that for any y and n we have $\tilde{b}^{\underline{w}}_{y,n} = b^{\underline{w}}_{y,n}$. On the other hand, applying f^* and using induction we have

$$\mathsf{B}_{\underline{w}} \cong f^*(\tilde{\mathsf{B}}_w) \oplus \bigoplus_{\substack{y \in \mathcal{W}, y<w \\ n \in \mathbb{Z}}} (\mathsf{B}_y(n))^{\tilde{b}^{\underline{w}}_{y,n}}.$$

Hence by the Krull–Schmidt property we must have $f^*(\tilde{\mathsf{B}}_w) \cong \mathsf{B}_w$, as desired. $\square$

It follows from Proposition 2.39 that in this setting we have

$$\underline{H}_w(V) = \underline{H}_w(\tilde{V}) \quad \text{for any } w \in \mathcal{W}. \tag{2.22}$$

2.11.5. *Independence.* We continue to assume that $(\mathcal{W}, \mathcal{S})$ does not admit any parabolic subgroup of type H_3. Consider a complete local domain $\Bbbk$ and a realization

$$(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S}))$$

of $(\mathcal{W}, \mathcal{S})$ over $\Bbbk$ which satisfies the conditions of §2.4. Let us assume furthermore that

$$\sum_{t \in \mathcal{S}} \Bbbk \cdot \langle \alpha_s, \alpha_t^\vee \rangle = \Bbbk \quad \text{for any } s \in \mathcal{S}. \tag{2.23}$$

(This condition holds automatically if 2 is invertible in $\Bbbk$.) Under this assumption, we will show that the basis

$$(\underline{H}_w(V) : w \in \mathcal{W})$$

only depends on the choice of $\Bbbk$ and of the matrix

$$(\langle \alpha_s, \alpha_t^\vee \rangle)_{s,t \in \mathcal{S}}$$

(sometimes called the *Cartan matrix* of the realization), but not on the full datum of the realization.

Consider the $\Bbbk$-module defined by $\tilde{V} = \Bbbk^{\oplus \mathcal{S}}$, with canonical basis denoted $(\widetilde{\alpha}_s^\vee : s \in \mathcal{S})$, and for $s \in \mathcal{S}$ denote by

$$\tilde{\alpha}_s : \tilde{V} \to \Bbbk$$

the morphism defined by

$$\langle \tilde{\alpha}_s, \tilde{\alpha}_t^\vee \rangle = \langle \alpha_s, \alpha_t^\vee \rangle$$

for any $t \in \mathcal{S}$. Then

$$(\tilde{V}, (\tilde{\alpha}_s : s \in \mathcal{S}), (\tilde{\alpha}_s^\vee : s \in \mathcal{S}))$$

is a realization of $(\mathcal{W}, \mathcal{S})$ which satisfies the conditions of §2.4. (In fact, the quantum numbers for this realization are the same as for the initial one, which justifies all the conditions except for Demazure surjectivity. The latter property holds by our assumption (2.23).) Moreover, the morphism of $\Bbbk$-modules

$$\tilde{V} \to V$$

sending $\tilde{\alpha}_s^\vee$ to $\alpha_s^\vee$ for any $s \in \mathcal{S}$ is a morphism of realizations. By (2.22) we deduce that $\underline{H}_w(V) = \underline{H}_w(\tilde{V})$ for any $w \in \mathcal{W}$, which justifies our assertion.

2.12. Relation with "usual" Soergel bimodules. One of the main motivations for the construction of the category $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ was the desire to describe the category of Soergel bimodules studied in Section 1 by generators and relations. We now explain how this goal can be achieved. Let $\Bbbk$ be a field of characteristic different from 2, and consider a balanced realization

$$(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S}))$$

of $(\mathcal{W}, \mathcal{S})$ over $\Bbbk$ which satisfies the condition related to type $\mathbf{H}_3$ discussed in §2.4. The condition on $\mathrm{char}(\Bbbk)$ implies that this realization also satisfies Demazure surjectivity. We will assume moreover that V is a reflection faithful representation of $(\mathcal{W}, \mathcal{S})$; then (2.4) is automatically satisfied. In fact $\Bbbk$ is a field with $\mathrm{char}(\Bbbk) \neq 2$, and for any $(s,t) \in \mathcal{S}_\circ^2$ the action of $\langle s,t \rangle$ on V^* is faithful. By Exercise 2.1(5) we also have $\ker(\alpha_s) \neq \ker(\alpha_t)$, hence $\Bbbk\alpha_s \neq \Bbbk\alpha_t$. The claim therefore follows from Lemma 2.5.

Then we can consider the categories $\mathsf{D}(\mathcal{W}, V)$ and $\mathsf{SBim}(\mathcal{W}, V)$. The following statement is proved in [**EW2**].

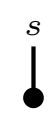
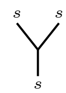
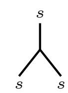
THEOREM 2.40. *Under the assumptions above, there exists a canonical equivalence of monoidal categories*

$$\mathsf{D}(\mathcal{W}, V) \xrightarrow{\sim} \mathsf{SBim}(\mathcal{W}, V).$$

The proof of this statement proceeds in two steps. First, one needs to construct a functor

$$\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V) \to \mathsf{SBim}(\mathcal{W}, V).$$

On objects, this functor will send $\mathsf{B}_{\underline{w}}(n)$ to $\mathsf{B}^{\mathrm{bim}}_{\underline{w}}(n)$ for any expression $\underline{w}$ and any $n \in \mathbb{Z}$. To define the functor on morphisms, one needs to describe the image of each generating morphism, and then check that these morphisms satisfy the appropriate relations. Here the image of a polynomial is defined to be multiplication by this polynomial on R, and the image of

 , resp.  , resp.  , resp. 

is given by

$$f \otimes g \mapsto fg, \quad \text{resp.} \quad f \mapsto f\delta_s \otimes 1 - f \otimes s(\delta_s),$$
$$\text{resp.} \quad f \otimes g \mapsto f \otimes 1 \otimes g, \quad \text{resp.} \quad f \otimes g \otimes h \mapsto f\partial_s(g) \otimes h$$

for $f, g, h \in R$. (Here, $\delta_s \in V^*$ is an element such that $\langle \delta_s, \alpha_s^\vee \rangle = 1$; the morphism described above does not depend on the choice of this element.) If $(s,t) \in \mathcal{S}^2_\circ$, the image of the corresponding $2m_{s,t}$-valent morphism is the unique morphism of graded bimodules

$$\mathsf{B}^{\mathrm{bim}}_{(s,t,\cdots)} \to \mathsf{B}^{\mathrm{bim}}_{(t,s,\cdots)}$$

sending the vector

$$1 \otimes 1 \otimes \cdots \in R \otimes_{R^s} R \otimes_{R^t} \cdots$$

to the vector

$$1 \otimes 1 \otimes \cdots \in R \otimes_{R^t} R \otimes_{R^s} \cdots$$

(The existence and unicity of such a morphism follows from [**Li1**, §§4.1–4.3]; see in particular [**Li1**, Proposition 4.3]. See also Exercise 2.3.) The verification that such morphisms satisfy the relations of $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ is explained in [**EW2**, Claim 5.14]. (This verification relies on the results of [**El**] and some computer computations. For a different approach to this question based on later work of Abe, see Remark 3.12 below.)

Once this functor is constructed, since $\mathsf{SBim}(\mathcal{W}, V)$ is additive and Krull–Schmidt we obtain a canonical "extension" to a fully faithful functor $\mathsf{D}(\mathcal{W}, V) \xrightarrow{\sim} \mathsf{SBim}(\mathcal{W}, V)$. To conclude it then suffices to prove that this functor induces an isomorphism

$$\mathrm{Hom}^\bullet_{\mathsf{D}(\mathcal{W},V)}(\mathsf{B}_{\underline{w}}, \mathsf{B}_{\underline{w}'}) \xrightarrow{\sim} \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{SBim}(\mathcal{W},V)}(\mathsf{B}^{\mathrm{bim}}_{\underline{w}}, \mathsf{B}^{\mathrm{bim}}_{\underline{w}'}(n))$$

for any expressions $\underline{w}, \underline{w}'$. (In fact, this will prove that this functor is fully faithful; essential surjectivity easily follows.) This follows from the fact that this functor sends the "double leaves" basis considered in §2.10 to the similar basis in $\mathsf{SBim}(\mathcal{W}, V)$ constructed by Libedinsky [**Li1**].

Remark 2.41. In the course of the proof of Theorem 2.40, it is claimed in [**EW2**] that, for any balanced realization satisfying Demazure surjectivity, there exists a monoidal functor

$$\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V) \to R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R$$

sending, for any expression $\underline{w} = (s_1, \cdots, s_r)$, the object $\mathsf{B}_{\underline{w}}$ to the graded bimodule

$$R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_r}} R(r).$$

Unfortunately, the proof of this claim is incomplete, as discussed in [**EW3**, §5.3]. Later work of Abe allows to complete the proof of this claim under the assumption that (2.4) is satisfied, which is sufficient for the proof of the theorem; see Remark 3.12 below for details. (As always, in case $(\mathcal{W}, \mathcal{S})$ admits a parabolic subgroup of type $\mathbf{H}_3$, one also needs to impose the extra assumption considered in §2.4.)

2.13. The p-canonical basis.

2.13.1. *Definition.* Consider a generalized Cartan matrix A, with rows and columns parametrized by a finite set I. Let $(\mathcal{W}, \mathcal{S})$ be the associated Coxeter system (see §1.2.3), and let p be either 0 or a prime number. In this subsection we explain the definition of the *p-canonical basis*

$$({}^p\underline{H}_w : w \in \mathcal{W})$$

of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$. This basis (for special choices of A) will play a major role in later chapters.

First, let us consider the case when either $p \neq 2$ or each column of A contains an odd number. Consider a Kac–Moody root datum

$$(\mathbf{X}, (\alpha_i : i \in I), (\alpha_i^\vee : i \in I))$$

associated with A; in case $p = 2$ we further assume that in the notation of §2.2.2 we have $\mathbb{Z}' = \mathbb{Z}$. If $\Bbbk$ is a field of characteristic p, we can consider the Cartan realization of $(\mathcal{W}, \mathcal{S})$ over $\Bbbk$ associated with $(\mathbf{X}, (\alpha_i : i \in I), (\alpha_i^\vee : i \in I))$, constructed in §2.2.2. This realization satisfies Demazure surjectivity by assumption, and it is balanced and satisfies (2.4) as explained in §2.2.2.

Our assumptions imply that the condition (2.23) is satisfied, hence the considerations in §2.11.5 show that the basis of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ produced from such a realization (see (2.15)) does not depend on the choice of the Kac–Moody root datum as above. (In this case, the realization used in §2.11.5 is the realization associated with the simply-connected datum from Example 2.6.) By (2.19), it does not depend on the choice of $\Bbbk$ either (but only on p). This basis is the p-canonical basis of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ associated with A.

In case $p = 2$ and A has a column containing only even numbers, one has to be more careful. We claim that the basis constructed as above is independent of the choice of a Kac–Moody root datum $(\mathbf{X}, (\alpha_i : i \in I), (\alpha_i^\vee : i \in I))$ which satisfies the following properties:

- $\mathbb{Z}' = \mathbb{Z}$;
- the vectors $(\alpha_i : i \in I)$ are linearly independent over $\mathbb{Z}$, and moreover $\mathbf{X}/(\sum_i \mathbb{Z}\alpha_i)$ has no torsion.

In fact, denote by $\mathbf{X}_{\mathrm{univ}}$ the underlying $\mathbb{Z}$-module of the universal datum from Example 2.6, and denote the bases of $\mathbf{X}_{\mathrm{univ}}$ and $\mathbf{X}_{\mathrm{univ}}^\vee$ considered in this example by $(\tilde{\alpha}_i, \beta_i)_{i \in I}$ and $(\beta_i^\vee, \tilde{\alpha}_i^\vee)_{i \in I}$. Our second assumption above ensures that for any $i \in I$ there exists $u_i \in \mathbf{X}^\vee$ such that $\langle \alpha_j, u_i \rangle = \delta_{i,j}$. We then consider the morphism of $\mathbb{Z}$-modules

$$f : \mathbf{X}_{\mathrm{univ}}^\vee \to \mathbf{X}^\vee$$

sending $\tilde{\alpha}_i^\vee$ to $\alpha_i^\vee$ and β_i to u_i, for any i. For any $i,j \in I$ we have

$$\langle \alpha_i \circ f, \tilde{\alpha}_j^\vee \rangle = \langle \alpha_i, \alpha_j^\vee \rangle = a_{j,i}, \quad \langle \alpha_i \circ f, \beta_j^\vee \rangle = \langle \alpha_i, u_j \rangle = \delta_{i,j},$$

hence $\alpha_i \circ f = \tilde{\alpha}_i$. This shows that the morphism $\Bbbk \otimes_{\mathbb{Z}} f$ is a morphism of realizations from the realization over $\Bbbk$ associated with the universal Kac–Moody root datum to that associated with our given datum. In view of (2.22) it follows that the associated bases of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ coincide. By (2.19), this basis does not depend on the choice of $\Bbbk$ either, which justifies our claim.

The first important property of the p-canonical basis is the following.

Proposition 2.42. *For $p = 0$, we have*

$$^0\underline{H}_w = \underline{H}_w \quad \textit{for any } w \in \mathcal{W}.$$

Proof. Consider a triple $(\mathfrak{h}, (\alpha_i : i \in I), (\alpha_i^\vee : i \in I))$ as in §1.2.3, and a lattice $\mathbf{X} \subset \mathfrak{h}^*$ as in §1.2.4. Then $(\mathbf{X}, (\alpha_i : i \in I), (\alpha_i^\vee : i \in I))$ is a Kac–Moody root datum for A, which can be used to compute the basis $({}^0\underline{H}_w : w \in \mathcal{W})$. More specifically, we will choose as base field (of characteristic 0) the field $\mathbb{R}$.

As explained in §1.2.3 the representation of $(\mathcal{W}, \mathcal{S})$ on $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{X}^\vee \xrightarrow{\sim} \mathfrak{h}$ is reflection faithful, hence so is the representation on $\mathbb{R} \otimes_{\mathbb{Z}} \mathbf{X}^\vee$. In view of Theorem 2.40, we deduce an equivalence of monoidal categories

$$\mathsf{D}(\mathcal{W}, \mathbb{R} \otimes_{\mathbb{Z}} \mathbf{X}^\vee) \xrightarrow{\sim} \mathsf{SBim}(\mathcal{W}, \mathbb{R} \otimes_{\mathbb{Z}} \mathbf{X}^\vee).$$

Now, as explained in §1.8, Soergel's conjecture is known in $\mathsf{SBim}(\mathcal{W}, \mathbb{R} \otimes_{\mathbb{Z}} \mathbf{X}^\vee)$, which implies that ${}^0\underline{H}_w = \underline{H}_w$ for any $w \in \mathcal{W}$, as desired. □

Corollary 2.43. *For any prime number p, there exist polynomials*

$$({}^p a_{y,w})_{y<w\in\mathcal{W}}$$

in $\mathbb{Z}_{\geq 0}[v, v^{-1}]$, invariant under the replacement of v by v^{-1}, and such that

$$^p\underline{H}_w = \underline{H}_w + \sum_{y<w} {}^p a_{y,w} \cdot \underline{H}_y$$

for any $w \in \mathcal{W}$.

Proof. To prove this property one can assume that our base field is $\mathbb{F}_p$. Then the claim follows from Proposition 2.42 and Remark 2.33 applied to the complete local domain $\mathbb{Z}_p$. □

Another important property of the p-canonical basis is that for any fixed $w \in \mathcal{W}$ we have

$$^p\underline{H}_w = \underline{H}_w \quad \text{if } p \gg 0.$$

(For a proof based on diagrammatic considerations, see [JW, Proposition 4.2(7)]. We will explain a proof based on geometry in REF.) In particular, if $\mathcal{W}$ is finite there are only a finite number of prime numbers for which the p-canonical basis differs from the Kazhdan–Lusztig basis. Determining exactly what these prime numbers are is however a very difficult problem, which is open in most cases. For many examples of computation of this basis, we refer to [JW] and [Je].

The p-canonical basis can be computed algorithmically using a procedure described in [GJW]. This algorithm becomes soon prohibitively heavy to run, but at least in some relatively small cases it can be used to describe this basis explicitly, and these cases already suggest that its behavior seems difficult to describe in general.

REMARK 2.44. We have explained above that the p-canonical basis only depends on A, and not on the choice of $\Bbbk$ or of the Kac–Moody root datum. But different generalized Cartan matrices can have the same associated Coxeter system, hence the same associated Hecke algebra. The corresponding p-canonical bases *can* differ. For an explicit example, see [**JW**, §5.4]. For another illustration of this idea, note that the p-canonical basis might *not* be stable under Coxeter group automorphisms which do not come from automorphisms of the associated Kac–Moody group; e.g. in types $\mathbf{B}_2$ or $\mathbf{G}_2$, these bases are not always invariant under the exchange of the two simple reflections; see [**JW**, §§5.1–5.2].

The coefficients in the expansion of ${}^p\underline{H}_w$ in the standard basis are called the *p-Kazhdan–Lusztig polynomials* and denoted $({}^p h_{y,w} : y, w \in \mathcal{W})$, with the convention that

$$
{}^p\underline{H}_w = \sum_y {}^p h_{y,w} \cdot H_y.
$$

Note that ${}^p h_{y,w}$ is a Laurent polynomial in v, but not necessarily a polynomial. Its coefficients are nonnegative by construction.

2.13.2. *The case of modular category $\mathcal{O}$.* Consider the setting of §1.11, assuming in addition that $p \notin \{2, 3\}$. In particular, $\mathcal{W} = W$ is now the Weyl group of $(\mathbf{G}, \mathbf{T})$. In view of Remark 1.31, Theorem 2.40 applies in this setting. Hence, for any $w \in W$, the element

$$
\sum_y (P_w : \mathsf{M}_y) \cdot y
$$

is ${}^p\underline{H}_{w|v=1}$, where we use the notation introduced at the end of §1.9. In other words, for $y, w \in W$ we have

$$
(P_w : \mathsf{M}_y) = {}^p h_{y,w}(1).
$$

Proposition 4.9 in Chapter 1 shows that, if Lusztig's character formula holds for a given p, then ${}^p h_{y,w}(1) = h_{y,w}(1)$ for any $y, w \in W$. In view of Corollary 2.43 and since Kazhdan–Lusztig polynomials have nonnegative coefficients, this in fact implies that

$$
{}^p\underline{H}_w = \underline{H}_w
$$

for any $w \in W$.

2.14. Examples.

2.14.1. *Type* $\mathbf{B}_2$. Consider the Cartan matrix of type $\mathbf{B}_2$, given by

$$
A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.
$$

We denote by s the reflection associated with the first line, and t the reflection associated with the second line. Then we have

$$
\langle \alpha_s, \alpha_s^\vee \rangle = \langle \alpha_s, \alpha_s^\vee \rangle = 2, \quad \langle \alpha_s, \alpha_t^\vee \rangle = -1, \quad \langle \alpha_t, \alpha_s^\vee \rangle = -2,
$$

and $m_{s,t} = 4$. We claim that

$$
{}^p\underline{H}_{sts} = \begin{cases} \underline{H}_{sts} + \underline{H}_s & \text{if } p = 2; \\ \underline{H}_{sts} & \text{if } p \neq 2. \end{cases} \tag{2.24}
$$

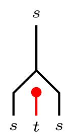
In fact, for any field $\Bbbk$, $\mathsf{B}^{\Bbbk}_{sts}$ is a direct summand of $\mathsf{B}^{\Bbbk}_{(s,t,s)}$, and we have

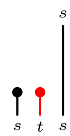
$$
\mathrm{ch}_{\mathsf{D}}(\mathsf{B}^{\Bbbk}_{(s,t,s)}) = \underline{H}_s \cdot \underline{H}_t \cdot \underline{H}_s = \underline{H}_{sts} + \underline{H}_s.
$$

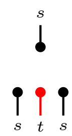
Using also Corollary 2.43, we deduce that for any p we have

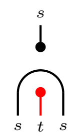
$$ {}^p\underline{H}_{sts} \in \{\underline{H}_{sts}, \underline{H}_{sts} + \underline{H}_s\}. $$

To determine what is the correct solution between the two options, one should determine if $\mathsf{B}_s^{\Bbbk}$ is a direct summand of $\mathsf{B}_{(s,t,s)}^{\Bbbk}$ or not. For that we consider the double leaves basis of $\mathrm{Hom}^\bullet(\mathsf{B}_{(s,t,s)}^{\Bbbk}, \mathsf{B}_s^{\Bbbk})$, see Theorem 2.30. In this case the natural choices lead to the basis consisting of the diagrams



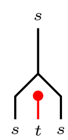
,


,


and


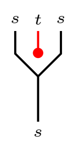
,

of respective degrees 0, 2, 4 and 2. Hence $\mathrm{Hom}(\mathsf{B}_{(s,t,s)}^{\Bbbk}, \mathsf{B}_s^{\Bbbk})$ is 1-dimensional, and spanned by the diagram



.

Applying the autoequivalence ι of Lemma 2.20, we deduce that $\mathrm{Hom}(\mathsf{B}_s^{\Bbbk}, \mathsf{B}_{(s,t,s)}^{\Bbbk})$ is also 1-dimensional, and spanned by the diagram

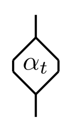


.

These considerations show that any composition of morphisms

$$ \mathsf{B}_s^{\Bbbk} \to \mathsf{B}_{(s,t,s)}^{\Bbbk} \to \mathsf{B}_s^{\Bbbk} \tag{2.25} $$

is a multiple of the morphism

(2.26)


.

To decide wether $\mathsf{B}_s^{\Bbbk}$ is a direct summand of $\mathsf{B}_{(s,t,s)}^{\Bbbk}$, we need to determine if the identity morphism can appear as a composition (2.25) or, in other words, if it is a multiple of (2.26). Now we observe that

$$ \partial_s(\alpha_t) = \langle \alpha_t, \alpha_s^\vee \rangle = -2. $$

The same computation as in (2.13) therefore shows that the morphism (2.26) equals $-2\,\mathrm{id}$. Of course, if $p \neq 2$ then id is a multiple of this morphism, but if $p = 2$ this is not the case, which justifies (2.24).

REMARK 2.45. See [**Ac**, Exercise 7.7.7] for a geometric computation of the same p-Kazhdan–Lusztig element. See [**JW**, §5.1] for a slightly different way of performing this computation, based on a more systematic method of computation of the p-canonical basis.

2.14.2. *Type* $\widetilde{\mathbf{A}}_1$. Consider now the Cartan matrix of type $\widetilde{\mathbf{A}}_1$, given by

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

We will denote by s and s_0 the elements of $\mathcal{S}$; then $\mathcal{W}$ is the infinite dihedral group with generators s and s_0. We are particularly interested in the element $s_0ss_0s \in \mathcal{W}$. Let $\Bbbk$ be a field, and p be its characteristic. By Exercise 2.12 we have ${}^p\underline{H}_{s_0s} = \underline{H}_{s_0s} = \underline{H}_{s_0}\underline{H}_s$; in other words, $\mathsf{B}^{\Bbbk}_{(s_0,s)}$ is indecomposable. A straightforward computation shows that

$$\underline{H}_{s_0}\underline{H}_s\underline{H}_{s_0}\underline{H}_s = \underline{H}_{s_0ss_0s} + 2\underline{H}_{s_0s}.$$

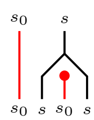
As in §2.14.1, this implies that ${}^p\underline{H}_{s_0ss_0s}$ is either $\underline{H}_{s_0ss_0s} + 2\underline{H}_{s_0s}$ (if $\mathsf{B}^{\Bbbk}_{(s_0,s,s_0,s)}$ is indecomposable), or $\underline{H}_{s_0ss_0s} + \underline{H}_{s_0s}$ (if $\mathsf{B}^{\Bbbk}_{(s_0,s,s_0,s)} \cong \mathsf{B}^{\Bbbk}_{s_0ss_0s} \oplus \mathsf{B}^{\Bbbk}_{s_0s}$), or $\underline{H}_{s_0ss_0s}$ (if $\mathsf{B}^{\Bbbk}_{(s_0,s,s_0,s)} \cong \mathsf{B}^{\Bbbk}_{s_0ss_0s} \oplus (\mathsf{B}^{\Bbbk}_{s_0s})^{\oplus 2}$). What we have to determine is therefore the multiplicity of $\mathsf{B}^{\Bbbk}_{(s_0,s)}$ as a direct summand of $\mathsf{B}^{\Bbbk}_{(s_0,s,s_0,s)}$.

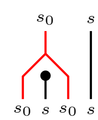
Using e.g. the light leaves basis, one sees that $\mathrm{End}(\mathsf{B}^{\Bbbk}_{(s_0,s)}) = \Bbbk \cdot \mathrm{id}$. One can therefore consider the bilinear form

$$\mathrm{Hom}(\mathsf{B}^{\Bbbk}_{(s_0,s,s_0,s)}, \mathsf{B}^{\Bbbk}_{(s_0,s)}) \times \mathrm{Hom}(\mathsf{B}^{\Bbbk}_{(s_0,s)}, \mathsf{B}^{\Bbbk}_{(s_0,s,s_0,s)}) \to \mathrm{End}(\mathsf{B}^{\Bbbk}_{(s_0,s)}) = \Bbbk \tag{2.27}$$

given by $(g,f) \mapsto g \circ f$. Since no composition $\mathsf{B}^{\Bbbk}_{s_0s} \to \mathsf{B}^{\Bbbk}_{s_0ss_0s} \to \mathsf{B}^{\Bbbk}_{s_0s}$ can be nonzero, we see that the multiplicity of $\mathsf{B}^{\Bbbk}_{(s_0,s)}$ as a direct summand of $\mathsf{B}^{\Bbbk}_{(s_0,s,s_0,s)}$ is the rank of (2.27).

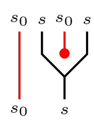
Using the light leaves basis one can check that $\mathrm{Hom}(\mathsf{B}^{\Bbbk}_{(s_0,s,s_0,s)}, \mathsf{B}^{\Bbbk}_{(s_0,s)})$ has dimension 2, and is spanned by

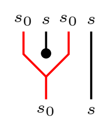
$p_1 :=$ 

and $p_2 :=$ 

.

Applying ι, we deduce that $\mathrm{Hom}(\mathsf{B}^{\Bbbk}_{(s_0,s)}, \mathsf{B}^{\Bbbk}_{(s_0,s,s_0,s)})$ also has dimension 2, and is spanned by

$i_1 :=$ 

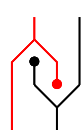
and $i_2 :=$ 

.

We next compute the compositions between these morphisms. First, using the same considerations as in §2.14.1 and the fact that $\partial_s(\alpha_{s_0}) = -2$ we find that

$$p_1 \circ i_1 = -2\,\mathrm{id}, \quad p_2 \circ i_2 = -2\,\mathrm{id}.$$

On the other hand we have

$p_2 \circ i_1 =$  $= \mathrm{id}$,

and similarly $p_1 \circ i_2 = \mathrm{id}$. Hence the matrix of our bilinear form (2.27) in the bases (p_1, p_2) and (i_1, i_2) is

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},$$

whose determinant is 3. If $p \neq 3$, this matrix has rank 2, so that $\mathsf{B}^{\Bbbk}_{(s_0,s,s_0,s)} \cong \mathsf{B}^{\Bbbk}_{s_0ss_0s} \oplus (\mathsf{B}^{\Bbbk}_{s_0s})^{\oplus 2}$, and hence

$${}^p\underline{H}_{s_0ss_0s} = \underline{H}_{s_0ss_0s}.$$

But if $p = 3$, the matrix has rank 1, so that $\mathsf{B}^{\Bbbk}_{(s_0,s,s_0,s)} \cong \mathsf{B}^{\Bbbk}_{s_0ss_0s} \oplus \mathsf{B}^{\Bbbk}_{s_0s}$, and hence

$${}^3\underline{H}_{s_0ss_0s} = \underline{H}_{s_0ss_0s} + \underline{H}_{s_0s}.$$

Remark 2.46. See [**JW**, §5.3] for another method of computation of the p-canonical basis in type $\widetilde{\mathsf{A}}_1$, based on the geometric Satake equivalence (see §5.1 in Chapter 3).

3. Abe's algebraic incarnation of the diagrammatic Hecke category

In this section we explain a different approach to the Hecke category, introduced by N. Abe in [**Ab1**]. This definition is closer to Soergel's original definition, and also solves the deficiencies of the latter approach when the representation under consideration is not reflection faithful.

3.1. Definition.

3.1.1. *Setup.* The starting data for Abe's construction are as follows. One considers a Coxeter system $(\mathcal{W}, \mathcal{S})$, a noetherian[11] integral domain $\Bbbk$, and a triple

$$\big(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S})\big)$$

where V is a free $\Bbbk$-module[12] of finite rank, $(\alpha_s : s \in S)$ is a collection of elements in $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$, and $(\alpha_s^\vee : s \in S)$ is a collection of elements in V, which satisfy the following conditions:

(1) for any $s \in \mathcal{S}$ we have $\langle \alpha_s, \alpha_s^\vee \rangle = 2$;
(2) the assignment
$$s \mapsto \big(v \mapsto v - \langle \alpha_s, v\rangle \alpha_s^\vee\big)$$
defines an action of $\mathcal{W}$ on V;
(3) for any $s \in \mathcal{S}$, $\alpha_s^\vee : V^* \to \Bbbk$ is surjective and $\alpha_s \neq 0$.

Here Condition (1) implies Condition (3) in case 2 is invertible in $\Bbbk$. Condition (3) ensures that Lemma 2.17 applies for any $s \in \mathcal{S}$. Of course, any realization in the sense of §2.2.1 which satisfies Demazure surjectivity gives rise to such data.

In this setting we will denote by R the symmetric algebra of V^* over $\Bbbk$ (a noetherian integral domain), which we will consider as a $\mathbb{Z}$-graded $\Bbbk$-algebra where V^* is concentrated in degree 2. We will also set

$$Q := R\left[\frac{1}{w(\alpha_s)} : s \in \mathcal{S}, w \in \mathcal{W}\right].$$

This localization makes sense thanks to the second part of Condition (3); it is endowed with a natural $\mathbb{Z}$-grading. (Note that $w(\alpha_s)$ only depends on the reflection wsw^{-1}, up to an invertible constant; see [**Ab1**, Lemma 2.1].) The $\mathcal{W}$-action on V from Condition (2) induces actions on R and on Q by graded algebra automorphisms.

[11] CHECK!

[12] Here, in order to make the comparison by Soergel's and Elias–Williamson's approaches easier, we deviate from Abe's notation: his "V" corresponds to V^* here.

REMARK 3.1. In [**Ab1**], Q is defined as the fraction field of R; however, with this definition it is not clear that the bimodules M_I and M^I introduced above [**Ab1**, Lemma 2.4] are graded. As explained to us by N. Abe, the modified definition of Q considered above solves this difficulty, so that all the statements from [**Ab1**] hold true after this modification. But in fact the two possible definitions of Q lead in the end to equivalent categories $\mathsf{D}_{\mathrm{BS}}^{\mathrm{Abe}}(\mathcal{W}, V)$ as in §3.1.5 below. Indeed there exists a natural fully faithful functor from the category C' of §3.1.2 to its analogue defined with Q replaced by the field of fractions of R, which restricts to a fully faithful functor on $\mathsf{D}_{\mathrm{BS}}^{\mathrm{Abe}}(\mathcal{W}, V)$.

3.1.2. *The category* C'. Given the data above, Abe defines a category C' with

- objects the triples $(M, (M_Q^w : w \in \mathcal{W}), \xi_M)$ where M is a graded R-bimodule, each M_Q^w is a graded (R, Q)-bimodule such that

$$m \cdot f = w(f) \cdot m \quad \text{for any } f \in R \text{ and } m \in M_Q^w, \tag{3.1}$$

  this bimodule being 0 for all but finitely many w's, and

$$\xi_M : M \otimes_R Q \xrightarrow{\sim} \bigoplus_{w \in \mathcal{W}} M_Q^w$$

  is an isomorphism of graded (R, Q)-bimodules;
- morphisms from $(M, (M_Q^w : w \in \mathcal{W}), \xi_M)$ to $(N, (N_Q^w : w \in \mathcal{W}), \xi_N)$ given by morphisms of graded R-bimodules $\varphi : M \to N$ such that

$$\xi_N \circ (\varphi \otimes_R Q) \circ \xi_M^{-1}(M_Q^w) \subset N_Q^w$$

  for any $w \in \mathcal{W}$.

Often the data of the collection $(M_Q^w : w \in \mathcal{W})$ and the isomorphism ξ_M will be omitted, and the triple $(M, (M_Q^w : w \in \mathcal{W}), \xi_M)$ will be simply denoted M. Note that if $(M, (M_Q^w : w \in \mathcal{W}), \xi_M) \in \mathsf{C}'$, (3.1) and the isomorphism ξ_M imply that any element $w(\alpha_s)$ acts invertibly on the left on $M \otimes_R Q$, so that this module becomes a graded Q-bimodule. Similarly each M_Q^w has a natural structure of graded Q-bimodule (such that the formula in (3.1) holds for any $f \in Q$), and ξ_M is an isomorphism of graded Q-bimodules.

These considerations allow to define a monoidal product $\star$ on C'. Namely, if $\mathcal{M} = (M, (M_Q^w : w \in \mathcal{W}), \xi_M)$ and $\mathcal{N} = (N, (N_Q^w : w \in \mathcal{W}), \xi_N)$ are objects of C', the object $\mathcal{M} \star \mathcal{N}$ is defined as the triple consisting of $M \otimes_R N$, the collection defined by

$$(M \otimes_R N)_Q^w = \bigoplus_{\substack{x,y \in \mathcal{W} \\ xy = w}} M_Q^x \otimes_Q N_Q^y$$

(where we use the left Q-module structure on N_Q^y explained above), and the isomorphism

$$(M \otimes_R N) \otimes_R Q = M \otimes_R (N \otimes_R Q) = (M \otimes_R Q) \otimes_Q (N \otimes_R Q) \cong \bigoplus_{w \in \mathcal{W}} (M \otimes_R N)_Q^w$$

induced by ξ_M and ξ_N (where in the second identification we use the left Q-module structure on $N \otimes_R Q$ explained above).

The unit for this monoidal product is the object R_e with underlying graded bimodule R (with the obvious structure), objects $((R_e)^w_Q : w \in \mathcal{W})$ defined by

$$(R_e)^w_Q = \begin{cases} Q & \text{if } w = e; \\ 0 & \text{otherwise,} \end{cases}$$

and the obvious morphism ξ_R.

REMARK 3.2. The assignment $(M, (M^w_Q : w \in \mathcal{W}), \xi_M) \mapsto M$ defines a faithful functor

$$\mathsf{C}' \to R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R.$$

This functor is however *not full* in general.

3.1.3. *The category* C. Next, Abe considers the full subcategory C of C' consisting of triples $(M, (M^w_Q : w \in \mathcal{W}), \xi_M)$ such that M is finitely generated as an R-bimodule and flat as a right R-module. These conditions have the following consequence.

LEMMA 3.3. *If* $(M, (M^w_Q : w \in \mathcal{W}), \xi_M) \in \mathsf{C}$*, then* M *is a finitely generated as a left* R*-module and as a right* R*-module.*

PROOF. Since M is flat as a right R-module, the natural morphism

$$M \to M \otimes_R Q$$

is injective. Now, ξ_M allows to identify the right-hand side with $\bigoplus_{w \in \mathcal{W}} M^w_Q$, and in this sum only finitely many terms are nonzero. For any $w \in W$, the image of M in M^w_Q is finitely generated as an R-bimodule, hence as a left R-module and as a right R-module in view of (3.1). Since M embeds in a direct sum of finitely many such modules, it is also finitely generated as a left R-module and as a right R-module. □

This property implies that the monoidal product $\star$ restricts to a monoidal product on C. In fact, if $\mathcal{M} = (M, (M^w_Q : w \in \mathcal{W}), \xi_M)$ and $\mathcal{N} = (N, (N^w_Q : w \in \mathcal{W}), \xi_N)$ belong to C, then $M \otimes_R N$ is finitely generated as a right R-module because M and N are, hence a fortiori it is finitely generated as an R-bimodule. And if $X \hookrightarrow Y \twoheadrightarrow Z$ is an exact sequence of left R-modules, then so is

$$N \otimes_R X \to N \otimes_R Y \to N \otimes_R Z$$

because N is flat as a right R-module, and then so is

$$(M \otimes_R N) \otimes_R X \to (M \otimes_R N) \otimes_R Y \to (M \otimes_R N) \otimes_R Z$$

since M is flat as a right R-module; this proves that $M \otimes_R N$ is flat as a right R-module.

For any $r \in \mathbb{Z}$, the shift-of-grading functor (r) induces in the natural way an autoequivalence of C' which stabilizes C. This autoequivalence will again be denoted (r).

3.1.4. *Some objects.* For any $s \in \mathcal{S}$, we define the object $\mathsf{B}_s^{\mathrm{Abe}}$ as follows. The underlying graded R-bimodule is $R \otimes_{R^s} R(1)$. Since e and s act differently on R, there exists at most one decomposition

$$(R \otimes_{R^s} R) \otimes_R Q \cong (\mathsf{B}_s^{\mathrm{Abe}})_Q^e \oplus (\mathsf{B}_s^{\mathrm{Abe}})_Q^s$$

such that the condition (3.1) is satisfied on $(\mathsf{B}_s^{\mathrm{Abe}})_Q^e$ and $(\mathsf{B}_s^{\mathrm{Abe}})_Q^s$. To prove that such a decomposition exists, we will use the following lemma.

Lemma 3.4. *Let $\delta_s \in V^*$ be such that $\langle \delta_s, \alpha_s^\vee \rangle = 1$. For any $f \in R$, in $R \otimes_{R^s} R$ we have*

$$\begin{aligned} f \cdot (\delta_s \otimes 1 - 1 \otimes s(\delta_s)) &= (\delta_s \otimes 1 - 1 \otimes s(\delta_s)) \cdot f, \\ f \cdot (\delta_s \otimes 1 - 1 \otimes \delta_s) &= (\delta_s \otimes 1 - 1 \otimes \delta_s) \cdot s(f). \end{aligned}$$

Proof. It suffices to prove the formulas when $f \in V^*$. Moreover, using (2.10) and the fact that the formulas are obvious if $f \in (V^*)^s$, it suffices to consider the case $f = \delta_s$. Now

$$\begin{aligned} \delta_s \cdot (\delta_s \otimes 1 - 1 \otimes s(\delta_s)) &= (\delta_s^2 \otimes 1 - \delta_s \otimes s(\delta_s)) \\ &= \delta_s \otimes (\delta_s + s(\delta_s)) - 1 \otimes s(\delta_s)\delta_s - \delta_s \otimes s(\delta_s) = (\delta_s \otimes 1 - 1 \otimes s(\delta_s)) \cdot \delta_s \end{aligned}$$

where the second equality uses (2.11), and similarly we have

$$\begin{aligned} \delta_s \cdot (\delta_s \otimes 1 - 1 \otimes \delta_s) &= (\delta_s^2 \otimes 1 - \delta_s \otimes \delta_s) \\ &= \delta_s \otimes (\delta_s + s(\delta_s)) - 1 \otimes s(\delta_s)\delta_s - \delta_s \otimes \delta_s = (\delta_s \otimes 1 - 1 \otimes \delta_s) \cdot s(\delta_s), \end{aligned}$$

which proves the desired formula. □

From Lemma 2.17 we obtain that $R \otimes_{R^s} R$ is free of rank 2 as a right R-module, with a basis consisting of $(\delta_s \otimes 1)$ and $(1 \otimes 1)$. Hence $(R \otimes_{R^s} R) \otimes_R Q$ has rank 2 as a right Q-module, with the same basis. The matrix of the family

$$(\delta_s \otimes 1 - 1 \otimes s(\delta_s), \delta_s \otimes 1 - 1 \otimes \delta_s)$$

in this basis is

$$\begin{pmatrix} 1 & 1 \\ -s(\delta_s) & -\delta_s, \end{pmatrix}$$

whose determinant is $s(\delta_s) - \delta_s = -\alpha_s$, hence it is invertible; it follows that this family is also a basis, or in other words that

$$(R \otimes_{R^s} R) \otimes_R Q = (\delta_s \otimes 1 - 1 \otimes s(\delta_s)) \cdot Q \oplus (\delta_s \otimes 1 - 1 \otimes \delta_s) \cdot Q.$$

Here Lemma 3.4 shows that the first, resp. second, factor satisfies the condition required for $(\mathsf{B}_s^{\mathrm{Abe}})_Q^e$, resp. $(\mathsf{B}_s^{\mathrm{Abe}})_Q^s$. Hence we can set

$$(\mathsf{B}_s^{\mathrm{Abe}})_Q^e = (\delta_s \otimes 1 - 1 \otimes s(\delta_s)) \cdot Q, \quad (\mathsf{B}_s^{\mathrm{Abe}})_Q^s = (\delta_s \otimes 1 - 1 \otimes \delta_s) \cdot Q.$$

We will denote by u_s the vector $(1 \otimes 1) \in R \otimes_{R^s} R$.

Once these objects are defined, we can extend the definition to expressions: if $\underline{w} = (s_1, \cdots, s_r)$ is an expression we set

$$\mathsf{B}_{\underline{w}}^{\mathrm{Abe}} := \mathsf{B}_{s_1}^{\mathrm{Abe}} \star \cdots \star \mathsf{B}_{s_r}^{\mathrm{Abe}}.$$

The underlying graded R-bimodule is

$$(R \otimes_{R^{s_1}} R) \otimes_R \cdots \otimes_R (R \otimes_{R^{s_r}} R)(r) = R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_r}} R(r).$$

In case $\underline{w}$ is the empty word, this is to be interpreted as the unit object R_e. We denote by $u_{\underline{w}}$ the vector $u_{s_1} \otimes \cdots \otimes u_{s_r}$.

3.1.5. *Definition.* We can finally define[13] the category $\mathsf{D}^{\mathrm{Abe}}_{\mathrm{BS}}(\mathcal{W}, V)$ as the monoidal $\Bbbk$-linear category with

- objects the pairs $(\underline{w}, n)$ where $\underline{w}$ is an expression for $(\mathcal{W}, \mathcal{S})$ and $n \in \mathbb{Z}$;
- morphisms from $(\underline{w}, n)$ to $(\underline{w}', n')$ given by $\mathrm{Hom}_{\mathsf{C}}(\mathsf{B}^{\mathrm{Abe}}_{\underline{w}}(n), \mathsf{B}^{\mathrm{Abe}}_{\underline{w}'}(n'))$.

This category admits a natural monoidal product, defined on objects by $(\underline{w}, n) \star (\underline{w}', n') = (\underline{ww}', n + n')$, and on morphisms using the obvious identification

$$\mathsf{B}^{\mathrm{Abe}}_{\underline{w}} \star \mathsf{B}^{\mathrm{Abe}}_{\underline{y}} = \mathsf{B}^{\mathrm{Abe}}_{\underline{wy}}.$$

By construction there exists a fully faithful monoidal functor

$$\mathsf{D}^{\mathrm{Abe}}_{\mathrm{BS}}(\mathcal{W}, V) \to \mathsf{C} \tag{3.2}$$

sending $(\underline{w}, n)$ to $\mathsf{B}^{\mathrm{Abe}}_{\underline{w}}(n)$. The autoequivalence (1) of C induces an autoequivalence of $\mathsf{D}^{\mathrm{Abe}}_{\mathrm{BS}}(\mathcal{W}, V)$, again denoted (1), and defined on objects by $(\underline{w}, n)(1) = (\underline{w}, n+1)$. We will also denote by $\mathsf{D}^{\mathrm{Abe},\oplus}_{\mathrm{BS}}(\mathcal{W}, V)$ the additive hull of $\mathsf{D}^{\mathrm{Abe}}_{\mathrm{BS}}(\mathcal{W}, V)$. The functor (3.2) (and the induced functor on $\mathsf{D}^{\mathrm{Abe},\oplus}_{\mathrm{BS}}(\mathcal{W}, V)$) will usually be omitted from notation, and $\mathsf{D}^{\mathrm{Abe}}_{\mathrm{BS}}(\mathcal{W}, V)$ and $\mathsf{D}^{\mathrm{Abe},\oplus}_{\mathrm{BS}}(\mathcal{W}, V)$ will usually be identified with their images in C.

In case $\Bbbk$ is a complete local domain, we will denote by $\mathsf{D}^{\mathrm{Abe}}(\mathcal{W}, V)$ the full subcategory of C whose objects are direct sums of direct summands of objects $\mathsf{B}^{\mathrm{Abe}}_{\underline{w}}(n)$; this category identifies with the Karoubian closure of $\mathsf{D}^{\mathrm{Abe},\oplus}_{\mathrm{BS}}(\mathcal{W}, V)$. It is not difficult to show that this category is Krull–Schmidt.

3.2. Abe's assumption. In order to analyze the categories $\mathsf{D}^{\mathrm{Abe}}_{\mathrm{BS}}(\mathcal{W}, V)$ and $\mathsf{D}^{\mathrm{Abe}}(\mathcal{W}, V)$, one needs one more assumption.

Assumption 3.5. For any pair $(s,t) \in \mathcal{S}^2_\circ$, there exists a morphism

$$\mathsf{B}^{\mathrm{Abe}}_{(s,t,\cdots)} \to \mathsf{B}^{\mathrm{Abe}}_{(t,s,\cdots)}$$

in C (where each word has length $m_{s,t}$) which sends the vector $u_{(s,t,\cdots)}$ to $u_{(t,s,\cdots)}$.

In this subsection we explain how this condition can be checked in practice.

First, Abe explains in [**Ab1**] that this assumption holds if $\Bbbk$ is a field and moreover, for any s, t as above, the representation of the subgroup $\langle s,t \rangle$ on V is reflection faithful. If fact, in this case, by Theorem 1.16 one can consider the indecomposable bimodule $\mathsf{B}^{\mathrm{bim}}_{w_{s,t}}$ associated with the longest element $w_{s,t}$ in $\langle s,t \rangle$. We have embeddings as direct summands

$$\mathsf{B}^{\mathrm{bim}}_{w_{s,t}} \subset \mathsf{B}^{\mathrm{bim}}_{(s,t,\cdots)} \quad \text{and} \quad \mathsf{B}^{\mathrm{bim}}_{w_{s,t}} \subset \mathsf{B}^{\mathrm{bim}}_{(t,s,\cdots)}.$$

By (1.13), the image of $\mathsf{B}^{\mathrm{bim}}_{w_{s,t}}$ in $\mathsf{B}^{\mathrm{bim}}_{(s,t,\cdots)}$ contains $u_{(s,t,\cdots)}$, and its image in $\mathsf{B}^{\mathrm{bim}}_{(t,s,\cdots)}$ contains $u_{(t,s,\cdots)}$. As a consequence, the composition

$$\mathsf{B}^{\mathrm{bim}}_{(s,t,\cdots)} \to \mathsf{B}^{\mathrm{bim}}_{w_{s,t}} \to \mathsf{B}^{\mathrm{bim}}_{(t,s,\cdots)}$$

where the first map is a projection on the direct summand $\mathsf{B}^{\mathrm{bim}}_{w_{s,t}}$ is a morphism of graded R-bimodules sending $u_{(s,t,\cdots)}$ to $u_{(t,s,\cdots)}$. By Exercise 2.16 this morphism defines a morphism $\mathsf{B}^{\mathrm{Abe}}_{(s,t,\cdots)} \to \mathsf{B}^{\mathrm{Abe}}_{(t,s,\cdots)}$ in C, which proves the desired claim.

[13]Of course this category depends on the choice of Coxeter generators $\mathcal{S}$. However it depends only on the $\mathcal{W}$-action on V, not on the choice of roots and coroots.

A more satisfactory solution to this problem is given in [**Ab3**], whose main result is the following proposition.

PROPOSITION 3.6. *Let* $(s,t) \in \mathcal{S}_\circ^2$. *If*

$$\begin{bmatrix} m_{s,t} \\ k \end{bmatrix}_{s,t} = \begin{bmatrix} m_{s,t} \\ k \end{bmatrix}_{t,s} = 0$$

for any $k \in \{1, \cdots, m_{s,t} - 1\}$, *then there exists a morphism*

$$\mathsf{B}^{\mathrm{Abe}}_{(s,t,\cdots)} \to \mathsf{B}^{\mathrm{Abe}}_{(t,s,\cdots)}$$

in C *(where each word has length* $m_{s,t}$*) which sends the vector* $u_{(s,t,\cdots)}$ *to* $u_{(t,s,\cdots)}$.

In particular, given a realization (in the sense of §2.2.1) which satisfies Demazure surjectivity and (2.4), we obtain the data needed to define Abe's category, and Assumption 3.5 is satisfied.

3.3. The character map. If Q' is the fraction field of R, it is clear from definitions that the assignment

$$[M] \mapsto \sum_{w \in \mathcal{W}} \dim_{Q'}(M_Q^w \otimes_Q Q') \cdot w$$

defines an algebra morphism

$$[\mathsf{D}^{\mathrm{Abe},\oplus}_{\mathrm{BS}}(\mathcal{W}, V)]_\oplus \to \mathcal{H}_{(\mathcal{W},\mathcal{S})}/v \cdot \mathcal{H}_{(\mathcal{W},\mathcal{S})} = \mathbb{Z}[\mathcal{W}].$$

One of the first important results of [**Ab1**] is that this morphism can be "lifted" to an algebra morphism

$$[\mathsf{D}^{\mathrm{Abe},\oplus}_{\mathrm{BS}}(\mathcal{W}, V)]_\oplus \to \mathcal{H}_{(\mathcal{W},\mathcal{S})},$$

provided Assumption 3.5 is satisfied.

More explicitly, for $M \in \mathsf{C}$ and $w \in \mathcal{W}$ we will denote by M^w the image of M under the composition

$$M \hookrightarrow M \otimes_R Q \xrightarrow{\xi_M} \bigoplus_{y \in \mathcal{W}} M_Q^y \to M_Q^w$$

where the rightmost map is projection on the factor parametrized by w. Then M^w is a graded R-bimodule. Next, given a finitely generated free graded R-module M we denote by $\mathrm{grk}(M) \in \mathbb{Z}[v, v^{-1}]$ its graded rank, with the following normalization: if M admits a homogenous basis $(m_i : i \in I)$, then

$$\mathrm{grk}(M) = \sum_{i \in I} v^{-\deg(m_i)}$$

where $\deg(m_i)$ is the degree of m_i. With this convention we have

$$\mathrm{grk}(M(1)) = v \cdot \mathrm{grk}(M).$$

The following statement is proved in [**Ab1**, Theorem 3.4]. (For a brief discussion of the proof, see §3.4 below.)

PROPOSITION 3.7. *Suppose that Assumption 3.5 is satisfied. For any expression* $\underline{x} = (s_1, \cdots, s_r)$ *and any* $w \in \mathcal{W}$, *the graded* R*-bimodule* $(\mathsf{B}^{\mathrm{Abe}}_{\underline{x}})^w$ *is free as a graded left module. Its graded rank is the coefficient of* H_w *in*

$$\underline{H}_{\underline{x}} = \underline{H}_{s_1} \cdots \underline{H}_{s_r}.$$

Proposition 3.7 allows to define a map

$$\mathrm{ch}_{\mathrm{Abe}} : [\mathsf{D}^{\mathrm{Abe},\oplus}_{\mathrm{BS}}(\mathcal{W}, V)]_{\oplus} \to \mathcal{H}_{(\mathcal{W},\mathcal{S})}$$

by the formula

$$\mathrm{ch}_{\mathrm{Abe}}([M]) = \sum_{w \in \mathcal{W}} \mathrm{grk}(M^w) \cdot H_w,$$

where we omit the functor (3.2). This proposition also implies that

$$\mathrm{ch}_{\mathrm{Abe}}([\mathsf{B}^{\mathrm{Abe}}_{\underline{x}}(n)]) = v^n \cdot \underline{H}_{\underline{x}}$$

for any $n \in \mathbb{Z}$ and any expression $\underline{x}$. Since the classes $[\mathsf{B}^{\mathrm{Abe}}_{\underline{x}}(n)]$ generate the $\mathbb{Z}$-module $[\mathsf{D}^{\mathrm{Abe},\oplus}_{\mathrm{BS}}(\mathcal{W}, V)]_{\oplus}$, it follows that $\mathrm{ch}_{\mathrm{Abe}}$ is an algebra morphism.

3.4. Indecomposable objects and categorification theorem. In this subsection we assume that Assumption 3.5 is satisfied, and that $\Bbbk$ is a complete local domain. The next important result of [**Ab1**] is a classification of indecomposable objects in $\mathsf{D}^{\mathrm{Abe}}(\mathcal{W}, V)$ under these assumptions.

THEOREM 3.8. *For any $w \in \mathcal{W}$ there exists a unique indecomposable object $\mathsf{B}^{\mathrm{Abe}}_w \in \mathsf{D}^{\mathrm{Abe}}(\mathcal{W}, V)$ which satisfies*

$$(\mathsf{B}^{\mathrm{Abe}}_w)^x \neq 0 \quad \Rightarrow \quad x \leq w$$

and $\mathsf{B}^{\mathrm{Abe}}_w \cong R(\ell(w))$ as left R-modules. Moreover:

- *the assignment $(w, n) \mapsto \mathsf{B}^{\mathrm{Abe}}_w(n)$ defines a bijection between $\mathcal{W} \times \mathbb{Z}$ and the set of isomorphism classes of indecomposable objects in $\mathsf{D}^{\mathrm{Abe}}(\mathcal{W}, V)$;*
- *for any reduced expression $\underline{w}$ for an element $w \in \mathcal{W}$, there exist nonnegative integers $c^w_{y,n}$ such that*

$$\mathsf{B}^{\mathrm{Abe}}_{\underline{w}} \cong \mathsf{B}^{\mathrm{Abe}}_w \oplus \bigoplus_{\substack{y \in \mathcal{W}, y < w \\ n \in \mathbb{Z}}} \mathsf{B}^{\mathrm{Abe}}_y(n)^{\oplus c^w_{y,n}}$$

and $c^w_{y,n} = c^w_{y,-n}$ for any $n \in \mathbb{Z}$.

From this theorem, we easily deduce the following analogue of Corollary 1.18 (see [**Ab1**, Theorem 4.3]).

COROLLARY 3.9. *The morphism $\mathrm{ch}_{\mathrm{Abe}}$ is an isomorphism. Moreover, for any $w \in \mathcal{W}$ we have*

$$\mathrm{ch}_{\mathrm{Abe}}([\mathsf{B}^{\mathrm{Abe}}_w]) \in H_w + \sum_{y < w} \mathbb{Z}_{\geq 0}[v, v^{-1}] \cdot H_y. \tag{3.3}$$

The proofs of Proposition 3.7 and Theorem 3.8 rely on the construction of a certain family of morphisms which adapts to this setting the construction of Libedinsky's "light leaves basis," see [**Li1**]. In particular Abe proves in this way that, for any expressions $\underline{w}$ and $\underline{y}$, the space

$$\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{D}^{\mathrm{Abe}}(\mathcal{W},V)}(\mathsf{B}^{\mathrm{Abe}}_{\underline{w}}, \mathsf{B}^{\mathrm{Abe}}_{\underline{y}}(n))$$

is free as a graded left R-module and as a graded right R-module, he gives a formula for its graded rank (see [**Ab1**, Theorem 4.6]), and constructs an explicit basis (see [**Ab1**, Theorem 5.5]).

REMARK 3.10. (1) The obvious analogue of Lemma 1.21 holds in the category $\mathsf{D}^{\mathrm{Abe}}(\mathcal{W}, V)$ under our present assumptions, with the same proof.
(2) In [**Ab2**], Abe develops an analogue of part of the theory of singular Soergel bimodules from [**W1**] (see Remarks 1.19 and 2.27).

3.5. Relation with the Elias–Williamson category. Consider a noetherian integral domain $\Bbbk$, and a balanced realization

$$\big(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S})\big)$$

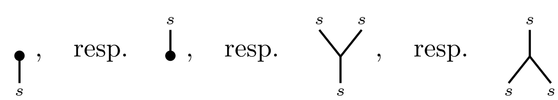
of $(\mathcal{W}, \mathcal{S})$ over $\Bbbk$ in the sense of §2.2.1. We assume furthermore that this realization satisfies Demazure surjectivity, together with the technical conditions considered in §2.4. We can therefore consider the category $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$, and also the category $\mathsf{D}_{\mathrm{BS}}^{\mathrm{Abe}}(\mathcal{W}, V)$,

The following result is [**Ab3**, Theorem 3.15].

THEOREM 3.11. *Under the assumptions above, there exists a canonical equivalence of monoidal categories*

$$\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V) \xrightarrow{\sim} \mathsf{D}_{\mathrm{BS}}^{\mathrm{Abe}}(\mathcal{W}, V).$$

The proof of Theorem 3.11 is similar to that of Theorem 2.40. It proceeds in two steps. First, one needs to define a monoidal functor from $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ to $\mathsf{D}_{\mathrm{BS}}^{\mathrm{Abe}}(\mathcal{W}, V)$. This functor will send $\mathsf{B}_{\underline{w}}(n)$ to $\mathsf{B}_{\underline{w}}^{\mathrm{Abe}}(n)$ for any expression $\underline{w}$ and $n \in \mathbb{Z}$; one therefore only needs to specify the images of the generating morphisms, and verify that these images satisfy the required relations. As for Theorem 3.11, the image of a polynomial is multiplication by this polynomial on R, and the image of



is given by

$$f \otimes g \mapsto fg, \quad \text{resp.} \quad f \mapsto f\delta_s \otimes 1 - f \otimes s(\delta_s),$$
$$\text{resp.} \quad f \otimes g \mapsto f \otimes 1 \otimes g, \quad \text{resp.} \quad f \otimes g \otimes h \mapsto f\partial_s(g) \otimes h$$

for $f, g, h \in R$, where δ_s is as in Lemma 3.4. (To justify that these morphisms indeed define morphisms in $\mathsf{D}_{\mathrm{BS}}^{\mathrm{Abe}}(\mathcal{W}, V)$, one can work in the full subcategory $\mathsf{D}_{\mathrm{BS}}^{\mathrm{Abe}}(\{e, s\}, V)$ and apply Exercise 2.16.) For $(s,t) \in \mathcal{S}_\circ^2$, the image of the $2m_{s,t}$-valent morphism attached to (s,t) is the morphism constructed in the proof of Proposition 3.6 in [**Ab3**]. The verification that these morphisms satisfy the required relations is essentially done in [**EW3**]; see [**Ab3**, Lemma 3.14] for details. (The verification of the Zamolodchikov relation partly relies on computer computations; see [**EW3**, §3.5].) Finally, one needs to prove that this functor is an equivalence of categories; this follows from the fact that by construction it sends the bases of morphism spaces in $\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V)$ considered in §2.10 to the similar bases in $\mathsf{D}_{\mathrm{BS}}^{\mathrm{Abe}}(\mathcal{W}, V)$ mentioned in §3.4.

Of course, in case $\Bbbk$ is a complete local domain, the equivalence of Theorem 3.11 induces an equivalence of monoidal categories

$$\mathsf{D}(\mathcal{W}, V) \xrightarrow{\sim} \mathsf{D}^{\mathrm{Abe}}(\mathcal{W}, V)$$

which, for any $w \in \mathcal{W}$, sends the object B_w to $\mathsf{B}_w^{\mathrm{Abe}}$.

Remark 3.12. Composing the equivalence of Theorem 3.11 with the functor of Remark 3.2 we obtain a monoidal functor

$$\mathsf{D}_{\mathrm{BS}}(\mathcal{W}, V) \to R\text{-}\mathsf{Mod}^{\mathbb{Z}}\text{-}R$$

sending, for any expression $\underline{w} = (s_1, \cdots, s_r)$, the object $\mathsf{B}_{\underline{w}}$ to the graded bimodule

$$R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_r}} R(r),$$

which solves the problem mentioned in Remark 2.41 under our present assumptions.

CHAPTER 3

Parity complexes

The formalism of parity complexes is due to Juteau–Mautner–Williamson, see [**JMW2**]. This formalism is extremely flexible, and can be adapted in many different settings; it is however difficult to explain it in a generality that encompasses all the known applications. Here we will present this theory in a setting that covers essentially all the sheaf-theoretic contexts where this formalism has found applications. For a presentation in a more abstract setting, which applies to different situations but not to all the contexts considered here, see e.g. [**AR4**].

The considerations in this chapter will involve the theory of perverse sheaves. For a very nice study of this theory (together with some of its main application to Representation Theory) we refer to the excellent book [**Ac**].

1. Motivation: Bott–Samelson sheaves on flag varieties

Before developing the general theory, we explain how one can compute the dimensions of stalks of intersection cohomology complexes with rational coefficients using parity considerations. This result was first proved by Kazhdan–Lusztig [**KL2**], but the presentation here follows Springer [**Sp1**]. This example was one of the motivations for developing the general theory of parity complexes.

1.1. Characters of Bruhat constructible sheaves on flag varieties. We fix a complex connected reductive algebraic group $\mathscr{G}$, and choose a Borel subgroup $\mathscr{B} \subset \mathscr{G}$ and a maximal torus $\mathscr{T} \subset \mathscr{B}$. (More generally, all the considerations in this section apply when $\mathscr{G}$ is a Kac–Moody group over $\mathbb{C}$; see REF below for details.) We will denote by $\mathcal{W} = N_{\mathscr{G}}(\mathscr{T})/\mathscr{T}$ the Weyl group of $(\mathscr{G}, \mathscr{T})$ and by $\mathcal{S} \subset \mathcal{W}$ the system of Coxeter generators associated with the choice of $\mathscr{B}$.

The *flag variety* of $\mathscr{G}$ is the projective complex algebraic variety

$$\mathscr{X} := \mathscr{G}/\mathscr{B}.$$

The Bruhat decomposition $\mathscr{G} = \bigsqcup_{w \in \mathcal{W}} \mathscr{B}w\mathscr{B}$ induces a stratification

$$\mathscr{X} = \bigsqcup_{w \in \mathcal{W}} \mathscr{X}_w \quad \text{with} \quad \mathscr{X}_w := \mathscr{B}w\mathscr{B}/\mathscr{B} \simeq \mathbb{A}^{\ell(w)}_{\mathbb{C}} \text{ for } w \in \mathcal{W}. \tag{1.1}$$

(Here ℓ is the length function in $\mathcal{W}$, with respect to the system of Coxeter generators $\mathcal{S}$).

Let $\Bbbk$ be a field, and let

$$D^{\mathrm{b}}_{(\mathscr{B})}(\mathscr{X}, \Bbbk)$$

be the derived category of Bruhat-constructible complexes of $\Bbbk$-sheaves on $\mathscr{X}$; in other words the full subcategory of the bounded derived category of the category of sheaves of $\Bbbk$-vector spaces on $\mathscr{X}$ consisting of complexes $\mathcal{F}$ such that the sheaf

$\mathsf{H}^i(\mathcal{F}_{|\mathscr{B}w\mathscr{B}/\mathscr{B}})$ is constant (or, equivalently, locally constant) for any $w \in \mathcal{W}$ and $i \in \mathbb{Z}$.

Recall the Hecke algebra attached to $(\mathcal{W}, \mathcal{S})$, see Definition 4.1 in Chapter 1. Out of objects of $D^{\mathrm{b}}_{(\mathscr{B})}(\mathscr{X}, \Bbbk)$ one can construct interesting elements in $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ as follows: for $\mathcal{F} \in D^{\mathrm{b}}_{(\mathscr{B})}(\mathscr{X}, \Bbbk)$ we set

$$\mathrm{ch}(\mathcal{F}) = \sum_{\substack{w \in W \\ k \in \mathbb{Z}}} \dim_{\Bbbk}\big(\mathsf{H}^{-\ell(w)-k}(\mathcal{F}_{w\mathscr{B}})\big) \cdot v^k H_w \quad \in \mathcal{H}_{(\mathcal{W},\mathcal{S})}$$

(where $\mathcal{F}_{w\mathscr{B}}$ denotes the stalk of the complex $\mathcal{F}$ at the point in $\mathscr{X}$ associated with w, a complex of $\Bbbk$-vector spaces). Note that $\mathrm{ch}(\mathcal{F}[1]) = v\,\mathrm{ch}(\mathcal{F})$ for any $\mathcal{F}$ in $D^{\mathrm{b}}_{(\mathscr{B})}(\mathscr{X}, \Bbbk)$.

1.2. Computation for Bott–Samelson sheaves. For $s \in \mathcal{S}$, we will denote by $\mathscr{P}_s \subset \mathscr{G}$ the associated minimal standard parabolic subgroup, and consider the associated partial flag variety

$$\mathscr{X}^s := \mathscr{G}/\mathscr{P}_s.$$

If we set $\mathcal{W}^s := \{w \in \mathcal{W} \mid \ell(ws) > \ell(w)\}$, then $\mathcal{W}^s$ is a set of representatives for the quotient $\mathcal{W}/\{e, s\}$, and the Bruhat decomposition provides a stratification

$$\mathscr{X}^s = \bigsqcup_{w \in \mathcal{W}^s} \mathscr{X}^s_w \quad \text{with} \quad \mathscr{X}^s_w := \mathscr{B}w\mathscr{P}_s/\mathscr{P}_s \simeq \mathbb{A}^{\ell(w)}_{\mathbb{C}} \text{ for } w \in \mathcal{W}^s.$$

We will denote by

$$D^{\mathrm{b}}_{(\mathscr{B})}(\mathscr{X}^s, \Bbbk)$$

the derived category of complexes of sheaves of $\Bbbk$-vector spaces on $\mathscr{X}^s$ constructible with respect to this stratification.

The natural projection morphism $\pi_s : \mathscr{X} \to \mathscr{X}^s$ induces (derived) functors

$$(\pi_s)_* : D^{\mathrm{b}}_{(\mathscr{B})}(\mathscr{X}, \Bbbk) \to D^{\mathrm{b}}_{(\mathscr{B})}(\mathscr{X}^s, \Bbbk), \quad (\pi_s)^* : D^{\mathrm{b}}_{(\mathscr{B})}(\mathscr{X}^s, \Bbbk) \to D^{\mathrm{b}}_{(\mathscr{B})}(\mathscr{X}, \Bbbk).$$

(We also have !-versions of these functors, but we have canonical identifications $(\pi_s)_* = (\pi_s)_!$ and $(\pi_s)^! = (\pi_s)^*[2]$ since π_s is proper and smooth.) For $s_1, \cdots, s_n \in \mathcal{S}$, we set

$$\mathcal{E}(s_1, \cdots, s_n) = (\pi_{s_n})^*(\pi_{s_n})_* \cdots (\pi_{s_1})^*(\pi_{s_1})_* \underline{\Bbbk}_{\mathscr{X}_e}[n],$$

We will call such complexes the *Bott–Samelson sheaves.*

Proposition 1.1. *For any $s_1, \cdots, s_n \in \mathcal{S}$ and $w \in \mathcal{W}$, we have*

$$\mathsf{H}^i\big(\mathcal{E}(s_1, \cdots, s_n)_{w\mathscr{B}}\big) = 0 \quad \textit{unless} \quad i \equiv n \pmod 2.$$

Moreover, we have

$$\mathrm{ch}\big(\mathcal{E}(s_1, \cdots, s_n)\big) = \underline{H}_{s_1} \cdots \underline{H}_{s_n} = (H_{s_1} + v) \cdots (H_{s_n} + v).$$

Proposition 1.1 is a direct consequence of the next lemma.

Lemma 1.2. *Let $\mathcal{F} \in D^{\mathrm{b}}_{(\mathscr{B})}(\mathscr{X}, \Bbbk)$ be such that $\mathsf{H}^k(\mathcal{F}) = 0$ unless k is even, and let $s \in \mathcal{S}$. Then $\mathsf{H}^k((\pi_s)^*(\pi_s)_*\mathcal{F}) = 0$ unless k is even, and*

$$\mathrm{ch}\big((\pi_s)^*(\pi_s)_*\mathcal{F}\big) = \mathrm{ch}(\mathcal{F}) \cdot v^{-1}\underline{H}_s.$$

PROOF. For $y \in \mathcal{W}$, we have

$$\mathsf{H}^k\big(((\pi_s)^*(\pi_s)_*\mathcal{F})_{y\mathscr{B}}\big) = \mathsf{H}^k\big(((\pi_s)_*\mathcal{F})_{y\mathscr{P}_s}\big) = \mathsf{H}^k\left(\pi_s^{-1}(y\mathscr{P}_s), \mathcal{F}_{|\pi_s^{-1}(y\mathscr{P}_s)}\right).$$

We distinguish two cases.

First case: $ys > y$. Fix $g \in \mathscr{P}_s$. Then we have

$$yx\mathscr{B} \in \begin{cases} \mathscr{B}ys\mathscr{B}/\mathscr{B} & \text{if } x \notin \mathscr{B}; \\ \mathscr{B}y\mathscr{B}/\mathscr{B} & \text{if } x \in \mathscr{B}. \end{cases}$$

Now, $\pi_s^{-1}(y\mathscr{P}_s) = \{yg\mathscr{B} : g \in \mathscr{P}_s\} \simeq \mathscr{P}_s/\mathscr{B} \simeq \mathbb{P}^1_{\mathbb{C}}$. We use the long exact sequence associated with the standard distinguished triangle

$$j_!j^* \to \mathrm{id} \to i_*i^* \xrightarrow{[1]}$$

for the decomposition of $\mathscr{P}_s/\mathscr{B}$ into the closed subset $\mathscr{B}/\mathscr{B}$ (whose embedding in $\mathscr{P}_s/\mathscr{B}$ is denoted i) and the open subset $\mathscr{P}_s/\mathscr{B} \smallsetminus (\mathscr{B}/\mathscr{B})$ (whose embedding in $\mathscr{P}_s/\mathscr{B}$ is denoted j, and which we identify with $\mathbb{A}^1_{\mathbb{C}}$) to obtain an exact sequence

$$\cdots \to \mathsf{H}^k_c(\mathbb{A}^1, \mathcal{F}_{|\mathbb{A}^1_{\mathbb{C}}}) \to \mathsf{H}^k\big(((\pi_s)^*(\pi_s)_*\mathcal{F})_{y\mathscr{B}}\big) \to \mathsf{H}^k(\mathrm{pt}, \mathcal{F}_{|\mathrm{pt}}) \to \cdots.$$

Note that $\mathcal{F}_{|\mathbb{A}^1_{\mathbb{C}}}$, resp. $\mathcal{F}_{|\mathrm{pt}}$, is constant with value $\mathcal{F}_{ys\mathscr{B}}$, resp. $\mathcal{F}_{y\mathscr{B}}$. Now we have $\mathsf{H}^k_c(\mathbb{A}^1_{\mathbb{C}}, \mathcal{F}_{|\mathbb{A}^1_{\mathbb{C}}}) \simeq \mathsf{H}^{k-2}(\mathcal{F}_{ys\mathscr{B}})$ because

$$\mathsf{H}^k_c(\mathbb{A}^1, \underline{\Bbbk}_{\mathbb{A}^1_{\mathbb{C}}}) = \begin{cases} \Bbbk & \text{if } k = 2; \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we have $\mathsf{H}^k(\mathrm{pt}, \mathcal{F}_{|\mathrm{pt}}) \simeq \mathsf{H}^k(\mathcal{F}_{y\mathscr{B}})$ and hence

$$\dim \mathsf{H}^k\big(((\pi_s)^*(\pi_s)_*\mathcal{F})_{y\mathscr{B}}\big) = \begin{cases} 0 & \text{if } k \text{ is odd;} \\ \dim \mathsf{H}^{k-2}(\mathcal{F}_{ys\mathscr{B}}) + \dim \mathsf{H}^k(\mathcal{F}_{y\mathscr{B}}) & \text{if } k \text{ is even.} \end{cases}$$

Second case: $ys < y$. In this case also we have $\pi_s^{-1}(y\mathscr{P}_s) = \pi_s^{-1}(ys\mathscr{P}_s) = \{ysg\mathscr{B} : g \in \mathscr{P}_s\} \simeq \mathscr{P}_s/\mathscr{B} \simeq \mathbb{P}^1$, with

$$ysg\mathscr{B} \in \begin{cases} \mathscr{B}y\mathscr{B}/\mathscr{B} & \text{if } x \notin \mathscr{B} \\ \mathscr{B}ys\mathscr{B}/\mathscr{B} & \text{if } x \in \mathscr{B}. \end{cases}$$

The same considerations as above show that we have

$$\dim \mathsf{H}^k\big(((\pi_s)^*(\pi_s)^*\mathcal{F})_{y\mathscr{B}}\big) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \dim \mathsf{H}^{k-2}(\mathcal{F}_{y\mathscr{B}}) + \dim \mathsf{H}^k(\mathcal{F}_{ys\mathscr{B}}) & \text{if } k \text{ is even.} \end{cases}$$

Now we consider the Hecke algebra side. One can easily check that

$$H_w(v^{-1}H_s + 1) = \begin{cases} v^{-1}H_{ws} + H_w & \text{if } ws > w; \\ v^{-2}H_w + v^{-1}H_{ws} & \text{if } ws < w. \end{cases}$$

Using this fact, one sees that $\mathrm{ch}(\mathcal{F}) \cdot (H_s + v)v^{-1}$ is equal to

$$\left(\sum_{\substack{w\in\mathcal{W}\\k\in\mathbb{Z}}} \dim \mathsf{H}^{-\ell(w)-k}(\mathcal{F}_{w\mathscr{B}})v^k H_w\right) \cdot (v^{-1}H_s + 1)$$
$$= \sum_{\substack{w\in\mathcal{W}\\ws>w\\k\in\mathbb{Z}}} \Big(\dim \mathsf{H}^{-\ell(w)-k}(\mathcal{F}_{w\mathscr{B}}) \cdot v^{k-1}H_{ws} + \dim \mathsf{H}^{-\ell(w)-k}(\mathcal{F}_{w\mathscr{B}}) \cdot v^k H_w\Big)$$
$$+ \sum_{\substack{w\in\mathcal{W}\\ws<w\\k\in\mathbb{Z}}} \Big(\dim \mathsf{H}^{-\ell(w)-k}(\mathcal{F}_{w\mathscr{B}}) \cdot v^{k-2}H_{w} + \dim \mathsf{H}^{-\ell(w)-k}(\mathcal{F}_{w\mathscr{B}}) \cdot v^{k-1} H_{ws}\Big)$$
$$= \sum_{\substack{y\in\mathcal{W}\\ys>y\\k\in\mathbb{Z}}} \Big(\dim \mathsf{H}^{-\ell(y)-k}(\mathcal{F}_{y\mathscr{B}})v^k + \dim \mathsf{H}^{-\ell(ys)-k}(\mathcal{F}_{ys\mathscr{B}})v^{k-1}\Big) \cdot H_y$$
$$+ \sum_{\substack{y\in\mathcal{W}\\ys<y\\k\in\mathbb{Z}}} \Big(\dim \mathsf{H}^{-\ell(y)-k}(\mathcal{F}_{y\mathscr{B}})v^{k-2} + \dim \mathsf{H}^{-\ell(ys)-k}(\mathcal{F}_{ys\mathscr{B}})v^{k-1}\Big) \cdot H_y$$
$$= \sum_{\substack{y\in\mathcal{W}\\ys>y\\j\in\mathbb{Z}}} \Big(\dim \mathsf{H}^{-\ell(y)-j}(\mathcal{F}_{y\mathscr{B}}) + \dim \mathsf{H}^{-\ell(y)-j-2}(\mathcal{F}_{y\mathscr{B}})\Big) \cdot v^j H_y$$
$$+ \sum_{\substack{y\in\mathcal{W}\\ys<y\\j\in\mathbb{Z}}} \Big(\dim \mathsf{H}^{-\ell(y)-j-2}(\mathcal{F}_{y\mathscr{B}}) + \dim \mathsf{H}^{-\ell(y)-j}(\mathcal{F}_{ys\mathscr{B}})\Big) \cdot v^j H_y,$$

which coincides with $\mathrm{ch}\big((\pi_s)^*(\pi_s)^*\mathcal{F}\big)$ by the above calculations. □

1.3. Application: computation of stalks of characteristic-0 intersection cohomology complexes on $\mathscr{X}$. In this subsection we choose $\Bbbk = \mathbb{Q}$. For $w \in \mathcal{W}$, we consider the simple perverse $\mathbb{Q}$-sheaf

$$\mathcal{IC}_w := j_{w!*}(\underline{\mathbb{Q}}_{\mathscr{X}_w}[\ell(w)]) \in \mathsf{Perv}_{(\mathscr{B})}(\mathscr{X}, \mathbb{Q}),$$

where $j_w : \mathscr{X}_w \hookrightarrow \mathscr{X}$ denotes the embedding.

The main result of this section is the following.

THEOREM 1.3. *We have $\mathsf{H}^k(\mathcal{IC}_w) = 0$ unless $k \equiv \ell(w) \pmod 2$. Moreover, for any $w \in \mathcal{W}$ we have*

$$\mathrm{ch}(\mathcal{IC}_w) = \underline{H}_w.$$

PROOF. The stalks condition in the characterization of intersection cohomology complexes shows that $\mathrm{ch}(\mathcal{IC}_w) \in H_w + \sum_{y<x} v\mathbb{Z}[v]H_y$. Below we will prove the parity vanishing condition and the fact that $\mathrm{ch}(\mathcal{IC}_w)$ is self-dual with respect to the Kazhdan–Lusztig involution; together, these facts will show that $\mathrm{ch}(\mathcal{IC}_w)$ satisfies the properties that characterize $\underline{H}_w$.

For any $s \in \mathcal{S}$, since the morphism π_s is smooth of relative dimension 1, with connected fibers, the functor $(\pi_s)^*$ sends intersection cohomology complexes to intersection cohomology complexes, see [**BBD**, p. 110]. On the other hand, since π_s is proper the functor $(\pi_s)^*$ sends intersection cohomology complexes to direct sums of cohomological shifts of intersection cohomology complexes[1] by the Decomposition Theorem, see [**BBD**, Théorème 6.2.5]. (Here we use our assumption that the field of coefficients has characteristic 0.)

Now, choose a reduced expression $w = s_1 \cdots s_n$. The considerations above show that $\mathcal{E}(s_1, \cdots, s_n)$ is a direct sum of cohomological shifts of intersection cohomology complexes. We have

$$\mathcal{E}(s_1, \cdots, s_n)_{|\mathscr{X}_w} \simeq \underline{\mathbb{Q}}_{\mathscr{X}_w}[\ell(w)],$$

and $\mathcal{E}(s_1, \cdots, s_n)$ is supported on $\overline{\mathscr{X}_w}$; hence $\mathcal{IC}_w$ is a direct summand of the complex $\mathcal{E}(s_1, \cdots, s_n)$. In view of Proposition 1.1, this shows that $\mathsf{H}^k(\mathcal{IC}_w) = 0$ unless k has the parity of $\ell(w)$.

Finally we show by induction on w that $\mathrm{ch}(\mathcal{IC}_w)$ is self-dual with respect to the Kazhdan–Lusztig involution. This claim is obvious if $w = e$. Now we assume that $\ell(w) > 0$, and that $\mathrm{ch}(\mathcal{IC}_y)$ is self-dual for any $y \in \mathcal{W}$ such that $y < w$. Once again we choose a reduced expression $w = s_1 \cdots s_n$. Proposition 1.1 shows that $\mathrm{ch}(\mathcal{E}(s_1, \ldots, s_n))$ is self dual. Now if $\mathbb{D}$ is the Verdier duality functor on $D^{\mathrm{b}}_{(\mathscr{B})}(\mathscr{X}, \mathbb{Q})$, then we have

$$\mathbb{D} \circ (\pi_s)_* = (\pi_s)_! \circ \mathbb{D} \cong (\pi_s)_* \circ \mathbb{D} \quad \text{and} \quad \mathbb{D} \circ (\pi_s)^* = (\pi_s)^! \circ \mathbb{D} \cong (\pi_s)^* \circ \mathbb{D}[2];$$

we deduce that

$$\mathbb{D}(\mathcal{E}(s_1, \cdots, s_n)) \cong \mathcal{E}(s_1, \cdots, s_n).$$

As explained above the complex $\mathcal{E}(s_1, \ldots, s_n)$ is a direct sum of $\mathcal{IC}_w$ and objects of the form $\mathcal{IC}_y[k]$ with $k \in \mathbb{Z}$ and $y \in \mathcal{W}$ satisfying $y < w$. By Verdier self-duality, for each such y and k the multiplicity of $\mathcal{IC}_y[k]$ as a direct summand of $\mathcal{E}(s_1, \ldots, s_n)$ is equal to that of $\mathcal{IC}_y[-k]$. This implies that $\mathrm{ch}(\mathcal{IC}_w)$ is self-dual, and finishes the proof. □

2. Parity complexes

2.1. Preliminaries. We start with some general considerations in Homological Algebra. Let $\Bbbk$ be a field, and let D be a $\Bbbk$-linear triangulated category endowed with a bounded t-structure whose heart will be denoted A, and whose cohomology functors will be denoted H. Let X be an object of A, and denote by $\langle X \rangle_\Delta$ the triangulated subcategory of D generated by X.

LEMMA 2.1. *Assume that*

$$\mathrm{End}(X) = \Bbbk \quad \textit{and} \quad \mathrm{Hom}(X, X[1]) = 0.$$

Then for Y in D the following conditions are equivalent:

(1) Y belongs to $\langle X \rangle_\Delta$;
(2) for any $n \in \mathbb{Z}$, the object $\mathsf{H}^n(Y)$ is isomorphic to a direct sum of copies of X.

PROOF. Using appropriate truncation triangles, one can easily check by induction on the cardinality of $\{n \in \mathbb{Z} \mid \mathsf{H}^n(Y) \neq 0\}$ that if each $\mathsf{H}^n(Y)$ is isomorphic to a direct sum of copies of X, then Y belongs to $\langle X \rangle_\Delta$. To prove the converse, it suffices to prove that if Y is an object such that each $\mathsf{H}^n(Y)$ is isomorphic to a direct sum of copies of X, and if we are given a distinguished triangle

$$Y \to Z \to X[m] \xrightarrow{[1]}$$

[1]An object of this form is called a *semisimple complex.*

for some $Z \in \mathsf{D}$ and $m \in \mathbb{Z}$, then each $\mathsf{H}^n(Z)$ is isomorphic to a direct sum of copies of X. The long exact sequence of cohomology associated with this triangle shows that

$$\mathsf{H}^n(Z) \cong \mathsf{H}^n(Y)$$

unless $n \in \{-m, -m+1\}$. For these values of n, there is therefore nothing to prove. Now, consider the following portion of this long exact sequence:

$$0 \to \mathsf{H}^{-m}(Y) \to \mathsf{H}^{-m}(Z) \to X \xrightarrow{f} \mathsf{H}^{-m+1}(Y) \to \mathsf{H}^{-m+1}(Z) \to 0.$$

If $f = 0$, then $\mathsf{H}^{-m+1}(Y) \cong \mathsf{H}^{-m+1}(Z)$ and $\mathsf{H}^{-m}(Z)$ is an extension of X by $\mathsf{H}^{-m}(Y)$. By assumption $\mathsf{H}^{-m}(Y)$ is a direct sum of copies of X; since $\mathrm{Hom}(X, X[1])$ we then have $\mathsf{H}^{-m}(Z) \cong \mathsf{H}^{-m}(Y) \oplus X$, so that the desired condition holds. On the other hand if $f \neq 0$, then since $\mathsf{H}^{-m+1}(Y)$ is a direct sum of copies of X and since $\mathrm{End}(X) = \Bbbk$, f is the embedding of a direct summand. Then we have $\mathsf{H}^{-m}(Z) \cong \mathsf{H}^{-m}(Y)$, and $\mathsf{H}^{-m+1}(Z)$ is isomorphic to a direct sum of copies of X. The desired condition is again satisfied in this case, which finishes the proof. □

We continue with the setting above.

Lemma 2.2. *Assume that*

$$\mathrm{End}(X) = \Bbbk \quad \textit{and} \quad \mathrm{Hom}(X, X[2n+1]) = 0 \textit{ for any } n \in \mathbb{Z}_{\geq 0}.$$

Then for Y in D the following conditions are equivalent:

(1) Y belongs to $\langle X \rangle_\Delta$ and $\mathsf{H}^m(Y) = 0$ for any $m \in \mathbb{Z}$ odd;
(2) there exist even integers $n_1, \cdots, n_r$ and an isomorphism

$$Y \cong \bigoplus_{i=1}^{r} X[n_i].$$

Proof. It is clear that if Y is isomorphic to a direct sum of even cohomological shifts of X, then Y belongs to $\langle X \rangle_\Delta$ and $\mathsf{H}^m(Y) = 0$ for any $m \in \mathbb{Z}$ odd. Conversely, we will prove that induction on the cardinality of $\{n \in \mathbb{Z} \mid \mathsf{H}^n(Y) \neq 0\}$ that if Y belongs to $\langle X \rangle_\Delta$ and satisfies $\mathsf{H}^m(Y) = 0$ for any $m \in \mathbb{Z}$ odd, then Y is isomorphic to a direct sum of even cohomological shifts of X. First, if this cardinality is 0 then $Y = 0$ and there is nothing to prove. Now, assume that this set is nonempty, and choose m maximal such that $\mathsf{H}^m(Y) \neq 0$. Then m is even, and we have a truncation triangle

$$Z \to Y \to \mathsf{H}^m(Y)[-m] \xrightarrow{[1]}$$

such that

$$\mathsf{H}^n(Z) = \begin{cases} \mathsf{H}^n(Y) & \text{if } n \neq -m; \\ 0 & \text{if } n = -m. \end{cases}$$

By induction, Z is then isomorphic to a direct sum of even shifts of X. Moreover, by Lemma 2.1 $\mathsf{H}^m(Y)$ is a direct sum of copies of X. Since $\mathrm{Hom}(X, X[n]) = 0$ for any n odd the transition morphism $\mathsf{H}^m(Y)[-m] \to Z[1]$ is our distinguished triangle vanishes; this implies that

$$Y \cong Z \oplus \mathsf{H}^m(Y)[-m],$$

and finishes the proof. □

2.2. Geometric setting. We consider an algebraically closed field $\mathbb{F}$, and an $\mathbb{F}$-algebraic variety X. We assume we are given a decomposition

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

where Λ is a finite set, each X_λ is a smooth connected locally closed subvariety in X, and for any $\lambda \in \Lambda$ the closure $\overline{X}_\lambda$ is a union of strata X_μ with $\mu \in \Lambda$. For any $\lambda \in \Lambda$, we will denote by j_λ the embedding of X_λ in X.

We will consider another field $\Bbbk$, and some categories of sheaves $\mathsf{D}(Y, \Bbbk)$ for each locally closed union of strata $Y \subset X$. The various settings we want to consider are the following. (For concrete examples in each of these settings, see Section 3 below.)

(1) (Analytic setting) Here $\mathbb{F} = \mathbb{C}$, $\Bbbk$ is arbitrary, and $\mathsf{D}(Y, \Bbbk)$ denotes the constructible derived category of $\Bbbk$-sheaves on Y with respect to the analytic topology.
(2) (Étale setting) Here $\mathbb{F}$ is arbitrary, $\Bbbk$ is a either a finite field of characteristic different from $\mathrm{char}(\mathbb{F})$ or a finite extension of $\mathbb{Q}_\ell$ for some prime number $\ell \neq \mathrm{char}(\mathbb{F})$, and $\mathsf{D}(Y, \Bbbk)$ denotes the constructible derived category of étale $\Bbbk$-sheaves on Y.
(3) (Equivariant analytic setting) Here $\mathbb{F}$ and $\Bbbk$ are as in (1), but we assume we are given an affine $\mathbb{C}$-algebraic group H acting on X and stabilizing each X_λ, and $\mathsf{D}(Y, \Bbbk)$ denotes the H-equivariant constructible derived category of $\Bbbk$-sheaves on Y (in the sense of Bernstein–Lunts [**BL**]) with respect to the analytic topology.
(4) (Equivariant étale setting) Here $\mathbb{F}$ and $\Bbbk$ are as in (2), but we assume we are given an affine $\mathbb{F}$-algebraic group H acting on X and stabilizing each X_λ, and $\mathsf{D}(Y, \Bbbk)$ denotes the H-equivariant constructible derived category of étale $\Bbbk$-sheaves on Y (in the sense of Bernstein–Lunts [**BL**]).

Recall that an additive category C is called *Krull–Schmidt* if any object has a decomposition as a direct sum of indecomposable objects with local endomorphism rings. Such a category is Karoubian (in other words, each idempotent splits), and any object admits a unique (up to isomorphisms and permutations of the factors) decomposition as a direct sum of indecomposable objects. Moreover, an object is indecomposable if and only if its endomorphism ring is Krull–Schmidt. It is noted in [**CYZ**, Corollary A.2] that if C is a $\Bbbk$-linear additive category (for some field bk) such that $\mathrm{Hom}(X, Y)$ is finite-dimensional for any objects X, Y, then C is Krull–Schmidt if and only if it is Karoubian. Since, on the other hand, a triangulated category which admits a bounded t-structure is Karoubian (by the main result of [**LC**]), in each of the settings above the categories $\mathsf{D}(Y, \Bbbk)$ are Krull–Schmidt.

In each of the settings considered above, for any $\lambda \in \Lambda$ we have (derived) functors

$$(j_\lambda)_*, (j_\lambda)_! : \mathsf{D}(X_\lambda, \Bbbk) \to \mathsf{D}(X, \Bbbk), \qquad (j_\lambda)^*, (j_\lambda)^! : \mathsf{D}(X, \Bbbk) \to \mathsf{D}(X_\lambda, \Bbbk).$$

We will additionally assume we are given, for any $\lambda \in \Lambda$, a local system $\mathcal{L}_\lambda$ on X_λ (assumed to be H-equivariant in settings (3) and (4)) such that

$$\mathrm{End}_{\mathsf{D}(X_\lambda, \Bbbk)}(\mathcal{L}_\lambda) = \Bbbk \text{ and } \mathrm{Hom}_{\mathsf{D}(X_\lambda, \Bbbk)}(\mathcal{L}_\lambda, \mathcal{L}_\lambda[2n+1]) = 0 \text{ for any } n \in \mathbb{Z}_{\geq 0}. \tag{2.1}$$

We will then set

$$\Delta_\lambda := (j_\lambda)_! \mathcal{L}_\lambda[\dim(X_\lambda)], \quad \nabla_\lambda := (j_\lambda)_* \mathcal{L}_\lambda[\dim(X_\lambda)].$$

REMARK 2.3. In the cases we will consider, the local systems $\mathcal{L}_\lambda$ will always have rank 1. In this case, the assumption (2.1) amounts to requiring that $\mathsf{H}^n(X_\lambda; \Bbbk) = 0$ for all odd integers n, where we consider:

- the ordinary (singular) cohomology in setting (1);
- the étale cohomology in setting (2);
- the H-equivariant cohomology in setting (3);
- the H-equivariant étale cohomology in setting (4).

In the first two cases, this condition is automatic if X_λ admits a paving by affine spaces. (In fact this condition is well known to guarantee that $\mathsf{H}^n_c(X_\lambda; \Bbbk) = 0$ for odd n's, and then one concludes by Poincaré duality, since X_λ is smooth.) In the last two cases, this condition holds e.g. if both $\mathsf{H}^n_H(\mathrm{pt}; \Bbbk)$ and $\mathsf{H}^n(X_\lambda, \Bbbk)$ vanish for odd n's. (To justify this one uses the standard spectral sequence

$$E_2^{p,q} = \mathsf{H}^p_H(\mathrm{pt}; \Bbbk) \otimes \mathsf{H}^q(X_\lambda; \Bbbk) \Rightarrow \mathsf{H}^{p+q}_H(X_\lambda, \Bbbk),$$

which degenerates since it vanishes "like a chessboard.") As above the second condition holds if X_λ admits a paving by affine spaces. The first condition holds if H is a torus, or if H is reductive and $\mathrm{char}(\Bbbk)$ avoids a few prime numbers (see [**JMW2**, §2.6] for details), or if H is a semidirect product of a group isomorphic (as a variety) to an affine space and a group which satisfies these conditions.

We will make the following additional assumption:

$$\text{for any } \lambda, \mu \in \Lambda \text{ we have } (j_\mu)^* \nabla_\lambda \in \langle \mathcal{L}_\mu \rangle_\Delta, \tag{2.2}$$

where we use the notation of §2.1 with respect to the triangulated category $\mathsf{D}(X_\mu, \Bbbk)$. Then, for any locally closed union of strata $Y \subset X$ we will denote by $\mathsf{D}_\Lambda(Y, \Bbbk)$ the triangulated subcategory of $\mathsf{D}(Y, \Bbbk)$ consisting of objects $\mathcal{F}$ such that $\mathcal{F}_{|X_\mu} \in \langle \mathcal{L}_\mu \rangle_\Delta$ for any $\mu \in \Lambda$. (Of course this subcategory depends on the choice of local systems $\mathcal{L}_\lambda$ and not only on the stratification, although this is not apparent in the notation.) With this notation, our assumption means that each ∇_λ belongs to $\mathsf{D}_\Lambda(X, \Bbbk)$. In fact it is not difficult to check that $\mathsf{D}_\Lambda(X, \Bbbk)$ is the triangulated subcategory of $\mathsf{D}(X, \Bbbk)$ generated by the objects $(\nabla_\lambda : \lambda \in \Lambda)$, and also the triangulated subcategory of $\mathsf{D}(X, \Bbbk)$ generated by the objects $(\Delta_\lambda : \lambda \in \Lambda)$. It is clear also that if the assumption (2.2) is satisfied, then the similar assumption is satisfied with any locally closed union of strata $Y \subset X$ (with respect to the stratification by strata contained in Y, and the local systems $\mathcal{L}_\lambda$ associated with these strata), and that $\mathsf{D}_\Lambda(Y, \Bbbk)$ is the same when considered with respect to the data relative to X or those relative to Y. One can also check that for any locally closed unions of strata $Y, Z \subset X$ with $Z \subset Y$, if we denote by $j : Y \to Z$ the embeddings then the functors j_*, $j_!$, j^* and $j^!$ induce functors

$$j_*, j_! : \mathsf{D}_\Lambda(Z, \Bbbk) \to \mathsf{D}_\Lambda(Y, \Bbbk), \qquad j^*, j^! : \mathsf{D}_\Lambda(Y, \Bbbk) \to \mathsf{D}_\Lambda(Z, \Bbbk).$$

Recall that for any $\lambda \in \Lambda$ the category $\mathsf{D}(X_\lambda, \Bbbk)$ is Krull–Schmidt. Our assumption (2.1) implies in particular that $\mathcal{L}_\lambda$ is indecomposable; it follows that a direct summand of an object which is a direct sum of copies if $\mathcal{L}_\lambda$ is itself a direct sum of copies of $\mathcal{L}_\lambda$; in view of Lemma 2.1 this shows that the subcategory $\mathsf{D}_\Lambda(X_\lambda, \Bbbk) \subset \mathsf{D}(X_\lambda, \Bbbk)$ is stable under direct summands, and then that the subcategory $\mathsf{D}_\Lambda(X, \Bbbk) \subset \mathsf{D}(X, \Bbbk)$ is stable under direct summands. We deduce that $\mathsf{D}_\Lambda(X, \Bbbk)$ is also a Krull–Schmidt category.

REMARK 2.4. Verdier duality will not restrict to an autoequivalence of the category $\mathsf{D}_\Lambda(X, \Bbbk)$ unless each $\mathcal{L}_\lambda$ is self-dual. However, in the general setting, if (2.1) is satisfied for a collection of local systems $(\mathcal{L}_\lambda : \lambda \in \Lambda)$, then it will also be satisfied for the collection $(\mathcal{L}_\lambda^\vee : \lambda \in \Lambda)$ where $\mathcal{L}^\vee$ is the local system dual to $\mathcal{L}$. We can then also consider the category $\mathsf{D}_{\Lambda,\mathrm{dual}}(X, \Bbbk)$ defined using the same stratification, but this new collection of localy systems, and $\mathbb{D}$ will induces equivalences

$$\mathsf{D}_\Lambda(X, \Bbbk) \xrightarrow{\sim} \mathsf{D}_{\Lambda,\mathrm{dual}}(X, \Bbbk), \quad \mathsf{D}_{\Lambda,\mathrm{dual}}(X, \Bbbk) \xrightarrow{\sim} \mathsf{D}_\Lambda(X, \Bbbk)$$

which will again be denoted $\mathbb{D}$.

2.3. Parity complexes. We consider one of the settings introduced in §2.2, assuming that conditions (2.1) and (2.2) hold.

We can now state the definition of the parity complexes, what are our main objects of study in this chapter.

DEFINITION 2.5. Let $\mathcal{F} \in \mathsf{D}_\Lambda(X, \Bbbk)$.

(1) $\mathcal{F}$ is said to be $*$-*even*, resp. $*$-*odd*, if for all $\lambda \in \Lambda$ we have

$$\mathsf{H}^n(j_\lambda^* \mathcal{F}) = 0$$

unless n is even, resp. odd.

(2) $\mathcal{F}$ is said to be $!$-*even*, resp. $!$-*odd*, if for all $\lambda \in \Lambda$ we have

$$\mathsf{H}^n(j_\lambda^! \mathcal{F}) = 0$$

unless n is even, resp. odd.

(3) $\mathcal{F}$ is said to be *even* if it is both $*$-even and $!$-even, and *odd* if it is both $*$-odd and $!$-odd.

(4) $\mathcal{F}$ is called a *parity complex* if it is isomorphic to the direct sum of an even object and an odd object.

It is clear that a direct summand of an even, resp. odd, resp. parity, complex is again even, resp. odd, resp. parity. Since the category $\mathsf{D}_\Lambda(X, \Bbbk)$ is Krull–Schmidt (see §2.2), this implies that its full subcategory $\mathsf{Parity}_\Lambda(X, \Bbbk)$ whose objects are the parity complexes is again Krull–Schmidt.

The following statement gathers some basic properties of parity complexes. (Here, Verdier duality should be interpreted in the sense of Remark 2.4: it takes values in a different category, but where the parity formalism still applies.)

LEMMA 2.6. *Let $\mathcal{F}$ in $\mathsf{D}_\Lambda(X, \Bbbk)$.*

(1) If $|\Lambda| = 1$, then the following are equivalent:

(a) $\mathcal{F}$ is $*$-even.

(b) $\mathcal{F}$ is $!$-even.

(c) $\mathcal{F}$ is even.

(d) $\mathcal{F}$ is a direct sum of objects $\mathcal{L}_\lambda[n]$ with n even, where λ is the only element in Λ.

Moreover, if $\mathcal{F}, \mathcal{G}$ are even, for $n \in \mathbb{Z}$ we have $\mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{F}, \mathcal{G}[n]) = 0$ unless n is even.

(2) $\mathcal{F}$ is $!$-even, resp. $!$-odd, if and only if $\mathbb{D}(\mathcal{F})$ is $*$-even, resp. $*$-odd. In particular, $\mathcal{F}$ is a parity complex if and only if $\mathbb{D}(\mathcal{F})$ is a parity complex.

(3) $\mathcal{F}$ is even, resp. odd, if and only if $\mathcal{F}[1]$ is odd, resp. even. In particular, $\mathcal{F}$ is a parity complex if and only if $\mathcal{F}[1]$ is a parity complex.

(4) $\mathcal{F}$ *is even if and only if* $\mathsf{H}^n(\mathcal{F}) = \mathsf{H}^n(\mathbb{D}(\mathcal{F})) = 0$ *for all odd integers* n.

PROOF. (1) The equivalence between the first three assertions is clear. The equivalence with the fourth one follows from Lemma 2.2. The concluding statement is clear from the fourth description and our assumption (2.1).

(2) We treat the even case; the odd case is similar. By (1), $\mathcal{F}$ is !-even if and only if for any $\lambda \in \Lambda$ the object $j_\lambda^! \mathcal{F}$ is a direct sum of objects $\mathcal{L}_\lambda[n]$ with n even. Now, if we denote by $\mathbb{D}_\lambda$ the Verdier duality functor in $\mathsf{D}(X_\lambda, \Bbbk)$, since $\mathbb{D}_\lambda(\mathcal{L}_\lambda) \cong \mathcal{L}_\lambda^\vee[2\dim(X_\lambda)]$, the latter condition is equivalent to the condition that for any $\lambda \in \Lambda$ the object $\mathbb{D}_\lambda(j_\lambda^! \mathcal{F})$ is a direct sum of objects $\mathcal{L}_\lambda^\vee[n]$ with n even. Since

$$\mathbb{D}_\lambda(j_\lambda^! \mathcal{F}) \cong j_\lambda^* \mathbb{D}(\mathcal{F}),$$

this proves the desired equivalence.

(3) This property is clear from definitions.

(4) A sheaf $\mathcal{G}$ on X is 0 iff $j_\lambda^*(\mathcal{G}) = 0$ for any $\lambda \in \Lambda$. Since $\mathsf{H}^n(j_\lambda^* \mathcal{F}) \cong j_\lambda^* \mathsf{H}^n(\mathcal{F})$ for any n, we deduce that $\mathcal{F}$ is $*$-even if and only iff $\mathsf{H}^n(\mathcal{F}) = 0$ for all odd integers n. We conclude using (2). □

We now state some immediate properties of compatibility with pushforwards and pullbacks.

LEMMA 2.7. *Let* $Y \subset X$ *be a locally closed union of strata and* $f : Y \hookrightarrow X$ *be the embedding.*

(1) If Y *is closed and* $\mathcal{F} \in \mathsf{D}_\Lambda(Y, \Bbbk)$ *is parity, then* $f_* \mathcal{F} \in \mathsf{D}_\Lambda(X, \Bbbk)$ *is parity.*
(2) If Y *is open and* $\mathcal{F} \in \mathsf{D}_\Lambda(X, \Bbbk)$ *is parity then* $f^* \mathcal{F} \in \mathsf{D}_\Lambda(Y, \Bbbk)$ *is parity.*

PROOF. (1) Let $\lambda \in \Lambda$. First, assume that $X_\lambda \subset X$ and denote by j_λ the embedding of X_λ in Y. Applying the base change theorem to the cartesian diagram

$$\begin{tikzcd} X_\lambda \arrow[r, "j'_\lambda"] \arrow[d, equal] & Y \arrow[d, "f"] \\ X_\lambda \arrow[r, "j_\lambda"] & X \end{tikzcd}$$

we see that $j_\lambda^! f_* \mathcal{F} \cong (j'_\lambda)^! \mathcal{F}$. Similarly, since $f_* = f_!$ we have $j_\lambda^* f_* \mathcal{F} \cong (j'_\lambda)^* \mathcal{F}$.

If X_λ is not contained in Y, since j_λ factors through the embedding of the open complement to Y we have $j_\lambda^! f_* \mathcal{F} = j_\lambda^* f_* \mathcal{F} = 0$.

These descriptions show that if $\mathcal{F}$ is $*$-even, resp. $*$-odd, resp. !-even, resp. !-odd, then so is $f_* \mathcal{F}$, which implies our claim.

(2) For any $\lambda \in \Lambda$ such that $X_\lambda \subset Y$, if we denote by j'_λ the embedding of X_λ in Y we have

$$(j'_\lambda)^* f^* \mathcal{F} \cong (f \circ j'_\lambda)^* \mathcal{F} \cong j_\lambda^* \mathcal{F},$$

which shows that if $\mathcal{F}$ is $*$-even, resp. $*$-odd, then so is $f^* \mathcal{F}$. Similarly, since $f^* \cong f^!$ we see that

$$(j'_\lambda)^! f^* \mathcal{F} \cong j_\lambda^! \mathcal{F},$$

which shows that if $\mathcal{F}$ is !-even, resp. !-odd, then so is $f^* \mathcal{F}$. The desired claim follows. □

2.4. Morphisms spaces between parity complexes. If $Y \subset X$ is a locally closed union of strata, for $\mathcal{F}, \mathcal{G} \in \mathsf{D}_\Lambda(Y, \Bbbk)$, we set

$$\mathrm{Hom}^\bullet_{\mathsf{D}_\Lambda(Y,\Bbbk)}(\mathcal{F}, \mathcal{G}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{D}_\Lambda(Y,\Bbbk)}(\mathcal{F}, \mathcal{G}[n]).$$

Depending on the context, this space will be considered either as a plain vector space, or as a graded vector space (with the grading provided by the right-hand description.) The following statement is not difficult, but turns out to be crucial for the study of parity complexes.

PROPOSITION 2.8. *Let $\mathcal{F}, \mathcal{G} \in \mathsf{D}_\Lambda(X, \Bbbk)$. If $\mathcal{F}$ is a direct sum of a $*$-even and a $*$-odd object, and if $\mathcal{G}$ is a direct sum of a $!$-even and a $!$-odd object, then there exists a (non-canonical) isomorphism of graded vector spaces*

$$\mathrm{Hom}^\bullet_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{F}, \mathcal{G}) \simeq \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}^\bullet_{\mathsf{D}_\Lambda(X_\lambda,\Bbbk)}(j_\lambda^* \mathcal{F}, j_\lambda^! \mathcal{G}).$$

PROOF. It suffices to prove the claim in case $\mathcal{F}$ is $*$-even and $\mathcal{G}$ is $!$-even. We proceed by induction on the number of strata contained in the support[2] of $\mathcal{F}$. Of course, if this number is 0 we have $\mathcal{F} = 0$, and there is nothing to prove.

Let Y be the support of $\mathcal{F}$, and let $X_\mu \subset Y$ be an open stratum. Let $j : X \smallsetminus (Y \smallsetminus X_\mu) \hookrightarrow X$ be the (open) embedding, and let i be the embedding of the complementary closed subvariety. We consider the associated distinguished triangle

$$j_! j^! \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \xrightarrow{[1]} .$$

Here we have $j_! j^! \mathcal{F} = j_! j^* \mathcal{F} = (j_\mu)_! j_\mu^* \mathcal{F}$. Applying the functor $\mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(-, \mathcal{G})$ we deduce a long exact sequence

$$\begin{aligned}(2.3) \qquad \cdots \to \mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(i_* i^* \mathcal{F}, \mathcal{G}[n]) &\to \mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{F}, \mathcal{G}[n]) \\ &\to \mathrm{Hom}_{\mathsf{D}_\Lambda(X_\mu,\Bbbk)}((j_\mu)_! j_\mu^* \mathcal{F}, \mathcal{G}[n]) \to \dots,\end{aligned}$$

where

$$\mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}((j_\mu)_! j_\mu^* \mathcal{F}, \mathcal{G}[n]) \cong \mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(j_\mu^* \mathcal{F}, j_\mu^! \mathcal{G}[n])$$

vanishes unless n is even (see Lemma 2.6(1)). By the induction hypothesis (applied to $i_* i^* \mathcal{F}$) we have

$$(2.4) \qquad \mathrm{Hom}^\bullet_{\mathsf{D}_\Lambda(X,\Bbbk)}(i_* i^* \mathcal{F}, \mathcal{G}) \simeq \bigoplus_{\substack{\lambda \in \Lambda \\ \lambda \neq \mu}} \mathrm{Hom}^\bullet_{\mathsf{D}_\Lambda(X_\lambda,\Bbbk)}(j_\lambda^* \mathcal{F}, j_\lambda^! \mathcal{G});$$

in particular, this graded vector space is concentrated in even degrees.These facts imply that the long exact sequence (2.3) breaks into short exact sequences

$$\begin{aligned}0 \to \mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(i_* i^* \mathcal{F}, \mathcal{G}[n]) &\to \mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{F}, \mathcal{G}[n]) \\ &\to \mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}((j_\mu)_! j_\mu^* \mathcal{F}, \mathcal{G}[n]) \to 0\end{aligned}$$

for any $n \in \mathbb{Z}$. It follows that

$$\mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{F}, \mathcal{G}[n]) = 0$$

if n is odd, and that

$$\mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{F}, \mathcal{G}[n]) \cong \mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(i_* i^* \mathcal{F}, \mathcal{G}[n]) \oplus \mathrm{Hom}_{\mathsf{D}_\Lambda(X_\mu,\Bbbk)}(j_\mu^* \mathcal{F}, j_\mu^! \mathcal{G}[n])$$

[2]By the *support* of a complex $\mathcal{H}$ we mean the closure of the union of the strata X_ν such that $j_\nu^* \mathcal{H} \neq 0$. In particular, this support is a *closed* union of strata.

if n is even. Using (2.4) once again, we deduce the isomorphism of the proposition. □

Proposition 2.8 has the following consequences.

COROLLARY 2.9. *(1) Let $\mathcal{F}, \mathcal{G} \in \mathsf{D}_\Lambda(X, \Bbbk)$. If $\mathcal{F}$ is $*$-even and $\mathcal{G}$ is $!$-odd, then* $\mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{F}, \mathcal{G}) = 0$.

(2) Let $\mathcal{F}, \mathcal{G} \in \mathsf{D}_\Lambda(X, \Bbbk)$ be parity complexes. Let $U \subset X$ be an open union of strata, and denote by $j : U \hookrightarrow X$ the embedding. Then the morphism

$$\mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathsf{D}_\Lambda(U,\Bbbk)}(j^*\mathcal{F}, j^*\mathcal{G})$$

induced by the functor j^* is surjective.

(3) Let $\mathcal{F} \in \mathsf{D}_\Lambda(X, \Bbbk)$ be an indecomposable parity complex. Let $U \subset X$ be an open union of strata, and denote by $j : U \hookrightarrow X$ the embedding. Then $j^*\mathcal{F}$ is either 0 or indecomposable.

PROOF. (1) By Proposition 2.8 we have

$$\mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{F}, \mathcal{G}) \simeq \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}_{\mathsf{D}_\Lambda(X_\lambda,\Bbbk)}(j_\lambda^*\mathcal{F}, j_\lambda^!\mathcal{G}).$$

Here for any $\lambda \in \Lambda$ the object $j_\lambda^*\mathcal{F}$ is even, and the object $j_\lambda^!\mathcal{G}$ is odd. Hence $\mathrm{Hom}_{\mathsf{D}_\Lambda(X_\lambda,\Bbbk)}(j_\lambda^*\mathcal{F}, j_\lambda^!\mathcal{G}) = 0$ by Lemma 2.6(1), which implies the desired vanishing.

(2) We can assume that $\mathcal{F}$ and $\mathcal{G}$ are even. Let $i : X \smallsetminus U \hookrightarrow X$ be the closed embedding. Then we have a distinguished triangle

$$\mathcal{G} \to j_*j^*\mathcal{G} \to i_!i^!\mathcal{G}[1] \xrightarrow{[1]},$$

where $i_!i^!\mathcal{G}$ is $!$-odd. Hence we get and exact sequence

$$\mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{F}, j_*j^*\mathcal{G}) \to \mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{F}, i_!i^!\mathcal{G}[1]).$$

Here the third term vanishes by (1), hence the first arrow is surjective. By adjunction, this morphism identifies with the morphism of the lemma, hence the claim is proved.

(3) By (2), the morphism

$$\mathrm{End}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{F}) \to \mathrm{End}_{\mathsf{D}_\Lambda(U,\Bbbk)}(j^*\mathcal{F})$$

induced by j^* is surjective. Since $\mathcal{F}$ is indecomposable, the left-hand side is a local ring. Hence the right-hand side is either 0 or a local ring, which implies the claim. □

2.5. Classification theorem – unicity. We can finally state the classification theorem for parity complexes.

THEOREM 2.10. *For each $\lambda \in \Lambda$, there exists at most one (up to isomorphism) indecomposable parity complex $\mathcal{E}_\lambda$ supported on $\overline{X_\lambda}$ and such that*

$$\mathcal{E}_{\lambda|X_\lambda} \cong \mathcal{L}_\lambda[\dim(X_\lambda)].$$

Moreover, any indecomposable parity complex is isomorphic to $\mathcal{E}_\lambda[n]$ for some unique $(\lambda, n) \in \Lambda \times \mathbb{Z}$.

PROOF. We first prove the unicity of $\mathcal{E}_\lambda$. Assume that we have two indecomposable parity complexes $\mathcal{E}_\lambda$ and $\mathcal{E}'_\lambda$ supported on $\overline{X_\lambda}$ and such that

$$\mathcal{E}_{\lambda|X_\lambda} \cong \mathcal{L}_\lambda[\dim(X_\lambda)] \cong \mathcal{E}'_{\lambda|X_\lambda}.$$

By Corollary 2.9(2), restriction induces a surjective morphism

$$\mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{E}_\lambda, \mathcal{E}'_\lambda) \twoheadrightarrow \mathrm{Hom}_{\mathsf{D}_\Lambda(X_\lambda,\Bbbk)}(\mathcal{E}_{\lambda|X_\lambda}, \mathcal{E}'_{\lambda|X_\lambda}) \cong \Bbbk.$$

In other words, there exists a morphism $f : \mathcal{E}_\lambda \to \mathcal{E}'_\lambda$ whose restriction to X_λ is an isomorphism. Similarly, there exists a morphism $g : \mathcal{E}'_\lambda \to \mathcal{E}_\lambda$ whose restriction to X_λ is an isomorphism. Then the element $g \circ f$ of the local ring $\mathrm{End}(\mathcal{E}_\lambda)$ is not nilpotent (since its restriction to X_λ is not nilpotent), hence is invertible. Similarly, $f \circ g$ is invertible. Hence f and g are isomorphisms, which proves that $\mathcal{E}_\lambda \cong \mathcal{E}'_\lambda$.

Now, let us prove that all indecomposable parity complexes are of the form $\mathcal{E}_\lambda[n]$. Let $\mathcal{F}$ be an indecomposable parity complex, and let Y be its support. First, we claim that there exists a unique $\lambda \in \Lambda$ such that X_λ is open in Y. Indeed, if X_λ and X_μ are distinct strata which are open in Y, then $X_\lambda \cup X_\mu$ is open in Y and the object $\mathcal{F}_{|X_\lambda \cup X_\mu} = \mathcal{F}_{|X_\lambda} \oplus \mathcal{F}_{|X_\mu}$ would be decomposable, contradicting Corollary 2.9(3). Then Y is the closure of X_λ, and Corollary 2.9(3) shows that $\mathcal{F}_{|X_\lambda}$ is indecomposable. In view of Lemma 2.6(1), this implies that $\mathcal{F}_{|X_\lambda} \cong \mathcal{L}_\lambda[\dim(X_\lambda) + n]$ for some $n \in \mathbb{Z}$; then the unicity of $\mathcal{E}_\lambda$ (already proved above) implies that $\mathcal{F} \cong \mathcal{E}_\lambda[n]$. □

REMARK 2.11.
(1) Let $\lambda \in \Lambda$, and assume that the indecomposable parity complex $\mathcal{E}_\lambda \in \mathsf{D}_\Lambda(X, \Bbbk)$ from Theorem 2.10 exists. Then the indecomposable parity complex $\mathcal{E}^\vee_\lambda \in \mathsf{D}_{\Lambda,\mathrm{dual}}(X, \Bbbk)$ associated with λ also exists, and we have $\mathbb{D}(\mathcal{E}_\lambda) \cong \mathcal{E}^\vee_\lambda$. (In fact, this follows from the unicity claim in Theorem 2.10.)
(2) The existence of $\mathcal{E}_\lambda$ might be a subtle question in general. In particular, there are examples where these objects do *not* exist; see [**JMW2**, §2.3.4]. In Setting (1) of §2.2, and if each X_λ is contractible (which forces $\mathcal{L}_\lambda \cong \underline{\Bbbk}_{X_\lambda}$ for any λ), then existence is guaranteed by [**JMW2**, Corollary 2.28].
(3) The objects $\mathcal{E}_\lambda$ are called *parity sheaves* in [**JMW2**]. In these notes we will avoid this terminology, which can sometimes be misleading.

2.6. Some comparison results. We will now show that the formalism of parity complexes is compatible in the most natural way with extension of scalars. We consider a field extension $\Bbbk \to \Bbbk'$. (If we are in Settings (2) or (4) of §2.2, we assume that this extension is finite.) Then we have a functor

$$\Bbbk' \otimes_\Bbbk (-) : \mathsf{D}(X, \Bbbk) \to \mathsf{D}(X, \Bbbk'),$$

such that the natural morphism

$$\Bbbk' \otimes_\Bbbk \mathrm{Hom}_{\mathsf{D}(X,\Bbbk)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathsf{D}(X,\Bbbk')}(\Bbbk' \otimes_\Bbbk \mathcal{F}, \Bbbk' \otimes_\Bbbk \mathcal{G})$$

is an isomorphism for any $\mathcal{F}, \mathcal{G}$ in $\mathsf{D}(X, \Bbbk)$. In fact there exists a similar functor for any locally closed union of strata in X, and these functors are compatible (in the obvious way) with pushforward and pullback under locally closed embeddings.

If we are given a collection $(\mathcal{L}_\lambda : \lambda \in \Lambda)$ of local systems which satisfy the assumptions (2.1) and (2.2) for the category $\mathsf{D}(X, \Bbbk)$, then the collection $(\Bbbk' \otimes_\Bbbk \mathcal{L}_\lambda : \lambda \in \Lambda)$ also satisfies these assumptions for the category $\mathsf{D}(X, \Bbbk')$. We can therefore consider the categories $\mathsf{D}_\lambda(X, \Bbbk)$ and $\mathsf{D}_\Lambda(X, \Bbbk')$, and the functor $\Bbbk' \otimes_\Bbbk (-)$ restricts to a functor

$$\mathsf{D}_\Lambda(X, \Bbbk) \to \mathsf{D}_\Lambda(X, \Bbbk'),$$

which will be denoted similarly. It is clear that this functor sends $*$-even complexes to $*$-even complexes, and similarly for all the notions introduced in Definition 2.5.

Proposition 2.12. *Let $\lambda \in \Lambda$, and assume that the object $\mathcal{E}_\lambda^{\Bbbk} \in \mathsf{D}_\Lambda(X, \Bbbk)$ associated to λ as in Theorem 2.10 exists. Then the object $\mathcal{E}_\lambda^{\Bbbk'} \in \mathsf{D}_\Lambda(X, \Bbbk')$ associated to λ as in Theorem 2.10 exists, and moreover we have*

$$\mathcal{E}_\lambda^{\Bbbk'} \cong \Bbbk' \otimes_{\Bbbk} \mathcal{E}_\lambda.$$

Proof. It is clear that $\Bbbk' \otimes_{\Bbbk} \mathcal{E}_\lambda$ is a parity complex supported on $\overline{X_\lambda}$, and that its restriction to X_λ is $\Bbbk' \otimes_{\Bbbk} \mathcal{L}_\lambda$. What remains to be proved is that this object is indecomposable. Now restriction induces a surjective $\Bbbk$-algebra morphism

$$\mathrm{End}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{E}_\lambda) \to \mathrm{End}_{\mathsf{D}_\Lambda(X_\lambda,\Bbbk)}(\mathcal{E}_{\lambda|X_\lambda}) = \Bbbk.$$

The kernel of this morphism is the unique maximal ideal in $\mathrm{End}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathcal{E}_\lambda)$; it therefore consists of nilpotent elements. Tensoring with $\Bbbk'$, we deduce a surjective $\Bbbk'$-algebra morphism

$$\mathrm{End}_{\mathsf{D}_\Lambda(X,\Bbbk')}(\Bbbk' \otimes_{\Bbbk} \mathcal{E}_\lambda) \to \Bbbk'$$

whose kernel consists of nilpotent elements. It follows that an element in the ring $\mathrm{End}_{\mathsf{D}_\Lambda(X,\Bbbk')}(\Bbbk' \otimes_{\Bbbk} \mathcal{E}_\lambda)$ is either nilpotent or invertible, hence that this ring is local. This shows that $\Bbbk' \otimes_{\Bbbk} \mathcal{E}_\lambda$ is indeed indecomposable, which finishes the proof. $\square$

We will now study the compatibility of the parity formalism with with forgetting the equivariance. We assume we are in settings (3) or (4) of §2.2, but we will also consider the corresponding non-equivariant category (which falls into setting (1) or (2) respectively). The equivariant category will be denoted $\mathsf{D}_H(X, \Bbbk)$, and the non-equivariant one will be denoted $\mathsf{D}(X, \Bbbk)$; we then have a canonical forgetful functor

$$\mathsf{For}^H : \mathsf{D}_H(X, \Bbbk) \to \mathsf{D}(X, \Bbbk),$$

and a similar functor for each locally closed union of strata in X.

We assume we are given a collection $(\mathcal{L}_\lambda : \lambda \in \Lambda)$ of local systems which satisfy the assumptions (2.1) and (2.2) for the category $\mathsf{D}_H(X, \Bbbk)$, and moreover that for any $\lambda \in \Lambda$ the morphism

$$\mathrm{Hom}^\bullet_{\mathsf{D}_H(X_\lambda,\Bbbk)}(\mathcal{L}_\lambda, \mathcal{L}_\lambda) \to \mathrm{Hom}^\bullet_{\mathsf{D}(X_\lambda,\Bbbk)}(\mathsf{For}^H(\mathcal{L}_\lambda), \mathsf{For}^H(\mathcal{L}_\lambda))$$

is surjective. Then the collection $(\mathsf{For}^H(\mathcal{L}_\lambda) : \lambda \in \Lambda)$ satisfies the assumptions (2.1) and (2.2) for the category $\mathsf{D}(X, \Bbbk)$. We can therefore consider the categories $\mathsf{D}_{H,\Lambda}(X, \Bbbk)$ and $\mathsf{D}_\Lambda(X, \Bbbk)$. The functor For^H restricts to a functor

$$\mathsf{D}_{H,\Lambda}(X, \Bbbk) \to \mathsf{D}_\Lambda(X, \Bbbk)$$

which will be denoted similarly. This functors sends $*$-even complexes to $*$-even complexes, and similarly for all the notions introduced in Definition 2.5.

Lemma 2.13. *Let $\mathcal{F}, \mathcal{G} \in \mathsf{D}_{H,\Lambda}(X, \Bbbk)$. If $\mathcal{F}$ is a direct sum of a $*$-even and a $*$-odd object, and if $\mathcal{G}$ is a direct sum of a $!$-even and a $!$-odd object, then the morphism*

$$\mathrm{Hom}_{\mathsf{D}_{H,\Lambda}(X,\Bbbk)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathsf{D}_\Lambda(X,\Bbbk)}(\mathsf{For}^H(\mathcal{F}), \mathsf{For}^H(\mathcal{G}))$$

is surjective.

Proof. It suffices to prove the claim in case $\mathcal{F}$ is $*$-even and $\mathcal{G}$ is $!$-even. We proceed by induction on the number of strata contained in the support of $\mathcal{F}$. Of course, if this number is 0 we have $\mathcal{F} = 0$, and there is nothing to prove.

Let Y be the support of $\mathcal{F}$, and let $X_\mu \subset Y$ be an open stratum. Let $j : X \smallsetminus (Y \smallsetminus X_\mu) \hookrightarrow X$ be the (open) embedding, and let i be the embedding of the complementary closed subvariety. Considering the associated distinguished triangle

$$j_! j^! \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \xrightarrow{[1]},$$

as in the proof of Proposition 2.8 we obtain exact sequences

$$\mathrm{Hom}_{\mathsf{D}_{H,\Lambda}(X,\Bbbk)}(i_* i^* \mathcal{F}, \mathcal{G}) \hookrightarrow \mathrm{Hom}_{\mathsf{D}_{H,\Lambda}(X,\Bbbk)}(\mathcal{F}, \mathcal{G}) \twoheadrightarrow \mathrm{Hom}_{\mathsf{D}_{H,\Lambda}(X,\Bbbk)}(j_! j^! \mathcal{F}, \mathcal{G})$$

and

$$\begin{aligned} \mathrm{Hom}_{\mathsf{D}_{\Lambda}(X,\Bbbk)}(i_* i^* \mathsf{For}^H(\mathcal{F}), \mathsf{For}^H(\mathcal{G})) \hookrightarrow \mathrm{Hom}_{\mathsf{D}_{H,\Lambda}(X,\Bbbk)}(\mathsf{For}^H(\mathcal{F}), \mathsf{For}^H(\mathcal{G})) \\ \twoheadrightarrow \mathrm{Hom}_{\mathsf{D}_{H,\Lambda}(X,\Bbbk)}(j_! j^! \mathsf{For}^H(\mathcal{F}), \mathsf{For}^H(\mathcal{G})) \end{aligned}$$

The functor For^H provides a morphism from the first of these exact sequences to the second one. By induction the morphism relating the first terms is surjective. Now by adjunction we have isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathsf{D}_{H,\Lambda}(X,\Bbbk)}(j_! j^! \mathcal{F}, \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{D}_{H,\Lambda}(X,\Bbbk)}(j^! \mathcal{F}, j^! \mathcal{G}), \\ \mathrm{Hom}_{\mathsf{D}_{\Lambda}(X,\Bbbk)}(j_! j^! \mathcal{F}, \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{D}_{\Lambda}(X,\Bbbk)}(j^! \mathcal{F}, j^! \mathcal{G}). \end{aligned}$$

Using Lemma 2.6(1), we see that our assumption implies that the morphism relating the third terms in our exact sequence is also surjective. By the four lemma, we deduce that the morphism relating the second terms is surjective, which finishes the proof. □

Using Lemma 2.13 we deduce the following claim.

PROPOSITION 2.14. *Let $\lambda \in \Lambda$, and assume that the object $\mathcal{E}_\lambda^H \in \mathsf{D}_{H,\Lambda}(X,\Bbbk)$ associated to λ as in Theorem 2.10 exists. Then the object $\mathcal{E}_\lambda \in \mathsf{D}_{\Lambda}(X,\Bbbk)$ associated to λ as in Theorem 2.10 exists, and moreover we have*

$$\mathcal{E}_\lambda \cong \mathsf{For}^H(\mathcal{E}_\lambda^H).$$

PROOF. As in the proof of Proposition 2.12, it suffices to prove that $\mathsf{For}^H(\mathcal{E}_\lambda^H)$ is indecomposable. Now by Lemma 2.13 the functor For^H induces a surjection

$$\mathrm{End}_{\mathsf{D}_{H,\Lambda}(X,\Bbbk)}(\mathcal{E}_\lambda^H) \to \mathrm{End}_{\mathsf{D}_{\Lambda}(X,\Bbbk)}(\mathsf{For}^H(\mathcal{E}_\lambda^H)).$$

Since $\mathrm{End}_{\mathsf{D}_{H,\Lambda}(X,\Bbbk)}(\mathcal{E}_\lambda^H)$ is a local ring, and since $\mathrm{End}_{\mathsf{D}_{\Lambda}(X,\Bbbk)}(\mathsf{For}^H(\mathcal{E}_\lambda^H))$ is nonzero, this ring is local, which implies that $\mathsf{For}^H(\mathcal{E}_\lambda^H)$ is indecomposable, as desired. □

3. The case of Kac–Moody flag varieties

In this section we apply the parity formalism in the setting of (possibly parabolic) flag varieties of Kac–Moody groups.

3.1. Flag varieties. There are several different things we might want to call "flag varieties," which we introduce now.

3.1.1. *"Finite" flag varieties.* Let $\mathbb{F}$ be an algebraically closed field, and let $\mathscr{G}$ be a connected reductive algebraic group over $\mathbb{F}$. Let $\mathscr{B}$ be a Borel subgroup of $\mathscr{G}$, and let $\mathscr{T}$ be a maximal torus contained in $\mathscr{G}$. Let $W = N_{\mathscr{G}}(\mathscr{T})/\mathscr{T}$ be the Weyl group of $(\mathscr{G}, \mathscr{T})$, and let $S \subset W$ be the subset of Coxeter generators determined by $\mathscr{B}$. As in §1.1 we can consider the flag variety

$$\mathscr{X} := \mathscr{G}/\mathscr{B},$$

a smooth projective algebraic variety over $\mathbb{F}$. The Bruhat decomposition determines a stratification

$$\mathscr{X} = \bigsqcup_{w \in W} \mathscr{X}_w \quad \text{with } \mathscr{X}_w = \mathscr{B}w\mathscr{B}/\mathscr{B} \cong \mathbb{A}_{\mathbb{F}}^{\ell(w)}.$$

More generally, for any subset $I \subset S$ we have a standard parabolic subgroup $\mathscr{P}_I$, and the corresponding parabolic flag variety

$$\mathscr{X}^I := \mathscr{G}/\mathscr{P}_I,$$

which is again a smooth projective algebraic variety over $\mathbb{F}$. If we denote by $W_I \subset W$ the subgroup generated by I, and by $W^I \subset W$ the subset of elements w which are minimal in the corresponding coset wW_I, then we have a stratification

$$\mathscr{X}^I = \bigsqcup_{w \in W^I} \mathscr{X}_w^I \quad \text{with } \mathscr{X}_w^I = \mathscr{B}w\mathscr{P}_I/\mathscr{P}_I \cong \mathbb{A}_{\mathbb{F}}^{\ell(w)}.$$

3.1.2. *"Kac–Moody" flag varieties.* [3]

Let A be a generalized Cartan matrix, whose rows and columns are parametrized by some finite set I, and let $(\mathcal{W}, \mathcal{S})$ be the associated Coxeter system; see §1.2.3 in Chapter 2. Let also

$$(\mathbf{X}, (\alpha_i : i \in I), (\alpha_i^\vee : i \in I))$$

be a Kac–Moody root datum associated with A; see §1.2.4 in Chapter 2.

3.2. p-Kazhdan–Lusztig polynomials (first incarnation).

3.3. To be added. Mention convolution.
Remarks: affine Schubert varieties for SL(2) are rationally smooth.
Schubert varieties are smooth in codimension 1, cf. Example 4.8

4. The case of affine flag varieties

4.1. Affine flag varieties. Let $\mathbb{F}$ be an algebraically closed field, and let G be a connected reductive algebraic group over $\mathbb{F}$. We fix a Borel subgroup $B \subset G$ and a maximal torus T contained in B. Let $\mathcal{R}$ be the root system of (G, T), and $\mathcal{R}_+ \subset \mathcal{R}$ be the system of positive roots consisting of the T-weights in $\mathrm{Lie}(G)/\mathrm{Lie}(B)$. Let also $\mathcal{R}^\vee$ be the corresponding system of coroots. We will denote by

$$W := N_G(T)/T$$

the Weyl group of (G, T), and by $S \subset W$ the system of Coxeter generated determined by B. We will consider the (extended) affine Weyl group

$$W_{\mathrm{ext}} := W \ltimes X_*(T).$$

[3] Intuitively, an ind-scheme is a formal inductive limit of schemes, with transition maps given by closed immersions. A very nice treatment of this subject is provided in the first section of [Rz].

Given $\lambda \in X_*(T)$, we will denote by t_λ the corresponding element in W_{ext}. Let also $\mathbb{Z}\mathcal{R}^\vee$ be the root lattice, i.e. the sublattice in $X_*(T)$ generated by $\mathcal{R}^\vee$. It is a standard fact that the subgroup

$$W_{\mathrm{aff}} := W \ltimes \mathbb{Z}\mathcal{R}^\vee$$

admits a natural system S_{aff} of Coxeter generators, consisting of S together with the elements of the form $t_{\beta^\vee} s_\beta$ where $\beta \in \mathcal{R}$ is a maximal root.

Let us consider the function

$$\ell : W \to \mathbb{Z}_{\geq 0}$$

given by

$$\ell(wt_\lambda) = \sum_{\substack{\alpha \in \mathcal{R}_+ \\ w(\alpha) \in \mathcal{R}_+}} |\langle \lambda, \alpha^\vee \rangle| + \sum_{\substack{\alpha \in \mathcal{R}_+ \\ w(\alpha) \in -\mathcal{R}_+}} |1 + \langle \lambda, \alpha^\vee \rangle|$$

Then it is known that the restriction of ℓ to W_{aff} is the length function for the Coxeter system $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$, and moreover that if we set

$$\Omega = \{w \in W \mid \ell(w) = 0\}$$

then Ω is a finitely generated abelian group, that the conjugation action of Ω on W_{ext} stabilizes Sa, so that this group acts on W_{aff} by Coxeter group automorphisms, and finally that multiplication induces a group isomorphism

$$\Omega \ltimes W_{\mathrm{aff}} \xrightarrow{\sim} W_{\mathrm{ext}}.$$

To G one associates two functors LG and L^+G from the category of $\mathbb{F}$-algebras to the category of set, by setting

$$LG(R) = G(R((z))), \quad L^+G(R) = G(R[[z]]).$$

(Here, for an $\mathbb{F}$-algebra A, we denote by $G(A)$ the group of A-points of G, or in other words of morphisms of $\mathbb{F}$-schemes from $\mathrm{Spec}(A)$ to G.) It is a standard fact that the functor L^+G is representable by a group scheme over $\mathbb{F}$ (not of finite type), and that LG is representable by a group ind-scheme. (See e.g. [**Rz**] for a proof of a much more general claim.)

To each finitary subset $A \subset S_{\mathrm{aff}}$ one can associate a "parahoric subgroup" $Q_A \subset LH$, and consider the functor from $\mathbb{F}$-algebras to sets given by

$$R \mapsto LH(R)/Q_A(R).$$

Again, it is a standard fact that the fppf sheafification of this functor is representable by an ind-scheme

$$\mathsf{Fl}_A$$

which is ind-projective.

EXAMPLE 4.1. Assume that $A \subset S$. Then A determines a parabolic subgroup $P_A \subset G$. We have a natural morphism $LG \to G$, induced by the morphisms $R[[z]] \to R$ sending z to 0. For such a subset, the parahoric subgroup Q_A is the preimage of P_A in LG. Two special cases of this setting will play important roles below:

- in case $A = \varnothing$, $Q_\varnothing$ is the standard Iwahori subgroup of LG; it will be denoted I, and we will write Fl for $\mathsf{Fl}_\varnothing$ (this is the familiar "affine flag variety" of G);
- in case $A = S$, $Q_S = LG$, and the ind-scheme Fl_S is called the *affine flag variety* of G and denoted Gr.

4.2. The equivariant case.

4.3. The Whittaker case.

5. The case of the affine Grassmannian

5.1. The geometric Satake equivalence. Let us consider the particular case of the affine Grassmannian Gr, and the natural action of L^+G on it. In this case the L^+G-orbits on Gr are parametrized in the obvious way by the set $X_*(T)^+$ of dominant cocharacters of T. We will denote by Gr^λ the L^+G-orbit associated with λ, so that we have a stratification

$$(\mathsf{Gr}^\lambda : \lambda \in X_*(T)^+)$$

of Gr. It is a standard fact that for any $\lambda \in X_*(T)^+$ we have

$$\dim(\mathsf{Gr}^\lambda) = \langle 2\rho, \lambda \rangle$$

where $2\rho \in X^*(T)$ is the sum of the positive roots, and that for $\lambda, \mu \in X_*(T)$ we have

$$\overline{\mathsf{Gr}^\lambda} \subset \overline{\mathsf{Gr}^\mu} \quad \text{iff } \mu - \lambda \text{ is a sum of positive coroots.} \tag{5.1}$$

It is known that the connected components of Gr are in a canonical bijection with the quotient $X_*(T)/\mathbb{Z}\mathcal{R}^\vee$, and that the orbit Gr^λ is included in the component corresponding to a coset $\Lambda \in X_*(T)/\mathbb{Z}\mathcal{R}^\vee$ if and only if $\lambda \in \Lambda$. This implies the following property, that is crucial for many considerations involving parity complexes on Gr.

Lemma 5.1. *Let $\lambda, \mu \in X_*(T)^+$. If Gr^λ and Gr^μ belong to the same connected component of Gr, then $\dim(\mathsf{Gr}^\lambda)$ and $\dim(\mathsf{Gr}^\mu)$ have the same parity.*

Let us now assume (for simplicity) that $\Bbbk$ is a field, and consider the L^+G-equivariant derived category

$$D^{\mathrm{b}}_{L^+G}(\mathsf{Gr}, \Bbbk)$$

of $\Bbbk$-sheaves on Gr. The same considerations as in REF above show that there exists a natural convolution product

$$(-) \star_{L^+G} (-) : D^{\mathrm{b}}_{L^+G}(\mathsf{Gr}, \Bbbk) \times D^{\mathrm{b}}_{L^+G}(\mathsf{Gr}, \Bbbk) \to D^{\mathrm{b}}_{L^+G}(\mathsf{Gr}, \Bbbk)$$

which endows $D^{\mathrm{b}}_{L^+G}(\mathsf{Gr}, \Bbbk)$ with the structure of a monoidal category.

Such a construction can be considered for any parahoric subgroup (or any parabolic subgroup of a Kac–Moody group), but there is a kind or "miracle" in this setting, which is that this convolution product is t-exact on both sides with respect to the perverse t-structure. In other words, if we denote by

$$\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk)$$

the heart of the perverse t-structure on $D^{\mathrm{b}}_{L^+G}(\mathsf{Gr}, \Bbbk)$, then for any objects $\mathcal{A}, \mathcal{B} \in \mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk)$ the product $\mathcal{A} \star_{L^+G} \mathcal{B}$ belongs to $\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk)$. Restricting the convolution product, we therefore obtain a monoidal category

$$\big(\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk), \star_{L^+G}\big).$$

To each $\lambda \in X_*(T)^+$ one can associate 3 natural objects in $\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk)$. Namely, denote by $j^\lambda : \mathsf{Gr}^\lambda \to \mathsf{Gr}$ the embedding. Then we set

$$\mathcal{J}_!(\lambda) := {}^{\mathrm{p}}\mathcal{H}^0(j^\lambda_! \underline{\Bbbk}_{\mathsf{Gr}^\lambda}[\langle 2\rho, \lambda \rangle]), \quad \mathcal{J}_*(\lambda) := {}^{\mathrm{p}}\mathcal{H}^0(j^\lambda_* \underline{\Bbbk}_{\mathsf{Gr}^\lambda}[\langle 2\rho, \lambda \rangle]).$$

There exists a canonical morphism of complexes

$$j_!^\lambda \underline{\Bbbk}_{\mathsf{Gr}^\lambda}[\langle 2\rho, \lambda\rangle] \to j_*^\lambda \underline{\Bbbk}_{\mathsf{Gr}^\lambda}[\langle 2\rho, \lambda\rangle],$$

which provides a canonical morphism

$$\mathcal{J}_!(\lambda) \to \mathcal{J}_*(\lambda),$$

whose image is denoted $\mathcal{J}_{!*}(\lambda)$; in fact the general theory of perverse sheaves guarantees that this image is simple; this is the intersection cohomology complex associated with the constant local system on the orbit Gr^λ.

Let us now denote by $G_\Bbbk^\vee$ a connected reductive algebraic group over $\Bbbk$ whose root datum is

$$(X_*(T), \mathcal{R}^\vee, X^*(T), \mathcal{R}).$$

This means that $G_\Bbbk^\vee$ is endowed with a maximal torus $T_\Bbbk^\vee$ whose character lattice is $X_*(T)$, the *co*character lattice of T, and that the root system of $(G_\Bbbk^\vee, T_\Bbbk^\vee)$ is $\mathcal{R}$. Our choice of Borel subgroup $B \subset G$ has provided us with a system of positive roots $\mathcal{R}_+ \subset \mathcal{R}$. The corresponding coroots $\mathcal{R}_+^\vee \subset \mathcal{R}^\vee$ define a system of positive roots for $G_\Bbbk^\vee$, and we denote by $B_\Bbbk^\vee \subset G_\Bbbk^\vee$ the Borel subgroup containing $T_\Bbbk^\vee$ and such that the set of $T_\Bbbk^\vee$-weights on $\mathrm{Lie}(G_\Bbbk^\vee)/\mathrm{Lie}(B_\Bbbk^\vee)$ is $\mathcal{R}_+^\vee$. Note that the set of dominant weights for this choice of positive roots is $X_*(T)^+$. Hence, for any $\lambda \in X_*(T)^+$ we have representations $\mathsf{M}(\lambda)$, $\mathsf{N}(\lambda)$ and $\mathsf{L}(\lambda)$ constructed as in Chapter 1.

The *geometric Satake equivalence*, first proved in this generality by Mirković–Vilonen [**MV**], is the following statement.

THEOREM 5.2. *There exists an equivalence of monoidal categories*

$$\big(\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk), \star_{L^+G}\big) \cong \big(\mathsf{Rep}(G_\Bbbk^\vee), \otimes\big)$$

which sends, for any $\lambda \in X_*(T)^+$, the perverse sheaf

$$\mathcal{J}_!(\lambda), \quad \textit{resp.} \quad \mathcal{J}_*(\lambda), \quad \textit{resp.} \quad \mathcal{J}_{!*}(\lambda),$$

to the representation

$$\mathsf{M}(\lambda), \quad \textit{resp.} \quad \mathsf{N}(\lambda), \quad \textit{resp.} \quad \mathsf{L}(\lambda).$$

In the course of the proof of Theorem 5.2, Mirković and Vilonen prove the following fact, which will be important for us. Denote by

$$D^{\mathrm{b}}_{(L^+G)}(\mathsf{Gr}, \Bbbk)$$

the full subcategory of the constructible derived category of $\Bbbk$-sheaves on Gr generated by the essential image of the functor

$$D^{\mathrm{b}}_{L^+G}(\mathsf{Gr}, \Bbbk) \to D^{\mathrm{b}}_c(\mathsf{Gr}, \Bbbk)$$

forgetting the equivariance. In other words, a complex of $\Bbbk$-vector spaces $\mathcal{A}$ on Gr belongs to $D^{\mathrm{b}}_{(L^+G)}(\mathsf{Gr}, \Bbbk)$ if and only if for any $n \in \mathbb{Z}$ and $\lambda \in X_*(T)^+$ the sheaf $\mathcal{H}^n(\mathcal{A}_{|\mathsf{Gr}^\lambda})$ is isomorphic to $(\Bbbk_{\mathsf{Gr}^\lambda})^{\oplus m}$ for some $m \in \mathbb{Z}_{\geq 0}$. It is easily seen that the perverse t-structure on $D^{\mathrm{b}}_c(\mathsf{Gr}, \Bbbk)$ restricts to a t-structure on $D^{\mathrm{b}}_{(L^+G)}(\mathsf{Gr}, \Bbbk)$, which will again be called the perverse t-structure. Its heart will be denoted

$$\mathsf{Perv}_{(L^+G)}(\mathsf{Gr}, \Bbbk).$$

It can be easily seen that $\mathsf{Perv}_{(L^+G)}(\mathsf{Gr}, \Bbbk)$ is the category of perverse sheaves on Gr all of whose composition factors are of the form $\mathcal{J}_{!*}(\lambda)$ with $\lambda \in X_*(T)^+$. By construction, forgetting the equivariance provides a canonical functor

$$\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk) \to \mathsf{Perv}_{(L^+G)}(\mathsf{Gr}, \Bbbk).$$

The general theory of perverse sheaves implies that this functor is fully faithful, but something even better occurs here: this functor is an equivalence of categories. In other words, a perverse sheaf all of whose composition factors are of the form $\mathcal{J}_{!*}(\lambda)$ is automatically L^+G-equivariant.

REMARK 5.3. If we work in the standard topological setting, then the category $D^{\mathrm{b}}_{(L^+G)}(\mathsf{Gr}, \Bbbk)$ is just the constructible derived category associated with the stratification $(\mathsf{Gr}^\lambda : \lambda \in X_*(T)^+)$.

5.2. Parity complexes and tilting modules. Let us now consider the setting of Section 2 for the stratification $(\mathsf{Gr}^\lambda : \lambda \in X_*(T)^+)$. This theory provides us with a collection $(\mathcal{E}^\lambda : \lambda \in X_*(T)^+)$ of normalized indecomposable parity complexes in $D^{\mathrm{b}}_{(L^+G)}(\mathsf{Gr}, \Bbbk)$.

The following theorem is due to Mautner and the author (see [**MR2**]) in full generality, after a proof under stronger assumptions (on $\mathrm{char}(\Bbbk)$) by Juteau–Mautner–Williamson [**JMW3**].

THEOREM 5.4. *If* $\mathrm{char}(\Bbbk)$ *is good for* G*, then* $\mathcal{E}^\lambda$ *is a perverse sheaf for any* $\lambda \in X_*(T)^+$.

The proof of this theorem in full generality requires a comparison with a category of equivariant coherent sheaves on the Springer resolution of the group $G^\vee_\Bbbk$. The proof in [**JMW3**] is more elementary: it proceeds by an explicit check in some "simple" cases (more explicitly when λ is either minuscule or quasi-minuscule), and then a proof that these cases suffice to imply the theorem for all λ. This second step is where stronger assumptions are necessary.

If the complex $\mathcal{E}^\lambda$ is perverse (which, by Theorem 5.4, is always true if $\mathrm{char}(\Bbbk)$ is good for G), the comments at the end of §5.1 show that this object "lifts" canonically to the category $\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk)$. One can therefore ask the question of understanding its image under the equivalence of Theorem 5.2. The answer turns out to be easy, and very interesting, as shown in [**JMW3**, Proposition 3.3].

PROPOSITION 5.5. *Let* $\lambda \in X_*(T)^+$*. If the complex* $\mathcal{E}^\lambda$ *is perverse, then its image in* $\mathsf{Rep}(G^\vee_\Bbbk)$ *is* $\mathsf{T}(\lambda)$.

PROOF. Using Propositions 4.1–4.3 in Chapter A, to prove that the image of $\mathcal{E}^\lambda$ is tilting it suffices to prove is that

$$\mathrm{Ext}^1_{\mathsf{Perv}_{L^+G}(\mathsf{Gr},\Bbbk)}(\mathcal{E}^\lambda, \mathcal{J}_*(\mu)) = 0 = \mathrm{Ext}^1_{\mathsf{Perv}_{L^+G}(\mathsf{Gr},\Bbbk)}(\mathcal{J}_!(\mu), \mathcal{E}^\lambda)$$

for any $\mu \in X_*(T)^+$, or equivalently (see §5.1) that

$$\mathrm{Ext}^1_{\mathsf{Perv}_{(L^+G)}(\mathsf{Gr},\Bbbk)}(\mathcal{E}^\lambda, \mathcal{J}_*(\mu)) = 0 = \mathrm{Ext}^1_{\mathsf{Perv}_{(L^+G)}(\mathsf{Gr},\Bbbk)}(\mathcal{J}_!(\mu), \mathcal{E}^\lambda)$$

for any $\mu \in X_*(T)^+$. The general theory of t-structures guarantees that for any $\mathcal{A}, \mathcal{B}$ in $\mathsf{Perv}_{(L^+G)}(\mathsf{Gr}, \Bbbk)$ we have an identification

$$\mathrm{Ext}^1_{\mathsf{Perv}_{(L^+G)}(\mathsf{Gr},\Bbbk)}(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_{D^{\mathrm{b}}_{(L^+G)}(\mathsf{Gr},\Bbbk)}(\mathcal{A}, \mathcal{B}[1]).$$

Hence what we have to prove is that

$$\mathrm{Hom}_{D^{\mathrm{b}}_{(L^+G)}(\mathsf{Gr},\Bbbk)}(\mathcal{E}^\lambda, \mathcal{J}_*(\mu)[1]) = 0 = \mathrm{Hom}_{D^{\mathrm{b}}_{(L^+G)}(\mathsf{Gr},\Bbbk)}(\mathcal{J}_!(\mu), \mathcal{E}^\lambda[1])$$

for any $\mu \in X_*(T)^+$. In view of Lemma 5.1, it suffices to prove this when $\langle 2\rho, \lambda\rangle$ and $\langle 2\rho, \mu\rangle$ have the same parity. We will prove the second equality (under this assumption); the first one can be proved similarly, or deduced using Verdier duality.

Since the complex $j_!^\lambda \underline{\Bbbk}_{\mathsf{Gr}^\lambda}[\langle 2\rho, \lambda\rangle]$ is concentrated in nonpositive degrees, we have a truncation triangle

$$\mathcal{A} \to j_!^\mu \underline{\Bbbk}_{\mathsf{Gr}^\mu}[\langle 2\rho, \mu\rangle] \to \mathcal{J}_!(\mu) \xrightarrow{[1]}$$

where $\mathcal{A}$ is concentrated in negative perverse degrees. We deduce an exact sequence

$$\mathrm{Hom}_{D^{\mathrm{b}}_{(L^+G)}(\mathsf{Gr},\Bbbk)}(\mathcal{A}, \mathcal{E}^\lambda) \to \mathrm{Hom}_{D^{\mathrm{b}}_{(L^+G)}(\mathsf{Gr},\Bbbk)}(\mathcal{J}_!(\mu), \mathcal{E}^\lambda[1]) \to \\ \mathrm{Hom}_{D^{\mathrm{b}}_{(L^+G)}(\mathsf{Gr},\Bbbk)}(j_!^\mu \underline{\Bbbk}_{\mathsf{Gr}^\mu}[\langle 2\rho, \mu\rangle], \mathcal{E}^\lambda[1]).$$

Here the first term vanishes by general properties of t-structures because $\mathcal{A}$ is concentrated in negative perverse degrees and $\mathcal{E}^\lambda$ is perverse. As explained above we assume that $\langle 2\rho, \lambda\rangle$ and $\langle 2\rho, \mu\rangle$ have the same parity. To fix notation, assume that these numbers are even. Then the complex $j_!^\mu \underline{\Bbbk}_{\mathsf{Gr}^\mu}[\langle 2\rho, \mu\rangle]$ is $*$-even, and $\mathcal{E}^\lambda[1]$ is odd (hence !-odd). By Corollary 2.9(1) this implies that the third term in our exact sequence vanishes, and finishes the proof of our claim.

We have now proved that the image of $\mathcal{E}^\lambda$ is parity. It is indecomposable because so is $\mathcal{E}^\lambda$. Since $\mathcal{E}^\lambda$ is supported on $\overline{\mathsf{Gr}^\lambda}$ and has nonzero restriction to Gr^λ, its composition factors are of the form $\mathcal{J}_{!*}(\mu)$ with $\lambda - \mu$ a sum of positive coroots, with the case $\lambda = \mu$ occurring (see (5.1)). We deduce that the composition factors of its image are of the form $\mathsf{L}(\mu)$ with μ as above, with the case $\mu = \lambda$ occurring. Hence this image is $\mathsf{T}(\lambda)$, as desired. □

In case $\mathrm{char}(\Bbbk)$ is bad for G, then it is known that not all of the complexes $\mathcal{E}^\lambda$ are perverse. But one can still consider, for any $n \in \mathbb{Z}$, the perverse sheaf ${}^{\mathrm{p}}\mathcal{H}^n(\mathcal{E}^\lambda)$, which defines defines an object in $\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk)$. The following result was conjectured in [JMW3], and proved in [BGMRR]

THEOREM 5.6. *For any $n \in \mathbb{Z}$ and $\lambda \in X_*(T)^+$, the image of the perverse sheaf ${}^{\mathrm{p}}\mathcal{H}^n(\mathcal{E}^\lambda)$ in $\mathsf{Rep}(G^\vee_\Bbbk)$ is a tilting module. Moreover, any tilting module occurs as a direct sum of direct summands of images of objects of the form ${}^{\mathrm{p}}\mathcal{H}^0(\mathcal{E}^\lambda)$ with $\lambda \in X_*(T)^+$.*

6. Mixed perverse sheaves and Koszul duality

CHAPTER 4

Tilting modules for reductive groups

In this chapter we will extensively use the notions and basic results about highest weights categories recalled in Appendix A.

1. Tilting modules for reductive groups

1.1. Definition. We use the setting and notation of Chapter 1. In particular, $\mathbf{G}$ is a connected reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p > 0$, $\mathbf{B} \subset \mathbf{G}$ is a Borel subgroup, and $\mathbf{T} \subset \mathbf{B}$ is a maximal torus.

Recall from §1.3 in Chapter 1 the induced modules $(\mathsf{N}(\lambda) : \lambda \in \mathbb{X}^+)$ and the Weyl modules $(\mathsf{M}(\lambda) : \lambda \in \mathbb{X}^+)$. By Theorem 2.3 from Chapter 1, these modules define a canonical structure of highest weight category on $\mathsf{Rep}(\mathbf{G})$, with weight poset $(\mathbb{X}^+, \preceq)$. As a special case of the theory recalled in §5 of Appendix A, one can consider the *tilting* objects in $\mathsf{Rep}(\mathbf{G})$, i.e. the finite-dimensional algebraic $\mathbf{G}$-modules which admit both a costandard filtration, i.e. a filtration with subquotients of the form $\mathsf{N}(\lambda)$ $(\lambda \in \mathbb{X}^+)$,[1] and a standard filtration, i.e. a filtration with subquotients of the form $\mathsf{M}(\lambda)$ $(\lambda \in \mathbb{X}^+)$.[2] We will denote by $\mathrm{Tilt}(\mathbf{G}) \subset \mathsf{Rep}(\mathbf{G})$ the full subcategory whose objects are the tilting $\mathbf{G}$-modules.

If $M \in \mathrm{Tilt}(\mathbf{G})$, then for any $\lambda \in \mathbb{X}^+$ the number of occurrences of the module $\mathsf{N}(\lambda)$ as a subquotient in a costandard filtration of M is independent of the choice of filtration, and will be denoted $(M : \mathsf{N}(\lambda))$; in fact we have

$$(M : \mathsf{N}(\lambda)) = \dim_{\Bbbk} \mathrm{Hom}_{\mathsf{Rep}(\mathbf{G})}(\mathsf{M}(\lambda), M), \tag{1.1}$$

see Exercise 7.5. With this notation, it is clear that in $K^0(\mathsf{Rep}(\mathbf{G}))$ we have

$$[M] = \sum_{\lambda \in \mathbb{X}^+} (M : \mathsf{N}(\lambda)) \cdot [\mathsf{N}(\lambda)]. \tag{1.2}$$

In particular, the coefficients in the expansion of the element $[M]$ in the basis $([\mathsf{N}(\lambda)] : \lambda \in \mathbb{X}^+)$ are *nonnegative*.

The multiplicities $(M : \mathsf{M}(\lambda))$ are defined similarly, considering now standard filtrations instead of costandard filtrations. In this case we have

$$(M : \mathsf{M}(\lambda)) = \dim_{\Bbbk} \mathrm{Hom}_{\mathsf{Rep}(\mathbf{G})}(M, \mathsf{N}(\lambda)),$$

and

$$[M] = \sum_{\lambda \in \mathbb{X}^+} (M : \mathsf{M}(\lambda)) \cdot [\mathsf{M}(\lambda)] \tag{1.3}$$

[1] In the algebraic groups literature, such a filtration is often called a "good filtration".

[2] In the algebraic groups literature, such a filtration is often called a "Weyl filtration".

in $K^0(\mathsf{Rep}(\mathbf{G}))$. In fact, since $[\mathsf{M}(\lambda)] = [\mathsf{N}(\lambda)]$ in $K^0(\mathsf{Rep}(\mathbf{G}))$ (see §1.9 in Chapter 1), comparing (1.2) and (1.3) we see that

$$(M : \mathsf{N}(\lambda)) = (M : \mathsf{M}(\lambda))$$

for any $M \in \mathrm{Tilt}(\mathbf{G})$ and $\lambda \in \mathbb{X}^+$.

General considerations for highest weight categories (see Exercise 7.5) show that for M and N in $\mathrm{Tilt}(\mathbf{G})$ we have

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathsf{Rep}(\mathbf{G})}(M, N) = \sum_{\lambda \in \mathbb{X}^+} (M : \mathsf{M}(\lambda)) \cdot (N : \mathsf{N}(\lambda)).$$

In this setting we therefore also have

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathsf{Rep}(\mathbf{G})}(M, N) = \sum_{\lambda \in \mathbb{X}^+} (M : \mathsf{N}(\lambda)) \cdot (N : \mathsf{N}(\lambda)).$$

1.2. Classification. As a special case of general results on highest weights category, it is known that any direct summand of a tilting $\mathbf{G}$-module is tilting. Therefore any tilting module can be written (in an essentially unique way) as a direct sum of indecomposable tilting modules. Hence, to describe *all* tilting $\mathbf{G}$-modules it suffices to describe the indecomposable ones.

The following theorem answers this question; it gathers the results from §5 in Appendix A, specialized to the case of the category $\mathsf{Rep}(\mathbf{G})$.

Theorem 1.1. *For any $\lambda \in \mathbb{X}^+$, there exists a unique (up to isomorphism) indecomposable tilting $\mathbf{G}$-module $\mathsf{T}(\lambda)$ such that $(\mathsf{T}(\lambda) : \mathsf{N}(\lambda)) = 1$ and $(\mathsf{T}(\lambda) : \mathsf{N}(\mu)) = 0$ unless $\mu \preceq \lambda$. Moreover, the assignment $\lambda \mapsto \mathsf{T}(\lambda)$ induces a bijection between $\mathbb{X}^+$ and the set of isomorphism classes of indecomposable tilting $\mathbf{G}$-modules, and any tilting $\mathbf{G}$-module is isomorphic to a direct sum of indecomposable tilting $\mathbf{G}$-modules.*

The indecomposable tilting $\mathbf{G}$-modules $(\mathsf{T}(\lambda) : \lambda \in \mathbb{X}^+)$ will be the main object of study in this chapter. Our goal will be to describe these modules in some simple special cases, and explain why understanding these modules is relevant for the question of computing characters of simple modules. Here, by "understanding" these modules we mean computing their characters, or equivalently (see (1.2)) computing the multiplicities $(\mathsf{T}(\lambda) : \mathsf{N}(\mu))$ for $\lambda, \mu \in \mathbb{X}^+$.

1.3. Stability by tensor product. The following theorem provides important properties of tilting modules, which are very useful when trying to construct new tilting modules out of known ones.

Theorem 1.2.
(1) For any tilting $\mathbf{G}$-modules M and N, the tensor product $M \otimes N$ is tilting.
(2) It $M \in \mathsf{T}(\mathbf{G})$ and $\mathbf{L} \subset \mathbf{G}$ is a Levi subgroup, then the restriction $M_{|\mathbf{L}}$ is a tilting $\mathbf{L}$-module.

This theorem is sometimes stated as saying that the tensor product of two modules admitting a costandard filtration (or the restriction to a Levi subgroup of a module admitting a costandard filtration) admits a costandard filtration, or similarly with standard filtrations. The various versions are in fact equivalent; see e.g. Exercise 4.2.

This theorem admits several independent proofs:

(1) The first general proof was found by O. Mathieu [**M1**], after earlier proofs imposing some technical assumptions by Wang and Donkin. This proof is based on geometric methods (and more precisely Frobenius splitting techniques); see e.g. [**J3**, §G.15] for an exposition.
(2) Lusztig later proved in [**L5**, §27.3] a result of crystal bases for quantum groups that has Theorem 1.2 as a corollary (see [**Kan**] for details).
(3) Another general proof based on the Geometric Satake Equivalence was recently found by R. Bezrukavnikov, D. Gaitsgory, I. Mirković, L. Rider and the author in [**BGMRR**], based on an idea of Juteau–Mautner–Williamson [**JMW3**].

See also [**J3**, §4.21] for other historical remarks. We will not review any of these proofs here.

1.4. Tilting modules, blocks, and translation functors. For any $\lambda \in \mathbb{X}^+$, since the tilting module $\mathsf{T}(\lambda)$ is indecomposable it must belong to the "block" $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \mu}$ for some $\mu \in \mathbb{X}$. Now, since $\mathrm{Hom}(\mathsf{M}(\lambda), \mathsf{T}(\lambda)) \neq 0$ (see (1.1)) we in fact have

$$\mathsf{T}(\lambda) \in \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}.$$

In particular, this implies that $(\mathsf{T}(\lambda) : \mathsf{N}(\mu)) = 0$ unless $\mu \in W_{\mathrm{aff}} \cdot_p \lambda$.

As for the study of simple modules, the main tool we will use in the study of indecomposable tilting modules are the translation functors. We begin with the following general property.

Proposition 1.3. *For any $\lambda, \mu \in \mathbb{X}$, the functor*

$$T_\lambda^\mu : \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda} \to \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \mu}$$

sends tilting modules to tilting modules.

Proof. This property can be obtained as a consequence of Theorem 1.2, since by Remark 2.18 in Chapter 1 the functor T_λ^μ can be described as the tensor product with the indecomposable tilting module with highest weight the dominant W-translate of $\mu - \lambda$, followed by projection on the block attached to $W_{\mathrm{aff}} \cdot_p \mu$.

In case $\lambda, \mu \in \overline{C}$ (which is the most interesting setting), this property can be proved more directly by observing that T_λ^μ sends each $\mathsf{N}(w \cdot_p \lambda)$ (with $w \in W_{\mathrm{aff}}$ such that $w \cdot_p \lambda \in \mathbb{X}^+$) to a module which admits a costandard filtration, and dually for Weyl modules; see Proposition 2.26(4) in Chapter 1. $\square$

As far as *indecomposable* tilting modules are concerned, the following result due to Andersen describes what happens when translating to or from a "more singular" weight. For details and references, see [**J3**, §II.E.11].

Proposition 1.4. *Let $\lambda, \mu \in \overline{C} \cap \mathbb{X}$, and assume that μ belongs to the closure of the facet of λ. Let $w \in W_{\mathrm{aff}}$ such that $w \cdot_p \mu \in \mathbb{X}^+$, and assume moreover that $w \cdot_p \lambda$ is maximal among the weights of the form $wx \cdot_p \lambda$ with $x \in \mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$. Then we have*

$$T_\mu^\lambda \mathsf{T}(w \cdot_p \mu) \cong \mathsf{T}(w \cdot_p \lambda),$$

and $T_\lambda^\mu \mathsf{T}(w \cdot_p \lambda)$ is a direct sum of $\#(\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)/\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\lambda))$ copies of $\mathsf{T}(w \cdot_p \mu)$.

Using the results of §2.8 of Chapter 1, one can translate this proposition in more Coxeter-theoretic terms.

PROPOSITION 1.5. *Let $\lambda, \mu \in \overline{C} \cap \mathbb{X}$, and assume that μ belongs to the closure of the facet of λ. Let $w \in W_{\mathrm{aff}}^{(\mu)}$. Then we have*

$$T_\mu^\lambda \mathsf{T}(w \cdot_p \mu) \cong \mathsf{T}(w \cdot_p \lambda),$$

and $T_\lambda^\mu \mathsf{T}(w \cdot_p \lambda)$ is a direct sum of $\#(\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)/\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\lambda))$ copies of $\mathsf{T}(w \cdot_p \mu)$.

REMARK 1.6. Let $w \in W_{\mathrm{aff}}^{(\mu)}$ and $x \in \mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$, and assume that $wx \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\lambda)}$. If $w' \in W_{\mathrm{aff}}^{(\mu)}$ and $x' \in \mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$ are such that $w'x' \leq wx$, then $w' \leq w$ by [Dou, Lemma 2.2]. In particular, if $\mathsf{N}(y \cdot_p \lambda)$ is such that $(\mathsf{T}(wx \cdot_p \lambda) : \mathsf{N}(y \cdot_p \lambda)) \neq 0$, using Proposition 2.36 in Chapter 1 we see that $T_\lambda^\mu \mathsf{N}(y \cdot_p \lambda)$ is either 0 or of the form $\mathsf{N}(y' \cdot_p \mu)$ for some $y' \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)}$ such that $y' \leq w$. We deduce that w is maximal (for the Bruhat order) in

$$\{z \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)} \mid (T_\lambda^\mu \mathsf{T}(w \cdot_p \lambda) : \nabla(z \cdot_p \mu)) \neq 0\},$$

which implies that $\mathsf{T}(w \cdot_p \mu)$ is a direct summand in $T_\lambda^\mu \mathsf{T}(w \cdot_p \lambda)$.

Assume for instance that $p \geq h$, and that $\lambda \in C$. As explained in §2.8 of Chapter 1, the weights in $(W_{\mathrm{aff}} \cdot_p \mu) \cap \mathbb{X}^+$ are in a natural bijection with the subset ${}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)} \subset W_{\mathrm{aff}}$ of elements w which are minimal in Ww and maximal in $w\mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$. For such w, the element $w \cdot_p \lambda$ is maximal among the weights of the form $wx \cdot_p \lambda$ with $x \in \mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$, so that Proposition 1.4 applies. We deduce the following formula.

COROLLARY 1.7. *Assume that $p \geq h$, and let $\lambda \in C \cap \mathbb{X}$ and $\mu \in \overline{C} \cap \mathbb{X}$. Then for any $w, y \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)}$ and any $x \in \mathrm{Stab}_{(W_{\mathrm{aff}}, \cdot_p)}(\mu)$ we have*

$$(\mathsf{T}(w \cdot_p \mu) : \mathsf{N}(y \cdot_p \mu)) = (\mathsf{T}(w \cdot_p \lambda) : \mathsf{N}(yx \cdot_p \lambda)).$$

PROOF. We have

$$\begin{aligned}(\mathsf{T}(w \cdot_p \mu) : \mathsf{N}(y \cdot_p \mu)) &\overset{(1.1)}{=} \dim_{\Bbbk} \mathrm{Hom}_{\mathsf{Rep}(\mathbf{G})}(\mathsf{M}(y \cdot_p \mu), \mathsf{T}(w \cdot_p \mu)) \\ &= \dim_{\Bbbk} \mathrm{Hom}_{\mathsf{Rep}(\mathbf{G})}(T_\lambda^\mu \mathsf{M}(yx \cdot_p \lambda), \mathsf{T}(w \cdot_p \mu))\end{aligned}$$

by Proposition 2.26 in Chapter 1. By adjunction we deduce that

$$(\mathsf{T}(w \cdot_p \mu) : \mathsf{N}(y \cdot_p \mu)) = \dim_{\Bbbk} \mathrm{Hom}_{\mathsf{Rep}(\mathbf{G})}(\mathsf{M}(yx \cdot_p \lambda), T_\mu^\lambda \mathsf{T}(w \cdot_p \mu)).$$

The claim then follows from Proposition 1.4. $\square$

Corollary 1.7 provides an explicit recipe for the computation of *all* multiplicities $(\mathsf{T}(\lambda) : \mathsf{N}(\mu))$ once one knows these data in the special case when $\lambda, \mu \in (W_{\mathrm{aff}} \cdot 0) \cap \mathbb{X}^+$ (assuming that $p \geq 0$).

1.5. First examples.

1.5.1. *Minimal weights.* As explained in §2.9 of Chapter 1, in case $\mu \in \mathbb{X}^+$ is minimal in $(W_{\mathrm{aff}} \cdot_p \mu) \cap \mathbb{X}^+$, we have isomorphisms

$$\mathsf{M}(\mu) \cong \mathsf{L}(\mu) \cong \mathsf{N}(\mu).$$

In particular, this module is tilting in this case, and of course indecomposable. We deduce that

$$\mathsf{T}(\mu) = \mathsf{L}(\mu).$$

This applies in particular if $\mu \in \overline{C} \cap \mathbb{X}^+$, and if $\mu = (p-1)\varsigma$ in case there exists $\varsigma \in \mathbb{X}$ such that $\langle \varsigma, \alpha^\vee \rangle = 1$ for any $\alpha \in \mathfrak{R}^{\mathrm{s}}$.

1.5.2. *The alcove above the fundamental one.* Now, assume that $p \geq h$, so that $0 \in C$, and that $\mathbf{G}$ is quasi-simple. Recall the simple reflection $s_\circ \in S_{\mathrm{aff}} \smallsetminus S$, and choose a weight μ on the wall of $\overline{C}$ associated with $s_\circ$. Then $\mu \in \mathbb{X}^+$, and the considerations above show that $\mathsf{T}(\mu) = \mathsf{L}(\mu)$. By Proposition 2.26(3) in Chapter 1, there exist short exact sequences

$$\mathsf{N}(0) \hookrightarrow T_\mu^0 \mathsf{L}(\mu) \twoheadrightarrow \mathsf{N}(s_\circ \cdot 0),$$
$$\mathsf{M}(s_\circ \cdot_p 0) \hookrightarrow T_\mu^0 \mathsf{L}(\mu) \twoheadrightarrow \mathsf{M}(0).$$

These exact sequences show that $T_\mu^0 \mathsf{L}(\mu)$ is tilting (which could also have been deduced from Proposition 1.3). In view of the description of $\mathsf{N}(s_\circ \cdot 0)$ in §2.9 of Chapter 1, we also know that $T_\mu^0 \mathsf{L}(\mu)$ has length three, with a filtration with successive subquotients $\mathsf{L}(0)$, $\mathsf{L}(s_\circ \cdot_p 0)$, $\mathsf{L}(0)$. Since

$$\mathrm{Hom}(T_\mu^0 \mathsf{L}(\mu), \mathsf{L}(s_\circ \cdot_p 0)) = \mathrm{Hom}(\mathsf{L}(\mu), T_0^\mu \mathsf{L}(s_\circ \cdot_p 0)) = 0$$

(because $T_0^\mu \mathsf{L}(s_\circ \cdot_p 0) = 0$) and

$$\mathrm{Hom}(T_\mu^0 \mathsf{L}(\mu), \mathsf{L}(0)) = \mathrm{Hom}(\mathsf{L}(\mu), T_0^\mu \mathsf{L}(0)) = \Bbbk,$$

the module $T_\mu^0 \mathsf{L}(\mu)$ has simple top (namely, $\mathsf{L}(0)$), hence is irreducible. We have thus proved that

$$\mathsf{T}(s_\circ \cdot 0) = T_\mu^0 \mathsf{L}(\mu).$$

(This could also have been obtained as an application of Proposition 1.4.) In particular, the nonzero multiplicities $(\mathsf{T}(s_\circ \cdot 0) : \mathsf{N}(\lambda))$ are

$$(\mathsf{T}(s_\circ \cdot 0) : \mathsf{N}(s_\circ \cdot 0)) = 1 = (\mathsf{T}(s_\circ \cdot 0) : \mathsf{N}(0)).$$

1.5.3. *The alcove above the Steinberg weight.* In case there exists $\varsigma \in \mathbb{X}$ such that $\langle \varsigma, \alpha^\vee \rangle = 1$ for any $\alpha \in \mathfrak{R}^{\mathrm{s}}$, considerations similar to those of §1.5.2 apply to show that

$$\mathsf{T}(p\varsigma) = T_{(p-1)\varsigma}^{p\varsigma} \mathsf{L}((p-1)\varsigma),$$

that we have

$$(\mathsf{T}(p\varsigma) : \mathsf{N}((p-1)\varsigma + x(\varsigma))) = 1$$

for any $x \in W$, and that these are the only nonzero multiplicities $(\mathsf{T}(p\varsigma) : \mathsf{N}(\lambda))$. (Here again $\mathsf{T}(p\varsigma)$ has simple top and socle, isomorphic to $\mathsf{L}(p\varsigma - 2\rho)$.)

This tilting module plays a crucial role in the study of Soergel's modular category $\mathscr{O}$ in [S5].

1.5.4. *Tilting modules in the extended Steinberg block.* We assume again that there exists $\varsigma \in \mathbb{X}$ such that $\langle \varsigma, \alpha^\vee \rangle = 1$ for any $\alpha \in \mathfrak{R}^{\mathrm{s}}$. Recall our conventions on representations of $\mathbf{G}^{(1)}$ from §2.4 in Chapter 1. For any dominant weight $\mu \in X^*(\mathbf{T}^{(1)})$, we will denote by $\mathsf{T}^{(1)}(\mu)$ the associated indecomposable tilting $\mathbf{G}^{(1)}$-module. Recall also the equivalence

$$\mathsf{Rep}(\mathbf{G}^{(1)}) \xrightarrow{\sim} \mathsf{Rep}_{\mathrm{Stein}}(\mathbf{G})$$

considered in Corollary 2.41 of Chapter 1. Since this equivalence sends induced, resp. Weyl, modules to induced, resp. Weyl, modules, it must send (indecomposable) tilting modules to (indecomposable) tilting modules. We deduce the following claim.

PROPOSITION 1.8. *For any $\mu \in X^*(\mathbf{T}^{(1)})$ dominant, we have*

$$\mathsf{T}((p-1)\varsigma) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{T}^{(1)}(\mu)) \cong \mathsf{T}((p-1)\varsigma + \mathrm{Fr}^*_{\mathbf{T}}(\mu)).$$

1.6. Examples for classical groups. We now describe some tilting modules for classical groups.

1.6.1. *Special linear groups.* Let us consider the setting of §1.4.1 of Chapter 1. In this case, for any $i \in \{1, \cdots, n-1\}$ we have seen that $\mathsf{N}(\omega_i) \cong \bigwedge^i V$, and that this module is also isomorphic to $\mathsf{M}(\omega_i)$ (see Exercise 1.6). It is therefore tilting. Since it is indecomposable, we deduce that

$$\mathsf{T}(\omega_i) \cong \bigwedge\nolimits^i V.$$

See Exercise 4.3 for more details.

1.6.2. *Symplectic groups.* Now we consider the setting of §1.4.2 of Chapter 1. The description of these induced modules show that for each $m \leq n$, the module $\bigwedge^m V$ admits a costandard filtrations. Since this module is self dual (see §1.1), it also admits a standard filtration, hence is tilting. Considering highest weights we see that this module admits $\mathsf{T}(\omega_m)$ as a direct summand. It is however not indecomposable in general. In fact, by [**McN2**, Proposition 6.3.5] we have

$$\bigwedge\nolimits^m V \cong \bigoplus_{e \in Y(m)} \mathsf{T}(\omega_e),$$

where

$$Y(m) = \left\{ a \in \{0, \cdots, m\} \;\middle|\; a \equiv m \pmod 2,\ p \nmid \binom{n-a}{(m-a)/2} \right\}.$$

(In case $e = 0$, ω_e should be interpreted as 0.)

1.6.3. *Odd orthogonal groups.* We turn to the setting of §1.4.4 of Chapter 1. In this case the modules $\bigwedge^m V$ ($m \leq n$) are simple induced modules; they are therefore tilting, and we have

$$\bigwedge\nolimits^m V \cong \begin{cases} \mathsf{T}(\omega_m) & \text{if } m \leq n-1; \\ \mathsf{T}(\varepsilon_1 + \cdots + \varepsilon_n) & \text{if } m = n. \end{cases}$$

1.6.4. *Even orthogonal groups.* Finally, in the setting of §1.4.3 of Chapter 1, as in §1.6.3 we have

$$\bigwedge\nolimits^m V \cong \begin{cases} \mathsf{T}(\omega_m) & \text{if } m \leq n-2; \\ \mathsf{T}(\varepsilon_1 + \cdots + \varepsilon_{n-1}) & \text{if } m = n-1. \end{cases}$$

The case of $\bigwedge^n V$ is a bit more subtle since this module is not simple, already in the characteristic-0 setting (see [**FH**, Theorem 19.2 and Exercise 24.43]). It can however be treated as follows, following [**McN1**, Remark 3.4]. The description of this module in characteristic 0 and Weyl's character formula (Theorem 1.20 of Chapter 1) imply that we have

$$\mathrm{ch}\left(\bigwedge\nolimits^n V\right) = \mathrm{ch}(\mathsf{N}(2\omega_{n-1})) + \mathrm{ch}(\mathsf{N}(2\omega_n)).$$

In particular, this module has at least two composition factors, namely $\mathsf{L}(2\omega_{n-1})$ and $\mathsf{L}(2\omega_n)$. Now, choose an orthogonal decomposition

$$V = L \oplus H$$

where L is a line and H an hyperplane. Then the special orthogonal group $\mathrm{SO}(H)$ associated with the restriction of our symmetric bilinear form to H is an odd orthogonal group which identifies with a subgroup of $\mathbf{G}$, and as $\mathrm{SO}(H)$-modules we have

$$\bigwedge^n V \cong \bigwedge^{n-1} H \oplus \bigwedge^n H.$$

The considerations of §1.6.3 and duality imply that both direct summands in the right-hand side are simple $\mathrm{SO}(H)$-modules; therefore $\bigwedge^n V$ has length 2 as an $\mathrm{SO}(H)$-module, hence has length at most 2 as a $\mathbf{G}$-module. Comparing these two informations we deduce that this module has length 2, with composition factors $\mathsf{L}(2\omega_{n-1})$ and $\mathsf{L}(2\omega_n)$, and that $\mathsf{N}(2\omega_{n-1})$ and $\mathsf{N}(2\omega_n)$ are simple. We therefore have

$$\mathsf{N}(2\omega_n) \cong \mathsf{M}(2\omega_n),$$

and using Proposition 2.5 of Chapter 1 we deduce that

$$\mathrm{Ext}^1_{\mathsf{Rep}(\mathbf{G})}(\mathsf{L}(2\omega_n), \mathsf{L}(2\omega_{n-1})) = 0.$$

It finally follows that

$$\bigwedge^n V \cong \mathsf{L}(2\omega_{n-1}) \oplus \mathsf{L}(2\omega_n) \cong \mathsf{T}(2\omega_{n-1}) \oplus \mathsf{T}(2\omega_n).$$

2. Tilting G-modules and injective bounded modules

In this section we explain the proof of a result of Jantzen [**J1**] which is the key to the relation between characters of indecomposable tilting modules and of simple modules. Our proof is essentially the same as that of [**J1**], rephrased in the language of highest weight categories (which, in our opinion, makes it clearer).

2.1. Representations of the group scheme $\mathbf{G}_1$. Recall the Frobenius morphism $\mathrm{Fr} : \mathbf{G} \to \mathbf{G}^{(1)}$, see §2.4 in Chapter 1. The Frobenius kernel $\mathbf{G}_1$ is the scheme-theoretic kernel of Fr. Then $\mathbf{G}_1$ is a finite affine group scheme over $\Bbbk$; in other words its algebra of functions $\mathscr{O}(\mathbf{G}_1)$ is a finite-dimensional Hopf algebra over $\Bbbk$. In fact this Hopf algebra has a very concrete description, as follows. Consider the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. As any Lie algebra of a group scheme over a field of characteristic p, $\mathfrak{g}$ has a "restricted p-th power" operation, denoted $X \mapsto X^{[p]}$, see [**J3**, §I.7.10]. (This operation is a nonlinear map from $\mathfrak{g}$ to itself.) In the universal enveloping algebra $\mathcal{U}\mathfrak{g}$, the elements of the form $X^p - X^{[p]}$ with $X \in \mathfrak{g}$ are central; they generate a subalgebra Z_{Fr} canonically isomorphism to $\mathscr{O}(\mathfrak{g}^{*(1)})$. The *restricted enveloping algebra* $\mathcal{U}_0\mathfrak{g}$ of $\mathfrak{g}$ is the quotient of $\mathcal{U}\mathfrak{g}$ by the ideal generated by the elements $X^p - X^{[p]}$ with $X \in \mathfrak{g}$. It is a finite-dimensional algebra, of dimension $p^{\dim(\mathbf{G})}$. Moreover, the natural Hopf algebra structure on $\mathcal{U}\mathfrak{g}$ induces a Hopf algebra structure on $\mathcal{U}_0\mathfrak{g}$, and there exists a canonical isomorphism of Hopf algebras

$$\mathscr{O}(\mathbf{G}_1) \cong (\mathcal{U}_0\mathfrak{g})^*; \tag{2.1}$$

see [**J3**, §I.9.6].

We will denote by $\mathsf{Rep}(\mathbf{G}_1)$ the category of finite-dimensional $\mathbf{G}_1$-modules. In fact representations of the group scheme $\mathbf{G}_1$ are the same as comodules over $\mathscr{O}(\mathbf{G}_1)$, which in turn are the same as modules for the algebra $\mathscr{O}(\mathbf{G}_1)^*$, hence (by (2.1)) as modules over the (finite-dimensional) algebra $\mathcal{U}_0\mathfrak{g}$. In particular, in this category we can consider the socle $\mathrm{soc}_{\mathbf{G}_1}(M)$ (i.e. the largest semisimple submodule) and the top $\mathrm{top}_{\mathbf{G}_1}(M)$ (i.e. the largest semisimple quotient) of a module M, and each simple object N admits an injective hull characterized as the unique (up to isomorphism) injective module I_N such that $\mathrm{soc}_{\mathbf{G}_1}(I_N) \cong N$, and a projective cover characterized as the unique projective module P_N such that $\mathrm{top}_{\mathbf{G}_1}(P_N) \cong N$.

Since $\mathbf{G}_1$ is a subgroup scheme of $\mathbf{G}$, there exists a canonical "restriction" functor from $\mathsf{Rep}(\mathbf{G})$ to $\mathsf{Rep}(\mathbf{G}_1)$, which will be denoted $M \mapsto M_{|\mathbf{G}_1}$. Using the isomorphism (2.1), this functor can alternatively be described as follows. Differentiating the $\mathbf{G}$-action we obtain, for any $V \in \mathsf{Rep}(\mathbf{G})$, an action of $\mathcal{U}\mathfrak{g}$ on (the underlying vector space of) V. The elements of the form $X^p - X^{[p]}$ act trivially for this structure; we therefore obtain an action of $\mathcal{U}_0\mathfrak{g}$ on V. The associated $\mathbf{G}_1$-module structure is the same as the one obtained by restriction along the embedding $\mathbf{G}_1 \subset \mathbf{G}$. Below we will use the important observation that for M in $\mathsf{Rep}(\mathbf{G})$ the socle $\mathrm{soc}_{\mathbf{G}_1}(M_{|\mathbf{G}_1})$ is stable under the $\mathbf{G}$-action, hence a $\mathbf{G}$-submodule.

By [**J3**, Proposition I.9.5] the Frobenius morphism $\mathrm{Fr}_{\mathbf{G}}$ induces an isomorphism of $\Bbbk$-group schemes

$$\mathbf{G}/\mathbf{G}_1 \xrightarrow{\sim} \mathbf{G}^{(1)}.$$

In particular, a $\mathbf{G}$-module is of the form $\mathrm{Fr}_{\mathbf{G}}^*(V)$ for some $\mathbf{G}^{(1)}$-module V if and only if its restriction to $\mathbf{G}_1$ is trivial.

One can classify the simple $\mathbf{G}_1$-modules in a way similar to the case of $\mathbf{G}$, replacing the Weyl modules by the *baby Verma modules*. Namely, let $\mathbf{B}^+$ be the Borel subgroup of $\mathbf{G}$ opposite to $\mathbf{B}$ with respect to $\mathbf{T}$, and let $\mathfrak{b}^+$ be the Lie algebra of $\mathbf{B}^+$. (In this way, the nonzero $\mathbf{T}$-weights of $\mathfrak{b}^+$ are the positive roots.) As for $\mathbf{G}$, we can consider the restricted enveloping algebra $\mathcal{U}_0\mathfrak{b}^+$ of $\mathfrak{b}^+$, and we have a canonical injective algebra morphism $\mathcal{U}_0\mathfrak{b}^+ \to \mathcal{U}_0\mathfrak{g}$. For any $\lambda \in \mathbb{X}$ the 1-dimensional $\mathbf{B}^+$-module $\Bbbk_{\mathbf{B}^+}(\lambda)$ defines, by differentiation, a $\mathcal{U}_0\mathfrak{b}^+$-module $\Bbbk_{\mathfrak{b}^+}(\lambda)$, which depends only on the image of $\lambda \in \mathbb{X}/p\mathbb{X}$. The *baby Verma module* associated with λ is

$$\mathsf{Z}(\lambda) := \mathcal{U}_0\mathfrak{g} \otimes_{\mathcal{U}_0\mathfrak{b}^+} \Bbbk_{\mathfrak{b}^+}(\lambda),$$

seen as a $\mathbf{G}_1$-module via the identification (2.1).

For the following theorem, we refer to [**J3**, Proposition II.3.10].

THEOREM 2.1. *For any $\lambda \in \mathbb{X}$, the top $\mathsf{L}_1(\lambda)$ of $\mathsf{Z}(\lambda)$ is simple. Moreover, $\mathsf{L}_1(\lambda)$ only depends on the class of λ in $\mathbb{X}/p\mathbb{X}$, and the assignment $\lambda \mapsto \mathsf{L}_1(\lambda)$ induces a bijection between $\mathbb{X}/p\mathbb{X}$ and the set of isomorphism classes of simple $\mathbf{G}_1$-modules.*

Recall the subset $\mathbb{X}^+_{\mathrm{res}} \subset \mathbb{X}^+$ of restricted dominant weights defined in §2.4 of Chapter 1. The relation between the simple $\mathbf{G}$-modules and the simple $\mathbf{G}_1$-modules is provided by the following classical result due to Curtis. (For a proof, see [**J3**, Proposition II.3.15].)

THEOREM 2.2. *For any $\lambda \in \mathbb{X}^+_{\mathrm{res}}$, the $\mathbf{G}_1$-module $\mathsf{L}(\lambda)_{|\mathbf{G}_1}$ is simple, and isomorphic to $\mathsf{L}_1(\lambda)$.*

In case $\mathscr{D}(\mathbf{G})$ is simply connected, the composition $\mathbb{X}^+_{\mathrm{res}} \hookrightarrow \mathbb{X} \to \mathbb{X}/p\mathbb{X}$ is surjective. (This follows from the existence of "fundamental weights," i.e. weights $(\varpi_\alpha : \alpha \in \mathfrak{R}^{\mathrm{s}})$ such that $\langle \varpi_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$ for $\alpha, \beta \in \mathfrak{R}^{\mathrm{s}}$.) In this case, any simple $\mathbf{G}_1$-module is therefore the restriction of a simple $\mathbf{G}$-module. In this case one can in fact describe the restriction of any simple $\mathbf{G}$-module to $\mathbf{G}_1$ as follows. Let $\lambda \in \mathbb{X}^+$, and write $\lambda = \mu + p\nu$ with $\mu \in \mathbb{X}^+_{\mathrm{res}}$. Then we have $\nu \in \mathbb{X}^+$. Moreover, the map $\mathrm{Fr}_{\mathbf{T}}^* : X^*(\mathbf{T}^{(1)}) \to \mathbb{X}$ given by the pullback under the Frobenius morphism $\mathrm{Fr}_{\mathbf{T}}$ is injective, with image $p\mathbb{X}$; it follows that there exists $\tilde{\nu} \in X^*(\mathbf{T}^{(1)})$ such that

$p\nu = \mathrm{Fr}^*_{\mathbf{T}}(\tilde{\nu})$. Since ν is dominant, so is $\tilde{\nu}$ (for the conventions chosen in §2.4 of Chapter 1). By Steinberg's tensor product theorem (Theorem 2.9 in Chapter 1) we then have

$$\mathsf{L}(\lambda) \cong \mathsf{L}(\mu) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{L}^{(1)}(\tilde{\nu})),$$

which implies that

$$\mathsf{L}(\lambda)_{|\mathbf{G}_1} \cong \mathsf{L}_1(\mu) \otimes \mathsf{L}^{(1)}(\tilde{\nu}),$$

where $\mathbf{G}_1$ acts trivially on $\mathsf{L}^{(1)}(\tilde{\nu})$. In particular, $\mathsf{L}(\lambda)_{|\mathbf{G}_1}$ is always semisimple (and more precisely a direct sum of copies of a single simple $\mathbf{G}_1$-module).

For later use, we note the following consequence.

LEMMA 2.3. *Assume that $\mathscr{D}(\mathbf{G})$ is simply connected. For $\lambda \in \mathbb{X}^+$, the $\mathbf{G}_1$-module $\mathsf{L}(\lambda)_{|\mathbf{G}_1}$ is simple iff $\lambda \in \mathbb{X}^+_{\mathrm{res}}$.*

PROOF. If $\lambda \in \mathbb{X}^+_{\mathrm{res}}$, then $\mathsf{L}(\lambda)_{|\mathbf{G}_1}$ is simple by Theorem 2.2. On the other hand, let $\lambda \in \mathbb{X}^+$ be such that $\mathsf{L}(\lambda)_{|\mathbf{G}_1}$ is simple. Write $\lambda = \mu + p\nu$ as above, so that

$$\mathsf{L}(\lambda)_{|\mathbf{G}_1} \cong \mathsf{L}_1(\mu) \otimes \mathsf{L}^{(1)}(\tilde{\nu}).$$

Then our assumption implies that $\dim(\mathsf{L}^{(1)}(\tilde{\nu})) = 1$. In view of Lemma 1.18 in Chapter 1 this implies that $\langle \nu, \alpha^\vee \rangle = 0$ for any $\alpha \in \mathfrak{R}$, hence that $\lambda \in \mathbb{X}^+_{\mathrm{res}}$. □

REMARK 2.4. If $\mathbf{G}$ is semisimple and simply connected the map $\mathbb{X}^+_{\mathrm{res}} \to \mathbb{X}/p\mathbb{X}$ considered above is a bijection. In general, this map induces a bijection

$$\mathbb{X}^+_{\mathrm{res}}/p\Pi \xrightarrow{\sim} \mathbb{X}/p\mathbb{X},$$

where $\Pi := \{\lambda \in \mathbb{X} \mid \forall \alpha \in \mathfrak{R}^{\mathrm{s}}, \langle \lambda, \alpha^\vee \rangle = 0\}$ and $p\Pi$ acts on $\mathbb{X}^+_{\mathrm{res}}$ by addition. (Here, restriction to $\mathbf{T}$ induces an isomorphism between the lattice of characters of $\mathbf{G}$ and Π.)

REMARK 2.5. An important difference between the representation theory of $\mathbf{G}$ and that of $\mathbf{G}_1$ is that the set $\mathbb{X}/p\mathbb{X}$ of labels of simple $\mathbf{G}_1$-modules has no partial order having a representation-theoretic meaning. For instance, it is known that for any $w \in W$ the $\mathbf{G}_1$-modules $\mathsf{Z}(\lambda)$ and $\mathsf{Z}(w \cdot_p \lambda)$ have the same composition factors. (REFERENCE?)

For $\lambda \in \mathbb{X}$ we will denote by $\mathsf{Q}(\lambda)$ the injective hull of the simple $\mathbf{G}_1$-module $\mathsf{L}_1(\lambda)$. (Once again, up to isomorphism, this module only depends on the class of λ in $\mathbb{X}/p\mathbb{X}$.) As a special case of a general result on finite group schemes, it is known that there exists an isomorphism of $\mathbf{G}_1$-modules $\mathscr{O}(\mathbf{G}_1) \cong \mathscr{O}(\mathbf{G}_1)^*$, see [J3, Lemma I.8.7]; as a consequence, a $\mathbf{G}_1$-module is injective if and only if it is projective. More precisely, by [J3, Equation II.11.5(4)], for any $\lambda \in \mathbb{X}$ the $\mathbf{G}_1$-module $\mathsf{Q}(\lambda)$ is also the projective cover of $\mathsf{L}(\lambda)$.

2.2. Representations of $\mathbf{G}_1\mathbf{T}$. The considerations of §2.1 have analogues for the larger subgroup scheme $\mathbf{G}_1\mathbf{T}$, defined as the preimage of $\mathbf{T}^{(1)}$ under the Frobenius morphism $\mathrm{Fr}_{\mathbf{G}} : \mathbf{G} \to \mathbf{G}^{(1)}$. Namely, the datum of a $\mathbf{G}_1\mathbf{T}$-module structure on a $\Bbbk$-vector space V is equivalent to that of a $\mathbf{G}_1$-module structure (in other words, an action of $\mathcal{U}_0\mathfrak{g}$) together with a $\mathbf{T}$-module structure (i.e. an $\mathbb{X}$-grading) such that the restricted enveloping algebra $\mathcal{U}_0\mathfrak{t}$ of $\mathfrak{t}$ acts on the λ-graded part of V via the character $\mathcal{U}_0\mathfrak{t} \to \Bbbk$ defined by the differential of λ, for any $\lambda \in \mathbb{X}$. In particular, each $\mathbf{G}_1\mathbf{T}$-module has an action of $\mathbf{T}$, hence we can speak of its $\mathbf{T}$-weights.

The category of finite-dimensional representations of $\mathbf{G}_1\mathbf{T}$ will be denoted $\mathsf{Rep}(\mathbf{G}_1\mathbf{T})$. We have natural "restriction" functors

$$\mathsf{Rep}(\mathbf{G}) \to \mathsf{Rep}(\mathbf{G}_1\mathbf{T}), \quad \mathsf{Rep}(\mathbf{G}_1\mathbf{T}) \to \mathsf{Rep}(\mathbf{G}_1),$$

which we will denote by $V \mapsto V_{|\mathbf{G}_1\mathbf{T}}$, resp. $V \mapsto V_{|\mathbf{G}_1}$. In terms of the description above, the second functor corresponds to forgetting the $\mathbf{T}$-action, i.e. the $\mathbb{X}$-grading. In this category we can again consider the socle $\mathrm{soc}_{\mathbf{G}_1\mathbf{T}}(M)$ and the top $\mathrm{top}_{\mathbf{G}_1\mathbf{T}}(M)$ of a $\mathbf{G}_1\mathbf{T}$-module M, and each simple objects has an injective hull and a projective cover characters as in the case of $\mathbf{G}_1$. Below we will use the fact that for $M \in \mathsf{Rep}(\mathbf{G}_1\mathbf{T})$ we have

$$\mathrm{soc}_{\mathbf{G}_1}(M_{|\mathbf{G}_1}) = \mathrm{soc}_{\mathbf{G}_1\mathbf{T}}(M), \tag{2.2}$$

see [J3, Remark in §II.9.6].

For any $\lambda \in \mathbb{X}$, the baby Verma module $\mathsf{Z}(\lambda)$ can be "lifted" to a $\mathbf{G}_1\mathbf{T}$-module $\widehat{\mathsf{Z}}(\lambda)$, defined also as

$$\widehat{\mathsf{Z}}(\lambda) := \mathcal{U}_0\mathfrak{g} \otimes_{\mathcal{U}_0\mathfrak{b}^+} \Bbbk_{\mathfrak{b}^+}(\lambda),$$

where $\mathbf{T}$ acts on $\Bbbk_{\mathfrak{b}^+}(\lambda)$ via the character λ, and $\mathcal{U}_0\mathfrak{g}$ by multiplication on the left. Now the $\mathbf{G}_1\mathbf{T}$-module $\widehat{\mathsf{Z}}(\lambda)$ really depends on λ, and not only on its class in $\mathbb{X}/p\mathbb{X}$. In fact, for any $\lambda \in \mathbb{X}$ and $\mu \in X^*(\mathbf{T}^{(1)})$ we have a canonical isomorphism

$$\widehat{\mathsf{Z}}(\lambda + \mathrm{Fr}^*_{\mathbf{T}}(\mu)) \cong \widehat{\mathsf{Z}}(\lambda) \otimes \Bbbk_{\mathbf{T}^{(1)}}(\mu),$$

where $\Bbbk_{\mathbf{T}^{(1)}}(\mu)$ is seen as a $\mathbf{G}_1\mathbf{T}$-module via the canonical morphism $\mathbf{G}_1\mathbf{T} \to \mathbf{T}^{(1)}$.

The following theorem is an analogue of Theorem 2.1 and Theorem 2.2. For a proof, see [J3, Proposition II.9.6].

Theorem 2.6. *(1) For any $\lambda \in \mathbb{X}$, the top $\widehat{\mathsf{L}}(\lambda)$ of $\widehat{\mathsf{Z}}(\lambda)$ is simple. Moreover, for any $\lambda \in \mathbb{X}$ and $\mu \in X^*(\mathbf{T}^{(1)})$ we have a canonical isomorphism*

$$\widehat{\mathsf{L}}(\lambda + \mathrm{Fr}^*_{\mathbf{T}}(\mu)) \cong \widehat{\mathsf{L}}(\lambda) \otimes \Bbbk_{\mathbf{T}^{(1)}}(\mu), \tag{2.3}$$

and the assignment $\lambda \mapsto \widehat{\mathsf{L}}(\lambda)$ induces a bijection between $\mathbb{X}$ and the set of isomorphism classes of simple $\mathbf{G}_1\mathbf{T}$-modules.

(2) For any $\lambda \in \mathbb{X}$, we have an isomorphism $\widehat{\mathsf{L}}(\lambda)_{|\mathbf{G}_1} \cong \mathsf{L}_1(\lambda)$.

(3) For any $\lambda \in \mathbb{X}^+_{\mathrm{res}}$, the $\mathbf{G}_1\mathbf{T}$-module $\mathsf{L}(\lambda)_{|\mathbf{G}_1\mathbf{T}}$ is simple, and isomorphic to $\widehat{\mathsf{L}}(\lambda)$.

As in §2.1, in case $\mathscr{D}(\mathbf{G})$ is simply connected, using Theorem 2.6 one can describe the restriction to $\mathbf{G}_1\mathbf{T}$ of any simple $\mathbf{G}$-module. Namely, let $\lambda \in \mathbb{X}^+$, and write $\lambda = \mu + p\nu$ with $\mu \in \mathbb{X}^+_{\mathrm{res}}$ and $\nu \in \mathbb{X}$. Then there exists a unique dominant weight $\tilde{\nu}$ for $\mathbf{G}^{(1)}$ such that $p\nu = \mathrm{Fr}^*_{\mathbf{T}}(\tilde{\nu})$, and we have

$$\mathsf{L}(\lambda)_{|\mathbf{G}_1\mathbf{T}} \cong \widehat{\mathsf{L}}(\mu) \otimes \mathsf{L}^{(1)}(\tilde{\nu})_{|\mathbf{T}^{(1)}},$$

where $\mathbf{G}_1\mathbf{T}$ acts on $\mathsf{L}^{(1)}(\tilde{\nu})_{|\mathbf{T}^{(1)}}$ via the morphism $\mathbf{G}_1\mathbf{T} \to \mathbf{T}^{(1)}$ induced by $\mathrm{Fr}_{\mathbf{G}}$. (In other words, $\mathcal{U}_0\mathfrak{g}$ acts trivially, and the $\mathbb{X}$-grading is obtained from the action of $\mathbf{T}^{(1)}$ on $\mathsf{L}^{(1)}(\tilde{\nu})$ by pullback along the map $X^*(\mathbf{T}^{(1)}) \to \mathbb{X}$ induced by $\mathrm{Fr}_{\mathbf{T}}$.) Again, this implies in particular that $\mathsf{L}(\lambda)_{|\mathbf{G}_1\mathbf{T}}$ is semisimple.

Remark 2.7. (1) Essentially, the representation theory does not change when replacing $\mathbf{G}_1$ by $\mathbf{G}_1\mathbf{T}$, except for the fact that the labelling set $\mathbb{X}/p\mathbb{X}$ is replaced by $\mathbb{X}$. This replacement corrects the difficulty mentioned in Remark 2.5, in that the order $\leq$ also has a representation-theoretic meaning for $\mathbf{G}_1\mathbf{T}$; for instance, all the composition factors of $\widehat{\mathsf{Z}}(\lambda)$ are of the form $\widehat{\mathsf{L}}(\mu)$ with $\mu \leq \lambda$. But this fact creates other difficulties, in particular because the poset $(\mathbb{X}, \leq)$ does not admit any minimal element.

(2) Any $\mathbf{G}_1\mathbf{T}$-module is in particular a $\mathbf{T}$-module, hence has a character. If $X \subset \mathbb{X}$ is such that the composition $X \to \mathbb{X} \to \mathbb{X}/p\mathbb{X}$ is surjective, the isomorphisms (2.3) reduce the question of determining the characters of all simple $\mathbf{G}_1\mathbf{T}$-modules to the case of modules parametrized by an element in X. Assuming that $\mathscr{D}\mathbf{G}$ is simply connected, and using this observation in case $X = \mathbb{X}^+_{\mathrm{res}}$, Theorem 2.6 shows that the question of determining the characters of all simple $\mathbf{G}_1\mathbf{T}$-modules is equivalent to the question of determining the characters of the simple $\mathbf{G}$-modules $\mathsf{L}(\lambda)$ for $\lambda \in \mathbb{X}^+_{\mathrm{res}}$, which is itself equivalent to the question of determining characters of all simple $\mathbf{G}$-modules by Steinberg's tensor product theorem (see §2.4 in Chapter 1).

We note the following fact for later use.

LEMMA 2.8. *For any $\lambda \in \mathbb{X}$, the simple $\mathbf{G}_1\mathbf{T}$-module $\widehat{\mathsf{L}}(2(p-1)\rho - \lambda)^*$ is a composition factor of $\widehat{\mathsf{Z}}(\lambda)$ with multiplicity 1.*

PROOF. By construction, $\widehat{\mathsf{Z}}(\lambda)$ admits $\lambda - 2(p-1)\rho$ as a minimal weight (with multiplicity 1). Hence $\widehat{\mathsf{Z}}(\lambda)^*$ admits $2(p-1)\rho - \lambda$ as a maximal weight, so that there exists a nonzero morphism of $\mathcal{U}_0\mathfrak{b}^+$-$\mathbf{T}$-modules $\Bbbk_{\mathfrak{b}^+}(2(p-1)\rho - \lambda) \to \widehat{\mathsf{Z}}(\lambda)^*$. Inducing to $\mathcal{U}_0\mathfrak{g}$ we deduce a nonzero morphism of $\mathbf{G}_1\mathbf{T}$-modules

$$\widehat{\mathsf{Z}}(2(p-1)\rho - \lambda) \to \widehat{\mathsf{Z}}(\lambda)^*.$$

This implies that the top $\widehat{\mathsf{L}}(2(p-1)\rho - \lambda)$ must appear as a composition factor of $\widehat{\mathsf{Z}}(\lambda)^*$ (with multiplicity 1 since the weight $2(p-1)\rho - \lambda$ has multiplicity 1). Dualizing, we deduce the desired claim. □

For any $\lambda \in \mathbb{X}$ we will denote by $\widehat{\mathsf{Q}}(\lambda)$ the injective hull of the simple $\mathbf{G}_1\mathbf{T}$-module $\widehat{\mathsf{L}}(\lambda)$ in the category $\mathsf{Rep}(\mathbf{G}_1\mathbf{T})$; by [J3, §II.11.3] we then have

$$\widehat{\mathsf{Q}}(\lambda)_{|\mathbf{G}_1} \cong \mathsf{Q}(\lambda). \tag{2.4}$$

Moreover, for $\lambda \in \mathbb{X}$ and $\mu \in X^*(\mathbf{T}^{(1)})$ we have a canonical isomorphism

$$\widehat{\mathsf{Q}}(\lambda + \mathrm{Fr}^*_{\mathbf{T}}(\mu)) \cong \widehat{\mathsf{Q}}(\lambda) \otimes \Bbbk_{\mathbf{T}^{(1)}}(\mu), \tag{2.5}$$

and $\widehat{\mathsf{Q}}(\lambda)$ is also the projective cover of $\widehat{\mathsf{L}}(\lambda)$ in the category of $\mathbf{G}_1\mathbf{T}$-modules. These objects behave in a way similar to injective objects in highest weight categories, as shown by the following result due to Humphreys. (For a proof, see [J3, Proposition II.11.4].)

PROPOSITION 2.9. *For any $\lambda \in \mathbb{X}$, the $\mathbf{G}_1\mathbf{T}$-module $\widehat{\mathsf{Q}}(\lambda)$ admits a filtration with subquotients of the form $\widehat{\mathsf{Z}}(\mu)$ with $\mu \in \mathbb{X}$. Moreover, the number of occurrences of $\widehat{\mathsf{Z}}(\mu)$ in such a filtration does not depend on the choice of filtration, and is equal to the multiplicity $[\widehat{\mathsf{Z}}(\mu) : \widehat{\mathsf{L}}(\lambda)]$ of $\widehat{\mathsf{L}}(\lambda)$ as a composition factor of $\widehat{\mathsf{Z}}(\mu)$.*

Below we will also use the following property, for which we refer to [J3, Proposition II.10.2]. Here we assume (as e.g. in §2.10) that there exists a weight $\varsigma \in \mathbb{X}$ such that $\langle \varsigma, \alpha^\vee \rangle = 1$ for any $\alpha \in \mathfrak{R}^{\mathrm{s}}$.

PROPOSITION 2.10. *The Steinberg module* $\mathsf{L}((p-1)\varsigma)$ *is injective and projective as a* $\mathbf{G}_1$*-module and as a* $\mathbf{G}_1\mathbf{T}$*-module. As a consequence we have*

$$\widehat{\mathsf{Q}}((p-1)\varsigma) = \widehat{\mathsf{Z}}((p-1)\varsigma) = \widehat{\mathsf{L}}((p-1)\varsigma) = \mathsf{L}((p-1)\varsigma)_{|\mathbf{G}_1\mathbf{T}}.$$

2.3. Existence of a G-module structure on injective hulls. For the rest of this section we assume that $\mathscr{D}(\mathbf{G})$ is simply connected, and we fix a weight $\varsigma \in \mathbb{X}$ such that $\langle \varsigma, \alpha^\vee \rangle = 1$ for any $\alpha \in \mathfrak{R}^{\mathrm{s}}$.

The following definition is ad-hoc, and will be used only in the current chapter. (This notion is taken from [**J1**], although our definition is slightly different.)

DEFINITION 2.11. We say that a $\mathbf{G}$-module V is *p-bounded* if for any weight μ of V and any dominant short root α we have $\langle \mu, \alpha^\vee \rangle \leq \langle (2p-1)\varsigma, \alpha^\vee \rangle$.

REMARK 2.12. The coroots of the form $\alpha^\vee$ with α a dominant short root are exactly the coroots which are maximal (for the standard order). There are as many such coroots as irreducible components in the root system of $(\mathbf{G}, \mathbf{T})$, and for such α the integer $\langle \varsigma, \alpha^\vee \rangle + 1$ is the Coxeter number of the corresponding component. The main property we will use is that if $\lambda \in \mathbb{X}^+ \cap \mathbb{Z}\mathfrak{R}$, then we have $\langle \lambda, \alpha^\vee \rangle \geq 2$ for some dominant short root α. (In fact, the nonzero dominant weights λ such that $\langle \lambda, \alpha^\vee \rangle \in \{0, 1\}$ for any $\alpha \in \mathfrak{R}^+$ are the *minuscule* dominant weights. It is a standard fact that these weights are representatives for the nontrivial cosets in $\mathbb{X}/\mathbb{Z}\mathfrak{R}$; in particular, none of them belongs to $\mathbb{Z}\mathfrak{R}$.)

We will denote by $\mathbb{X}^+_{\mathrm{b}}$ the subset of $\mathbb{X}^+$ consisting of dominant weights μ which satisfy $\langle \mu, \alpha^\vee \rangle \leq \langle (2p-1)\varsigma, \alpha^\vee \rangle$ for any dominant short root α. We will also denote by $\mathsf{Rep}_{\mathrm{b}}(\mathbf{G})$ the category of finite-dimensional p-bounded $\mathbf{G}$-modules, i.e. the Serre subcategory of $\mathsf{Rep}(\mathbf{G})$ generated by the simple modules $\mathsf{L}(\mu)$ with $\mu \in \mathbb{X}^+_{\mathrm{b}}$. The subset $\mathbb{X}^+_{\mathrm{b}} \subset \mathbb{X}^+$ is an ideal for $\leq$; therefore the category $\mathsf{Rep}_{\mathrm{b}}(\mathbf{G})$ has a natural highest weight structure, see Lemma 1.3(2) in Appendix A. Moreover, each block in $\mathsf{Rep}(\mathbf{G})$ has only finitely many p-bounded simple modules; therefore this category has enough injective (and projective) objects, see Theorem 2.1 in Appendix A. For $\lambda \in \mathbb{X}^+_{\mathrm{b}}$, we will denote by $\mathsf{R}(\lambda)$ the injective hull of $\mathsf{L}(\lambda)$ in $\mathsf{Rep}_{\mathrm{b}}(\mathbf{G})$.

Note that if λ is a restricted dominant weight then $\langle \lambda, \alpha^\vee \rangle \leq \langle (p-1)\varsigma, \alpha^\vee \rangle$ for any simple root, hence for any positive root. As a consequence, we have $\mathbb{X}^+_{\mathrm{res}} \subset \mathbb{X}^+_{\mathrm{b}}$.

The main result of the present subsection is the following.

THEOREM 2.13. *Assume that* $p \geq 2h-2$. *For any* $\lambda \in \mathbb{X}^+_{\mathrm{res}}$*, we have an isomorphism of* $\mathbf{G}_1$*-modules*

$$\mathsf{R}(\lambda)_{|\mathbf{G}_1} \cong \mathsf{Q}(\lambda).$$

In particular, $\mathsf{Q}(\lambda)$ *admits a structure of* $\mathbf{G}$*-module.*

2.4. Preliminaries. We start with some preliminary results. As in §2.4 of Chapter 1, in addition to $\mathbf{G}$-modules we will consider $\mathbf{G}^{(1)}$-modules, and we will use the same conventions and notation as in this subsection. We will also denote by $\mathfrak{R}^{(1)} \subset X^*(\mathbf{T}^{(1)})$ the root system of $(\mathbf{G}^{(1)}, \mathbf{T}^{(1)})$, by $\rho^{(1)}$ the halfsum of the positive roots, by $C^{(1)} \subset \mathbb{R} \otimes_{\mathbb{Z}} X^*(\mathbf{T}^{(1)})$ the corresponding fundamental alcove, and $W^{(1)}_{\mathrm{aff}}$ the associated affine Weyl group. Here the Weyl group of $(\mathbf{G}^{(1)}, \mathbf{T}^{(1)})$ identifies canonically with W, so that we have $W^{(1)}_{\mathrm{aff}} = W \ltimes \mathbb{Z}\mathfrak{R}^{(1)}$.

Let M, N be $\mathbf{G}$-modules. Then the vector space $\mathrm{Hom}_{\mathbf{G}_1}(M, N)$ admits a natural structure of module over $\mathbf{G}/\mathbf{G}_1 \cong \mathbf{G}^{(1)}$. By the linkage principle (see Corollary 2.13 in Chapter 1) for the group $\mathbf{G}^{(1)}$, we have a canonical decomposition as

$\mathbf{G}^{(1)}$-modules

$$\mathrm{Hom}_{\mathbf{G}_1}(M,N) = \bigoplus_{\nu \in \overline{C^{(1)}} \cap X^*(\mathbf{T}^{(1)})} \mathrm{Hom}^{\nu}_{\mathbf{G}_1}(M,N)$$

where all the composition factors of $\mathrm{Hom}^{\nu}_{\mathbf{G}_1}(M,N)$ are of the form $\mathsf{L}^{(1)}(\lambda)$ with λ in the orbit of ν under the action of $W^{(1)}_{\mathrm{aff}}$. Of course, we then have an inclusion

$$\mathrm{Hom}_{\mathbf{G}}(M,N) = (\mathrm{Hom}_{\mathbf{G}_1}(M,N))^{\mathbf{G}^{(1)}} \subset \mathrm{Hom}^{0}_{\mathbf{G}_1}(M,N).$$

The following lemma is the key step for the later proofs in this section.

LEMMA 2.14. *Assume that $p \geq 2h-2$. Let M, N be $\mathbf{G}$-modules, such that M is p-bounded and $\mathrm{soc}_{\mathbf{G}_1}(N)$, resp. $\mathrm{top}_{\mathbf{G}_1}(N)$, is simple. Then the embedding*

$$\mathrm{Hom}_{\mathbf{G}}(M,N) \subset \mathrm{Hom}^{0}_{\mathbf{G}_1}(M,N), \quad \textit{resp.} \quad \mathrm{Hom}_{\mathbf{G}}(N,M) \subset \mathrm{Hom}^{0}_{\mathbf{G}_1}(N,M),$$

is an equality.

PROOF. We explain the proof if the first variant; the other variant follows by duality (or directly by similar arguments). We therefore assume that $\mathrm{soc}_{\mathbf{G}_1}(N)$ is simple. As explained in §2.1, this socle is a sub-$\mathbf{G}$-module of N, which has to be simple as $\mathbf{G}$-module, hence isomorphic to $\mathsf{L}(\lambda)$ for some $\lambda \in \mathbb{X}^+$. By Lemma 2.3 we have $\lambda \in \mathbb{X}^+_{\mathrm{res}}$, and then by Theorem 2.2 and Theorem 2.6 (see also (2.2)) we have

$$\mathrm{soc}_{\mathbf{G}_1}(N) \cong \mathsf{L}_1(\lambda), \quad \mathrm{soc}_{\mathbf{G}_1\mathbf{T}}(N) \cong \widehat{\mathsf{L}}(\lambda).$$

We have

$$\mathrm{Hom}_{\mathbf{G}}(M,N) = (\mathrm{Hom}_{\mathbf{G}_1}(M,N))^{\mathbf{G}^{(1)}} = (\mathrm{Hom}^{0}_{\mathbf{G}_1}(M,N))^{\mathbf{G}^{(1)}};$$

since $\mathsf{L}^{(1)}(0)$ has no nontrivial self-extensions (as follows e.g. from Proposition 2.5 in Chapter 1), to prove our claim it therefore suffices to show that all the composition factors of the $\mathbf{G}^{(1)}$-module $\mathrm{Hom}^{0}_{\mathbf{G}_1}(M,N)$ are isomorphic to $\mathsf{L}^{(1)}(0)$. Assume the contrary; then this module admits a composition factor of the form $\mathsf{L}^{(1)}(\mu)$ with $\mu \in W^{(1)}_{\mathrm{aff}} \cdot_p 0 \smallsetminus \{0\}$. Write $\mu = (t_{\mu'}v) \cdot_p 0$ with $\mu' \in \mathbb{Z}\mathfrak{R}^{(1)}$ and $v \in W$. Then the considerations of §2.8 in Chapter 1 show that μ' is dominant and nonzero. By Remark 2.12, it follows that $\langle \mu', \alpha^\vee \rangle \geq 2$ for some short dominant root α of $\mathbf{G}^{(1)}$. We deduce that

$$\begin{aligned} \text{(2.6)} \quad \langle \mu, \alpha^\vee \rangle &= \langle p\mu' + v\rho^{(1)} - \rho^{(1)}, \alpha^\vee \rangle \\ &\geq 2p + \langle \rho^{(1)}, v^{-1}\alpha^\vee \rangle - \langle \rho^{(1)}, \alpha^\vee \rangle \geq 2p - 2\langle \rho^{(1)}, \alpha^\vee \rangle, \end{aligned}$$

because $v^{-1}\alpha^\vee$ is a coroot in the same component as $\alpha^\vee$.

On the other hand, the $\mathbf{T}$-module $\mathrm{Hom}^{0}_{\mathbf{G}_1}(M,N)$ has a nonzero weight space for the weight $\mathrm{Fr}^*_{\mathbf{T}}(w_0(\mu)) = w_0(\mathrm{Fr}^*_{\mathbf{T}}(\mu))$. Hence there exists a nonzero morphism of $\mathbf{G}_1\mathbf{T}$-modules $M \otimes \Bbbk_{\mathbf{T}^{(1)}}(w_0(\mu)) \to N$. The image of this morphism must contain $\mathrm{soc}_{\mathbf{G}_1\mathbf{T}}(N)$; therefore M admits a nonzero vector of $\mathbf{T}$-weight $\lambda - w_0\mathrm{Fr}^*_{\mathbf{T}}(\mu)$. The image of $\mathfrak{R}^{(1)}$ under the embedding $\mathrm{Fr}^*_{\mathbf{T}} : X^*(\mathbf{T}^{(1)}) \to \mathbb{X}$ is $p\mathfrak{R}$. If we denote by $\beta \in \mathfrak{R}$ the dominant short root such that $\mathrm{Fr}^*_{\mathbf{T}}(\alpha)$ is $p\beta$, then the image of $\beta^\vee$ under the morphism $X_*(\mathbf{T}) \to X_*(\mathbf{T}^{(1)})$ is $p\alpha^\vee$. Since M is p-bounded, we then have

$$\langle \lambda - w_0\mathrm{Fr}^*_{\mathbf{T}}(\mu), -w_0\beta^\vee \rangle \leq (2p-1)\langle \rho, -w_0\beta^\vee \rangle.$$

Since λ is dominant, we deduce that

$$p\langle \mu, \alpha^\vee \rangle = \langle \mathrm{Fr}^*_{\mathbf{T}}(\mu), \beta^\vee \rangle \leq (2p-1)\langle \rho, \beta^\vee \rangle = (2p-1)\langle \rho^{(1)}, \alpha^\vee \rangle.$$

This inequality contradicts (2.6) as soon as $p \geq 2\langle \rho^{(1)}, \alpha^\vee \rangle$, which is automatic if $p \geq 2h-2$ (since the Coxeter number of $\mathbf{G}^{(1)}$ is the same as that of $\mathbf{G}$). $\square$

The first consequence of Lemma 2.14 we will consider is the following.

Corollary 2.15. *Assume that $p \geq 2h-2$. Let M be a p-bounded $\mathbf{G}$-module, and assume that M admits a unique simple sub-$\mathbf{G}$-module, isomorphic to $\mathsf{L}(\lambda)$ for some $\lambda \in \mathbb{X}^+_{\mathrm{res}}$. Then $\mathrm{soc}_{\mathbf{G}_1}(M) = \mathsf{L}_1(\lambda)$; in particular, M is indecomposable as a $\mathbf{G}_1$-module.*

Proof. Our assumption implies that any nonzero sub-$\mathbf{G}$-module of M admits a unique simple submodule, hence is indecomposable. This applies in particular to the submodule $\mathrm{soc}_{\mathbf{G}_1}(M)$ (see §2.1). If we choose a subset $\Lambda \subset \mathbb{X}^+_{\mathrm{res}}$ of representatives for the quotient $\mathbb{X}/p\mathbb{X}$ which contains λ (see Remark 2.4), then as $\mathbf{G}$-modules we have

$$\mathrm{soc}_{\mathbf{G}_1}(M) = \bigoplus_{\mu \in \Lambda} \mathrm{Hom}_{\mathbf{G}_1}(\mathsf{L}(\mu), M) \otimes \mathsf{L}(\mu)$$

$$= \bigoplus_{\mu \in \Lambda} \bigoplus_{\nu \in \overline{C^{(1)}} \cap X^*(\mathbf{T}^{(1)})} \mathrm{Hom}^\nu_{\mathbf{G}_1}(\mathsf{L}(\mu), M) \otimes \mathsf{L}(\mu).$$

By indecomposability, there exists exactly one pair (μ, ν) such that

$$\mathrm{Hom}^\nu_{\mathbf{G}_1}(\mathsf{L}(\mu), M) \neq 0.$$

Since $\mathrm{Hom}_{\mathbf{G}}(\mathsf{L}(\lambda), M) \neq 0$, this pair must be $(\lambda, 0)$, and by Lemma 2.14 (in its second variant) we have

$$\mathrm{Hom}^0_{\mathbf{G}_1}(\mathsf{L}(\lambda), M) = \mathrm{Hom}_{\mathbf{G}}(\mathsf{L}(\lambda), M) = \Bbbk.$$

The claim follows. $\square$

2.5. Proof of Theorem 2.13. We can now give the proof of Theorem 2.13.

Proof of Theorem 2.13. We assume that $p \geq 2h-2$, and fix $\lambda \in \mathbb{X}^+_{\mathrm{res}}$. Let us assume that there exists a $\mathbf{G}$-module M which is p-bounded, which contains $\mathsf{L}(\lambda)$ as a $\mathbf{G}$-submodule, and which is injective as a $\mathbf{G}_1$-module. Then since $\mathsf{R}(\lambda)$ is injective in $\mathsf{Rep}_{\mathrm{b}}(\mathbf{G})$ the embedding $\mathsf{L}(\lambda) \hookrightarrow \mathsf{R}(\lambda)$ factors through a $\mathbf{G}$-module morphism $M \to \mathsf{R}(\lambda)$. On the other hand, since M is injective as a $\mathbf{G}_1$-module and contains $\mathsf{L}(\lambda)$ in its socle as a $\mathbf{G}_1$-module, it contains $\mathsf{Q}(\lambda)$ as a direct summand.

Consider now the composition $\mathsf{Q}(\lambda) \hookrightarrow M \to \mathsf{R}(\lambda)$, a morphism of $\mathbf{G}_1$-modules. This morphism is injective on the $\mathbf{G}_1$-socle of $\mathsf{Q}(\lambda)$, hence is injective. Since $\mathsf{Q}(\lambda)$ is injective, its image must be a direct summand of $\mathsf{R}(\lambda)$ as a $\mathbf{G}_1$-module. On the other hand, it follows from Corollary 2.15 that $\mathsf{R}(\lambda)$ is indecomposable as a $\mathbf{G}_1$-module; hence our morphism $\mathsf{Q}(\lambda) \to \mathsf{R}(\lambda)$ is an isomorphism, which proves the desired isomorphism.

To finish the proof, it remains to show the existence of a $\mathbf{G}$-module M as above. Since $\lambda \in \mathbb{X}^+_{\mathrm{res}}$, we have $(p-1)(2\rho - \varsigma) + w_0\lambda \in \mathbb{X}^+$. Hence we can consider the $\mathbf{G}$-module

$$M = \mathsf{L}\big((p-1)(2\rho - \varsigma) + w_0\lambda\big) \otimes \mathsf{L}\big((p-1)\varsigma\big).$$

We claim that this module satisfies the desired properties. Indeed we have

$$\mathrm{Hom}_{\mathbf{G}}(\mathsf{L}(\lambda), M) \cong \mathrm{Hom}_{\mathbf{G}}\big(\mathsf{L}(\lambda) \otimes \mathsf{L}((p-1)\varsigma - \lambda), \mathsf{L}((p-1)\varsigma)\big)$$

$$\cong \mathrm{Hom}_{\mathbf{B}}\big(\mathsf{L}(\lambda) \otimes \mathsf{L}((p-1)\varsigma - \lambda), \Bbbk_{\mathbf{B}}((p-1)\varsigma)\big)$$

by Frobenius reciprocity, since $\mathsf{L}((p-1)\varsigma) = \mathsf{N}((p-1)\varsigma)$ (see §2.10 in Chapter 1). Now $(p-1)\varsigma$ is maximal among the $\mathbf{T}$-weights of $\mathsf{L}(\lambda) \otimes \mathsf{L}((p-1)\varsigma - \lambda)$; hence there exists a nonzero morphism of $\mathbf{B}$-modules $\mathsf{L}(\lambda) \otimes \mathsf{L}((p-1)\varsigma - \lambda) \to \Bbbk_{\mathbf{B}}((p-1)\varsigma)$, from which we obtain a nonzero (hence injective) morphism of $\mathbf{G}$-modules $\mathsf{L}(\lambda) \to M$. On the other hand, for any $\mu \in \mathrm{wt}(M)$ and any dominant short root α we have

$$\langle \mu, \alpha^\vee \rangle \leq \langle 2(p-1)\rho + w_0\lambda, \alpha^\vee \rangle \leq \langle (2p-1)\rho, \alpha^\vee \rangle$$

since λ is dominant, so that M is p-bounded. Finally, M is injective as a $\mathbf{G}_1$-module because so is $\mathsf{L}((p-1)\varsigma)$, see Proposition 2.10. □

REMARK 2.16. In the proof above, the morphism of $\mathbf{G}$-module $M \to \mathsf{R}(\lambda)$ is surjective, so that each weight of $\mathsf{R}(\lambda)$ is also a weight of M. Using the choice of M considered in this proof, we deduce that

$$\mu \in \mathrm{wt}(\mathsf{R}(\lambda)) \quad \Rightarrow \quad \mu \preceq 2(p-1)\rho + w_0\lambda.$$

Once Theorem 2.13 is proven we obtain the following slightly more precise claim.

COROLLARY 2.17. *Assume that $p \geq 2h-2$. For any $\lambda \in \mathbb{X}^+_{\mathrm{res}}$ we have an isomorphism of $\mathbf{G}_1\mathbf{T}$-modules*

$$\mathsf{R}(\lambda)_{|\mathbf{G}_1\mathbf{T}} \cong \widehat{\mathsf{Q}}(\lambda).$$

PROOF. Since $\mathsf{R}(\lambda)$ is injective as a $\mathbf{G}_1$-module, it is also injective as a $\mathbf{G}_1\mathbf{T}$-module (see [**J3**, Lemma II.9.4]). Since there exists an embedding of $\mathbf{G}$-modules $\mathsf{L}(\lambda) \hookrightarrow \mathsf{R}(\lambda)$, in view of Theorem 2.6 there also exists an embedding of $\mathbf{G}_1\mathbf{T}$-modules $\widehat{\mathsf{L}}(\lambda) \hookrightarrow \mathsf{R}(\lambda)_{|\mathbf{G}_1\mathbf{T}}$, so that $\mathsf{R}(\lambda)_{|\mathbf{G}_1\mathbf{T}}$ contains $\widehat{\mathsf{Q}}(\lambda)$ as a direct summand. Finally, since

$$\dim(\mathsf{R}(\lambda)) = \dim(\mathsf{Q}(\lambda)) = \dim(\widehat{\mathsf{Q}}(\lambda))$$

(see (2.4)), we deduce the desired claim. □

2.6. Relation with tilting modules. We now explain the connection of Theorem 2.13 with the main topic of this chapter, namely tilting modules.

For any $\lambda \in \mathbb{X}^+_{\mathrm{b}}$, since the object $\mathsf{R}(\lambda)$ is injective in the highest weight category $\mathsf{Rep}_{\mathrm{b}}(\mathbf{G})$, it admits a costandard filtration, and satisfies the reciprocity formula

$$(\mathsf{R}(\lambda) : \mathsf{N}(\mu)) = [\mathsf{M}(\mu) : \mathsf{L}(\lambda)]$$

for any $\mu \in \mathbb{X}^+_{\mathrm{b}}$, see Theorem 2.1 in Appendix A.

The subset $\mathbb{X}^+_{\mathrm{b}} \subset \mathbb{X}^+$ is stable under the operation $\mu \mapsto -w_0\mu$. In view of (1.4) in Chapter 1, the subcategory $\mathsf{Rep}_{\mathrm{b}}(\mathbf{G}) \subset \mathsf{Rep}(\mathbf{G})$ is therefore stable under the duality operation $V \mapsto V^*$.

LEMMA 2.18. *Assume that $p \geq 2h-2$. Let $M \in \mathsf{Rep}_{\mathrm{b}}(\mathbf{G})$ and $\mu \in \mathbb{X}^+_{\mathrm{res}}$, and assume that we have $M_{|\mathbf{G}_1\mathbf{T}} \cong \widehat{\mathsf{Q}}(\mu)$. Then $M \cong \mathsf{R}(\mu)$ (as $\mathbf{G}$-modules).*

PROOF. Since $M_{|\mathbf{G}_1\mathbf{T}} \cong \widehat{\mathsf{Q}}(\mu)$, we have $M_{|\mathbf{G}_1} \cong \mathsf{Q}(\mu)$ (see (2.4)), and in particular $\mathrm{soc}_{\mathbf{G}_1}(M)$ is simple. By Lemma 2.14 (in its first variant) we deduce that

$$\mathrm{Hom}_{\mathbf{G}}(\mathsf{L}(\mu), M) \cong \mathrm{Hom}^0_{\mathbf{G}_1}(\mathsf{L}(\mu), M), \quad \mathrm{Hom}_{\mathbf{G}}(\mathsf{R}(\mu), M) \cong \mathrm{Hom}^0_{\mathbf{G}_1}(\mathsf{R}(\mu), M).$$

Since M is injective as a $\mathbf{G}_1$-module, the embedding $\mathsf{L}(\mu) \hookrightarrow \mathsf{R}(\mu)$ induces a surjection

$$\mathrm{Hom}_{\mathbf{G}_1}(\mathsf{R}(\mu), M) \twoheadrightarrow \mathrm{Hom}_{\mathbf{G}_1}(\mathsf{L}(\mu), M),$$

hence a surjection

$$\mathrm{Hom}^0_{\mathbf{G}_1}(\mathsf{R}(\mu), M) \twoheadrightarrow \mathrm{Hom}^0_{\mathbf{G}_1}(\mathsf{L}(\mu), M),$$

so that finally the induced morphism

$$\mathrm{Hom}_{\mathbf{G}}(\mathsf{R}(\mu), M) \to \mathrm{Hom}_{\mathbf{G}}(\mathsf{L}(\mu), M)$$

is surjective. Now since the socle of M as a $\mathbf{G}_1\mathbf{T}$-module is $\widehat{\mathsf{L}}(\mu) = \mathsf{L}(\mu)_{|\mathbf{G}_1\mathbf{T}}$, this $\mathbf{G}$-module contains a unique simple sub-$\mathbf{G}$-module, isomorphic to $\mathsf{L}(\mu)$. The surjectivity proved above implies that the embedding $\mathsf{L}(\mu) \hookrightarrow M$ factors through a morphism of $\mathbf{G}$-modules $\mathsf{R}(\mu) \to M$. Since this morphism is injective on the unique simple submodule $\mathsf{L}(\mu)$ of $\mathsf{R}(\mu)$, it is injective; comparing the dimensions of $\mathsf{R}(\mu)$ and M we conclude that it is an isomorphism, which finishes the proof. □

Corollary 2.19. *Assume that $p \geq 2h-2$. For $\lambda \in \mathbb{X}^+_{\mathrm{res}}$ we have $\mathsf{R}(\lambda)^* \cong \mathsf{R}(-w_0\lambda)$.*

Proof. Fix $\lambda \in \mathbb{X}^+_{\mathrm{res}}$. Since, by Corollary 2.17, $\mathsf{R}(\lambda)$ is isomorphic to $\widehat{\mathsf{Q}}(\lambda)$ as a $\mathbf{G}_1\mathbf{T}$-module, $\mathsf{R}(\lambda)^*$ is the projective cover of $\widehat{\mathsf{L}}(\lambda)^* = \widehat{\mathsf{L}}(-w_0\lambda)$ as a $\mathbf{G}_1\mathbf{T}$-module, hence is isomorphic (as a $\mathbf{G}_1\mathbf{T}$-module) to $\widehat{\mathsf{Q}}(-w_0\lambda)$, see §2.2. By Lemma 2.18 we deduce that $\mathsf{R}(\lambda)^* \cong \mathsf{R}(-w_0\lambda)$, as desired. □

We finally obtain the desired relation between the modules considered above and tilting modules.

Proposition 2.20. *Assume that $p \geq 2h-2$. For any $\lambda \in \mathbb{X}^+_{\mathrm{res}}$, the $\mathbf{G}$-module $\mathsf{R}(\lambda)$ is tilting, and isomorphic to $\mathsf{T}(2(p-1)\rho + w_0\lambda)$.*

Proof. Fix $\lambda \in \mathbb{X}^+_{\mathrm{res}}$. As explained at the beginning of this subsection $\mathsf{R}(\lambda)$ admits a costandard filtration. On the other hand, it follows from Corollary 2.19 that $\mathsf{R}(\lambda)^*$ admits a costandard filtration; hence $\mathsf{R}(\lambda)$ admits a standard filtration, and is therefore tilting. This $\mathbf{G}$-module is indecomposable by assumption; hence to conclude it only remains to determine its highest weight.

First, we have observed in Remark 2.16 that all the weights μ of $\mathsf{R}(\lambda)$ satisfy $\mu \preceq 2(p-1)\rho + w_0\lambda$. On the other hand, we have $\mathsf{R}(\lambda) \cong \widehat{\mathsf{Q}}(\lambda)$ as $\mathbf{G}_1\mathbf{T}$-modules by Corollary 2.17. By Lemma 2.8, the baby Verma module $\widehat{\mathsf{Z}}(2(p-1)\rho + w_0\lambda)$ admits $\widehat{\mathsf{L}}(-w_0\lambda)^*$ as a composition factor. Now we have

$$\widehat{\mathsf{L}}(-w_0\lambda)^* = (\mathsf{L}(-w_0\lambda)_{|\mathbf{G}_1\mathbf{T}})^* \cong (\mathsf{L}(-w_0\lambda)^*)_{|\mathbf{G}_1\mathbf{T}} \cong \mathsf{L}(\lambda)_{\mathbf{G}_1\mathbf{T}} \cong \widehat{\mathsf{L}}(\lambda)$$

by Theorem 2.6 and (1.4) in Chapter 1. By reciprocity (see Proposition 2.9) we deduce that $\widehat{\mathsf{Z}}(2(p-1)\rho + w_0\lambda)$ appears as a subquotient in a filtration of $\widehat{\mathsf{Q}}(\lambda)$, hence that $2(p-1)\rho + w_0\lambda$ is a $\mathbf{T}$-weight of $\widehat{\mathsf{Q}}(\lambda)$. Corollary 2.17 then implies that $2(p-1)\rho + w_0\lambda$ occurs as a $\mathbf{T}$-weight of $\mathsf{R}(\lambda)$, hence is its highest weight, which finishes the proof. □

2.7. Donkin's conjecture. Combining Proposition 2.20 and Theorem 2.13, resp. Corollary 2.17, we obtain that for $\lambda \in \mathbb{X}^+_{\mathrm{res}}$, if $p \geq 2h-2$ we have

$$\mathsf{T}(2(p-1)\rho + w_0\lambda)_{|\mathbf{G}_1} \cong \mathsf{Q}(\lambda), \tag{2.7}$$

resp.

$$\mathsf{T}(2(p-1)\rho + w_0\lambda)_{|\mathbf{G}_1\mathbf{T}} \cong \widehat{\mathsf{Q}}(\lambda). \tag{2.8}$$

(In fact, since we know a priori that $\widehat{\mathsf{Q}}(\lambda)$ has highest weight $2(p-1)\rho + w_0\lambda$, see the proof of Proposition 2.20, these two properties are equivalent.)

It has been conjectured by Donkin in [D1] that (2.7) holds for any $\lambda \in \mathbb{X}^+_{\mathrm{res}}$, for any value of p. A counterexample to this conjecture has recently been found by Bendel–Nakano–Pillen–Sobaje [BNPS] for the simple group of type $\mathbf{G}_2$ in characteristic 2. In fact it is always true, and easy to see, that for any $\mu \in (p-1)\varsigma + \mathbb{X}^+$ (hence, in particular, when $\mu = 2(p-1)\rho + w_0\lambda$ for some $\lambda \in \mathbb{X}^+_{\mathrm{res}}$) the restriction $\mathsf{T}(\mu)_{|\mathbf{G}_1}$ is injective: this follows from the fact that the tensor product

$$\mathsf{T}\big(\mu - (p-1)\varsigma\big) \otimes \mathsf{T}\big((p-1)\varsigma\big)$$

is tilting by Theorem 1.2, and has μ as its highest weight, hence admits $\mathsf{T}(\mu)$ as a direct summand. Now this tensor product is injective as a $\mathbf{G}_1$-module because so it $\mathsf{T}((p-1)\varsigma) \cong \mathsf{L}((p-1)\varsigma)$ (see §1.5) by Proposition 2.10, hence the same holds for $\mathsf{T}(\mu)$. The more delicate question is wether or not $\mathsf{T}(2(p-1)\rho + w_0\lambda)$ is indecomposable as a $\mathbf{G}_1$-module when $\lambda \in \mathbb{X}^+_{\mathrm{res}}$; the precise condition on p which guarantees that this property holds is unclear at this point.

3. Applications

In this section we continue to assume (for simplicity) that $\mathscr{D}\mathbf{G}$ is simply connected, and fix a weight $\varsigma \in \mathbb{X}$ such that $\langle \varsigma, \alpha^\vee \rangle = 1$ for any $\alpha \in \mathfrak{R}^{\mathrm{s}}$.

3.1. Donkin's tensor product formula. The first application we will consider is some sort of analogue of Steinberg's tensor product formula (Theorem 2.9 in Chapter 1), due to Donkin. Here we continue with our conventions on representations of $\mathbf{G}^{(1)}$ from §1.5.4, §2.4 and §2.4 of Chapter 1.

THEOREM 3.1. *For any $\lambda \in \mathbb{X}^+_{\mathrm{res}}$ and any $\mu \in X^*(\mathbf{T}^{(1)})$ dominant, the $\mathbf{G}$-module*

$$\mathsf{T}((p-1)\varsigma + \lambda) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{T}^{(1)}(\mu))$$

is tilting, of highest weight $(p-1)\varsigma + \lambda + \mathrm{Fr}^*_{\mathbf{T}}(\mu)$. If $\mathsf{T}((p-1)\varsigma+\lambda)$ is indecomposable as a $\mathbf{G}^{(1)}$-module (which is automatic provided $p \geq 2h-2$), then we have

$$\mathsf{T}((p-1)\varsigma + \lambda) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{T}^{(1)}(\mu)) \cong \mathsf{T}\big((p-1)\varsigma + \lambda + \mathrm{Fr}^*_{\mathbf{T}}(\mu)\big).$$

REMARK 3.2. For $\mu \in X^*(\mathbf{T}^{(1)})$ dominant, it is *not* true in general that $\mathrm{Fr}^*_{\mathbf{G}}(\mathsf{T}^{(1)}(\mu))$ is tilting; it is only itself tensor product with each $\mathsf{T}((p-1)\varsigma + \lambda)$ ($\lambda \in \mathbb{X}^+_{\mathrm{res}}$) which has this property.

PROOF. By Theorem 1.2 the module $\mathsf{T}((p-1)\varsigma) \otimes \mathsf{T}(\lambda)$ is tilting, and its admits $(p-1)\varsigma + \lambda$ as its highest weight; it must therefore admit $\mathsf{T}((p-1)\varsigma+\lambda)$ as a direct summand. Now we have

$$\mathsf{T}((p-1)\varsigma) \otimes \mathsf{T}(\lambda) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{T}^{(1)}(\mu)) \cong \mathsf{T}((p-1)\varsigma + \mathrm{Fr}^*_{\mathbf{T}}(\mu)) \otimes \mathsf{T}(\lambda)$$

by Proposition 1.8, and the right-hand side is tilting by Theorem 1.2. It follows that $\mathsf{T}((p-1)\varsigma+\lambda) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{T}^{(1)}(\mu))$ is tilting, as desired. It is clear that this module has highest weight $(p-1)\varsigma + \lambda + \mathrm{Fr}^*_{\mathbf{T}}(\mu)$.

Now, let us assume that $\mathsf{T}((p-1)\varsigma+\lambda)$ is indecomposable as a $\mathbf{G}^{(1)}$-module. Then we have algebra isomorphisms

$$\begin{aligned}&\mathrm{End}_{\mathbf{G}}(\mathsf{T}((p-1)\varsigma+\lambda)\otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{T}^{(1)}(\mu)))\\ &\qquad = \big(\mathrm{End}_{\mathbf{G}_1}(\mathsf{T}((p-1)\varsigma+\lambda)\otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{T}^{(1)}(\mu)))\big)^{\mathbf{G}^{(1)}}\\ &\qquad\qquad \cong \big(\mathrm{End}_{\mathbf{G}_1}(\mathsf{T}((p-1)\varsigma+\lambda))\otimes \mathrm{End}_{\Bbbk}(\mathsf{T}^{(1)}(\mu))\big)^{\mathbf{G}^{(1)}}.\end{aligned}$$

Our assumption implies that $\mathrm{End}_{\mathbf{G}_1}(\mathsf{T}((p-1)\varsigma+\lambda))$ is a local algebra, hence that its Jacobson radical is a nilpotent $\mathbf{G}^{(1)}$-stable ideal which satisfies $\mathrm{End}_{\mathbf{G}_1}(\mathsf{T}((p-1)\varsigma+\lambda)) = \Bbbk\cdot\mathrm{id}\oplus I$. We deduce that

$$\begin{aligned}&\big(\mathrm{End}_{\mathbf{G}_1}(\mathsf{T}((p-1)\varsigma+\lambda))\otimes \mathrm{End}_{\Bbbk}(\mathsf{T}(\mu))\big)^{\mathbf{G}^{(1)}}\\ &\qquad = \mathrm{End}_{\mathbf{G}^{(1)}}(\mathsf{T}^{(1)}(\mu))\oplus \big(I\otimes \mathrm{End}_{\Bbbk}(\mathsf{T}(\mu))\big)^{\mathbf{G}^{(1)}},\end{aligned}$$

where $(I\otimes \mathrm{End}_{\Bbbk}(\mathsf{T}(\mu)))^{\mathbf{G}^{(1)}}$ is a nilpotent ideal. Since $\mathrm{End}_{\mathbf{G}^{(1)}}(\mathsf{T}^{(1)}(\mu))$ is local, this implies that $\mathrm{End}_{\mathbf{G}}(\mathsf{T}((p-1)\varsigma+\lambda)\otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{T}^{(1)}(\mu)))$ is local, i.e. that $\mathsf{T}((p-1)\varsigma+\lambda)\otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{T}^{(1)}(\mu))$ is indecomposable, as desired.

Finally, if $p\geq 2h-2$, the module $\mathsf{T}((p-1)\varsigma+\lambda)$ is indecomposable as a $\mathbf{G}_1$-module by (2.7). (Note that $\{2(p-1)\rho+w_0\mu : \mu\in\mathbb{X}^+_{\mathrm{res}}\} = \{(p-1)\varsigma+\lambda : \lambda\in\mathbb{X}^+_{\mathrm{res}}\} = \{\nu\in\mathbb{X}\mid \forall\alpha\in\mathfrak{R}^{\mathrm{s}},\ p-1\leq\langle\nu,\alpha^\vee\rangle\leq 2(p-1)\}$.) □

As noted above, Theorem 3.1 can be considered as an analogue of Steinberg's tensor product formula. There is however one important difference: while Steinberg's formula reduces the determination of characters of simple modules to the case the highest weight is restricted, which only involves a finite number of closures of alcoves (see §2.4 of Chapter 1), the analogous comment does *not* apply to tilting modules; in fact this formula reduces the question of describing characters of indecomposable tilting modules to the case when the highest weight belongs to $\mathbb{X}^+\smallsetminus((p-1)\varsigma+\mathbb{X}^+)$, which involves infinitely many closures of alcoves unless $\mathbf{G}$ is of type $\mathbf{A}_1$.

3.2. The case when $\mathbf{G}=\mathrm{SL}_2(\Bbbk)$. In this subsection we consider the case $\mathbf{G}=\mathrm{SL}_2$, and use the notation from Example 2.10 of Chapter 1. In particular, we identify $\mathbf{G}^{(1)}$ with $\mathbf{G}$ in the natural way. In this case $h=2$, so that the condition $p\geq 2h-2$ in Theorem 3.1 is always satisfied, and this result *does* allow to describe all indecomposable tilting modules, as we will explain.

Recall that we have $\mathbb{X}^+=\{r\varpi_1 : r\in\mathbb{Z}_{\geq 1}\}$. The considerations of §1.5.1 show that for $r\in\{0,\cdots,p-1\}$ we have

$$\mathsf{T}(r\varpi_1)=\mathsf{N}(r\varpi_1)=\mathsf{L}(r\varpi_1),$$

so that this module is described in §1.4.1 of Chapter 1.

Next, the considerations of §1.5.2 show that we have an exact sequence

$$\mathsf{N}(0)\hookrightarrow \mathsf{T}((2p-2)\varpi_1)\twoheadrightarrow \mathsf{N}((2p-2)\varpi_1).$$

Using an appropriate translation functor, we deduce that for any $j\in\{0,\cdots,p-2\}$ the module $\mathsf{T}((p+j)\varpi_1)$ fits in a short exact sequence

$$\mathsf{N}((p-2-j)\varpi_1)\hookrightarrow \mathsf{T}((p+j)\varpi_1)\twoheadrightarrow \mathsf{N}((p+j)\varpi_1). \tag{3.1}$$

EXAMPLE 3.3. In case $j = 0$, one has an even more explicit description of $\mathsf{T}(p\varpi_1)$: one can easily check that

$$\mathsf{T}(p\varpi_1) = \mathsf{N}(\varpi_1) \otimes \mathsf{N}((p-1)\varpi_1).$$

If now $r \geq 2p-1$, we can write $r = (p-1) + s + pt$ with $s \in \{0, \cdots, p-1\}$ and $t \in \mathbb{Z}_{\geq 0}$. Theorem 3.1 then says that

$$\mathsf{T}(r\varpi_1) \cong \mathsf{T}((p-1+s)\varpi_1) \otimes \mathsf{T}(t\varpi_1)^{(1)}, \tag{3.2}$$

where $\mathsf{T}((p-1+s)\varpi_1)$ is described above and $\mathsf{T}(t\varpi_1)$ can be considered known since $t < r$. This provides an inductive description of tilting modules.

REMARK 3.4. If $r \in \{0, \cdots, p-2\}$, then $\dim \mathsf{T}(r\varpi_1) = r+1$; in particular this dimension is not divisible by p. On the other hand, we have $\dim \mathsf{T}((p-1)\varpi_1) = p$, and if $j \in \{0, \cdots, p-2\}$ the exact sequence (3.1) shows that

$$\dim \mathsf{T}((p+j)\varpi_1) = (p+j+1) + (p-2-j+1) = 2p;$$

in particular, this dimension is divisible by p. Once this is known, (3.2) implies that p divides $\dim \mathsf{T}(r\varpi_1)$ for any $r \geq p-1$. In conclusion, p divides $\dim \mathsf{T}(r\varpi_1)$ if and only if $r \geq p-1$. This property is in fact a special case of a property proved by Georgiev–Mathieu and saying that for a general reductive group $\mathbf{G}$, if $p \geq h$ then $\dim(\mathsf{T}(\lambda))$ is divisible by p if and only if $\lambda \notin C$; see [**M2**, Lemma 9.3] or [**AHR**, Proposition 7.9].

To make this description more explicit, one proceeds as follows, following [**EH**]. Let us start with an elementary combinatorial lemma.

LEMMA 3.5. *(1) Any $n \in \mathbb{Z}_{\geq 0}$ can be written uniquely as*

$$n = \sum_{i=0}^{r} n_i p^i \text{ with } p-1 \leq n_i \leq 2p-2 \text{ for } i \in \{0, \cdots, r-1\} \text{ and } 0 \leq n_r \leq p-1.$$

(2) Write $n = \sum_{i=0}^{r} n_i p^i$ with $p-1 \leq n_i \leq 2p-2$ for $i \in \{0, \cdots, r-1\}$ and $0 \leq n_r \leq p-1$. Then the numbers of the form

$$m = \sum_{i=0}^{r-1} m_i p^i + n_r p^r$$

with $m_i \in \{n_i, 2p-2-n_i\}$ for $i \in \{0, \cdots, r-1\}$ are all distinct for distinct choices of the m_i's.

PROOF. (1) Let us first prove existence by induction. If $n \in \{0, \cdots, p-1\}$ we set $r = 0$ and $n_0 = n$. Then if $n > p-1$ one writes $n = s+pb$ with $s \in \{0, \cdots, p-1\}$ and $b \in \mathbb{Z}_{\geq 1}$. If $s = p-1$ one sets $n_0 = p-1$ and uses the decomposition of b. Otherwise we write $n = (s+p) + p(b-1)$, choose $n_0 = s+p$, and use the decomposition of $b-1$.

For unicity, we again argue by induction. If $n \in \{0, \cdots, p-1\}$, it is clear that the only possible choice is $r = 0$ and $n_0 = n$. Then if $p > 1$, n_0 is determined by the remainder of n modulo p, and the claim follows.

(2) Again, we argue by induction on n. First, assume that $p \neq 2$. If $n \in \{0, \cdots, p-1\}$ there is only one such integer. If $n > 1$ and $n_0 = p-1$, then there is only one choice for m_0, and we conclude using the claim for $(n-n_0)/p$. If $n_0 \neq p-1$, then there are two choices for m_0. But n_0 and $2p-2-n_0$ have different remainders modulo p, so that the numbers produced out of these choices must be distinct.

Finally, assume that $p = 2$. In this case, there is one choice for m_i (namely, $m_i = 1$) if $n_i = 1$, and two choices (namely, $m_i = 0$ or $m_i = 2$) if $n_i = 2$. What we have to observe is therefore that for any finite subset $I \subset \mathbb{Z}_{\geq 0}$, the numbers of the form $\sum_{i \in I} m_i 2^i$ with $m_i \in \{0, 2\}$ are all distinct. $\square$

Writing $n = \sum_{i=0}^{r} n_i p^i$ with $p - 1 \leq n_i \leq 2p - 2$ for $i \in \{0, \cdots, r-1\}$ and $0 \leq n_r \leq p - 1$, the formula (3.2) implies that

$$\mathsf{T}(n\varpi_1) \cong \bigotimes_{i=0}^{r} \mathsf{T}(n_i \cdot \varpi_1)^{(i)}.$$

To deduce information about multiplicities, we will use the following lemma.

Lemma 3.6. *Let $n \in \mathbb{Z}_{\geq 0}$.*

(1) We have $\mathsf{T}((p-1)\varpi_1) \otimes \mathsf{N}(n\varpi_1)^{(1)} \cong \mathsf{N}((p-1+pn)\varpi_1)$.
(2) For any $j \in \{0, \cdots, p-2\}$ there exists a short exact sequence of $\mathbf{G}$-modules

$$\mathsf{N}((p-2-j+pn)\varpi_1) \hookrightarrow \mathsf{T}((p+j)\varpi_1) \otimes \mathsf{N}(n\varpi_1)^{(1)} \twoheadrightarrow \mathsf{N}((p+j+pn)\varpi_1).$$

Proof. The isomorphism (1) is a special case of Proposition 2.39 in Chapter 1, since $\mathsf{T}((p-1)\varpi_1) = \mathsf{N}((p-1)\varpi_1)$.

To deduce (2), one needs a preliminary remark.[3] The W_{aff}-orbits in $\mathbb{X}$ are parametrized by $\mathbb{X} \cap \overline{C} = \{-\varpi_1, 0, \cdots, (p-1)\varpi_1\}$. Let us denote by σ the permutation of this set defined by $\sigma(j\varpi_1) = (p-2-j)\varpi_1$. Now we remark that for $\lambda \in \mathbb{X} \cap \overline{C}$, we have:

- if n is even and M belongs to $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot \lambda}$, then $M \otimes \mathsf{N}(n\varpi_1)^{(1)}$ belongs to $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot \lambda}$;
- if n is odd and M belongs to $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot \lambda}$, then $M \otimes \mathsf{N}(n\varpi_1)^{(1)}$ belongs to $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot \sigma(\lambda)}$.

In fact it suffices to prove these claims in case M is simple, and in this case they follow from Steinberg's tensor product theorem (Theorem 2.9 in Chapter 1.) For instance, if $j \in \{0, \cdots, p-2\}$ and $\lambda = j\varpi_1$, then a simple object in $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot \lambda}$ has the form $\mathsf{L}(\lambda) \otimes \mathsf{L}(m\varpi_1)^{(1)}$ with m even or $\mathsf{L}(\sigma(\lambda)) \otimes \mathsf{L}(m\varpi_1)^{(1)}$ with m odd. In case n is even the composition factors of $\mathsf{L}(m\varpi_1) \otimes \mathsf{N}(n\varpi_1)$ will be of the form $\mathsf{L}(m'\varpi_1)$ with $m' \equiv m \mod 2$, which implies the claim. The other cases can be treated similarly.

This claim implies that for $\lambda \in \mathbb{X} \cap \overline{C}$ and M in $\mathsf{Rep}(\mathbf{G})_{(p-1)\varpi_1}$, in case n is even we have

$$T^{\lambda}_{(p-1)\varpi_1}(M \otimes \mathsf{N}(n\varpi_1)^{(1)}) \cong T^{\lambda}_{(p-1)\varpi_1}(M) \otimes \mathsf{N}(n\varpi_1)^{(1)},$$

and in case n is odd we have

$$T^{\sigma(\lambda)}_{-\varpi_1}(M \otimes \mathsf{N}(n\varpi_1)^{(1)}) \cong T^{\lambda}_{(p-1)\varpi_1}(M) \otimes \mathsf{N}(n\varpi_1)^{(1)}.$$

Now we fix $j \in \{0, \cdots, p-2\}$. Using this observation and the fact that $\mathsf{T}((p+j)\varpi_1) = T^{(p-2-j)\varpi_1}_{(p-1)\varpi_1}\mathsf{T}((p-1)\varpi_1)$, we obtain that

$$\mathsf{T}((p+j)\varpi_1) \otimes \mathsf{N}(n\varpi_1)^{(1)} \cong \begin{cases} T^{(p-2-j)\varpi_1}_{(p-1)\varpi_1}(\mathsf{N}((p-1+pn)\varpi_1)) & \text{if } n \text{ is even;} \\ T^{j\varpi_1}_{-\varpi_1}(\mathsf{N}((p-1+pn)\varpi_1)) & \text{if } n \text{ is odd.} \end{cases}$$

We conclude using Proposition 2.26(3) in Chapter 1. $\square$

[3]EXPLAIN THIS IN GENERAL IN CHAPTER I!

We can finally deduce the desired multiplicities, as follows.

PROPOSITION 3.7. *Let $n \in \mathbb{Z}_{\geq 0}$, and write $n = \sum_{i=0}^{r} n_i p^i$ with $p-1 \leq n_i \leq 2p-2$ for $i \in \{0, \cdots, r-1\}$ and $0 \leq n_r \leq p-1$. Then for $m \geq 0$ the multiplicity $(\mathsf{T}(n\varpi_1) : \mathsf{N}(m\varpi_1))$ is 1 if*

$$m = \sum_{i=0}^{r-1} m_i p^i + n_r p^r$$

with $m_i \in \{n_i, 2p-2-n_i\}$ for all $i \in \{0, \cdots, r-1\}$, and is 0 otherwise.

PROOF. If $n \in \{0, \cdots, p-1\}$, then $\mathsf{T}(n\varpi_1) = \mathsf{N}(n\varpi_1)$, so that the claim holds. Otherwise we have $r \geq 1$. We set $n' = \sum_{i=0}^{r-1} n_{i+1} p^i$, so that $n = n_0 + p'$. By (3.2) we have

$$\mathsf{T}(n\varpi_1) \cong \mathsf{T}(n_0\varpi_1) \otimes \mathsf{T}(n'\varpi_1)^{(1)}.$$

By induction, $\mathsf{T}(n'\varpi_1)$ has a costandard filtration, with subquotients $\mathsf{N}(m'\varpi_1)$ for

$$m' = \sum_{i=0}^{r-2} m_{i+1} p^i + n_r p^{r-1}$$

with $m_i \in \{n_i, 2p-2-n_i\}$ for all $i \in \{1, \cdots, r-1\}$. We deduce a filtration of $\mathsf{T}(n\varpi_1)$ with subquotients $\mathsf{T}(n_0\varpi_1) \otimes \mathsf{N}(m'\varpi_1)^{(1)}$ with m' as above. Now by Lemma 3.6 $\mathsf{T}(n_0\varpi_1) \otimes \mathsf{N}(m'\varpi_1)^{(1)}$ has a costandard filtration with subquotients $\mathsf{N}((m_0 + pm')\varpi_1)$ with $m_0 \in \{n_0, 2p-2-n_0\}$. Since the numbers $m_0 + pm'$ are distinct for distinct choices of (m_0, m') as above by Lemma 3.5, we deduce the desired claim. $\square$

For a picture illustrating this proposition, see [**JW**, Figure 1]. For an application of this description of tilting modules to the determination of dimensions of some simple modules for symmetric groups, we refer to [**Er**].

3.3. Tilting characters determine simple characters: first method. We now explain (following Andersen) how the results of Section 2 can be used to compute characters of simple $\mathbf{G}$-modules, provided we know the characters of the indecomposable tilting $\mathbf{G}$-modules. In fact there are two different ways of making this computation, which we explain in this and the next subsection.

We will assume that $p \geq 2h-2$. (Then $p \geq h$ since $h \geq 2$ unless $\mathbf{G}$ is a torus.) Let us consider the bijection

$$(-)^{\blacktriangledown} : (p-1)\varsigma + \mathbb{X}^+ \xrightarrow{\sim} \mathbb{X}^+$$

defined as follows. Let $\lambda \in (p-1)\varsigma + \mathbb{X}^+$, and write $\lambda = (p-1)\varsigma + p\gamma + \eta$, where $\eta \in \mathbb{X}^+_{\mathrm{res}}$ and $\gamma \in \mathbb{X}^+$. (This is always possible, although not uniquely if $\mathbf{G}$ is not semisimple.) Then the weight $(p-1)\varsigma + p\gamma + w_0\eta$ does not depend on the choice of η and γ, and is chosen as the definition of $\lambda^{\blacktriangledown}$. The inverse bijection will be denoted $\mu \mapsto \mu^{\blacktriangle}$. (It can be easily seen that these bijections do not depend on the choice of ς.)

EXAMPLE 3.8. Assume that $\mu \in \mathbb{X}^+_{\mathrm{res}}$. Then the weight $w_0(\mu - (p-1)\varsigma)$ also belongs to $\mathbb{X}^+_{\mathrm{res}}$; it follows that

$$\mu^{\blacktriangle} = (p-1)\varsigma + w_0(\mu - (p-1)\varsigma) = 2(p-1)\rho + w_0\mu.$$

Let us consider the subset $\mathbb{X}^+_{\mathrm{bb}} \subset \mathbb{X}^+$ consisting of weights λ which satisfy

$$\langle \lambda, \alpha^\vee \rangle \leq (p-1)\langle \varsigma, \alpha^\vee \rangle$$

for any dominant short root α. Then $\mathbb{X}^+_{\mathrm{bb}}$ is an ideal with respect to the order $\preceq$, and we have

$$\mathbb{X}^+_{\mathrm{res}} \subset \mathbb{X}^+_{\mathrm{bb}} \subset \mathbb{X}^+_{\mathrm{b}},$$

where $\mathbb{X}^+_{\mathrm{b}}$ is as in §2.3.

The main result of the present subsection is the following.

Proposition 3.9. *For any $\lambda, \mu \in \mathbb{X}^+_{\mathrm{bb}}$ we have*

$$[\mathsf{M}(\lambda) : \mathsf{L}(\mu)] = (\mathsf{T}(\mu^\blacktriangle) : \mathsf{N}(\lambda)).$$

This proposition implies that if we know the characters $(\mathrm{ch}(\mathsf{T}(\mu^\blacktriangle)) : \mu \in \mathbb{X}^+_{\mathrm{bb}})$, or in other words the multiplicities $((\mathsf{T}(\mu^\blacktriangle) : \mathsf{N}(\lambda)) : \mu \in \mathbb{X}^+_{\mathrm{bb}} : \lambda \in \mathbb{X}^+)$, then we can in theory determine the characters of all simple $\mathbf{G}$-modules. In fact, assume more specifically that we know the multiplicities $(\mathsf{T}(\mu^\blacktriangle) : \mathsf{N}(\lambda))$ for any $\lambda, \mu \in \mathbb{X}^+_{\mathrm{bb}}$; then using Proposition 3.9 we obtain the multiplicities $([\mathsf{M}(\lambda) : \mathsf{L}(\mu)] : \lambda, \mu \in \mathbb{X}^+_{\mathrm{bb}})$. Note that $\mathbb{X}^+_{\mathrm{bb}}$ is an ideal with respect to the order $\preceq$. Hence, inverting the appropriate matrix we can then express the characters of the modules $(\mathsf{L}(\mu) : \mu \in \mathbb{X}^+_{\mathrm{bb}})$ in terms of those of the modules $(\mathsf{M}(\lambda) : \lambda \in \mathbb{X}^+_{\mathrm{bb}})$, which are given by Weyl's character formula (see §1.9 in Chapter 1). Since $\mathbb{X}^+_{\mathrm{bb}}$ contains $\mathbb{X}^+_{\mathrm{res}}$, one can then deduce characters of *all* simple $\mathbf{G}$-modules using Steinberg's tensor product formula (Theorem 2.9 of Chapter 1).

Remark 3.10.
(1) The bijection $\lambda \mapsto \lambda^\blacktriangle$ sends regular weights to regular weights. Hence, assuming we only have an explicit formula for the characters of the tilting modules $\mathsf{T}(\nu)$ with $\nu \in \mathbb{X}$ *regular* of the form $\mu^\blacktriangle$ with $\mu \in \mathbb{X}^+_{\mathrm{bb}}$, the method above still provides a way to compute characters of simple modules with a regular highest weight.
(2) The proof of Proposition 3.9 given below shows that the characters of the modules $\mathsf{T}(\mu^\blacktriangle)$ with $\mu \in \mathbb{X}^+_{\mathrm{bb}}$ can be computed provided we know them in the special case when $\mu \in \mathbb{X}^+_{\mathrm{res}}$.
(3) The formula of Proposition 3.9 is stated in [**A2**] under the assumption that $\mu \in \mathbb{X}^+_{\mathrm{res}}$ and $\lambda \in \mathbb{X}^+_{\mathrm{b}}$. Indeed it is true in this generality, and the proof below simplifies drastically in this case. However, as was pointed out to us by Jantzen, from the equalities in this case one cannot a priori deduce characters of simple modules, because the subset $\mathbb{X}^+_{\mathrm{res}} \subset \mathbb{X}^+$ is not an ideal for the order $\preceq$.

We now explain how to deduce Proposition 3.9 from Proposition 2.20. As explained above $\mathbb{X}^+_{\mathrm{bb}}$ is an ideal with respect to the order $\preceq$, so that the Serre subcategory $\mathsf{Rep}_{\mathrm{bb}}(\mathbf{G})$ generated by the simple modules $\mathsf{L}(\lambda)$ with $\lambda \in \mathbb{X}^+_{\mathrm{bb}}$ has a natural structure of highest weight category, see Lemma 1.3(2) in Appendix A. Moreover, by Proposition 3.2 in Appendix A, the natural functor

$$\imath : D^{\mathrm{b}}\mathsf{Rep}_{\mathrm{bb}}(\mathbf{G}) \to D^{\mathrm{b}}\mathsf{Rep}(\mathbf{G})$$

admits a right adjoint $\imath^{\mathrm{R}}$. This functor sends the induced module $\mathsf{N}(\lambda)$ to itself if $\lambda \in \mathbb{X}^+_{\mathrm{bb}}$ and to 0 otherwise; in particular it sends any object of $\mathsf{Rep}(\mathbf{G})$ which admits a costandard filtration to an object of $\mathsf{Rep}_{\mathrm{bb}}(\mathbf{G})$.

Proposition 3.9 will be deduced from the following claim.

LEMMA 3.11. *For any* $\mu \in \mathbb{X}^+_{\mathrm{bb}}$, $\imath^{\mathrm{R}}(\mathsf{T}(\mu^{\blacktriangle}))$ *is the injective hull of* $\mathsf{L}(\mu)$ *in* $\mathsf{Rep}_{\mathrm{bb}}(\mathbf{G})$.

PROOF. Write $\mu = \mu_0 + p\mu_1$ with $\mu_0 \in \mathbb{X}^+_{\mathrm{res}}$ and $\mu_1 \in \mathbb{X}^+$. Then $\mu^{\blacktriangle} = \mu_0^{\blacktriangle} + p\mu_1$. If α is a dominant short root, then we have

$$p\langle \mu_1 + \varsigma, \alpha^\vee \rangle \leq \langle \mu, \alpha^\vee \rangle + p\langle \varsigma, \alpha^\vee \rangle \leq (2p-1)\langle \varsigma, \alpha^\vee \rangle,$$

hence

$$\langle \mu_1 + \varsigma, \alpha^\vee \rangle < 2\langle \varsigma, \alpha^\vee \rangle \leq 2(h-1) \leq p.$$

If $\tilde{\mu}_1 \in X^*(\mathbf{T}^{(1)})$ is the only dominant weight such that $\mathrm{Fr}^*_{\mathbf{T}}(\tilde{\mu}_1) = p\mu_1$, then $\tilde{\mu}_1$ belongs to the fundamental alcove of $\mathbf{G}^{(1)}$, so that

$$\mathsf{M}^{(1)}(\tilde{\mu}_1) = \mathsf{N}^{(1)}(\tilde{\mu}_1) = \mathsf{L}^{(1)}(\tilde{\mu}_1) = \mathsf{T}^{(1)}(\tilde{\mu}_1),$$

see §1.5.1. Using Donkin's formula (Theorem 3.1) we deduce that

$$\mathsf{T}(\mu^{\blacktriangle}) \cong \mathsf{T}(\mu_0^{\blacktriangle}) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{L}^{(1)}(\tilde{\mu}_1)).$$

On the other hand we have $\mu_0^{\blacktriangle} = 2(p-1)\rho + w_0\mu_0$, see Example 3.8 hence

$$\mathsf{T}(\mu_0^{\blacktriangle}) \cong \mathsf{R}(\mu_0)$$

by Proposition 2.20. We deduce that

$$\mathsf{T}(\mu^{\blacktriangle}) \cong \mathsf{R}(\mu_0) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{L}^{(1)}(\tilde{\mu}_1)).$$

We claim that $\mathsf{T}(\mu^{\blacktriangle})$ admits a unique simple submodule, isomorphic to $\mathsf{L}(\mu)$. Indeed, for $\lambda \in \mathbb{X}^+$, written as $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in \mathbb{X}^+_{\mathrm{res}}$ and $\lambda_1 \in \mathbb{X}^+$, if we denote by $\tilde{\lambda}_1 \in X^*(\mathbf{T}^{(1)})$ the only dominant weight such that $\mathrm{Fr}^*_{\mathbf{T}}(\tilde{\lambda}_1) = p\lambda_1$, using Steinberg's tensor product formula we see that

$$\begin{aligned}\mathrm{Hom}_{\mathbf{G}}(\mathsf{L}(\lambda), \mathsf{T}(\mu^{\blacktriangle})) &\cong \mathrm{Hom}_{\mathbf{G}}(\mathsf{L}(\lambda_0) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{L}^{(1)}(\tilde{\lambda}_1)), \mathsf{R}(\mu_0) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{L}^{(1)}(\tilde{\mu}_1))) \\ &\cong \mathrm{Hom}_{\mathbf{G}^{(1)}}(\mathsf{L}^{(1)}(\tilde{\lambda}_1), \mathrm{Hom}_{\mathbf{G}_1}(\mathsf{L}(\lambda_0), \mathsf{R}(\mu_0)) \otimes \mathsf{L}^{(1)}(\tilde{\mu}_1)).\end{aligned}$$

Here Theorem 2.13 implies that $\mathrm{Hom}_{\mathbf{G}_1}(\mathsf{L}(\lambda_0), \mathsf{R}(\mu_0)) = 0$ unless λ_0 and μ_0 have the same image in $\mathbb{X}/p\mathbb{X}$, i.e. unless there exists $\eta \in \mathbb{X}$ such that $\langle \eta, \alpha^\vee \rangle = 0$ for any $\alpha \in \mathfrak{R}$ and $\mu_0 = \lambda_0 + p\eta$ (see Remark 2.4). Let us assume that this condition is satisfied. Then there exists a unique character $\tilde{\eta}$ of $\mathbf{G}^{(1)}$ whose restriction to $\mathbf{T}^{(1)}$ has pullback $p\eta$, and

$$\mathrm{Hom}_{\mathbf{G}_1}(\mathsf{L}(\lambda_0), \mathsf{R}(\mu_0)) = \Bbbk_{\mathbf{G}^{(1)}}(\tilde{\eta}).$$

We deduce that

$$\mathrm{Hom}_{\mathbf{G}}(\mathsf{L}(\lambda), \mathsf{T}(\mu^{\blacktriangle})) \cong \mathrm{Hom}_{\mathbf{G}^{(1)}}(\mathsf{L}^{(1)}(\tilde{\lambda}_1), \mathsf{L}^{(1)}(\tilde{\mu}_1 + \tilde{\eta}));$$

here the right-hand side vanishes unless $\tilde{\lambda}_1 = \tilde{\mu}_1 + \tilde{\eta}$, and is equal to $\Bbbk$ in this case. If this further condition is satisfied we have

$$\lambda = \lambda_0 + p\lambda_1 = (\mu_0 - p\eta) + p\lambda_1 = \mu_0 + p\mu_1 = \mu,$$

which finishes the proof of our claim. Using adjunction, this claim implies that $\imath^{\mathrm{R}}(\mathsf{T}(\mu^{\blacktriangle}))$ also has a unique simple submodule, which is isomorphic to $\mathsf{L}(\mu)$.

To conclude the proof, it now suffices to show that $\imath^{\mathrm{R}}(\mathsf{T}(\mu^{\blacktriangle}))$ is injective in $\mathsf{Rep}_{\mathrm{bb}}(\mathbf{G})$. However, if $\lambda \in \mathbb{X}^+_{\mathrm{bb}}$ we have

$$\begin{aligned}\mathrm{Ext}^1_{\mathsf{Rep}_{\mathrm{bb}}(\mathbf{G})}(\mathsf{L}(\lambda), \imath^{\mathrm{R}}(\mathsf{T}(\mu^{\blacktriangle}))) &\cong \mathrm{Ext}^1_{\mathsf{Rep}(\mathbf{G})}(\mathsf{L}(\lambda), \mathsf{T}(\mu^{\blacktriangle})) \\ &\cong \mathrm{Ext}^1_{\mathsf{Rep}(\mathbf{G})}(\mathsf{L}(\lambda), \mathsf{R}(\mu_0) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{L}^{(1)}(\tilde{\mu}_1))) \\ &\cong \mathrm{Ext}^1_{\mathsf{Rep}(\mathbf{G})}(\mathsf{L}(\lambda) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{L}^{(1)}(-w_0\tilde{\mu}_1)), \mathsf{R}(\mu_0)).\end{aligned}$$

Now $\mathsf{L}(\lambda) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{L}^{(1)}(-w_0\tilde{\mu}_1))$ has highest weight $\lambda + p(-w_0\mu_1)$, and for any dominant short root α we have

$$\langle \lambda + p(-w_0\mu_1), \alpha^\vee \rangle \leq (p-1)\langle \varsigma, \alpha^\vee \rangle + p\langle \mu_1, \alpha^\vee \rangle \leq 2(p-1)\langle \varsigma, \alpha^\vee \rangle,$$

hence $\mathsf{L}(\lambda) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{L}^{(1)}(-w_0\tilde{\mu}_1))$ belongs to $\mathsf{Rep}_{\mathrm{b}}(\mathbf{G})$, so that

$$\mathrm{Ext}^1_{\mathsf{Rep}(\mathbf{G})}(\mathsf{L}(\lambda) \otimes \mathrm{Fr}^*_{\mathbf{G}}(\mathsf{L}^{(1)}(-w_0\tilde{\mu}_1)), \mathsf{R}(\mu_0)) = 0.$$

The proof is now complete. □

We can finally give give the proof of Proposition 3.9.

Proof of Proposition 3.9. Let $\lambda, \mu \in \mathbb{X}^+_{\mathrm{bb}}$. Since $\imath^{\mathrm{R}}(\mathsf{T}(\mu^{\blacktriangle}))$ is the injective hull of $\mathsf{L}(\mu)$ in $\mathsf{Rep}_{\mathrm{bb}}(\mathbf{G})$, we have

$$[\mathsf{M}(\lambda) : \mathsf{L}(\mu)] = \dim_{\Bbbk} \mathrm{Hom}(\mathsf{M}(\lambda), \imath^{\mathrm{R}}(\mathsf{T}(\mu^{\blacktriangle}))) = \dim_{\Bbbk} \mathrm{Hom}(\mathsf{M}(\lambda), \mathsf{T}(\mu^{\blacktriangle})).$$

Now the right-hand side is equal to $(\mathsf{T}(\mu^{\blacktriangle}) : \mathsf{N}(\lambda))$ by (1.1), hence the desired equality is proved. □

3.4. Tilting characters determine simple characters: second method. We now explain another way of deducing a character formula for simple $\mathbf{G}$-modules out of a character formula for indecomposable tilting modules. In fact this second method only uses the property (2.8), hence might work under assumptions weaker than $p \geq 2h-2$; see §2.7. Instead of working with $\mathbf{G}$-modules, in this case we work with $\mathbf{G}_1\mathbf{T}$-modules.

In fact, assume that we know the characters $\mathrm{ch}(\mathsf{T}(2(p-1)\rho + w_0\lambda))$ for any $\lambda \in \mathbb{X}^+_{\mathrm{res}}$. Then using (2.8) we know the characters of the injective $\mathbf{G}_1\mathbf{T}$-module $\widehat{\mathsf{Q}}(\lambda)$ for any $\lambda \in \mathbb{X}^+_{\mathrm{res}}$, hence for any $\lambda \in \mathbb{X}$ using (2.5). The characters of baby Verma modules are easy to compute (see Exercise 4.7), hence this knowledge is equivalent to that of the multiplicities of baby Verma modules as subquotients of a filtration of an injective $\mathbf{G}_1\mathbf{T}$-module as in Proposition 2.9, or in other words of the multiplicities

$$\left([\widehat{\mathsf{Z}}(\lambda) : \widehat{\mathsf{L}}(\mu)] : \lambda, \mu \in \mathbb{X}\right).$$

Now we claim that if we know these multiplicities then we can compute the characters of all the modules $\widehat{\mathsf{L}}(\lambda)$, hence of all the simple $\mathbf{G}$-modules by Remark 2.7. Indeed, by the same remark one can assume that $\lambda \in \mathbb{X}^+_{\mathrm{res}}$. In this case $\widehat{\mathsf{L}}(\lambda)$ is the restriction of a $\mathbf{G}$-module by Theorem 2.6, hence its character is W-invariant. It follows that to determine the character of $\widehat{\mathsf{L}}(\lambda)$ it suffices to compute $\dim \widehat{\mathsf{L}}(\lambda)_\mu$ when $\mu \in \mathbb{X}^+$. Now for any $\nu \in \mathbb{X}$, all the $\mathbf{T}$-weights of $\widehat{\mathsf{Z}}(\nu)$ are $\preceq \nu$, so that the multiplicity $[\widehat{\mathsf{Z}}(\nu) : \widehat{\mathsf{L}}(\eta)]$ vanishes unless $\eta \preceq \nu$ (and is equal to 1 in case $\nu = \eta$).

Using this property, from the datum of the multiplicities $([\widehat{\mathsf{Z}}(\nu) : \widehat{\mathsf{L}}(\eta)] : \mu, \nu \in \mathbb{X})$ one can obtain an expression

$$\mathrm{ch}(\widehat{\mathsf{L}}(\lambda)) = \sum_{\nu \in X_\lambda} m_\nu \cdot \mathrm{ch}(\widehat{\mathsf{Z}}(\nu)) + \sum_{\nu \in Y_\lambda} m'_\nu \cdot \mathrm{ch}(\widehat{\mathsf{L}}(\nu))$$

for some finite subsets $X_\lambda, Y_\lambda \subset \mathbb{X}$ and some integers $m_\nu, m'_\nu \in \mathbb{Z}$, such that there exists no dominant weight μ such that $\mu \preceq \nu$ for some $\nu \in Y_\lambda$. Then we have

$$\dim(\widehat{\mathsf{L}}(\lambda)_\mu) = \sum_{\nu \in X_\lambda} m_\nu \cdot \dim(\widehat{\mathsf{Z}}(\nu)_\mu)$$

for any $\mu \in \mathbb{X}^+$.

4. Andersen's conjecture

4.1. Antispherical Kazhdan–Lusztig polynomials. Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system, and let $I \subset \mathcal{S}$ be a subset. Recall (see e.g. §2.8 in Chapter 1) that to I we attach a standard parabolic subgroup $\mathcal{W}_I \subset \mathcal{W}$. The Hecke algebra $\mathcal{H}_{(\mathcal{W}_I, I)}$ (see Definition 4.1 in Chapter 1) embeds naturally in $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$. Let us consider the "sign" right module $\mathbb{Z}[v, v^{-1}]_{\mathrm{sgn}}$ for $\mathcal{H}_{(\mathcal{W}_I, I)}$ which is equal to $\mathbb{Z}[v, v^{-1}]$ with the natural action of $\mathbb{Z}[v, v^{-1}]$, and with H_s acting by multiplication by $-v$ for any $s \in I$. (It is easily seen that these rules uniquely extend to a right $\mathcal{H}_{(\mathcal{W}_I, I)}$-action.) Then the corresponding "antispherical" right $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$-module is defined by

$$\mathcal{N}^I_{(\mathcal{W},\mathcal{S})} := \mathbb{Z}[v, v^{-1}]_{\mathrm{sgn}} \otimes_{\mathcal{H}_{(\mathcal{W}_I, I)}} \mathcal{H}_{(\mathcal{W},\mathcal{S})}.$$

Let us denote by ${}^I\mathcal{W} \subset \mathcal{W}$ the subset of elements w that are minimal in their coset $\mathcal{W}_I w$. From the "standard" basis $(H_w : w \in \mathcal{W})$ of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ we deduce a standard basis $(N^I_w : w \in {}^I\mathcal{W})$ of $\mathcal{N}^I_{(\mathcal{W},\mathcal{S})}$ (as a $\mathbb{Z}[v, v^{-1}]$-module), where for $w \in {}^I\mathcal{W}$ we set

$$N^I_w := 1 \otimes H_w.$$

In terms of this basis, the action of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ on $\mathcal{N}^I_{(\mathcal{W},\mathcal{S})}$ is determined by the following rule for $w \in {}^I\mathcal{W}$ and $s \in \mathcal{S}$ (see [S3, §3]):

$$N^I_w \cdot \underline{H}_s = \begin{cases} N^I_{ws} + vN^I_w & \text{if } ws \in {}^I\mathcal{W} \text{ and } ws > w; \\ N^I_{ws} + v^{-1}N^I_w & \text{if } ws \in {}^I\mathcal{W} \text{ and } ws < w; \\ 0 & \text{if } ws \notin {}^I\mathcal{W}. \end{cases} \tag{4.1}$$

Recall the Kazhdan–Lusztig involution ι of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$, see §4.2 in Chapter 1. Then the assignment

$$a \otimes H \mapsto \iota(a) \otimes \iota(H)$$

for $a \in \mathbb{Z}[v, v^{-1}]$ and $H \in \mathcal{H}_{(\mathcal{W},\mathcal{S})}$ defines an involution ι_I of $\mathcal{N}^I_{(\mathcal{W},\mathcal{S})}$, which satisfies

$$\iota_I(N \cdot H) = \iota_I(N) \cdot \iota(H)$$

for any $N \in \mathcal{N}^I_{(\mathcal{W},\mathcal{S})}$ and $H \in \mathcal{H}_{(\mathcal{W},\mathcal{S})}$.

The following theorem is due to Deodhar [De]. For an easy proof, we refer to [S3, Theorem 3.1].

Theorem 4.1. *For any $w \in {}^I\mathcal{W}$, there exists a unique element $\underline{N}_w \in \mathcal{N}^I_{(\mathcal{W},\mathcal{S})}$ such that*

$$\iota_I(\underline{N}_w) = \underline{N}_w, \qquad \underline{N}_w \in N_w + \sum_{y \in {}^I\mathcal{W}} v\mathbb{Z}[v] N_y.$$

The elements $(\underline{N}_w : w \in {}^I\mathcal{W})$ *form a basis of* $\mathcal{N}^I_{(\mathcal{W},\mathcal{S})}$*, called the* Kazhdan–Lusztig basis *of* $\mathcal{N}^I_{(\mathcal{W},\mathcal{S})}$*. If one writes*

$$\underline{N}_x = \sum_{y \in {}^I\mathcal{W}} n^I_{y,x} \cdot N_y,$$

then the polynomials $(n^I_{y,x} : y, x \in {}^I\mathcal{W})$ *are called the* antispherical Kazhdan–Lusztig polynomials *attached to* I*.*

This basis has a simple relation with the Kazhdan–Lusztig basis of $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$, as explained e.g. in [S3, Proposition 3.4 and its proof]: by definition we have a canonical surjective morphism of right $\mathcal{H}_{(\mathcal{W},\mathcal{S})}$-modules

$$\xi : \mathcal{H}_{(\mathcal{W},\mathcal{S})} \to \mathcal{N}^I_{(\mathcal{W},\mathcal{S})},$$

sending an element H to $1 \otimes H$. For any $w \in \mathcal{W}$, this morphism satisfies

$$\xi(\underline{H}_w) = \begin{cases} \underline{N}_w & \text{if } w \in {}^I\mathcal{W}; \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, we obtain that for any $y, w \in {}^I\mathcal{W}$ we have

$$n^I_{y,w} = \sum_{x \in \mathcal{W}_I} (-1)^{\ell(x)} h_{xy,w}.$$

4.2. Andersen's conjecture. We now specialize the considerations of §4.1 to the case $(\mathcal{W}, \mathcal{S}) = (W_{\mathrm{aff}}, S_{\mathrm{aff}})$, with $I = W$. Recall that in this case, the subset of minimal elements is denoted ${}^{\mathrm{f}}W_{\mathrm{aff}} \subset W_{\mathrm{aff}}$. We will also write $\mathcal{N}_{\mathrm{aff}}$ for the corresponding antispherical module, $(N_w : w \in {}^{\mathrm{f}}W_{\mathrm{aff}})$ for its standard basis, $(\underline{N}_w : w \in {}^{\mathrm{f}}W_{\mathrm{aff}})$ for its Kazhdan–Lusztig basis, and $(n_{y,w} : {}^{\mathrm{f}}W_{\mathrm{aff}})$ for the associated Kazhdan–Lusztig polynomials. If we consider $\mathbb{Z}$ as a $\mathbb{Z}[v, v^{-1}]$-module with v acting as the identity, then we can consider the right module

$$\mathcal{N}^0_{\mathrm{aff}} := \mathbb{Z}[v, v^{-1}] \otimes_{\mathbb{Z}} \mathcal{N}_{\mathrm{aff}}.$$

over $\mathbb{Z}[v, v^{-1}] \otimes_{\mathbb{Z}} \mathcal{H}_{\mathrm{aff}} \cong \mathbb{Z}[W_{\mathrm{aff}}]$, see (4.4) in Chapter 1. Setting $N^0_w := 1 \otimes N_w$, in view of (4.1) the action of $\mathbb{Z}[W_{\mathrm{aff}}]$ if determined by the following rule for $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ and $s \in S_{\mathrm{aff}}$:

$$N^0_w \cdot (1+s) = \begin{cases} N^0_w + N^0_{ws} & \text{if } ws \in {}^{\mathrm{f}}W_{\mathrm{aff}}; \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Now assume that $\mathscr{D}\mathbf{G}$ is simply connected, that $p \geq h$, and fix $\lambda \in C \cap \mathbb{X}$ Recall that the Grothendieck group $[\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}]$ has as basis $([\mathsf{N}(\mu)] : \mu \in \mathbb{X}^+ \cap (W_{\mathrm{aff}} \cdot_p \lambda))$, and that $\mathbb{X}^+ \cap (W_{\mathrm{aff}} \cdot_p \lambda)$ is identified with ${}^{\mathrm{f}}W_{\mathrm{aff}}$ via $w \mapsto w \cdot_p \lambda$, see §2.8 in Chapter 1. In particular, we have a $\mathbb{Z}$-module isomorphism

$$\mathcal{N}^0_{\mathrm{aff}} \cong [\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}] \tag{4.3}$$

which identifies N^0_w with $[\mathsf{N}(w \cdot_p \lambda)]$ for any $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$.

Now, fix for any $s \in S_{\mathrm{aff}}$ a weight $\mu_s \in \mathbb{X}$ on the wall of $\overline{C}$ associated with s. (Such a weight always exists, see Remark 2.27 in Chapter 1.) Then we can consider the endomorphism $[T^\lambda_{\mu_s} T^{\mu_s}_\lambda]$ of $[\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}]$ induced by the (exact) functor $T^\lambda_{\mu_s} T^{\mu_s}_\lambda$. The considerations of §2.8 in Chapter 1 show that for $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, the weight $w \cdot_p \mu$ is dominant iff ws belongs to ${}^{\mathrm{f}}W_{\mathrm{aff}}$. With this in mind, the formulas in Proposition 2.26 in Chapter 1 show that for $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ we have

$$[T_{\mu_s}^{\lambda} T_{\lambda}^{\mu_s}(\mathsf{N}(w \cdot_p \lambda))] = \begin{cases} [\mathsf{N}(w \cdot_p \lambda)] + [\mathsf{N}(ws \cdot_p \lambda)] & \text{if } ws \in {}^{\mathrm{f}}W_{\mathrm{aff}}; \\ 0 & \text{otherwise.} \end{cases}$$

Comparing with (4.2), we deduce that via the identification (4.3) the endomorphism $[T_{\mu_s}^{\lambda} T_{\lambda}^{\mu_s}]$ identifies with the (right) action of $1+s$.

Inspired by Lusztig's conjecture (see Conjecture 4.6 in Chapter 1) and work of Soergel in the setting of quantum groups at a root of unity (see [S4]), Andersen has proposed in [A2] the following conjecture, which compares the basis $([\mathsf{T}(w \cdot_p \lambda)] : w \in {}^{\mathrm{f}}W_{\mathrm{aff}})$ of $[\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}]$ with the basis of $\mathcal{N}^0_{\mathrm{aff}}$ obtained as the image of the Kazhdan–Lusztig basis $(\underline{N}_w : w \in {}^{\mathrm{f}}W_{\mathrm{aff}})$ of $\mathcal{N}_{\mathrm{aff}}$.

CONJECTURE 4.2 (Andersen's conjecture). *Assume that $p \geq h$, and let $\lambda \in C \cap \mathbb{X}$. Then for any $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ such that*

$$\langle w \cdot_p \lambda, \alpha^\vee \rangle < p^2 \quad \textit{for any } \alpha \in \mathfrak{R}^+$$

and any $y \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ we have

$$(\mathsf{T}(w \cdot_p \lambda) : \mathsf{N}(y \cdot_p \lambda)) = n_{y,w}(1).$$

4.3. Andersen's conjecture implies Lusztig's conjecture. In this subsection we assume that $\mathscr{D}\mathbf{G}$ is simply connected and that $p \geq 2h-2$. Our goal is to explain that if $p > 3h$ and if Conjecture 4.2 holds, then Lusztig's conjecture (Conjecture 4.6 in Chapter 1) holds.

More specifically, consider the set

$$W_{\mathrm{aff}}^{\mathrm{bb}} = \{w \in W_{\mathrm{aff}} \mid w \cdot_p 0 \in \mathbb{X}^+_{\mathrm{bb}}\}.$$

Then $W_{\mathrm{aff}}^{\mathrm{bb}}$ is a finite subset of W_{aff}, contained in ${}^{\mathrm{f}}W_{\mathrm{aff}}$, and independent of p. For any $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, we denote by $w^\blacktriangle \in W_{\mathrm{aff}}$ the unique element such that $(w \cdot_p 0)^\blacktriangle = w^\blacktriangle \cdot_p 0$.

PROPOSITION 4.3. *Assume that for any $w, y \in W_{\mathrm{aff}}^{\mathrm{bb}}$ we have*

$$(\mathsf{T}(w^\blacktriangle \cdot_p 0) : \mathsf{N}(y \cdot_p 0)) = n_{y,w^\blacktriangle}(1).$$

Then for any $w \in W_{\mathrm{aff}}^{\mathrm{bb}}$ we have

$$[\mathsf{L}(w \cdot_p 0)] = \sum_{y \in {}^{\mathrm{f}}W_{\mathrm{aff}}} (-1)^{\ell(w)+\ell(y)} h_{w_0 y, w_0 w}(1) \cdot [\mathsf{N}(y \cdot_p 0)].$$

To justify the claim at the beginning of the subsection, we observe that for w as in the proposition the weight $w^\blacktriangle \cdot_p 0 = (w \cdot_p 0)^\blacktriangle$ satisfies

$$\langle (w \cdot_p 0)^\blacktriangle, \alpha^\vee \rangle \leq 3(p-1)h$$

for any dominant short root α. Assuming that $p > 3h$, the element $w^\blacktriangle$ therefore satisfies the assumption of Conjecture 4.2. Assuming that the latter conjecture holds, we therefore obtain the formula in Lusztig's conjecture for any $w \in W_{\mathrm{aff}}^{\mathrm{bb}}$. Since $\mathbb{X}^+_{\mathrm{bb}}$ contains $\mathbb{X}^+_{\mathrm{res}}$, results of Kato (see §4.4 in Chapter 1) then imply that the formula applies to all w's as in the conjecture

Proof of Proposition 4.3. By Proposition 3.9, our assumption implies that for any $w, y \in W_{\mathrm{aff}}^{\mathrm{bb}}$ we have

$$[\mathsf{M}(y \cdot_p 0) : \mathsf{L}(w \cdot_p 0)] = n_{y,w^{\blacktriangle}}(1).$$

To deduce an expression for $[\mathsf{L}(w \cdot_p 0)]$ we need to invert the matrix $([\mathsf{M}(y \cdot_p 0) : \mathsf{L}(w \cdot_p 0)] : y, w \in W_{\mathrm{aff}}^{\mathrm{bb}})$. However, the result of this inversion is given by [S3, Theorem 5.1], and precisely gives the formula in Lusztig's conjecture.

More specifically, in [S3, Theorem 3.6] (see also [S3, Proposition 3.4]) Soergel introduces some polynomials $(m^{y,w} : y, w \in {}^{\mathrm{f}}W_{\mathrm{aff}})$ which satisfy

$$\sum_z (-1)^{\ell(z)+\ell(y)} m^{x,z} h_{w_0 y, w_0 z} = \delta_{x,y}$$

for $x, y \in {}^{\mathrm{f}}W_{\mathrm{aff}}$. In terms of these data, [S3, Theorem 5.1] implies that for $y, w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ we have

$$m^{y,w}(1) = n_{y,w^{\blacktriangle}}(1).$$

If we set for $w \in W_{\mathrm{aff}}^{\mathrm{bb}}$

$$L_w = \sum_{y \in {}^{\mathrm{f}}W_{\mathrm{aff}}} (-1)^{\ell(w)+\ell(y)} h_{w_0 y, w_0 w}(1) \cdot [\mathsf{N}(y \cdot_p 0)],$$

then we deduce that for any $y \in W_{\mathrm{aff}}^{\mathrm{bb}}$ we have

$$\begin{aligned}\sum_{w \in W_{\mathrm{aff}}^{\mathrm{bb}}} n_{y,w^{\blacktriangle}}(1) \cdot L_w &= \sum_{w \in W_{\mathrm{aff}}^{\mathrm{bb}}} m^{y,w}(1) \cdot L_w \\ &= \sum_{w \in W_{\mathrm{aff}}^{\mathrm{bb}}} \sum_{z \in {}^{\mathrm{f}}W_{\mathrm{aff}}} (-1)^{\ell(w)+\ell(z)} m^{y,w}(1) h_{w_0 z, w_0 w}(1) \cdot [\mathsf{N}(z \cdot_p \lambda)] = [\mathsf{N}(y \cdot_p 0)],\end{aligned}$$

which finishes the proof. $\square$

CHAPTER 5

Williamson's counterexamples

CHAPTER 6

Tilting modules and the p-canonical basis

1. Hecke action on regular blocks

1.1. Regular and subregular blocks. We continue with the setting and notation of Chapter 4, assuming in addition that $\mathbf{G}$ has simply connected derived subgroup and that $p \geq h$ (so that regular weights exist, see §2.7 of Chapter 1). We fix a weight $\lambda \in C \cap \mathbb{X}$, and set

$$\mathsf{Rep}_0(\mathbf{G}) := \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}.$$

As explained in §2.6 of Chapter 4, this category admits a canonical structure of highest weight category, with weight poset $\big((W_{\mathrm{aff}} \cdot_p \lambda) \cap \mathbb{X}^+, \uparrow\big)$. In fact, as explained in §2.8 of Chapter 4, there exists a canonical bijection

$${}^{\mathrm{f}}W_{\mathrm{aff}} \xrightarrow{\sim} (W_{\mathrm{aff}} \cdot_p \lambda) \cap \mathbb{X}^+$$

which identifies the order $\uparrow$ on the right-hand side with the (restriction of the) Bruhat order on the left-hand side. To simplify notation, for $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ we will set

$$\mathsf{N}_w := \mathsf{N}(w \cdot_p \lambda), \quad \mathsf{M}_w := \mathsf{M}(w \cdot_p \lambda), \quad \mathsf{T}_w := \mathsf{T}(w \cdot_p \lambda),$$

and consider $\mathsf{Rep}_0(\mathbf{G})$ as a highest weight category with weight poset ${}^{\mathrm{f}}W_{\mathrm{aff}}$, standard objects $(\mathsf{M}_w : w \in {}^{\mathrm{f}}W_{\mathrm{aff}})$, and costandard objects $(\mathsf{N}_w : w \in {}^{\mathrm{f}}W_{\mathrm{aff}})$.

On the other hand, recall that the walls contained in $\overline{C}$ are in a canonical bijection with S_{aff}. For any $s \in S_{\mathrm{aff}}$, it is known that there exists a weight $\mu_s \in \mathbb{X}$ which belongs to the corresponding wall, see Remark 2.27 in Chapter 1. We fix such a weight, and set

$$\mathsf{Rep}_s(\mathbf{G}) := \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \mu_s}.$$

As explained in §2.6 of Chapter 4, this category admits a canonical structure of highest weight category, with weight poset $\big((W_{\mathrm{aff}} \cdot_p \mu_s) \cap \mathbb{X}^+, \uparrow\big)$. In fact, as explained in §2.8 of Chapter 4, setting ${}^{\mathrm{f}}W^s_{\mathrm{aff}} := {}^{\mathrm{f}}W^{(\mu_s)}_{\mathrm{aff}}$, there exists a canonical bijection

$${}^{\mathrm{f}}W^s_{\mathrm{aff}} \xrightarrow{\sim} (W_{\mathrm{aff}} \cdot_p \mu_s) \cap \mathbb{X}^+$$

which identifies the order $\uparrow$ on the right-hand side with the (restriction of the) Bruhat order on the left-hand side. In more concrete terms, we have

$${}^{\mathrm{f}}W^s_{\mathrm{aff}} := \{w \in {}^{\mathrm{f}}W_{\mathrm{aff}} \mid ws < w\};$$

in particular, this set does not depend on the choice of μ_s. To simplify notation, for $w \in {}^{\mathrm{f}}W^s_{\mathrm{aff}}$ we will set

$$\mathsf{N}^s_w := \mathsf{N}(w \cdot_p \mu_s), \quad \mathsf{M}^s_w := \mathsf{M}(w \cdot_p \mu_s), \quad \mathsf{T}^s_w := \mathsf{T}(w \cdot_p \mu_s),$$

and consider $\mathsf{Rep}_s(\mathbf{G})$ as a highest weight category with weight poset ${}^{\mathrm{f}}W^s_{\mathrm{aff}}$, standard objects $(\mathsf{M}^s_w : w \in {}^{\mathrm{f}}W^s_{\mathrm{aff}})$, and costandard objects $(\mathsf{N}^s_w : w \in {}^{\mathrm{f}}W^s_{\mathrm{aff}})$.

We then have translation functors

$$T^s := T_\lambda^{\mu_s} : \mathsf{Rep}_0(\mathbf{G}) \to \mathsf{Rep}_s(\mathbf{G}), \qquad T_s := T_{\mu_s}^{\lambda} : \mathsf{Rep}_s(\mathbf{G}) \to \mathsf{Rep}_0(\mathbf{G}).$$

We set

$$\Theta_s := T_s \circ T^s : \mathsf{Rep}_0(\mathbf{G}) \to \mathsf{Rep}_0(\mathbf{G}).$$

REMARK 1.1. As explained in Remark 2.18 in Chapter 1, translations functors are defined only up to isomorphism. In this section we fix arbitrary choices for the functors T^s and T_s.

Recall from §4.2 in Chapter 4 that there exists an action of the Hecke algebra $\mathcal{H}_{\mathrm{aff}}$ on the Grothendieck group $[\mathsf{Rep}_0(\mathbf{G})]$ such that $\underline{H}_s$ acts via the morphism induced by the functor Θ_s for any $s \in S_{\mathrm{aff}}$. (Of course, this action factors through the algebra morphism $\mathcal{H}_{\mathrm{aff}} \to \mathbb{Z}[W_{\mathrm{aff}}]$ sending v to 1.) The basic idea that underlies the constructions of the present chapter is that this action "lifts" to the categorical level (in other words, "categorifies"), and that this has strong implications for the structure of the category $\mathsf{Rep}(\mathbf{G})$.

1.2. The Hecke category. We will now consider a particular Hecke category (in the sense of Chapter 2) associated with the Coxeter system $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$, over the field $\Bbbk$. To define the associated realization, we will use the fact that the roots and coroots of $(\mathbf{G}^{(1)}, \mathbf{T}^{(1)})$ are in a canonical bijection with those of $(\mathbf{G}, \mathbf{T})$. Namely, there exists a unique isomorphism $\mathbb{X} = X^*(\mathbf{T}^{(1)})$ which identifies the pullback morphism $X^*(\mathbf{T}^{(1)}) \to X^*(\mathbf{T})$ with the morphism $\lambda \mapsto p\lambda$. Under this isomorphism, the roots and coroots of $(\mathbf{G}^{(1)}, \mathbf{T}^{(1)})$ coincide with those of $(\mathbf{G}, \mathbf{T})$.

The realization that we will consider is constructed as follows:

- the underlying $\Bbbk$-module is $\mathfrak{h} = \Bbbk \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\mathfrak{R}^\vee, \mathbb{Z})$;
- if $s \in S \subset S_{\mathrm{aff}}$, the associated "root" $\alpha_s \in \mathfrak{h}^* = \Bbbk \otimes_{\mathbb{Z}} \mathbb{Z}\mathfrak{R}^\vee$ (resp. "coroot" $\alpha_s^\vee \in \mathfrak{h}$) is the image of the simple coroot (resp. root) of $(\mathbf{G}^{(1)}, \mathbf{T}^{(1)})$ associated with s;
- if $s \in S_{\mathrm{aff}} \smallsetminus S$, then the image of s under the natural projection $W_{\mathrm{aff}} \to W$ is a reflection s_γ for some positive root γ for $(\mathbf{G}^{(1)}, \mathbf{T}^{(1)})$; the "root" $\alpha_s \in \mathfrak{h}^*$ (resp. "coroot" $\alpha_s^\vee \in \mathfrak{h}$) is defined as the image of $-\gamma^\vee$ (resp. of $-\gamma$).

The assumption that $\mathbf{G}$ has simply-connected derived subgroup implies that for any $s \in S_{\mathrm{aff}}$ the morphism $\alpha_s : \mathfrak{h} \to \Bbbk$ is nonzero, i.e. surjective. We will also assume that each $\alpha_s^\vee \in \mathfrak{h}$ is nonzero. This implies that our realization satisfies Demazure surjectivity. The other technical conditions imposed on realizations are automatically satisfied. (REFERENCES??)

Note that the realization is "degenerate" in the sense that the action of W_{aff} on $\mathfrak{h}$ is very far from faithful: in fact it factors through the natural action of W (identified with the Weyl group of $(\mathbf{G}^{(1)}, \mathbf{T}^{(1)})$) on $\Bbbk \otimes_{\mathbb{Z}} X^*(\mathbf{T}^{(1)})$.

The Hecke category associated with this realization will be denoted $\mathsf{D}_{\mathrm{aff}}^{\mathrm{BS}}$, and the Karoubian closure of its additive envelope will be denoted $\mathsf{D}_{\mathrm{aff}}$.

REMARK 1.2. (1) In view of the "barbell" relation in the Hecke category, the morphisms in $\mathsf{D}_{\mathrm{aff}}^{\mathrm{BS}}$ are generated by the upper and lower dot morphisms together with the $2m_{s,t}$-valent morphisms. In other words, one does not need to consider the "box" morphisms: they can be expressed in terms of the other generating morphisms.

(2) The identification of $\mathbb{X}$ with characters of $\mathbf{T}^{(1)}$ rather than $\mathbf{T}$ is mainly a matter of esthetics. Of course it is not necessary to make sense of the definition above, but we believe that this is the most natural way to think about it. (This feeling is somewhat justified by the discussion of the Finkelberg–Mirković conjecture in §3.2 below.)

(3) There are other "natural" choices for the $\Bbbk$-module $\mathfrak{h}$ in the definition above, which give rise to different Hecke categories but the same canonical basis, and for which Conjecture 1.3 below seems reasonable. For instance, one could take $\mathfrak{h} = \Bbbk \otimes_{\mathbb{Z}} X^*(\mathbf{T}^{(1)})$. Our choice is so that the Hecke category is "as small as possible," so that the requirement in Conjecture 1.3 is as mild as possible. For instance, if one denotes by $\mathsf{D}^{\mathrm{BS}\prime}_{\mathrm{aff}}$ the Hecke category defined using $\mathfrak{h} = \Bbbk \otimes_{\mathbb{Z}} X^*(\mathbf{T}^{(1)})$, then there exists a canonical faithful monoidal functor $\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}} \to \mathsf{D}^{\mathrm{BS}\prime}_{\mathrm{aff}}$ (induced by the natural morphism $\mathrm{S}(\Bbbk \otimes \mathbb{Z}\mathfrak{R}^\vee) \to \mathrm{S}(\Bbbk \otimes_{\mathbb{Z}} X_*(\mathbf{T}^{(1)})$, see §2.11.4 in Chapter 2), so that any action of $\mathsf{D}^{\mathrm{BS}\prime}_{\mathrm{aff}}$ gives rise to an action of $\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}$.

Recall that there exists a canonical algebra isomorphism

$$\mathcal{H}_{\mathrm{aff}} \xrightarrow{\sim} [\mathsf{D}_{\mathrm{aff}}]_{\oplus} \tag{1.1}$$

(where in the right-hand side we consider the split Grothendieck ring of the additive monoidal category $\mathsf{D}_{\mathrm{aff}}$) which sends, for any $s \in S_{\mathrm{aff}}$, the element $\underline{H}_s$ to the class $[\mathsf{B}_s]$; see REFERENCE. Recall also that there exists, for any $w \in W_{\mathrm{aff}}$, a canonical object $\mathsf{B}_w \in \mathsf{D}_{\mathrm{aff}}$ such the assignment $(w, n) \mapsto \mathsf{B}_w(n)$ induces a bijection between $W_{\mathrm{aff}} \times \mathbb{Z}$ and the set of isomorphism classes of indecomposable objects in $\mathsf{D}_{\mathrm{aff}}$. The classes

$$([\mathsf{B}_w] : w \in W_{\mathrm{aff}})$$

then form a $\mathbb{Z}[v, v^{-1}]$-basis of $[\mathsf{D}_{\mathrm{aff}}]_{\oplus}$. For any $w \in W_{\mathrm{aff}}$, we will denote by ${}^p\underline{H}_w$ the inverse image of $[\mathsf{B}_w]$ under the isomorphism (1.1).

1.3. The categorical conjecture. The following is a slight variant of a conjecture formulated and studied in [**RW1**].

CONJECTURE 1.3. *Assume that $p \geq h$, and let $\lambda \in C$. There exists a right action*[1] *of the monoidal category $\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}$ on the category $\mathsf{Rep}_0(\mathbf{G})$ such that:*

(1) the functor (1) acts by the identity.
(2) for any $s \in S_{\mathrm{aff}}$ the object B_s acts via a functor isomorphic to Θ_s.

REMARK 1.4. (1) In Conjecture 1.3 we required the existence of a *right* action of $\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}$. The reason for that is that it makes the comparison with the combinatorics of tilting $\mathbf{G}$-modules transparent, as we will see below. However, the existence of a left action or of a right action is equivalent, since $\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}$ admits an equivalence which switches the order in products and fixes each of the generating objects.

(2) The original formulation of this conjecture in [**RW1**, §5.1] included a requirement that the images of the dot and trivalent morphisms are provided by fixed choices of adjunctions (T^s, T_s) and (T_s, T^s). Although it seems

[1] By an action of a monoidal category A on a category C we mean a monoidal functor from A to the category $\mathsf{EndoFun}_{\mathsf{C}}$ of endofunctors of C. By a *right* action of A on C we mean an action of A° on C, where A° is the monoidal category with the same underlying category as A, and monoidal product $\bullet$ defined by $A \bullet B = B \cdot A$.

very natural, this condition is in fact not needed for the main application of this construction, as noted in [RW1, Remark 5.1.2(3)] (and as we will see below). In practice, this extra condition is not always easy to check.

In Section 2, we explain how this conjecture (if true) solves the question of computing characters of indecomposable tilting $\mathbf{G}$-modules. In Section 3 we discuss several proofs of this conjecture and of its main application.

2. Consequences on tilting characters

2.1. The tilting character formula. Recall (see REF) that to the data above we can associate a family

$$({}^p h_{y,w} : y, w \in W_{\mathrm{aff}})$$

of p-Kazhdan–Lusztig polynomials. Recall also the subset ${}^{\mathrm{f}}W_{\mathrm{aff}} \subset W_{\mathrm{aff}}$ of elements in W_{aff} which are minimal in their right coset relative to the parabolic subgroup $W \subset W_{\mathrm{aff}}$ (see §2.8.1 in Chapter 1). Relative to this choice of parabolic subgroup we also have the "antispherical" polynomials

$$({}^p n_{y,w} : y, w \in {}^{\mathrm{f}}W_{\mathrm{aff}}),$$

which are related to the previous polynomials by the following formula:

$${}^p n_{y,w} = \sum_{z \in W} (-v)^{\ell(z)} \cdot {}^p h_{zy,w} \quad \text{for } y, w \in {}^{\mathrm{f}}W_{\mathrm{aff}}.$$

Recall also the indecomposable tilting modules $(\mathsf{T}(\lambda) : \lambda \in \mathbb{X}^+)$ introduced in Chapter 4.

Assume now that $p \geq h$, and fix $\lambda \in C \cap \mathbb{X}$. We have explained in Chapter 4 the importance of determining the multiplicities $(\mathsf{T}(w \cdot_p \lambda) : \mathsf{N}(y \cdot_p \lambda))$ for $y, w \in W_{\mathrm{aff}}$ such that $w \cdot_p \lambda$ and $y \cdot_p \lambda$ are dominant, i.e. (see Proposition 2.29 in Chapter 1) for $y, w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$. The following conjecture (first stated in [RW1]) proposes an answer to this question.

Conjecture 2.1. *Assume that $p \geq h$, and let $\lambda \in C$. For any $y, w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ we have*

$$(\mathsf{T}(w \cdot_p \lambda) : \mathsf{N}(y \cdot_p \lambda)) = {}^p n_{y,w}(1).$$

As we will see below, Conjecture 2.1 is in fact a consequence of Conjecture 1.3. However, due to its importance (in fact this consequence was the main motivation behind the formulation of Conjecture 1.3), and since it can in fact be attacked by other methods, we state this formula as an independent conjecture.

Of course, the formula in Conjecture 2.1 looks quite similar to that in Andersen's conjecture (Conjecture 4.2 in Chapter 4). There are however two important differences. The first one is that it involves the p-Kazhdan–Lusztig polynomials rather than the "ordinary" Kazhdan–Lusztig polynomials. The second one is that it does *not* require any bound on the elements y, w. This is related to the fact that the p-Kazhdan–Lusztig basis "sees" all iterations of Donkin's formula (see §3.1 in Chapter 4), which is not the case of the ordinary Kazhdan–Lusztig basis. (For elaborations on this idea, see [AR4].)

Remark 2.2. Recall that for fixed $y, w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, for $p \gg 0$ we have ${}^p n_{y,w} = n_{y,w}$ where the right-hand side is as in §4.2 in Chapter 4. Hence, for fixed $y, w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, the formula in Conjecture 2.1 implies that in Andersen's conjecture for large p.

However, it is *not* the case that Conjecture 2.1 implies Andersen's conjecture for large p. This is due to the fact that the number of elements w which satisfy the bound in Andersen's conjecture grows with p.

2.2. A singular variant. Since it involves a regular weight, the statement of Conjecture 2.1 makes sense only under the assumption that $p \geq h$. We now explain how a modification of this conjecture makes sense for any value of p. Namely, let us now drop the assumption on p. As explained in §2.7.2 in Chapter 1, the set $\overline{C} \cap \mathbb{X}$ is a set of representatives for the W_{aff}-orbits in $\mathbb{X}$. Fix $\mu \in \overline{C} \cap \mathbb{X}$; as explained in §2.8.1 in Chapter 1, the set $W_{\mathrm{aff}} \cdot_p \mu \cap \mathbb{X}^+$ is then in a canonical bijection with ${}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)}$. (This subset only depends on the facet containing μ.)

The following conjecture is a "singular" variant of Conjecture 2.1, which makes sense for any p, and gives (if true) a general answer to the question of computing characters of all indecomposable tilting $\mathbf{G}$-modules. (This variant was also stated in [RW1].)

CONJECTURE 2.3. *Let $\mu \in \overline{C} \cap \mathbb{X}$. For any $y, w \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{(\mu)}$ we have*

$$(\mathsf{T}(w \cdot_p \mu) : \mathsf{N}(y \cdot_p \mu)) = {}^{p}n_{y,w}(1).$$

REMARK 2.4. It is clear that Conjecture 2.1 is a special case of Conjecture 2.3. In fact, it is not difficult to check that if $p \geq h$, the two conjectures are equivalent. (See Exercise 6.1.)

2.3. Translation functors and ∇-sections.

2.3.1. *Setting.* The rest of this section is devoted to the proof that Conjecture 1.3 implies Conjecture 2.1, following [RW1, Part I]. The statements in the next three subsections are independent of any conjecture. They could have been stated (and proved) in Chapter 4; however, since their only application so far is to the question considered in the present chapter, we have chosen to explain them here.

We start with some generalities regarding the ∇-sections (defined in §5.4 of Appendix A) for the highest weight categories introduced in §1.1. We will also consider the categories $\mathsf{Rep}_0(\mathbf{G})^{\geq w}$ involved in this definition (for any $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$).

First we fix some data. By Proposition 2.19 in Chapter 1, there exist adjunctions (T_s, T^s) and (T^s, T_s). For simplicity, we fix a choice for such adjunctions; this gives rise to adjunction morphisms

$$\mathrm{id} \xrightarrow{\mathrm{adj}} T^s T_s, \quad \mathrm{id} \xrightarrow{\mathrm{adj}} T_s T^s. \tag{2.1}$$

For any $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}^s$, by Proposition 1.5 in Chapter 4 we have

$$T_s \mathsf{T}_w^s \cong \mathsf{T}_w. \tag{2.2}$$

We fix a choice of such an isomorphism. Similarly, by Remark 1.6 in Chapter 4, T_w^s is a direct summand in $T^s \mathsf{T}_{ws}$. In fact it is easily seen that T_w^s is a direct summand with multiplicity 1, and that all other indecomposable direct summands have a label y which satisfies $y < w$. We fix a split embedding and a split surjection

$$\mathsf{T}_w^s \hookrightarrow T^s \mathsf{T}_{ws}, \quad T^s \mathsf{T}_{ws} \twoheadrightarrow \mathsf{T}_w^s. \tag{2.3}$$

2.3.2. *Translation to a wall.* We now fix an object $M \in \mathsf{Rep}_0(\mathbf{G})$ which admits a costandard filtration, and a ∇-section $(\Pi, e, (\varphi_\pi : \pi \in \Pi)$ of M. We set

$$\Pi' := \{\pi \in \Pi \mid e(\pi)s \in {}^{\mathrm{f}}W_{\mathrm{aff}}\}.$$

We also define a map

$$e' : \Pi' \to {}^{\mathrm{f}}W_{\mathrm{aff}}^s$$

by defining $e'(\pi)$ as the maximal element in $\{e(\pi), e(\pi)s\}$. (Here, by definition of Π' both elements belong to ${}^{\mathrm{f}}W_{\mathrm{aff}}$, so that the maximal element among them indeed belongs to ${}^{\mathrm{f}}W_{\mathrm{aff}}^s$.) Next we explain how to define, for any $\pi \in \Pi'$, a morphism

$$\psi_\pi : \mathsf{T}_{e'(\pi)} \to T^s M.$$

- First, let us assume that $e(\pi)s < e(\pi)$. Then $e'(\pi) = e(\pi)$, and we have an isomorphism $T_s \mathsf{T}^s_{e'(\pi)} \cong \mathsf{T}_{e(\pi)}$, see (2.2). We define ψ_π as the composition
$$\mathsf{T}^s_{e'(\pi)} \xrightarrow{\mathrm{adj}} T^s T_s \mathsf{T}^s_{e'(\pi)} \cong T^s \mathsf{T}_{e(\pi)} \xrightarrow{T^s\varphi_\pi} T^s M,$$
where the first morphism is as in (2.1). In other words, ψ_π is the image of φ_π under the series of isomorphisms
$$\mathrm{Hom}(\mathsf{T}_{e(\pi)}, M) \cong \mathrm{Hom}(T_s \mathsf{T}^s_{e'(\pi)}, M) \cong \mathrm{Hom}(\mathsf{T}^s_{e'(\pi)}, T^s M).$$
- Next, assume that $e(\pi) < e(\pi)s$. Then $e'(\pi) = e(\pi)s$, and we have a split embedding $\mathsf{T}^s_{e'(\pi)} \hookrightarrow T^s \mathsf{T}_{e(\pi)}$, see (2.3). In this case, we define ψ_π as the composition
$$\mathsf{T}^s_{e'(\pi)} \hookrightarrow T^s \mathsf{T}_{e(\pi)} \xrightarrow{T^s\varphi_\pi} T^s M.$$

PROPOSITION 2.5. *The triple $(\Pi', e', (\psi_\pi : \pi \in \Pi'))$ is a ∇-section of $T^s M$.*

PROOF. We will prove the proposition in 3 steps: first if $M = \mathsf{N}_w$ for some $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, then if M is a direct sum of copies of an object N_w for some $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, and finally in general. (The general case will be reduced to the special case treated before using the "truncation" functors considered in Exercise 7.7 and some compatibility property of our construction with respect to these functors.)

First, we assume that $M = \mathsf{N}_w$. In this case a ∇-flag of M consists of one nonzero (hence surjective) morphism $f : \mathsf{T}_w \to \mathsf{N}_w$. If $\{w, ws\} \cap {}^{\mathrm{f}}W_{\mathrm{aff}}^s = \varnothing$, then $T^s M = 0$ by Proposition 2.36 in Chapter 1, and the datum constructed above is empty, so that the claim is clear. Next we assume that $\{w, ws\} \cap {}^{\mathrm{f}}W_{\mathrm{aff}}^s \neq \varnothing$, and denote by y the largest element among w and ws. Then $T^s M = \mathsf{N}^s_y$ (again by Proposition 2.36 in Chapter 1), and the datum constructed above consists of one morphism $\mathsf{T}^s_y \to T^s M$. What we have to prove is that this morphism is nonzero. If $y = w$ then this is clear since our morphism is the image of f under some isomorphism. If $y = ws$, we use the fact that $T^s f$ is surjective (hence nonzero) by exactness of T^s. Since its restriction to any indecomposable direct summand of $T^s \mathsf{T}_w$ distinct from T^s_{ws} vanishes (because the multiplicity of M^s_{ws} in such a module vanishes), its restriction to T^s_{ws} is nonzero, as desired.

Next, we assume that M is a direct sum of copies of N_w. In this case, $e(\pi) = w$ for any $\pi \in \Pi$. Fixing a surjection $p : \mathsf{T}_w \to \mathsf{N}_w$, each φ_π factors through p, and this collection determines an isomorphism

$$M \cong (\mathsf{N}_w)^\Pi,$$

and our ∇-section consists of a union of ∇-sections of each factor. This reduces this case to the one treated above.

Finally we consider the general case. What we have to prove is that for any $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}^s$, the image of the collection $(\psi_{\pi'} : \pi' \in (e')^{-1}(w))$ forms a basis of $\mathrm{Hom}_{\mathsf{Rep}_s(\mathbf{G})^{\geq w}}(\mathsf{T}_w^s, T^s M)$. Here we have

$$(e')^{-1}(w) = e^{-1}(w) \sqcup e^{-1}(ws).$$

Fix such a w, and choose an ideal $\Omega \subset {}^{\mathrm{f}}W_{\mathrm{aff}}$ such that

- $\Omega \cap \{w, ws\} = \{ws\}$;
- $\Omega' := \Omega \smallsetminus \{ws\}$ is an ideal;
- $\Omega'' := \Omega \cup \{w\}$ is an ideal.

(For instance, $\Omega = \{y \in {}^{\mathrm{f}}W_{\mathrm{aff}} \mid y < w\}$ satifies these conditions.) Then we have embeddings

$$\Gamma_{\Omega'}(M) \hookrightarrow \Gamma_{\Omega}(M) \hookrightarrow \Gamma_{\Omega''}(M) \hookrightarrow M \tag{2.4}$$

where we use the notation of Exercise 7.7. Let us denote by e_Ω and $e_{\Omega''}$ the restrictions of e to $e^{-1}(\Omega)$ and $e^{-1}(\Omega'')$ respectively. By Exercise 7.8, if $\pi \in e^{-1}(\Omega)$, resp. $\pi \in e^{-1}(\Omega'')$, then φ_π factors through a morphism

$$\varphi_\pi^\Omega : \mathsf{T}_{e(\pi)} \to \Gamma_\Omega(M), \quad \text{resp.} \quad \varphi_\pi^{\Omega''} : \mathsf{T}_{e(\pi)} \to \Gamma_{\Omega''}(M),$$

and the collection $(e^{-1}(\Omega), e_\Omega, (\varphi_\pi^\Omega : \pi \in e^{-1}(\Omega)))$, resp. $(e^{-1}(\Omega''), e_{\Omega''}, (\varphi_\pi^{\Omega''} : \pi \in e^{-1}(\Omega'')))$ is a ∇-section of $\Gamma_\Omega(M)$, resp. $\Gamma_{\Omega''}(M)$.

By exactness, applying T^s to (2.4) we obtain embeddings

$$T^s(\Gamma_{\Omega'}(M)) \hookrightarrow T^s(\Gamma_\Omega(M)) \hookrightarrow T^s(\Gamma_{\Omega''}(M)) \hookrightarrow T^s(M).$$

By construction, if $\pi \in e^{-1}(\Omega)$, resp. $\pi \in e^{-1}(\Omega'')$, then the morphism ψ_π factors through a morphism

$$\psi_\pi^\Omega : \mathsf{T}_{e'(\pi)}^s \to T^s(\Gamma_\Omega M), \quad \text{resp.} \quad \psi_\pi^{\Omega''} : \mathsf{T}_{e'(\pi)}^s \to T^s(\Gamma_{\Omega''} M).$$

Moreover, these morphisms coincide with those obtained by the procedure above applied to the ∇-section $(e^{-1}(\Omega), e_\Omega, (\varphi_\pi^\Omega : \pi \in e^{-1}(\Omega)))$ of $\Gamma_\Omega(M)$, resp. to the ∇-section $(e^{-1}(\Omega''), e_{\Omega''}, (\varphi_\pi^{\Omega''} : \pi \in e^{-1}(\Omega'')))$ of $\Gamma_{\Omega''}(M)$.

Note that

$$(T^s(M/\Gamma_{\Omega''}(M)) : \mathsf{N}_w^s) = 0,$$

so that the natural morphism

$$\mathrm{Hom}_{\mathsf{Rep}_s(\mathbf{G})^{\geq w}}(\mathsf{T}_w^s, T^s(\Gamma_{\Omega''}(M))) \to \mathrm{Hom}_{\mathsf{Rep}_s(\mathbf{G})^{\geq w}}(\mathsf{T}_w^s, T^s(M))$$

is an isomorphism. As a consequence, to finish the proof it suffices to check that the image of the family $(\psi_\pi^{\Omega''} : \pi \in e^{-1}(w) \sqcup e^{-1}(ws))$ is a basis of the vector space $\mathrm{Hom}_{\mathsf{Rep}_s(\mathbf{G})^{\geq w}}(\mathsf{T}_w^s, T^s(M))$.

Next, if $\pi \in e^{-1}(ws)$, resp. $\pi \in e^{-1}(w)$, we consider the composition

$$\varphi_\pi^{\Omega,\Omega'} : \mathsf{T}_{ws} \xrightarrow{\varphi_\pi^\Omega} \Gamma_\Omega(M) \twoheadrightarrow \Gamma_\Omega(M)/\Gamma_{\Omega'}(M),$$

resp.

$$\varphi_\pi^{\Omega'',\Omega} : \mathsf{T}_w \xrightarrow{\varphi_\pi^{\Omega''}} \Gamma_{\Omega''}(M) \twoheadrightarrow \Gamma_{\Omega''}(M)/\Gamma_\Omega(M).$$

Again by Exercise 7.8, if we denote by $e_{ws} : e^{-1}(ws) \to \{ws\}$, resp. $e_w : e^{-1}(w) \to \{w\}$ the unique map, then the collection $(e^{-1}(ws), e_{ws}, (\varphi_\pi^{\Omega,\Omega'} : \pi \in e^{-1}(ws)))$, resp. $(e^{-1}(w), e_w, (\varphi_\pi^{\Omega'',\Omega} : \pi \in e^{-1}(w)))$, is a ∇-section of $\Gamma_\Omega(M)/\Gamma_{\Omega'}(M)$, resp. of

$\Gamma_{\Omega''}(M)/\Gamma_{\Omega}(M)$. Moreover, the morphisms obtained by the procedure above applied to these ∇-sections are the compositions

$$\psi_\pi^{\Omega,\Omega'} : \mathsf{T}^s_{e'(\pi)} \xrightarrow{\psi_\pi^{\Omega}} T^s(\Gamma_\Omega M) \twoheadrightarrow T^s(\Gamma_\Omega(M)/\Gamma_{\Omega'}(M)),$$

resp.

$$\psi_\pi^{\Omega'',\Omega} : \mathsf{T}^s_{e'(\pi)} \xrightarrow{\psi_\pi^{\Omega''}} T^s(\Gamma_{\Omega''} M) \twoheadrightarrow T^s(\Gamma_{\Omega''}(M)/\Gamma_{\Omega}(M)).$$

Note that $\Gamma_\Omega(M)/\Gamma_{\Omega'}(M)$, resp. $\Gamma_{\Omega''}(M)/\Gamma_\Omega(M)$, is isomorphic to a direct sum of copies of N_{ws}, resp. N_w. By the special case treated above, we deduce that the images of the families $(\psi_\pi^{\Omega,\Omega'} : \pi \in e^{-1}(ws))$ and $(\psi_\pi^{\Omega'',\Omega} : \pi \in e^{-1}(w))$ are bases of the vector spaces

$$\mathrm{Hom}_{\mathsf{Rep}_s(\mathbf{G})^{\geq w}}(\mathsf{T}^s_w, T^s(\Gamma_\Omega(M)/\Gamma_{\Omega'}(M)))$$

and

$$\mathrm{Hom}_{\mathsf{Rep}_s(\mathbf{G})^{\geq w}}(\mathsf{T}^s_w, T^s(\Gamma_{\Omega''}(M)/\Gamma_{\Omega}(M)))$$

respectively.

Finally, if $\pi \in e^{-1}(ws)$, resp. $\pi \in e^{-1}(w)$, we consider the composition

$$\varphi_\pi^{\Omega'',\Omega'} : \mathsf{T}_{ws} \xrightarrow{\varphi_\pi^{\Omega,\Omega'}} \Gamma_\Omega(M)/\Gamma_{\Omega'}(M) \hookrightarrow \Gamma_{\Omega''}(M)/\Gamma_{\Omega'}(M),$$

resp.

$$\varphi_\pi^{\Omega'',\Omega'} : \mathsf{T}_w \xrightarrow{\varphi_\pi^{\Omega''}} \Gamma_{\Omega''}(M) \twoheadrightarrow \Gamma_{\Omega''}(M)/\Gamma_{\Omega'}(M).$$

Once again these morphisms constitute a ∇-section of $\Gamma_{\Omega''}(M)/\Gamma_{\Omega'}(M)$, and the procedure above provides morphisms

$$\psi_\pi^{\Omega'',\Omega'} : \mathsf{T}^s_w \to T^s(\Gamma_{\Omega''}(M)/\Gamma_{\Omega'}(M))$$

for $\pi \in e^{-1}(ws) \sqcup e^{-1}(w)$. If $e(\pi) = ws$ then $\psi_\pi^{\Omega'',\Omega'}$ is the composition of $\psi_\pi^{\Omega,\Omega'}$ with the embedding

$$T^s(\Gamma_\Omega(M)/\Gamma_{\Omega'}(M)) \hookrightarrow T^s(\Gamma_{\Omega''}(M)/\Gamma_{\Omega'}(M)),$$

and if $e(\pi) = w$ then the composition of $\psi_\pi^{\Omega'',\Omega'}$ with the surjection

$$T^s(\Gamma_{\Omega''}(M)/\Gamma_{\Omega'}(M)) \twoheadrightarrow T^s(\Gamma_{\Omega''}(M)/\Gamma_{\Omega}(M))$$

is $\psi_\pi^{\Omega'',\Omega}$.

Consider the exact sequence

$$\Gamma_\Omega(M)/\Gamma_{\Omega'}(M) \hookrightarrow \Gamma_{\Omega''}(M)/\Gamma_{\Omega'}(M) \twoheadrightarrow \Gamma_{\Omega''}(M)/\Gamma_{\Omega}(M)$$

and the exact sequence

$$\begin{aligned}&\mathrm{Hom}_{\mathsf{Rep}_s(\mathbf{G})^{\geq w}}(\mathsf{T}^s_w, T^s(\Gamma_\Omega(M)/\Gamma_{\Omega'}(M)))\\ &\qquad\hookrightarrow \mathrm{Hom}_{\mathsf{Rep}_s(\mathbf{G})^{\geq w}}(\mathsf{T}^s_w, T^s(\Gamma_{\Omega''}(M)/\Gamma_{\Omega'}(M)))\\ &\qquad\qquad\twoheadrightarrow \mathrm{Hom}_{\mathsf{Rep}_s(\mathbf{G})^{\geq w}}(\mathsf{T}^s_w, T^s(\Gamma_{\Omega''}(M)/\Gamma_{\Omega}(M)))\end{aligned}$$

obtained by applying the functor $\mathrm{Hom}_{\mathsf{Rep}_s(\mathbf{G})^{\geq w}}(\mathsf{T}^s_w, T^s(-))$. (This sequence is exact because the image of T^s_w, resp. $T^s(\Gamma_\Omega(M)/\Gamma_{\Omega'}(M))$, admits a standard, resp. costandard, filtration in the highest weight category $\mathsf{Rep}_s(\mathbf{G})^{\geq w}$.) Here we have the family $(\psi_\pi^{\Omega'',\Omega'} : \pi \in e^{-1}(ws) \sqcup e^{-1}(w)$ in the middle term. By the comments above, the part of this family parametrized by $e^{-1}(ws)$ is the image of a basis of the first term, and the part parametrized by $e^{-1}(w)$ maps to a basis of the third term. This family is therefore a basis.

To conclude, we remark that we have

$$(T^s(\Gamma_{\Omega'}(M)) : \mathsf{N}^s_w) = 0,$$

hence the natural morphism

$$\mathrm{Hom}_{\mathsf{Rep}_s(\mathbf{G})^{\geq w}}(\mathsf{T}^s_w, T^s(\Gamma_{\Omega''}(M))) \to \mathrm{Hom}_{\mathsf{Rep}_s(\mathbf{G})^{\geq w}}(\mathsf{T}^s_w, T^s(\Gamma_{\Omega''}(M)/\Gamma_{\Omega'}(M)))$$

is an isomorphism. This isomorphism sends the image of the family $(\psi_\pi^{\Omega''} : \pi \in e^{-1}(w) \sqcup e^{-1}(ws))$ to the image of the family $(\psi_\pi^{\Omega'',\Omega'} : \pi \in e^{-1}(ws) \sqcup e^{-1}(w))$, which implies that the former image is a basis and concludes the proof. $\square$

2.3.3. *Translation from a wall.* We now fix $s \in S_{\mathrm{aff}}$, an object $M \in \mathsf{Rep}_s(\mathbf{G})$ which admits a costandard filtration, and a ∇-section $(\Pi, e, (\varphi_\pi)_{\pi\in\Pi})$ of M. We set

$$\Pi' := \Pi \times \{0, 1\},$$

and define a map $e' : \Pi' \to {}^{\mathrm{f}}W_{\mathrm{aff}}$ by setting $e'(\pi, \varepsilon) = e(\pi)s^{1-\varepsilon}$ for $\varepsilon \in \{0, 1\}$. Finally, for $\pi \in \Pi$ we define the morphisms $\psi_{(\pi,0)}$ and $\psi_{(\pi,1)}$ as follows.

- Since $e'(\pi, 1) \in {}^{\mathrm{f}}W^s_{\mathrm{aff}}$, we have a fixed isomorphism $T_s\mathsf{T}^s_{e'(\pi,1)} \cong \mathsf{T}_{e(\pi)}$, see (2.2). We define $\psi_{(\pi,1)}$ as the composition
$$\mathsf{T}_{e'(\pi,1)} \cong T_s\mathsf{T}^s_{e(\pi)} \xrightarrow{T_s\varphi_\pi} T_sM.$$
- In (2.3) we have fixed a split surjection $T^s\mathsf{T}_{e'(\pi,0)} \twoheadrightarrow \mathsf{T}_{e(\pi)}$. We define $\psi_{(\pi,0)}$ as the composition
$$\mathsf{T}_{e'(\pi,0)} \xrightarrow{\mathrm{adj}} T_sT^s\mathsf{T}_{e'(\pi,0)} \twoheadrightarrow T_s\mathsf{T}_{e(\pi)} \xrightarrow{T_s\varphi_\pi} T_sM,$$
where the first morphism is as in (2.1). In other words, $\psi_{(\pi,0)}$ is the image of the composition
$$T^s\mathsf{T}_{e'(\pi,0)} \twoheadrightarrow \mathsf{T}_{e(\pi)} \xrightarrow{\varphi_\pi} M$$
under the isomorphism
$$\mathrm{Hom}_{\mathsf{Rep}_s(\mathbf{G})}(T^s\mathsf{T}_{e'(\pi,0)}, M) \cong \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{T}_{e'(\pi,0)}, T_sM)$$
provided by adjunction.

PROPOSITION 2.6. *The triple $(\Pi', e', (\psi_{\pi'})_{\pi'\in\Pi'})$ is a ∇-section of T_sM.*

The proof of Proposition 2.6 is very similar to those of Proposition 2.5. It will use the following easy lemma.

LEMMA 2.7. *For any M in $\mathsf{Rep}_s(\mathbf{G})$ which admits a costandard filtration and any $y \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, if $(T_sM : \mathsf{N}_w) \neq 0$ then $\{w, ws\} \cap {}^{\mathrm{f}}W^s_{\mathrm{aff}} \neq \varnothing$.*

PROOF. This property follows from the exactness of the functor T_s and Proposition 2.36(2) in Chapter 1. $\square$

PROOF OF PROPOSITION 2.6. We first consider the case when $M = \mathsf{N}^s_w$ for some $w \in {}^{\mathrm{f}}W^s_{\mathrm{aff}}$. Then the ∇-section consists of a single nonzero (hence surjective) morphism $f : \mathsf{T}^s_w \to \mathsf{N}^s_w$. By Proposition 2.36 in Chapter 1, there exists an exact sequence

$$\mathsf{N}_{ws} \hookrightarrow T_sM \twoheadrightarrow \mathsf{N}_w,$$

and it is easily seen that the natural morphisms

$$\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{T}_{ws}, \mathsf{N}_{ws}) \to \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{T}_{ws}, \mathsf{N}_{ws}) \to \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{T}_{ws}, T_sM)$$

are isomorphisms, as well as the natural morphisms

$$\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq w}}(\mathsf{T}_w, T_s M) \to \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq w}}(\mathsf{T}_w, \mathsf{N}_w)$$

and

$$\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{T}_w, \mathsf{N}_w) \to \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq w}}(\mathsf{T}_w, \mathsf{N}_w),$$

and that all of these spaces are 1-dimensional. Our construction provides morphisms

$$g : \mathsf{T}_{ws} \to T_s M \quad \text{and} \quad h : \mathsf{T}_w \to T_s M.$$

Since $\mathrm{Hom}(\mathsf{T}_{ws}, \mathsf{N}_w) = 0$ (because $(\mathsf{T}_{ws} : \mathsf{M}_w) = 0$), g must factor through a morphism $g' : \mathsf{T}_{ws} \to \mathsf{N}_{ws}$, and we denote by h' the composition of h with the surjection $T_s M \to \mathsf{N}_w$. With this notation, to conclude it suffices to prove that g' and h' are nonzero. By construction h' is surjective, hence nonzero. On the other hand g is nonzero, as the image of a surjective (hence nonzero) morphism under an isomorphism, which implies that g' is nonzero as well.

Once this case is known, we deduce the case when M is isomorphic to a direct sum of copies of some module N_w^s as in the proof of Proposition 2.5.

Finally we treat the general case. We need to show that for any $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, the image of the family $(\psi_\pi : \pi \in (e')^{-1}(w))$ forms a basis of $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq w}}(\mathsf{T}_w, T_s M)$. If $\{w, ws\} \cap {}^{\mathrm{f}}W_{\mathrm{aff}}^s = \varnothing$ then we have

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq w}}(\mathsf{T}_w, T_s M) = (T_s M : \mathsf{N}_w) = 0$$

by Lemma 2.7, and $(e')^{-1}(w) = \varnothing$, so that there is nothing to prove in this case. We will now show that for any $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}^s$ the claim holds both for w and for ws. Here, by construction we have canonical bijections

$$e^{-1}(w) \xrightarrow{\sim} (e')^{-1}(w) \quad \text{and} \quad e^{-1}(w) \xrightarrow{\sim} (e')^{-1}(ws)$$

given by $\pi \mapsto (\pi, 1)$ and $\pi \mapsto (\pi, 0)$ respectively.

Let $\Omega \subset {}^{\mathrm{f}}W_{\mathrm{aff}}^s$ be an ideal containing w and in which w is maximal. Then $\Omega' := \Omega \smallsetminus \{w\}$ is also an ideal in ${}^{\mathrm{f}}W_{\mathrm{aff}}^s$. We have embeddings

$$\Gamma_{\Omega'}(M) \hookrightarrow \Gamma_\Omega(M) \hookrightarrow M,$$

and $\Gamma_\Omega(M)/\Gamma_{\Omega'}(M)$ is a direct sum of copies of N_w^s. By Exercise 7.8, for any $\pi \in e^{-1}(w)$ the morphism φ_π factors through a morphism

$$\varphi_\pi^\Omega : \mathsf{T}_w^s \to \Gamma_\Omega(M),$$

and moreover the compositions

$$\varphi_\pi^{\Omega,\Omega'} : \mathsf{T}_w^s \xrightarrow{\varphi_\pi^\Omega} \Gamma_\Omega(M) \to \Gamma_\Omega(M)/\Gamma_{\Omega'}(M)$$

for $\pi \in e^{-1}(w)$ constitute a ∇-section of $\Gamma_\Omega(M)/\Gamma_{\Omega'}(M)$. By the case treated above, applying the construction above we obtain morphisms

$$\psi_{(\pi,1)}^{\Omega,\Omega'} : \mathsf{T}_w \to T_s(\Gamma_\Omega(M)/\Gamma_{\Omega'}(M))$$

whose images constitute a basis of $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq w}}(\mathsf{T}_w, T_s(\Gamma_\Omega(M)/\Gamma_{\Omega'}(M)))$, and morphisms

$$\psi_{(\pi,0)}^{\Omega,\Omega'} : \mathsf{T}_{ws} \to T_s(\Gamma_\Omega(M)/\Gamma_{\Omega'}(M))$$

whose images constitute a basis of $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{T}_{ws}, T_s(\Gamma_\Omega(M)/\Gamma_{\Omega'}(M)))$.

Now we have

$$(T_s(\Gamma_{\Omega'}(M)) : \mathsf{N}_w) = (T_s(M/\Gamma_\Omega(M)) : \mathsf{N}_w) = 0,$$

so that the natural morphisms

$$\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq w}}(\mathsf{T}_w, T_s(\Gamma_\Omega(M))) \to \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq w}}(\mathsf{T}_w, T_s M)$$

and

$$\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq w}}(\mathsf{T}_w, T_s(\Gamma_\Omega(M))) \to \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq w}}(\mathsf{T}_w, T_s(\Gamma_\Omega(M)/\Gamma_{\Omega'}(M)))$$

are isomorphisms. From the construction we see that under the identification

$$\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq w}}(\mathsf{T}_w, T_s M) \cong \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq w}}(\mathsf{T}_w, T_s(\Gamma_\Omega(M)/\Gamma_{\Omega'}(M)))$$

the image of the family $(\psi_{(\pi,1)} : \pi \in e^{-1}(w))$ corresponds to the image of the family $(\psi^{\Omega,\Omega'}_{(\pi,1)} : \pi \in e^{-1}(w))$, so that the former family constitutes a basis. Similar arguments prove the desired claim for ws, which concludes the proof. □

2.4. Bott–Samelson type tilting modules and morphisms between them. If $\underline{w} = (s_1, \cdots, s_r)$ is a word in S_{aff}, we set

$$\mathsf{T}_{\underline{w}} := \Theta_{s_r} \circ \cdots \circ \Theta_{s_1}(\mathsf{T}(\lambda)).$$

This object is a tilting module. Recall that the Grothendieck group $[\mathsf{Rep}_0(\mathbf{G})]$ identifies with $\mathcal{N}^0_{\mathrm{aff}}$, see §4.2 in Chapter 4. Through this identification the morphism induced by Θ_s corresponds to right multiplication by $\underline{H}_s$. Moreover, for a tilting module $M \in \mathsf{Rep}_0(\mathbf{G})$ we have

$$[M] = \sum_{y \in {}^{\mathrm{f}}W_{\mathrm{aff}}} (M : \mathsf{N}_y) \cdot N^0_y. \tag{2.5}$$

We deduce that the multiplicity of N_y in $[\mathsf{T}_{\underline{w}}]$ is equal to the coefficient of N^0_y in the expansion of the element

$$N^0_e \cdot \underline{H}_{s_1} \cdot (\cdots) \cdot \underline{H}_{s_r}$$

in the basis $(N^0_x : x \in {}^{\mathrm{f}}W_{\mathrm{aff}})$. In particular, if $\underline{w}$ is a reduced expression for some element $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, the indecomposable tilting module T_w appears as a direct summand of $\mathsf{T}_{\underline{w}}$ with multiplicity 1, and all the other direct summands are of the form T_y with $y \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ which satisfies $y < w$.

Proposition 2.8. *Let $\underline{x}$ and $\underline{y}$ be words in S_{aff}, and assume that $\underline{x}$ is a reduced expression for some element $x \in {}^{\mathrm{f}}W_{\mathrm{aff}}$. Let also $s \in S_{\mathrm{aff}}$.*

(1) Assume that $x < xs$. Let $(f_i : i \in I)$ be a family of morphisms in $\mathrm{Hom}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$ whose images span the vector space $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$, and let $(g_j : j \in J)$ be a family of morphisms in $\mathrm{Hom}(\mathsf{T}_{\underline{x}s}, \mathsf{T}_{\underline{y}})$ whose images span the vector space $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq xs}}(\mathsf{T}_{\underline{x}s}, \mathsf{T}_{\underline{y}})$. There exist morphisms $f'_i : \mathsf{T}_{\underline{x}} \to \mathsf{T}_{\underline{x}s}$ (for $i \in I$) and $g'_j : \mathsf{T}_{\underline{x}} \to \mathsf{T}_{\underline{x}ss}$ (for $j \in J$) such that the images of the compositions

$$\mathsf{T}_{\underline{x}} \xrightarrow{f'_i} \mathsf{T}_{\underline{x}s} = \Theta_s \mathsf{T}_{\underline{x}} \xrightarrow{\Theta_s(f_i)} \Theta_s \mathsf{T}_{\underline{y}} = \mathsf{T}_{\underline{y}s}$$

and the compositions

$$\mathsf{T}_{\underline{x}} \xrightarrow{g'_j} \mathsf{T}_{\underline{x}ss} = \Theta_s \mathsf{T}_{\underline{x}s} \xrightarrow{\Theta_s(g_j)} \Theta_s \mathsf{T}_{\underline{y}} = \mathsf{T}_{\underline{y}s}$$

span $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}s})$.

(2) *Assume that* $\underline{x} = \underline{z}s$ *for some word* $\underline{z}$ *which is a reduced expression for some element* $z \in {}^{\mathrm{f}}W_{\mathrm{aff}}$. *Let* $(f_i : i \in I)$ *be a family of morphisms in* $\mathrm{Hom}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$ *whose images span the vector space* $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$, *and let* $(g_j : j \in J)$ *be a family of morphisms in* $\mathrm{Hom}(\mathsf{T}_{\underline{z}}, \mathsf{T}_{\underline{y}})$ *whose images span the vector space* $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq z}}(\mathsf{T}_{\underline{z}}, \mathsf{T}_{\underline{y}})$. *There exist morphisms* $f'_i : \mathsf{T}_{\underline{x}} \to \mathsf{T}_{\underline{x}s}$ *(for* $i \in I$*) and* $g'_j : \mathsf{T}_{\underline{x}} \to \mathsf{T}_{\underline{x}}$ *(for* $j \in J$*) such that the images of the compositions*

$$\mathsf{T}_{\underline{x}} \xrightarrow{f'_i} \mathsf{T}_{\underline{x}s} = \Theta_s \mathsf{T}_{\underline{x}} \xrightarrow{\Theta_s f_i} \Theta_s \mathsf{T}_{\underline{y}} = \mathsf{T}_{\underline{y}s}$$

and the compositions

$$\mathsf{T}_{\underline{x}} \xrightarrow{g'_j} \mathsf{T}_{\underline{x}} = \Theta_s \mathsf{T}_{\underline{z}} \xrightarrow{\Theta_s g_j} \Theta_s \mathsf{T}_{\underline{y}} = \mathsf{T}_{\underline{y}s}$$

span $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}s})$.

PROOF. (1) Note that here xs belongs to ${}^{\mathrm{f}}W_{\mathrm{aff}}$ by Lemma 2.30 in Chapter 1. As a consequence, we have $xs \in {}^{\mathrm{f}}W^s_{\mathrm{aff}}$.

Omitting some of the morphisms we can assume that the image of $(f_i : i \in I)$ is a basis of $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$ and that the image of $(g_j : j \in J)$ is a basis of $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq xs}}(\mathsf{T}_{\underline{x}s}, \mathsf{T}_{\underline{y}})$. As explained above, T_x is a direct summand of $\mathsf{T}_{\underline{x}}$; we can therefore choose a split embedding $\mathsf{T}_x \to \mathsf{T}_{\underline{x}}$. Similarly, we can choose a split embedding $\mathsf{T}_{xs} \to \mathsf{T}_{\underline{x}s}$. Then, since these embeddings are isomorphisms in $\mathsf{Rep}_0(\mathbf{G})^{\geq x}$ and $\mathsf{Rep}_0(\mathbf{G})^{\geq xs}$ respectively, we can complete the compositions

$$\mathsf{T}_x \hookrightarrow \mathsf{T}_{\underline{x}} \xrightarrow{f_i} \mathsf{T}_{\underline{y}} \quad \text{and} \quad \mathsf{T}_{xs} \hookrightarrow \mathsf{T}_{\underline{x}s} \xrightarrow{g_j} \mathsf{T}_{\underline{y}}$$

to a ∇-section of $\mathsf{T}_{\underline{y}}$. Starting with this ∇-section, Proposition 2.5 provides a ∇-section of $T^s\mathsf{T}_{\underline{y}}$ whose morphisms $\mathsf{T}^s_{xs} \to T^s\mathsf{T}_{\underline{y}}$ are parametrized by $I \sqcup J$ in such a way that the morphism corresponding to $i \in I$ factors through the morphism $T^s(f_i) : T^s\mathsf{T}_{\underline{x}} \to T^s\mathsf{T}_{\underline{y}}$ and the morphism corresponding to $j \in J$ factors through the morphism $T^s(g_j) : T^s\mathsf{T}_{\underline{x}s} \to T^s\mathsf{T}_{\underline{y}}$. Next we apply Proposition 2.6, which provides a ∇-section of $T_sT^s\mathsf{T}_{\underline{y}} = \mathsf{T}_{\underline{y}s}$ whose morphisms $\mathsf{T}_x \to \mathsf{T}_{\underline{y}s}$ are parametrized by $I \sqcup J$ in such a way that the morphism corresponding to $i \in I$ factors through the morphism $\Theta_s(f_i) : \mathsf{T}_{\underline{x}s} \to \mathsf{T}_{\underline{y}s}$ and the morphism corresponding to $j \in J$ factors through the morphism $\Theta_s(g_j) : \mathsf{T}_{\underline{x}ss} \to \mathsf{T}_{\underline{y}s}$. Composing these morphisms with a split surjection $\mathsf{T}_{\underline{x}} \to \mathsf{T}_x$ and forgetting the rest of the ∇-section, we obtain the desired data.

(2) The proof is similar. In this case we have $x = zs \in {}^{\mathrm{f}}W^s_{\mathrm{aff}}$. Omitting some morphisms we can assume that the images of our families are bases of $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$ and $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq z}}(\mathsf{T}_{\underline{z}}, \mathsf{T}_{\underline{y}})$ respectively. Then we compose these morphisms with some split embeddings

$$\mathsf{T}_x \hookrightarrow \mathsf{T}_{\underline{x}} \quad \text{and} \quad \mathsf{T}_z \hookrightarrow \mathsf{T}_{\underline{z}},$$

and complete these data to a ∇-section of $\mathsf{T}_{\underline{y}}$. Applying Proposition 2.5 and then Proposition 2.6 we obtain a ∇-section of $\mathsf{T}_{\underline{y}s}$ whose morphisms $\mathsf{T}_x \to \mathsf{T}_{\underline{y}s}$ are parametrized by $I \sqcup J$ in such a way that the morphism corresponding to $i \in I$ factors through $\Theta_s(f_i)$ and the morphism corresponding to $j \in J$ factors through $\Theta_s(g_j)$. Finally, forgetting the rest of the data and composing our morphisms with a split surjection $\mathsf{T}_{\underline{x}} \to \mathsf{T}_x$ we obtain the desired claim. $\square$

2.5. More preliminaries. This subsection and the next one gather a number of technical statements that will be required below. We start with a few statements that are independent of Conjecture 1.3.

LEMMA 2.9. *Let $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{s}$. If $y \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ satisfies $y < ws$, then the image of $\Theta_s(\mathsf{L}_y)$ in $\mathsf{Rep}_0(\mathbf{G})^{\geq ws}$ vanishes.*

PROOF. We have a surjection $\mathsf{N}_y \twoheadrightarrow \mathsf{L}_y$; it follows that to prove the claim it suffices to prove that the image of $\Theta_s(\mathsf{N}_y)$ in $\mathsf{Rep}_0(\mathbf{G})^{\geq ws}$ vanishes. However, by Proposition 2.36 in Chapter 1, if $\Theta_s(\mathsf{N}_y)$ is nonzero then y and ys both belong to ${}^{\mathrm{f}}W_{\mathrm{aff}}$, and this object admits a filtration with subquotients N_y and N_{ys}. None of these objects admits a composition factor of the form L_z with $z \geq ws$; they therefore vanish in $\mathsf{Rep}_0(\mathbf{G})^{\geq ws}$, which implies our claim. □

LEMMA 2.10. *Let $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}^{s}$. The morphism*

$$\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{M}_{ws}, \Theta_s(\mathsf{M}_{ws})) \to \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{M}_{ws}, \Theta_s(\mathsf{M}_{ws}))$$

induced by the quotient functor is an isomorphism, and both spaces are 1-dimensional.

PROOF. By adjunction and Proposition 2.36 in Chapter 1, we have

$$\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{M}_{ws}, \Theta_s(\mathsf{M}_{ws})) \cong \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{M}^s_w, \mathsf{M}^s_w).$$

The right-hand side is 1-dimensional, hence so is the left-hand side. On the other hand, since N_{ws} has head L_{ws}, the image of any nonzero morphism with domain N_{ws} admits L_{ws} as a composition factor. We deduce that for any object M the morphism

$$\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{M}_{ws}, M) \to \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{M}_{ws}, M)$$

induced by the quotient functor is injective. To conclude, it therefore suffices to prove that

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{M}_{ws}, \Theta_s(\mathsf{M}_{ws})) = 1. \tag{2.6}$$

Let us fix some nonzero morphisms

$$\mathsf{M}_{ws} \twoheadrightarrow \mathsf{L}_{ws} \hookrightarrow \mathsf{N}_{ws}.$$

Lemma 2.9 implies that the images in $\mathsf{Rep}_0(\mathbf{G})^{\geq ws}$ of the induced morphisms

$$\Theta_s(\mathsf{M}_{ws}) \twoheadrightarrow \Theta_s(\mathsf{L}_{ws}) \hookrightarrow \Theta_s(\mathsf{N}_{ws})$$

are isomorphisms. They therefore induce isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{M}_{ws}, \Theta_s(\mathsf{M}_{ws})) &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{M}_{ws}, \Theta_s(\mathsf{L}_{ws})) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{M}_{ws}, \Theta_s(\mathsf{N}_{ws})).\end{aligned}$$

Now by Proposition 2.36 in Chapter 1 $\Theta_s(\mathsf{N}_{ws})$ admits a costandard filtration with subquotients N_w and N_{ws}. Using Lemma 3.1 in Appendix A we deduce a similar claim in the highest weight category $\mathsf{Rep}_0(\mathbf{G})^{\geq ws}$, which implies (2.6) and finishes the proof. □

2.6. Even more preliminaries. From now on, and until the end of this section, we assume that Conjecture 1.3 holds. We therefore have an action of the monoidal category $\mathsf{D}_{\mathrm{aff}}^{\mathrm{BS}}$ on the category $\mathsf{Rep}_0(\mathbf{G})$, given by a bifunctor

$$\mathsf{Rep}_0(\mathbf{G}) \times \mathsf{D}_{\mathrm{aff}}^{\mathrm{BS}} \to \mathsf{Rep}_0(\mathbf{G})$$

which we will denote by $(M, B) \mapsto M \cdot B$, and we can consider the functor

$$\Psi : \mathsf{D}_{\mathrm{aff}}^{\mathrm{BS}} \to \mathsf{Rep}_0(\mathbf{G})$$

defined by

$$\Psi(B) = \mathsf{T}(\lambda) \cdot B.$$

By construction, this functors satisfies

$$\Psi(\mathsf{B}_{\underline{w}}) = \mathsf{T}_{\underline{w}}$$

for any word $\underline{w}$ in S_{aff}.

Since the category $\mathsf{Rep}_0(\mathbf{G})$ is additive and Karoubian (as an abelian category), the functor Ψ can be "extended" to an additive functor

$$\mathsf{D}_{\mathrm{aff}} \to \mathsf{Rep}_0(\mathbf{G}),$$

which will still be denoted Ψ. Since any direct sum or direct summand of tilting modules is tilting (see Exercise 7.3), this functor takes values in the full subcategory whose objects are the tilting modules.

Consider the morphism

$$[\mathsf{D}_{\mathrm{aff}}]_{\oplus} \to [\mathsf{Rep}_0(\mathbf{G})] \tag{2.7}$$

induced by Ψ on Grothendieck groups. Here the left-hand side has been identified with $\mathcal{H}_{\mathrm{aff}}$, see (1.1), and the right-hand side has been identified with $\mathcal{N}_{\mathrm{aff}}^0$, see §2.4. The morphism (2.7) therefore defines a morphism

$$\mathcal{H}_{\mathrm{aff}} \to \mathcal{N}_{\mathrm{aff}}^0. \tag{2.8}$$

Here both sides have natural structures of right $\mathcal{H}_{\mathrm{aff}}$-modules. The functor Ψ satisfies

$$\Psi(B \cdot \mathsf{B}_s) \cong \Theta_s(\Psi(B))$$

for any B in $\mathsf{D}_{\mathrm{aff}}$ and any $s \in S_{\mathrm{aff}}$. It follows that (2.8) commutes with the actions of each element $\underline{H}_s \in \mathcal{H}_{\mathrm{aff}}$. It also commutes with the action of v (because $\Psi \circ (1) \cong \Psi$), hence is a morphism of right $\mathcal{H}_{\mathrm{aff}}$-modules. In view of (2.5), we deduce the following property.

LEMMA 2.11. *For any $B \in \mathsf{D}_{\mathrm{aff}}$ and any $y \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, the multiplicity $(\Psi(B) : \mathsf{N}_y)$ is the coefficient of N_y^0 in the expansion of the element $N_e^0 \cdot [B]$ in the basis $(N_x^0 : x \in {}^{\mathrm{f}}W_{\mathrm{aff}})$, where $[B]$ is the class of B in $[\mathsf{D}_{\mathrm{aff}}]_{\oplus} = \mathcal{H}_{\mathrm{aff}}$.*

The following statement involves the notion of *rex move* from §2.9 in Chapter 2.

LEMMA 2.12. *Let $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, and consider a rex move in Γ_w from $\underline{w}$ to $\underline{w}'$. Then the image of the associated morphism $\mathsf{B}_{\underline{w}} \to \mathsf{B}_{\underline{w}'}$ under the composition of Ψ with the quotient functor $\mathsf{Rep}_0(\mathbf{G}) \to \mathsf{Rep}_0(\mathbf{G})^{\geq w}$ is an isomorphism, with inverse the image of the morphism induced by the reversed rex move.*

PROOF. Consider the reversed rex move, and the associated morphism $\mathsf{B}_{\underline{w}'} \to \mathsf{B}_{\underline{w}}$. By Proposition 2.29 in Chapter 2, there exist words $\underline{x}_1, \cdots, \underline{x}_r$ of length at most $\ell(w) - 2$ and morphism $f_1, \cdots, f_r : \mathsf{B}_{\underline{w}} \to \mathsf{B}_{\underline{w}}$ where each f_i factors through a shift of $\mathsf{B}_{\underline{x}_i}$ such that the composition

$$\mathsf{B}_{\underline{w}} \to \mathsf{B}_{\underline{w}'} \to \mathsf{B}_{\underline{w}}$$

equals $\mathrm{id} + \sum_{i=1}^r f_i$. For any i, by Lemma 2.11 the costandard objects occuring in a costandard filtration of $\Psi(\mathsf{B}_{\underline{x}_j})$ have labels of length at most $\ell(w) - 2$; it follows that the image of $\Psi(\mathsf{B}_{\underline{x}_j})$ in $\mathsf{Rep}_0(\mathbf{G})^{\geq w}$ vanishes. It follows that the image of our morphism $\mathsf{B}_{\underline{w}} \to \mathsf{B}_{\underline{w}}$ in $\mathsf{Rep}_0(\mathbf{G})^{\geq w}$ is the identity. Similar comments apply to the composition of our morphisms in the other order, which proves our claim. □

LEMMA 2.13. *Let $M \in \mathsf{Rep}_0(\mathbf{G})$ be an object such that $\Theta_s(M) \neq 0$. Then the morphism*

$$M \to \Theta_s(M)$$

induced by the lower dot morphism

$$\overset{s}{\bullet} : \mathsf{B}_\varnothing \to \mathsf{B}_s(1)$$

is nonzero.

PROOF. Recall that we have an adjunction

$$\big((-) \cdot \mathsf{B}_s, (-) \cdot \mathsf{B}_s\big),$$

which is defined by morphisms $\mathsf{B}_\varnothing \to \mathsf{B}_{ss}$ and $\mathsf{B}_{ss} \to \mathsf{B}_\varnothing$ constructed using the dot and trivalent morphisms. Using our action on $\mathsf{Rep}_0(\mathbf{G})$ we deduce morphisms

$$\mathrm{id} \to \Theta_s\Theta_s, \quad \Theta_s\Theta_s \to \mathrm{id}$$

which define an adjunction (Θ_s, Θ_s). Since $\Theta_s(M) \neq 0$, the first of these morphisms induces a nonzero morphism

$$M \to \Theta_s\Theta_s(M).$$

This morphism factors through the morphism considered in the statement; the latter morphism is therefore nonzero. □

2.7. A surjectivity claim. The crucial observation we will need is the following.

PROPOSITION 2.14. *For any words $\underline{x}, \underline{y}$ in S_{aff}, the morphism*

$$\mathrm{Hom}^\bullet_{\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}}(\mathsf{B}_{\underline{x}}, \mathsf{B}_{\underline{y}}) \to \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$$

induced by the functor Ψ is surjective.

The proof of this proposition will rely on the following more technical statement.

LEMMA 2.15. *Let $\underline{x}, \underline{y}$ be words in S_{aff}, and assume that $\underline{x}$ is a reduced expression for some element $x \in {}^{\mathrm{f}}W_{\mathrm{aff}}$. Then the morphism*

$$\mathrm{Hom}^\bullet_{\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}}(\mathsf{B}_{\underline{x}}, \mathsf{B}_{\underline{y}}) \to \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$$

induced by the composition of Ψ with the quotient functor $\mathsf{Rep}_0(\mathbf{G}) \to \mathsf{Rep}_0(\mathbf{G})^{\geq x}$ is surjective.

Note that in the setting of Lemma 2.15, by the comments at the beginning of §2.4, the image of $\mathsf{T}_{\underline{x}}$ in $\mathsf{Rep}_0(\mathbf{G})^{\geq x}$ coincides with the image of T_x. As a consequence, and in view of the comments in §5.4 of Appendix A, we have

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}}) = (\mathsf{T}_{\underline{y}} : \mathsf{N}_x). \tag{2.9}$$

We start by proving some particular cases.

Lemma 2.16. *Let $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}^s$, and let $\underline{z}$ be a reduced expression for ws. Then Lemma 2.15 holds when*

$$(\underline{x}, \underline{y}) \in \{(\underline{z}, \underline{z}s), (\underline{z}, \underline{z}ss), (\underline{z}s, \underline{z}s), (\underline{z}s, \underline{z}ss)\}.$$

Proof. By Lemma 2.22 there exists an isomorphism

$$\mathsf{B}_{\underline{z}ss} \cong \mathsf{B}_{\underline{z}s}(1) \oplus \mathsf{B}_{\underline{z}s}(-1).$$

This isomorphism reduces the proof to the cases of the pairs $(\underline{z}, \underline{z}s)$ and $(\underline{z}s, \underline{z}s)$. The case of $(\underline{z}s, \underline{z}s)$ is obvious, since the codomain of our morphism is spanned by the identity morphism (see (2.9)).

We now consider the case of the pair $(\underline{z}, \underline{z}s)$. In this case we will show more explicitly that $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{T}_{\underline{z}}, \mathsf{T}_{\underline{z}s})$ is 1-dimensional, and spanned by the image of the morphism

$$\mathrm{id}_{\mathsf{B}_{\underline{z}}} \cdot \overset{s}{\bullet} : \mathsf{B}_{\underline{z}} \to \mathsf{B}_{\underline{z}s}(1).$$

Fix a nonzero (and necessarily injective) morphism $f : \mathsf{M}_{ws} \to \mathsf{T}_{\underline{z}}$. We then have morphisms

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{T}_{\underline{z}}, \mathsf{T}_{\underline{z}s}) \arrow[dr, "(-)\circ f"] & \\ & \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{M}_{ws}, \mathsf{T}_{\underline{z}s}), \\ \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{M}_{ws}, \Theta_s(\mathsf{M}_{ws})) \arrow[ur, "\Theta_s(f)\circ(-)"'] & \end{tikzcd}$$

and the images of the morphisms induced by the lower dot morphism in the spaces in the left column coincide. The images of f and $\Theta_s(f)$ in $\mathsf{Rep}_0(\mathbf{G})^{\geq ws}$ are isomorphisms: for f this is clear, and for $\Theta_s(f)$ this follows from Lemma 2.9. It follows that both morphisms in our diagram are invertible.

These comments show that our desired claim is equivalent to the claim that $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq ws}}(\mathsf{M}_{ws}, \Theta_s(\mathsf{M}_{ws}))$ is 1-dimensional, and spanned by the morphism induced by the lower dot morphism. These facts follow from Lemma 2.10 and Lemma 2.13. □

Proof of Lemma 2.15. To simplify notation, if $\underline{x}, \underline{y}$ are as in the statement we will denote by

$$\gamma_{\underline{x},\underline{y}} : \mathrm{Hom}^{\bullet}_{\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}}(\mathsf{B}_{\underline{x}}, \mathsf{B}_{\underline{y}}) \to \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$$

the morphism induced by Ψ. We will prove by induction on the length of $\underline{y}$ that the statement holds for all reduced expressions $\underline{x}$.

First, if $\underline{y}$ is the empty word, then $\mathsf{T}_{\underline{y}} = \mathsf{T}(\lambda) = \mathsf{L}(\lambda)$, and the image of this object in $\mathsf{Rep}_0(\mathbf{G})^{\geq x}$ vanishes (so that the claim is obvious) unless $x = e$. On the other hand, if $x = e$ then $\underline{x}$ is the empty word, and the claim is clear in this case too.

Now, we assume that $\underline{y}$ has positive length, and write $\underline{y} = \underline{z}s$ for some word $\underline{z}$ and some $s \in S_{\mathrm{aff}}$. We also assume that the claim is known for the word $\underline{z}$. Let now $\underline{x}$ and x be as in the statement. We distinguish three cases.

Case 1: $\{x, xs\} \cap {}^{\mathrm{f}}W^s_{\mathrm{aff}} = \varnothing$. In this case, by (2.9) and Lemma 2.7 the right-hand space vanishes, so that there is nothing to prove.

Case 2: $xs \in {}^{\mathrm{f}}W^s_{\mathrm{aff}}$. By induction there exists a family $(f_i : i \in I)$ of elements of $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{z}})$ which belong to the image of $\gamma_{\underline{x},\underline{z}}$ and whose images span $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{z}})$, and a family $(g_j : j \in J)$ of elements in $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{T}_{\underline{x}s}, \mathsf{T}_{\underline{z}})$ which belong to the image of $\gamma_{\underline{x}s,\underline{z}}$ and whose images span $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq xs}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{z}})$. Then by Proposition 2.8(1) there exist morphisms $f'_i : \mathsf{T}_{\underline{x}} \to \mathsf{T}_{\underline{x}s}$ (for $i \in I$) and $g'_j : \mathsf{T}_{\underline{x}} \to \mathsf{T}_{\underline{x}ss}$ (for $j \in J$) such that the images of the compositions

$$\mathsf{T}_{\underline{x}} \xrightarrow{f'_i} \mathsf{T}_{\underline{x}s} = \Theta_s \mathsf{T}_{\underline{x}} \xrightarrow{\Theta_s f_i} \Theta_s \mathsf{T}_{\underline{z}} = \mathsf{T}_{\underline{y}}$$

and the compositions

$$\mathsf{T}_{\underline{x}} \xrightarrow{g'_j} \mathsf{T}_{\underline{x}ss} = \Theta_s \mathsf{T}_{\underline{x}s} \xrightarrow{\Theta_s g_j} \Theta_s \mathsf{T}_{\underline{z}} = \mathsf{T}_{\underline{y}}$$

span $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$. By Lemma 2.16, for any $i \in I$ there exists a morphism $f''_i : \mathsf{T}_{\underline{x}} \to \mathsf{T}_{\underline{x}s}$ in the image of $\gamma_{\underline{x},\underline{x}s}$ whose image in $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{x}s})$ coincides with that of f'_i, and for any $j \in J$ there exists a morphism $g''_j : \mathsf{T}_{\underline{x}} \to \mathsf{T}_{\underline{x}ss}$ in the image of $\gamma_{\underline{x},\underline{x}ss}$ whose image in $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{x}ss})$ coincides with that of g'_j. Then the family

$$\{\Theta_s(f_i) \circ f''_i : i \in I\} \cup \{\Theta_s(g_j) \circ g''_j : j \in J\}$$

consists of morphisms in the image of $\gamma_{\underline{x},\underline{y}}$, and its image spans the vector space $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$. We deduce the desired surjectivity.

Case 3: $x \in {}^{\mathrm{f}}W^s_{\mathrm{aff}}$. Since $xs < x$, x admits a reduced expression $\underline{x}'$ which finishes with s. Since the rex graph of x is connected (see §2.9 in Chapter 2), there exists a rex move from $\underline{x}$ and $\underline{x}'$. Choosing such a rex move we obtain morphisms $\mathsf{B}_{\underline{x}} \to \mathsf{B}_{\underline{x}'}$ and $\mathsf{B}_{\underline{x}'} \to \mathsf{B}_{\underline{x}}$, whose images in $\mathsf{Rep}_0(\mathbf{G})^{\geq x}$ are inverse to each other by Lemma 2.12. We deduce a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^\bullet_{\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}}(\mathsf{B}_{\underline{x}}, \mathsf{B}_{\underline{y}}) \arrow[r, shift left] \arrow[d] & \mathrm{Hom}^\bullet_{\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}}(\mathsf{B}_{\underline{x}'}, \mathsf{B}_{\underline{y}}) \arrow[l, shift left] \arrow[d] \\
\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}}) \arrow[r, shift left] & \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}'}, \mathsf{T}_{\underline{y}}) \arrow[l, shift left]
\end{tikzcd}$$

where the vertical arrows are the morphisms of the lemma for the pairs $(\underline{x}, \underline{y})$ and $(\underline{x}', \underline{y})$ and the lower horizontal arrows are inverse to each other. It therefore suffices to prove the statement for the pair $(\underline{x}', \underline{y})$.

The rest of the proof is similar to Case 2. Write $\underline{x}' = \underline{v}s$. By induction there exists a family $(f_i : i \in I)$ of elements of $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{z}})$ which belong to the image of $\gamma_{\underline{x},\underline{z}}$ and whose images span $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{z}})$, and a family $(g_j : j \in J)$ of elements in $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{T}_{\underline{v}}, \mathsf{T}_{\underline{z}})$ which belong to the image of $\gamma_{\underline{v},\underline{z}}$ and whose images span $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq xs}}(\mathsf{T}_{\underline{v}}, \mathsf{T}_{\underline{z}})$. Then by Proposition 2.8(2) there exist morphisms $f'_i : \mathsf{T}_{\underline{x}} \to \mathsf{T}_{\underline{x}s}$ (for $i \in I$) and $g'_j : \mathsf{T}_{\underline{x}} \to \mathsf{T}_{\underline{x}}$ (for $j \in J$) such that the images of the compositions

$$\mathsf{T}_{\underline{x}} \xrightarrow{f'_i} \mathsf{T}_{\underline{x}s} = \Theta_s \mathsf{T}_{\underline{x}} \xrightarrow{\Theta_s f_i} \Theta_s \mathsf{T}_{\underline{z}} = \mathsf{T}_{\underline{y}}$$

and the compositions

$$\mathsf{T}_{\underline{x}} \xrightarrow{g'_j} \mathsf{T}_{\underline{x}} = \Theta_s \mathsf{T}_{\underline{v}} \xrightarrow{\Theta_s g_j} \Theta_s \mathsf{T}_{\underline{z}} = \mathsf{T}_{\underline{y}}$$

span $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$. By Lemma 2.16, for any $i \in I$ there exists a morphism $f''_i : \mathsf{T}_{\underline{x}} \to \mathsf{T}_{\underline{x}s}$ in the image of $\gamma_{\underline{x},\underline{x}s}$ whose image in $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{x}s})$ coincides with that of f'_i, and for any $j \in J$ there exists a morphism $g''_j : \mathsf{T}_{\underline{x}} \to \mathsf{T}_{\underline{x}}$ in the image of $\gamma_{\underline{x},\underline{x}}$ whose image in $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{x}})$ coincides with that of g'_j. Then the family

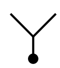
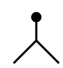
$$\{\Theta_s(f_i) \circ f''_i : i \in I\} \cup \{\Theta_s(g_j) \circ g''_j : j \in J\}$$

consists of morphisms in the image of $\gamma_{\underline{x},\underline{y}}$, and its image spans the vector space $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})^{\geq x}}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$. We deduce the desired surjectivity. $\square$

We can finally prove Proposition 2.14.

Proof of Proposition 2.14. We proceed by induction on the length of the word $\underline{x}$. If this length is 0, then $\underline{x}$ is the empty word, and our claim is a particular case of Lemma 2.15.

Now, assume that $\underline{x}$ has positive length, and write $\underline{x} = \underline{z}s$ for some words $\underline{z}$ and some $s \in S_{\mathrm{aff}}$. Assume that the claim is known for the word $\underline{z}$ (and any word $\underline{y}$). Consider the morphisms

 $: \mathsf{B}_\varnothing \to \mathsf{B}_{(s,s)}$ and  $: \mathsf{B}_{(s,s)} \to \mathsf{B}_\varnothing.$

As explained in REF, these morphisms define an adjunction

$$\big((-)\cdot \mathsf{B}_s, (-)\cdot \mathsf{B}_s\big).$$

Their images define morphisms of functors

$$\mathrm{id} \to \Theta_s\Theta_s \quad \text{and} \quad \Theta_s\Theta_s \to \mathrm{id}$$

which satisfy the zigzag relations, hence define an adjunction (Θ_s, Θ_s). Using these adjunctions we obtain the horizontal isomorphisms in the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}^{\bullet}_{\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}}(\mathsf{B}_{\underline{x}}, \mathsf{B}_{\underline{y}}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}^{\bullet}_{\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}}(\mathsf{B}_{\underline{z}}, \mathsf{B}_{\underline{y}s}) \arrow[d] \\
\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}}) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{T}_{\underline{z}}, \mathsf{T}_{\underline{y}s}).
\end{tikzcd}$$

Here the vertical morphisms are induced by the functor Ψ. This diagram commutes, and its right vertical arrow is surjective by assumption. We deduce that its left vertical arrow is surjective as well, which finishes the proof. $\square$

2.8. Completion of the proof. We finally prove Conjecture 2.1, under the assumption that Conjecture 1.3 holds. Recall the indecomposable objects $(\mathsf{B}_w : w \in W_{\mathrm{aff}})$ in $\mathsf{D}_{\mathrm{aff}}$ (see REFERENCE). By definition the class $[\mathsf{B}_w]$ in $[\mathsf{D}_{\mathrm{aff}}]_{\oplus} = \mathcal{H}_{\mathrm{aff}}$ is ${}^p\underline{H}_w$. On the other hand, by definition of the module $\mathcal{N}_{\mathrm{aff}}$, for any $y \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ and $x \in W$ we have

$$N_e \cdot H_{xy} = (-v)^{\ell(x)} N_y.$$

If follows that for any $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ we have

$$N_e \cdot {}^p\underline{H}_w = \sum_{y \in {}^{\mathrm{f}}W_{\mathrm{aff}}} {}^p n_{y,w} \cdot N_y,$$

hence finally that

$$N_e^0 \cdot {}^p\underline{H}_w = \sum_{y \in {}^{\mathrm{f}}W_{\mathrm{aff}}} {}^p n_{y,w}(1) \cdot N_y^0.$$

This formula and Lemma 2.11 show that the formula in Conjecture 2.1 will follow from the following statement.

PROPOSITION 2.17. *Let $w \in W_{\mathrm{aff}}$. We have*

$$\Psi(\mathsf{B}_w) \cong \begin{cases} \mathsf{T}_w & \text{if } w \in {}^{\mathrm{f}}W_{\mathrm{aff}}; \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. We prove the proposition by induction on $\ell(w)$. The case $\ell(w) = 0$ is obvious since $\Psi(\mathsf{B}_e) = \Psi(\mathsf{B}_{\varnothing}) = \mathsf{T}(\lambda) = \mathsf{T}_e$.

Now, assume that $\ell(w) > 0$ and that the claim is known for shorter elements. If $w \notin {}^{\mathrm{f}}W_{\mathrm{aff}}$, then w admits a reduced expression $\underline{w}$ whose first letter s belongs to S. The object B_w is a direct summand of $\mathsf{B}_{\underline{w}}$, and $\Psi(\mathsf{B}_{\underline{w}}) = 0$ since $\Theta_s(\mathsf{T}(\lambda)) = 0$. It follows that $\Psi(\mathsf{B}_w) = 0$, as desired.

Finally, assume that $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$, and choose a reduced expression $\underline{w}$ for w. Then B_w is a direct summand of $\mathsf{B}_{\underline{w}}$, and all the other direct summands of this object are of the form $\mathsf{B}_y(n)$ with $\ell(y) < \ell(w)$. By induction, $\Psi(\mathsf{B}_{\underline{w}})$ is therefore the direct sum of $\Psi(\mathsf{B}_w)$ and some objects T_y with $y \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ which satisfies $\ell(y) < \ell(w)$. On the other hand, it follows from Lemma 2.11 that

$$(\Psi(\mathsf{B}_{\underline{w}}) : \mathsf{N}_w) = 1.$$

Since $(\mathsf{T}_y, \mathsf{N}_w) = 0$ for any $y \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ such that $\ell(y) < \ell(w)$, we deduce that $(\Psi(\mathsf{B}_w) : \mathsf{N}_w) = 1$; in particular, $\Psi(\mathsf{B}_w) \neq 0$. Similarly, for any $z \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ we have

$$(\Psi(\mathsf{B}_{\underline{w}}) : \mathsf{N}_z) \neq 0 \quad \Rightarrow \quad z \leq w.$$

We deduce that

$$(\Psi(\mathsf{B}_w) : \mathsf{N}_z) \neq 0 \quad \Rightarrow \quad z \leq w.$$

These properties show that if $\Psi(\mathsf{B}_w)$ is indecomposable, then it is isomorphic to T_w. To conclude the proof, it therefore suffices to prove that $\Psi(\mathsf{B}_w)$ is indecomposable.

It follows from Proposition 2.14 that the morphism

$$\mathrm{Hom}^{\bullet}_{\mathsf{D}_{\mathrm{aff}}}(\mathsf{B}_w, \mathsf{B}_w) \to \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\Psi(\mathsf{B}_w), \Psi(\mathsf{B}_w))$$

is surjective. Since $\Theta_s(\mathsf{T}(\lambda)) = 0$ for any $s \in S$, this morphism factors through the quotient morphism

$$\mathrm{Hom}^{\bullet}_{\mathsf{D}_{\mathrm{aff}}}(\mathsf{B}_w, \mathsf{B}_w) \twoheadrightarrow \mathrm{Hom}^{\bullet}_{\mathsf{D}_{\mathrm{aff}}}(\mathsf{B}_w, \mathsf{B}_w) \otimes_{\mathscr{O}(\mathfrak{h}^*)} \Bbbk$$

where in the right-hand side $\Bbbk$ is the trivial $\mathscr{O}(\mathfrak{h}^*)$-module. The right-hand side is a finite-dimensional graded $\Bbbk$-algebra whose degree-0 component is local (since it is a quotient of the local algebra $\mathrm{Hom}_{\mathsf{D}_{\mathrm{aff}}}(\mathsf{B}_w, \mathsf{B}_w)$). By [GG, Theorem 3.1] this algebra is therefore local (as a non-graded algebra). We deduce that the algebra $\mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\Psi(\mathsf{B}_w), \Psi(\mathsf{B}_w))$ is local, hence that $\Psi(\mathsf{B}_w)$ is indecomposable, which finishes the proof. □

REMARK 2.18. One can make Proposition 2.14 more precise, by showing that the kernel of the morphism involved in this statement is the subspace spanned by morphisms which factor through an object of the form $\mathsf{B}_{\underline{z}}(n)$ where $\underline{z}$ is a word in S_{aff} starting with an element of S and $n \in \mathbb{Z}$. In fact, since $\Theta_s(\mathsf{T}(\lambda)) = 0$ for $s \in S$, these morphisms belong to the kernel of this morphism. What remains to be proved is that the dimension of the quotient of $\mathrm{Hom}^{\bullet}_{\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}}(\mathsf{B}_{\underline{x}}, \mathsf{B}_{\underline{y}})$ by this subspace is at most $\dim \mathrm{Hom}_{\mathsf{Rep}_0(\mathbf{G})}(\mathsf{T}_{\underline{x}}, \mathsf{T}_{\underline{y}})$. (Here the latter dimension can be expressed in terms of the combinatorics of the Hecke algebra using Exercise 6.5.) This is checked in [**RW1**, §4.5] using diagrammatical considerations.

3. Proofs of the tilting character formula

3.1. The case of $\mathrm{GL}(n)$**.** First we consider the case $\mathbf{G} = \mathrm{GL}_n(\Bbbk)$, assuming that $p > n \geq 3$. In this case, Conjecture 1.3 was proved in [**RW1**, Part II] using the theory of categorical actions of Lie algebras due to Rouquier [**Ro2**] and Khovanov–Lauda [**KhL1, KhL2**]. (The two definitions given—almost simultaneously—by these authors are similar but a priori different. The fact that they give rise to the same category was later proved by Brundan [**Br2**].) In this subsection we outline this proof.

3.1.1. *The Lie algebra* $\widehat{\mathfrak{gl}}_N$ *and its natural module.* We start by giving a (slightly non-standard) definition of the Lie algebra $\widehat{\mathfrak{gl}}_N$. Let $N \geq 3$. First we set

$$\widehat{\mathfrak{sl}}_N := \mathfrak{sl}_N\left(\mathbb{C}[t, t^{-1}]\right) \oplus \mathbb{C}K \oplus \mathbb{C}d$$

with Lie bracket defined by

$$\begin{aligned} [x \otimes t^m, y \otimes t^n] &= [x, y] \otimes t^{m+n} + m\delta_{m,-n}\,\mathrm{Tr}(xy)K, \\ [d, x \otimes t^m] &= mx \otimes t^m, \\ [K, \widehat{\mathfrak{sl}}_N] &= 0. \end{aligned}$$

Then we set $\widehat{\mathfrak{gl}}_N = \widehat{\mathfrak{sl}}_N \oplus \mathbb{C}$, with $(0, 1)$ identified with the identity matrix in $\mathfrak{gl}_N(\mathbb{C})$. Denote by $\mathfrak{h}_{\mathrm{f}} \subset \mathfrak{gl}_N(\mathbb{C})$ the Cartan subalgebra of diagonal matrices, and set

$$\mathfrak{h} := \mathfrak{h}_{\mathrm{f}} \oplus \mathbb{C}K \oplus \mathbb{C}d$$

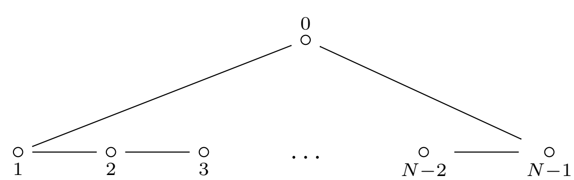
Let us denote by $\varepsilon_1, \cdots, \varepsilon_N$ the obvious basis of $\mathfrak{h}_{\mathrm{f}}^*$. Any element $\lambda \in \mathfrak{h}_{\mathrm{f}}^*$ can be "extended" to a linear form on $\mathfrak{h}$ by setting $\langle \lambda, K \rangle = \langle \lambda, d \rangle = 0$. If we denote by K^*, resp. δ, the linear forms on $\mathfrak{h}$ vanish on $\mathfrak{h}_{\mathrm{f}}$ and satisfy

$$K^*(K) = \delta(d) = 1, \quad K^*(d) = \delta(K) = 0,$$

then we have

$$\mathfrak{h}^* = \mathfrak{h}_{\mathrm{f}}^* \oplus \mathbb{C}c^* \oplus \mathbb{C}\delta.$$

REMARK 3.1. The Lie algebra $\widehat{\mathfrak{gl}}_N$ is the Kac-Moody algebra associated with the Dynkin diagram



and the realization with underlying vector space $\mathfrak{h}$, with simple roots

$$\alpha_0 = \delta - (\varepsilon_N - \varepsilon_1), \quad \alpha_i = \varepsilon_{i+1} - \varepsilon_i \ (i \in \{1, \dots, N-1\})$$

and simple coroots

$$h_0 = K + e_{1,1} - e_{N,N}, \quad h_i = e_{i+1,i+1} - e_{i,i} \ (i \in \{1, \dots, N-1\}).$$

(Here, $e_{i,j}$ is the matrix unit with coefficient 1 in position (i, j).)

We define "Chevalley elements" in $\widehat{\mathfrak{gl}}_N$ by setting, for $i \in \{0, \cdots, N-1\}$,

$$e_i = \begin{cases} e_{i+1,i} & \text{if } i \geq 1; \\ te_{1,N} & \text{if } i = 0, \end{cases} \qquad f_i = \begin{cases} e_{i,i+1} & \text{if } i \geq 1; \\ t^{-1}e_{N,1} & \text{if } i = 0. \end{cases}$$

We now define the "natural" representation nat_N of $\widehat{\mathfrak{gl}}_N$. Let

$$A = \mathbb{C}^N = \oplus_{1 \leq i \leq N} \mathbb{C} a_i$$

be the natural representation of $\mathfrak{gl}_N(\mathbb{C})$. We set

$$\mathrm{nat}_N = A \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$$

with $\widehat{\mathfrak{sl}}_N$ acting via

$$(x \otimes t^m) \cdot (a \otimes t^n) = x(a) \otimes t^{m+n} \quad \text{for } x \in \mathfrak{sl}_N(\mathbb{C}),\ a \in A,\ m, n \in \mathbb{Z};$$
$$d \cdot (a \otimes t^n) = na \otimes t^n \quad \text{for } a \in A,\ n \in \mathbb{Z};$$
$$c \cdot (a \otimes t^n) = 0 \quad \text{for } a \in A,\ n \in \mathbb{Z}.$$

If $\lambda \in \mathbb{Z}$, write $\lambda = \mu N + \nu$ with $\mu \in \mathbb{Z}$ and $1 \leq \nu \leq N$, and set

$$m_\lambda := a_\nu \otimes t^\mu.$$

Then

$$\mathrm{nat}_N = \oplus_{\lambda \in \mathbb{Z}} \mathbb{C} m_\lambda,$$

where m_λ is a weight vector with weight $\varepsilon_\nu + \mu\delta$, with the convention above.

3.1.2. *Representations of* $\mathrm{GL}_n(\Bbbk)$. We now fix $n \geq 3$, and assume that $p > n$. We set $\mathbf{G} = \mathrm{GL}_n(\Bbbk)$, and choose as $\mathbf{T}$, resp. $\mathbf{B}$, the maximal torus of diagonal matrices, resp. the Borel subgroup of lower triangular matrices. We have a canonical identification

$$\mathbb{X} = \mathbb{Z}^n,$$

where $(\lambda_1, \cdots, \lambda_n)$ corresponds to the character sending a diagonal matrix with coefficients $x_1, \cdots, x_n$ to $\prod_i x_i^{\lambda_i}$. With this identification we have

$$\mathbb{X}^+ = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}.$$

We set $V = \Bbbk^n$ (the natural representation of $\mathbf{G}$) and define

$$E := V \otimes_{\Bbbk} (-) : \mathsf{Rep}(\mathbf{G}) \to \mathsf{Rep}(\mathbf{G}),$$
$$F := V^* \otimes_{\Bbbk} (-) : \mathsf{Rep}(\mathbf{G}) \to \mathsf{Rep}(\mathbf{G}).$$

We define an endomorphism $X \in \mathrm{End}(E)$ as follows: for $M \in \mathsf{Rep}(\mathbf{G})$, the $\mathbf{G}$-action induces a morphism

$$V \otimes V^* \otimes M = \mathfrak{gl}_n(\Bbbk) \otimes M \to M;$$

by adjunction we deduce a morphism

$$X_M : EM = V \otimes M \to V \otimes M = EM.$$

Then we have a decomposition of E into generalized eigenspaces for the action of X:

$$E = \oplus_{a \in \Bbbk} E_a$$

Since F is right adjoint to E, the endomorphism X of E also determines an endomorphism of F. With respect to this endomorphism, we similarly obtain a decomposition into generalized eigenspaces:

$$E = \oplus_{a \in \Bbbk} E_a.$$

The following proposition is due to Chuang–Rouquier, see [**CR**, §7.5]. (See also [**RW1**, Proposition 6.3.4] for a review of the proof.)

Proposition 3.2. *(1) We have $E_a = 0$ and $F_a = 0$ unless a belongs to the prime subfield $\mathbb{F}_p \subset \Bbbk$.*

(2) The isomorphism of $\mathbb{C}$-vector spaces

$$\mathbb{C} \otimes_{\mathbb{Z}} [\mathsf{Rep}(\mathbf{G})] \xrightarrow{\sim} \bigwedge^n \mathrm{nat}_p$$

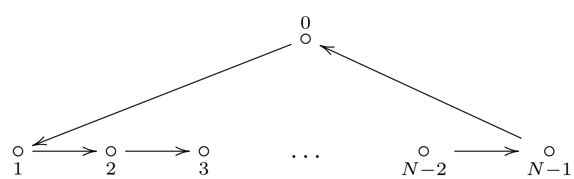
sending $[\mathsf{M}(\lambda)]$ to

$$m_{\lambda_1} \wedge m_{\lambda_2 - 1} \wedge \cdots \wedge m_{\lambda_n - n - 1}$$

identifies the action of $[E_a]$, resp. $[F_a]$, on the left-hand side with the action of e_a, resp. f_a, on the right-hand side, for any $a \in \{0, \cdots, p-1\} = \mathbb{F}_p$.

(3) Under the isomorphism above, the decomposition of $\mathbb{C} \otimes_{\mathbb{Z}} [\mathsf{Rep}(\mathbf{G})]$ induced by the linkage principle (see Corollary 2.13 in Chapter 1) corresponds to the weight space decomposition of $\bigwedge^n \mathrm{nat}_p$.

3.1.3. *Categorification.* It turns out that the picture presented in §3.1.2 "lifts" to the categorical level, as follows. We view the Dynkin diagram of $\widehat{\mathfrak{gl}}_N$ as a quiver with the following orientation:



Then we set

$$t_{ij} = \begin{cases} -1 & \text{if } i \to j, \\ 1 & \text{otherwise} \end{cases}, \quad P = \{\lambda \in \mathfrak{h}^* : \langle h_i, \lambda \rangle \in \mathbb{Z}, \forall i \in \{0, 1, \ldots, N-1\}\}.$$

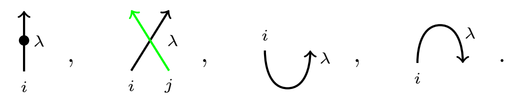
As mentioned above, Rouquier and Khovanov–Lauda have defined a "categorical incarnation" and each Kac–Moody algebra. Here we will follow the notations and conventions of Brundan [**Br2**].[2] In the particular case under considerations, and with the appropriate choice of structure constants, we obtain the strict additive $\Bbbk$-linear 2-category

$$\mathbb{U}(\widehat{\mathfrak{gl}}_N)$$

defined by generators and relations as follows. Its objects consist of P, its 1-morphisms are generated by $E_i 1_\lambda : \lambda \to \lambda + \alpha_i$ (which we will depict as an upward

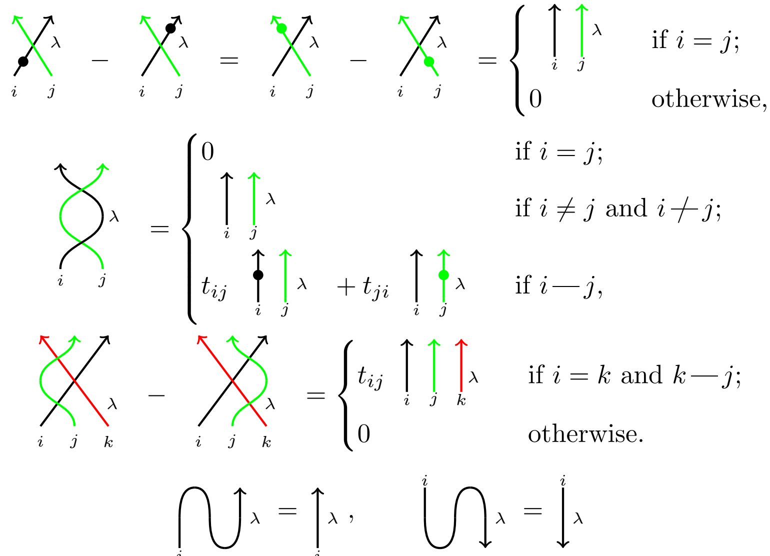
[2] In fact, we have even followed this reference in copying its source file for most of the drawings below...

arrow decorated by λ in the right region) and $F_i 1_\lambda \colon \lambda \to \lambda - \alpha_i$ (which we will depict as a downward arrow decorated by λ in the right region), and its generating 2-morphisms are

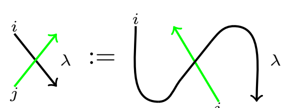


(As in Chapter 2 these diagrams are to be read from bottom to top. Hence these morphisms are morphisms from $E_i 1_\lambda$ to itself, from $E_i E_j 1_\lambda := (E_i 1_{\lambda+\alpha_j}) \circ (E_j 1_\lambda)$ to $E_j E_i 1_\lambda$, from id_λ to $F_i E_i 1_\lambda$, and from $E_i F_i 1_\lambda$ to id_λ respectively.

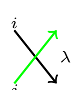
These 2-morphisms are required to satisfy the following 4 sets of relations. Here we write $i — j$ if $i \to j$ or $j \to i$, and $i \not— j$ if neither $i \to j$ nor $j \to i$.

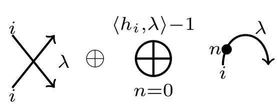


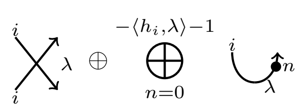
Finally, one has to impose that certain 2-morphisms are invertible. These 2-morphisms involve the new diagram

 $: E_j F_i 1_\lambda \to F_i E_j 1_\lambda.$

With this notation the following 2-morphisms are required to be isomorphisms:

 $: E_j F_i 1_\lambda \xrightarrow{\sim} F_i E_j 1_\lambda \quad \text{if } i \neq j,$

 $: E_i F_i 1_\lambda \xrightarrow{\sim} F_i E_i 1_\lambda \oplus 1_\lambda^{\oplus \langle h_i, \lambda \rangle} \quad \text{if } \langle h_i, \lambda \rangle \geq 0,$

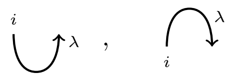
 $: E_i F_i 1_\lambda \oplus 1_\lambda^{\oplus -\langle h_i, \lambda \rangle} \xrightarrow{\sim} F_i E_i 1_\lambda \quad \text{if } \langle h_i, \lambda \rangle \leq 0.$

(Here, a dot with a nonnegative integer n means n successive dots.)

REMARK 3.3. It is important to note that the 2-morphisms in $\mathbb{U}(\widehat{\mathfrak{gl}}_N)$ are *not* invariant by isotopy of diagrams.

The following theorem is essentially due to Chuang–Rouquier. For the details of its proof, we refer to [**RW1**, Theorem 6.4.6].

THEOREM 3.4. *There exists an action of the* 2*-category* $\mathbb{U}(\widehat{\mathfrak{gl}}_p)$ *on the category* $\mathsf{Rep}(\mathbf{G})$ *such that* λ *is sent to the "block" of* $\mathsf{Rep}(\mathbf{G})$ *corresponding to the* λ*-weight space in* $\bigwedge^n \mathrm{nat}_p$ *(see Proposition 3.2), each* $E_i 1_\lambda$ *acts via the functor* E_i*, each* $F_i 1_\lambda$ *acts via the functor* F_i*, and the morphisms*



are sent to the unit and counit morphisms of the natural adjunction (E_i, F_i).

REMARK 3.5. The image of the other generating morphisms can also be described explicitly; see [**RW1**, §6.4.7] for details.

3.1.4. *Application to the proof of Conjecture 1.3.* Now that the required data have defined introduced, we can outline the proof of Conjecture 1.3 in this case. It will consist in 2 steps.

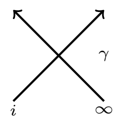
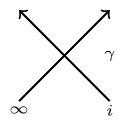
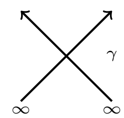
Let
$$\omega = \varepsilon_1 + \cdots + \varepsilon_n \in P.$$
Under the action of Theorem 3.4, the weight ω is mapped to the block corresponding to the $(W_{\mathrm{aff}}, \cdot_p)$-orbit of
$$\lambda_0 = (n, \cdots, n) \in \mathbb{X},$$
a regular block. With this choice of weight "λ," the category $\mathsf{Rep}_0(\mathbf{G})$ is therefore a "weight space" for the action of $\mathbb{U}(\widehat{\mathfrak{gl}}_p)$.

Step 1: Restriction of the action to $\widehat{\mathfrak{gl}}_p$ *to* $\widehat{\mathfrak{gl}}_n$*.* In addition to the Lie algebra $\widehat{\mathfrak{gl}}_p$ (and its associated 2-category) considered in Theorem 3.4, let us now also consider the Lie algebra $\widehat{\mathfrak{gl}}_n$ (and its associated 2-category). To distinguish the 2-cases, for $\widehat{\mathfrak{gl}}_n$ the "affine" vertex of the Dynkin diagram will be denoted ∞. We set
$$P_n := \left\{ \sum_{i=1}^{n} n_i \varepsilon_i + m\delta : n_i \in \mathbb{Z}_{\geq 0} \text{ s.t. } \sum_{i=1}^{n} n_i = n,\ m \in \mathbb{Z} \right\}.$$
These weights are weights for both $\widehat{\mathfrak{gl}}_p$ and $\widehat{\mathfrak{gl}}_n$. We also denote by $\mathsf{Rep}^{[n]}(\mathbf{G})$ the sum of the "blocks" in $\mathsf{Rep}(\mathbf{G})$ corresponding to weights in P_n.

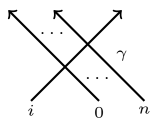
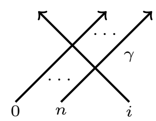
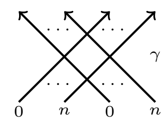
In [**RW1**, Theorem 7.4.1] it is shown that one can "restrict" the action of $\widehat{\mathfrak{gl}}_p$ on $\mathsf{Rep}(\mathbf{G})$ to an action of $\widehat{\mathfrak{gl}}_n$ on $\mathsf{Rep}^{[n]}(\mathbf{G})$ by sending:

- λ to the "block" as before if $\lambda \in P_n$, and to 0 otherwise;
- E_i to
$$\begin{cases} E_i & \text{if } 1 \leq i \leq n-1, \\ E_0 E_{p-1} \cdots E_n & \text{if } i = \infty; \end{cases}$$
- F_i to
$$\begin{cases} F_i & \text{if } 1 \leq i \leq n-1 \\ E_n \cdots F_{p-1} F_0 & \text{if } i = \infty; \end{cases}$$

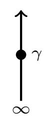
- each 2-morphism not involving ∞ to the 2-morphism corresponding to the same diagram in $\widehat{\mathfrak{gl}}_p$;
- the 2-morphism corresponding to

 , resp.  , resp. 

to

 , resp.  , resp. $(-1)^{p-n}$  ;

- the 2-morphism



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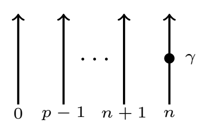
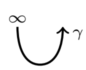
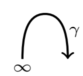
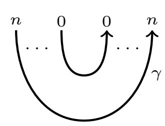
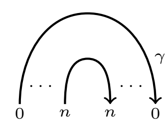
 ;

- the 2-morphism

 , resp. 

to

 , resp.  .

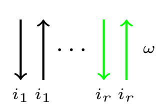
REMARK 3.6. This step of the proof is a special case of a result of Maksimau on restriction of certain actions of 2-Kac–Moody algebras; see [**Ma**].

Step 2: Relating $\mathbb{U}(\widehat{\mathfrak{gl}}_n)$ *to the Hecke category.* Let us now denote by $\mathbb{U}^{[n]}(\widehat{\mathfrak{gl}}_n)$ the quotient of $\mathbb{U}(\widehat{\mathfrak{gl}}_n)$ by the span of 2-morphisms which contain a weight not in P_n. Clearly, the action of $\mathbb{U}(\widehat{\mathfrak{gl}}_n)$ on $\mathsf{Rep}^{[n]}(\mathbf{G})$ considered in Step 1 factors through an action of $\mathbb{U}^{[n]}(\widehat{\mathfrak{gl}}_n)$.

In [**RW1**, Theorem 8.1.1] it is shown that there exists a strict monoidal functor

$$\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}} \to \mathrm{End}_{\mathbb{U}^{[n]}(\widehat{\mathfrak{gl}}_n)}(\omega).$$

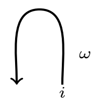
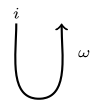
Explicitly, this morphism is constructed as follows. It sends the object $\mathsf{B}_{(i_1,\cdots,i_r)}\langle k\rangle$ to the functor

 .

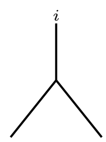
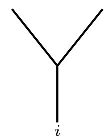
It sends the morphism

 resp. 

to

 , resp.  ,

and the morphism

 , resp. 

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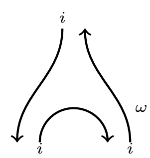
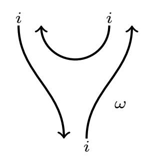
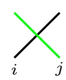
 , resp.  .

For $i, j \in \{1, \cdots, n-1, \infty\}$ with $i \not\!- j$, the functor sends



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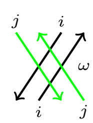
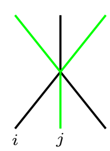
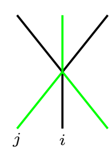


Finally, for $i, j \in \{1, \cdots, n-1, \infty\}$ with $j \to i$, the functor sends

 , resp. 

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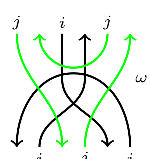
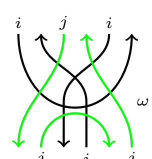
 , resp.  .

REMARK 3.7. This step of the proof is closely related to (and inspired by) the earlier works [**MSV, MT1, MT2**].

For both steps, the proof consists of manipulations with the diagrams in $\mathbb{U}(\widehat{\mathfrak{gl}}_p)$ or $\mathbb{U}(\widehat{\mathfrak{gl}}_n)$. Combining these two steps one obtains an action of $\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}$ on the category $\mathsf{Rep}_0(\mathbf{G})$, and one can easily check that each object B_i acts via a wall crossing functor associated with the wall corresponding to i; see [**RW1**, §6.4.8] for details. This provides the desired proof of Conjecture 1.3 in this special case.

3.2. The Finkelberg–Mirković conjecture. Most of the general proofs of Conjecture 1.3 or Conjecture 2.1 involve the *Finkelberg–Mirković conjecture* (formulated in [FM]) in some form. Hence we start by explaining this highly influential conjecture.

3.2.1. *Statement.* Consider a connected reductive algebraic group G over an algebraically closed field $\mathbb{F}$ of characteristic ℓ, with a choice of Borel subgroup $B \subset G$ and maximal torus $T \subset B$. Consider the groups LG, L^+G and I and the affine Grassmannian Gr as in Sections 4–5 of Chapter 3. As in Chapter 3 we work either in the "topological case" where $\mathbb{F} = \mathbb{C}$ (and then consider sheaves for the analytic topology) or in the "étale case" (and consider étale sheaves). Next, let $\Bbbk$ be an algebraic closure of a finite field of characteristic $p \neq \ell$. (In the "topological" setting, $\Bbbk$ can in fact be an arbitrary algebraically closed field.) Then we can consider the category $\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk)$ of L^+G-equivariant $\Bbbk$-perverse sheaves of Gr, which admits a natural structure of monoidal category with monoidal product $\star_{L^+G}$. As explained in §5.1 of Chapter 3, the geometric Satake equivalence provides an equivalence of monoidal categories

$$\mathsf{Sat} : (\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk), \star_{L^+G}) \xrightarrow{\sim} (\mathsf{Rep}(G^\vee_\Bbbk), \otimes)$$

where $G^\vee_\Bbbk$ is a split connected reductive algebraic group over $\Bbbk$, with a maximal torus $T^\vee_\Bbbk$ whose lattice of characters is $X_*(T)$ and such that the root datum of $(G^\vee_\Bbbk, T^\vee_\Bbbk)$ is dual to that of (G, T). We will also denote by $B^\vee_\Bbbk \subset G^\vee_\Bbbk$ the Borel subgroup whose roots are the negative coroots of (G, T) (with respect to our choice of B, considered as a negative Borel subgroup in G).

We will assume that we have

$$\mathbf{G}^{(1)} = G^\vee_\Bbbk, \quad \mathbf{B}^{(1)} = B^\vee_\Bbbk, \quad \mathbf{T}^{(1)} = T^\vee_\Bbbk.$$

We will identify the character lattice $\mathbb{X} = X^*(\mathbf{T})$ with $X^*(\mathbf{T}^{(1)}) = X_*(T)$ in such a way that the pullback under the Frobenius morphism $\mathbf{T} \to \mathbf{T}^{(1)}$ corresponds to the morphism $\lambda \mapsto p\lambda$ on $\mathbb{X}$. In this way the root system $\mathcal{R}$ of (G, T) identifies with $\mathfrak{R}^\vee$, $\mathcal{R}^\vee$ identifies with $\mathfrak{R}$, and the subset $\mathbb{X}^+ \subset \mathbb{X}$ of dominant weights identifies with the subset $X_*(T)^+ \subset X_*(T)$ of dominant coweights. Similarly, the affine Weyl group W_{aff} therefore identifies with $W \ltimes \mathbb{Z}\mathcal{R}^\vee$, and the extended Weyl group W_{ext} with $W \ltimes X_*(T)$, so that the notation of Chapter 1 matches that of Sections 4–5 of Chapter 3.

It will also be convenient to consider the quotient

$$\mathsf{Gr}' := L^+G \backslash LG,$$

with its action of L^+G induced by multiplication on the right in LG. Of course we have a canonical isomorphism

$$\mathsf{Gr}' \xrightarrow{\sim} \mathsf{Gr}$$

induced by the assignment $g \mapsto g^{-1}$, which commutes with the actions of L^+G on both sides.

Let I_{u} be the prounipotent radical of I, i.e. the preimage of the unipotent radical of B under the canonical morphism $L^+G \to G$; then the I_{u}-orbits on Gr' are in a canonical bijection with the subset ${}^{\mathrm{f}}W_{\mathrm{ext}} \subset W_{\mathrm{ext}}$ of elements w which have minimal length in the coset Ww. Here multiplication in W_{ext} induces a bijection

$${}^{\mathrm{f}}W_{\mathrm{aff}} \times \Omega \xrightarrow{\sim} {}^{\mathrm{f}}W_{\mathrm{ext}}.$$

Let us assume now that $\ell \geq h$, and fix a weight $\lambda \in C$. For technical reasons, we will assume that the stabilizer of λ in W_{ext} (for the action $\cdot_p$) intersects Ω trivially.[3] Then we can consider the extended principal block $\mathsf{Rep}_{[0]}(\mathbf{G})$ in the category $\mathsf{Rep}(\mathbf{G})$, namely the Serre subcategory generated by the simple $\mathbf{G}$-modules of the form $\mathsf{L}(w \cdot_\ell \lambda)$ with $w \in {}^{\mathrm{f}}W_{\mathrm{ext}}$. In terms of the "blocks" considered in §2.5 of Chapter 1, we have

$$\mathsf{Rep}_{[0]}(\mathbf{G}) = \bigoplus_{\omega \in \Omega} \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p (\omega \cdot_p \lambda)}.$$

(Here we have $\{\omega \cdot_p \lambda : \omega \in \Omega\} = C \cap (W_{\mathrm{ext}} \cdot_p \lambda)$, and this set is in bijection with Ω by assumption.)

On the other side we consider the category $\mathsf{Perv}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk)$ of I_{u}-equivariant $\Bbbk$-perverse sheaves on Gr'. We have a natural convolution product

$$D^{\mathrm{b}}_{L^+G}(\mathsf{Gr}, \Bbbk) \times D^{\mathrm{b}}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk) \to D^{\mathrm{b}}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk)$$

which defines an action of the monoidal triangulated category $(D^{\mathrm{b}}_{L^+G}(\mathsf{Gr}, \Bbbk), \star_{L^+G})$ on $D^{\mathrm{b}}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk)$. As in the case of the geometric Satake equivalence, it turns out that this bifunctor is t-exact on both sides, hence defines an action of the monoidal abelian category $(\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk), \star_{L^+G})$ on $\mathsf{Perv}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk)$. The simple I_{u}-equivariant perverse sheaf supported on the closure of the I_{u}-orbit labeled by $w \in {}^{\mathrm{f}}W_{\mathrm{ext}}$ will be denoted by $\mathcal{IC}_w$.

Conjecture 3.8 (Finkelberg–Mirković conjecture). *Assume that $p \geq h$ and that $\mathrm{Stab}_{(\Omega, \cdot_p)}(0) = \{e\}$. There exists an equivalence of categories*

$$\mathsf{FM} : \mathsf{Perv}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk) \xrightarrow{\sim} \mathsf{Rep}_{[0]}(\mathbf{G})$$

which identifies the natural highest weight structures on both sides, and satisfies

$$\mathsf{FM}(\mathcal{IC}_w) \cong \mathsf{L}(w \cdot_p \lambda) \qquad \text{for any } w \in {}^{\mathrm{f}}W_{\mathrm{ext}}.$$

Moreover, for $\mathcal{F}$ in $\mathsf{Perv}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk)$ and $\mathcal{G}$ in $\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk)$ there exists a bifunctorial isomorphism

$$\mathsf{FM}(\mathcal{G} \star_{L^+G} \mathcal{F}) \cong \mathsf{FM}(\mathcal{F}) \otimes \mathrm{Fr}^*\big(\mathsf{Sat}(\mathcal{G})\big).$$

Remark 3.9. The Finkelberg–Mirković conjecture is often stated in terms of I_{u}-equivariant perverse sheaves on Gr rather than Gr'. The formulation involving Gr' allows to avoid the "swapping" equivalence sw from [**AR5**, Conjecture 1].

3.2.2. *A "singular" version.* The formulation of Conjecture 3.8 requires the assumption that $p \geq h$. One can however state a "singular" variant which makes sense in larger generality. Namely, recall that $X_*(T)$ identifies with $\mathbb{X}$. We consider some $\mu \in \overline{C} \cap \mathbb{X}$, and assume that $\mathrm{Stab}_{(\Omega, \cdot_p)}(\mu) = \{e\}$. Let also $A \subset S_{\mathrm{aff}}$ be the subset consisting of the elements fixing μ, which is a finitary subset of S_{aff}. Consider the subgroup I_{u}^A as in §4.3 of Chapter 3, and the local system $\mathcal{X}_A$. Then we can consider the $(I_{\mathrm{u}}^A, \mathcal{X}_A)$-equivariant derived category $D^{\mathrm{b}}_{(I_{\mathrm{u}}^A, \mathcal{X}_A)}(\mathsf{Gr}', \Bbbk)$, and its subcategory $\mathsf{Perv}_{(I_{\mathrm{u}}^A, \mathcal{X}_A)}(\mathsf{Gr}', \Bbbk)$ of perverse sheaves. It can be checked that the simple objects in this category are in a natural bijection with

$${}^{\mathrm{f}}W_{\mathrm{ext}}^{(\mu)} = \bigsqcup_{\omega \in \Omega} {}^{\mathrm{f}}W_{\mathrm{aff}}^{\omega \cdot_p \mu} \cdot \omega,$$

[3] By Exercise 1.15, this condition is satisfied e.g. if $X_*(T)/\mathbb{Z}\mathcal{R}^\vee$ has no p-torsion, which is automatic if $p > h$. But it fails e.g. for $G = \mathrm{PGL}_p$ in characteristic p.

and we will denote by $\mathcal{IC}^A_w$ the simple object attached to w.

One the other hand, denote by $\mathsf{Rep}_{[\mu]}(\mathbf{G})$ the Serre subcategory in $\mathsf{Rep}(\mathbf{G})$ generated by the simple $\mathbf{G}$-modules of the form $\mathsf{L}(w \cdot_\ell \mu)$ with $w \in {}^{\mathrm{f}}W^{(\mu)}_{\mathrm{ext}}$. (Here, by definition of ${}^{\mathrm{f}}W^{(\mu)}_{\mathrm{ext}}$ and Proposition 2.29 in Chapter 1, we have $w \cdot_\ell \mu \in \mathbb{X}^+$ for any $w \in {}^{\mathrm{f}}W^{(\mu)}_{\mathrm{ext}}$.) In terms of the "blocks" considered in §2.5 of Chapter 1, we have

$$\mathsf{Rep}_{[\mu]}(\mathbf{G}) = \bigoplus_{\omega \in \Omega} \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p (\omega \cdot_p \mu)}.$$

CONJECTURE 3.10 (Singular Finkelberg–Mirković conjecture). *Let $\mu \in \mathbb{X} \cap \overline{C}$, and assume that $\mathrm{Stab}_{(\Omega, \cdot_p)}(\mu) = \{e\}$. There exists an equivalence of categories*

$$\mathsf{FM}_\mu : \mathsf{Perv}_{(I^A_{\mathrm{u}}, \mathcal{X}_A)}(\mathsf{Gr}', \Bbbk) \xrightarrow{\sim} \mathsf{Rep}_{[\mu]}(\mathbf{G})$$

which identifies the natural highest weight structures on both sides, and satisfies

$$\mathsf{FM}(\mathcal{IC}^A_w) \cong \mathsf{L}(w \cdot_p \mu) \qquad \text{for any } w \in {}^{\mathrm{f}}W^{(\mu)}_{\mathrm{ext}}.$$

Moreover, for $\mathcal{F}$ in $\mathsf{Perv}_{(I^A_{\mathrm{u}}, \mathcal{X}_A}(\mathsf{Gr}', \Bbbk)$ and $\mathcal{G}$ in $\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk)$ there exists a bifunctorial isomorphism

$$\mathsf{FM}_\mu(\mathcal{G} \star_{L^+G} \mathcal{F}) \cong \mathsf{FM}_\mu(\mathcal{F}) \otimes \mathrm{Fr}^*\big(\mathsf{Sat}(\mathcal{G})\big).$$

Of course, in case $p \geq h$ and $\lambda \in C$, Conjecture 3.10 boils down to Conjecture 3.8.

3.2.3. *The Iwahori–Whittaker model of the Satake category.* At this point there is no proof of Conjecture 3.8. There is a special case of Conjecture 3.10 which is known, however. Namely, assume that there exists $\varsigma \in \mathbb{X}$ such that

$$\langle \varsigma, \alpha^\vee \rangle = 1$$

for any $\alpha \in \mathfrak{R}^\vee$. (Such a weight exists at least if $\mathbf{G}$ is semisimple and simply connected; in this case we necessarily have $\varsigma = \rho$. But it might also exist when $\rho \notin \mathbb{X}$, e.g. if $\mathbf{G} = \mathrm{GL}_n(\Bbbk)$.) Then we consider the case $\mu = -\varsigma$. We have $A = S$, and the facet of $-\varsigma$ has the smallest possible dimension. The assumption that $\mathrm{Stab}_{(\Omega, \cdot_p)}(-\varsigma) = \{e\}$ is automatic, since W acts trivially on $-\varsigma$ and the projection $W \to \mathbb{X}$ is injective on Ω.

The category $\mathsf{Rep}_{[-\varsigma]}(\mathbf{G})$ has been studied in §2.10 of Chapter 1. (It was denoted $\mathsf{Rep}_{\mathrm{Stein}}(\mathbf{G})$ there.) As explained in Corollary 2.41 of Chapter 1, the functor $V \mapsto \mathsf{L}((p-1)\varsigma)$ induces an equivalence of highest weight categories

$$\mathsf{Rep}(G^\vee_\Bbbk) \xrightarrow{\sim} \mathsf{Rep}_{\mathrm{Stein}}(\mathbf{G}).$$

On the other hand, as in (2.13) in Chapter 1 we have

$${}^{\mathrm{f}}W^{(-\varsigma)}_{\mathrm{ext}} = \{t_\lambda w_0 : \lambda \in \varsigma + \mathbb{X}^+\}.$$

Since $\mathsf{Rep}(G^\vee_\Bbbk)$ identifies with $\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk)$ by the geometric Satake equivalence (Theorem 5.2 in Chapter 3), the singular Finkelberg–Mirković conjecture in this special case predicts an equivalence of highest weight categories

$$\mathsf{Perv}_{L^+G}(\mathsf{Gr}, \Bbbk) \xrightarrow{\sim} \mathsf{Perv}_{(I^S_{\mathrm{u}}, \mathcal{X}_S)}(\mathsf{Gr}, \Bbbk).$$

In fact, the compatibility of this equivalence with the geometric Satake equivalence forces this functor to be given by

$$\mathcal{F} \mapsto \mathcal{F} \star_{L^+G} \mathcal{IC}^S_{t_\varsigma w_0}.$$

The fact that this functor is an equivalence is the main result of [**BGMRR**]. (Here the element $t_\varsigma w_0$ is minimal in ${}^{\mathrm{f}}W_{\mathrm{ext}}^{(-\varsigma)}$, so that that $\mathcal{IC}^S_{t_\varsigma w_0}$ coincides with the !-extension and the $*$-extension of the rank-1 Iwhaori–Whittaker local system on the orbit labelled by $t_\varsigma w_0$.)

3.3. Proof of the tilting character formula via Koszul duality. The first proof of Conjecture 2.1 for a general reductive group was obtained (under the assumption that $p > h$) in [**AMRW**], building on the earlier works [**AR3, ARi2, MR2**]. This proof is very indirect, and inspired by work of Bezrukavnikov and several collaborators on representations of Lusztig's quantum groups at a root of unity (see in particular [**ABG, BY**]). More specifically we will assume that $\mathbf{G}$ is semisimple and simply-connected[4] (which is sufficient to imply the general case). In this case, the idea is to rephrase the question by building functors as follows:

$$D^{\mathrm{b}}\mathsf{Rep}_{[0]}(\mathbf{G}) \xleftarrow{F} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \xrightarrow{\sim} D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk) \xrightarrow{\sim} D^{\mathrm{mix}}_{(I^S_{\mathrm{u}},\mathcal{X}_S)}(\mathsf{Fl}, \Bbbk). \tag{3.1}$$

In the next few paragraphs we explain the meaning of each of these categories and functors, and why they allow to solve our problem.

3.3.1. *The functor F.* The variety $\widetilde{\mathcal{N}}$ is the Springer resolution of $G^\vee_{\Bbbk} = \mathbf{G}^{(1)}$, i.e. the cotangent bundle to its flag variety $G^\vee_{\Bbbk}/B^\vee_{\Bbbk}$. This variety admits a natural action of $G^\vee_{\Bbbk}$ induced by the obvious action on $G^\vee_{\Bbbk}$, and an action of the multiplicative group $\mathbb{G}_{\mathrm{m}}$ by dilation along the cotangent direction of the cotangent bundle. (More precisely, $z \in \Bbbk^\times$ acts by multiplication by z^{-2} on each cotangent fiber.) The category

$$\mathsf{Coh}^{G^\vee_{\Bbbk}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$$

is the abelian category of $(G^\vee_{\Bbbk}\times\mathbb{G}_{\mathrm{m}})$-equivariant coherent sheaves on $\widetilde{\mathcal{N}}$, i.e. coherent sheaves endowed with isomorphisms between their pullbacks under the two natural morphisms

$$(G^\vee_{\Bbbk}\times\mathbb{G}_{\mathrm{m}}) \times \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$$

(namely, the action and projection morphisms, respectively) which satisfy a natural cocycle condition. (We refer to [**MR1**, Appendix] for details and references on equivariant coherent sheaves.) We have a natural "shift" autoequivalence

$$\langle 1\rangle : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$$

given by tensoring with the tautological $\mathbb{G}_{\mathrm{m}}$-module. For any $n \in \mathbb{Z}$, we will denote by $\langle n\rangle$ the n-th power of $\langle 1\rangle$.

The functor F in (3.1) is a triangulated functor which is not an equivalence of categories, but it is "as close as possible to an equivalence given the difference of nature between the categories involved." Namely, it is a "degrading functor" with respect to the autoequivalence $\langle 1\rangle[1]$, which means that there exists a canonical isomorphism $F \circ \langle 1\rangle[1] \cong F$ such that

- for any $\mathcal{F}, \mathcal{G}$ in $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_{\Bbbk}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$, the functor F induces an isomorphism
$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(\mathcal{F}, \mathcal{G}\langle n\rangle[n]) \xrightarrow{\sim} \mathrm{Hom}(F(\mathcal{F}), F(\mathcal{G}));$$
- the essential image of F generates the category $D^{\mathrm{b}}\mathsf{Rep}_{[0]}(\mathbf{G})$ as a triangulated category.

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[4] In fact, the results described in §3.3.1–3.3.2 are proved in [**AR3**] for reductive groups with simply-connected derived subgroups.

This functor is also compatible with the natural actions of $\mathsf{Rep}(G^\vee_\Bbbk)$, in the sense that for any $\mathcal{F} \in D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ and $V \in \mathsf{Rep}(G^\vee_\Bbbk)$ there exists a bifunctorial isomorphism

$$F(\mathcal{F} \otimes V) \cong F(\mathcal{F}) \otimes \mathrm{Fr}^*(V).$$

The construction of this functor proceeds in 2 steps, called the "induction theorem" and the "formality theorem," and follow a pattern similar to that in [**ABG**] (although some of the proofs require different arguments). We refer to the introduction of [**AR3**] for more details.

The Weyl and induced modules in $\mathsf{Rep}_{[0]}(\mathbf{G})$ are images under F of the standard and costandard objects involved with the construction of the "exotic t-structure" on $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$. (The definition of this t-structure is due to Bezrukavnikov. See [**MR1**] for a review of its main properties.) The heart of this t-structure has a canonical structure of highest weight categories, so in particular their are indecomposable tilting objects in this category, but these are *not* sent to tilting modules under the functor F. In fact, the indecomposable tilting modules are images of the objects characterized by some parity vanishing conditions similar to those involved in the definition of parity complexes in Chapter 3. (This fact was not explicitly stated in [**AR3**]; it was made explicit later in [**AHR**].) The functor F therefore "sends parity objects to tilting objects," which is a property one should expect from a "Koszul duality functor" in a context of representations in positive characteristic; see [**AR4**] for more on this point of view.

3.3.2. *The middle arrow.* Let us now consider the middle arrow in (3.1). Here the category $D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk)$ is the "mixed derived category of I_{u}-equivariant sheaves on Gr'," defined more formally as the homotopy category of the category of I_{u}-equivariant parity complexes on Gr'. We want to think of this category as a "mixed version" of the derived category $D^{\mathrm{b}}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk)$. In fact there exists no formal relation between these two categories (in particular, from the definition we do not have any "natural" forgetful functor $D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk) \to D^{\mathrm{b}}(\mathsf{Gr}', \Bbbk)$), but the category $D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk)$ has the same kind of structure as $D^{\mathrm{b}}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk)$ (in particular, a "perverse" t-structure, whose heart $\mathsf{Perv}^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk)$ is a graded highest weight category[5]), plus a "Tate twist" autoequivalence $\langle 1 \rangle$. Such categories were introduced and studied (in a larger generality) in [**AR2**].

The second arrow in (3.1) is an equivalence of triangulated category

$$\Phi : D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \xrightarrow{\sim} D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk)$$

endowed with an isomorphism of functor $\Phi \circ \langle 1 \rangle \cong \langle 1 \rangle [-1] \circ \Phi$, and which sends the standard, resp. costandard, objects involved in the construction of the exotic t-structure to the standard, resp. costandard, objects involved in the construction of the perverse t-structure on $D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk)$. Again it can be thought of a some example of "modular Koszul duality" in that it sends the (normalized) indecomposable parity complexes in $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ to the (normalized) indecomposable tilting objects in $D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}', \Bbbk)$. This construction has two variants: on constructed in [**MR2**], and the other one in [**ARi2**]. The latter construction provides a functors which is compatible with the Satake equivalence in the sense that for a natural

[5]We will not give the definition of a graded highest weight category; informally, this a highest weight category endowed with a compatible "grading shift" autoequivalence.

action $\star_{L^+G}$ of the category $\mathsf{Perv}_{L^+G}(\mathsf{Gr},\Bbbk)$ on $D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}',\Bbbk)$ we have a bifunctorial isomorphism

$$\Phi(\mathcal{F}\otimes\mathrm{Sat}(\mathcal{G}))\cong\mathcal{G}\star_{L^+G}\Phi(\mathcal{F})$$

for $\mathcal{F}\in D^{\mathrm{b}}\mathsf{Coh}^{G^\vee_\Bbbk\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ and $\mathcal{G}\in\mathsf{Perv}_{L^+G}(\mathsf{Gr},\Bbbk)$. (The former construction has other advantages; in particular, it is involved in the proof of Theorem 5.4 in Chapter 3 under the optimal assumptions.)

Combining the two steps reviewed so far, we obtain a triangulated functor

$$D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}',\Bbbk)\to D^{\mathrm{b}}\mathsf{Rep}_{[0]}(\mathbf{G})$$

which is degrading with respect to $\langle 1\rangle$, and sends standard objects to Weyl modules and costandard objects to induced modules. It is easy to see that such a functor is necessarily t-exact, hence restricts to an exact degrading functor

$$\mathsf{Perv}^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}',\Bbbk)\to D^{\mathrm{b}}\mathsf{Rep}_{[0]}(\mathbf{G}). \tag{3.2}$$

This functor is compatible with the Satake equivalence in an appropriate way; this construction therefore produces a "mixed analogue" of Conjecture 3.8.

3.3.3. *Koszul duality and proof of the tilting character formula.* The functor considered in (3.2) sends tilting modules to tilting modules. Being a degrading functor, it also sends indecomposable objects to indecomposable objects by [**GG**, Theorem 3.1]. The question of computing multiplicities of standard objects in indecomposable tilting modules in $\mathsf{Rep}_{[0]}(\mathbf{G})$ is therefore reduced to the similar problem in the category $\mathsf{Perv}^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}',\Bbbk)$.

This question is solved in [**AMRW**], by constructing a "Koszul duality" equivalence

$$D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}',\Bbbk)\xrightarrow{\sim}D^{\mathrm{mix}}_{(I^S_{\mathrm{u}},\mathcal{X}_S)}(\mathsf{Fl},\Bbbk), \tag{3.3}$$

where the right-hand side is defined as for the left-hand side, in terms of the bounded homotopy category of the category of parity complexes. It has the same structure as $D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}',\Bbbk)$; in particular we have a perverse t-structure, and standard and costandard objects. This equivalence exchanges standard, resp. costandard, objects in both categories, intertwines the equivalence $\langle 1\rangle$ on the left-hand side with the equivalence $\langle -1\rangle[1]$ on the right-hand side, and sends normalized indecomposable tilting perverse sheaves to images of normalized indecomposable parity complexes in $D^{\mathrm{mix}}_{(I^S_{\mathrm{u}},\mathcal{X}_S)}(\mathsf{Fl},\Bbbk)$. This implies that multiplicities of standard objects in indecomposable tilting objects in $D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Gr}',\Bbbk)$ can be expressed as dimensions of stalks of parity complexes in $D^{\mathrm{b}}_{(I^S_{\mathrm{u}},\mathcal{X}_S)}(\mathsf{Fl},\Bbbk)$, which are known to be given by antispherical p-Kazhdan–Lusztig polynomials, as explained in §4.3 of Chapter 3. This therefore solves the question considered in Conjecture 2.1.

The equivalence (3.3) is a "parabolic-singular" analogue of a similar Koszul duality

$$D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Fl}',\Bbbk)\xrightarrow{\sim}D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Fl},\Bbbk)$$

where $\mathsf{Fl}'=I\backslash LG$, which has similar properties. This construction has variants for flag varieties of Kac–Moody groups (also treated in [**AMRW**]). At the combinatorial level, it shows that multiplicities of standard objects in indecomposable tilting objects in mixed derived categories of sheaves on flag varieties are also computed by p-Kazhdan–Lusztig polynomials. The ideas behind this construction go back to [**BGS**], which treated the case of characteristic-0 coefficients for flag varieties of reductive groups. This construction was later generalized to arbitrary Kac–Moody groups in [**BY**], and to the modular setting (but still for flag varieties of reductive groups) in [**AR2**].

Remark 3.11. As explained above, there is a priori no formal relation between the categories $D^{\mathrm{mix}}_{I_{\mathrm{u}}}(\mathsf{Fl}, \Bbbk)$ and $D^{\mathrm{b}}_{I_{\mathrm{u}}}(\mathsf{Fl}, \Bbbk)$, and similarly for flag varieties of Kac–Moody groups. Therefore, the computation of multiplicities of standard objects in indecomposable tilting objects in these two contexts are distinct problems. We however expect the two questions to give the same answer. In the case of affine flag varieties, this was recently confirmed in [**BR2**] under appropriate assumptions on the coefficients.

3.4. Proof of the tilting character formula via Smith–Treumann theory. A second proof of Conjecture 2.1, which in fact establishes Conjecture 2.3 for any group in any characteristic, was later found in [**RW3**]. This proof is in a sense more direct. It relies on the geometric Satake equivalence (Theorem 5.2 in Chapter 3), or more specifically on the composition of this equivalence with that presented in §3.2.3. Namely, composing these equivalences we obtain an equivalence of highest weight categories

$$\mathsf{Rep}(G^\vee_\Bbbk) \cong \mathsf{Perv}_{(I^A_{\mathrm{u}}, \mathcal{X}_S)}(\mathsf{Gr}', \Bbbk).$$

One next applies "Smith–Tremann theory" in the left-hand side, which produces some kind of "localization functor" relating sheaves on Gr' to sheaves on the fixed points under the group of p-th roots of unity in $\mathbb{F}$ (acting via loop rotation). One then observes that these fixed points identify with a disjoint union of partial flag varieties for the "p-dilated" loop group of G, and that the localization functor is fully faithful on tilting modules. This allows to compute dimensions of morphisms between indecomposable tilting modules, and hence to compute multiplicities using the ideas of Exercise 7.9. In practice this involves many ingredients not covered in this book, so that we will not explain it more.

3.5. Constructions of the categorical action. Conjecture 1.3 also now 2 independent proofs for simply-connected semisimple groups, both assuming that $p > h$.

3.5.1. *Hecke category action via completed Harish-Chandra bimodules.* One of these proofs was found in [**BR1**]. The idea there is to consider a larger category, involving equivariant modules over the universal enveloping algebra $\mathcal{U}\mathbf{g}$ of the Lie algebra $\mathbf{g}$ of $\mathbf{G}$. Before explaining this, we need to recall a few facts about the structure of the center of $\mathcal{U}\mathbf{g}$ (valid for semisimple groups in very good characteristic). First, as in the characteristic-0 setting, the subalgebra

$$Z_{\mathrm{HC}} := (\mathcal{U}\mathbf{g})^{\mathbf{G}}$$

is central in $\mathcal{U}\mathbf{g}$, and identifies canonically with the algebra

$$\mathscr{O}(\mathbf{t}^*/(W, \bullet)) = \mathscr{O}(\mathbf{t}^*)^{(W,\bullet)}$$

of functions on the quotient $\mathbf{t}^*/(W, \bullet)$, where $\mathbf{t}$ is the Lie algebra of $\mathbf{T}$ and the action $\bullet$ of W on $\mathbf{t}^*$ is defined by $w \bullet \xi = w(\xi + \rho) - \rho$ where we write ρ for the diffierential of this character of $\mathbf{T}$. (This subalgebra is called the "Harish-Chandra center"). On the other hand, we have the "Frobenius center"

$$Z_{\mathrm{Fr}} \subset \mathcal{U}\mathbf{g},$$

which is generated by elements of the form $x^p - x^{[p]}$ with $x \in \mathbf{g}$. (Here, $(\cdot)^{[p]}$ is the restricted p-th power map, which is available on the Lie algebra of any algebraic group defined over a field of characteristic p.) This subalgebra can also be described geometrically: it identifies with

$$\mathscr{O}(\mathbf{g}^{*(1)}).$$

(Here, $(\cdot)^{(1)}$ is the Frobenius twist of $\Bbbk$-schemes, as defined in §2.4 of Chapter 1.) This subalgebra has the property that if M is a $\mathbf{G}$-module, then for the $\mathcal{U}\mathbf{g}$-action obtained by differentiation the subalgebra Z_{Fr} acts via the "trivial character"

$$\mathscr{O}(\mathbf{g}^{*(1)}) \to \Bbbk$$

corresponding to the 0-vector in $\mathbf{g}^{*(1)}$. Another important property (which follows from the Poincaré–Birkhoff–Witt theorem) is that $\mathcal{U}\mathbf{g}$ is finite as a module over Z_{Fr}.

Denote by $\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U}\mathbf{g})$ the category of finitely generated $\mathbf{G}$-equivariant $\mathcal{U}\mathbf{g}$-modules, i.e. finitely generated $\mathbf{G}$-equivariant $\mathcal{U}\mathbf{g}$-modules M endowed with a structure of (rational) $\mathbf{G}$-module which satisfy

$$g \cdot (x \cdot m) = (g \cdot x) \cdot (g \cdot m)$$

for $g \in \mathbf{G}$, $x \in \mathcal{U}\mathbf{g}$ and $m \in M$. There exists a natural fully faithful functor

$$\mathsf{Rep}(\mathbf{G}) \to \mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U}\mathbf{g}),$$

which sends a $\mathbf{G}$-module to itself, with the $\mathcal{U}\mathbf{g}$-action obtained by differentiating the $\mathbf{G}$-action. (In fact, the essential image of this functor exactly consists of $\mathbf{G}$-equivariant $\mathcal{U}\mathbf{g}$-modules which have the property that the $\mathcal{U}\mathbf{g}$-action is the differential of the $\mathbf{G}$-action.)

We will also set

$$\mathcal{U} := (\mathcal{U}\mathbf{g}) \otimes_{Z_{\mathrm{Fr}}} (\mathcal{U}\mathbf{g})^{\mathrm{op}},$$

endowed with the diagonal action of $\mathbf{G}$, and consider the category $\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U})$ of finitely generated $\mathbf{G}$-equivariant $\mathcal{U}$-modules. We will call *Harish-Chandra bimodule* an object of $\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U})$ such that the differential of the $\mathbf{G}$-action is given by the restriction of the $\mathcal{U}$-action along the "diagonal embedding"

$$\mathcal{U}\mathbf{g} \to \mathcal{U}$$

induced by the assignment $x \mapsto (x \otimes 1) - (1 \otimes x)$ for $x \in \mathbf{g}$. The full subcategory of $\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U})$ consisting of Harish-Chandra bimodules will be denoted HC. The tensor product functor

$$(-) \otimes_{\mathcal{U}\mathbf{g}} (-) : \mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U}) \times \mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U}) \to \mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U})$$

defines a monoidal structure of the category $\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U})$, with unit object the natural module $\mathcal{U}\mathbf{g}$, and the subcategory HC is monoidal. Moreover, the tensor product

$$(-) \otimes_{\mathcal{U}\mathbf{g}} (-) : \mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U}) \times \mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U}\mathbf{g}) \to \mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U}\mathbf{g})$$

defines an action of $\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U})$ on $\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U}\mathbf{g})$, and the action of the subcategory $\mathsf{HC} \subset \mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U})$ stabilizes the subcategory $\mathsf{Rep}(\mathbf{G}) \subset \mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U}\mathbf{g})$.

To proceed further one needs to consider characters of the Harish-Chandra center. Namely, if $\lambda \in \mathbb{X}$, by differentiation we deduce a linear form on $\mathbf{t}$, hence a character of $\mathscr{O}(\mathbf{t}^*)$, and finally of Z_{HC}. The kernel of this character will be denoted $\mathfrak{m}_\lambda$. We now assume that $\lambda \in C$, and denote by $\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg},\lambda}(\mathcal{U}\mathbf{g})$ the full subcategory of $\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U}\mathbf{g})$ consisting of modules on which $\mathfrak{m}_\lambda$ acts nilpotently. It is easy to see that the image of the principal block $\mathsf{Rep}_0(\mathbf{G})$ (defined with respect to the character *lambda*) in $\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U}\mathbf{g})$ lies in the subcategory $\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg},\lambda}(\mathcal{U}\mathbf{g})$. Next, let us denote by

$$\mathcal{U}^{\hat{\lambda},\hat{\lambda}}$$

the completion of the algebra $\mathcal{U}$ with respect to the ideal

$$\mathfrak{m}_\lambda \cdot \mathcal{U} + \mathcal{U} \cdot \mathfrak{m}_\lambda.$$

Then $\mathcal{U}^{\hat{\lambda},\hat{\lambda}}$ is a noetherian algebra, endowed with a structure of rational $\mathbf{G}$-module. We can therefore consider the category

$$\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg}}(\mathcal{U}^{\hat{\lambda},\hat{\lambda}})$$

of $\mathbf{G}$-equivariant finitely generated modules over this algebra, and the full subcategory

$$\mathsf{HC}^{\hat{\lambda},\hat{\lambda}}$$

of Harish-Chandra bimodules. These categories can be endowed with structures of monoidal categories, and with actions on the category $\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg},\lambda}(\mathcal{U}\mathbf{g})$.

It turns out that for any $s \in S_{\mathrm{aff}}$ the action of the functor Θ_s on $\mathsf{Rep}_0(\mathbf{G})$ is the restriction of the action on $\mathsf{Mod}^{\mathbf{G}}_{\mathrm{fg},\lambda}(\mathcal{U}\mathbf{g})$ of a certain Harish-Chandra bimodule P_s. To prove Conjecture 1.3 it therefore suffices to construct a monoidal functor

$$\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}} \to \mathsf{HC}^{\hat{\lambda},\hat{\lambda}}$$

which intertwines the shift functor (1) with the identity functor, and sends for any $s \in S_{\mathrm{aff}}$ the object B_s to the bimodule P^s. The construction of such a functor is the main result of [**BR1**].

The proof uses in a crucial way Abe's bimodule incarnation of the Hecke category (see Section 3 of Chapter 2). Namely, using a variant of the "localization theory" of [**BMR**] one relates the category $\mathsf{HC}^{\hat{\lambda},\hat{\lambda}}$ with a certain category of representations of a group scheme defined in terms of the "universal centralizer" for $\mathbf{g}$. One then observes that the "extra data" attached to bimodules in Abe's category exactly encode an action of this group scheme, which provides the desired functor. One next has to check that the image of each B_s is the corresponding bimodule P_s. This is checked relatively explicitly in case $s \in S$ (using "singular" variants of the same theory), and the general case is reduced to this one using in particular Exercise 1.14.

3.5.2. *Hecke category action via Smith–Treumann theory.* A completely different proof of Conjecture 1.3 was found (more or less simultaneously) by J. Ciappara, see [**Ci**]. This proof is based on the construction presented in §3.4. Namely, these constructions provide an action of $\mathsf{D}^{\mathrm{BS}}_{\mathrm{aff}}$ on the principal block, but it is not clear at first that the object B_s acts via the wall crossing functor Θ_s. Checking this fact is the main result of [**Ci**].

APPENDIX A

Highest weight categories

The theory of highest weight categories was initially studied by Cline–Parshall–Scott in connection with the theory of quasi-hereditary algebras, see [CPS]. However we prefer to use a different, more "categorical," point of view introduced in [BGS, §3.2]. In this appendix we gather references or proofs for some standard results on these categories using this point of view. (These results are sometimes available in the literature only in the Cline–Parshall–Scott setting, which seems to justify a complete treatment from the Beĭlinson–Ginzburg–Soergel perspective.) For a detailed treatment of these questions from the original "algebraic" point of view, see e.g. [D2, Appendix A].

1. Definitions

Throughout the appendix, $\Bbbk$ will be a field, and $\mathcal{A}$ will be a finite-length $\Bbbk$-linear abelian category such that $\mathrm{Hom}_{\mathcal{A}}(M, N)$ is finite-dimensional for any M, N in $\mathcal{A}$. Note that such a category is Krull–Schmidt,[1] see [CYZ, Remark A.2].

Let $\mathscr{S}$ be the set of isomorphism classes of irreducible objects of $\mathcal{A}$. Assume that $\mathscr{S}$ is equipped with a partial order $\leq$, and that for each $s \in \mathscr{S}$ we have a fixed representative simple object L_s. Assume also we are given, for any $s \in \mathscr{S}$, objects Δ_s and ∇_s, and morphisms $\Delta_s \to \mathsf{L}_s$ and $\mathsf{L}_s \to \nabla_s$. For $\mathscr{T} \subset \mathscr{S}$, we denote by $\mathcal{A}_{\mathscr{T}}$ the Serre subcategory[2] of $\mathcal{A}$ generated by the objects L_t for $t \in \mathscr{T}$, i.e. the full subcategory whose objects are those all of whose composition factors are labelled by elements of $\mathscr{T}$. We write $\mathcal{A}_{\leq s}$ for $\mathcal{A}_{\{t \in \mathscr{S} \mid t \leq s\}}$, and similarly for $\mathcal{A}_{<s}$. Finally, recall that an *ideal* of $\mathscr{S}$ is a subset $\mathscr{T} \subset \mathscr{S}$ such that if $t \in \mathscr{T}$ and $s \in \mathscr{S}$ are such that $s \leq t$, then $s \in \mathscr{T}$.

Definition 1.1. The category $\mathcal{A}$ (together with the above data) is said to be a *highest weight category* if the following conditions hold:

(1) for any $s \in \mathscr{S}$, the set $\{t \in \mathscr{S} \mid t \leq s\}$ is finite;
(2) for each $s \in \mathscr{S}$, we have $\mathrm{Hom}_{\mathcal{A}}(\mathsf{L}_s, \mathsf{L}_s) = \Bbbk$;
(3) for any $s \in \mathscr{S}$ and any ideal $\mathscr{T} \subset \mathscr{S}$ such that $s \in \mathscr{T}$ is maximal, $\Delta_s \to \mathsf{L}_s$ is a projective cover in $\mathcal{A}_{\mathscr{T}}$ and $\mathsf{L}_s \to \nabla_s$ is an injective envelope in $\mathcal{A}_{\mathscr{T}}$;
(4) the kernel of $\Delta_s \to \mathsf{L}_s$ and the cokernel of $\mathsf{L}_s \to \nabla_s$ belong to $\mathcal{A}_{<s}$;

[1] An additive category $\mathcal{C}$ is called *Krull–Schmidt* if any object X has a decomposition $X = X_1 \oplus \cdots \oplus X_n$, such that each X_i is indecomposable with local endomorphism ring. An additive category $\mathcal{C}$ is Krull–Schmidt if and only if any idempotent in $\mathcal{C}$ splits, and $\mathrm{End}_{\mathcal{C}} X$ is semiperfect for any $X \in \mathcal{C}$. In this case, any object has a unique (up to order) direct decomposition into indecomposables.

[2] A *Serre subcategory* of an abelian category is a nonempty full subcategory stable under subquotients and extensions.

(5) we have $\mathrm{Ext}^2_{\mathcal{A}}(\Delta_s, \nabla_t) = 0$ for all $s, t \in \mathscr{S}$.

In this case, the poset $(\mathscr{S}, \leq)$ is called the *weight poset* of $\mathcal{A}$.

If $\mathcal{A}$ satisfies Definition 1.1, the objects Δ_s are called *standard objects*, and the objects ∇_s are called *costandard objects*. We say that an object M *admits a* Δ*-filtration*, resp. *admits a* ∇*-filtration*, if there exists a finite filtration of M whose subquotients are standard objects, resp. costandard objects.

From the axioms (3) and (4) we see in particular that

(1.1) Δ_s and ∇_s belong to $\mathcal{A}_{\leq s}$ and satisfy $[\Delta_s : \mathsf{L}_s] = [\nabla_s : \mathsf{L}_s] = 1$.

Remark 1.2. (1) The axioms in Definition 1.1 are exactly those in [**BGS**, §3.2], except that we replace the condition that $\mathscr{S}$ is finite by the weaker condition (1).

(2) In [**AR2**] we used the term *quasihereditary category* instead of *highest weight category*. We now believe that the latter term is more appropriate than the former, and we changed our terminology in [**MR1, AR3**].

(3) The axioms in Definition 1.1 can be easily modified to define a *graded highest weight category*, where we consider in addition a "shift" autoequivalence $\langle 1 \rangle$ of $\mathcal{A}$; see [**AR2**, Appendix A] for details. All the statements below have analogues in this context, but for simplicity we will not state them explicitly.

We start with the following observations.

Lemma 1.3. *Let $\mathcal{A}$ be a highest weight category, with weight poset $(\mathscr{S}, \leq)$, standard objects $\{\Delta_s : s \in \mathscr{S}\}$ and costandard objects $\{\nabla_s : s \in \mathscr{S}\}$.*

(1) The category $\mathcal{A}^{\mathrm{op}}$ is a highest weight category, with weight poset $(\mathscr{S}, \leq)$, standard objects $\{\nabla_s : s \in \mathscr{S}\}$, and costandard objects $\{\Delta_s : s \in \mathscr{S}\}$.

(2) If $\mathscr{T} \subset \mathscr{S}$ is an ideal, then $\mathcal{A}_{\mathscr{T}}$ is a highest weight category with weight poset $(\mathscr{T}, \leq)$, standard objects $\{\Delta_t : t \in \mathscr{T}\}$ and costandard objects $\{\nabla_t : t \in \mathscr{T}\}$.

Proof. Part (1) is clear. In part (2), the only axiom which might not be clear is (5). However, this axiom for $\mathcal{A}_{\mathscr{T}}$ follows from the similar axiom for $\mathcal{A}$ using [**BGS**, Lemma 3.2.3]. □

Lemma 1.4. *For any $s, t \in \mathscr{S}$, we have*

$$\mathrm{Hom}_{\mathcal{A}}(\Delta_s, \nabla_t) \cong \begin{cases} \Bbbk & \text{if } s = t; \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Ext}^1_{\mathcal{A}}(\Delta_s, \nabla_t) = \{0\}.$$

Proof. If $s \not< t$, then s is maximal in the ideal $\mathscr{T} = \{u \in \mathscr{S} \mid u \leq s \text{ or } u \leq t\}$, and both Δ_s and ∇_t belong to $\mathcal{A}_{\mathscr{T}}$ by (1.1). Then we have $\mathrm{Hom}_{\mathcal{A}}(\Delta_s, \nabla_t) = \mathrm{Hom}_{\mathcal{A}_{\mathscr{T}}}(\Delta_s, \nabla_t)$ and $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_s, \nabla_t) = \mathrm{Ext}^1_{\mathcal{A}_{\mathscr{T}}}(\Delta_s, \nabla_t)$, and the claim follows from axiom (3) and (1.1).

If $s < t$, then t is maximal in the ideal $\mathscr{T} = \{u \in \mathscr{S} \mid u \leq t\}$, and both Δ_s and ∇_t belong to $\mathcal{A}_{\mathscr{T}}$ by (1.1); then the claim follows again from axiom (3) and (1.1). □

From Lemma 1.4 we see that if M is an object of $\mathcal{A}$ which admits a Δ-filtration, then the number of times Δ_s appears as a subquotient in such a filtration is equal to $\dim_{\Bbbk}(\mathrm{Hom}_{\mathcal{A}}(M, \nabla_s))$. In particular this number does not depend on the filtration, and will be denoted $(M : \Delta_s)$. Similarly, if M admits a ∇-filtration, then the number of times ∇_s appears as a subquotient in such a filtration is well defined, and will be denoted $(M : \nabla_s)$.

2. Existence of projectives and some consequences

The following result is proved in [**BGS**, Theorem 3.2.1 & Remarks following the theorem].

THEOREM 2.1. *Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$ and assume that $\mathscr{S}$ is finite. Then $\mathcal{A}$ has enough projective objects, and any projective object admits a Δ-fitration. Moreover, if P_s is the projective cover of L_s, we have*

$$(\mathsf{P}_s : \Delta_t) = [\nabla_t : \mathsf{L}_s]. \tag{2.1}$$

REMARK 2.2. The formula (2.1) shows that in the setting of Theorem 2.1, for $s, t \in \mathscr{S}$ we have

$$(\mathsf{P}_s : \Delta_t) \neq 0 \quad \Rightarrow \quad s \leq t.$$

This observation shows that one can "detect" some indecomposable direct summands of a projective object from its standard multiplicities. More explicitly, if P is projective and if s is minimal among the elements $t \in \mathscr{S}$ such that $(P : \Delta_t) \neq 0$, then P_s is a direct summand of P, with multiplicity $(P : \Delta_s)$.

Applying Theorem 2.1 to the category $\mathcal{A}^{\mathrm{op}}$ (see Lemma 1.3(1)), we see that if $\mathscr{S}$ is finite, then $\mathcal{A}$ also has enough injective objects, and any injective object admits a ∇-filtration.

COROLLARY 2.3. *Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$. Then for any $s, t \in \mathscr{S}$ we have*

$$\mathrm{Ext}^i_{\mathcal{A}}(\Delta_s, \nabla_t) = \begin{cases} \Bbbk & \text{if } s = t \text{ and } i = 0; \\ \{0\} & \text{otherwise.} \end{cases}$$

PROOF. The case when $i \in \{0, 1\}$ is proved in Lemma 1.4, and we only have to prove the vanishing when $i \geq 2$.

First, we assume that $\mathscr{S}$ is finite, and prove the claim by descending induction on s. If s is maximal in $\mathscr{S}$, then Δ_s is a projective cover of L_s in $\mathcal{A}$ by axiom (3), and the claim follows. In general, consider the projective cover P_s of L_s. By Theorem 2.1, this object admits a Δ-filtration. Moreover, the last term in such a filtration must be Δ_s, since the top of P_s is L_s. In particular, we have an exact sequence

$$\ker \hookrightarrow \mathsf{P}_s \twoheadrightarrow \Delta_s$$

where ker admits a Δ-filtration. Moreover, (2.1) and (1.1) imply that if $(\ker : \Delta_t) \neq 0$, then $t > s$. Then the desired vanishing follows from induction and a long exact sequence consideration.

Now we prove the general case. Let $i \geq 2$, and consider a morphism $f : \Delta_s \to \nabla_t[i]$ in $D^{\mathrm{b}}(\mathcal{A})$. This morphism is represented by a fraction $\frac{g}{h}$, where M is a bounded complex of objects of $\mathcal{A}$, $h : M \xrightarrow{\mathrm{qis}} \Delta_s$ is a quasi-isomorphism of complexes, and $g : M \to \nabla_t[i]$ is a morphism of complexes. Choose a finite ideal

$\mathscr{S}' \subset \mathscr{S}$ which contains s, t, and the isomorphism classes of all composition factors of nonzero terms of M. (Such an ideal exists thanks to axiom (1).) Then $\frac{g}{h}$ defines a morphism in $D^{\mathrm{b}}(\mathcal{A}_{\mathscr{S}'})$, which must be the 0 morphism by Lemma 1.3(2) and the case of finite weight posets. We deduce that f is also 0 in $D^{\mathrm{b}}(\mathcal{A})$, which concludes the proof. ☐

REMARK 2.4. Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$. Let $\preceq$ be the preorder generated by the relation

$$s \preceq t \quad \text{if} \quad [\Delta_t : \mathsf{L}_s] \neq 0 \text{ or } [\nabla_t : \mathsf{L}_s] \neq 0.$$

Then (1.1) implies that $\preceq$ is an order such that $\leq$ refines $\preceq$. We claim that $\mathcal{A}$ is also a highest weight category for the poset $(\mathscr{S}, \preceq)$. Indeed, the only axiom which might not be clear is (3). However, as in the proof of Corollary 2.3, to check this axiom we can assume that $\mathscr{S}$ is finite. Then $\mathcal{A}$ has enough projective objects by Theorem 2.1, and the reciprocity formula (2.1) ensures that, if P_t is the projective cover of L_t in $\mathcal{A}$, then we have an exact sequence

$$\ker \hookrightarrow \mathsf{P}_t \twoheadrightarrow \Delta_t \tag{2.2}$$

where ker admits a Δ-filtration such that if $(\ker : \Delta_s) \neq 0$, then $s \succ t$. Now if $u \in \mathscr{S}$, considering the long exact sequence associated with (2.2) we obtain a surjection

$$\mathrm{Hom}_{\mathcal{A}}(\ker, \mathsf{L}_u) \twoheadrightarrow \mathrm{Ext}^1_{\mathcal{A}}(\Delta_t, \mathsf{L}_u).$$

Hence if $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_t, \mathsf{L}_u) \neq \{0\}$ then $\mathrm{Hom}_{\mathcal{A}}(\ker, \mathsf{L}_u) \neq \{0\}$, so that there exists $s \in \mathscr{S}$ such that $(\ker : \Delta_s) \neq 0$ and $\mathrm{Hom}_{\mathcal{A}}(\Delta_s, \mathsf{L}_u) \neq \{0\}$. Then $u = s$, so that $u \succ t$. From this it is easy to see that if $\mathscr{T}$ is an ideal in $(\mathscr{S}, \preceq)$ in which t is maximal, then Δ_t is projective in $\mathcal{A}_{\mathscr{T}}$, hence the projective cover of L_t.

More generally, the same considerations show that if $\leq'$ is any order which satisfies

$$s \preceq t \quad \Rightarrow \quad s \leq' t,$$

then $\mathcal{A}$ is a highest weight category for the poset $(\mathscr{S}, \leq')$. These comments show that it makes sense to say that a category is highest weight without specifying the order $\leq$ (if one specifies the standard and costandard objects).

3. Ideals and associated subcategories and quotients

3.1. Serre quotients of abelian categories. The next property we will see uses the notion of Serre quotient of an abelian category. Before stating this property, let us recall this construction.

Let $\mathcal{A}$ be an abelian category, and $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}$, see [Ga]. Namely, the objects of the quotient category $\mathcal{A}/\mathcal{B}$ are defined as those of $\mathcal{A}$. Given objects M, N in $\mathcal{A}$, the morphism space $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M, N)$ is defined as the inductive limit

$$\varinjlim_{M', N'} \mathrm{Hom}_{\mathcal{A}}(M', N/N')$$

where here M' runs over the subobjects of M such that the quotient M/M' (in the abelian category $\mathcal{A}$) belongs to $\mathcal{B}$, and N' runs over the subobjects of N which belong to $\mathcal{B}$. The composition law is defined as follows. Consider objects M, N, P in $\mathcal{A}$, and morphisms $f : M \to N$, $g : N \to P$ in $\mathcal{A}/\mathcal{B}$. By definition, there exist subobjects $M' \subset M$, $N', N'' \subset N$, $P' \subset P$ such that M/M', N', N/N'' and P' belong to $\mathcal{B}$, and such that f, resp. g, is the image of a morphism $\tilde{f} : M' \to N/N'$, resp. $\tilde{g} : N'' \to P/P'$. Let M'' be the preimage by $\tilde{f}$ of the image $(N' + N'')/N'$ of N'' in N/N'; then M/M'' belongs to $\mathcal{B}$ (because it embeds in $N/(N' + N'')$, which is a quotient of N/N''), and $\tilde{f}$ induces a morphism $\tilde{f}' : M'' \to (N'+N'')/N'$. Similarly, the image $\tilde{g}(N' \cap N'')$ of $N' \cap N''$ under $\tilde{g}$ belongs to $\mathcal{B}$, hence so does the sum $P'' := \tilde{g}(N' \cap N'') + P'$, and $\tilde{g}$ induces a morphism $\tilde{g}' : N''/(N' \cap N'') \to P/P''$. We can finally consider the composition

$$M'' \xrightarrow{\tilde{f}'} (N' + N'')/N' \cong N''/(N' \cap N'') \xrightarrow{\tilde{g}'} P/P'';$$

one can check that the class of this morphism in $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M, P)$ does not depend on the choice of $\tilde{f}$ and $\tilde{g}$ (but only on their classes f and g), hence can serve as the definition of the composition $g \circ f$. It can also be checked that the operation $\circ$ is associative, and that the class of id_M is an identity for the object M; this construction therefore indeed defines a category $\mathcal{A}/\mathcal{B}$. We also have a canonical functor $\Pi : \mathcal{A} \to \mathcal{A}/\mathcal{B}$, sending an object M to itself and a morphism $f : M \to N$ to its class in $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M, N)$.

As explained in [**Ga**, Proposition 1 on p. 367], the category $\mathcal{A}/\mathcal{B}$ is abelian, and the functor Π is exact. Moreover, these data have the following universal property (see [**Ga**, Corollaire 2 on p. 368]): if $\mathcal{C}$ is an abelian category and $F : \mathcal{A} \to \mathcal{C}$ is an exact functor such that $F(M) = 0$ for any M in $\mathcal{B}$, there exists a unique functor $G : \mathcal{A}/\mathcal{B} \to \mathcal{C}$ such that $F = G \circ \Pi$. In this setting, the functor G is moreover exact by [**Ga**, Corollaire 3 on p. 369].

3.2. Statement. The following results show that highest weight categories satisfy some "gluing" formalism which turns out to be very useful to run inductive arguments.

LEMMA 3.1. *Let $\mathcal{A}$ be a highest weight category, with weight poset $(\mathscr{S}, \leq)$, standard objects $\{\Delta_s : s \in \mathscr{S}\}$ and costandard objects $\{\nabla_s : s \in \mathscr{S}\}$. If $\mathscr{T} \subset \mathscr{S}$ is an ideal, then the Serre quotient $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$ is a highest weight category with weight poset $(\mathscr{S} \smallsetminus \mathscr{T}, \leq)$, standard objects $\{\pi_{\mathscr{T}}(\Delta_s) : s \in \mathscr{S} \smallsetminus \mathscr{T}\}$, and costandard objects $\{\pi_{\mathscr{T}}(\nabla_s) : s \in \mathscr{S} \smallsetminus \mathscr{T}\}$, where $\pi_{\mathscr{T}} : \mathcal{A} \to \mathcal{A}/\mathcal{A}_{\mathscr{T}}$ is the quotient functor.*

PROOF. It is clear that the category $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$ satisfies axioms (1), (2) and (4).

Now we check axiom (3) in the case of Δ_s; the case of ∇_s is similar. First, we claim that for any $s \in \mathscr{S} \smallsetminus \mathscr{T}$ and N in $\mathcal{A}$, the morphism

$$\mathrm{Hom}_{\mathcal{A}}(\Delta_s, N) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(N)) \tag{3.1}$$

induced by $\pi_{\mathscr{T}}$ is an isomorphism. Indeed, consider a morphism $f : \pi_{\mathscr{T}}(\Delta_s) \to \pi_{\mathscr{T}}(N)$. By definition, this morphism is represented by a morphism $f' : M' \to N/N'$ in $\mathcal{A}$, where $M' \subset \Delta_s$ and $N' \subset N$ are subobjects such that Δ_s/M' and N' belong to $\mathcal{A}_{\mathscr{T}}$. Since the head of Δ_s is L_s and $s \notin \mathscr{T}$, we have necessarily $M' = \Delta_s$. And since $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_s, N') = \{0\}$, the morphism f' factors through a morphism $f'' : \Delta_s \to N$. These arguments show that (3.1) is surjective. Since the image of any nonzero morphism from Δ_s to N contains L_s as a composition factor, its image under $\pi_{\mathscr{T}}$ is nonzero, hence the image of the morphism itself is nonzero. This shows that (3.1) is also injective, hence an isomorphism.

Now, let $\mathscr{U} \subset \mathscr{S} \smallsetminus \mathscr{T}$ be an ideal, and let $s \in \mathscr{U}$ be maximal. The isomorphisms (3.1) show that the top of $\pi_{\mathscr{T}}(\Delta_s)$ is $\pi_{\mathscr{T}}(\mathsf{L}_s)$. It remains to prove that this object is projective. If $f : \pi_{\mathscr{T}}(M) \to \pi_{\mathscr{T}}(N)$ is a surjection with $\pi_{\mathscr{T}}(M)$ and $\pi_{\mathscr{T}}(N)$ in $(\mathcal{A}/\mathcal{A}_{\mathscr{T}})_{\mathscr{U}}$, then M and N belong to $\mathcal{A}_{\mathscr{U} \sqcup \mathscr{T}}$, and f is represented by a morphism $f' : M' \to N/N'$ in $\mathcal{A}$ whose cokernel C belongs to $\mathcal{A}_{\mathscr{T}}$, where $M' \subset M$ and $N' \subset N$ are subobjects such that M/M' and N' belong to $\mathcal{A}_{\mathscr{T}}$. Then using isomorphisms (3.1) we see that we have

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(M)) \cong \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(M')) \cong \mathrm{Hom}_{\mathcal{A}}(\Delta_s, M')$$

and

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(N)) \cong \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(N/N')) \cong \mathrm{Hom}_{\mathcal{A}}(\Delta_s, N/N'),$$

and that the morphism

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(M)) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(N))$$

induced by f coincides with the morphism

$$\mathrm{Hom}_{\mathcal{A}}(\Delta_s, M') \to \mathrm{Hom}_{\mathcal{A}}(\Delta_s, N/N')$$

induced by f'. Hence the desired surjectivity follows from the facts that Δ_s is projective in $\mathcal{A}_{\mathscr{U} \sqcup \mathscr{T}}$ and that $\mathrm{Hom}_{\mathcal{A}}(\Delta_s, C) = \{0\}$.

Finally, we need to check axiom (5). For this we first assume that $\mathscr{S}$ is finite. Then $\mathcal{A}$ has enough projective objects by Theorem 2.1. Moreover, the proof of Corollary 2.3 shows that to prove the desired vanishing it suffices to prove that for any $s \in \mathscr{S} \smallsetminus \mathscr{T}$ there exists a projective object P in $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$ and a surjection $P \twoheadrightarrow \pi_{\mathscr{T}}(\Delta_s)$ whose kernel admits a filtration with subquotients $\pi_{\mathscr{T}}(\Delta_t)$ with $t > s$. We claim that $P = \pi_{\mathscr{T}}(\mathsf{P}_s)$ satisfies these properties. In fact, the only property which is not clear is that P is projective. If this were not the case, there would exist a non-split and non-trivial surjection $f : \pi_{\mathscr{T}}(M) \to \pi_{\mathscr{T}}(\mathsf{P}_s)$ for some M in $\mathcal{A}$. This morphism is represented by a morphism $f' : M' \to \mathsf{P}_s/N'$ whose cokernel D belongs to $\mathcal{A}_{\mathscr{T}}$, where $M' \subset M$ and $N' \subset \mathsf{P}_s$ are subobjects such that M/M' and N' belong to $\mathcal{A}_{\mathscr{T}}$. Now D is a quotient of P_s; hence if it belongs to $\mathcal{A}_{\mathscr{T}}$ it must be 0, so that f' is surjective. Since P_s is projective, there exists a morphism $g' : \mathsf{P}_s \to M'$ such that $f' \circ g'$ is the quotient morphism $\mathsf{P}_s \twoheadrightarrow \mathsf{P}_s/N'$. Then $\pi_{\mathscr{T}}(f') \circ \pi_{\mathscr{T}}(g')$ is an isomorphism in $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$, so that $\pi_{\mathscr{T}}(f')$ is split. This is absurd, and finishes the proof of axiom (5) in the case $\mathscr{S}$ is finite.

Property (5) in the general case follows from the same property for finite weight posets using the same arguments as in the proof of Corollary 2.3. □

Proposition 3.2. *Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$ and let $\mathscr{T} \subset \mathscr{S}$ be an ideal.*

(1) The functor $\imath_{\mathscr{T}} : D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}}) \to D^{\mathrm{b}}(\mathcal{A})$ induced by the embedding $\mathcal{A}_{\mathscr{T}} \to \mathcal{A}$ is fully faithful.

(2) The quotient functor $\pi_{\mathscr{T}} : \mathcal{A} \to \mathcal{A}/\mathcal{A}_{\mathscr{T}}$ induces an equivalence of categories

$$D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}}) \xrightarrow{\sim} D^{\mathrm{b}}(\mathcal{A}/\mathcal{A}_{\mathscr{T}}),$$

where $D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ is the Verdier quotient.

(3) The functor $\imath_{\mathscr{T}}$ and the quotient functor $\Pi_{\mathscr{T}} : D^{\mathrm{b}}(\mathcal{A}) \to D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ admit (triangulated) left and right adjoints $\imath^{\mathrm{L}}_{\mathscr{T}}$, $\imath^{\mathrm{R}}_{\mathscr{T}}$ and $\Pi^{\mathrm{L}}_{\mathscr{T}}$, $\Pi^{\mathrm{R}}_{\mathscr{T}}$ respectively. Moreover, we have isomorphisms

$$\imath^{\mathrm{R}}_{\mathscr{T}} \circ \imath_{\mathscr{T}} \cong \mathrm{id}_{D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})} \cong \imath^{\mathrm{L}}_{\mathscr{T}} \circ \imath_{\mathscr{T}}$$

$$\text{and} \quad \Pi_{\mathscr{T}} \circ \Pi^{\mathrm{R}}_{\mathscr{T}} \cong \mathrm{id}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})} \cong \Pi_{\mathscr{T}} \circ \Pi^{\mathrm{L}}_{\mathscr{T}},$$

for any $s \in \mathscr{S} \smallsetminus \mathscr{T}$ we have

$$\Pi^{\mathrm{L}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(\Delta_s) \cong \Delta_s, \quad \Pi^{\mathrm{R}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(\nabla_s) \cong \nabla_s,$$
$$\imath^{\mathrm{L}}_{\mathscr{T}}(\Delta_s) = 0, \quad \imath^{\mathrm{R}}_{\mathscr{T}}(\nabla_s) = 0,$$

and for any M in $D^{\mathrm{b}}(\mathcal{A})$ there exist functorial distinguished triangles

$$\Pi^{\mathrm{L}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(M) \to M \to \imath_{\mathscr{T}} \circ \imath^{\mathrm{L}}_{\mathscr{T}}(M) \xrightarrow{[1]}$$
$$\text{and} \quad \imath_{\mathscr{T}} \circ \imath^{\mathrm{R}}_{\mathscr{T}}(M) \to M \to \Pi^{\mathrm{R}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(M) \xrightarrow{[1]}$$

where the first and second morphisms are induced by adjunction.

PROOF. This result is proved in [**AR3**, Lemma 2.2]. Here we explain the construction in more detail.

For part (1), we remark that the category $D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ is generated (as a triangulated category) by the objects $\{\Delta_t : t \in \mathscr{T}\}$ as well as by the objects $\{\nabla_t : t \in \mathscr{T}\}$. Hence to prove the claim if suffices to prove that for $s, t \in \mathscr{T}$ the morphism

$$\mathrm{Ext}^i_{\mathcal{A}_{\mathscr{T}}}(\Delta_s, \nabla_t) \to \mathrm{Ext}^i_{\mathcal{A}}(\Delta_s, \nabla_t)$$

induced by $\imath_{\mathscr{T}}$ is an isomorphism. This follows from Corollary 2.3 (applied to $\mathcal{A}$ and $\mathcal{A}_{\mathscr{T}}$).

Then we prove part (3). Consider the full triangulated subcategory $\mathcal{D}^{\nabla}_{\mathscr{S} \smallsetminus \mathscr{T}}$ of $D^{\mathrm{b}}(\mathcal{A})$ generated by the objects ∇_s with $s \in \mathscr{S} \smallsetminus \mathscr{T}$. Then for M in $D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ and N in $\mathcal{D}^{\nabla}_{\mathscr{S} \smallsetminus \mathscr{T}}$, by Corollary 2.3 we have $\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(M, N) = 0$. From this one can deduce that for any M in $D^{\mathrm{b}}(\mathcal{A})$ and N in $\mathcal{D}^{\nabla}_{\mathscr{S} \smallsetminus \mathscr{T}}$, the morphism

$$\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(M, N) \to \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Pi_{\mathscr{T}}(M), \Pi_{\mathscr{T}}(N))$$

induced by $\Pi_{\mathscr{T}}$ is an isomorphism.

Now the category $D^{\mathrm{b}}(\mathcal{A})$ is generated, as a triangulated category, by (the essential image of) $D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ and by $\mathcal{D}^{\nabla}_{\mathscr{S} \smallsetminus \mathscr{T}}$. Using the octahedral axiom, we deduce that for any M in $D^{\mathrm{b}}(\mathcal{A})$ there exists a distinguished triangle

$$M' \to M \to M'' \xrightarrow{[1]} \tag{3.2}$$

where M' belongs to $D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ and M'' belongs to $\mathcal{D}^{\nabla}_{\mathscr{S} \smallsetminus \mathscr{T}}$. Moreover, [**BBD**, Proposition 1.1.9] implies that this triangle is unique and functorial.

These facts show that the restriction of $\Pi_{\mathscr{T}}$ to $\mathcal{D}^{\nabla}_{\mathscr{S} \smallsetminus \mathscr{T}}$ is an equivalence, and that if we define $\Pi^{\mathrm{R}}_{\mathscr{T}} : D^{\mathrm{b}}(\mathcal{A}) \to D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ as the composition of the inverse equivalence with the embedding $\mathcal{D}^{\nabla}_{\mathscr{S} \smallsetminus \mathscr{T}} \to D^{\mathrm{b}}(\mathcal{A})$, then $\Pi^{\mathrm{R}}_{\mathscr{T}}$ is right adjoint to $\Pi_{\mathscr{T}}$. (In more concrete terms, $\Pi^{\mathrm{R}}_{\mathscr{T}}$ sends an object M to the object M'' in (3.2).)

Finally we define the functor $\imath^{\mathrm{R}}_{\mathscr{T}}$ as the functor sending an object M to the object M' in (3.2). Again, it is easily checked that this functor is right adjoint to $\imath_{\mathscr{T}}$. The isomorphisms $\imath^{\mathrm{R}}_{\mathscr{T}} \circ \imath_{\mathscr{T}} \cong \mathrm{id}_{D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}$, $\Pi_{\mathscr{T}} \circ \Pi^{\mathrm{R}}_{\mathscr{T}} \cong \mathrm{id}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}$, $\Pi^{\mathrm{R}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(\nabla_s) \cong \nabla_s$ and $\imath^{\mathrm{R}}_{\mathscr{T}}(\nabla_s) = 0$, and the existence of the functorial triangles $\imath_{\mathscr{T}} \circ \imath^{\mathrm{R}}_{\mathscr{T}}(M) \to M \to \Pi^{\mathrm{R}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(M) \xrightarrow{[1]}$, are clear from the construction of $\Pi^{\mathrm{R}}_{\mathscr{T}}$ and $\imath^{\mathrm{R}}_{\mathscr{T}}$.

The construction of the functors $\Pi^{\mathrm{L}}_{\mathscr{T}}$ and $\imath^{\mathrm{L}}_{\mathscr{T}}$ is completely similar, using the full triangulated subcategory $\mathcal{D}^{\Delta}_{\mathscr{S} \smallsetminus \mathscr{T}}$ generated by the objects Δ_s with $s \in \mathscr{S} \smallsetminus \mathscr{T}$ instead of $\mathcal{D}^{\nabla}_{\mathscr{S} \smallsetminus \mathscr{T}}$.

Finally we prove part (2). The universal property of the Verdier quotient guarantees the existence of a natural functor $D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}}) \to D^{\mathrm{b}}(\mathcal{A}/\mathcal{A}_{\mathscr{T}})$, and what we have to prove is that this functor is an equivalence of categories. Both $D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})$ and $D^{\mathrm{b}}(\mathcal{A}/\mathcal{A}_{\mathscr{T}})$ are generated, as triangulated categories, by the images of the objects Δ_s with $s \in \mathscr{S} \smallsetminus \mathscr{T}$, as well as by the images of the objects ∇_s with $s \in \mathscr{S} \smallsetminus \mathscr{T}$. Hence what we have to prove is that for any $s, t \in \mathscr{S} \smallsetminus \mathscr{T}$ the induced morphism

$$\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Pi_{\mathscr{T}}(\Delta_s), \Pi_{\mathscr{T}}(\nabla_t)[i]) \to \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A}/\mathcal{A}_{\mathscr{T}})}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(\nabla_t)[i])$$

is an isomorphism. However we have

$$\begin{aligned}\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Pi_{\mathscr{T}}(\Delta_s), \Pi_{\mathscr{T}}(\nabla_t)[i]) &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Delta_s, \Pi_{\mathscr{T}}^{\mathrm{R}} \circ \Pi_{\mathscr{T}}(\nabla_t)[i]) \\ &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Delta_s, \nabla_t[i]),\end{aligned}$$

and then the claim follows from Corollary 2.3 applied to the highest weight categories $\mathcal{A}$ and $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$, see Lemma 3.1. □

Remark 3.3. Assume that $\mathscr{S}$ is finite; in this case we can consider the indecomposable projective objects $(\mathsf{P}_s : s \in \mathscr{S})$, see Theorem 2.1. Remark 2.2 implies that if $s \in \mathscr{S}$, then $(\mathsf{P}_s : \Delta_t) = 0$ for any $t \in \mathscr{T}$. Proposition 3.2(3) therefore implies that for any $s \in \mathscr{S}$ the natural morphism

$$\Pi_{\mathscr{T}}^{\mathrm{L}} \circ \Pi_{\mathscr{T}}(\mathsf{P}_s) \to \mathsf{P}_s$$

is an isomorphism, which using adjunction implies that for any $M \in \mathcal{A}$ the functor $\Pi_{\mathscr{T}}$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\mathsf{P}_s, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\Pi_{\mathscr{T}}(\mathsf{P}_s), \Pi_{\mathscr{T}}(M)).$$

Since any exact sequence in $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$ is the image under $\Pi_{\mathscr{T}}$ of an exact sequence in $\mathcal{A}$ (see [Ga, Corollaire 1 on p. 368]) this shows that $\Pi_{\mathscr{T}}(\mathsf{P}_s)$ is projective. This property also implies that $\Pi_{\mathscr{T}}(\mathsf{P}_s)$ is indecomposable; it is therefore the projective cover of $\Pi_{\mathscr{T}}(\mathsf{L}_s)$.

4. Criterion for the existence of standard and costandard filtrations

4.1. Costandard filtrations.

Proposition 4.1. *Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$, and let M be in $\mathcal{A}$. Then the following conditions are equivalent:*

(1) M admits a ∇-filtration;
(2) for any $s \in \mathscr{S}$ and $i \in \mathbb{Z}_{>0}$, we have $\mathrm{Ext}^i_{\mathcal{A}}(\Delta_s, M) = \{0\}$;
(3) for any $s \in \mathscr{S}$, we have $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_s, M) = \{0\}$.

Remark 4.2. It follows in particular from Proposition 4.1 that a direct summand of an object which admits a ∇-filtration also admits a ∇-filtration.

Proof. The fact that (1) ⇒ (2) follows from Corollary 2.3, and the implication (2) ⇒ (3) is clear. It remains to prove that (3) ⇒ (1). For this we can assume that $\mathscr{S}$ is finite, and argue by induction on $\#\mathscr{S}$, the case $\#\mathscr{S} = 1$ being obvious.

Assume that $\#\mathscr{S} > 1$, let $t \in \mathscr{S}$ be a minimal element, and let $\mathscr{T} = \{t\}$. Let M be an object in $\mathcal{A}$ such that $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_s, M) = 0$ for all $s \in \mathscr{S}$. Then for any $s \in \mathscr{S} \smallsetminus \mathscr{T}$, using Proposition 3.2 we see that

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(M)) &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Pi_{\mathscr{T}}(\Delta_s), \Pi_{\mathscr{T}}(M)[1]) \\ &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Pi_{\mathscr{T}}^{\mathrm{L}} \circ \Pi_{\mathscr{T}}(\Delta_s), M[1]) \cong \mathrm{Ext}^1_{\mathcal{A}}(\Delta_s, M) = \{0\}.\end{aligned}$$

Hence, by induction, $\pi_{\mathscr{T}}(M)$ admits a ∇-filtration in the highest weight category $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$. Using again Proposition 3.2, it follows that $\Pi_{\mathscr{T}}^{\mathrm{R}} \circ \Pi_{\mathscr{T}}(M)$ belongs to $\mathcal{A}$, and admits a ∇-filtration.

Consider now the distinguished triangle

$$\imath_{\mathscr{T}} \circ \imath_{\mathscr{T}}^{\mathrm{R}}(M) \to M \to \Pi_{\mathscr{T}}^{\mathrm{R}} \circ \Pi_{\mathscr{T}}(M) \xrightarrow{[1]} \tag{4.1}$$

provided once again by Proposition 3.2. Since the second and third terms belong to $\mathcal{A}$, the first term can have nonzero cohomology objects only in degrees 0 and 1. Moreover, we have

$$\begin{aligned}\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Delta_t, \imath_{\mathscr{T}} \circ \imath_{\mathscr{T}}^{\mathrm{R}}(M)[1]) &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\imath_{\mathscr{T}}^{\mathrm{L}} \circ \imath_{\mathscr{T}}(\Delta_t), \imath_{\mathscr{T}}^{\mathrm{R}}(M)[1]) \\ &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Delta_t, \imath_{\mathscr{T}}^{\mathrm{R}}(M)[1]) \cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Delta_t, M[1]),\end{aligned}$$

hence

$$\mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Delta_t, \imath_{\mathscr{T}} \circ \imath_{\mathscr{T}}^{\mathrm{R}}(M)[1]) = \{0\}. \tag{4.2}$$

We claim that $\imath_{\mathscr{T}} \circ \imath_{\mathscr{T}}^{\mathrm{R}}(M)$ belongs to $\mathcal{A}$. Indeed, consider the truncation distinguished triangle

$$H^0(\imath_{\mathscr{T}} \circ \imath_{\mathscr{T}}^{\mathrm{R}}(M)) \to \imath_{\mathscr{T}} \circ \imath_{\mathscr{T}}^{\mathrm{R}}(M) \to H^1(\imath_{\mathscr{T}} \circ \imath_{\mathscr{T}}^{\mathrm{R}}(M))[-1] \xrightarrow{[1]} .$$

Since the category $\mathcal{A}_{\mathscr{T}}$ is semisimple, this triangle is split. Hence if $H^1(\imath_{\mathscr{T}} \circ \imath_{\mathscr{T}}^{\mathrm{R}}(M))$ were nonzero there would exist a nonzero morphism $\Delta_t[-1] \to \imath_{\mathscr{T}} \circ \imath_{\mathscr{T}}^{\mathrm{R}}(M)$, which would contradict (4.2).

Finally, since the functor $\imath_{\mathscr{T}}$ is exact and does not kill any object (since it is fully-faithful), we deduce that $\imath_{\mathscr{T}}^{\mathrm{R}}(M)$ belongs to $\mathcal{A}_{\mathscr{T}}$, hence that $\imath_{\mathscr{T}} \circ \imath_{\mathscr{T}}^{\mathrm{R}}(M)$ is a direct sum of copies of ∇_t. Then the distinguished triangle (4.1) is an exact sequence in $\mathcal{A}$, and shows that M admits a ∇-filtration. □

4.2. Standard filtrations. Applying Proposition 4.1 to the opposite category $\mathcal{A}^{\mathrm{op}}$, we obtain the following "dual" statement.

Proposition 4.3. *Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$, and let M be in $\mathcal{A}$. Then the following conditions are equivalent:*

(1) M admits a Δ-filtration;
(2) for any $s \in \mathscr{S}$ and $i \in \mathbb{Z}_{>0}$, we have $\mathrm{Ext}^i_{\mathcal{A}}(M, \nabla_s) = \{0\}$;
(3) for any $s \in \mathscr{S}$, we have $\mathrm{Ext}^1_{\mathcal{A}}(M, \nabla_s) = \{0\}$.

5. Tilting objects

In this section we fix a highest weight category $\mathcal{A}$ with weight poset $(\mathscr{S}, \leq)$.

5.1. Definition. The following definition (stated in the different, but closely related, language of quasi-hereditary algebras) is due to Ringel [Rin].

Definition 5.1. An object M in $\mathcal{A}$ is said to be *tilting* if admits both a Δ-filtration and a ∇-filtration.

It follows from Remark 4.2 that any direct summand of a tilting object is tilting. Since $\mathcal{A}$ is Krull–Schmidt, this implies that any tilting objects is a direct sum of indecomposable tilting objects.

5.2. Classification. The main result of this section is the following theorem, which provides a classification of the indecomposable tilting objects.

THEOREM 5.2. *For any $s \in \mathscr{S}$, there exists (up to isomorphism) a unique indecomposable tilting object T_s such that*

$$[\mathsf{T}_s : \mathsf{L}_s] = 1 \quad \text{and} \quad [\mathsf{T}_s : \mathsf{L}_t] \neq 0 \;\Rightarrow\; t \leq s. \tag{5.1}$$

Moreover there exists an embedding $\Delta_s \hookrightarrow \mathsf{T}_s$ whose cokernel admits a Δ-filtration, and a surjection $\mathsf{T}_s \twoheadrightarrow \nabla_s$ whose kernel admits a ∇-filtration. Finally, any indecomposable tilting object is isomorphic to T_s for a unique $s \in \mathscr{S}$.

Our proof is inspired by the proof of [S4, Proposition 3.1] (where the author considers a much more general setting). We begin with the following preliminary result.

LEMMA 5.3. *For any $s \in \mathscr{S}$, there exists a tilting object T endowed with an embedding $\Delta_s \hookrightarrow T$ whose cokernel admits a Δ-filtration with subquotients Δ_t with $t < s$.*

PROOF. We proceed by induction on $\#\{t \in \mathscr{S} \mid t \leq s\}$. If s is minimal then we can take $T = \Delta_s = \nabla_s$. Otherwise, consider some minimal $t \in \mathscr{S}$ with $t < s$. We set $\mathscr{T} = \{t\}$. By induction, we have an object M in $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$ with the desired properties, and we consider $M' := \Pi^{\mathrm{L}}_{\mathscr{T}}(M)$. Using Proposition 3.2, we see that there exists an embedding from $\Delta_s = \Pi^{\mathrm{L}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(\Delta_s)$ to M', whose cokernel admits a Δ-filtration with subquotients Δ_u with $u < s$. Moreover, for any $u \neq t$ we have

$$\begin{aligned}
\mathrm{Ext}^1_{\mathcal{A}}(\Delta_u, M') &\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})}(\Pi^{\mathrm{L}}_{\mathscr{T}} \circ \Pi_{\mathscr{T}}(\Delta_u), M'[1]) \\
&\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Pi_{\mathscr{T}}(\Delta_u), \Pi_{\mathscr{T}}(M')[1]) \\
&\cong \mathrm{Hom}_{D^{\mathrm{b}}(\mathcal{A})/D^{\mathrm{b}}(\mathcal{A}_{\mathscr{T}})}(\Pi_{\mathscr{T}}(\Delta_u), M[1]) = \{0\}.
\end{aligned}$$

Now, let $E := \mathrm{Ext}^1_{\mathcal{A}}(\Delta_t, M')$. Consider the image of id_E in

$$\mathrm{Hom}_{\Bbbk}(E, E) \cong E^* \otimes_{\Bbbk} E \cong \mathrm{Ext}^1_{\mathcal{A}}(E \otimes_{\Bbbk} \Delta_t, M').$$

This element corresponds to a short exact sequence

$$M' \hookrightarrow T \twoheadrightarrow E \otimes_{\Bbbk} \Delta_t. \tag{5.2}$$

Clearly, there exists an embedding $\Delta_s \hookrightarrow T$ whose cokernel admits a Δ-filtration with subquotients Δ_u with $u < s$. Hence to conclude our construction we only have to prove that T also admits a ∇-filtration. By Proposition 4.1, for this it suffices to prove that

$$\mathrm{Ext}^1_{\mathcal{A}}(\Delta_u, T) = \{0\}$$

for any $u \in \mathscr{S}$. If $u \neq t$, this property follows from the similar vanishing for M' proved above and the fact that $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_u, \Delta_t) = \mathrm{Ext}^1_{\mathcal{A}}(\Delta_u, \nabla_t) = \{0\}$. And to prove it for $u = t$ we consider the following part of the long exact sequence obtained by applying $\mathrm{Hom}_{\mathcal{A}}(\Delta_t, -)$ to (5.2):

$$\mathrm{Hom}_{\mathcal{A}}(\Delta_t, E \otimes_{\Bbbk} \Delta_t) \to \mathrm{Ext}^1_{\mathcal{A}}(\Delta_t, M') \to \mathrm{Ext}^1_{\mathcal{A}}(\Delta_t, T) \to \mathrm{Ext}^1_{\mathcal{A}}(\Delta_t, E \otimes_{\Bbbk} \Delta_t).$$

Here by construction the first morphism is the identity of E, and the fourth term vanishes; hence the third term vanishes also, as desired. □

Now we prove Theorem 5.2.

PROOF OF THEOREM 5.2. For any $s \in \mathscr{S}$ there exists an indecomposable tilting object T_s endowed with an embedding $\Delta_s \hookrightarrow \mathsf{T}_s$ whose cokernel admits a Δ-filtration with subquotients Δ_t with $t < s$. Indeed, Lemma 5.3 provides an object T with such properties, which is not necessarily indecomposable. But then T admits an indecomposable direct summand T_s with $(\mathsf{T}_s : \Delta_s) = 1$. The composition $\Delta_s \hookrightarrow T \twoheadrightarrow \mathsf{T}_s$ is still injective, and its cokernel still admits the required filtration, since there exists no nonzero morphism from Δ_s to any other direct summand of T.

We fix such objects (and the corresponding embeddings), and now prove that any indecomposable tilting object is isomorphic to T_s for some $s \in \mathscr{S}$. Indeed, let T be an indecomposable tilting object, and choose $t \in \mathscr{S}$ and an embedding $\Delta_t \hookrightarrow T$ whose cokernel admits a Δ-filtration. Consider the diagram

$$\begin{tikzcd} \Delta_t \arrow[r, hook] \arrow[d, equal] & T \arrow[r, two heads] \arrow[d, dashed] & \mathrm{coker} \\ \Delta_t \arrow[r, hook] & \mathsf{T}_t \arrow[r, two heads] & \mathrm{coker}'. \end{tikzcd}$$

Since coker admits a Δ-filtration and T_t is tilting, we have $\mathrm{Ext}^1_{\mathcal{A}}(\mathrm{coker}, \mathsf{T}_t) = 0$. Hence there exists a morphism $\varphi : \mathsf{T}_t \to T$ which restricts to the identity on Δ_t. Similarly, there exists $\psi : T \to \mathsf{T}_t$ which restricts to the identity on Δ_t. Then $\varphi \circ \psi$ is an element of the artinian local ring $\mathrm{End}_{\mathcal{A}}(T)$ which is not nilpotent, hence invertible by Fitting's lemma. Similarly $\psi \circ \varphi$ is invertible, hence φ and ψ are isomorphisms.

We have proved that the objects $\{\mathsf{T}_s : s \in \mathscr{S}\}$ constructed above provide representatives for all isomorphism classes of indecomposable tilting objects in $\mathcal{A}$. Among these objects, it is clear that T_s is characterized by (5.1). Hence to conclude it suffices to prove that there exists a surjection $\mathsf{T}_s \twoheadrightarrow \nabla_s$ whose kernel admits a ∇-filtration. However, Lemma 5.3 applied to $\mathcal{A}^{\mathrm{op}}$ guarantees the existence, for any $s \in \mathscr{S}$, of a tilting object T'_s with a surjection $\mathsf{T}'_s \twoheadrightarrow \nabla_s$ whose kernel admits a ∇-filtration with subquotients of the form ∇_t with $t < s$. Moreover, as above this object can be assumed to be indecomposable. This object satisfies the conditions (5.1); hence it must be isomorphic to T_s. $\square$

REMARK 5.4. The proof of Theorem 5.2 shows also that if T is an indecomposable tilting object in $\mathcal{A}$, then the first term in any Δ-filtration of T is Δ_s, where s is the (unique) maximal element of $\mathscr{S}$ such that $[T : \mathsf{L}_s] \neq 0$. In particular this first term does not depend on the chosen Δ-filtration, and characterizes T up to isomorphism.

5.3. Describing a highest weight category in terms of its tilting objects. We denote by $\mathrm{Tilt}(\mathcal{A})$ the additive full subcategory of $\mathcal{A}$ whose objects are the tilting objects. The following is an easy but very useful observation.

PROPOSITION 5.5. *The natural functor*

$$K^{\mathrm{b}}\mathrm{Tilt}(\mathcal{A}) \to D^{\mathrm{b}}(\mathcal{A})$$

is an equivalence of triangulated categories.

PROOF. The category $D^{\mathrm{b}}(\mathcal{A})$ is generated as a triangulated category by the objects Δ_s for $s \in \mathscr{S}$, hence also (using Theorem 5.2) by the tilting objects. So, to prove the proposition it suffices to prove that our functor is fully-faithful. However, this follows directly from the observation that

$$\mathrm{Ext}^i_{\mathcal{A}}(T, T') = 0 \quad \text{for all } i > 0$$

if T and T' are tilting objects, as follows from Corollary 2.3. □

5.4. ∇-sections. We now fix, for any $s \in \mathscr{S}$, an indecomposable object T_s as in Theorem 5.2. The following notion was introduced (using a slightly different terminology) in [**RW1**]. (This definition has antecedents in the literature; see e.g. [**RW1**, Remark 2.3.3].) Here, for $s \in \mathscr{S}$ we consider the ideal $\{t \in \mathscr{S} \mid t \not\geq s\}$, and the quotient category

$$\mathcal{A}^{\geq s} := \mathcal{A}/\mathcal{A}_{\{t \in \mathscr{S} \mid t \not\geq s\}}.$$

Definition 5.6. Let X be an object in $\mathcal{A}$ which admits a costandard filtration. A ∇-*section* of X is a triple $(\Pi, e, (\varphi_\pi : \pi \in \Pi))$ where

- Π is a finite set;
- $e : \Pi \to \mathscr{S}$ is a map;
- for each $\pi \in \Pi$, φ_π is an element in $\mathrm{Hom}_{\mathcal{A}}(\mathsf{T}_{e(\pi)}, X)$

such that for any $s \in \mathscr{S}$ the images of the morphisms

$$(\varphi_\pi : \mathsf{T}_s \to X : \pi \in e^{-1}(s))$$

form a basis of the $\Bbbk$-vector space $\mathrm{Hom}_{\mathcal{A}^{\geq s}}(\mathsf{T}_s, X)$.

In this definition and below, we omit the notation for the obvious quotient functor $\mathcal{A} \to \mathcal{A}^{\geq s}$. Note that, since s is minimal in $\mathscr{S} \smallsetminus \{t \in \mathscr{S} \mid t \not\geq s\}$, the images of the objects Δ_s, ∇_s, L_s and T_s in $\mathcal{A}^{\geq s}$ all coincide. In particular, for any X in $\mathcal{A}$ we have

$$\mathrm{Hom}_{\mathcal{A}^{\geq s}}(\mathsf{T}_s, X) \cong \mathrm{Hom}_{\mathcal{A}^{\geq s}}(\Delta_s, X).$$

On the other hand, by Exercise 7.6, if X admits a costandard filtration the natural morphism

$$\mathrm{Hom}_{\mathcal{A}}(\Delta_s, X) \to \mathrm{Hom}_{\mathcal{A}^{\geq s}}(\Delta_s, X)$$

is an isomorphism. Finally, after a choice of a (necessarily injective) morphism $\Delta_s \to \mathsf{T}_s$, the induced morphism

$$\mathrm{Hom}_{\mathcal{A}}(\mathsf{T}_s, X) \to \mathrm{Hom}_{\mathcal{A}}(\Delta_s, X)$$

is surjective (because the cokernel Y of the morphism $\Delta_s \to \mathsf{T}_s$ admits a standard filtration, which implies that $\mathrm{Ext}^1_{\mathcal{A}}(Y, X) = 0$). These comments show that any object which admits a costandard filtration admits a ∇-section: more specifically, a choice of such a datum is equivalent to a choice, for any $s \in \mathscr{S}$, of a family of vectors in $\mathrm{Hom}_{\mathcal{A}}(\mathsf{T}_s, X)$ whose image in $\mathrm{Hom}_{\mathcal{A}}(\Delta_s, X)$ is a basis. (This family is necessarily of cardinality $(X : \nabla_s)$, where we use the notation of Exercise 7.5.)

APPENDIX B

Exercises

1. Exercises for Chapter 1

Exercise 1.1 (Representations of SL_2). This exercise aims at proving the Steinberg tensor product formula "by hand" for the group $\mathrm{SL}_2(\Bbbk)$. The reader is supposed not to use the general theory to treat it.

Consider the ring $\Bbbk[x, y]$ of polynomials in two variables. The group $\mathrm{SL}_2(\Bbbk)$ acts on this ring by linear substitutions in the variables:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(x, y) = f(ax + by, cx + dy).$$

Let $M_n \subset \Bbbk[x, y]$ be the space of homogeneous polynomials of degree n (i.e., the span of the polynomials $x^n, x^{n-1}y, \ldots, y^n$). This space is preserved by the action of $\mathrm{SL}_2(\Bbbk)$. Let p be the characteristic of $\Bbbk$. Let L_n denote the irreducible $\mathrm{SL}_2(\Bbbk)$-representation of highest weight $n \cdot \varpi_1$.

(1) Show that if $0 \leq n < p$, then M_n is irreducible, so $L_n \cong M_n$. In particular, we have

$$\operatorname{ch} L_n = e^{-n} + e^{-n+2} + \cdots + e^n \qquad \text{if } 0 \leq n < p.$$

(2) (Frobenius twist) For any representation V, let $V^{(1)}$ be the representation on the same underlying vector space, but with a modified action of $\mathrm{SL}_2(\Bbbk)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot_{\text{new}} v = \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix} \cdot_{\text{old}} v.$$

Show that if V is irreducible, then $V^{(1)}$ is irreducible. Show that $L_n^{(1)} \cong L_{pn}$.

(3) (Steinberg tensor product theorem) Show that if $0 \leq a < p$, and if n is any nonnegative integer, then $L_a \otimes L_n^{(1)}$ is irreducible. As a consequence, $L_a \otimes L_n^{(1)} \cong L_{a+pn}$.

(4) (Character formula) Now let n be any nonnegative integer. Write down its "p-adic expansion" as

$$n = \sum_{i \geq 0} a_i p^i \qquad \text{where } 0 \leq a_i < p \text{ for each } i.$$

Then show that

$$\operatorname{ch} L_n = \prod_{i \geq 0} (\operatorname{ch} L_{a_i})|_{e \mapsto e^{p^i}}.$$

Exercise 1.2 (Root system of $\mathrm{GL}_n(\Bbbk)$). Let $\Bbbk$ be an algebraically closed field. We fix $n \geq 1$, and consider the group $\mathbf{G} = \mathrm{GL}_n(\Bbbk)$ of invertible $n \times n$-matrices.

(1) Let $\mathbf{T} \subset \mathbf{G}$ be the subgroup of diagonal matrices. Show that $\mathbf{T}$ is a maximal torus in $\mathbf{G}$.
(2) Fox $i \in \{1, \cdots, n\}$ we denote by $\varepsilon_i : \mathbf{T} \to \Bbbk^\times$ the character sending an invertible diagonal matrix to the i-th entry on its diagonal. Show that the root system of $(\mathbf{G}, \mathbf{T})$ is
$$\Phi := (\varepsilon_i - \varepsilon_j : i \neq j \in \{1, \cdots, n\}),$$
and describe the corresponding root spaces in the Lie algebra of $\mathbf{G}$.
(3) Fox $i \in \{1, \cdots, n\}$ we denote by $\varepsilon_i^\vee : \Bbbk^\times \to \mathbf{T}$ the cocharacter sending t to the diagonal matrix with i-th coefficient t, and all other (diagonal) coefficients equal to 1. Show that the coroot system of $(\mathbf{G}, \mathbf{T})$ is
$$\Phi^\vee := (\varepsilon_i^\vee - \varepsilon_j^\vee : i \neq j \in \{1, \cdots, n\}).$$
(4) Show that $\Phi_+ := \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\}$ is a positive system in Φ.
(5) Determine the basis of Φ associated with Φ_+, and write every root as a linear combination of simple roots.
(6) Determine the highest root in Φ.
(7) Determine the positive and negative Borel subgroups associated with our choice of Φ^+, and their respective unipotent radicals.
(8) Describe the standard parabolic subgroup of $\mathbf{G}$ (with respect to the negative Borel subgroup) associated with each subset of the set of simple roots.
(9) Determine the Weyl group of $(\mathbf{G}, \mathbf{T})$.
(10) For $i \in \{1, \ldots, n\}$ we set $\omega_i := \sum_{j=1}^i \varepsilon_j$. Show that the dominant weights for the choice of Φ_+ as above are the weights of the form
$$k_1\omega_1 + \cdots + k_{n-1}\omega_{n-1} + k_n\omega_n$$
with $k_1, \ldots, k_{n-1} \in \mathbb{Z}_{\geq 0}$ and $k_n \in \mathbb{Z}$.

EXERCISE 1.3 (Root system of $\mathrm{Sp}_{2n}(\Bbbk)$). Let $\Bbbk$ be an algebraically closed field. We fix $n \geq 1$, and consider the matrix
$$J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$
(of size $2n$). In this exercise we consider the group $\mathbf{G} = \mathrm{Sp}_{2n}(\Bbbk)$ of matrices $X \in \mathrm{SL}_{2n}(\Bbbk)$ which satisfy ${}^{\mathrm{t}}XJX = J$. Its Lie algebra is
$$\mathfrak{g} = \mathfrak{sp}_{2n}(\Bbbk) = \{X \in \mathfrak{sl}_{2n}(\Bbbk) \mid {}^{\mathrm{t}}XJ + JX = 0\}.$$
(1) We let $\mathbf{T} \subset \mathbf{G}$ be the subgroup of diagonal matrices. Show that
$$\mathbf{T} = \{\mathrm{diag}(t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1}) : t_1, \cdots, t_n \in \Bbbk^\times\},$$
and that $\mathbf{T}$ is a maximal torus in $\mathbf{G}$.
(2) Show that the matrices
$$\begin{aligned} &E_{i,j} - E_{n+j,n+i} && (1 \leq i, j \leq n), \\ &E_{i,n+j} + E_{j,n+i} && (1 \leq i \leq j \leq n), \\ &E_{n+j,i} + E_{n+i,j} && (1 \leq i \leq j \leq n) \end{aligned}$$
form a $\Bbbk$-basis of $\mathfrak{g}$.

(3) For $i \in \{1, \cdots, n\}$, we denote by $\varepsilon_i : \mathbf{T} \to \Bbbk^\times$ the character sending a matrix to its i-th diagonal entry. Using the preceding question, show that the root system Φ of $(\mathbf{G}, \mathbf{T})$ consists of the characters

$$\varepsilon_i - \varepsilon_j \ (i \neq j), \quad \varepsilon_i + \varepsilon_j \ (i < j), \quad -(\varepsilon_i + \varepsilon_j) \ (i < j), \quad 2\varepsilon_i, \quad -2\varepsilon_i$$

with $i, j \in \{1, \cdots, n\}$.

(4) Fox $i \in \{1, \cdots, n\}$, we denote by $\varepsilon_i^\vee : \Bbbk^\times \to \mathbf{T}$ the cocharacter sending t to the diagonal matrix with i-th coefficient t, $n+i$-th coefficient t^{-1}, and all other (diagonal) coefficients equal to 1. Show that the coroot system of $(\mathbf{G}, \mathbf{T})$ consists of the cocharacters

$$\varepsilon_i^\vee - \varepsilon_j^\vee \ (i \neq j), \quad \varepsilon_i^\vee + \varepsilon_j^\vee \ (i < j), \quad -(\varepsilon_i^\vee + \varepsilon_j^\vee) \ (i < j), \quad \varepsilon_i^\vee, \quad -\varepsilon_i^\vee$$

with $i, j \in \{1, \cdots, n\}$.

(5) For $i \in \{1, \cdots, n\}$ we set

$$\alpha_i = \begin{cases} \varepsilon_i - \varepsilon_{i+1} & \text{if } i \neq n; \\ 2\varepsilon_n & \text{if } i = n. \end{cases}$$

Show that $(\alpha_1, \cdots, \alpha_n)$ is a basis of Φ.

(6) Determine the system of positive roots associated with the basis of the preceding question, and the associated positive and negative Borel subgroups.

(7) Determine the Weyl group of $(\mathbf{G}, \mathbf{T})$.

(8) For $i \in \{1, \dots, n\}$ we set $\omega_i := \sum_{j=1}^{i} \varepsilon_j$. Show that the dominant weights for the choice of basis of Φ as above are the weights of the form

$$k_1\omega_1 + \cdots + k_n\omega_n$$

with $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$.

Exercise 1.4 (Root system of $\mathrm{SO}_{2n}(\Bbbk)$). Let $\Bbbk$ be an algebraically closed field of odd characteristic. We fix $n \geq 1$, and consider the matrix

$$J := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

(of size $2n$). In this exercise we consider the group $\mathbf{G} = \mathrm{SO}_{2n}(\Bbbk)$ of matrices $X \in \mathrm{SL}_{2n}(\Bbbk)$ which satisfy ${}^{\mathrm{t}}XJX = J$. Its Lie algebra is

$$\mathfrak{g} = \mathfrak{so}_{2n}(\Bbbk) = \{X \in \mathfrak{sl}_{2n}(\Bbbk) \mid {}^{\mathrm{t}}XJ + JX = 0\}.$$

(1) We let $\mathbf{T} \subset \mathbf{G}$ be the subgroup of diagonal matrices. Show that

$$\mathbf{T} = \{\mathrm{diag}(t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1}) : t_1, \cdots, t_n \in \Bbbk^\times\},$$

and that $\mathbf{T}$ is a maximal torus in $\mathbf{G}$.

(2) Show that the matrices

$$\begin{aligned} &E_{i,j} - E_{n+j,n+i} && (1 \leq i, j \leq n), \\ &E_{i,n+j} - E_{j,n+i} && (1 \leq i < j \leq n), \\ &E_{n+j,i} - E_{n+i,j} && (1 \leq i < j \leq n) \end{aligned}$$

form a $\Bbbk$-basis of $\mathfrak{g}$.

(3) For $i \in \{1, \cdots, n\}$, we denote by $\varepsilon_i : \mathbf{T} \to \Bbbk^\times$ the character sending a matrix to its i-th diagonal entry. Using the preceding question, show that the root system Φ of $(\mathbf{G}, \mathbf{T})$ consists of the characters

$$\varepsilon_i - \varepsilon_j \ (i \neq j), \quad \varepsilon_i + \varepsilon_j \ (i < j), \quad -(\varepsilon_i + \varepsilon_j) \ (i < j)$$

with $i, j \in \{1, \cdots, n\}$.

(4) Fox $i \in \{1, \cdots, n\}$, we denote by $\varepsilon_i^\vee : \Bbbk^\times \to \mathbf{T}$ the cocharacter sending t to the diagonal matrix with i-th coefficient t, $n+i$-th coefficient t^{-1}, and all other (diagonal) coefficients equal to 1. Show that the coroot system of $(\mathbf{G}, \mathbf{T})$ consists of the cocharacters

$$\varepsilon_i^\vee - \varepsilon_j^\vee \ (i \neq j), \quad \varepsilon_i^\vee + \varepsilon_j^\vee \ (i < j), \quad -(\varepsilon_i^\vee + \varepsilon_j^\vee) \ (i < j)$$

with $i, j \in \{1, \cdots, n\}$.

(5) For $i \in \{1, \cdots, n\}$ we set

$$\alpha_i = \begin{cases} \varepsilon_i - \varepsilon_{i+1} & \text{if } i \neq n; \\ \varepsilon_{n-1} + \varepsilon_n & \text{if } i = n. \end{cases}$$

Show that $(\alpha_1, \cdots, \alpha_n)$ is a basis of Φ.

(6) Determine the system of positive roots associated with the basis of the preceding question, and the associated positive and negative Borel subgroups.

(7) Determine the Weyl group of $(\mathbf{G}, \mathbf{T})$.

(8) For $i \in \{1, \ldots, n\}$ we set

$$\omega_i := \begin{cases} \sum_{j=1}^{i} \varepsilon_j & \text{if } i \leq n-2 \ ; \\ \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_{n-1} - \varepsilon_n) & \text{if } i = n-1 \ ; \\ \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_{n-1} + \varepsilon_n) & \text{if } i = n. \end{cases}$$

Show that the dominant weights for the choice of basis of Φ as above are the weights of the form

$$k_1\omega_1 + \cdots + k_n\omega_n$$

with $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$ and k_{n-1}, k_n of the same parity.

EXERCISE 1.5 (Root system of $\mathrm{SO}_{2n+1}(\Bbbk)$). Let $\Bbbk$ be an algebraically closed field of odd characteristic. We fix $n \geq 1$, and consider the matrix

$$J := \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(of size $2n+1$). In this exercise we consider the group $\mathbf{G} = \mathrm{SO}_{2n+1}(\Bbbk)$ of matrices $X \in \mathrm{SL}_{2n+1}(\Bbbk)$ which satisfy ${}^{\mathrm{t}}XJX = J$. Its Lie algebra is

$$\mathfrak{g} = \mathfrak{so}_{2n+1}(\Bbbk) = \{X \in \mathfrak{sl}_{2n+1}(\Bbbk) \mid {}^{\mathrm{t}}XJ + JX = 0\}.$$

(1) We let $\mathbf{T} \subset \mathbf{G}$ be the subgroup of diagonal matrices. Show that

$$\mathbf{T} = \{\mathrm{diag}(t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1}, 1) : t_1, \cdots, t_n \in \Bbbk^\times\},$$

and that $\mathbf{T}$ is a maximal torus in $\mathbf{G}$.

(2) Show that the matrices

$$\begin{aligned} E_{i,j} - E_{n+j,n+i} \quad & (1 \le i,j \le n), \\ E_{i,n+j} - E_{j,n+i} \quad & (1 \le i < j \le n), \\ E_{n+j,i} - E_{n+i,j} \quad & (1 \le i < j \le n), \\ E_{i,2n+1} - E_{2n+1,n+i} \quad & (1 \le i \le n), \\ E_{n+i,2n+1} - E_{2n+1,i} \quad & (1 \le i \le n) \end{aligned}$$

form a $\Bbbk$-basis of $\mathfrak{g}$.

(3) For $i \in \{1, \cdots, n\}$, we denote by $\varepsilon_i : \mathbf{T} \to \Bbbk^\times$ the character sending a matrix to its i-th diagonal entry. Using the preceding question, show that the root system Φ of $(\mathbf{G}, \mathbf{T})$ consists of the characters

$$\varepsilon_i - \varepsilon_j \ (i \neq j), \quad \varepsilon_i + \varepsilon_j \ (i < j), \quad -(\varepsilon_i + \varepsilon_j) \ (i < j), \quad \varepsilon_i, \quad -\varepsilon_i$$

with $i, j \in \{1, \cdots, n\}$.

(4) Fox $i \in \{1, \cdots, n\}$, we denote by $\varepsilon_i^\vee : \Bbbk^\times \to \mathbf{T}$ the cocharacter sending t to the diagonal matrix with i-th coefficient t, $n+i$-th coefficient t^{-1}, and all other (diagonal) coefficients equal to 1. Show that the coroot system of $(\mathbf{G}, \mathbf{T})$ consists of the cocharacters

$$\varepsilon_i^\vee - \varepsilon_j^\vee \ (i \neq j), \quad \varepsilon_i^\vee + \varepsilon_j^\vee \ (i < j), \quad -(\varepsilon_i^\vee + \varepsilon_j^\vee) \ (i < j), \quad 2\varepsilon_i^\vee, \quad -2\varepsilon_i^\vee$$

with $i, j \in \{1, \cdots, n\}$.

(5) For $i \in \{1, \cdots, n\}$ we set

$$\alpha_i = \begin{cases} \varepsilon_i - \varepsilon_{i+1} & \text{if } i \neq n; \\ \varepsilon_n & \text{if } i = n. \end{cases}$$

Show that $(\alpha_1, \cdots, \alpha_n)$ is a basis of Φ.

(6) Determine the system of positive roots associated with the basis of the preceding question, and the associated positive and negative Borel subgroups.

(7) Determine the Weyl group of $(\mathbf{G}, \mathbf{T})$.

(8) For $i \in \{1, \ldots, n-1\}$ we set $\omega_i := \sum_{j=1}^{i} \varepsilon_j$, and for $i = n$ we set

$$\omega_n := \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n).$$

Show that the dominant weights for the choice of basis of Φ as above are the weights of the form

$$k_1\omega_1 + \cdots + 2k_n\omega_n$$

with $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$.

Exercise 1.6 (Some induced and Weyl modules for $\mathrm{SL}_n(\Bbbk)$). In this exercise we consider the setting of Example 1.1, and denote by $V = \Bbbk^n$ the natural module for $\mathbf{G} = \mathrm{SL}_n(\Bbbk)$.

(1) For any $i \in \{1, \ldots, n-1\}$, show that the only element $\lambda \in \mathbb{X}^+$ such that $\lambda \preceq \omega_i$ is $\lambda = \omega_i$.

(2) Deduce that for $i \in \{1, \ldots, n-1\}$ we have

$$\mathrm{wt}(\mathsf{N}(\omega_i)) = \{w\omega_i : w \in W\}.$$

(Hint: use Lemma 1.11 from Chapter 1.)

(3) For any $i \in \{1, \dots, n-1\}$, show that there exists a unique (up to an invertible scalar) nonzero morphism of $\mathbf{G}$-modules
$$\bigwedge^i V \to \mathsf{N}(\omega_i),$$
and that this morphism is an isomorphism.

(4) Show that for $i \in \{1, \dots, n-1\}$ we also have
$$\mathsf{M}(\omega_i) \cong \mathsf{L}(\omega_i) \cong \bigwedge^i V.$$

EXERCISE 1.7 (Divisibility of dimensions). This exercise is taken from [M2, Lemma 10.1]. Here we assume that $p \geq h$.

(1) Show that if $\lambda \in \mathbb{X}^+$, we have $p \mid \dim(\mathsf{N}(\lambda))$ iff λ is regular. (*Hint*: use the formula from Remark 1.21.)
(2) Deduce that if $\mu \in \mathbb{X}$ is singular, then $p \mid \dim(M)$ for any module M in $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \mu}$.

EXERCISE 1.8 (Coxeter groups). We recall that if W is a group and $S \subset W$ is a subset consisting of involutions generating W, then (W, S) is a Coxeter system iff it satisfies the *exchange condition*, i.e. iff for any reduced expression $s_1 \cdots s_k$ of an element $w \in W$ and any $s \in S$ such that $\ell(sw) < \ell(w)$, there exists $i \in \{1, \cdots, k\}$ such that $sw = s_1 \cdots \widehat{s_i} \cdots s_k$. (For this, see e.g. [Mi, Theorem 4.2].)

Let W be a group and $S \subset W$ a subset consisting of involutions generating W. Let also $(D_s : s \in S)$ be a set of subsets of W such that

(1) $e \in D_s$ for any $s \in S$;
(2) $D_s \cap sD_s = \varnothing$ for any $s \in S$;
(3) if $s, s' \in S$, $w \in D_s$ and $ws' \notin D_s$, then $ws' = sw$.

Show that (W, S) is a Coxeter system and that moreover for any $s \in S$ we have $D_s = \{w \in W \mid \ell(sw) > \ell(w)\}$. (*Hint*: if $w \in W \smallsetminus D_s$ and $w = s_1 \cdots s_k$ is a reduced expression, by considering the smallest i such that $s_1 \cdots s_i \notin D_s$, show that there exists j such that $sw = s_1 \cdots \widehat{s_j} \cdots s_k$, so that in particular $\ell(sw) < \ell(w)$. Deduce that (W, S) satisfies the exchange condition.)

EXERCISE 1.9. Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system. Let $w, x, y \in \mathcal{W}$ be such that $\ell(wy) = \ell(w) + \ell(y)$ and $\ell(xy) = \ell(x) + \ell(y)$. Show that $w \leq x$ if and only if $wy \leq xy$ (for the Bruhat order). (*Hint*: Use the characterization of the Bruhat order in terms of reduced expressions and the exchange condition.)[1]

EXERCISE 1.10. Show that if $w \in {}^{\mathrm{f}}W_{\mathrm{aff}}$ and if $y \in W_{\mathrm{aff}}$ satisfies $wy < w$, then $wy \in {}^{\mathrm{f}}W_{\mathrm{aff}}$. (*Hint*: argue by induction on $\ell(y)$ and use Lemma 2.30.)

EXERCISE 1.11 (Dihedral groups). Let $n \geq 2$, and let W be the set of symmetries of the regular n-gon in $\mathbb{R}^2$.

(1) If $S \subset W$ consists of two reflections whose axes differ by an angle of $\frac{\pi}{n}$, show that (W, S) is a Coxeter system.
(2) Write all the elements of W as products of simple reflections, and determine the longest element in W.
(3) If n is twice an odd integer and $S = \{s, t\}$, show that $(W, \{s, w_0 t, w_0\})$ is also a Coxeter system.[2]

[1] This statement can be found in [AR5, Lemma 2.1].

[2] In particular, this example shows that given a group W, there might exist essentially different subsets $S \subset W$ such that the pair (W, S) is a Coxeter system.

(4) Show that for any $w \in W$ we have

$$\underline{H}_w = \sum_{x \leq w} v^{\ell(w)-\ell(x)} H_x.$$

(For this question, in case of difficulties the reader might consult [H3, §7.12].)

(5) Decompose, for any $s \in S$ and $w \in W$, the element $\underline{H}_s \cdot \underline{H}_w$ in the Kazhdan–Lusztig basis.

(6) Show that

$$\varepsilon(\underbrace{b_s \cdot b_t \cdot b_s \cdots}_{2n \text{ terms}}) \in 1 + v^2\mathbb{Z}[v^2].$$

Exercise 1.12 (Coxeter groups of type $\mathbf{A}$). Let $W = \mathfrak{S}_n$ be the symmetric group on n letters.

(1) Let $S = \{s_1, \ldots, s_{n-1}\}$, where s_i is the permutation that swaps i and $i+1$. Show that (W, S) is a Coxeter system. (*Hint*: Use the criterion from Exercise 1.8 above with $D_{s_i} = \{w \in W \mid w^{-1}(i) < w^{-1}(i+1)\}$.)

(2) Choose some permutations, and write down reduced expressions for them.

(3) What is the longest element in W?

Exercise 1.13 (Coxeter groups of type $\mathbf{B}$). We continue with the notation $W = \mathfrak{S}_n$, $s_i = (i, i+1)$.

(1) Let W act on $(\mathbb{Z}/2\mathbb{Z})^n$ by permuting the coordinates, and let $W' = W \ltimes (Z/2\mathbb{Z})^n$. Show that W' can be identified with the "group of permutations with sign changes," i.e., the group of bijections $\sigma : \{\pm 1, \ldots, \pm n\} \to \{\pm 1, \ldots, \pm n\}$ such that $\sigma(-i) = -\sigma(i)$.

(2) Let s_0 be the element $(-1, 1, \ldots, 1) \in (\mathbb{Z}/2\mathbb{Z})^n$, regarded as an element of W', and let $S' = \{s_0, \ldots, s_{n-1}\}$. Show that (W', S') is a Coxeter systrem. (This group is the Weyl group of SO_{2n+1} or of Sp_{2n}.) (*Hint*: Use the criterion from Exercise 1.8 above with $D_{s_i} = \{w \in W' \mid w^{-1}(i) < w^{-1}(i+1)\}$, where by convention $w(0) = 0$ for any $w \in W'$.)

(3) What is the longest element in W'?

Exercise 1.14 (Conjugacy of simple reflections in W_{ext}). Show that if $\mathbf{G}$ has simply-connected derived subgroup, every element in S_{aff} is conjugate *in* W_{ext} to an element in S. (*Hint*: you might check the case when $\mathbf{G}$ is quasi-simple and simply connected by case-by-case considerations, then deduce the case when $\mathbf{G}$ is semisimple and simply connected, and finally the general case using restriction to the derived subgroup. In case of difficulties, consult [BR2, Lemma 3.1].)

Exercise 1.15 (Elements of length 0 in W_{ext}). In this exercise we consider the group Ω of Remark 2.25.

(1) Show that the composition

$$W_{\mathrm{ext}} \to \mathbb{X} \to \mathbb{X}/\mathbb{Z}\mathfrak{R}$$

restricts to an isomorphism $\Omega \xrightarrow{\sim} \mathbb{X}/\mathbb{Z}\mathfrak{R}$.

(2) Show that if $\mathbb{X}/\mathbb{Z}\mathfrak{R}$ has no p-torsion, then the action of Ω on $\mathbb{X}$ (via $\cdot_p$) is free.

EXERCISE 1.16 (Kazhdan–Lusztig element associated with the longest element). Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system such that $\mathcal{W}$ is finite, and let $w_\circ$ be the longest element in $\mathcal{W}$. We set

$$R = \sum_{y \in W} v^{\ell(w_\circ)-\ell(y)} \cdot H_y.$$

(1) Show that

$$\{h \in \mathcal{H}_{(W,\mathcal{S})} \mid \forall s \in S,\ h \cdot \underline{H}_s = (v + v^{-1}) \cdot h\} = \mathbb{Z}[v, v^{-1}] \cdot R.$$

(2) Deduce that R is stable under the Kazhdan–Lusztig involution, and then that $R = \underline{H}_{w_\circ}$.

EXERCISE 1.17 (Longest representatives and the Kazhdan–Lusztig basis). Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system. For $h \in \mathcal{H}_{(\mathcal{W},\mathcal{S})}$ we denote by $a_w(h)$ the coefficient of H in the basis $(H_w : w \in \mathcal{W})$, so that $h = \sum_w a_w(h) \cdot H_w$.

(1) Show that if $s \in \mathcal{S}$ and $w \in \mathcal{W}$ we have

$$\underline{H}_s \cdot H_w = \begin{cases} H_{sw} + vH_w & \text{if } sw > w; \\ H_{sw} + v^{-1}H_w & \text{if } sw < w. \end{cases}$$

(2) For $s \in \mathcal{S}$ we set

$${}^s\mathcal{H}_{(\mathcal{W},\mathcal{S})} := \{h \in \mathcal{H}_{(\mathcal{W},\mathcal{S})} \mid \underline{H}_s \cdot h = (v + v^{-1})h\}.$$

Show that for $h \in \mathcal{H}_{(\mathcal{W},\mathcal{S})}$ we have $h \in {}^s\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ iff for any $y \in \mathcal{W}$ such that $sy > y$ we have $a_y(h) = v \cdot a_{sy}(h)$.

(3) Deduce that ${}^s\mathcal{H}_{(\mathcal{W},\mathcal{S})} = \underline{H}_s \cdot \mathcal{H}_{(\mathcal{W},\mathcal{S})}$.

(4) Our goal now is to show that

$${}^s\mathcal{H}_{(\mathcal{W},\mathcal{S})} = \bigoplus_{\substack{w \in \mathcal{W} \\ sw < w}} \mathbb{Z}[v, v^{-1}] \cdot \underline{H}_w.$$

(a) Show that in the formula (4.6), if $p_y(0) \neq 0$ then $y \leq w$ and $sy < y$.

(b) Show that for any $y \in \mathcal{W}$ such that $sy < y$ we have $\underline{H}_y \in {}^s\mathcal{H}_{(\mathcal{W},\mathcal{S})}$. (*Hint*: proceed by induction).

(c) Conclude.

(5) If $I \subset \mathcal{S}$ is a finitary subset (i.e. a subset such that $\mathcal{W}_I$ is finite) we set

$${}^I\mathcal{H}_{(\mathcal{W},\mathcal{S})} = \bigcap_{s \in I} {}^s\mathcal{H}_{(\mathcal{W},\mathcal{S})}.$$

We also denote by ${}^I\mathcal{W} \subset \mathcal{W}$ the subset of longest right coset representatives; in other words, ${}^I\mathcal{W} = \{w \in \mathcal{W} \mid \forall x \in \mathcal{W}_I,\ xw < w\}$.

(a) Show that for $h \in \mathcal{H}_{(\mathcal{W},\mathcal{S})}$ we have $h \in {}^I\mathcal{H}_{(\mathcal{W},\mathcal{S})}$ iff for any $y \in {}^I\mathcal{W}$ and $x \in \mathcal{W}_I$ we have $a_{xy}(h) = v^{\ell(x)} \cdot a_y(h)$.

(b) Show that

$${}^I\mathcal{H}_{(\mathcal{W},\mathcal{S})} = \bigoplus_{w \in {}^I\mathcal{W}} \mathbb{Z}[v, v^{-1}] \cdot \underline{H}_w.$$

(c) Show (without using Exercise 1.16) that we have

$$\underline{H}_{w_I} = \sum_{x \in \mathcal{W}_I} v^{\ell(x)} \cdot H_{xw_I}.$$

(d) Show that ${}^I\mathcal{H}_{(\mathcal{W},\mathcal{S})} = \underline{H}_{w_I} \cdot \mathcal{H}_{(\mathcal{W},\mathcal{S})}$.

EXERCISE 1.18 (Alcoves). For the indecomposable root systems of rank 2 (i.e. of type A_2, B_2 or G_2):

(1) draw the roots in the plane $\mathbb{R}^2$, and the corresponding hyperplanes;
(2) choose a basis (α, β), and write ρ in terms of α and β;
(3) draw the corresponding decomposition of $\mathbb{X} \otimes_{\mathbb{Z}} \mathbb{R}$ into facets (in the spirit of Examples 2.21–2.22);
(4) determine $\overline{C} \cap \mathbb{X}$ (depending on p);
(5) determine the facets contained in $\overline{C}$, and which of them contain elements of $\mathbb{X}$ (the answer will depend on p).

EXERCISE 1.19 (Translation functors and quotient functors). Let $\lambda, \mu \in \overline{C}$, and assume that μ belongs to the closure of the facet containing λ.

(1) Denote by $\overline{\mathsf{Rep}}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}$ the quotient of $\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda}$ by the Serre subcategory generated by the simple objects $\mathsf{L}(w \cdot_p \lambda)$ where $w \in W_{\mathrm{aff}}$ is such that $w \cdot_p \lambda \in \mathbb{X}^+$ and $w \cdot_p \mu$ does not belong to the upper closure of the facet of $w \cdot_p \lambda$. Show that there exists a unique functor
$$\overline{T}_\lambda^\mu : \overline{\mathsf{Rep}}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda} \to \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \mu}$$
such that the composition
$$\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda} \to \overline{\mathsf{Rep}}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \lambda} \xrightarrow{\overline{T}_\lambda^\mu} \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p \mu}$$
(where the first arrow is the canonical quotient functor) is T_λ^μ.
(2) Show that $\overline{T}_\lambda^\mu$ is exact, and that it does not kill any object.
(3) Deduce that $\overline{T}_\lambda^\mu$ is faithful.
(4) We now restrict to the case $\mathbf{G} = \mathrm{SL}_2$, $\lambda = 0$ and $\mu = p-1$. Our goal in this question is to show that in this case the functor $\overline{T}_\lambda^\mu$ is *not* a equivalence of categories.
 (a) Show that the object $T_{p-1}^0\big(\mathsf{L}(p-1)\big)$ is indecomposable of length 3, with socle and top $\mathsf{L}(0)$, and the "middle" composition factor being $\mathsf{L}(2p-2)$. (*Hint*: use Proposition 2.26(3).)
 (b) Deduce that
$$\dim \mathrm{End}_{\mathsf{Rep}(\mathbf{G})}\big(T_{p-1}^0(\mathsf{L}(p-1))\big) = 2,$$
 and that the morphism
$$\mathrm{End}_{\mathsf{Rep}(\mathbf{G})}\big(T_{p-1}^0(\mathsf{L}(p-1))\big) \to \mathrm{End}_{\overline{\mathsf{Rep}}(\mathbf{G})_{W_{\mathrm{aff}} \cdot_p 0}}\big(T_{p-1}^0(\mathsf{L}(p-1))\big)$$
 induced by the quotient functor is an isomorphism.
 (c) Show that
$$T_0^{p-1} T_{p-1}^0(\mathsf{L}(p-1)) \cong \mathsf{L}(p-1)^{\oplus 2},$$
 and deduce that
$$\dim \mathrm{End}_{\mathsf{Rep}(\mathbf{G})}\big(T_0^{p-1} T_{p-1}^0(\mathsf{L}(p-1))\big) = 4.$$
 (d) Conclude

EXERCISE 1.20 (A weak form of the strong linkage principal[3]). Consider a $\Bbbk$-algebraic group $\mathbf{H}$, and the action of $\mathbf{H}$ on itself by left multiplication. Let $\mathfrak{h}$ be

[3]The argument in the exercise is adapted from the proof of [GS, Lemma 4].

the Lie algebra of $\mathbf{H}$. Recall that for any open subvariety $V \subset \mathbf{H}$ the space $\mathscr{O}(V)$ has a canonical structure of $\mathcal{U}\mathfrak{h}$-module such that the restriction morphism

$$\mathscr{O}(\mathbf{H}) \to \mathscr{O}(V)$$

is $\mathcal{U}\mathfrak{h}$-equivariant (where $\mathcal{U}\mathfrak{h}$ acts on $\mathscr{O}(\mathbf{H})$ via the differential of the $\mathbf{H}$-action). For instance, if V is the principal open subvariety defined by an element $f \in \mathscr{O}(\mathbf{H})$, then $\mathscr{O}(V) = \mathscr{O}(\mathbf{H})[\frac{1}{f}]$; the action of $\mathcal{U}\mathfrak{h}$ on $\mathscr{O}(\mathbf{H})$ is by derivations, hence it extends naturally to an action on $\mathscr{O}(V)$ by derivations.

(1) Let $\mathbf{K} \subset \mathbf{H}$ be a subgroup. Show that for any $M \in \mathsf{Rep}^\infty(\mathbf{K})$ and for any open subvariety $V \subset \mathbf{H}/\mathbf{K}$, the space $\Gamma(V, \mathscr{L}_{\mathbf{H}/\mathbf{K}}(M))$ admits a natural structure of $\mathcal{U}\mathfrak{h}$-module.

(2) Fix an open affine cover $\mathfrak{U}$ of the noetherian separated scheme $\mathbf{H}/\mathbf{K}$, and recall the associated Čech cohomology groups $\check{\mathsf{H}}^i(\mathfrak{U}, \mathscr{F})$, see [Ha, §III.4]. Show that for any $M \in \mathsf{Rep}^\infty(\mathbf{K})$ and $i \geq 0$ the space $\check{\mathsf{H}}^i(\mathfrak{U}, \mathscr{L}_{\mathbf{H}/\mathbf{K}}(M))$ has a canonical structure of $\mathcal{U}\mathfrak{h}$-module, and that we have a canonical isomorphism of $\mathcal{U}\mathfrak{h}$-modules

$$R^i \operatorname{Ind}_{\mathbf{K}}^{\mathbf{H}}(M) \cong \check{\mathsf{H}}^i(\mathfrak{U}, \mathscr{L}_{\mathbf{H}/\mathbf{K}}(M)).$$

(3) Show that if the center $Z(\mathbf{H})$ of $\mathbf{H}$ is contained in $\mathbf{K}$, then each space $\check{\mathsf{H}}^i(\mathfrak{U}, \mathscr{L}_{\mathbf{H}/\mathbf{K}}(M))$ also has a canonical structure of $Z(\mathbf{H})$-module, and that the isomorphism in (2) is also $Z(\mathbf{H})$-equivariant.

(4) Now we assume that $\mathbf{H} = \mathbf{G}$ and $\mathbf{K} = \mathbf{B}$. Show that for any $\lambda \in \mathbb{X}$ and any open subvariety $V \subset \mathbf{G}/\mathbf{B}$, the central subalgebra $(\mathcal{U}\mathfrak{g})^{\mathbf{G}}$ acts on $\Gamma(V, \mathscr{L}_\lambda)$ via the character defined by the image of the differential of λ in $\mathfrak{t}^*/(W, \bullet)$. (*Hint*: use a variant of [DM, II, §6, Corollaire 1.5].)

(5) Assume that the conditions considered in §2.5 are satisfied. Show that if $\lambda \in \mathbb{X}$ and $i \in \mathbb{Z}$, any composition factor of $R^i \operatorname{Ind}_{\mathbf{B}}^{\mathbf{G}}(\Bbbk_{\mathbf{B}}(\lambda))$ is of the form $\mathsf{L}(\nu)$ with $\nu \in W_{\mathrm{aff}} \cdot_p \lambda$.

EXERCISE 1.21 (Computation of simple characters for SL_4). This exercise will use the following result: for each $\lambda \in \mathbf{X}^+$, the Weyl module $\mathsf{M}(\lambda)$ of highest weight λ admits a finite decreasing filtration

$$\mathsf{M}(\lambda) = \mathsf{M}(\lambda)^0 \supset \mathsf{M}(\lambda)^1 \supset \mathsf{M}(\lambda)^2 \supset \cdots$$

such that

- we have
$$\sum_{i>0} \mathrm{ch}\big(\mathsf{M}(\lambda)^i\big) = \sum_{\alpha \in R^+} \sum_{0<mp<\langle \lambda+\rho, \alpha^\vee\rangle} \nu_p(mp) \cdot \chi(s_{\alpha,mp} \cdot_p \lambda)$$
where ν_p is the p-adic valuation,
$$\chi(\mu) = \frac{\sum_{w \in W_{\mathrm{f}}} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\sum_{w \in W_{\mathrm{f}}} (-1)^{\ell(w)} e^{w(\rho)-\rho}}$$
and $s_{\alpha,mp}(\mu) = \mu - (\langle \mu, \alpha^\vee \rangle - mp)\alpha$;
- $\mathsf{M}(\lambda)/\mathsf{M}(\lambda)^1 = \mathsf{L}(\lambda)$.

(This filtration is called *Jantzen's filtration*. For the construction of this filtration, and the proof of its properties, see [J3, Proposition II.8.19].)

In this exercise we assume that $\mathbf{G} = \mathrm{SL}_4(\Bbbk)$, and that $p \geq 5$. We denote by ϖ_1, ϖ_2, ϖ_3 the 3 fundamental weights, numbered in the obvious way. We will write (r, s, t) for the weight $r\varpi_1 + s\varpi_2 + t\varpi_3$.

(1) Show that the restricted dominant weights in the dot-orbit of 0 are:

$$\lambda_0 = 0, \quad \lambda_1 = (p-3, 0, p-3), \quad \lambda_2 = (p-2, 1, p-4),$$
$$\lambda_3 = (p-4, 1, p-2), \quad \lambda_4 = (p-3, 2, p-3), \quad \lambda_5 = (p-2, 2, p-2).$$

(2) Show that $\mathrm{ch}\big(\mathsf{L}(\lambda_0)\big) = \chi(\lambda_0)$.

(3) Using Jantzen's filtration, show that

$$\mathrm{ch}\big(\mathsf{L}(\lambda_1)\big) = \chi(\lambda_1) - \chi(\lambda_0), \quad \mathrm{ch}\big(\mathsf{L}(\lambda_2)\big) = \chi(\lambda_2) - \chi(\lambda_1) + \chi(\lambda_0),$$
$$\mathrm{ch}\big(\mathsf{L}(\lambda_3)\big) = \chi(\lambda_3) - \chi(\lambda_1) + \chi(\lambda_0)$$

(4) Using the fact that $[\mathsf{M}(\lambda_4) : \mathsf{L}(\lambda_1)] = [\mathsf{M}(\lambda_3) : \mathsf{L}(\lambda_1)]$ (which is a special case of [**J3**, Proposition II.7.18]) and Jantzen's filtration, show that

$$\mathrm{ch}\big(\mathsf{L}(\lambda_4)\big) = \chi(\lambda_4) - \chi(\lambda_3) - \chi(\lambda_2) + \chi(\lambda_1) - 2\chi(\lambda_0).$$

(5) Using the fact that $[\mathsf{M}(\lambda_5) : \mathsf{L}(\lambda_3)] = [\mathsf{M}(\lambda_4) : \mathsf{L}(\lambda_3)]$ and that $[\mathsf{M}(\lambda_5) : \mathsf{L}(\lambda_2)] = [\mathsf{M}(\lambda_4) : \mathsf{L}(\lambda_2)]$, prove that

$$\mathrm{ch}\big(\mathsf{L}(\lambda_5)\big) = \chi(\lambda_5) - \chi(\lambda_4) - \chi(\lambda_0') - \chi(\lambda_0'') + \chi(\lambda_3) + \chi(\lambda_2) - 2\chi(\lambda_1) + 3\chi(\lambda_0),$$

where

$$\lambda_0' = (p, 0, p-4), \quad \lambda_0'' = (p-4, 0, p).$$

(*Hint*: Start by computing $\mathrm{ch}\big(\mathsf{L}(\lambda_0')\big)$ and $\mathrm{ch}\big(\mathsf{L}(\lambda_0'')\big)$.)

(6) Check that these computations agree with Lusztig's conjecture.

(In case of difficulties with this exercise, see [**J3**, §II.8.20].)

EXERCISE 1.22 (Characters of simple modules in rank 2). Using the method of Exercise 1.21 above, for the groups of type B_2 and G_2, and assuming that $p \geq h$, compute the character of each simple module whose highest weight is restricted and in the dot-orbit of 0, and compare with Lusztig's formula.

EXERCISE 1.23 (Lusztig's conjecture for SL_2). We consider the case $\mathbf{G} = \mathrm{SL}_2(\Bbbk)$, and assume that p is odd.

(1) Show that

$$(W_{\mathrm{aff}} \cdot_p 0) \cap \mathbb{X}^+ = \{2jp\varpi_1 : j \in \mathbb{Z}_{\geq 0}\} \cup \{(2jp-2)\varpi_1 : \mathbb{Z}_{\geq 1}\}.$$

(2) Show that Lusztig's conjecture says in this case that for $j \in \{0, \cdots, \frac{p-1}{2}\}$ we have

$$[\mathsf{L}(2jp)] = \sum_{i=1}^{j} \big([\mathsf{N}(2ip)] - [\mathsf{N}(2ip-2)]\big) + [\mathsf{N}(0)]$$

and that for $j \in \{1, \cdots, \frac{p-1}{2}\}$ we have

$$\mathsf{L}(2jp-2) = [\mathsf{N}(2jp-2)] - \left(\sum_{i=1}^{j-1} \big([\mathsf{N}(2ip)] - [\mathsf{N}(2ip-2)]\big)\right) - [\mathsf{N}(0)].$$

(Here the reader might want to use Exercise 1.11.)

(3) Show that these formulas indeed hold.

(4) Show that
$$\dim(\mathsf{L}(p^2+p-2)) = 2p-2$$
and that
$$\dim(\mathsf{N}(p^2+p-2)) - \left(\sum_{i=1}^{\frac{p-1}{2}} \big(\dim(\mathsf{N}(2ip)) - \dim(\mathsf{N}(2ip-2))\big)\right) - \dim(\mathsf{N}(0))$$
is equal to p^2-1. Deduce that the formula (4.8) from Chapter 1 does not hold when $w = t_{(p+1)\varpi_1}s$ where $s \in W$ is the unique simple reflection. (In this case the condition (4.7) is not satisfied, so that there is no contradiction here!)

2. Exercises for Chapter 2

The book [EMTW] countains a large collection of exercises on the subject of this chapter. Our advice to readers willing to understand this material better is to try solving (part of) them.

EXERCISE 2.1. Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system, and let V be a reflection faithful representation of $\mathcal{W}$, with defining morphism $\varrho : \mathcal{W} \to \mathrm{End}(V)$. For any $t \in \mathcal{T}$, we denote by V^{-t} the eigenspace of the action of t for the eigenvalue -1. The goal of this exercise is to show (following [S7, Bemerkung 1.6]) that for $t, t' \in \mathcal{T}$ we have
$$V^{-t} = V^{-t'} \quad \Leftrightarrow \quad t = t'.$$
Of course the implication "$\Leftarrow$" is obvious. We therefore fix $t, t' \in \mathcal{T}$ such that $V^{-t} = V^{-t'}$.

(1) Show that tt' acts trivially on V/V^{-t}.
(2) Deduce that $\ker(\rho(tt') - \mathrm{id})$ contains a hyperplane. (*Hint*: use that the kernel of a matrix and of its transpose have the same dimension.)
(3) Show that $tt' \notin \mathcal{T}$. (*Hint*: consider the determinant.)
(4) Deduce that tt' acts trivially on V, and conclude.
(5) Similarly, for $t \in \mathcal{T}$ we denote by $V^t \subset V$ the subspace of vectors fixed by t. Show that
$$V^t = V^{t'} \quad \Leftrightarrow \quad t = t'.$$

EXERCISE 2.2. [4] Let (W, S) be a Coxeter system, and let (V, ρ) be a reflection-faithful representation of (W, S) over an infinite field k of caracteristic $\neq 2$. We denote by R the symmetric algebra of V and, for $s \in S$, by B_s the associated Soergel bimodule.

We fix $s \in S$.

(1) Show that for any graded R-bimodule M there exists an isomorphism of graded R-bimodules
$$B_s \otimes_R M \cong R \otimes_{R^s} M(1).$$
and an isomorphism of left R-modules
$$B_s \otimes_R M \cong M(-1) \oplus M(1).$$

[4]The proof in this exercise is taken from [Li1].

(2) The goal of this question is to construct construire, for any graded R-bimodules M and N, a natural isomorphism
$$\mathrm{Hom}(B_s \otimes_R M, N) \cong \mathrm{Hom}(M, B_s \otimes_R N)$$
where the Hom spaces are spaces of graded R-bimodules. (In other words, we will show that the functor $M \mapsto B_s \otimes_R M$ is self-adjoint.)
  (a) Show that the map
  $$F : \mathrm{Hom}(B_s \otimes_R M, N) \to \mathrm{Hom}(M, B_s \otimes_R N)$$
  given by
  $$F(f)(m) = v_s \otimes f(1 \otimes m) + 1 \otimes f(1 \otimes v_s m)$$
  is well defined.
  (b) Show that if $g : M \to B_s \otimes_R N$is a morphism of graded R-bimodules, there exist unique morphisms of graded (R^s, R)-bimodules $g_1 : M \to N(1)$ and $g_2 : M \to N(-1)$ such that for any $m \in M$ we have
  $$g(m) = 1 \otimes g_1(m) + v_s \otimes g_2(m)$$
  where we use the identification of (1).
  (c) With the notation of the previous section, show that the map
  $$G : \mathrm{Hom}(M, B_s \otimes_R N) \to \mathrm{Hom}(B_s \otimes_R M, N)$$
  sending a morphism $g : M \to B_s \otimes_R N$ to the morphism
  $$B_s \otimes_R M = R \otimes_{R^s} M(1) \to N$$
  given for $r \in R$ and $m \in M$ by $G(g)(r \otimes m) = r \cdot g_2(m)$ is well defined.
  (d) Show that $G \circ F = \mathrm{id}$.
  (e) Show that $F \circ G = \mathrm{id}$.
  (f) Conclude.

EXERCISE 2.3. We fix a Coxeter system $(\mathcal{W}, \mathcal{S})$, and set $\mathcal{T} = \{xsx^{-1} : x \in \mathcal{W},\ s \in \mathcal{S}\}$. We also fix a reflection faithful representation (V, ρ) of $(\mathcal{W}, \mathcal{S})$ over an infinite field of caracteristic $\neq 2$, of (finite) dimension $n \geq 1$.

(1) Show that for $w \in W$ the following properties are equivalent:
  (i) $w \in \mathcal{T}$;
  (ii) the endomorphism $\rho(w)$ of V is diagonalizable, of eigenvalues 1 with multiplicity $n-1$, and -1 with multiplicity 1.
  (iii) the endomorphism ${}^{\mathrm{t}}\rho(w)$ of V^* is diagonalizable, of eigenvalues 1 with multiplicity $n-1$, and -1 with multiplicity 1.
(2) Let $s \in S$, and fix an eigenvector ξ_s of $\rho(s)$ associated with the eigenvalue -1.
  (a) Show that, for $r \in R$, the exists a unique element $\partial_s(r) \in R^s$ such that $r - \xi_s \cdot \partial_s(r) \in R^s$.
  (b) Show that the map ∂_s considered in the previous question defines a morphism of graded R^s-modules $R \to R^s(-2)$, which restricts to a linear form on $V^* \subset R$.
  (c) Show that for $\xi \in V^*$ we have $\rho(s)(\xi) = \xi - \partial_s(\xi) \cdot \xi_s$.
  (d) Show that there exist morphisms of graded R-bimodules
  $$f_s : R \to B_s(1), \quad g_s : B_s \to R(1),$$
  $$h_s : B_s \to B_s \otimes_R B_s(-1), \quad i_s : B_s \otimes_R B_s \to B_s(-1)$$

which satisfy

$$f_s(1) = \xi_s \otimes 1 + 1 \otimes \xi_s, \quad g_s(r \otimes r') = rr',$$
$$h_s(1 \otimes 1) = 1 \otimes 1 \otimes 1, \quad i_s(r \otimes r' \otimes r'') = (r\partial_s(r')) \otimes r''$$

pour $r, r', r'' \in R$, where we used the natural identification $B_s \otimes_R B_s = R \otimes_{R^s} R \otimes_{R^s} R(2)$.

(3) We fix now $s, t \in S$ such that $s \neq t$ and st has finite order m. We also choose vectors ξ_s and ξ_t as in question (2) (for s and t), and we consider the associated maps ∂_s and ∂_t.
 (a) Show that there exists up to scalar a unique nonzero morphism of graded R-bimodules

$$\varphi_{s,t} : \underbrace{B_s \otimes_R B_t \otimes_R \cdots}_{m \text{ terms}} \to \underbrace{B_t \otimes_R B_s \otimes_R \cdots}_{m \text{ terms}}.$$

 (*Hint*: use Exercise 1.11 and Remark 1.17(6).)
 (b) Show that the restriction of $\varphi_{s,t}$ to the components of degree $-m$ is an isomorphism. (*Hint*: use the fact that the component of the indecomposable bimodule $\mathsf{B}^{\mathrm{bim}}_{w_{s,t}}$ in degree $-m$ has dimension 1, where $w_{s,t}$ is the longest element in the subgroup of W generated by s and t; see (1.13).)
 (c) In this question we assume that $m = 2$.
  (i) Show that $B_s \otimes_R B_t$ is indecomposable. (*Hint*: first show that R is generated by R^s and R^t, and then that $B_s \otimes_R B_t$ is generated as a bimodule by its component of degree -2.)
  (ii) Deduce that $\varphi_{s,t}$ is an isomorphism.
 (d) In this question we assume that $m \geq 3$.
  (i) Show that $\partial_t(v_s) \neq 0$. (*Hint* : consider the endomorphism $\rho(st)$.)
  (ii) Show that $\mathsf{B}^{\mathrm{bim}}_s$ is a direct summand in $\mathsf{B}^{\mathrm{bim}}_s \otimes_R \mathsf{B}^{\mathrm{bim}}_t \otimes_R \mathsf{B}^{\mathrm{bim}}_s$. (*Hint*: use the morphisms of question (2d).)
  (iii) Deduce that $\varphi_{s,t}$ is neither injective not surjective.

EXERCISE 2.4. Prove the inequality mentioned in Remark 1.24, and show that this inequality can be strict. (*Hint*: consider the case of type $\mathbf{A}_2$.)

EXERCISE 2.5. Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system, and let V be a reflection faithful representation of $\mathcal{W}$. Show that if w is the longest element in a finite parabolic subgroup of $\mathcal{W}$ we have $\varepsilon(\underline{H}_w) = [\mathsf{B}^{\mathrm{bim}}_w]$. (*Hint*: use Exercise 1.16.)

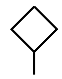
EXERCISE 2.6. Check that the Kazhdan–Lusztig conjecture as stated in [**Ac**, Remark 7.3.10] or [**HTT**] is indeed equivalent to the formula (1.22).

EXERCISE 2.7. Let A be a commutative ring and $\varphi : \mathbb{Z}[x, y] \to A$ be a morphism. Show that if $\varphi([2]_x[2]_y) = 4$, then for any $n \geq 0$ we have

$$\varphi([2n]_x) = \varphi([2]_x) \cdot n, \quad \varphi([2n]_y) = \varphi([2]_y) \cdot n, \quad \varphi([2n+1]) = 2n + 1.$$

EXERCISE 2.8. Prove Lemma 2.12. (*Hint*: check that the formulas in the lemma produce morphisms which are killed by composition with (2.7) or (2.8).)

EXERCISE 2.9. Show that, under the assumption that the other 1-color relations hold, the needle relation (relation (8) in §2.5) is equivalent to the relation

 $= 0.$

(*Hint*: add a trivalent vertex under the diagram.)

EXERCISE 2.10. Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system, let $\Bbbk$ be a complete local domain, and let $(V, (\alpha_s : s \in \mathcal{S}), (\alpha_s^\vee : s \in \mathcal{S}))$ be a realization satisfying the technical conditions of §2.4.

(1) Consider the functor ι of Lemma 2.20. Show that, under the isomorphism ch_{D} (see Corollary 2.24), the induced automorphism of $[\mathsf{D}(V, \mathcal{W})]_{\oplus}$ identifies with the Kazhdan–Lusztig involution (see §4.2 in Chapter 1).
(2) Show that for any $w \in \mathcal{W}$ we have satisfies $\iota(\mathsf{B}_w) \cong \mathsf{B}_w$.
(3) Deduce that for any $w \in \mathcal{W}$, the element $\mathrm{ch}_{\mathsf{D}}(\mathsf{B}_w)$ is fixed by the Kazhdan–Lusztig involution, and that the integers in (2.14) satisfy $b_{y,n}^{w} = b_{y,-n}^{w}$.
(4) In the setting of Remark 2.33, show that $a_{y,w,n} = a_{y,w,-n}$ for any $n \in \mathbb{Z}$.

EXERCISE 2.11. Let A be a generalized Cartan matrix and $(\mathcal{W}, \mathcal{S})$ the associated Coxeter system.

(1) Consider the polynomials $({}^p a_{y,w})_{y<w\in\mathcal{W}}$ of Corollary 2.43. Show that if y, w are such that ${}^p a_{y,w} \neq 0$, and if $s \in \mathcal{S}$ is such that $sw < w$, then $sy < y$. (*Hint*: use (2.16) and Exercise 1.17(4).)
(2) Deduce that if $\mathcal{W}$ is finite and w_0 is its longest element we have ${}^p\underline{H}_{w_0} = \underline{H}_{w_0}$ for any p.

EXERCISE 2.12. (1) Show that for any p and for any expression $\underline{w}$, the coefficients of the expansion of the element $\underline{H}_{\underline{w}}$ in the p-canonical basis are Laurent polynomials with nonnegative coefficients, which are moreover invariant under the replacement of v by v^{-1}.
(2) Deduce that if $w \in \mathcal{W}$ admits a reduced expression $\underline{w}$ such that $\underline{H}_{\underline{w}} = \underline{H}_w$, then ${}^p\underline{H}_w = \underline{H}_w$ for any p.
(3) Show that if $w \in \mathcal{W}$ satisfies $\ell(w) \leq 2$, then ${}^p\underline{H}_w = \underline{H}_w$ for any p.

EXERCISE 2.13. In the case of Cartan realizations, write down explicitly the Jones–Wenzl relations (see (12) in §2.5).

EXERCISE 2.14. Check the assertions of §§2.3.3–2.3.4 regarding Cartan realizations.

EXERCISE 2.15. Let V, $\mathcal{W}$ be as in §2.11.3, and assume that $\Bbbk$ is a field. Recall the category $\overline{\mathsf{D}}_{\mathrm{BS}}(V, \mathcal{W})$ defined in this subsection. Let also $I \subset \mathcal{S}$ be a subset. We define $\overline{\mathsf{D}}_{\mathrm{BS}}^{I}(V, \mathcal{W})$ as the category with objects in bijection with expressions (via $\underline{w} \mapsto \overline{\mathsf{B}}_{\underline{w}}^{I}$), and with morphisms from $\overline{\mathsf{B}}_{\underline{w}}^{I}$ to $\overline{\mathsf{B}}_{\underline{w}'}^{I}$ by by the quotient of $\mathrm{Hom}_{\overline{\mathsf{D}}_{\mathrm{BS}}(V,\mathcal{W})}(\overline{\mathsf{B}}_{\underline{w}}, \overline{\mathsf{B}}_{\underline{w}'})$ by the subspace spanned by morphisms which factor through an object $\overline{\mathsf{B}}_{\underline{y}}$ where $\underline{y}$ is an expression starting by an element of I. Let also $\overline{\mathsf{D}}^{I}(V, \mathcal{W})$ be the Karoubian envelope of the additive hull of $\overline{\mathsf{D}}_{\mathrm{BS}}^{I}(V, \mathcal{W})$.

(1) Show that there exists a canonical full functor $p_I : \overline{\mathsf{D}}(V, \mathcal{W}) \to \overline{\mathsf{D}}^{I}(V, \mathcal{W})$.
(2) Show that if $w \in \mathcal{W} \smallsetminus {}^I\mathcal{W}$, then the image of $\overline{\mathsf{B}}_w$ under p_I vanishes.

(3) Show that if $w \in {}^I\mathcal{W}$, then the image of $\overline{\mathsf{B}}_w$ under p_I is a nonzero indecomposable object.
(4) Show that the assignment $(w,n) \mapsto p_I(\overline{\mathsf{B}}_w)(n)$ induces a bijection between ${}^I\mathcal{W} \times \mathbb{Z}$ and the set of isomorphism classes of indecomposable objects in $\overline{\mathsf{D}}^I(V,\mathcal{W})$.
(5) Recall the antispherical module $\mathcal{N}^I_{(\mathcal{W},\mathcal{S})}$ from §4.1 in Chapter 4. Show that there exists a canonical isomorphism

$$[\overline{\mathsf{D}}^I(V,\mathcal{W})]_\oplus \cong \mathcal{N}^I_{(\mathcal{W},\mathcal{S})}.$$

The category $\overline{\mathsf{D}}^I(V,\mathcal{W})$ is an incarnation of the *antispherical category* associated to I. For more on this category, see [**RW1**, §§4.4–4.5] and [**LW**].

Exercise 2.16. Let $(\mathcal{W},\mathcal{S})$ and V be as in Section 3. Show that if the $\mathcal{W}$-action on V is faithful, the functor of Remark 3.2 is fully faithful. Deduce analogues of the results of §1.4 in this setting.

3. Exercises for Chapter 3

Parity sheaves for parabolic stratifications.
Parity sheaves and pullback.
Affine Schubert varieties for SL2 are rationally smooth.

Exercise 3.1. Let $\mathscr{G}$ be a complex semisimple algebraic group with a choice of Borel subgroup $\mathscr{B}$ and maximal torus $\mathscr{T} \subset \mathscr{B}$. Let W be the associated Weyl group. Show that there exists a t-exact auto-equivalence of $D^{\mathrm{b}}_{(B)}(\mathscr{X},\Bbbk)$ sending the simple perverse sheaf, resp. standard perverse sheaf, resp. costandard perverse sheaf, resp. normalized indecomposable parity complex, labelled by w to the similar object labelled by w_0ww_0. (*Hint*: use an automorphism of $\mathscr{G}$ exchanging $\mathscr{B}$ with the opposite Borel subgroup; see [**J3**, Proof of Corollary II.1.16].)

4. Exercises for Chapter 4

Exercise 4.1. Show without using Theorem 2.2 or Steinberg's tensor product theorem that if $V \in \mathsf{Rep}(\mathbf{G})$ is semisimple, then $V_{|\mathbf{G}_1}$ is a semisimple $\mathbf{G}_1$-module. (*Hint*: use that the socle of V as a $\mathbf{G}_1$-module is $\mathbf{G}$-stable.)

Exercise 4.2. Deduce from Theorem 1.2 that if M, N are objects of $\mathsf{Rep}(\mathbf{G})$ which admit a costandard (resp. standard) filtration, then so does $M \otimes N$. (*Hint* : use Exercise 7.4 below.)

In case of difficulties, the reader might consult [**JMW3**, §5].

Exercise 4.3 (Tilting modules for $\mathrm{SL}_n(\Bbbk)$). In this exercise we consider the setting of Example 1.1, and denote by $V = \Bbbk^n$ the natural module for $\mathbf{G} = \mathrm{SL}_n(\Bbbk)$.

(1) Show that for any $i \in \{1, \cdots, n-1\}$ we have

$$\mathsf{T}(\omega_i) = \bigwedge^i V.$$

(Hint: use Exercise 1.6.)
(2) Deduce that each indecomposable tilting module appears as a direct summand of a module of the form

$$V^{\otimes k_1} \otimes \left(\bigwedge^2 V\right)^{\otimes k_2} \otimes \cdots \otimes \left(\bigwedge^{n-1} V\right)^{\otimes k_{n-1}}$$

for some $k_1, \ldots, k_{n-1} \in \mathbb{Z}_{\geq 0}$. (Hint: use Theorem 1.2.)

(3) Show that if $p \geq \lfloor n/2 \rfloor$, then the tilting modules for $\mathbf{G}$ are exactly the direct sums of direct summands of tensor powers of V.

EXERCISE 4.4 (Tilting tensor product theorem for $\mathrm{SL}_n(\Bbbk)$). This exercise will use the property that if $\lambda \in \mathbb{X}$ satisfies $\langle \lambda, \alpha^\vee \rangle = -1$ for some $\alpha \in \mathfrak{R}^{\mathrm{s}}$, then $R^i \operatorname{Ind}_{\mathbf{B}}^{\mathbf{G}}(\lambda) = 0$ for any $i \geq 0$; see [**J3**, Proposition II.5.4(a)]. Our goal in this exercise is to prove Theorem 1.2 by elementary methods in the special case $\mathbf{G} = \mathrm{SL}_n(\Bbbk)$, assuming $p \geq \lfloor n/2 \rfloor$. (Only the last question will use the assumption on p.)

(1) Let $V = \Bbbk^n$ by the natural representation of $\mathbf{G}$. Show that for any $\lambda \in \mathbb{X}^+$, the module $V \otimes \mathsf{N}(\lambda)$ admits a costandard filtration. (*Hint*: use the tensor identity and Kempf's vanishing theorem.)
(2) Deduce that for any $n \geq 0$, the $\mathbf{G}$-module $V^{\otimes n}$ is tilting.
(3) Conclude. (*Hint*: a look at Exercise 4.3 might help.)

EXERCISE 4.5 (Tilting modules for products of groups). (1) Show that if $\mathbf{G}_1$ and $\mathbf{G}_2$ are connected reductive groups, then the indecomposable tilting $\mathbf{G}_1 \times \mathbf{G}_2$-modules are exactly the modules $V_1 \otimes V_2$ where V_1, resp. V_2, is an indecomposable tilting $\mathbf{G}_1$-module, resp. $\mathbf{G}_2$-module.
(2) In case $\mathbf{G}_1 = \mathbf{G}_2 (= \mathbf{G})$, deduce that for any tilting $\mathbf{G} \times \mathbf{G}$-module M, the restriction of M to the diagonal copy of $\mathbf{G}$ is tilting. (Hint: use Theorem 1.2.)

EXERCISE 4.6 (Restriction of tilting modules to subgroups). This exercise is taken from [**Br1**, Proposition 3.3].

(1) Let $(\mathbf{G}, \mathbf{H})$ be one the pairs

$$(\mathrm{SL}_{2n}(\Bbbk), \mathrm{Sp}_{2n}(\Bbbk)), \quad (\mathrm{SL}_{2n}(\Bbbk), \mathrm{SO}_{2n}(\Bbbk)), \quad (\mathrm{SL}_{2n+1}(\Bbbk), \mathrm{SO}_{2n+1}(\Bbbk))$$

(for some $n \geq 1$). Show that for any tilting $\mathbf{G}$-module M, the restriction $M_{|\mathbf{H}}$ is tilting. (Hint: use Exercise 4.3 and the examples in §1.6 of Chapter 4.)
(2) Consider $V = \Bbbk^{2n}$ with its standard basis $(e_1, \cdots, e_{2n})$. Fix $m < n$, and write

$$V = V_1 \oplus V_2$$

where

$$V_1 = \mathrm{span}(e_1, \cdots, e_m, e_{n+1}, \cdots, e_{n+m}),$$
$$V_2 = \mathrm{span}(e_{m+1}, \cdots, e_n, e_{n+m+1}, \cdots, e_{2n}).$$

Identifying V_1 and V_2 with the spaces $\Bbbk^{2m}$ and $\Bbbk^{2(n-m)}$ with the standard alternating form as in Exercise 1.3, this decomposition provides embeddings

$$\mathrm{Sp}_{2m}(\Bbbk) \times \mathrm{Sp}_{2(n-m)}(\Bbbk) \subset \mathrm{Sp}_{2n}(\Bbbk) \subset \mathrm{SL}_{2n}(\Bbbk)$$

and

$$\mathrm{Sp}_{2m}(\Bbbk) \times \mathrm{SL}_{2(n-m)}(\Bbbk) \subset \mathrm{SL}_{2m}(\Bbbk) \times \mathrm{SL}_{2(n-m)}(\Bbbk) \subset \mathrm{SL}_{2n}(\Bbbk).$$

(a) Show that for any tilting $\mathrm{SL}_{2n}(\Bbbk)$-module M, the restriction

$$M_{|\mathrm{SL}_{2m}(\Bbbk) \times \mathrm{SL}_{2(n-m)}(\Bbbk)}$$

is tilting.

(b) Deduce that for any tilting $\mathrm{Sp}_{2n}(\Bbbk)$-module M, the restriction

$$M_{|\mathrm{Sp}_{2m}(\Bbbk)\times\mathrm{Sp}_{2(n-m)}(\Bbbk)}$$

is tilting. (Hint: use (1), Exercise 7.3 below and the examples in §1.6.2 of Chapter 4.)

(3) Assume that $p \neq 2$. Let V be a $\Bbbk$-vector space endowed with a nondegenerate symmetric bilinear form, and consider the associated special orthogonal group $\mathrm{SO}(V)$. Consider an orthogonal decomposition

$$V = V_1 \oplus V_2,$$

and the corresponding embedding of groups

$$\mathrm{SO}(V_1) \times \mathrm{SO}(V_2) \subset \mathrm{SO}(V).$$

Show that for any tilting $\mathrm{SO}(V)$-module M, the restriction

$$M_{|\mathrm{SO}(V_1)\times\mathrm{SO}(V_2)}$$

is tilting. (Hint: use the same strategy as in (2).)

Exercise 4.7 (Characters of baby Verma modules). Show that for any $\lambda \in \mathbb{X}$ we have

$$\mathrm{ch}(\widehat{\mathsf{Z}}(\lambda)) = e^{\lambda} \cdot \prod_{\alpha\in\mathfrak{R}^+} \frac{1-e^{-p\alpha}}{1-e^{-\alpha}}.$$

Exercise 4.8 ($\mathbf{G}_1\mathbf{T}$-modules for $\mathrm{SL}_2(\Bbbk)$). In this exercise we assume that $\mathbf{G} = \mathrm{SL}_2(\Bbbk)$.

(1) Let $n \in \mathbb{Z}$, and let $r \in \{0, \cdots, p-1\}$ be the residue of n modulo p. Show that if $r = p-1$ then $\widehat{\mathsf{Z}}(n\varpi_1)$ is simple, and that otherwise there exists a nonsplit short exact sequence

$$\widehat{\mathsf{L}}((n-2r-2)\varpi_1) \hookrightarrow \widehat{\mathsf{Z}}(n\varpi_1) \twoheadrightarrow \widehat{\mathsf{L}}(n\varpi_1).$$

(2) Let $n \in \mathbb{Z}$, and let $r \in \{0, \cdots, p-1\}$ be the residue of n modulo p. Show that if $r = p-1$ then we have $\widehat{\mathsf{Q}}(n\varpi_1) = \widehat{\mathsf{Z}}(n\varpi_1)$, and that otherwise there exists a nonsplit short exact sequence

$$\widehat{\mathsf{Z}}((n+2(p-r)-2)\varpi_1) \hookrightarrow \widehat{\mathsf{Q}}(n\varpi_1) \twoheadrightarrow \widehat{\mathsf{Z}}(n\varpi_1).$$

Exercise 4.9. This exercise is taken from [RW2, Lemma 5.6]. We assume that $\mathscr{D}\mathbf{G}$ is simply connected and $p \geq h$, and fix $\varsigma \in \mathbb{X}$ such that $\langle \varsigma, \alpha^\vee\rangle = 1$ for any $\alpha \in \mathfrak{R}^{\mathrm{s}}$. We will use the fact that for any $\lambda, \mu \in \mathbb{X}$ there exists an exact functor

$$\widehat{T}_\lambda^\mu : \mathsf{Rep}(\mathbf{G}_1\mathbf{T}) \to \mathsf{Rep}(\mathbf{G}_1\mathbf{T})$$

such that the diagram

$$\begin{tikzcd}
\mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}}\cdot_p\lambda} \arrow[r, "\widehat{T}_\lambda^\mu"] \arrow[d] & \mathsf{Rep}(\mathbf{G})_{W_{\mathrm{aff}}\cdot_p\mu} \arrow[d] \\
\mathsf{Rep}(\mathbf{G}_1\mathbf{T}) \arrow[r, "T_\lambda^\mu"] & \mathsf{Rep}(\mathbf{G}_1\mathbf{T})
\end{tikzcd}$$

commutes, and that for any $\lambda, \mu \in \mathbb{X}$ the functor $\widehat{T}_\lambda^\mu$ is both left and right adjoint to $\widehat{T}_\mu^\lambda$, see [J3, §9.22].

(1) Show that the $\mathbf{G}_1\mathbf{T}$-module $\widehat{T}_{(p-1)\varsigma}^\varsigma \widehat{\mathsf{Z}}((p-1)\varsigma)$ is injective.

(2) Show that the socle of this module is $\widehat{\mathsf{L}}(p\varsigma - 2\rho)$.
(3) Deduce that $\widehat{T}^{\varsigma}_{(p-1)\varsigma}\widehat{\mathsf{Z}}((p-1)\varsigma) \cong \widehat{\mathsf{Q}}(p\varsigma - 2\rho)$.
(4) Show that $\mathsf{T}(p\varsigma)_{|\mathbf{G}_1\mathbf{T}} \cong \widehat{\mathsf{Q}}(p\varsigma - 2\rho)$.

EXERCISE 4.10 (Tilting characters and Kazhdan–Lusztig combinatorics for $\mathrm{SL}_2(\Bbbk)$). In this exercise we assume that $\mathbf{G} = \mathrm{SL}_2(\Bbbk)$. Recall that in this case W_{aff} is the infinite dihedral group; the unique element in S will be denoted s, and the unique element in $S_{\mathrm{aff}} \smallsetminus S$ will be denoted s_0.

(1) Show that for $n \geq 1$ we have
$$\underline{H}_{(s_0 s)^n} = H_{(s_0 s)^n} + vH_{(s_0 s)^{n-1}s_0} + v\underline{H}_{s(s_0 s)^{n-1}},$$
and deduce that
$$\underline{N}_{(s_0 s)^n} = N_{(s_0 s)^n} + vN_{(s_0 s)^{n-1}s_0}.$$
(*Hint*: use Exercise 1.11.)
(2) Show that for $n \geq 1$ we have
$$\underline{H}_{(s_0 s)^n s_0} = H_{(s_0 s)^n s_0} + vH_{(s_0 s)^n} + v\underline{H}_{(s s_0)^n},$$
and deduce that
$$\underline{N}_{(s_0 s)^n s_0} = N_{(s_0 s)^n s_0} + vN_{(s_0 s)^n}.$$
(*Hint*: use Exercise 1.11.)
(3) Check Andersen's conjecture (Conjecture 4.2) in this case using the formulas above and Proposition 3.7.
(4) Show that $\mathsf{T}((p+1)p\varpi_1)$ has 4 nonzero costandard objects in any of its costandard filtrations, of highest weights $(p^2+p)\varpi_1$, $(p^2+p-2)\varpi_1$, $(p^2-p)\varpi_1$ and $(p^2-p-2)\varpi_1$.
(5) Show that the formula in Conjecture 4.2 does *not* hold for the weight $(p+1)p\varpi_1$. (This weight does not satisfy the assumption in this conjecture, so that there is no contradiction here.)

EXERCISE 4.11 (Dimensions). (1) Show that for any injective $\mathbf{G}_1$-module M, $\dim(M)$ is divisible by $p^{\#\mathfrak{R}^+}$. (*Hint*: use Proposition 2.9.)
(2) Let us assume that $\mathscr{D}\mathbf{G}$ is simply connected, and fix $\varsigma \in \mathbb{X}$ such that $\langle \varsigma, \alpha^\vee \rangle = 1$ for any $\alpha \in \mathfrak{R}^{\mathrm{s}}$. Show that for any $\mu \in (p-1)\varsigma + \mathbb{X}^+$, $\dim(\mathsf{T}(\mu))$ is divisible by $p^{\#\mathfrak{R}^+}$.

5. Exercises for Chapter 5

6. Exercises for Chapter 6

EXERCISE 6.1. Show that if $p \geq h$, Conjecture 2.1 implies Conjecture 2.3. (*Hint*: use Corollary 1.7 in Chapter 4.)

EXERCISE 6.2. Show that if Conjecture 2.1 is true for one choice of $\lambda \in C$, then it is true for *any* choice of such a weight.

EXERCISE 6.3. (1) Show that in Conjecture 1.3 one can equivalently require that there exists a *left* action of D^{BS} on $\mathsf{Rep}_0(\mathbf{G})$.
(2) Show that in Conjecture 1.3 one can equivalently require that there exists a right action of D^{BS} on the subcategory of tilting objects in $\mathsf{Rep}_0(\mathbf{G})$. (*Hint*: use Proposition 5.5 in Appendix A.)

EXERCISE 6.4. Consider the case $\mathbf{G} = \mathrm{SL}_2$ with $p = 3$, and identify $\mathbb{X}$ with $\mathbb{Z}$ in the natural way. Denote by s the unique element in S, and by s_0 the other element in S_{aff}. Show that

$$s_0 s \cdot_3 0 = 6, \quad s_0 s s_0 s \cdot_3 0 = 12,$$

that

$$(\mathsf{T}(12) : \mathsf{N}(6)) = 1,$$

and that

$$n_{s_0 s, s_0 s s_0 s}(v) = 0, \quad {}^3 n_{s_0 s, s_0 s s_0 s}(v) = 1.$$

(For the second case, use the computation in §2.14.2 of Chapter 2.) Discuss this example in light of Andersen's conjecture and Conjecture 2.1.

EXERCISE 6.5. Identify $[\mathsf{Rep}_0(\mathbf{G})]$ with $\mathcal{N}^0_{\mathrm{aff}}$ as in §2.4. Consider the bilinear pairing $\langle -, - \rangle$ which satisfies

$$\langle N^0_x, N^0_y \rangle = \delta_{x,y}$$

for $x, y \in {}^{\mathrm{f}}W_{\mathrm{aff}}$. Show that for $M, N \in \mathsf{Rep}_0(\mathbf{G})$ tilting we have

$$\dim_{\Bbbk} \mathrm{Hom}(M, N) = \langle [M], [N] \rangle.$$

(*Hint*: use Exercise 7.5.)

7. Exercises for Appendix A

EXERCISE 7.1 (Finiteness of Ext^1-spaces in highest weight categories). Show that if $\mathcal{A}$ is a highest weight category, then for any $M, N \in \mathcal{A}$ the vector space $\mathrm{Ext}^1_{\mathcal{A}}(M, N)$ is finite-dimensional. (*Hint*: Reduce to the case M, N are simple, and then use the standard/costandard objects associated with these simple modules.)

EXERCISE 7.2 (Projective objects in highest weight categories). We consider a highest weight category $\mathcal{A}$ with weight poset $\mathscr{S}$. The goal of this exercise is to prove that if $\mathscr{S}$ is finite then $\mathcal{A}$ has enough projective objects, and moreover that these projective objects admit a standard filtration. The proof proceeds by induction on the cardinality of $\mathscr{S}$; so we assume the result is known for highest weight categories whose weight poset is strictly smaller than $\mathscr{S}$. We fix $s \in \mathscr{S}$ maximal, and set $\mathscr{T} := \mathscr{S} \smallsetminus \{s\}$. Then by induction we know the result for the category $\mathcal{A}_{\mathscr{T}}$.

(1) Show that Δ_s is a projective cover of L_s in $\mathcal{A}$.
(2) Let $t \in \mathscr{T}$, and consider a projective cover P'_t of L_t in $\mathcal{A}_{\mathscr{T}}$. We consider the finite-dimensional vector space $E := \mathrm{Ext}^1_{\mathcal{A}}(P'_t, \Delta_s)$ (see Exercise 7.1 above). The identity of E defines a canonical short exact sequence

$$0 \to E^* \otimes \Delta_s \to P_t \to P'_t \to 0$$

for some object $P_t \in \mathcal{A}$. We now want to show that P_t is a projective cover of L_t in $\mathcal{A}$. First, show that for $r \in \mathscr{T}$ we have

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(P_t, L_r) = \begin{cases} 1 & \text{if } r = t; \\ 0 & \text{otherwise} \end{cases}$$

and that $\mathrm{Ext}^1_{\mathcal{A}}(P_t, L_r) = \mathrm{Ext}^2_{\mathcal{A}}(P_t, L_r) = 0$.
(3) Show that $\mathrm{Hom}_{\mathcal{A}}(P_t, L_s) = 0$. (*Hint*: Consider the long exact sequences obtained from the exact sequence $\ker \hookrightarrow \Delta_s \twoheadrightarrow L_s$ by applying $\mathrm{Hom}(P_t, -)$ and $\mathrm{Hom}(P'_t, -)$.)

(4) Show that $\mathrm{Ext}^1_{\mathcal{A}}(P_t, \Delta_s) = 0$.
(5) Deduce that $\mathrm{Ext}^1_{\mathcal{A}}(P_t, L_s) = 0$. (*Hint*: Consider once again the long exact sequence obtained from the exact sequence $\ker \hookrightarrow \Delta_s \twoheadrightarrow L_s$ by applying $\mathrm{Hom}(P_t, -)$.)
(6) Conclude.
(7) For general $s, t \in \mathscr{S}$, show that the multiplicity of a standard object Δ_t in a standard filtration of P_s does not depend on the choice of filtration, and equals $[\nabla_t : L_s]$.
(8) Show dually that (under the same assumptions) $\mathcal{A}$ has enough injective objects, and that any injective object admits a costandard filtration.

Exercise 7.3 ((Co)standard filtrations and subobjects/quotients). Let $\mathcal{A}$ be a highest weight category. Let M be an object in $\mathcal{A}$, and let $N \subset M$ be a subobject.

(1) Show that if N and M admit costandard filtrations, then so does M/N.
(2) Show that if M and M/N admit standard filtrations, then so does N.
(3) Show that if N is a direct summand of M, then M is tilting iff N and M/N are tilting.

Exercise 7.4 ((Co)standard filtrations and tilting resolutions). Let $\mathcal{A}$ be a highest weight category, with weight poset $\mathscr{S}$.

(1) The goal of this question is to prove that an object M of $\mathcal{A}$ admits a costandard filtration iff it admits a "left tilting resolution", i.e. iff there exist tilting objects $T_1, \cdots, T_n$ and an exact sequence
$$0 \to T_1 \to \cdots \to T_n \to M \to 0.$$
   (a) Show that if M admits a left tilting resolution, then it admits a costandard filtration. (Hint: use induction on n and Exercise 7.3.)
   (b) Let $M \in \mathcal{A}$ be an object admitting a costandard filtration, and let $\mathscr{T} \subset \mathscr{S}$ be an ideal containing the labels of all costandard objects appearing in a costandard filtration of M. Let $s \in \mathscr{T}$ be maximal. Show that there exists $n \in \mathbb{Z}_{\geq 0}$ and a surjection $M \twoheadrightarrow \nabla_s^{\oplus n}$ whose kernel admits a costandard filtration, all of whose labels belong to $\mathscr{T} \smallsetminus \{s\}$.
   (c) In the setting of (1b), show that there exists a tilting object T and a surjection $T \twoheadrightarrow M$ whose kernel admits a costandard filtration, all of whose labels belong to $\mathscr{T} \smallsetminus \{s\}$. (Hint: reduce the claim to the case $\mathscr{T}$ is finite, and then use induction on $\#\mathscr{T}$.)
   (d) If $M \in \mathcal{A}$ admits a costandard filtration, show that it admits a left tilting resolution. (Hint: argue again by induction on $\#\mathscr{T}$, where $\mathscr{T}$ is a finite ideal as above.)
(2) Show dually that an object M of $\mathcal{A}$ admits a standard filtration iff it admits a "right tilting resolution", i.e. iff there exist tilting objects $T_1, \cdots, T_n$ and an exact sequence
$$0 \to M \to T_1 \to \cdots \to T_n \to 0.$$

Exercise 7.5 ((Co-)standard multiplicities). Let $\mathcal{A}$ be a highest weight category, with weight poset $\mathscr{S}$.

(1) Show that if M admits a costandard filtration
$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M,$$

then for any $s \in \mathscr{S}$ we have

$$\#\{i \in \{1, \cdots, n\} \mid M_i/M_{i-1} \cong \nabla_s\} = \dim_{\Bbbk} \operatorname{Hom}_{\mathcal{A}}(\Delta_s, M).$$

In particular, the number in the left-hand side is independent of the choice of filtration, and is denoted $(M : \nabla_s)$.

(2) Show that if M admits a standard filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M,$$

then for any $s \in \mathscr{S}$ we have

$$\#\{i \in \{1, \cdots, n\} \mid M_i/M_{i-1} \cong \Delta_s\} = \dim_{\Bbbk} \operatorname{Hom}_{\mathcal{A}}(M, \nabla_s).$$

In particular, the number in the left-hand side is independent of the choice of filtration, and is denoted $(M : \Delta_s)$.

(3) Show that if M and N are tilting objects, then we have

$$\dim_{\Bbbk} \operatorname{Hom}_{\mathcal{A}}(M, N) = \sum_{s \in \mathscr{S}} (M : \Delta_s) \cdot (N : \nabla_s).$$

Exercise 7.6 ((Co-)standard filtrations and quotient functors). Let $\mathcal{A}$ be a highest weight category with weight poset $\mathscr{S}$, and let $\mathscr{T} \subset \mathscr{S}$ be an ideal. Consider the quotient functor $\pi_{\mathscr{T}} : \mathcal{A} \to \mathcal{A}/\mathcal{A}_{\mathscr{T}}$, and the highest weight structure on $\mathcal{A}/\mathcal{A}_{\mathscr{T}}$ considered in Lemma 3.1.

(1) Show that if X admits a standard, resp. costandard, filtration, then so does $\pi_{\mathscr{T}}(X)$.
(2) Show that if X admits a costandard filtration, for any $s \in \mathscr{S} \smallsetminus \mathscr{T}$ the natural morphism

$$\operatorname{Hom}_{\mathcal{A}}(\Delta_s, X) \to \operatorname{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(\Delta_s), \pi_{\mathscr{T}}(X))$$

is an isomorphism.

(3) Show that if X, resp. Y, admits a standard, resp. costandard, filtration, then the morphism

$$\operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{A}/\mathcal{A}_{\mathscr{T}}}(\pi_{\mathscr{T}}(X), \pi_{\mathscr{T}}(Y))$$

induced by the functor $\pi_{\mathscr{T}}$ is surjective.

Exercise 7.7 ((Co-)standard filtrations and ideals). Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$.

(1) Show that if $s, t \in \mathscr{S}$ are such that $\operatorname{Ext}^1(\nabla_s, \nabla_t) \neq 0$, then $s \geq t$.
(2) Let $X \in \mathcal{A}$ be an object which admits a costandard filtration, and set $\mathscr{U} := \{s \in \mathscr{S} \mid (X : \nabla_s) \neq 0\}$. (See Exercise 7.5 for the notation $(X : \nabla_s)$.) Choose an enumeration $s_1, \cdots, s_r$ of the elements of $\mathscr{U}$ such that $s_i \leq s_j \Rightarrow i \leq j$. Show that there exists a filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{r-1} \subset X_r = X$$

such that for any i the object X_i/X_{i-1} is isomorphic to $(\nabla_{s_i})^{\oplus (X:\nabla_{s_i})}$.

(3) Show that if $s, t \in \mathscr{S}$ are such that $\operatorname{Hom}(\nabla_s, \nabla_t) \neq 0$, then $t \leq s$.
(4) Let $X \in \mathcal{A}$ be an object which admits a costandard filtration. Show that for any ideal $\mathscr{T} \subset \mathscr{S}$, there exists a unique subobject $\Gamma_{\mathscr{T}}(X) \subset X$ which admits a costandard filtration and such that

$$(\Gamma_{\mathscr{T}}(X) : \nabla_s) \neq 0 \Rightarrow s \in \mathscr{T};$$

$$(X/\Gamma_{\mathscr{T}}(X) : \nabla_s) \neq 0 \Rightarrow s \in \mathscr{S} \smallsetminus \mathscr{T}.$$

(Note that in this setup $X/\Gamma_{\mathscr{T}}(X)$ automatically admits a costandard filtration by Exercise 7.3.)

(5) Let $\mathscr{T} \subset \mathscr{S}$ be an ideal. Denote by $\mathcal{A}_\nabla$ the full subcategory of $\mathcal{A}$ whose objects are those which admit a costandard filtration, and by $\mathcal{A}_{\nabla,\mathscr{T}} \subset \mathcal{A}_\nabla$ the full subcategory whose objects are those which satisfy $(X : \nabla_s) = 0$ for any $s \in \mathscr{S} \smallsetminus \mathscr{T}$. Show that the assignment $X \mapsto \Gamma_{\mathscr{T}}(X)$ extends to a functor from $\mathcal{A}_\nabla$ to $\mathcal{A}_{\nabla,\mathscr{T}}$ which is right adjoint to the natural embedding $\mathcal{A}_{\nabla,\mathscr{T}} \to \mathcal{A}_\nabla$.

(6) State and prove dual properties for standard filtrations.

Exercise 7.8. Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$. Consider an ideal $\mathscr{T} \subset \mathscr{S}$ and the functor $\Gamma_{\mathscr{T}}$ introduced in Exercise 7.7. We fix an object M in $\mathcal{A}$ which admits a costandard filtration, and a ∇-section $(\Pi, e, (\varphi_\pi : \pi \in \Pi))$ for M.

(1) Show that for any $t \in \mathscr{T}$ and any $\pi \in e^{-1}(t)$ the morphism $\varphi_\pi : \mathsf{T}_t \to M$ factors through a morphism $\varphi'_\pi : \mathsf{T}_t \to \Gamma_{\mathscr{T}}(M)$.

(2) Set $\Pi_{\mathscr{T}} = e^{-1}(\mathscr{T})$, and denote by $e_{\mathscr{T}}$ the restriction of e to $\Pi_{\mathscr{T}}$. Show that $(\Pi_{\mathscr{T}}, e_{\mathscr{T}}, (\varphi'_\pi : \pi \in \Pi_{\mathscr{T}})$ is a ∇-section of $\Gamma_{\mathscr{T}}(M)$.

(3) Set $\Pi^{\mathscr{T}} := e^{-1}(\mathscr{S} \smallsetminus \mathscr{T})$, and denote by $e^{\mathscr{T}}$ the restriction of e to $\Pi^{\mathscr{T}}$. For any $\pi \in \Pi^{\mathscr{T}}$, denote by φ''_t the composition

$$\mathsf{T}_t \to M \to M/\Gamma_{\mathscr{T}}(M).$$

Show that $(\Pi^{\mathscr{T}}, e^{\mathscr{T}}, (\varphi''_\pi : \pi \in \Pi^{\mathscr{T}})$ is a ∇-section of $M/\Gamma_{\mathscr{T}}(M)$.

Exercise 7.9. Let $\mathcal{A}$ be a highest weight category with weight poset $(\mathscr{S}, \leq)$. Assume that $\mathcal{A}$ has a "duality", i.e. that there exists an functor $d : \mathcal{A} \to \mathcal{A}$ which satisfies $d \circ d = \mathrm{id}$ and $d(\Delta_s) \cong \nabla_s$ for any $s \in \mathscr{S}$.

(1) Show that $d(\mathsf{T}_s) \cong \mathsf{T}_s$ for any $s \in \mathscr{S}$.

(2) Show that

$$(\mathsf{T}_s : \Delta_t) = (\mathsf{T}_s : \nabla_t)$$

for any $s, t \in \mathscr{S}$.

(3) Show that if $(a_{s,t} : s, t \in \mathscr{S})$ is a sequence of integers such that $a_{s,t} = 0$ unless $s \leq t$, $a_{s,s} = 1$ for any $s \in \mathscr{S}$, and for any $s, t \in \mathscr{S}$ we have

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(\mathsf{T}_s, \mathsf{T}_t) = \sum_{u \in \mathscr{S}} a_{u,s} a_{u,t},$$

then for any $s, t \in \mathscr{S}$ we have

$$(\mathsf{T}_t : \nabla_s) = a_{s,t}.$$

To see this idea used in practice, see [**AR1**, §6.2] or [**RW3**, Proof of Theorem 8.9].

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