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Contributions en cohomologie de Hochschild et en théorie des représentations des algèbres de dimension finie

Synthèse des travaux en vue de l'obtention de l'Habilitation à Diriger des Recherches

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Table des matières

Throughout this document, citations with numbers ([2] etc.) refer to those on pages 65 to 72, while those with letters ([AGST16] etc.) refer to those for which I am a (co-)author and that are presented in this text ; they appear on this page.

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Introduction

Dans cette synthèse, je présente l'essentiel des travaux de recherche que j'ai effectués depuis ma thèse. Ces travaux sont centrés principalement sur la cohomologie de Hochschild et sur des problèmes de classification d'algèbres de dimension finie, mais je vais également décrire le travail que j'ai fait sur les doubles de Drinfeld de certaines algèbres de Hopf et sur le calcul de Koszul des algèbres préprojectives.

La cohomologie de Hochschild joue un rôle important dans la théorie des représentations des algèbres associatives. Elle est munie d'une structure très riche : celle d'algèbre de Gerstenhaber, c'est-à-dire que c'est une algèbre associative graduée commutative (pour le cup-produit) équipée d'un crochet de Lie gradué, et elle est fortement liée à la théorie des déformations des algèbres associatives. La cohomologie de Hochschild a été utilisée par Snashall et Solberg pour définir des variétés de support pour les algèbres associatives, ce qui a soulevé un intérêt pour certaines questions de finitude associées. La cohomologie de Hochschild est aussi un invariant important pour plusieurs types d'équivalences entre algèbres associatives comme nous le verrons dans ce texte.

La question de la classification d'algèbres de dimension finie, telles que les blocs d'algèbres de groupe, les algèbres de Hopf, les algèbres de graphes de Brauer, etc., a intéressé de nombreux auteurs comme Białkowski, Bocian, Erdmann, Holm, Skowroński, Zhou, Zimmermann, etc. Nous présenterons ici quelques outils liés à la cohomologie de Hochschild, et en particulier ceux qui ont été développés par Holm et Zimmermann, qui nous ont été utiles pour obtenir de tels résultats de classification. Nous donnerons également d'autres résultats de classification, dans le contexte des algèbres de graphes de Brauer et des algèbres héréditaires par morceaux.

Je décris ici très brièvement les sujets que j'ai étudiés ; ils sont présentés et mis dans leur contexte de façon plus détaillée dans les chapitres qui suivent.

1 Cohomologie de Hochschild

Une grande partie de mon travail concerne la cohomologie de Hochschild.

Variétés de support pour les algèbres de dimension finie (Chapitre 1).

Les variétés de support pour les algèbres associatives ont été définies par Snashall et Solberg à l'aide de la cohomologie de Hochschild ; ils étaient inspirés par la théorie des variétés de support pour les groupes finis qui a fait preuve d'une utilité remarquable dans l'étude des représentations modulaires de groupes finis.

Dans un travail en commun avec Erdmann, Holloway, Snashall et Solberg [EHSST04], nous avons démontré en particulier que de nombreux résultats qui sont valables pour les variétés de support des groupes finis le sont aussi dans le cadre des algèbres auto-injectives de dimension finie satisfaisant à certaines conditions de finitude appelées **(Fg)** :

$$\textbf{(Fg)} \quad \begin{array}{l} \text{HH}^*(\Lambda) \text{ est nœthérienne et le } \text{HH}^*(\Lambda)\text{-module} \\ \text{Ext}_\Lambda^*(\Lambda/\operatorname{rad}(\Lambda), \Lambda/\operatorname{rad}(\Lambda)) \text{ est de type fini.} \end{array}$$

Ces conditions **(Fg)** jouent un rôle important dans l'étude des variétés de support des algèbres de dimension finie et ont fait l'objet de plusieurs articles dernièrement.

Cohomologie de Hochschild pour des familles d'algèbres de dimension finie (Chapitre 2).

Lorsqu'ils ont introduit les variétés de support pour les algèbres, Snashall et Solberg ont été menés à faire la conjecture suivante.

Conjecture. *[135] Soit Λ une k-algèbre de dimension finie. Alors $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ est un anneau de type fini.*

Bien que cette conjecture soit fausse en général (voir [152]), elle a été démontrée pour de nombreux types d'algèbres et a motivé de nombreuses recherches. En particulier, Snashall et moi l'avons démontrée pour plusieurs familles d'algèbres de dimension finie.

La première famille d'algèbres provient d'une étude de doubles de Drinfeld que nous avons faite avec Erdmann et Green (voir plus bas). Pour ces algèbres, nous avons entièrement donné la structure d'algèbre de leur cohomologie de Hochschild et nous avons démontré que leur cohomologie de Hochschild est une algèbre de type fini [ST10-1], donc en particulier que la conjecture est vérifiée pour ces algèbres.

Au sein de cette famille, certaines sont Koszul. Pour celles-ci, nous avons déterminé toutes les socle-déformations qui proviennent d'éléments dans le deuxième espace de cohomologie de Hochschild, et donné explicitement le quotient de la cohomologie de Hochschild de ces socle-déformations par l'idéal des éléments nilpotents, via le centre gradué de leur algèbre de Yoneda [ST10-2], démontrant ainsi que la conjecture est vérifiée pour ces algèbres. De plus, nous avons démontré que ces déformations satisfont aux conditions **(Fg)**.

Cohomologie de Hochschild d'extensions triviales (Chapitre 2).

Dans un travail en commun avec Assem, Gatica et Schiffler dans [AGST16], nous avons comparé la cohomologie de Hochschild d'une algèbre de dimension finie C avec celle d'une extension scindée B de C par un C-bimodule E par le biais d'un morphisme $\varphi^* : \mathrm{HH}^*(B) \to \mathrm{HH}^*(C)$. Nous avons étudié des propriétés de ce morphisme, et nous avons déterminé en particulier sa surjectivité dans le cas où B est une extension triviale de C (c'est-à-dire que $E^2 = 0$ dans B). Nous avons ensuite appliqué ces résultats aux extensions par relations et généralisé des résultats d'Assem, Bustamante, Igusa, Redondo and Schiffler lorsque B est une algèbre inclinée amassée.

Déformations PBW d'algèbres 3-Calabi-Yau (Chapitre 4).

Les algèbres Calabi-Yau au sens de Kontsevich trouvent leur origine en géométrie algébrique. Elles ont été redéfinies par Ginzburg [68] en termes de cohomologie de Hochschild, inspiré par le théorème de dualité de Van den Bergh entre homologie et cohomologie de Hochschild.

Berger et moi étions intéressés dans [BT07] par la conjecture suivante de Ginzburg et de Van den Bergh.

Conjecture. *[68, 143] Toute algèbre 3-Calabi-Yau dérive d'un potentiel.*

Cette conjecture n'est pas vraie en général (voir [41]), mais Bocklandt l'a démontrée pour les algèbres graduées dans [20]. Ces algèbres 3-Calabi-Yau graduées de Bocklandt sont toutes N-Koszul, pour un N qui dépend du potentiel. Il est donc naturel de considérer leurs déformations PBW au sens de Berger et Ginzburg [12].

Soit A une algèbre 3-Calabi-Yau graduée, de potentiel W_{N+1}. Dans [BT07], nous avons démontré que si on ajoute un potentiel de longueur plus petite à W_{N+1}, on obtient une déformation PBW de A; nous avons donné une condition nécessaire et des conditions suffisantes pour qu'une déformation PBW de A dérive d'un potentiel; et nous avons démontré qu'une déformation PBW de A qui dérive d'un potentiel est 3-Calabi-Yau.

2 Résultats de classification d'algèbres de dimension finie

Je me suis aussi intéressée à des problèmes de classification.

Classification des algèbres de graphes de Brauer 2-d-Koszul et $\mathcal{K}_2$ (Chapitre 3).

Dans un travail en commun avec Green, Schroll et Snashall [GSST17], nous avons considéré plusieurs généralisations des algèbres Koszul : les algèbres 2-d-Koszul de Green et Marcos [70] et les algèbres $\mathcal{K}_2$ de Cassidy et Shelton [30], parmi les algèbres de graphes de Brauer. Nous avons donné une classification des algèbres de graphes de Brauer qui sont Koszul, d-Koszul, 2-d-Koszul et $\mathcal{K}_2$ en termes du graphe de Brauer. Cela a nécessité de déterminer (une partie de) l'algèbre de Yoneda d'une algèbre de graphe de Brauer.

Classification de familles d'algèbres à équivalence stable à la Morita près (Chapitre 5).

Snashall et moi avons étudié dans [ST15] les algèbres bisérielles spéciales basiques et indécomposables qui ont au plus un module projectif indécomposable qui n'est pas unisériel. Cette famille contient et étend la famille des algèbres de Nakayama basiques et symétriques.

Nous les avons décrites par carquois et relations, et classifié à équivalence stable à la Morita près, en utilisant plusieurs outils : la cohomologie de Hochschild, les idéaux de Külshammer et les algèbres d'arbres de Brauer généralisés en particulier.

J'ai aussi travaillé sur la classification des algèbres dociles de type diédral, semi-diédral et quaternionique à équivalence stable à la Morita près. Ces algèbres ont été introduites par Erdmann [44], généralisant les algèbres qu'elle a obtenues lors de son étude des blocs dociles d'algèbres de groupe. Elle les a décrites à équivalence Morita près, puis plusieurs auteurs (Holm, Zhou, Zimmermann) ont étudié leurs classes d'équivalence dérivée. Plus récemment, Zhou et Zimmermann [154] ont établi la plus grande partie de leur classification à équivalence stable à la Morita près, mais quelques questions restaient sans réponse.

En utilisant le premier espace de cohomologie de Hochschild et sa structure d'algèbre de Lie, j'ai pu répondre à certaines de ces questions dans [T19]. En particulier, la classification des algèbres de type diédral à équivalence stable à la Morita près est maintenant complète.

Classification combinatoire des algèbres héréditaires par morceaux (Chapitre 6).

Happel a décrit la trace de la matrice de Coxeter d'une algèbre de dimension finie et de dimension globale finie sur un corps algébriquement clos en termes de sa cohomologie de Hochschild [81]. Il a ensuite utilisé cette trace pour donner une classification partielle des algèbres héréditaires par morceaux [81, 82]. Cependant, pour les deux types d'algèbres héréditaires par morceaux (de type héréditaire ou canonique), cette trace peut valoir -1.

Dans un travail en commun avec Lanzilotta et Redondo [LRT11], nous avons étudié les autres coefficients du polynôme caractéristique de la matrice de Coxeter. Cela nous a permis de compléter la classification combinatoire des algèbres héréditaires par morceaux.

3 Autres résultats

Doubles de Drinfeld d'algèbres de Taft généralisées (Chapitre 7).

Dans un travail en commun avec Erdmann, Green et Snashall, nous avons étudié dans [EGST06] les doubles de Drinfeld d'algèbres de Taft généralisées en tant qu'algèbres : nous avons déterminé leur présentation par carquois et relations ainsi que toutes leurs représentations (ce sont des algèbres symétriques, dociles et bisérielles spéciales). Nous avons ensuite étudié les produits tensoriels de ces représentations, d'abord dans [EGST06] (mais nous y avons fait des

erreurs), puis dans [EGST19] d'une manière plus homologique, où nous avons décrit tous les produits dans les anneaux de Green de ces doubles de Drinfeld.

J'ai ensuite appliqué certains de ces résultats (ainsi qu'un résultat de ma thèse) dans [T07] pour déterminer la cohomologie de Gerstenhaber-Schack des algèbres de Taft généralisées.

Calcul de Koszul des algèbres préprojectives (Chapitre 8).

Berger et moi avons étudié le calcul de Koszul des algèbres préprojectives [BT]. Le calcul de Koszul a été introduit par Berger, Lambre et Solotar dans [13] pour les algèbres quadratiques connexes ; il coïncide avec le calcul de Hochschild pour les algèbres Koszul, mais diffère en général.

Nous avons étendu dans [BT] leurs définitions et certains de leurs résultats aux algèbres quadratiques qui sont des quotients d'algèbres de chemins. Nous avons ensuite étudié ce calcul de Koszul pour les algèbres préprojectives, démontrant certains propriétés de dualité et de type Calabi-Yau. Finalement, nous avons déterminé complètement le calcul de Koszul pour les algèbres préprojectives qui ne sont pas Koszul.

Notations

Dans toute cette synthèse, les algèbres sont des algèbres associatives unitaires sur un corps k. Les modules sur une algèbre sont de type fini et en général sont des modules à gauche sauf mention du contraire. La catégorie des A-modules à gauche de type fini est notée $A\text{-mod}$.

Etant donnée une algèbre A, nous notons $A^e = A \otimes_k A^{op}$ son algèbre enveloppante ; ainsi, les A-bimodules sont indentifiés aux A^e-modules à gauche.

Le produit tensoriel $\otimes_k$ sur le corps k est noté $\otimes$ et le dual d'un module M est noté $DM = \text{Hom}_k(M, k)$.

Etant donné un carquois $\mathcal{Q}$, on note e_i le chemin de longueur 0 au sommet i et l'idempotent de l'algèbre de chemins qui correspond. L'ensemble des chemins de longueur j dans $\mathcal{Q}$ est noté $\mathcal{Q}_j$ et l'ensemble des chemins de longueur inférieure ou égale à j est noté $\mathcal{Q}_{\leqslant j}$. Le chemins sont écrits de droite à gauche.

Etant donné un module M sur une algèbre A, on note $\text{p.dim}_A M$ sa dimension projective et $\text{i.dim}_A M$ sa dimension injective. La dimension globale de A est notée $\text{gldim} A$.

Introduction

In this thesis, I present most of the work that I have done since my PhD. The main focus is on Hochschild cohomology and on classification problems for finite dimensional algebras, but I shall also describe some work that I have done on the Drinfeld doubles of some Hopf algebras and on Koszul calculus for preprojective algebras.

Hochschild cohomology plays an important role in the representation theory of associative algebras. It is endowed with a very rich structure: that of a Gerstenhaber algebra, that is, a graded-commutative associative algebra (for the cup-product) endowed with a graded Lie bracket, and it is strongly related to the deformation theory of associative algebras. Hochschild cohomology has been used by Snashall and Solberg to define support varieties for associative algebras, which has led to an interest in certain related finite generation questions. Hochschild cohomology is also an important invariant of equivalences of several kinds between associative algebras as we shall see in this text.

The question of classifying finite dimensional algebras, such as blocks of group algebras, Hopf algebras, Brauer graph algebras, etc., has interested many authors, such as Białkowski, Bocian, Erdmann, Holm, Skowroński, Zhou, Zimmermann, etc. We shall present here some tools related to Hochschild cohomology, and in particular some that have been developed by Holm and Zimmermann, that have been helpful to us in obtaining some such classification results. We will also give some other classification results in the context of Brauer graph algebras and of piecewise hereditary algebras.

Here I describe very briefly the topics that I have studied; they are presented and contextualised in more detail in the next chapters.

1 Hochschild cohomology

Much of my work has revolved around Hochschild cohomology.

Support varieties for finite dimensional algebras (Chapter 1).

Support varieties for associative algebras were defined by Snashall and Solberg using Hochschild cohomology; they were inspired by support varieties for finite groups that have proved remarkably useful in the study of modular representations of finite groups.

In joint work with Erdmann, Holloway, Snashall and Solberg [EHSST04], we have proved in particular that many of the results that hold for support varieties of finite groups are also true in the case of finite dimensional selfinjective algebras when some finite generation conditions, called **(Fg)**, are assumed:

$$\textbf{(Fg)} \quad \begin{array}{l} \mathrm{HH}^*(\Lambda) \text{ is noetherian and the } \mathrm{HH}^*(\Lambda)\text{-module} \\ \mathrm{Ext}^*_\Lambda(\Lambda/\operatorname{rad}(\Lambda), \Lambda/\operatorname{rad}(\Lambda)) \text{ is finitely generated.} \end{array}$$

These conditions **(Fg)** play an important role in the study of support varieties for finite dimensional algebras and have been the object of several papers lately.

Hochschild cohomology algebra of some families of finite dimensional algebras (Chapter 2).

When they introduced support varieties for algebras, Snashall and Solberg were led to make the following conjecture.

Conjecture. *[135] Let Λ be a finite dimensional k-algebra. Then $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is finitely generated as a ring.*

Although this conjecture does not hold in general (see [152]), it has been proved for many types of algebras and has motivated much research since. In particular, Snashall and I have proved it for several families of finite dimensional algebras.

The first family of algebras arose from a study of Drinfeld doubles that we did with Erdmann and Green (see below). For these algebras, we have given the complete algebra structure of their Hochschild cohomology and proved that the whole Hochschild cohomology is a finitely generated algebra [ST10-1], so that in particular the conjecture holds.

Within this family, some algebras are Koszul. For those, we have determined all the socle deformations that arise from elements in the second Hochschild cohomology space, and given explicitly the Hochschild cohomology modulo nilpotents of these socle deformations, via the graded centre of their Yoneda algebra [ST10-2], this showing that the conjecture is true for these algebras. Moreover, we have shown that these deformations satisfy conditions **(Fg)**.

Hochschild cohomology of trivial extensions (Chapter 2).

In joint work with Assem, Gatica and Schiffler in [AGST16], we have compared the Hochschild cohomology of a finite dimensional algebra C with that of a split extension B of C by a C-bimodule E by means of a morphism $\varphi^* : \mathrm{HH}^*(B) \to \mathrm{HH}^*(C)$. We studied some properties of this map, and in particular we determined when it is surjective in the case where B is a trivial extension of C (that is, when $E^2 = 0$ in B). We then applied our results to relation extensions and generalise results of Assem, Bustamante, Igusa, Redondo and Schiffler when B is a cluster-tilted algebra.

PBW-deformations of 3-Calabi-Yau algebras (Chapter 4).

Calabi-Yau algebras in the sense of Kontsevich originated from algebraic geometry. They were re-defined by Ginzburg [68] in terms of Hochschild cohomology, inspired by Van den Bergh's duality theorem for Hochschild homology and cohomology.

Berger and I were interested in [BT07] in the following conjecture made by Ginzburg and by Van den Bergh.

Conjecture. *[68, 143] Every 3-Calabi-Yau algebra derives from a potential.*

This conjecture is not true in general (see [41]), but it was proved for graded algebras by Bocklandt [20]. Bocklandt's 3-Calabi-Yau graded algebras are all N-Koszul, for some N depending on the potential. It is therefore natural to consider their PBW-deformations in the sense of Berger and Ginzburg [12].

Let A be a graded 3-Calabi-Yau algebra, with potential W_{N+1}. In [BT07], we showed that adding a potential of smaller length to W_{N+1} produced a PBW-deformation of A, we gave a necessary condition and some sufficient conditions for a PBW-deformation of A to derive from a potential, and we proved that a PBW-deformation of A that derives from a potential is 3-Calabi-Yau.

2 Classification results for finite dimensional algebras

I have also been interested in classification problems.

Classification of 2-d-Koszul and $\mathcal{K}_2$ Brauer graph algebras (Chapter 3).

In joint work with Green, Schroll and Snashall [GSST17], we considered several generalisations of Koszul algebras: the 2-d-Koszul algebras of Green and Marcos [70] and the $\mathcal{K}_2$ algebras of Cassidy and Shelton [30], within the family of Brauer graph algebras. We gave a classification of Koszul, d-Koszul, 2-d-Koszul and $\mathcal{K}_2$ Brauer graph algebras in terms of the Brauer graph. This required the determination of (part of) the Yoneda algebra of Brauer graph algebras.

Classification of some families of algebras up to stable equivalence of Morita type (Chapter 5).

Snashall and I studied in [ST15] the basic indecomposable special biserial algebras such that at most one indecomposable projective is not uniserial. This family contains and generalises the symmetric basic Nakayama algebras.

We described them by quiver and relations, and classified them up to derived equivalence and stable equivalence of Morita type, using several tools: Hochschild cohomology, Külshammer ideals and generalised Brauer tree algebras in particular.

I also worked on the classification of the tame algebras of dihedral, semi-dihedral and quaternion types up to stable equivalence of Morita type. These algebras were introduced by Erdmann [44], extending the algebras she had obtained in her study of tame blocks of group algebras. She described them up to Morita equivalence, then several authors (Holm, Zhou, Zimmermann) studied their derived equivalence classes. More recently, Zhou and Zimmermann [154] established most of their classification up to stable equivalence of Morita type, although some questions remained.

Using the first Hochschild cohomology space and its Lie algebra structure, I was able in [T19] to answer some of these questions. In particular, the classification of the algebras of dihedral type up to stable equivalence of Morita type is now complete.

Combinatorial classification of piecewise hereditary algebras (Chapter 6).

Happel described the trace of the Coxeter matrix of a finite dimensional algebra of finite global dimension over an algebraically closed field in terms of its Hochschild cohomology [81]. He then used this trace to give a partial classification of piecewise hereditary algebras [81, 82]. However, in the case where the trace is -1, both types of piecewise hereditary algebras (hereditary type or canonical type) occur.

In joint work with Lanzilotta and Redondo [LRT11], we studied the other coefficients of the characteristic polynomial of the Coxeter matrix. This enabled us to complete the combinatorial classification of piecewise hereditary algebras.

3 Other results

Drinfeld doubles of generalised Taft algebras (Chapter 7).

In joint work with Erdmann, Green and Snashall, we have studied the Drinfeld doubles of generalised Taft algebras as algebras in [EGST06]: we determined the presentation by quiver and relations and all the representations of these algebras (they are symmetric, tame and special biserial as algebras). We then studied the tensor products of these representations, first in [EGST06] (but we made some mistakes), then in [EGST19] in a more homological way, where we described all the products in the stable Green rings of these Drinfeld doubles.

I then applied some of these results (as well as a result in my PhD thesis) in [T07] to compute the Gerstenhaber-Schack cohomology of the generalised Taft algebras.

Koszul calculus of preprojective algebras (Chapter 8).

Berger and I studied the Koszul calculus of preprojective algebras [BT]. Koszul calculus was introduced by Berger, Lambre and Solotar in [13] for quadratic connected algebras; it coincides with the Hochschild calculus for Koszul algebras, but differs in general.

We extended in [BT] their definitions and some of their results to quadratic quotients of path algebras. We then studied this calculus for the preprojective algebras, proving some duality and Calabi-Yau type properties. Finally we determined the Koszul calculus completely for the non Koszul preprojective algebras.

Notation

Throughout this report, algebras will be associative algebras with unit over a field k. Modules over an algebra will be finitely generated, and generally they will be left modules unless otherwise specified. The category of left A-modules will be denoted by A-mod.

Given an algebra A, we shall denote by $A^e = A \otimes_k A^{op}$ its enveloping algebra, so that A-bimodules are identified with left A^e-modules.

The tensor product $\otimes$ will denote the tensor product $\otimes_k$ over the base field k and the k-dual of a module M will be denoted by $DM = \text{Hom}_k(M, k)$.

Given a quiver $\mathcal{Q}$, we shall denote by e_i the path of length 0 at vertex i and the corresponding idempotent in the path algebra. The set of paths of length j in $\mathcal{Q}$ will be denoted by $\mathcal{Q}_j$, the set of paths of length at most j by $\mathcal{Q}_{\leqslant j}$. Paths are written from right to left.

Given a module M over an algebra A, we shall denote by $\text{p.dim}_A M$ its projective dimension and by $\text{i.dim}_A M$ its injective dimension. The global dimension of A will written $\text{gldim} A$.

Chapter 1

Hochschild cohomology as a tool: support varieties

Summary

We describe the theory of support varieties for finite dimensional algebras introduced by Snashall and Solberg [135] using Hochschild cohomology, then present the results Erdmann, Holloway, Snashall, Solberg and I obtained in [EHSST04] when some finite generation conditions were added, especially for selfinjective algebras.

1.1 Motivation

Let G be a finite group, and let k be an algebraically closed field. Consider the representation theory of the group algebra kG. If $\mathrm{char}(k)$ does not divide the order of G, then kG is semisimple, which implies that the representation theory is well-known.

However, if $\mathrm{char}(k)$ divides the order of G, then the theory is more complicated. Quillen, Carlson, Avrunin and Scott [124, 26, 9] among others introduced varieties in order to study this representation theory. This means that we can use tools from geometry and commutative algebra to study the representations of a finite group.

This theory has then been transposed and adapted to many other contexts, for instance p-Lie algebras (see for instance [58]), finite dimensional cocommutative algebras (see for instance [59, 139]), quantum groups (see for instance [54, 60, 115]).

Before describing the work that Erdmann, Holloway, Snashall, Solberg and I have done on support varieties for finite dimensional algebras, I shall give an overview of some the theory of varieties for group algebras, since it inspired our work.

Support varieties of a finite group G are defined using the group cohomology of G, that can be defined by $\mathrm{H}^*(G) = \mathrm{Ext}^*_{kG}(k,k)$. This is a graded-commutative algebra that is finitely generated [51, 145].

We can therefore consider the commutative, graded, finitely generated algebra

$$\mathrm{H}^{\mathrm{ev}}(G) = \begin{cases} \displaystyle\bigoplus_{n \geqslant 0} \mathrm{H}^n(G) & \text{if } \mathrm{char}(k) = 2 \\ \displaystyle\bigoplus_{n \geqslant 0} \mathrm{H}^{2n}(G) & \text{if } \mathrm{char}(k) \neq 2. \end{cases}$$

Definition 1.1.1. The variety V_G is the maximal spectrum of $\mathrm{H}^{\mathrm{ev}}(G)$.

Now given a finite dimensional kG-module M, we describe the definition of a subvariety of V_G associated with M. Consider the map:

$$\mathrm{H}^*(G) \xrightarrow{\gamma_M} \mathrm{Ext}^*_{kG}(M,M)$$

defined by tensoring by M over k, that is, induced by the map

$$(0 \to k \to R_n \to \ldots \to R_1 \to k \to 0) \mapsto (0 \to M \to R_n \otimes M \to \ldots \to R_1 \otimes M \to M \to 0)$$

on extensions. This map is a morphism of graded rings. Therefore we can view $\text{Ext}^*_{kG}(M, M)$ as a module over $H^*(G)$. As such it is finitely generated [51].

Definition 1.1.2. The support variety $V_G(M)$ of M is defined to be the set of maximal ideals in $H^*(G)$ which contain the annihilator of $\text{Ext}^*_{kG}(M, M)$ as an $H^*(G)$-module.

We now give a few properties of these varieties in order to illustrate the theory, and to compare with our situation later.

Properties 1.1.3. *Let M, M_1, M_2, M_3, M' be finite dimensional G-modules. The following properties hold.*

(P1) *M is projective if, and only if, $V_G(M)$ is trivial (that is, it contains only the ideal $H^{\geq 1}(G)$).*

(P2) *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence, then $V_G(M_i) \subseteq V_G(M_j) \cup V_G(M_\ell)$ where $\{i, j, \ell\} = \{1, 2, 3\}$.*

(P3) *$V_G(M \oplus M') = V_G(M) \cup V_G(M')$*

(P4) *$V_G(M) = V_G(DM) = V_G(\Omega M)$ where $DM = \text{Hom}_k(M, k)$ is the usual dual and ΩM is the kernel of a projective cover $P \xrightarrow{\pi} M \to 0$ of M.*

(P5) *We can define inductively $\Omega^n M$. We say that M is periodic if there exists $n \geq 1$ such that $\Omega^n(M) \cong M$.*

   Assume that M is indecomposable. Then M is periodic if, and only if, $\dim V_G(M) = 1$.

(P6) *Let I be any finitely generated ideal in $H^*(G)$. Then there exists a module M such that $V_G(M) = V(I) := \{\mathfrak{m} \in V_G \mid I \subseteq \mathfrak{m}\}$.*

(P7) *If $V_G(M) = V_1 \cup V_2$ with V_i closed homogeneous (that is, of the form $V(\eta_1, \ldots, \eta_m)$ for some homogeneous elements $\eta_1, \ldots, \eta_m$ in $H^*(G)$) and such that $V_1 \cap V_2$ is trivial, then $M \cong M_1 \oplus M_2$ where M_1 and M_2 are G-modules such that $V_G(M_i) = V_i$ for $i = 1, 2$.*

(P8) *If $V_G(M) \cap V_G(N)$ is trivial, then $\text{Ext}^n_{kG}(M, N) = 0$ for all $i \geq 1$.*

(P9) *$V_G(M) \cap V_G(N) = V_G(M \otimes_k N)$.*

1.2 Varieties for modules over finite dimensional algebras

Varieties in the case of group algebras have proved to be very useful in the study of the representation theory of a group algebra in the non-commutative case. There is therefore a strong motivation to introduce varieties in other contexts as we mentioned earlier.

The theory we have worked on in [EHSST04] was defined by Snashall and Solberg in [135]. They defined support varieties for an algebra Λ over a commutative ring k over which Λ is flat. To simplify, we shall assume that k is an algebraically closed field and that Λ is a finite dimensional algebra over k (this is the context of [EHSST04]). Moreover, all Λ-modules will be finite dimensional over k.

As in the case of group algebras, the definition of a variety relies on a cohomology for Λ. The natural choice is Hochschild cohomology, which can be defined by $HH^*(\Lambda) := \text{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$ where $\Lambda^e = \Lambda \otimes \Lambda^{op}$. This is a graded-commutative ring [65].

Note that in the case of group algebras, we can also consider $HH^*(kG)$, and we have a factorisation:

$$H^*(G) \xrightarrow{\gamma_M = - \otimes_k M} \text{Ext}^*_{kG}(M, M)$$

$$HH^*(kG) \xrightarrow{-\otimes_{kG} M}$$

It is therefore natural to consider, for a Λ-module M, the map:

$$\varphi_M = - \otimes_\Lambda M \colon \mathrm{HH}^*(\Lambda) \to \mathrm{Ext}^*_\Lambda(M, M).$$

This is a homomorphism of graded rings, so that we can view $\mathrm{Ext}^*_\Lambda(M, M)$ as a module over $\mathrm{HH}^*(\Lambda)$.

The definition of support varieties depends on a subalgebra of the Hochschild cohomology of Λ.

Definition 1.2.1. [135] Let H^* be a commutative graded Noetherian subalgebra of $\mathrm{HH}^*(\Lambda)$. The support variety of a Λ-module M is defined using the spectrum of maximal ideals in H^* by

$$V_{H^*}(M) = \{\mathfrak{m} \mid \mathfrak{m} \in \mathrm{MaxSpec}(H^*) \text{ such that } \mathrm{Ann}_{H^*}(\mathrm{Ext}^*_\Lambda(M, M)) \subseteq \mathfrak{m}\}.$$

Snashall and Solberg proved in [135, Propositions 3.4 and 3.5] that Properties (P2), (P3) and (P4) hold in this context.

Now I shall describe joint work with Erdmann, Holloway, Snashall and Solberg [EHSST04].

In order to be able to use commutative algebra (or geometry) in the theory of support varieties, we require some finiteness conditions.

The first assumption that we made is on the subalgebra H^*:

Assumption 1.2.2. [EHSST04] **(Fg1)** Fix a graded subalgebra H^* of $\mathrm{HH}^*(\Lambda)$ such that

(i) H is a commutative Noetherian ring;

(ii) $H^0 = \mathrm{HH}^0(\Lambda) = Z(\Lambda)$.

This assumption allowed us to define the affine variety $V_H = \mathrm{MaxSpec}(H^*)$ in which to consider the support varieties of finite dimensional Λ-modules as introduced in [135]. By [135, Proposition 4.6(b)], we have $V_H = V_H(\Lambda/\mathfrak{r})$ where $\mathfrak{r}$ is the Jacobson radical of Λ.

As we mentioned before, if $\Lambda = kG$ for a finite group G, then $H^* = \mathrm{H}^{\mathrm{ev}}(G) \cdot \mathrm{HH}^0(kG)$ satisfies **(Fg1)**.

We also need finite generation of $\mathrm{Ext}^*_\Lambda(M, M)$ as an H^*-module. Let $E(\Lambda) = \mathrm{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ be the Yoneda algebra of Λ. The following result holds.

Proposition 1.2.3. *[EHSST04, Proposition 2.4] The H^*-module $E(\Lambda)$ is finitely generated if, and only if, $\mathrm{Ext}^*_\Lambda(M, N)$ is a finitely generated H^*-module for any Λ-modules M, N.*

This led us to the following assumption.

Assumption 1.2.4. [EHSST04] **(Fg2)** $E(\Lambda)$ is a finitely generated H^*-module.

Note that this forces H^* to be sufficiently large. When $H^* = \mathrm{H}^{\mathrm{ev}}(G) \cdot \mathrm{HH}^0(kG)$ where G is a finite group, **(Fg2)** holds [51, 145].

Moreover, if both **(Fg1)** and **(Fg2)** hold, then $\mathrm{HH}^*(\Lambda)$ and $E(\Lambda)$ are finitely generated rings [EHSST04].

This finite generation condition does not always hold: for instance, if Λ is the algebra $k\langle x, y\rangle / (x^2, yx + qxy, y^2)$ where q is in k and is not a root of unity, then it is known that the image of $\varphi_{\Lambda/\mathfrak{r}}$ is contained in the graded centre of $E(\Lambda)$, which is k. The algebra Λ is Koszul, therefore $E(\Lambda) \cong k\langle x, y\rangle / (yx - qxy)$ is an infinitely generated module over any subalgebra H.

With these assumptions, we could now prove some results on varieties. Recall that the **complexity** $c_\Lambda(M)$ of a Λ-module is the rate of growth of a minimal projective Λ-module resolution $\mathbf{P}^\bullet$ of M, that is, the smallest non-negative integer s such that there is a constant $\kappa > 0$ with $\dim P_n \leqslant \kappa n^{s-1}$ for all positive integers n, or ∞ if no such s exists.

Theorem 1.2.5. *[EHSST04, Theorem 2.5] Let Λ be a finite dimensional algebra such that the assumptions **(Fg1)** and **(Fg2)** hold. Then*

(a) *The algebra Λ is Gorenstein, that is, the injective dimensions of Λ viewed as left and right Λ-modules are finite.*

(b) *The following are equivalent:*

   (i) *$V_H(M)$ is trivial*

  (ii) *the projective dimension $\mathsf{p.dim}_\Lambda(M)$ is finite*

 (iii) *the injective dimension $\mathsf{i.dim}_\Lambda(M)$ is finite.*

(c) *For any module M in $A\text{-}\mathsf{mod}$, we have $\dim V_H(M) = c_\Lambda(M)$.*

These properties are weaker than Property (P1) we had for group algebras. However, group algebras are selfinjective, so that every projective module is also injective, and any module with finite projective dimension is in fact projective. It follows that Property (P1) holds when Λ is a selfinjective algebra such that **(Fg1)** and **(Fg2)** are satisfied.

Property (P6) also holds whenever **(Fg1)** and **(Fg2)** are satisfied by [EHSST04, Theorem 4.4]. For Property (P8), we have the following generalisation.

Proposition 1.2.6. *[EHSST04, Proposition 7.1] Suppose that **(Fg1)** and **(Fg2)** hold. If $V_H(M) \cap V_H(N)$ is trivial, then $\mathrm{Ext}^n_\Lambda(M, N) = 0$ for all $n > \mathsf{i.dim}\,\Lambda_\Lambda$.*

If moreover Λ is selfinjective, then $\mathrm{Ext}^n_\Lambda(M, N) = 0$ for all $n \geqslant 1$.

We could also prove Property (P7), that is, that the variety of an indecomposable module is connected.

Proposition 1.2.7. *[EHSST04, Theorem 7.3] Suppose that **(Fg1)** and **(Fg2)** hold and that Λ is selfinjective. If $V_H(M) = V_1 \cup V_2$ with V_i closed homogeneous such that $V_1 \cap V_2$ is trivial, then $M \cong M_1 \oplus M_2$ with $V_H(M_i) = V_i$ for $i = 1, 2$.*

Finally, we also have the equivalent of Property (P5).

Proposition 1.2.8. *[EHSST04, Theorem 5.3] Suppose we have **(Fg1)** and **(Fg2)** and that Λ is selfinjective. Suppose that M is an indecomposable Λ-module. Then M is periodic if, and only if, $V_H(M)$ is a line.*

We then used support varieties and complexity to obtain some structural information on the module category of Λ.

It follows from Theorem 1.2.5 and Proposition 1.2.8 that periodic modules for selfinjective algebras are characterised as modules with complexity one. Using this, we obtained a generalisation of Webb's theorem [147] on the stable Auslander-Reiten quiver of a finite group algebra.

Theorem 1.2.9. *[EHSST04, Theorem 5.6] Suppose that the Nakayama functor is of finite order on any finite-dimensional indecomposable Λ-module. Then the tree class of a component of the stable Auslander-Reiten quiver of Λ is one of the following: a finite Dynkin diagram ($\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_{6,7,8}$), an infinite Dynkin diagram of the type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$, $\mathbb{A}_\infty^\infty$ or a Euclidean diagram.*

This theorem applies in particular to finite dimensional cocommutative Hopf algebras (they are selfinjective, satisfy **(Fg1)** and **(Fg2)** by [61] and the Nakayama automorphism has finite order [56]).

Remark 1.2.10. Our results strongly depend on Conditions **(Fg1)** and **(Fg2)**. These have therefore been much studied since, as they are crucial to a good theory of support varieties, usually in the equivalent form of Condition **(Fg)** obtained in [136]:

(Fg) $HH^*(\Lambda)$ is noetherian and the $HH^*(\Lambda)$-module $E(\Lambda)$ is finitely generated.

Erdmann and Solberg have studied it in [48, 49] for algebras with radical cube zero (also in relation with the generalised Auslander-Reiten condition); Furuya and Snashall used it in their study of varieties for (D, A)-stacked monomial algebras [62]; Skartsæterhagen has studied its invariance under various constructions with several co-authors [134, 101, 123] (derived equivalence and singular equivalence of Morita type in particular); Itaba studied its relationship with cogeometric pairs [92] and Erdmann proved that **(Fg)** cannot hold for finite dimensional selfinjective algebras with a non-periodic bounded module [45]. Similar conditions for Hopf algebras have also been considered in [111, 54].

Remark 1.2.11. There is no general statement corresponding to Property (P9) for finite dimensional algebras. Some difficulties arise in this question. The first is that it does not always make sense to consider the tensor product over k of two Λ-modules: this is well defined if Λ is a Hopf algebra, which is the case of group algebras, but not in general. The other difficulty is that we cannot simply translate the proof from the group case, as this used rank varieties for elementary abelian groups, which we do not have here in general.

However, some work has been done since on this question, see for instance [14] where the authors show that such a property with the tensor product taken over Λ cannot hold in general, and also [149] in the context of Hopf algebras when we can take the tensor product over k.

Remark 1.2.12. Complexity is strongly related to representation type. Recall that finite dimensional algebras over an algebraically closed field split into three disjoint classes: finite representation type, tame or wild. Rickard [129] stated and partially proved a wildness criterion for selfinjective algebras. The gap in his proof has been filled in several contexts. For instance, if Λ is an algebra satisfying **(Fg)** that is selfinjective [55, 15] or a (block of a) Hopf algebra [54], such that there exists a Λ-module M with $c_\Lambda(M) \geqslant 3$, then Λ is wild. In the case of selfinjective algebras, Bergh and Solberg [15] used a theory of relative support varieties that they introduce and actually require weaker assumptions than this, and Feldvoss and Witherspoon [55] use the theory developed in [EHSST04].

Chapter 2

Hochschild cohomology as an object of study

Summary

In this chapter, we present some work on Hochschild cohomology. The first topic is Snashall and Solberg's conjecture, that states that the Hochschild cohomology ring of a finite dimensional algebra up to nilpotents is finitely generated. Snashall and I proved it for several families of algebras in [ST10-1, ST10-2]. The second topic, that I studied in collaboration with Assem, Gatica and Schiffler in [AGST16], is the study of the relationship between the Hochschild cohomology of a trivial extension algebra and that of the original algebra, in particular in the case of relation extensions (related to cluster tilting).

2.1 Snashall and Solberg's conjecture

2.1.1 Introduction

When Snashall and Solberg introduced support varieties of modules over a finite dimensional algebra Λ, they studied the homogeneous nilpotent elements in the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$. They remarked that nilpotent elements in $\mathrm{HH}^*(\Lambda)$ do not contribute to the variety, since the ideal $\mathcal{N}$ generated by the homogeneous nilpotent elements and the radical of $\mathrm{HH}^0(\Lambda)$ is contained in all the maximal ideals. Their study and this remark lad them to state the following:

Conjecture 2.1.1. *[135] Let Λ be a finite dimensional k-algebra. Then $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is finitely generated as a ring.*

This conjecture is not true in general [152] but it is known to hold for many algebras (for the first four listed, the whole Hochschild cohomology ring is finitely generated):

➢ any block of a group ring of a finite group [51, 145].

➢ any block of a finite-dimensional cocommutative Hopf algebra [61].

➢ any finite dimensional pointed Hopf algebra with abelian group of grouplike elements [111].

➢ algebras of finite global dimension.

➢ Ω-periodic algebras, that is, algebras Λ such that $\Omega^n_{\Lambda^e}(\Lambda) \cong \Lambda$ for some positive integer n [78].

➢ finite-dimensional selfinjective algebras of finite representation type over an algebraically closed field [78].

➢ finite-dimensional monomial algebras [79] (they are generally not selfinjective); Green, Snashall and Solberg show that the Hochschild cohomology ring modulo nilpotents is a commutative finitely generated k-algebra of Krull dimension at most one. Moreover, for some specific classes of monomial algebras (Koszul, D-Koszul and (D, A)-stacked [77, 71, 76]), the Hochschild cohomology ring has been completely determined.

I shall now present some families of algebras that Snashall and I have studied.

2.1.2 A class of special biserial algebras $A_{m,N}$

We were interested in [ST10-1] in the Hochschild cohomology of the algebras defined by quiver and relations as follows.

Let $A_{m,N} = k\mathcal{Q}/I_N$ where $\mathcal{Q}$ is the quiver with m vertices and $2m$ arrows (indexed by the cyclic group $\mathbb{Z}_m$):
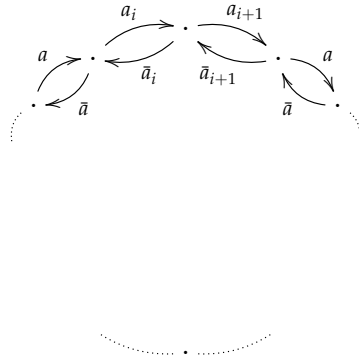
and $I_N = \langle a_{i+1}a_i, \bar{a}_i\bar{a}_{i+1}, (a_i\bar{a}_i)^N - (\bar{a}_{i+1}a_{i+1})^N; i \in \mathbb{Z}_m \rangle$.

For $N = 1$ and m even, this algebra arises in the presentation of the Drinfeld double $\mathcal{D}(\Lambda_{n,d})$ of the Hopf algebra $\Lambda_{n,d}$, where $d|n$, that we obtained in [EGST06] (see Chapter 7); the algebra $\Lambda_{n,d}$ is the quotient of the path algebra of the oriented cycle with n vertices by the ideal generated by all paths of length d.

The algebras $A_{m,N}$ are special biserial selfinjective algebras of tame (not finite) representation type.

The final result is as follows and shows in particular that Conjecture 2.1.1 holds for the algebras $A_{m,N}$.

Theorem 2.1.2. *[ST10-1] The algebra* $\mathrm{HH}^*(A_{m,N})$ *is finitely generated algebra and* $\mathrm{HH}^*(A_{m,N})/\mathcal{N}$ *is a commutative finitely generated k-algebra of Krull dimension two.*

In order to prove this result, we gave a minimal bimodule projective resolution of $A_{m,N}$ explicitly, which is different depending on whether $N = 1$ (when the algebras $A_{m,1}$ are Koszul) or $N > 1$ (when they are not). The main difficulty was to find the maps in the resolution, since the projective bimodules are easy to find using Happel's theorem [80].

Once we had the resolution, we were able to describe the Hochschild cohomology spaces $\mathrm{HH}^n(A_{m,N})$, and the algebra structure on $\mathrm{HH}^*(A_{m,N})$ (the cup-products). We had to separate many cases depending on N (either equal to or greater than 1), m (equal to 1, 2, even or odd), and the characteristic of the base field (equal to 2 or not).

In each case, the Hochschild cohomology ring $\mathrm{HH}^*(A_{m,N})$ is finitely generated and we gave a complete description by generators and relations.

Remark 2.1.3. Since then, the Gerstenhaber bracket on $\mathrm{HH}^*(A_{m,N})$ has also been computed [112].

2.1.3 Socle deformations of the algebras $A_{m,1}$

Fix an integer $m \geqslant 1$ and set $\Lambda = A_{m,1}$. This is a Koszul algebra. Since we have computed its Hochschild cohomology, we have in particular a description of $\mathrm{HH}^2(\Lambda)$. This enabled us in [ST10-2] to determine the algebras $\Lambda(t)$ arising from formal deformations of Λ by a specific element π in $\mathrm{HH}^2(\Lambda)$, depending on a parameter $t \in k, t \neq 0$.

We were interested in socle deformations of Λ, that is, algebras Λ' that are selfinjective and such that $\Lambda'/\mathrm{soc}(\Lambda') \cong \Lambda/\mathrm{soc}(\Lambda)$. Therefore we assumed that $t \neq 1$ since $\Lambda(1)$ is not selfinjective. Then, for $t \neq 0, 1$, the algebra $\Lambda(t)$ is indeed a socle deformation of Λ. We then checked that this element π is the only element in $\mathrm{HH}^2(\Lambda)$ that gives rise to a socle deformation of Λ in this way.

The algebra $\Lambda(t)$ is isomorphic to the algebra $\Lambda_{\mathbf{q}}$ defined by the quiver Q above, with m vertices, and the relations $a_{i+1}a_i$, $\bar{a}_{i-2}\bar{a}_{i-1}$ and $q_i\bar{a}_ia_i - a_{i-1}\bar{a}_{i-1}$ for $i = 0, \ldots, m-1$ and where $\mathbf{q} = (q_0, \ldots, q_{m-1})$ and $t = 1 - q_0q_1\cdots q_{m-1}$.

Set $E^n(\Lambda_{\mathbf{q}}) = \operatorname{Ext}^n_{\Lambda_{\mathbf{q}}}(\Lambda_{\mathbf{q}}/\operatorname{rad}(\Lambda_{\mathbf{q}}), \Lambda_{\mathbf{q}}/\operatorname{rad}(\Lambda_{\mathbf{q}}))$ and let $E(\Lambda_{\mathbf{q}}) = \bigoplus_{n \geqslant 0} E^n(\Lambda_{\mathbf{q}})$ be the Yoneda algebra of $\Lambda_{\mathbf{q}}$. Let $Z_{\mathrm{gr}}(E(\Lambda_{\mathbf{q}}))$ be the graded centre of $E(\Lambda_{\mathbf{q}})$, that is, the ideal generated by all homogeneous elements $z \in E^n(\Lambda_{\mathbf{q}})$ for which $zg = (-1)^{mn}gz$ for all $g \in E^m(\Lambda_{\mathbf{q}})$.

The algebra $\Lambda_{\mathbf{q}}$ is Koszul, therefore by [25, 135], we have $\operatorname{HH}^*(\Lambda_{\mathbf{q}})/\mathcal{N} \cong Z_{\mathrm{gr}}(E(\Lambda_{\mathbf{q}}))/\mathcal{N}_Z$, where $\mathcal{N}_Z$ denotes the ideal of $Z_{\mathrm{gr}}(E(\Lambda_{\mathbf{q}}))$ which is generated by all nilpotent elements.

Moreover, the Yoneda algebra of a Koszul algebra is known by [73], so that we could determine the graded centre $Z_{\mathrm{gr}}(E(\Lambda_{\mathbf{q}}))$ of the Yoneda algebra $E(\Lambda_{\mathbf{q}})$ as follows.

Theorem 2.1.4. *[ST10-2, Theorem 2.6] Let $\mathbf{q} = (q_0, q_1, \ldots, q_{m-1}) \in (k^*)^m$ and let $\zeta = q_0q_1\cdots q_{m-1}$. If ζ is not a root of unity then $Z_{\mathrm{gr}}(E(\Lambda_{\mathbf{q}})) = k$. Now suppose that ζ is a primitive d-th root of unity.*

➢ *If m is even or if $\operatorname{char} k = 2$, then*

$$Z_{\mathrm{gr}}(E(\Lambda_{\mathbf{q}})) = k[x, y, w]/\langle w^m - \varepsilon xy \rangle,$$

where $\varepsilon = (-1)^{md/2} \prod_{l=1}^{m-1} \prod_{j=1}^{ld}(q_j \cdots q_{j+d-1})^{-1}$.

➢ *If m is odd and $\operatorname{char} k \neq 2$, then*

$$Z_{\mathrm{gr}}(E(\Lambda_{\mathbf{q}})) = \begin{cases} k[x, y, w]/\langle w^m - \varepsilon xy \rangle & \text{if } d \equiv 0 \ (\mathrm{mod}\ 4) \text{ or } d \text{ is odd} \\ k[x, y, w]/\langle w^{2m} - \varepsilon xy \rangle & \text{if } d \equiv 2 \ (\mathrm{mod}\ 4) \end{cases}$$

where $\varepsilon = \begin{cases} \prod_{l=1}^{m-1} \prod_{j=1}^{ld}(q_j \cdots q_{j+d-1})^{-1} & \text{if } d \equiv 0 \ (\mathrm{mod}\ 4) \\ \prod_{l=1}^{2m-1} \prod_{j=1}^{ld/2}(q_j \cdots q_{j+d-1})^{-1} & \text{if } d \equiv 2 \ (\mathrm{mod}\ 4) \\ \prod_{l=1}^{m-1} \prod_{j=1}^{2ld}(q_j \cdots q_{j+2d-1})^{-1} & \text{if } d \text{ is odd.} \end{cases}$

It is clear that $\mathcal{N}_Z = 0$, therefore this theorem gives a description of $\operatorname{HH}^*(\Lambda_{\mathbf{q}})/\mathcal{N}$ and proves that Conjecture 2.1.1 is true for $\Lambda_{\mathbf{q}}$.

Remark 2.1.5. It has been shown since by Parker and Snashall [117] that when ζ is not a root of unity, then $\operatorname{HH}^*(\Lambda_{\mathbf{q}})$ is finite dimensional.

2.2 Hochschild cohomology of trivial extension algebras

2.2.1 Introduction and motivation

This section is devoted to the presentation of joint work with Assem, Gatica and Schiffler [AGST16].

Definition 2.2.1. Let C be an algebra and let E be a C-C-bimodule endowed with a product $E \otimes_C E \to E$. Then the vector space $B = C \oplus E$ is an algebra for the multiplication

$$(c, x)(c, x') = (cc', cx' + xc' + xx').$$

This algebra B is called a **split extension** of C by E. In the case where the ideal E in B satisfies $E^2 = 0$, the algebra B is called a **trivial extension** of C by E and is denoted by $B = C \ltimes E$.

If moreover the algebra C has global dimension at most two and $E = E_2 := \operatorname{Ext}^2_C(DC, C)$ then B was called the **relation extension** of C by Assem, Brüstle and Schiffler [3].

Split-by-nilpotent extensions, when E is nilpotent, have been much studied, in particular by Assem and collaborators (see for instance [5, 8, 4]), who have related many properties of the algebras B and C with those of their representations.

Relation extension algebras are of interest because of their close relationship with cluster-tilted algebras. Cluster-tilted algebras were defined by Buan, Marsh and Reiten [24] as a by-product of the now extensive theory of cluster algebras of Fomin and Zelevinsky. They are the subject of many investigations.

Recall that a tilted algebra is an algebra of the form $C = \text{End}_{kQ}(T)$ where Q is a quiver and T is a tilting kQ-module, that is, $\text{Ext}^1_{kQ}(T, T) = 0$ and T has $\#Q_0$ indecomposable summands. The definition of a cluster-tilted algebra is similar, starting with a hereditary algebra and a tilting object in its cluster category. Assem, Brüstle and Schiffler proved the following result.

Theorem 2.2.2. *[3] If C is a tilted algebra, then its relation extension is cluster-tilted and, conversely, every cluster-tilted algebra is of this form. Of course, there exist relation extensions which are not cluster-tilted algebras.*

Their aim was to describe the construction of the quiver of a cluster-tilted algebra from the quiver and relations of a tilted algebra (previously observed in some special cases).

The first Hochschild cohomology groups of a tilted algebra C and the corresponding cluster-tilted algebra B were compared by means of a linear map $\varphi \colon \text{HH}^1(B) \to \text{HH}^1(C)$ in several papers by Assem, Bustamante, Igusa, Redondo and Schiffler [6, 2, 7]. In each of these papers, the map φ is surjective, but the proofs in each case were long and combinatorial and strongly relied on the structure of the algebra C.

Our objective in [AGST16] was to find a purely homological proof of this fact, which we did in a more general situation.

We introduced an algebra morphism $\varphi^* \colon \text{HH}^*(B) \to \text{HH}^*(C)$ relating the Hochschild cohomologies of the two algebras (such that φ^1 is the map φ above). We then turned to the case of a trivial extension of C by E; we introduced necessary and sufficient conditions for each φ^n to be surjective and used them to prove that φ^1 is surjective when $E = \text{Ext}^m_{C\text{-}C}(DC, C)$. We then applied this to relation extensions and generalised results of Assem, Bustamante, Igusa, Redondo and Schiffler [2, 7] when B is a cluster-tilted algebra.

2.2.2 Definition of the Hochschild projection morphisms and first properties

Given a split extension B of C by E, there is an exact sequence of vector spaces

$$0 \to E \xrightarrow{i} B \underset{q}{\overset{p}{\rightleftarrows}} C \to 0$$

in which $p : (c, x) \mapsto c$ and $i : x \mapsto (0, x)$. Clearly, p is an algebra morphism which has a section $q : c \mapsto (c, 0)$.

Now let $d : B \to B$ be a k-linear map. Then $pdq : C \to C$ is also k-linear. It was shown by Assem and Redondo [6] that this correspondence induces a map $\text{HH}^1(B) \to \text{HH}^1(C)$. We extended this to the nth Hochschild cohomology groups.

Proposition 2.2.3. *[AGST16] Let B be the split extension of C by E. Then there exists a k-linear map $\varphi^n \colon \text{HH}^n(B) \to \text{HH}^n(C)$ given by $[f] \mapsto [pfq^{\otimes n}]$, where [?] denotes the cohomology class of a cochain.*

*The map φ^n is called the nth **Hochschild projection morphism**.*

Theorem 2.2.4. *[AGST16] The map $\varphi^* \colon \text{HH}^*(B) \to \text{HH}^*(C)$ is a morphism of graded algebras.*

Remark 2.2.5. However, φ^* is not a morphism of graded Lie algebras in general.

We were interested in the surjectivity of φ^* and more specifically φ^1. This map φ^1 is not surjective in general, not even for trivial extensions.

2.2.3 Trivial extensions

In the case of trivial extensions, we built on the work of [39]. We assume from now on that $B = C \ltimes E$ is the trivial extension of C by E, that is, that $E^2 = 0$.

Applying $\mathrm{Hom}_{B\text{-}B}(B, -)$ to the exact sequence $0 \to E \to B \to C \to 0$ induces a long cohomological exact sequence

$$0 \to \mathrm{Hom}_{B\text{-}B}(B, E) \to \mathrm{Hom}_{B\text{-}B}(B, B) \to \mathrm{Hom}_{B\text{-}B}(B, C) \xrightarrow{\delta_B^0} \mathrm{H}^1(B, E) \to \mathrm{H}^1(B, B) \to \mathrm{H}^1(B, C) \to \cdots$$

$$\cdots \to \mathrm{HH}^n(B, C) \xrightarrow{\delta_B^n} \mathrm{H}^{n+1}(B, E) \to \mathrm{H}^{n+1}(B, B) \to \mathrm{H}^{n+1}(B, C) \cdots$$

Among other things, Cibils, Marcos, Redondo and Solotar gave a description of the connecting morphism δ_B^n.

Let $E^{s,r}$ be the sub-vector space of $B^{\otimes(r+s)}$ spanned by the $x_1 \otimes \cdots \otimes x_n$ such that exactly s of the x_i are in E and r are in C. Since $B = C \oplus E$ as C-C-bimodules, we have $B^{\otimes n} = \bigoplus_{r+s=n} E^{s,r}$. Therefore, if X is any B-B-bimodule, the Hochschild complex $\mathrm{Hom}_k(B^{\otimes *}, X)$ organises into a double complex $\mathrm{Hom}_k(E^{*,*}, X)$.

We shall denote the sth column in this double complex (with vertical differential) by $\mathcal{C}^s(X)$.

Proposition 2.2.6. *[39] The connecting morphism $\delta_B^n : \mathrm{H}^n(B, C) \to \mathrm{H}^{n+1}(B, E)$ decomposes into $\delta_B^n = \bigoplus_{s+r=n} \delta^{s,r}$ where $\delta^{s,r} : \mathrm{H}^r(\mathcal{C}^s(C)) \to \mathrm{H}^r(\mathcal{C}^{s+1}(E))$. Moreover, $\delta^{s,r}([\zeta]) = [\mathrm{id}_E \smile \zeta] + (-1)^{s+r+1}[\zeta \smile \mathrm{id}_E]$ where, if $f : B^{\otimes s} \to C$ and $g : B^{\otimes r} \to C$, $f \smile g : B^{\otimes(s+r)} \xrightarrow{f \otimes g} C \otimes C \to C \otimes_B C$.*

We now give the connection with the Hochschild projection morphisms.

Proposition 2.2.7. *[AGST16] For any $n \geqslant 0$, the nth Hochschild projection morphism $\varphi^n : \mathrm{HH}^n(B) \to \mathrm{HH}^n(C)$ is surjective if, and only if, $\delta^{0,n} = 0$.*

As a consequence, we have a way of characterising explicitly when φ^n is surjective.

Corollary 2.2.8. *[AGST16]*

(a) *The Hochschild projection morphism φ^0 is surjective if, and only if, E is symmetric over $Z(C)$.*

(b) *For $n \geqslant 1$, the Hochschild projection morphism φ^n is surjective if, and only if, for any Hochschild cocycle $\zeta \in \mathrm{Hom}_k(C^{\otimes n}, C)$, there exists a morphism $\alpha \in \mathrm{Hom}_k(E^{1,n-1}, E)$ that satisfies the following three conditions for all $\theta \in E$ and all $\underline{c} = c_1 \otimes \cdots \otimes c_n \in C^{\otimes n}$:*

$$\theta \cdot \zeta(\underline{c}) = -\alpha(\theta \cdot c_1 \otimes c_2 \otimes \cdots \otimes c_n)$$

$$+ \sum_{i=1}^{n-1} (-1)^{i+1} \alpha(\theta \otimes c_1 \otimes \cdots \otimes c_i c_{i+1} \otimes \cdots \otimes c_n) \qquad (2.2.1)$$

$$+ (-1)^{n+1} \alpha(\theta \otimes c_1 \otimes \cdots \otimes c_{n-1}) \cdot c_n$$

$$(-1)^{n+1} \zeta(\underline{c}) \cdot \theta = c_1 \cdot \alpha(c_2 \otimes c_3 \otimes \cdots \otimes c_n \otimes \theta)$$

$$+ \sum_{i=1}^{n-1} (-1)^i \alpha(c_1 \otimes \cdots \otimes c_i c_{i+1} \otimes \cdots \otimes c_n \otimes \theta) \qquad (2.2.2)$$

$$+ (-1)^n \alpha(c_1 \otimes \cdots \otimes c_{n-1} \otimes c_n \cdot \theta)$$

$$0 = d_v \alpha(c_1 \otimes \cdots \otimes c_i \otimes \theta \otimes c_{i+1} \otimes \cdots \otimes c_n)$$

$$\text{for all } i = 1, 2, \ldots, n-1. \qquad (2.2.3)$$

Examples 2.2.9.   ➤ *If $B = C \ltimes DC$ is the trivial extension algebra, then φ^* is surjective. This follows from [39].*

➢ If $B = C \ltimes C$, then φ^* is surjective.

We now consider another family of examples, namely when $E = E_m := \text{Ext}_C^m(DC, C)$. When C is tilted, the trivial extensions $C \ltimes E_m$ are related to $(m-1)$-cluster-tilted algebras.

Theorem 2.2.10. *[AGST16] Let $B = C \ltimes E_m$ be the trivial extension of C by the C-C-bimodule $E_m = \text{Ext}_C^m(DC, C)$, for some integer $m \geqslant 0$. Then the first Hochschild projection morphism $\varphi^1 : \text{HH}^1(B) \to \text{HH}^1(C)$ is surjective.*

Corollary 2.2.11. *[AGST16] If B is the relation extension of C (case $m = 2$), then $\varphi^1 : \text{HH}^1(B) \to \text{HH}^1(C)$ is surjective.*

We then analysed the kernel of φ^1 when φ^1 is surjective, and applied our results to relation extensions.

Theorem 2.2.12. *[AGST16] Let B be the relation extension of a triangular algebra C of global dimension at most two by the C-C-bimodule E_2. Then we have short exact sequences*

(a) $0 \to H^0(B, E_2) \longrightarrow \text{HH}^0(B) \xrightarrow{\varphi^0} \text{HH}^0(C) \to 0.$

(b) $0 \to H^1(B, E_2) \longrightarrow \text{HH}^1(B) \xrightarrow{\varphi^1} \text{HH}^1(C) \to 0.$

As an application, we get the following result when C is a tilted algebra (so that $\text{HH}^n(C) = 0$ for $n \geqslant 2$).

Corollary 2.2.13. *[AGST16] Let B be a cluster-tilted algebra and let C be a tilted algebra such that $B = C \ltimes E_2$. Then the algebra morphism $\varphi^* : \text{HH}^*(B) \to \text{HH}^*(C)$ is surjective. Moreover, there is an exact sequence*

$$0 \to H^0(B, E_2) \oplus H^1(B, E_2) \to \text{HH}^*(B) \xrightarrow{\varphi^*} \text{HH}^*(C) \to 0.$$

It should be noted that even in the case of a relation extension, when φ^1 is surjective, the higher Hochschild projection morphisms need not be surjective.

Chapter 3

The Yoneda algebra of Brauer graph algebras and the Brauer graph algebras that are 2-d-*Koszul and* $\mathcal{K}_2$

Summary

We present joint work with Green, Schroll and Snashall in [GSST17] on 2-d-determined, 2-d-Koszul and $\mathcal{K}_2$ Brauer graph algebras.

3.1 Introduction

Koszul algebras are a well-known and much studied class of algebras. These were generalised in 2001 by Berger to N-Koszul algebras. They are the algebras $T_E(V)/I$ over a semisimple k-algebra E whose ideal I can be generated by elements of degree N and such that the projective modules in a minimal graded projective A-module resolution of k can be generated in specific degrees depending on N. Moreover, the Yoneda algebra of A is generated in degrees 0, 1 and 2.

This notion has been generalised since in several ways. We are interested in two of them:

➢ an algebra is called $\mathcal{K}_2$ if it is graded and if its Yoneda algebra is generated in degrees 0, 1 and 2 (Cassidy and Shelton [30]);

➢ an algebra $A = T_E(V)/I$ is called **2-d-determined** if the ideal I can be generated by elements of degrees 2 and d, where $d > 2$ is an integer, and the projective modules in a minimal graded projective resolution of E can be generated in specific degrees depending on 2 and d (Green and Marcos [70]).

The aim of the work Green, Schroll, Snashall and I did in [GSST17] was to give examples of such algebras, within the class of Brauer graph algebras, and to compare and characterise $\mathcal{K}_2$ Brauer graph algebras and 2-d-determined Brauer graph algebras.

Throughout this chapter, k is a field, E is a semisimple k-algebra (in the sequel, we will have $E = k\mathcal{Q}_0$ for some quiver $\mathcal{Q}$), $\Lambda = T_E(V)/I$ is a finite-dimensional graded k-algebra and R is a set of homogeneous generators for I. We write Λ_d for the summand of degree d of Λ.

Let $\boldsymbol{P}_* : \ \cdots \to P_1 \to P_0 \to \Lambda_0 \to 0$ be a minimal graded projective resolution of Λ_0.

Recall that Λ is **Koszul** if P_n is generated in degree n for all $n \geqslant 0$.

Let $E(\Lambda) = \mathrm{Ext}^*_\Lambda(\Lambda_0, \Lambda_0)$ be the Yoneda algebra of Λ. Recall the following characterisations of Koszul algebras.

Theorem 3.1.1. *[72] The following are equivalent:*

(i) Λ *is Koszul*

(ii) $E(\Lambda)$ *is generated in degrees* 0 *and* 1.

(iii) $E(E(\Lambda)) \cong \Lambda$ *as graded algebras.*

In particular, Λ can be recovered from its Yoneda algebra. Another characterisation of Koszul algebras in terms of the A_∞-structure of $E(\Lambda)$ was given by Keller [94].

Theorem 3.1.2. *[72] If Λ is Koszul, then Λ is quadratic. If Λ is quadratic and has global dimension 2, then Λ is Koszul.*

There are many examples of Koszul algebras, and they occur in many places, such as topology, commutative and non-commutative algebra, commutative and non-commutative geometry.

The notion of Koszul algebra was generalised to homogeneous algebras with relations that are not (necessarily) quadratic by Berger [10] when $E = k$ (see [71] for the non-local case). He was motivated by Artin-Schelter regular algebras of global dimension 3 and generated in degree 1, which are non-commutative analogues of polynomial algebras and are examples of 3-Koszul algebras.

Define $\delta : \mathbb{N} \to \mathbb{N}$ by $\delta(n) = \begin{cases} \dfrac{n}{2}d & \text{if } n \text{ is even} \\ \dfrac{n-1}{2}d + 1 & \text{if } n \text{ is odd.} \end{cases}$

Definition 3.1.3. [10] Λ is d-**Koszul** if P_n is generated in degree $\delta(n)$ for all n.

When $d = 2$, $\delta(n) = n$ so that a 2-Koszul algebra is precisely a Koszul algebra.

Property 3.1.4. *[71] If Λ is d-Koszul, then the ideal I can be generated in degree d (that is, $R \subset V^{\otimes d}$). We say that Λ is d-**homogeneous** in this case.*

Green, Marcos, Martínez-Villa and Zhang then gave a characterisation of d-Koszul algebras using their Yoneda algebras.

Theorem 3.1.5. *[71] The following are equivalent:*

(i) Λ *is d-Koszul*

(ii) Λ *is d-homogeneous and $E(\Lambda)$ is generated in degrees 0, 1 and 2.*

Again, there are many examples of such algebras as well as Artin-Schelter regular algebras of global dimension 3, such as truncated quiver algebras, anti-symmetriser algebras, 3-Calabi-Yau algebras.

These algebras in turn were generalised in several directions (see for instance [103, 86, 108, 104]). We are interested here in weakly δ-determined and 2-d-determined/Koszul algebras, that were introduced by Green and Marcos [70].

The general definition is as follows. Let $F : \mathbb{N} \to \mathbb{N}$ be a function. They considered algebras as above such that P_n is generated

➢ in degree $F(n)$, which they called *F-**determined*** algebras;

➢ in degree at most $F(n)$, which they called **weakly *F*-determined** algebras.

A d-Koszul algebra is therefore a δ-determined algebra.

Weakly δ-determined algebras are difficult to study, so we consider only **2-d-homogeneous** algebras, that is, algebras such that all elements in R are of degree 2 or d but that are neither quadratic nor d-homogeneous.

Definition 3.1.6. [70]

➢ Λ is **2-d-determined** if it is weakly δ-determined and 2-d-homogeneous.

➢ Λ is **2-d-Koszul** if Λ is 2-d-determined and $E(\Lambda)$ is finitely generated.

Cassidy and Shelton also considered a generalisation of d-Koszul algebras, defined in terms of the Yoneda algebra.

Definition 3.1.7. [30] Λ is $\mathcal{K}_2$ if Λ is graded and $E(\Lambda)$ is generated in degrees 0, 1 and 2.

Green and Marcos proved a number of properties of 2-d-determined and 2-d-Koszul algebras, analysing a minimal resolution and using Gröbner bases. In particular, they proved that 2-d-determined *monomial* algebras are 2-d-Koszul. But this left a number of questions open. Assume that the global dimension of Λ is infinite.

(i) If Λ is 2-d-determined, is it 2-d-Koszul?

(ii) If Λ is 2-d-Koszul, is it $\mathcal{K}_2$?

(iii) If Λ is 2-d-homogeneous and $\mathcal{K}_2$, is it 2-d-Koszul?

They were answered negatively in general by Cassidy and Phan [29], who gave explicit examples of 2-4-determined algebras with 9 (*resp.* 14) generators and whose Ext algebra is not finitely generated (*resp.* whose Ext algebra is finitely generated but not in degrees 0, 1 and 2).

Another important question is to find examples of such algebras. Such examples can be found within Brauer graph algebras, and for these algebras we will answer positively the questions above.

3.2 Brauer graph algebras (BGA)

To simplify, we now fix an algebraically closed field k.

In order to fix notation and conventions, let us recall some definitions.

Definition 3.2.1. A **Brauer graph** is a triple $(\Gamma, \mathfrak{o}, \mathfrak{m})$ where:

➢ Γ is a finite connected graph (a local embedding of a graph in the plane), with vertex set Γ_0 and edge set Γ_1;

➢ $\mathfrak{m} : \Gamma_0 \to \mathbb{N}^*$ is a **multiplicity function**;

➢ $\mathfrak{o}$ is the choice of a cyclic ordering/orientation around each vertex in Γ_0.

Given an edge t attached to a vertex α, the next edge in the cyclic ordering around α is called the **successor of** t **at** α. If t is the only edge attached to α and $\mathfrak{m}(\alpha) > 1$, then t is its own successor. A loop has two successors (and two predecessors), one of which is the loop itself.

If t is the only edge attached to α and $\mathfrak{m}(\alpha) = 1$, then we say that t is **truncated at** α.

To each Brauer graph, we can associate algebras defined by quiver and relations, $\mathcal{A}_\Gamma = k\mathcal{Q}_\Gamma / I_\Gamma$.

In the special case where Γ is the graph $\alpha - \beta$, we take $\mathcal{Q}_\Gamma$ to be the quiver $\cdot \, \overset{\curvearrowright}{} \, x$ and $I_\Gamma = (x^2)$ so that the algebra is $\mathcal{A}_\Gamma = k[x]/(x^2)$. We now exclude this case.

Definition 3.2.2. To a Brauer graph $(\Gamma, \mathfrak{o}, \mathfrak{m})$, we associate the quiver $\mathcal{Q}_\Gamma$, where

➢ Γ_1 is in one-to-one correspondence with $(\mathcal{Q}_\Gamma)_0$ via $\{t; t \in \Gamma_1\} \overset{1-1}{\longleftrightarrow} \{v_t \in (\mathcal{Q}_\Gamma)_0\}$.

➢ if the edge s is not truncated at α and t is the successor of s at α, then there is an arrow from v_s to v_t.

Given an edge s which is not truncated at one of its endpoints α, if $s = s_0, s_1, \ldots, s_n$ are the edges incident with α, where s_{i+1} is the successor of s_i at α for all i and s_0 is the successor of s_n at α, then this defines a cycle $C_{s,\alpha}$ in the quiver.

In order to define I_Γ, we need more data.

Definition 3.2.3. If s is an edge which is not truncated at either of its endpoints α and β, we associate two non-zero scalars $\mathfrak{q}_{s,\alpha}$ and $\mathfrak{q}_{s,\beta}$.

With this additional data we call $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ a **quantised Brauer graph**, and $\mathfrak{q}$ is a **quantising function**.

If the field is algebraically closed and if either the Brauer graph is a tree or the Brauer graph algebra is symmetric, then we may assume that $\mathfrak{q} \equiv 1$.

We can now define the set ρ_Γ such that $I_\Gamma = (\rho_\Gamma)$.

Definition 3.2.4. The set ρ_Γ contains three types of relations.

➢ Type I. If s is an edge which is not truncated at either of its endpoints α and β, then the relation $\mathfrak{q}_{s,\alpha} C_{s,\alpha}^{\mathfrak{m}(\alpha)} - \mathfrak{q}_{s,\beta} C_{s,\beta}^{\mathfrak{m}(\beta)}$ or its opposite is in ρ_Γ.

➢ Type II. Let s be an edge that is truncated at one of its endpoints α. By assumption, s is not truncated at its other endpoint β. Write $C_{s,\beta} = b_n \cdots b_1 b_0$. Then the relation $b_0 C_{s,\beta}^{\mathfrak{m}(\beta)}$ is in ρ_Γ.

➢ Type III. They are the relations of type ab, where ab is a path of length 2 which is not a sub-path of any $C_{s,\alpha}^{m(\alpha)}$.

3.3 'Koszul' Brauer graph algebras

Which Brauer graph algebras are d-Koszul? 2-d-Koszul?

3.3.1 d-homogeneous and 2-d-homogeneous Brauer graph algebras.

The set ρ_Γ is not a minimal set of relations for I_Γ, some of the relations of type II are superfluous. A relation of type II associated to an edge s truncated at α is in a minimal set of relations if, and only if, the successor of s at its other endpoint is also truncated.

Example 3.3.1. *We give an example of two different embeddings in the plane of the same 3-dimensional graph; the Brauer graphs are therefore different and so are the corresponding Brauer graph algebras.*

In both examples, the multiplicity at every vertex is 1.

| Brauer graph | construction of quiver | $\mathcal{Q}_1$ |

| Brauer graph | construction of quiver | $\mathcal{Q}_2$ |
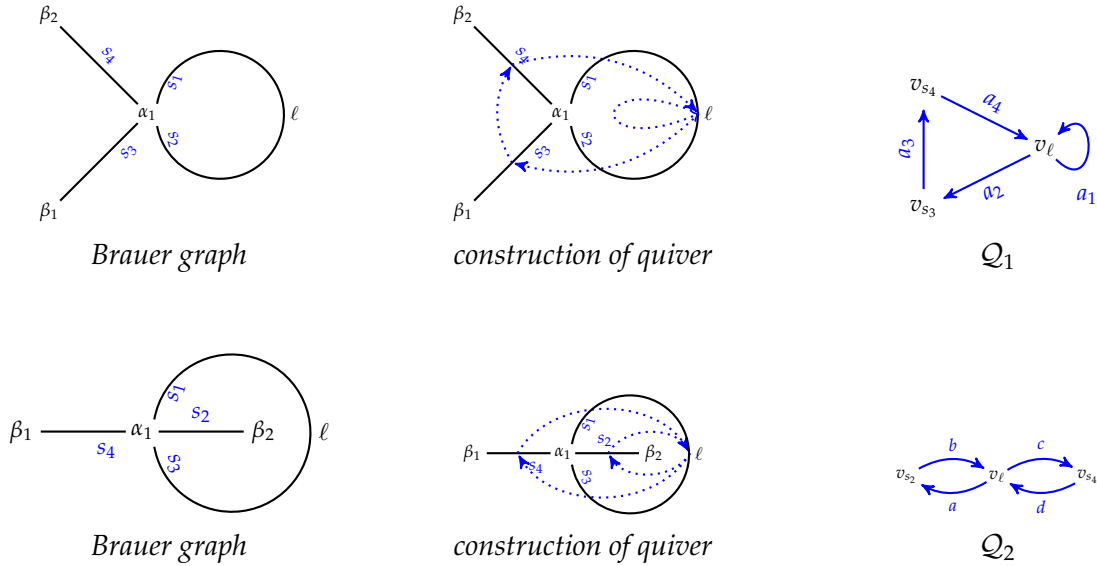
The relations in the first Brauer graph algebra are

➢ *Type I.* $a_4a_3a_2a_1 - a_1a_4a_3a_2$ *(at s_1 and s_2)*

➢ *Type II.* $a_3a_2a_1a_3$ *at s_3 and* $a_4a_3a_2a_1a_4$ *at s_4*

➢ *Type III.* a_2a_4.

The only superfluous relation is $a_4a_3a_2a_1a_4$ (the successor ℓ of s_4 at α_1 is not truncated at its other endpoint α_1).

The relations in the second Brauer graph algebra are

➢ *Type I.* $dcba - badc$ *at ℓ*

➢ *Type II.* $badcb$ *at s_2 and $dcbad$ at s_4*

➢ *Type III.* ab and cd.

Neither of the relations of type II are in a minimal set of generators.

In order to find out which Brauer graph algebras are d-Koszul, we first determined all the Brauer graph algebras that are d-homogeneous ([GSST17, Propositions 3.2 and 3.4]). We then examined each one to determine whether or not it is d-Koszul. The list of d-Koszul Brauer graph algebras is then as follows.

Theorem 3.3.2. *[GSST17, Theorem 3.4] Let $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ be a quantised Brauer graph and let $\mathcal{A}_\Gamma$ be the associated Brauer graph algebra. Then $\mathcal{A}_\Gamma$ is 2-Koszul if and only if it is quadratic and either $\Gamma = \mathbb{A}_2$ or Γ has no truncated edges. For $d \geqslant 3$, the Brauer graph algebra $\mathcal{A}_\Gamma$ is d-Koszul if and only if it is d-homogeneous.*

Note that the quadratic Koszul Brauer graph algebras include the algebra $\Lambda_{\mathbf{q}}$ of Section 2.1.2. The d-Koszul Brauer graph algebras for $d \geqslant 3$ are precisely the symmetric Nakayama algebras whose quiver is a cycle of length n with $n \mid (d-1)$ and its ideal I_Γ is generated by all paths of length d.

We now turn to 2-d-homogeneous Brauer graph algebras. They are characterised as follows.

Proposition 3.3.3. *[GSST17, Proposition 3.5] Let $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ be a quantised Brauer graph and let $\mathcal{A}_\Gamma$ be the associated Brauer graph algebra. Then $\mathcal{A}_\Gamma$ is 2-d-homogeneous if and only if $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ satisfies one of the following conditions.*

(1) *For all vertices α in Γ, the number of edges incident with α is $\dfrac{d}{\mathfrak{m}(\alpha)}$.*

(2) *Γ has a truncated edge, $\Gamma \neq \mathbb{A}_2$, no two successors are truncated, and for every vertex α in Γ, the number of edges incident with α is either $\dfrac{d}{\mathfrak{m}(\alpha)}$ or $\dfrac{1}{\mathfrak{m}(\alpha)}$.*

3.3.2 The Yoneda algebra

The question is now which of these 2-d-homogeneous Brauer graph algebras are 2-d-Koszul? That is, for which of these algebras is the Ext algebra finitely generated? We studied the Yoneda algebra of a Brauer graph algebra in general.
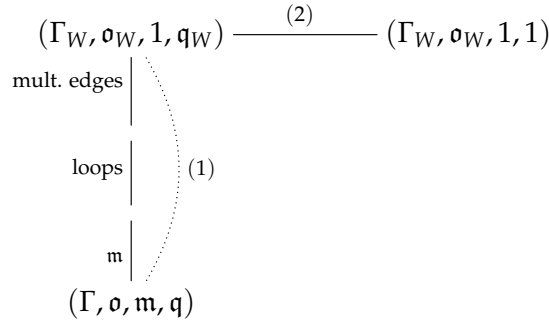
The first step was to simplify the Brauer graph algebras we needed to consider. We used the theory of coverings, which was studied for Brauer graph algebras by Green, Schroll and Snashall in [74]. Using this, we could prove the following.

Proposition 3.3.4. *[GSST17, Propositions 4.1 and 4.2] Let $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ be a quantised Brauer graph. Then there exists a Brauer graph $(\Gamma_W, \mathfrak{o}_W, \mathfrak{m}_W, \mathfrak{q}_W)$ such that*

➤ *Γ_W has no loops or multiple edges;*

➤ *$\mathfrak{m}_W \equiv 1$;*

➤ *the Ext algebras $\mathrm{Ext}^*_{\mathcal{A}_{\Gamma_W}}(\mathcal{A}_{\Gamma_W}/\mathfrak{r}\mathcal{A}_{\Gamma_W}, \mathcal{A}_{\Gamma_W}/\mathfrak{r}\mathcal{A}_{\Gamma_W})$ and $\mathrm{Ext}^*_{\mathcal{A}_{\Gamma}}(\mathcal{A}_{\Gamma}/\mathfrak{r}\mathcal{A}_{\Gamma}, \mathcal{A}_{\Gamma}/\mathfrak{r}\mathcal{A}_{\Gamma})$ are generated in the same degrees.*

If moreover k is algebraically closed and $\mathcal{A}_{\Gamma_W}$ is length graded, then the number of generators and their degrees do not depend on $\mathfrak{q}_W$.

Principle:

$$(\Gamma_W, \mathfrak{o}_W, 1, \mathfrak{q}_W) \xrightarrow{\quad (2) \quad} (\Gamma_W, \mathfrak{o}_W, 1, 1)$$

$$\text{mult. edges} \Big| \quad \vdots$$

$$\text{loops} \Big| \quad \vdots \ (1)$$

$$\mathfrak{m} \Big|$$

$$(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$$

where

➤ (1) is a tower of coverings so that Γ_W has no loops or multiple edges (and $\mathfrak{m}_W = 1$).

These coverings preserves finite generation of the Ext algebra and the set of degrees of the generators.

➤ (2) is obtained by twisting $\mathcal{A}_{\Gamma_W}$ by a *graded* automorphism, adapting an idea of Cassidy and Shelton [30].

This construction preserves the number of generators and their degrees. It requires some assumptions, either that k be algebraically closed, or some weaker conditions on k and stronger conditions on the Brauer graph. (Note that if k is algebraically closed and if Γ_W is a tree or $\mathcal{A}_{\Gamma_W}$ is symmetric, we can always replace $\mathfrak{q}_W$ by 1 by re-scaling the arrows in the quiver).

Assumption 3.3.5. We will therefore assume from now on that Γ has no loops or multiple edges, that $\mathfrak{m} \equiv 1$ and that $\mathfrak{q} \equiv 1$ (however, the results will be valid in general).

With these assumptions, we then studied the structure of $\mathcal{A}_\Gamma$-modules; there are two types of indecomposable $\mathcal{A}_\Gamma$-modules: uniserial modules and string modules, and they are described in terms of the Brauer graph ([GSST17, Sections 5-7]).

Then, in the case where the Brauer graph has no truncated edges, we gave a minimal projective resolution for each of the simple $\mathcal{A}_\Gamma$-modules in [74, Theorem 8.4]. This enabled us to prove

Theorem 3.3.6. *[GSST17, Theorem 9.1] Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ be a Brauer graph with no truncated edges, and let $\mathcal{A}_\Gamma$ denote the associated Brauer graph algebra. Then the Yoneda algebra $E(\mathcal{A}_\Gamma)$ is finitely generated in degrees $0, 1$ and 2.*

As a consequence, when $\mathcal{A}_\Gamma$ is graded, we got the following result.

Corollary 3.3.7. *[GSST17, Corollary 9.2] Let k be an algebraically closed field and let $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ be a quantised Brauer graph with no truncated edges. Suppose $\mathcal{A}_\Gamma$ is length graded. Then $\mathcal{A}_\Gamma$ is a $\mathcal{K}_2$ algebra.*

In the case where the Brauer graph has truncated edges, we were not able to describe explicitly a minimal projective resolution of the simple modules, however, we obtained some information on some of their syzygies ([GSST17, Corollary 10.02]) and on the Yoneda algebra of $\mathcal{A}_\Gamma$ when the Brauer graph has both truncated and non truncated edges ([GSST17, Theorem 10.3]).

We were then able to describe exactly which Brauer graph algebras are $\mathcal{K}_2$.

Theorem 3.3.8. *[GSST17, Corollary 10.5] Let $(\Gamma, \mathfrak{o}, \mathfrak{m})$ be a Brauer graph. Then the Yoneda algebra $E(\mathcal{A}_\Gamma)$ is finitely generated in degrees $0, 1$ and 2 if and only if Γ does not have both truncated and non-truncated edges.*

If moreover $\mathcal{A}_\Gamma$ is length graded, then $\mathcal{A}_\Gamma$ is $\mathcal{K}_2$ if and only if Γ does not have both truncated and non-truncated edges.

Our final result gives necessary and sufficient conditions for a Brauer graph algebras to be $\mathcal{K}_2$ and to be 2-d-Koszul. In particular, the questions asked by Green and Marcos all have a positive answer for Brauer graph algebras.

Theorem 3.3.9. *[GSST17, Theorem 10.6] Let $(\Gamma, \mathfrak{o}, \mathfrak{m}, \mathfrak{q})$ be a quantised Brauer graph, and assume that either $\mathfrak{q} \equiv 1$ or the field k is algebraically closed (different conditions work). Let $d \geqslant 3$ and suppose that $\mathcal{A}_\Gamma$ is 2-d-homogeneous. Then the following are equivalent:*

(1) *Γ has no truncated edges,*

(2) *$\mathcal{A}_\Gamma$ is 2-d-determined,*

(3) *$\mathcal{A}_\Gamma$ is 2-d-Koszul,*

(4) *$\mathcal{A}_\Gamma$ is $\mathcal{K}_2$.*

(5) *For all vertices α in Γ, the number of edges incident with α is $\dfrac{d}{\mathfrak{m}(\alpha)}$.*

Since one of the aims of our study was to find explicit examples of 2-d-Koszul algebras, let us give some here.

Examples 3.3.10.

Example 1. *The algebras $A_{m,N}$ of Chapter 2 are Brauer graph algebras (the Brauer graph is the oriented cycle with m edges and the multiplicity at every vertex is N). They are 2-2N-Koszul by Theorem 3.3.9.*

Example 2. *Let us give a smaller example, to illustrate the fact that further examples are easy to construct.*
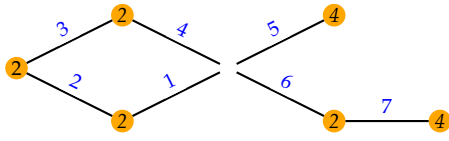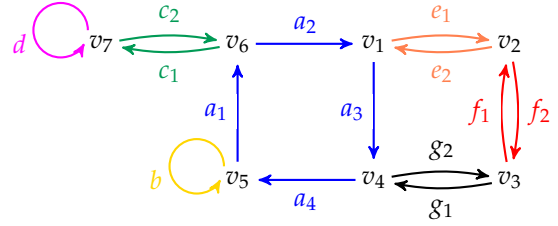

Brauer graph Γ *Quiver of* $\mathcal{A}_\Gamma$

Relations:

➤ *Type one (up to scalars* $\mathfrak{q}$*).*

$$(g_1g_2)^2 - a_3a_2a_1a_4 \qquad a_4a_3a_2a_1 - b^4 \qquad a_1a_4a_3a_2 - (c_2c_1)^2$$
$$(c_1c_2)^2 - d^4 \qquad a_2a_1a_4a_3 - (e_2e_1)^2 \qquad (e_1e_2)^2 - (f_1f_2)^2$$
$$(f_1f_2)^2 - (g_1g_2)^2.$$

➤ *Type two. None.*

➤ *Type three.*

e_1a_2	f_2e_1	e_2f_1	a_3e_2	g_1f_2
f_1g_2	a_4g_1	ba_4	a_1b	c_1a_1
d_1c_1	c_2d	a_2c_2		

Since the algebra is 2-4-*homogeneous and none of the edges in the Brauer graph are truncated, the algebra* $\mathcal{A}_\Gamma$ *is* 2-4-*Koszul.*

Chapter 4

Poincaré-Birkhoff-Witt deformations of 3-Calabi-Yau algebras

Summary

In this chapter, we discuss work that Berger and I did in [BT07] on a conjecture of Van den Bergh for non-graded 3-Calabi-Yau algebras.

4.1 Introduction

Since their introduction by Kontsevich (inspired by Calabi-Yau varieties in complex algebraic geometry), the study of Calabi-Yau categories in representation theory has expanded rapidly in recent years (especially in the context of cluster categories) and inspired a non-commutative version of Calabi-Yau algebras whose precise definition is due to Ginzburg.

Let A be an algebra over a field k. The vector space $A \otimes A$ is an A-bimodule in two different ways:

➢ for the *outer* structure ($\lambda \cdot a \otimes b \cdot \mu = a\mu \otimes \lambda b$) and

➢ for the *inner* structure ($\lambda \cdot a \otimes b \cdot \mu = \lambda a \otimes b\mu$).

Therefore $\operatorname{Hom}_{A-A}(A, A \overset{\text{out}}{\otimes} A)$ is again an A-bimodule, using the inner structure of $A \otimes A$.

Consequently, the Hochschild cohomology groups $\operatorname{H}^n(A, A \overset{\text{out}}{\otimes} A)$ are A-bimodules using the inner structure on $A \otimes A$. The following definition is due to Ginzburg.

Definition 4.1.1. [68] An algebra A which has a finite resolution by projective A-bimodules of finite type is **Calabi-Yau of dimension** $d \geqslant 1$ or d-**Calabi-Yau** if there are A-bimodule isomorphisms

$$\operatorname{H}^n(A, A \otimes A) \cong \begin{cases} A & \text{if } n = d \\ 0 & \text{if } n \neq d. \end{cases}$$

Proposition 4.1.2. *[BT07] Let A be a Calabi-Yau algebra of dimension d. The Hochschild dimension of A (p. $\dim_{A^e} A$) is equal to d. Moreover, if there exists a nonzero finite dimensional A-module, then the global dimension of A is d.*

We were interested in [BT07] in a conjecture of Van den Bergh's. In order to state it, we need the following definition. From now on, $A = kQ/I$ where Q is a quiver and I is an admissible ideal.

Definition 4.1.3. ([68], first introduced in [97]) Let Q be a quiver.

➢ A **potential** in Q is an element in the vector space $\operatorname{Pot}(Q) := kQ/[kQ, kQ]$, that is, the class of a linear combination of cycles. It can be viewed as an element in kQ via the map $\mathfrak{c} : \operatorname{Pot}(Q) \to kQ$ that sends a cycle $a_n \ldots a_1$ to $\sum_{i=1}^n a_{i-1} \ldots a_1 a_n \ldots a_i$.

➢ If p is a path and b is an arrow, then pb^{-1} denotes the path $a_n \cdots a_2$ if $b = a_1$, and is 0 otherwise. Define similarly the path $b^{-1}p$. For each $a \in Q_1$, we shall consider the map $\partial_a : \mathrm{Pot}(Q) \to kQ$ that sends an element p in $\mathrm{Pot}(Q)$ to $\mathfrak{c}(p)a^{-1} = a^{-1}\mathfrak{c}(p)$.

➢ If W is a potential, we shall denote by $A(Q, W)$ the **potential algebra** $kQ/(\partial_a W \mid a \in Q_1)$ (also called Jacobian algebra following [42]). We shall say that an algebra A **derives from a potential** if it is isomorphic to a potential algebra.

Ginzburg and Van den Bergh conjectured that

Conjecture 4.1.4. *[68, 143] Every 3-Calabi-Yau algebra derives from a potential.*

Bocklandt [20] proved it for a graded algebra A, that is, when I is a homogeneous ideal. Denote by Q_j the set of paths of length j in Q and let $\Delta : A \to A \underset{kQ_0}{\otimes} kQ_1 \underset{kQ_0}{\otimes} A$ be the map defined on a path $a_n \cdots a_1 \in Q_n$ by $\Delta(a_n \cdots a_1) = \sum_{i=1}^{n} a_k \cdots a_{i+1} \underset{kQ_0}{\otimes} a_i \underset{kQ_0}{\otimes} a_{i-1} \cdots a_1$ and by $\Delta(1) = 0$.

Theorem 4.1.5. *[20] If a quiver algebra $A = kQ/I$ is a graded Calabi-Yau algebra of dimension 3, then there exists a (homogeneous) potential W such that $A = A(Q, W)$. Moreover, if W is a potential, the algebra $A(Q, W)$ is Calabi-Yau of dimension 3 if and only if the complex C_W below is exact:*

$$C_W : \bigoplus_{e \in Q_0} A \underset{kQ_0}{\otimes} k e \mathfrak{c}(W) \underset{kQ_0}{\otimes} A \xrightarrow{\delta_3} \bigoplus_{a \in Q_1} A \underset{kQ_0}{\otimes} k \partial_a W \underset{kQ_0}{\otimes} A \xrightarrow{\delta_2} A \underset{kQ_0}{\otimes} kQ_1 \underset{kQ_0}{\otimes} A \xrightarrow{\delta_1} A \underset{kQ_0}{\otimes} A \to 0$$

where

$$\delta_3(e\mathfrak{c}W) = \sum_{a \in eQ_1} a \underset{kQ_0}{\otimes} \partial_a W \underset{kQ_0}{\otimes} 1 - \sum_{b \in Q_1 e} 1 \underset{kQ_0}{\otimes} \partial_b W \underset{kQ_0}{\otimes} a \quad \text{for } e \in Q_0$$

$$\delta_2(1 \underset{kQ_0}{\otimes} \partial_a W \underset{kQ_0}{\otimes} 1) = \Delta(\partial_a W)$$

$$\delta_1(1 \underset{kQ_0}{\otimes} a \underset{kQ_0}{\otimes} 1) = a \underset{kQ_0}{\otimes} 1 - 1 \underset{kQ_0}{\otimes} a \quad \text{for } a \in Q_1.$$

Remark 4.1.6. Bocklandt worked with a stronger definition of a Calabi-Yau algebra. However, the properties of Calabi-Yau algebras which he actually used are also true with our definition.

Starting from Bocklandt's result Theorem 4.1.5, we have considered the non-homogeneous situation.

4.2 PBW-deformations of N-Koszul algebras (Berger-Ginzburg)

The following result defines and characterises Poincaré-Birkhoff-Witt deformations of N-Koszul algebras (in the sense of Berger [10]). We state it in the context of our work, but the assumptions that A is N-Koszul and that k is a field can be weakened.

Theorem 4.2.1. *[12] (see also [57]) Let Q be a quiver, $N \geqslant 2$ an integer, P a kQ_0-sub-bimodule of $kQ_{\leqslant N}$, $\pi : kQ_{\leqslant N} \to kQ_N$ the projection and $R = \pi(P)$. Consider the algebras $U = kQ/(P)$ and $A = kQ/(R)$. Assume that A is N-Koszul. Then*

$$A = \mathrm{gr}(U) \iff \begin{cases} P \cap kQ_{\leqslant N-1} = 0 & \textbf{(PBW1)} \\ (P \underset{kQ_0}{\otimes} kQ_1 + kQ_1 \underset{kQ_0}{\otimes} P) \cap kQ_{\leqslant N} \subseteq P & \textbf{(PBW2)} \end{cases}$$

*We say that U is a **PBW deformation** of A (by analogy with the classical PBW theorem which states that $\mathrm{gr}\,\mathcal{U}(\mathfrak{g}) \cong \mathcal{S}(\mathfrak{g})$).*

Lemma 4.2.2. *[BT07] The 3-Calabi-Yau algebras considered by Bocklandt are N-Koszul, where N is the length of the relations in I or, equivalently, $N + 1$ is the length of the cycles occurring in W.*

We can consider PBW-deformations of these algebras in the sense of Berger-Ginzburg.

4.3 Non-homogeneous 3-Calabi-Yau and potential algebras

We first show that 'deforming' the potential of a graded potential 3-Calabi-Yau algebra A produces a PBW-deformation of A.

Theorem 4.3.1. *[BT07] Let $A = A(Q, W_{N+1})$ be a graded Calabi-Yau algebra of dimension 3, where W_{N+1} is a homogeneous potential of degree $N + 1$, and let $W = W_{N+1} + W' = W_{N+1} + W_N + \cdots + W_0$ be a potential with $\deg W_j = j$ for each $0 \leqslant j \leqslant N + 1$. Then $A' := A(Q, W)$ is a PBW deformation of A.*

In fact, the proof shows that A' satisfies a condition stronger than **(PBW2)**:

$$\textbf{(PBW2')} \quad (P \underset{kQ_0}{\otimes} kQ_1 + kQ_1 \underset{kQ_0}{\otimes} P) \cap kQ_{\leqslant N} = 0.$$

We also have a partial converse.

Theorem 4.3.2. *[BT07] Let $A = A(Q, W_{N+1})$ be a graded Calabi-Yau algebra of dimension 3, and let A' be a PBW deformation of A.*

  (i) *If A' can be obtained from a potential of the form $W = W_{N+1} + W'$ with $\deg W' \leqslant N$ then condition **(PBW2')** holds.*

  (ii) *Assume that $\mathrm{char}(k)$ does not divide $N!$ and that condition **(PBW2')** holds. Then A' can be obtained from a potential of the form $W = W_{N+1} + W'$ with $\deg W' \leqslant N$.*

Our proof gives an explicit construction of a potential. The condition on the characteristic of k is necessary: there are examples when the conclusion is not true without this assumption.
Then we showed that such PBW deformations are also 3-Calabi-Yau.

Theorem 4.3.3. *[BT07] Let $A = A(Q, W_{N+1})$ be a graded Calabi-Yau algebra of dimension 3. Let $A' = A(Q, W)$ be a PBW deformation of A defined by a potential $W = W_{N+1} + W'$ with $\deg W' \leqslant N$. Then A' is Calabi-Yau of dimension 3.*

We finally gave several examples illustrating our results.

Examples 4.3.4. **(1)** *Yang-Mills algebras. They are graded, 3-Koszul, AS-Gorenstein of global dimension 3 and 3-Calabi-Yau. Therefore they derive from a potential, which we give explicitly.*

  *Their PBW deformations are known from [11]. Not all of them satisfy **(PBW2')**, we give explicitly those that do.*

  (2) *Artin-Schelter regular algebras of global dimension 3. We prove that those that are Calabi-Yau are precisely those of type A in Artin-Schelter's classification [1]. They all derive from a potential.*

  *In the cubic case, all their PBW deformations were given in [57] and they all satisfy **(PBW2')**, therefore they all derive from non homogeneous potentials, which we give explicitly.*

  (3) *Antisymmetriser algebras A_n. We prove that A_n is Calabi-Yau if and only n is odd. We then characterise all the PBW deformations of A_n (n odd) that satisfy **(PBW2')** and show that not all PBW deformations do.*

(4) *Some non-connected examples (quiver algebras with several vertices). We first consider some examples taken from [20] and check that all their PBW deformations satisfy **(PBW2')**.*

We also consider some small examples:

➢ *examples such that if* char $k \nmid (N + 1)!$*, all PBW deformations satisfy **(PBW2')** and hence derive from a potential,*

➢ *examples such that if* char $k \mid N!$ *we can have PBW deformations that do not satisfy **(PBW2')**,*

➢ *examples that satisfy **(PBW2')** but do not derive from a potential.*

Remark 4.3.5. Ginzburg and Van den Bergh's Conjecture 4.1.4 is not true in general: Davison [41] gave some counterexamples arising from hyperbolic manifolds.

Remark 4.3.6. Generalisations of Conjecture 4.1.4 has been studied by several authors (before and since), and proved assuming some conditions on A [1, 68, 42]. The most general result in this direction was obtained by Van den Bergh in [144], who proved that any complete d-Calabi-Yau algebra is derived from a potential. In a different direction, Bocklandt, Schedler and Wemyss proved in [21] that N-Koszul twisted Calabi-Yau algebras correspond to algebras that derive from a potential using higher order derivations and a condition on a complex.

In the opposite direction, Berger and Solotar [19] wondered which connected graded potential algebras A are 3-Calabi-Yau and gave a necessary and sufficient condition on the potential W when A is assumed to be N-Koszul.

Chapter 5

Classification of families of algebras up to stable equivalence of Morita type

Summary

We describe several invariants of stable equivalence of Morita type, then present two instances in which they have been used to separate families of algebras up to stable equivalence of Morita type: in joint work with Snashall on generalised Nakayama algebras in [ST15], and in the case of tame symmetric algebras of dihedral, semi-dihedral and quaternion type in [T19].

5.1 Introduction

Throughout this chapter, *k* is an algebraically closed field and all algebras are finite dimensional indecomposable *k*-algebras.

Many authors, such as Skowroński, Bocian, Holm, Białkowski, Zimmermann... are interested in tame finite dimensional algebras, in particular those that are selfinjective, for instance

- ➤ blocks of group algebras

- ➤ Hopf algebras,

- ➤ Brauer graph algebras,

- ➤ Erdmann's algebras, that are characterised by properties of the Auslander-Reiten quiver, the Cartan matrix and the representation type; they contain all tame blocks of group algebras.

Their aim is to classify them, up to equivalences of categories, such as Morita equivalence, derived equivalence, stable equivalence. Some of the questions and methods in this subject can be found in [160].

In this chapter, I will describe some invariants of equivalences of categories associated to Hochschild cohomology and give some applications obtained in [ST15, T19].

5.2 Invariants

5.2.1 Hochschild cohomology

It is well known that $\mathrm{HH}^0(A)$ identifies with the centre $Z(A)$ of A and that $\mathrm{HH}^1(A)$ is the quotient of the set of derivations of A by the inner derivations of A. Moreover, $\mathrm{HH}^1(A)$ is a Lie algebra.

In fact, this is part of a larger structure. As we have mentioned in previous chapters, the Hochschild cohomology $\mathrm{HH}^*(A) = \bigoplus_{n \in \mathbb{N}} \mathrm{HH}^n(A)$ is a graded algebra, whose product is the cup-product $\smile$.

It is moreover endowed with a graded Lie bracket, for a shifted grading (on $\mathrm{HH}^{*+1}(A)$): $[,] : \mathrm{HH}^p(A) \times \mathrm{HH}^q(A) \to \mathrm{HH}^{p+q-1}(A)$. This graded Lie bracket induces the Lie bracket on the first cohomology group $\mathrm{HH}^1(A)$.

These two structures are compatible and $\mathrm{HH}^*(A)$ is then called a Gerstenhaber algebra.

When we want to compute Hochschild cohomology explicitly, the Hochschild complex is usually too large. Therefore we use other constructions.

The Hochschild cohomology of A is the cohomology of the complex $(\mathrm{Hom}_{A^e}(P^\bullet, A), \delta^\bullet_*)$ where $(P^\bullet, \delta^\bullet)$ is a (preferably minimal) projective A-bimodule resolution of A.

There are methods to compute the first few terms of such minimal projective resolutions for general basic algebras (see for instance [75]), but they do not generalise well for higher HH^n, except in some cases (monomial algebras for example [79]). There are also methods to compute whole minimal projective resolutions for algebras satisfying some conditions (see work by Chouhy and Solotar for instance [37]).

Both $\smile$ and $[,]$ were originally defined in terms of the bar resolution, but whereas there are several ways of determining $\smile$ when $\mathrm{HH}^*(A)$ is described in terms of another projective resolution (see for instance [ST10-1, Paragraph 2.1.2]), the situation is more difficult for $[,]$. Given two projective resolutions $(P^\bullet, \delta^\bullet)$ and $(Q^\bullet, \partial^\bullet)$ of A, there always exist comparison morphisms $f^\bullet : P^\bullet \to Q^\bullet$ and $g^\bullet : Q^\bullet \to P^\bullet$ such that $f \circ g$ and $g \circ f$ are quasi-isomorphisms. If we know such comparison morphisms between $(P^\bullet, \delta^\bullet)$ and the bar resolution explicitly, or at least for small values of n, then we can transport the Lie algebra structure on $\mathrm{HH}^1(A)$ so that it is described in terms of cocycles in $\mathrm{Hom}_{A-A}(P_1, A)$ instead of derivations. (A different approach to finding the graded Lie bracket on $\mathrm{HH}^*(A)$ was given in [114]).

This Lie algebra has been studied in particular by C. Strametz [138] (using a minimal projective resolution) in the case of a basic monomial algebra. She gives an explicit combinatorial description of the bracket in terms of paths in the quiver. This description was used in particular by Bessenrodt and Holm [16] in their study of gentle algebras.

5.2.2 Equivalences of categories

We shall use Hochschild cohomology to distinguish algebras up to some equivalences of categories, which I describe briefly here.

Derived equivalences.

Let A be a finite dimensional k-algebra. Consider the category $\mathcal{K}$ of complexes of A-modules whose homology vanishes for sufficiently large positive and negative degrees. The **bounded derived category** $\mathcal{D}^b(A)$ of A is the largest quotient of $\mathcal{K}$ such that quasi-isomorphisms become isomorphisms. This category is naturally a triangulated category.

Two algebras A and B are **derived equivalent** if their bounded derived categories are equivalent as triangulated categories.

Theorem 5.2.1. *[130] If A and B are derived equivalent, then their Hochschild cohomologies $\mathrm{HH}^*(A)$ and $\mathrm{HH}^*(B)$ are isomorphic. Moreover, the Lie algebras $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$ are also isomorphic, as well as the algebras $\mathrm{HH}^0(A)$ and $\mathrm{HH}^0(B)$.*

Holm in particular has used this invariant in order to classify some of Erdmann's algebras up to derived equivalence [87, 88].

Remark 5.2.2. In fact, the whole of the Gerstenhaber structure is invariant under derived equivalence [93].

Stable equivalences.

Let A be a finite dimensional k-algebra. The **stable category** $A\text{-}\underline{\mathrm{mod}}$ associated to A has the same objects as $A\text{-}\mathrm{mod}$ and $\underline{\mathrm{Hom}}_A(X, Y) = \mathrm{Hom}_A(X, Y)/\mathcal{P}(X, Y)$ where $\mathcal{P}(X, Y)$ is the space

of morphisms of A-modules from X to Y that factor through a projective module.

$$X \xrightarrow{f \in \mathcal{P}(X,Y)} Y$$
$$P$$

Two algebras are said to be **stably equivalent** if their stable module categories are equivalent.

We shall be interested in a special kind of stable equivalence, called **stable equivalence of Morita type**, introduced by Broué, that is often known to be induced by an exact functor between the module categories of the algebras ([131, 43, 159]).

Definition 5.2.3. [23] Two algebras A and B are stably equivalent of Morita type of there exist left-right projective modules ${}_B M_A$ and ${}_A N_B$ such that $M \otimes_A N \cong B \oplus Q$ and $N \otimes_B M \cong A \oplus P$ where ${}_A P_A$ and ${}_B Q_B$ are projective bimodules.

Such an equivalence induces a stable equivalence, in the same way as a Morita equivalence, via $M \otimes_A -$ and $N \otimes_B -$. Stable equivalences of Morita type occur naturally, as the following result shows.

Theorem 5.2.4. *[128, 95] Let A and B be two* selfinjective *algebras that are derived equivalent. Then they are stably equivalent of Morita type.*

This kind of equivalence has been much studied, for instance in [105, 43, 91, 102]; some invariants for stable equivalences of Morita type have been developed [157, 96, 116]; it has been shown for example that they preserve representation type [98], the property of an algebra of being selfinjective, symmetric or indecomposable [106, 105] but not tensor products or trivial extensions [107], and various families of algebras have been classified up to stable equivalence of Morita type [154, 155, 158]. Such equivalences have also been generalised, to singular equivalences of Morita type [156] (where the projectives P and Q are replaced with modules with finite projective dimension) and to singular equivalence of Morita type with level [146, 134] (here, in the conditions on $M \otimes_A N$ and $N \otimes_B M$, the algebras A and B are replaced by a bimodule syzygy).

5.2.3 Invariants of derived and stable equivalences of Morita type associated with Hochschild cohomology

We start with the Hochschild cohomology spaces.

Theorem 5.2.5. *[150] Let A and B be two algebras that are stably equivalent of Morita type. Then for $n \geqslant 1$ we have $\mathrm{HH}^n(A) \cong \mathrm{HH}^n(B)$ (as k-vector spaces).*

This is not necessarily true for HH^0 or for the Lie structure of HH^1 in general, I shall come back to this.

However, Hochschild cohomology is not easy to compute in general. It is therefore useful to have invariants that are finer and easier to compute that the $\mathrm{HH}^n(A)$.

We now describe these invariants and fix notation.

Invariants in the centre in positive characteristic

$\mathrm{HH}^0(A)$, the centre $Z(A)$ of A, is invariant under derived equivalence.

Assume here that k is a perfect (or algebraically closed) field of characteristic $p > 0$.

Definition 5.2.6. Set

$$[A, A] := \text{span}\{ab - ba; a, b \in A\},$$
$$T_n(A) := \left\{ a \in A; a^{p^n} \in [A, A] \right\} \quad \text{for } n \in \mathbb{N}.$$

Brauer proved that $(a + b)^{p^n} \equiv a^{p^n} + b^{p^n} \pmod{[A, A]}$ so that $T_n(A)$ is a subspace of A. It is even a $Z(A)$-module and $T_n(A) \subset T_{n+1}(A)$.

Now assume that A is symmetric, that is, A is isomorphic to its k-dual DA as an A-bimodule or, equivalently, there exists a non-degenerate associative symmetric bilinear form $(,) : A \times A \to k$.

Let $M^\perp$ be the orthogonal of a subset M of A for this bilinear form. Then $[A, A]^\perp = Z(A)$.

There is a sequence of ideals in $Z(A)$:

$$Z(A) = [A, A]^\perp = T_0(A)^\perp \supseteq T_1(A)^\perp \supseteq T_2(A)^\perp \supseteq \cdots \supseteq T_n(A)^\perp \supseteq \cdots$$

called **Külshammer ideals** or **generalised Reynolds ideals**. They do not depend on the choice of bilinear form on A.

They were defined by Külshammer, who also proved that they are Morita invariant. Then Zimmermann considered derived invariance.

Theorem 5.2.7. *[157] If B is derived equivalent to A, then B is necessarily symmetric and the isomorphism between $Z(A)$ and $Z(B)$ induces isomorphisms between $T_n(A)^\perp$ and $T_n(B)^\perp$ for all n.*

However, the centre is not preserved under stable equivalence of Morita type in general. But we have the following.

Proposition 5.2.8. *[107] Let A and B be two algebras that are stably equivalent of Morita type. If A is symmetric then B is symmetric by [106], and we have $\dim Z(A) = \dim Z(B)$ if, and only if, the number of non-projective simple modules is the same for A and for B.*

The Auslander-Reiten conjecture states that the number of non-projective simple modules is preserved under stable equivalence.

Pogorzały [122] proved this conjecture for selfinjective special biserial algebras.

Definition 5.2.9. An algebra kQ/I is called **special biserial** if

- each vertex in Q is the source of at most two arrows and the target of at most two arrows,

- given an arrow β, there is at most one arrow α whose target is the source of β and such that $\alpha\beta \notin I$, and there is at most one arrow γ whose source is the target of β and such that $\beta\gamma \notin I$.

In particular, two special biserial algebras such that one of them is symmetric and indecomposable and that are stably equivalent of Morita type have isomorphic centres.

Another option in general in the case of stable equivalences of Morita type is to replace the centre by the stable centre. Recall that $Z(A) \cong \text{End}_{A^e}(A)$.

Definition 5.2.10. The **stable centre** is $Z^{st}(A) = \underline{\text{End}}_{A^e}(A) = \text{End}_{A^e}(A)/Z^{pr}(A)$ where $Z^{pr}(A) = \mathcal{P}(A, A)$ is the **projective centre** of A, formed by the endomorphisms of A that factor through a projective.

Theorem 5.2.11. **(a)** *[23] The stable centre is invariant under stable equivalence of Morita type.*

(b) *[107] If A is symmetric, the ideals $T_n^{st}(A)^\perp := T_n(A)^\perp / Z^{pr}(A)$ of $Z^{st}(A)$ are invariant under stable equivalence of Morita type.*

Zhou and Zimmermann [154] used these invariants in the classification of algebras of dihedral, semi-dihedral and quaternion type up to stable equivalence of Morita type.

Note that the algebra $HH^*(A)/Z^{pr}(A)$ is invariant under stable equivalence of Morita type [116].

The Lie algebra $HH^1(A)$

We have already mentioned that the Lie algebra structure of $HH^1(A)$ is preserved under derived equivalence (as well as the Gerstenhaber algebra structure on $HH^*(A)$ [93]).

It is not known in general whether the Lie algebra structure of $HH^1(A)$ is preserved under stable equivalence of Morita type. However, for symmetric algebras, the stable Hochschild cohomology $HH^*(A)/Z^{pr}(A)$ is a Gerstenhaber algebra (in fact, it is a Batalin-Vilkovisky algebra), and this structure is preserved under stable equivalence of Morita type [96]. In particular, so is the Lie algebra $HH^1(A)$.

We shall now use these invariants in order to classify some algebras up to stable equivalence of Morita type.

5.3 Generalised Nakayama algebras

This is joint work with Nicole Snashall [ST15].

Definition 5.3.1. Nakayama algebras are the algebras such that for any indecomposable projective or injective module M, the sequence

$$M \supset \mathrm{rad}(M) \supset \mathrm{rad}^2(M) \supset \cdots$$

is a composition series (M is said to be **uniserial**).

They have been much studied, are of finite representation type, their module categories are well known, they are the basic algebras such that $\Omega^2_{A^e}(A) \cong A_\sigma$ for some automorphism σ [46]. The Nakayama algebras are the distinct representatives of the stable equivalence classes of Brauer tree algebras [63]. Moreover, the basic Nakayama algebras are special biserial.

Isomorphism classes, quivers and relations

Since k is algebraically closed, every basic finite dimensional algebra is isomorphic to an algebra of the form kQ/I where I is a two-sided ideal in the path algebra kQ of a quiver Q.

The basic symmetric Nakayama algebras are the algebras $N_m^n = k\Delta_n/(\text{paths of length} \geqslant nm + 1)$.
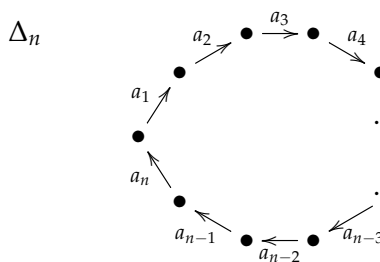

They are indeed symmetric, with bilinear form defined on the paths p in Q by $(p, 1) =$
$\begin{cases} 1 & \text{if } p \text{ has length } nm \\ 0 & \text{if } p \text{ has length } < nm. \end{cases}$

The paths of length nm are the cyclic permutations of $(a_n \cdots a_1)^m$, and it is then easy to check that $(,)$ is symmetric. The fact that this bilinear form is non-degenerate and associative is a consequence of a theorem of Holm and Zimmermann [90].

Set $\Lambda = kQ/I$ and let $1, \ldots, n$ be the vertices of Q. Then the indecomposable projective left Λ-modules are the Λe_i. Moreover, $\mathrm{rad}^k(\Lambda e_i)$ is the vector space generated by the paths of length at least k that start at i. Since injective modules are projective (the algebra is symmetric), the module Λe_i is projective if, and only if, for each k there is at most one path of length k that starts at i. This is clearly true for the Nakayama algebras N_m^n.

We were interested in symmetric special biserial algebras with at most one non-uniserial indecomposable projective module. It is easy to see that, if they are not Nakayama algebras, their quiver must necessarily be

$$Q_{(p,q)}, \; 1 \leqslant p \leqslant q$$
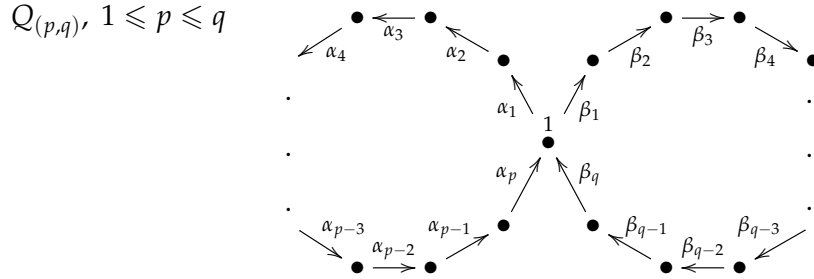
Define two ideals in $kQ_{(p,q)}$:

- the ideal I_r, for $r \in \mathbb{N}^{>0}$, generated by

$$\alpha_1 \alpha_p, \qquad \beta_1 \beta_q, \qquad (\beta_q \cdots \beta_1 \alpha_p \cdots \alpha_1)^r - (\alpha_p \cdots \alpha_1 \beta_q \cdots \beta_1)^r,$$
$$\alpha_i (\alpha_{i-1} \cdots \alpha_1 \beta_q \cdots \beta_1 \alpha_p \cdots \alpha_i)^r \text{ for all } 2 \leqslant i \leqslant p-1,$$
$$\beta_j (\beta_{j-1} \cdots \beta_1 \alpha_p \cdots \alpha_1 \beta_q \cdots \beta_j)^r \text{ for all } 2 \leqslant j \leqslant q-1;$$

- the ideal $J_{(s,t)}$, for s, t in $\mathbb{N}^{>0}$, generated by

$$\beta_1 \alpha_p, \qquad \alpha_1 \beta_q, \qquad (\alpha_p \cdots \alpha_1)^s - (\beta_q \cdots \beta_1)^t,$$
$$\alpha_i (\alpha_{i-1} \cdots \alpha_1 \alpha_p \cdots \alpha_i)^s \text{ for all } 2 \leqslant i \leqslant p-1,$$
$$\beta_j (\beta_{j-1} \cdots \beta_1 \beta_q \cdots \beta_j)^t \text{ for all } 2 \leqslant j \leqslant q-1,$$

with, if $p = 1$ then $s \geqslant 2$, and, if $q = 1$ then $p = 1, s \geqslant 2$ and $t \geqslant 2$.

The algebras $kQ_{(p,q)}/I_r$ and $kQ_{(p,q)}/J_{(s,t)}$ are clearly special biserial. The algebras $kQ_{(p,q)}/I_1$ and $kQ_{1,n}/J_{(2,2)}$ occur as two of the three families of selfinjective algebras of Euclidean type up to derived and stable equivalence by [18]. Moreover, some of these algebras are derived equivalent to algebras of dihedral type in the classification of Holm [87]: $kQ_{(1,1)}/I_r = D(1\mathcal{A})_1^r$, $kQ_{(1,2)}/I_r$ that is derived equivalent to $D(2\mathcal{B})^{1,r}(0)$, and $kQ_{(2,2)}/I_r$ that is derived equivalent to $D(3\mathcal{K})^{r,1,1}$, all three of which come from tame blocks of finite groups when $\mathrm{char}(k) = 2$ and r is a power of 2, as well as $kQ_{(2,2)}/J_{(s,t)}$ that is derived equivalent to $D(2\mathcal{R})^{1,s,t,1}$ and which does not come from blocks, see [88, 87, 106].

Theorem 5.3.2. [ST15] *The algebras $kQ_{(p,q)}/I_r$ and $kQ_{(p,q)}/J_{(s,t)}$ are symmetric, special biserial, and have at most one non-uniserial indecomposable projective module.*

Moreover, every basic, indecomposable, finite dimensional, symmetric, special biserial algebra with at most one non-uniserial indecomposable projective module is isomorphic to one of the algebras N_m^n, $kQ_{(p,q)}/I_r$ and $kQ_{(p,q)}/J_{(s,t)}$.

Classification up to derived equivalence and stable equivalence of Morita type

Theorem 5.3.3. *[ST15, Theorems 4.3 and 5.1]*

(1) *An algebra of the form $kQ_{(p,q)}/J_{(s,t)}$ (with $1 \leqslant p \leqslant q$) is derived equivalent (resp. stably equivalent of Morita type) to exactly one algebra in the following list:*

    (a) $kQ_{(1,p+q-1)}/J_{(s,t)}$ *with $2 \leqslant s \leqslant t$,*

    (b) N_M^{p+q-1} *with $p + q > 2$ and $\min(s,t) = 1$, $\max(s,t) = M$.*

(2) *An algebra of the form $kQ_{(p,q)}/I_r$ (with $1 \leqslant p \leqslant q$) is derived equivalent (resp. stably equivalent of Morita type) to an algebra of the form $kQ_{(p,q)}/J_{(s,t)}$ if and only if they are isomorphic. This is only the case for $kQ_{(1,1)}/I_1 \cong kQ_{(1,1)}/J_{(2,2)}$ and $\operatorname{char} k \neq 2$.*

(3) *The algebras $kQ_{(p,q)}/I_r$ and $kQ_{(p',q')}/I_{r'}$ (with $1 \leqslant p \leqslant q$ and $1 \leqslant p' \leqslant q'$) are derived equivalent (resp. stably equivalent of Morita type) if and only if $(p,q,r) = (p',q',r')$.*

For the proof, we used the following invariants:

➢ The number of simple modules (n and $p + q - 1$). It is derived invariant, and also invariant under stable equivalence of Morita type for special biserial selfinjective algebras [122].

➢ The centre of the algebra, as an algebra, for derived equivalence.

➢ $\dim \mathrm{HH}^1$.

➢ The determinant of the Cartan matrix $C_A = (c_{ij})$ where $c_{ij} = e_j A e_i$ (it is a derived invariant [19]) and its absolute value (which is an invariant of stable equivalence of Morita type by [150]).

➢ $\dim \mathrm{HH}^{2i}$ for $i < p$, computed for $kQ_{(p,q)}/I_r$.

➢ Generalised Brauer tree algebras (see below). We also used generalised Brauer tree algebras for stable equivalence of Morita type, but only for the Nakayama algebras.

➢ Külshammer ideals and the stable equivalence of Morita type invariants $Z^{st}(A)/T_n^{st}(A)^\perp \cong Z(A)/T_n(A)^\perp$.

A generalised Brauer tree is a Brauer graph algebra whose underlying graph is a tree. The generalised Brauer tree algebras are completely determined, up to derived equivalence, by the number of edges and the set of multiplicities [113]. The algebras $kQ_{(p,q)}/J_{(s,t)}$ are such algebras. This enabled us to prove the first part of the theorem.

A Brauer tree is a generalised Brauer tree for which all multiplicities except possibly one of them are equal to 1. The corresponding algebras are precisely the algebras which are stably equivalent to a symmetric non-simple Nakayama algebra N_m^n [63] (the multiplicity is m).

5.4 Tame algebras of dihedral, semi-dihedral and quaternion types

Let k be an algebraically closed field of characteristic $p \geqslant 0$ and let G be a finite group. Then the group algebra kG is the direct sum of indecomposable algebras, the blocks of kG. Each block is a symmetric algebra, and when p divides the order of G they are not semisimple in general. To each block B of kG is associated a defect group.

Erdmann [44] studied the representations of blocks of kG, in particular when the blocks are tame. In this case, the defect groups are dihedral, semi-dihedral or generalised quaternion and $p = 2$. She then studied finite dimensional symmetric algebras of dihedral, semi-dihedral and quaternion type, which she describes up to Morita equivalence, giving a list of representatives

by quiver and relations. They are characterised by properties of their Auslander-Reiten quiver, their Cartan matrix and their representation type. They contain in particular all tame blocks of group algebras.

Derived equivalences. Holm [87] classified these algebras up to derived equivalence, except some special cases depending on a scalar that he could not separate.

Using Külshammer ideals, Holm and Zimmermann [90] continued this classification in the case of algebras of dihedral and semi-dihedral type (the case of algebras of dihedral type, which they complete, was also completed by less elementary methods by Kauer). For this, they defined a specific associative symmetric non-degenerate bilinear form on these algebras (valid for any symmetric algebra defined by quiver and relations). In their computations, it is enough to consider $T_1(A)^\perp$.

Moreover, Holm and Zhou [89] used Külshammer ideals to prove that a family of algebras in Erdmann's list was indeed in the same derived equivalence class as blocks of group algebras in characteristic 2.

Stable equivalences. Zhou and Zimmermann next studied these algebras up to stable equivalence of Morita type using the invariants in the centre and the stable centre (algebra structure of $Z^{st}(A)/T_n^{st}(A)^\perp \cong Z(A)/T_n(A)^\perp$).

But this wasn't enough to separate some families. I then used the Lie algebra $\mathrm{HH}^1(A)$ to make some progress in separating some of these families. The bracket in each case is determined using a minimal projective resolution.

The algebras in question all have one, two or three simple modules. It is known that there are then 9 subfamilies, each of which is stable under stable equivalent of Morita type (algebras of dihedral, semi-dihedral and quaternion type with one, two or three simple modules).

These algebras are described by quiver and relations as follows.

Quivers.
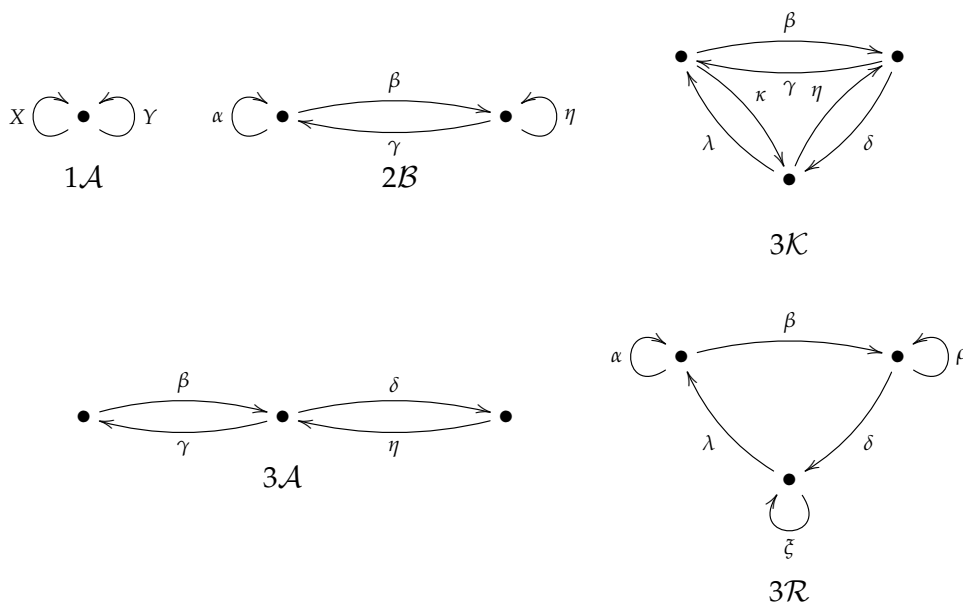
Relations.

$$D(1\mathcal{A})_1^\ell \;:\;\; X^2, Y^2, (XY)^\ell - (YX)^\ell;$$

$$D(1\mathcal{A})_2^\ell(d) \;:\;\; X^2 - (XY)^\ell, Y^2 - d(XY)^\ell, (XY)^\ell - (YX)^\ell, (XY)^\ell X, (YX)^\ell Y;$$

$$SD(1\mathcal{A})_1^\ell \;:\;\; (XY)^\ell - (YX)^\ell, (XY)^\ell X, Y^2, X^2 - (YX)^{\ell-1}Y;$$

$$SD(1\mathcal{A})_2^\ell(c,d) \;:\;\; (XY)^\ell - (YX)^\ell, (XY)^\ell X, Y^2 - d(XY)^\ell, X^2 - (YX)^{\ell-1}Y + c(XY)^\ell;$$

$$Q(1\mathcal{A})_1^\ell \;:\;\; (XY)^\ell - (YX)^\ell, (XY)^\ell X, Y^2 - (XY)^{\ell-1}X, X^2 - (YX)^{\ell-1}Y;$$

$$Q(1\mathcal{A})_2^\ell(c,d) \;:\;\; X^2 - (YX)^{\ell-1}Y - c(XY)^\ell, Y^2 - (XY)^{\ell-1}X - d(XY)^\ell,$$
$$(XY)^\ell - (YX)^\ell, (XY)^\ell X, (YX)^\ell Y;$$

$$D(2\mathcal{B})^{\ell,s}(c) \;:\;\; \eta\beta, \gamma\eta, \beta\gamma, \alpha^2 - c(\gamma\beta\alpha)^\ell, (\gamma\beta\alpha)^\ell - (\alpha\gamma\beta)^\ell, \eta^s - (\beta\alpha\gamma)^\ell;$$

$$SD(2\mathcal{B})_1^{\ell,t}(c) \;:\;\; \beta\gamma, \gamma\eta, \eta\beta, \alpha^2 - \gamma\beta(\alpha\gamma\beta)^{\ell-1} - c(\gamma\beta\alpha)^\ell, \eta^t - (\beta\alpha\gamma)^\ell,$$
$$(\gamma\beta\alpha)^\ell - (\alpha\gamma\beta)^\ell;$$

$$SD(2\mathcal{B})_2^{\ell,t}(c) \;:\;\; \eta\beta - \beta\alpha(\gamma\beta\alpha)^{\ell-1}, \gamma\eta - \alpha\gamma(\beta\alpha\gamma)^{\ell-1}, \beta\gamma - \eta^{t-1},$$
$$\alpha^2 - c(\beta\beta\alpha)^\ell, \eta^2\beta, \gamma\eta^2;$$

$$Q(2\mathcal{B})_1^{\ell,s}(a,c) \;:\;\; \beta\gamma - \eta^{s-1}, \eta\beta - \beta\alpha(\gamma\beta\alpha)^{\ell-1}, \gamma\eta - \alpha\gamma(\beta\alpha\gamma)^{\ell-1},$$
$$\alpha^2 - a\gamma\beta(\alpha\gamma\beta)^{\ell-1} - c(\alpha\gamma\beta)^\ell, \beta\alpha^2, \alpha^2\gamma;$$

$$D(3\mathcal{K})^{a,b,c} \;:\;\; \delta\beta, \lambda\delta, \beta\lambda, \kappa\gamma, \eta\kappa, \gamma\eta, (\gamma\beta)^a - (\lambda\kappa)^b, (\kappa\lambda)^b - (\delta\eta)^c, (\eta\delta)^c - (\beta\gamma)^a;$$

$$D(3\mathcal{R})^{\ell,s,t,u} \;:\;\; \beta\alpha, \rho\beta, \delta\rho, \xi\delta, \lambda\xi, \alpha\lambda, \alpha^s - (\lambda\delta\beta)^\ell, \rho^t - (\beta\gamma\delta)^\ell, \xi^u - (\delta\beta\lambda)^\ell;$$

$$SD(3\mathcal{K})^{a,b,c} \;:\;\; \eta\kappa, \gamma\eta, \kappa\gamma, \lambda\delta - \gamma(\beta\gamma)^{a-1}, \delta\beta - \kappa(\lambda\kappa)^{b-1}, \beta\lambda - \eta(\delta\eta)^{c-1};$$

$$Q(3\mathcal{K})^{a,b,c} \;:\;\; \delta\beta - \kappa(\lambda\kappa)^{a-1}, \gamma\eta - \lambda(\kappa\lambda)^{a-1}, \lambda\delta - \gamma(\beta\gamma)^{b-1}, \eta\kappa - \beta(\gamma\beta)^{b-1},$$
$$\beta\lambda - \eta(\delta\eta)^{c-1}, \kappa\gamma - \delta(\eta\delta)^{c-1}, \delta\beta\gamma, \gamma\eta\delta, \varepsilon\kappa\lambda;$$

$$Q(3\mathcal{A})_1^{2,2}(d) \;:\;\; \varepsilon\delta\beta - \beta\gamma\beta, \beta\eta\delta - \gamma\beta\gamma, \beta\gamma\eta - d\eta\delta\eta, \delta\beta\gamma - d\delta\eta\delta, \delta\eta\delta\beta, \gamma\beta\gamma\eta.$$

Next, we summarise in Table 5.1 the results of [154] on the stable equivalence of Morita type classification of these algebras, using also [158] for the algebras of quaternion type with two simples.

For the algebras of dihedral type, the only remaining question was answered negatively in [T19]: $D(1\mathcal{A})^\ell(0)$ and $D(1\mathcal{A})^\ell(1)$ are not stably equivalent of Morita type since their first Hochschild cohomology groups have different dimensions. The classification of algebras of dihedral type up to stable equivalence of Morita type is now complete.

Now consider algebras of semi-dihedral and quaternion type. For those which have one simple module, some progress could be made:

Proposition 5.4.1. *[T19] The algebras with the following parameters are not stably equivalent of Morita type;*

(a) **Semi-dihedral.** *If $c \neq 0$: $SD(1\mathcal{A})^\ell(0,0)$, $SD(1\mathcal{A})^\ell(0,1)$, $SD(1\mathcal{A})^\ell(1,0)$ and $SD(1\mathcal{A})^\ell(c,1)$ are not stably equivalent of Morita type.*

(b) **Quaternion.** *If $cd \neq 0$: $Q(1\mathcal{A})^\ell(0,0)$, $Q(1\mathcal{A})^\ell(0,d)$ and $Q(1\mathcal{A})^\ell(c,d)$ are not stably equivalent of Morita type. (NB. $Q(1\mathcal{A})^\ell(c,d) \cong Q(1\mathcal{A})^\ell(d,c)$).*

For those with two simple modules, some progress could be made also.

In the case of algebras of quaternion type with two simple modules, we avoided the computation of $HH^1(A)$ (which gives the same result) and separated some algebras with the help of a result of Zimmermann's [158] and using stable Külshammer ideals.

Table 5.1: List of algebras up to stable equivalence of Morita type [154, 158]

	Dihedral	Semi-dihedral	Quaternion
1 simple	• $k[X,Y]/(XY, X^m - Y^n), m \geq 2, n \geq 2, m+n > 4$ • $D(1A)_1^1$ • char$(k) = 2, k[X,Y]/(X^2, YX - Y^2)$ • $D(1A)_1^\ell, \ell \geq 2$ • char$(k) = 2, D(1A)_2^\ell(d), \ell \geq 2, d \in \{0,1\}$ *Pairwise not SEMT except perhaps $D(1A)_2^\ell(0) \sim D(1A)_2^\ell(1)$ if char$(k) = 2$*	• $SD(1A)_1^\ell, \ell \geq 2$ • char$(k) = 2, SD(1A)_2^\ell(c,d), \ell \geq 2, (c,d) \neq (0,0)$ *Pairwise not SEMT except perhaps $SD(1A)_2^\ell(c,d) \sim SD(1A)_2^\ell(c',d')$ (c,d)* *(includes $SD(1A)_1^\ell$ identified with $SD(1A)_2^\ell(0,0)$ if char$(k) = 2$)*	• $Q(1A)_1^\ell, \ell \geq 2$ • char$(k) = 2, Q(1A)_2^\ell(c,d), \ell \geq 2, (c,d) \neq (0,0)$ *Pairwise not SEMT except perhaps $Q(1A)_2^\ell(c,d) \sim Q(1A)_2^\ell(c',d')$* *(includes $Q(1A)_1^\ell$ identified with $Q(1A)_2^\ell(0,0)$ if char$(k) = 2$)*
2 simples	$D(2B)_1^{\ell,s}(c), \ell \geq s \geq 1, c \in \{0,1\}$ *Pairwise not SEMT*	• $SD(2B)_1^{\ell,t}(c), t \geq 2, \ell \geq 1, c \in \{0,1\}$ • $SD(2B)_2^{\ell,t}(c), t \geq 2, \ell \geq 1, \ell+t > 4, c \in \{0,1\}$ SEMT $\Rightarrow \{\ell,t\} = \{\ell',t'\}$ *Question: $SD(2B)_\varepsilon^{\ell,t}(c) \sim SD(2B)_{\varepsilon'}^{\ell',t'}(c')$ for $\varepsilon, \varepsilon', c, c' \in \{0,1\}$ and $\{\ell,t\} = \{\ell',t'\}$?* *Except known when char$(k) = 2$:* $SD(2B)_1^{\ell,t}(0) \not\sim SD(2B)_1^{\ell,t}(1)$ if ℓ and t odd, or if $\ell = 2$ and $t \geq 3$ odd	• if char$(k) \neq 2, Q(2B)_1^{\ell,s}(1,0), s \geq 3, \ell \geq 2$ • if char$(k) = 2, Q(2B)_1^{\ell,s}(1,c), s \geq 3, \ell \geq 2, c \in k$ SEMT $\Rightarrow \{\ell,s\} = \{\ell',s'\}$ *Pairwise not SEMT except perhaps $Q(2B)_1^{\ell,s}(1,c) \sim Q(2B)_1^{\ell',s'}(1,c')$ with $\{\ell,s\} = \{\ell',s'\}$ if char$(k) = 2$*
3 simples	• $D(3K)^{a,b,c}, a,b,c \geq 1$ • $D(3R)^{\ell,s,t,u}, s \geq t \geq u \geq \ell \geq 1, t \geq 2$ *Pairwise not SEMT*	$SD(3K)^{a,b,c}, a \geq b \geq c \geq 1, a \geq 2$ *Pairwise not SEMT*	• $Q(3K)^{a,b,c}, a \geq b \geq c \geq 1, b \geq 2, (a,b,c) \neq (2,2,1)$ • $Q(3A)_1^{2,2}(d), d \notin \{0,1\}$ *Pairwise not SEMT except perhaps $Q(3A)_1^{2,2}(d) \sim Q(3A)_1^{2,2}(d')$*

For algebras of semi-dihedral type with two simple modules, many (but far from all) of the algebras can be separated using the Lie algebra structure of $\mathrm{HH}^1(A)$; more precisely, the lower central series, the derived series, the nilradical, the Killing form and some generalised derivations (on the Lie algebra $\mathrm{HH}^1(A)$).

The details are quite technical and I shall not give them here. They can be found in [T19].

Finally, for algebras with three simple modules (the only remaining question is for algebras of quaternion type), the study of $\mathrm{HH}^1(A)$ does not give any new information.

Remark 5.4.2. In some cases, the whole algebra $\mathrm{HH}^*(A)$ is known from results of Generalov, Ivanov and Ivanov. However, for the purpose of distinguishing the algebras up to stable equivalence of Morita type, it does not give any more information than the Lie structure of $\mathrm{HH}^1(A)$, whose determination is more elementary.

Chapter 6

Combinatorial classification of piecewise hereditary algebras

Summary

We use the coefficients of the characteristic polynomial of the Coxeter matrix of a piecewise hereditary algebra to complete a combinatorial classification (involving the trace of the Coxeter matrix) given by Happel [81, 82].

6.1 Introduction

In [81] Happel gave a homological formula for the trace of the Coxeter matrix ϕ_A of an algebra A. Among other consequences of this result, he gave a combinatorial criterion in terms of the trace of ϕ_A to distinguish most piecewise hereditary algebras in [81, 82]. However, for two different types of piecewise hereditary algebras, the trace of the Coxeter matrix is the same.

In [LRT11], with Lanzilotta and Redondo, we considered the other coefficients of the Coxeter polynomial, that is, of the characteristic polynomial of the Coxeter matrix ϕ_A. We obtained combinatorial criteria to characterise the cases that the trace did not separate. We also gave a homological interpretation of these coefficients.

Throughout this chapter, k is an algebraically closed field and all algebras are finite dimensional indecomposable k-algebras, so that for our purposes we can assume that they are of the form kQ/I for some connected quiver Q and some admissible ideal I. Modules are finite-dimensional over k. We also assume that all our algebras have finite global dimension.

If A is an algebra, let $P(1), \ldots, P(n)$ denote the representatives of the distinct isomorphism classes of indecomposable projective left A-modules. If M is a finite-dimensional left A-module, the **dimension vector** of M is the vector $\underline{\dim}_A(M) \in \mathbb{Z}^n$ whose i^{th} component is $(\underline{\dim}_A(M))_i = \dim_k(\text{Hom}_k(P(i), M))$. The **Cartan matrix** of A is the $n \times n$ matrix $C_A = (c_{ij})$ whose j^{th} column is the transpose of $\underline{\dim}_A(P(j))$, so that $c_{ij} = \dim_k \text{Hom}_A(P(i), P(j))$.

It is well known that since A has finite global dimension, C_A is invertible over $\mathbb{Z}$ (see [132]). We may therefore consider the following:

- ➢ The **Coxeter matrix** $\phi_A = -C_A^{-t} C_A$ of A.

- ➢ The **Euler form** associated to A, $\langle -, - \rangle_A : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ defined by $\langle x, y \rangle_A = x C_A^{-t} y^t$. It is known that for two finite dimensional A-modules X and Y such that X has finite projective dimension or Y has finite injective dimension we have $\langle \underline{\dim}_A X, \underline{\dim}_A Y \rangle_A = \sum_{i \geqslant 0} (-1)^i \dim_k \text{Ext}_A^i(X, Y)$.

We were interested in the characteristic polynomial $\chi_A(x) = \det(x I_n - \phi_A)$ of ϕ_A, that is, the **Coxeter polynomial** of A. Its first non-trivial coefficient was studied by Happel.

Theorem 6.1.1. *[81] Let $A^e = A \otimes_k A^{op}$ be the enveloping algebra of A and let $\text{HH}^i(A)$ be the ith Hochschild cohomology group of A. Then $\text{tr}\, \phi_A = -\langle \underline{\dim}_{A^e} A, \underline{\dim}_{A^e} A \rangle = -\sum_{i \geqslant 0} \dim_k \text{HH}^i(A)$.*

Using this result, he obtained some information on simply connected algebras.

6.2 Classification of piecewise hereditary algebras

Definition 6.2.1. An algebra A is said to be **piecewise hereditary of type** $\mathcal{H}$ if its bounded derived category is triangle equivalent to the bounded derived category of a hereditary abelian k-category $\mathcal{H}$.

Such algebras have been much studied.

Happel and Reiten [85, Lemma 1.5] have shown that if A is a piecewise hereditary algebra of type $\mathcal{H}$, then the category $\mathcal{H}$ has a tilting object, and therefore Happel's classification result [83, Theorem 3.1] applies: either $\mathcal{H}$ is derived equivalent to H-mod for some finite dimensional hereditary k-algebra H, or $\mathcal{H}$ is derived equivalent to some category of coherent sheaves on a weighted projective line. It then follows that A is derived equivalent to either a finite dimensional hereditary algebra (that is, a path algebra $k\vec{\Delta}$ for a quiver $\vec{\Delta}$) or a canonical algebra. In the first case we say that A is of **type** $k\vec{\Delta}$ and in the second case we say that A is of **canonical type**.

Definition 6.2.2. A **canonical algebra** is a one-point extension of the path algebra of the quiver

$$\vec{Q} := \qquad \omega$$
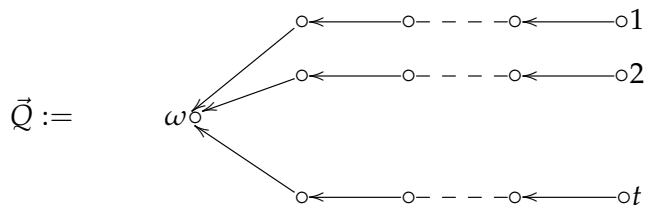
by a certain indecomposable B-module whose dimension vector is $m := \underline{\dim}_{k\vec{Q}}(M) = (1, \ldots, 1, 2)$ when we order the vertices from left to right and from top to bottom, the last vertex being ω.

Recall that the one-point extension of an algebra B by a B-module M is the algebra $B[M] = \begin{pmatrix} B & M \\ 0 & k \end{pmatrix}$ (with usual matrix addition and multiplication).

The quiver of $B[M]$ contains the quiver of B as a full subquiver and there is an additional vertex, the *extension vertex*. When the global dimension of B is finite, so is the global dimension of $B[M]$, so that no further restrictions on $B[M]$ are required.

Let $m = \underline{\dim}_B M$ denote the dimension vector of the B-module M. Then the Cartan matrix of $B[M]$ is $\begin{pmatrix} 1 & 0 \\ m^t & C_B \end{pmatrix}$ (if the extension vertex has index 0 so is placed 'before' the others) and the Coxeter matrix of $B[M]$ is $\begin{pmatrix} \langle m, m \rangle_B - 1 & -m\phi_B \\ -C_B^{-t} m^t & \phi_B \end{pmatrix}$.

Assume now that $A = B[M]$ is a one-point extension. Let n be the number of isomorphism classes of indecomposable projectives for B, so that the number of isomorphism classes of indecomposable projectives for A is $n + 1$. Denote by $\chi_A(x) = \sum_{i=0}^{n+1} \lambda_i^A x^i$ the Coxeter polynomial of A and by $\chi_B(x) = \sum_{i=0}^{n} \lambda_i^B x^i$ the Coxeter polynomial of B.

Theorem 6.2.3. *[84] With the notation above, for any integer ℓ with $0 \leqslant \ell \leqslant n$, we have*

$$\lambda_{n+1-\ell}^A = \lambda_{n-\ell}^B - (\langle m, m \rangle_B - 1)\lambda_{n-(\ell-1)}^B - \sum_{i=1}^{\ell-1} \lambda_{n-\ell+i+1}^B \langle m\phi_B^i, m \rangle_B.$$

Happel has given the following combinatorial characterisation of some piecewise hereditary algebras, using the trace of the Coxeter matrix ϕ_A of A.

Theorem 6.2.4. *[81, 82] Let A be an indecomposable piecewise hereditary algebra over an algebraically closed field. Then*

(1) *A is of type $k\vec{\Delta}$ where the underlying graph Δ is not a tree if and only if $\operatorname{tr}\phi_A > -1$,*

(2) *A is of canonical type with $t > 3$ branches if and only if $\operatorname{tr}\phi_A < -1$,*

(3) *A is of canonical type with $t = 3$ branches or of type $k\vec{\Delta}$ where the underlying graph Δ is a tree if and only if $\operatorname{tr}\phi_A = -1$.*

Our aim was to separate the algebras of canonical type and tree type when the trace is -1.

Remark 6.2.5. The Coxeter polynomial is a derived invariant, so we need only consider the Coxeter polynomials of path algebras of trees and of canonical algebras with three branches.

Moreover, let $\vec{\Delta}_1$ and $\vec{\Delta}_2$ be two quivers with the same underlying tree graph Δ. Let $A_1 = k\vec{\Delta}_1$ and $A_2 = k\vec{\Delta}_2$ be the corresponding path algebras. It is known that A_1 and A_2 are derived equivalent. Therefore the Coxeter polynomials of A_1 and A_2 are equal and we shall also denote them by χ_Δ.

Remark 6.2.6. Let $\mathscr{C}$ be a canonical algebra with 3 branches. If $0 < a \leqslant b \leqslant c$ are the numbers of vertices in the three branches, then the sign of $\delta := 1 - \frac{1}{a+1} - \frac{1}{b+1} - \frac{1}{c+1}$ gives us information on the representation type of $\mathscr{C}$ [64, Section 5]; in particular, if $\delta < 0$ then $\mathscr{C}$ is derived equivalent to the module category of a tame hereditary algebra. Therefore, in order to distinguish piecewise hereditary algebras of canonical type which are not of hereditary type, we need only consider the case $\delta \geqslant 0$, so that $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \leqslant 1$.

We first obtained some information on coefficients of the Coxeter polynomial for trees (it was already known for type $k\vec{\mathbb{A}}$, the rest we obtained using a result of Boldt [22, Corollary 3.2]). We denote by $\mathbb{T}_{a,b,c}$ the star with 3 branches, a vertices on the first branch, b vertices on the second and c on the third, so that there are $a + b + c + 1$ vertices in the graph.

Proposition 6.2.7. *[LRT11]*

➢ *The Coxeter polynomial of $k\vec{\mathbb{A}}_{n+1}$ for the Dynkin graph $\mathbb{A}_{n+1}$ is $\sum_{i=0}^{n+1} x^i$.*

➢ *Consider the path algebra $k\vec{\mathbb{T}}_{a,b,c}$ with $a \leqslant b \leqslant c$. Then*

$$\lambda^{k\vec{\mathbb{T}}_{a,b,c}}_{(a+b+c)-\ell} = \frac{(1-\ell)(2+\ell)}{2} \quad \text{for } 0 \leqslant \ell \leqslant a.$$

Moreover, if $a = 1$ then $\chi_{\mathbb{T}_{1,b,c}}(x) = x^{b+c+2} + \sum_{j=c+1}^{b+c+1}(j - b - c)x^j + \sum_{j=b+1}^{c}(1 - b)x^j + \sum_{j=1}^{b}(2 - j)x^j + 1$. In particular, for the Dynkin graph $\mathbb{D}_{n+1} = \mathbb{T}_{1,1,n-2}$ we get $\chi_{\mathbb{D}_{n+1}}(x) = x^{n+1} + x^n + x + 1$.

➢ *Let Δ be a tree with $n + 1$ vertices which is neither $\mathbb{A}_{n+1}$ nor $\mathbb{T}_{a,b,c}$. Then $\lambda^\Delta_{n-1} \leqslant -1$.*

Then we used Happel's induction result (Theorem 6.2.3) as well as a result of Boldt [22] giving $\phi_{k\vec{\mathbb{T}}_{a,b,c}}$ to obtain information on some coefficients of the Coxeter polynomial of canonical algebras and our main result.

Theorem 6.2.8. *[LRT11] Let A be an indecomposable piecewise hereditary algebra over an algebraically closed field with $n + 1$ isomorphism classes of indecomposable projective modules and $\operatorname{tr}\phi_A = -1$. Let $\chi_A(x) = \sum_{i=0}^{n+1} \lambda_i^A x^i$ be the characteristic polynomial of the Coxeter matrix ϕ_A of A. Then A is of canonical type (with $t = 3$ branches) if and only if one of the three following sets of conditions holds:*

(i) $\lambda^A_{n-1} = 0$, $\lambda^A_{n-2} = -1 = \lambda^A_{n-3}$ *and* $\lambda^A_{n-4} = 0$.

(ii) $\lambda^A_{n-1} = 0 = \lambda^A_{n-2}$ *and* $\lambda^A_{n-\ell} \leqslant -1$ *for some* $\ell \geqslant 3$.

(iii) $\lambda^A_{n-1} = 1$ *and* $\lambda^A_{n-\ell} \leqslant 0$ *for some* $\ell \geqslant 2$.

Otherwise it is of tree type.

6.3 Cohomological interpretation of the coefficients of the Coxeter polynomial

We now assume that the characteristic of k is 0.

Proposition 6.3.1. *[LRT11] Let ϕ be a matrix, and let $\chi(x) = \det(x \operatorname{id} - \phi)$ be its characteristic polynomial. Write $\chi(x) = x^n + \lambda_{n-1}x^{n-1} + \cdots + \lambda_1 x + \lambda_0$. Then*

$$\lambda_{n-\ell} = \sum (-1)^{\sigma(\underline{p})} \alpha_{\underline{p}} \operatorname{tr}(\phi^{p_1}) \cdots \operatorname{tr}(\phi^{p_r}),$$

where the sum is taken over all partitions $\underline{p} = (p_1, \ldots, p_r)$ of ℓ, $\sigma(\underline{p})r$ and $\alpha_{\underline{p}} = $

$$\frac{1}{p_1 p_2 \ldots p_r} \prod_{a=0}^{\ell} \frac{1}{(\#\{i \mid p_i = a\})!}.$$

Theorem 6.3.2. *[LRT11] Let DA denote the k-dual of A, viewed as a bimodule over A. Then*

- $\operatorname{tr}(\phi_A^2) = \langle \underline{\dim}_{A^e} DA, \underline{\dim}_{A^e} A \rangle$.

- *If $k \geqslant 3$, then we have an explicit formula for $\operatorname{tr}(\phi_A^k)$ in terms of the dimension vectors of the indecomposable projective and injective A-modules, of the simple A-modules, of the A^e-module DA and of the simple A^e-modules.*

Chapter 7

The stable Green ring of the Drinfeld doubles of generalised Taft algebras; application to the Gerstenhaber-Schack cohomology of the generalised Taft algebras

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Summary

We describe the quiver and representations of the Drinfeld doubles $\mathcal{D}(\Lambda_{n,d})$ of generalised Taft algebras obtained in [EGST06], the decompositions of the tensor products of the indecomposable $\mathcal{D}(\Lambda_{n,d})$-modules as a direct sum of indecomposable modules up to projectives obtained in [EGST06, EGST19], with an application to the classification of endotrivial and algebraic $\mathcal{D}(\Lambda_{n,d})$-modules, and finally we use the description of $\mathcal{D}(\Lambda_{n,d})$ by quiver and relations (up to Morita equivalence) to determine the Gerstenhaber-Schack cohomology of the algebras $\Lambda_{n,d}$ [T07].

7.1 Introduction

In joint work with Erdmann, Green and Snashall [EGST06, EGST19], we studied the representation theory and the stable Green ring of some Drinfeld doubles. The Drinfeld double of a Hopf algebra was introduced by Drinfeld in order to produce solutions to the quantum Yang-Baxter equation (arising from Statistical Mechanics).

The isomorphism classes of representations of a Hopf algebra form a ring, called the Green ring, in which the product is given by the tensor product over the base field. In the case of a quasi-triangular Hopf algebra, such as the Drinfeld double of a Hopf algebra, this ring is commutative.

We wanted to understand the Drinfeld double as an algebra, through its representations. Not much was known in general. For instance, the Drinfeld double of a group algebra in positive characteristic has been studied by Witherspoon in [148], where she proves in particular that its Green ring decomposes as a product of ideals associated to some subgroups of the original group; Chen gave a complete list of simple modules over the Drinfeld doubles of the Taft algebras in [33]; Radford has given a characterisation of the simple modules in some cases (which include the Taft algebras and quiver Hopf algebras) [126]; Chen [32] has studied the indecomposable representations of the Taft algebras. Since then, further work has been done on the representation theory of Drinfeld doubles and generalised Drinfeld doubles and on the tensor products of their representations, see for instance [99, 100, 127, 120, 34, 153, 36, 121].

In [EGST06, EGST19], we have considered a family of Hopf algebras $\Lambda_{n,d}$, the duals of the generalised Taft algebras, described all the representations of their Drinfeld doubles and described the product in the stable Green ring. Some of these Drinfeld doubles occur elsewhere. For instance, the finite-dimensional quotients of $U_q(\mathfrak{sl}_2)$ studied by Suter, Xiao, Patra in [140, 151, 119] are quotients of some of these Drinfeld doubles, and the algebras Λ_{2p^r,p^n} in characteristic p also arose in the classification of Farnsteiner and Skowroński of some restricted Lie algebras [53]. A description of the decomposition of the Drinfeld doubles of the Taft algebras (the case where $n = d$) has also been determined in [36] by other methods.

As an application, using results from my thesis, I was able to compute the Gerstenhaber-Schack cohomology of the duals of the generalised Taft algebras in [T07].

7.2 Description of the algebras and of their representations.

7.2.1 The algebras $\Lambda_{n,d}$ and $\mathcal{D}(\Lambda_{n,d})$

The algebra $\Lambda_{n,d}$ is described by quiver and relations; the quiver is cyclic, with n vertices $0, \ldots, n-1$ and n arrows $a_0, \ldots, a_{n-1}$, where the arrow a_i goes from the vertex i to the vertex $i+1$, and we factor by the ideal generated by all paths of length $d \geqslant 2$. The indices are viewed as elements in the cyclic group $\mathbb{Z}_n$ and are written mod n. We shall denote by γ_i^m the path $a_{i+m-1} \ldots a_{i+1} a_i$ (read from right to left), that is, the path of length m starting at the vertex i. In particular, $\gamma_i^0 = e_i$ and $\gamma_i^1 = a_i$.

When d divides n, this algebra is a Hopf algebra, and in fact the condition $d \mid n$ is a necessary and sufficient condition for $\Lambda_{n,d}$ to be a Hopf algebra when char $k = 0$ (see [38, 35]). This Hopf algebra can actually be considered over more general fields, and here we assume only that the characteristic of k does not divide n (to ensure existence of roots of unity).

Fix a primitive d^{th} root of unity q in k. The formulae

$$
\begin{array}{lll}
\varepsilon(e_i) = \delta_{i0} & \Delta(e_i) = \sum_{j+\ell=i} e_j \otimes e_\ell & S(e_i) = e_{-i} \\
\varepsilon(a_i) = 0 & \Delta(a_i) = \sum_{j+\ell=i}(e_j \otimes a_\ell + q^\ell a_j \otimes e_\ell) & S(a_i) = -q^{i+1} a_{-i-1}
\end{array}
$$

determine the Hopf algebra structure of $\Lambda_{n,d}$ (the indices are written in $\mathbb{Z}_n$). It was shown in [38] that when $n = d$, the Hopf algebra $\Lambda_{n,n}$ is self-dual and isomorphic to the Taft algebra.

In general, the dual $\Lambda_{n,d}^{*cop}$ of $\Lambda_{n,d}$ is isomorphic as an algebra to the extended Taft algebra

$$
\langle G, X \mid G^n = 1, X^d = 0, GX = q^{-1} XG \rangle
$$

(see [38] for the case $n = d$); if $\{ \breve{\gamma}_i^m \mid i \in \mathbb{Z}_n, 0 \leqslant m \leqslant d-1 \}$ denotes the dual basis of $\Lambda_{n,d}^{*cop}$, the correspondence is determined by $\breve{e}_i \mapsto G^i$ and $\breve{a}_i \mapsto G^i X$). Its Hopf algebra structure is determined by

$$
\begin{array}{lll}
\varepsilon(G) = 1 & \Delta(G) = G \otimes G & S(G) = G^{-1} \\
\varepsilon(X) = 0 & \Delta(X) = X \otimes G + 1 \otimes X & S(X) = -XG^{-1} = -q^{-1} G^{-1} X.
\end{array}
$$

Recall that, if H is a Hopf algebra, the **Drinfeld double** of H is the Hopf algebra which is equal to $H^{*cop} \otimes H$ as a coalgebra (ordinary tensor product of coalgebras), and whose product is defined by

$$
(\alpha \otimes h)(\beta \otimes g) = \alpha\beta(S^{-1}h^{(3)} ? h^{(1)}) \otimes h^{(2)} g,
$$

where $\beta(S^{-1}h^{(3)} ? h^{(1)})$ is the map which sends $x \in H$ to $\beta(S^{-1}h^{(3)} x h^{(1)}) \in k$ and where we have used the Sweedler notation $\Delta(h) = h^{(1)} \otimes h^{(2)}$ for the comultiplication.

The description of the Drinfeld double of $\Lambda_{n,d}$ is then as follows.

Proposition 7.2.1. *[EGST06, Proposition 2.4] The Drinfeld double $\mathcal{D}(\Lambda_{n,d})$ is $\Lambda_{n,d}^{*cop} \otimes \Lambda_{n,d}$ as a coalgebra; we write the basis elements $G^i X^j \gamma_\ell^m$, with $i, \ell \in \mathbb{Z}_n$ and $0 \leqslant j, m \leqslant d-1$ (that is, we do not write the tensor product symbol). The following relations determine the algebra structure completely:*

$$
G^n = 1, \quad X^d = 0, \quad GX = q^{-1} XG,
$$

the product of elements γ_ℓ^m is the usual product of paths,

$$
\gamma_\ell^m G = q^{-m} G \gamma_\ell^m, \quad \text{and} \quad \gamma_\ell^m X = q^{-m} X \gamma_{\ell+1}^m - q^{-m}(m)_q \gamma_{\ell+1}^{m-1} + q^{\ell+1-m}(m)_q G \gamma_{\ell+1}^{m-1}
$$

where the q-integers $(m)_q$ are defined by $(0)_q = 0$ and $(m)_q = \sum_{i=1}^{m-1} q^i$ if $m > 0$.

Notation 7.2.2. *Since some indices are described modulo d and others modulo n, we need to make a distinction. If j is an element in $\mathbb{Z}_n$, we shall denote its representative modulo d in $\{1, \ldots, d\}$ by $\langle j \rangle$ and its representative modulo d in $\{0, \ldots, d-1\}$ by $\langle j \rangle^-$.*

We shall also need the permutation σ_u of $\mathbb{Z}_n$ defined by

$$\sigma_u(j) = d + j - \langle 2j + u - 1 \rangle.$$

Note that if d does not divide $2j + u - 1$, then the orbit of j under the action of σ_u has $2\frac{n}{d}$ elements and we have $\sigma_u^{2t}(j) = j + td$ and $\sigma_u^{2t+1}(j) = \sigma_u(j) + td$ in $\mathbb{Z}_n$; if d divides $2j + u - 1$, then $\sigma_u^t(j) = j$ for all $t \in \mathbb{Z}$.

7.2.2 The quiver of $\mathcal{D}(\Lambda_{n,d})$

In order to find the quiver of $\mathcal{D}(\Lambda_{n,d})$, we determined all the projective $\mathcal{D}(\Lambda_{n,d})$-modules.

Proposition 7.2.3. *[EGST06, Subsection 2.2.3] The representatives of the isomorphism classes of all the indecomposable projective $\mathcal{D}(\Lambda_{n,d})$-modules are indexed by pairs (u, j) of elements in $\mathbb{Z}_n$. They are described as follows.*

(1) *Assume that $2j + u - 1 \equiv 0 \ (d)$. Then $P(u, j)$ has dimension d and has the following structure:*

(2) *Assume that $2j + u - 1 \not\equiv 0 \ (d)$. Then $P(u, j)$ has dimension 2d and has the following structure:*

In these diagrams, the arrows drawn represent the actions of X and of the arrows a in the original quiver, up to a nonzero scalar; the basis vectors are eigenvectors for the action of G.

Remark 7.2.4. In order to determine the relations, we used an important property of the algebra underlying the Drinfeld double of a finite dimensional Hopf algebra H that can be deduced from work of Radford and Farnsteiner ([125, Corollary 2 and Theorem 4] and [52, Proposition 2.3]): the Drinfeld double $\mathcal{D}(H)$ is a symmetric algebra.

Using this, we were able to describe all the simple $\mathcal{D}(\Lambda_{n,d})$-modules: they are the $L(u,j) = P(u,j)/\operatorname{rad}(P(u,j))$ for $(u,j) \in \mathbb{Z}_n^2$. If $2j + u \equiv 1\,(d)$, then $L(u,j) = P(u,j)$ is projective, but not otherwise. We have $\dim L(u,j) = d$ if $2j + u \equiv 1\,(d)$ and $\dim L(u,j) = \sigma_u(j) - j$ otherwise. Moreover, if $2j + u \not\equiv 1\,(d)$, then $\operatorname{top}(P(u,j)) = L(u,j) \cong \operatorname{soc}(P(u,j))$ and $\operatorname{rad}(P(u,j))/\operatorname{soc}(P(u,j)) \cong L(u,\sigma_u^1(j)) \oplus L(u,\sigma_u(j))$.

The simple modules are characterised as follows.

Proposition 7.2.5. *[EGST06, Proposition 2.21] Let S be a simple module. Set $E_u = \frac{1}{n}\sum_{i,j\in\mathbb{Z}_n} q^{-i(u+j)} G^i e_j$ for $u \in \mathbb{Z}_n$. Then S is isomorphic to $L(u,i)$ if and only if the three following properties hold:*

(a) $\dim S = \dim L(u,i)$.

(b) *The idempotent $E_u := \frac{1}{n}\sum_{i,j\in\mathbb{Z}^n} q^{-i(u+j)} G^i e_j$ acts as the identity on S, and E_v acts as zero on S if $v \neq u$.*

(c) *Let Y be the generator of S which is in the kernel of the action of X (this is well-defined up to a non-zero scalar and corresponds to $\tilde{H}_{u,i}$). Then the vertex e_i acts as the identity on Y, and the other vertices act as zero.*

We then described completely the quiver of $\mathcal{D}(\Lambda_{n,d})$.

Theorem 7.2.6. *[EGST06, Theorem 2.26] We have $\mathcal{D}(\Lambda_{n,d}) \cong \prod_{u\in\mathbb{Z}_n} \mathcal{D}(\Lambda_{n,d})E_u$ and the algebras $\mathcal{D}(\Lambda_{n,d})E_u$ decompose into blocks as follows: if $j_{u,1},\dots,j_{u,r_u}$ are the representatives of the orbits of σ_u in $\mathbb{Z}_n$, then $\mathcal{D}(\Lambda_{n,d})E_u = \bigoplus_{i=1}^{r_u} \mathbb{B}_{u,i}$ where*

$$\mathbb{B}_{u,i} = \bigoplus_t \bigoplus_{h=0}^{d-\langle 2j+u-1\rangle^- -1} P(u,\sigma^t(j_{u,i}))X^h,$$

where t ranges from 0 to $\frac{2n}{d} - 1$ if $2j_{u,i} + u - 1 \not\equiv 0$ and $t = 0$ if $2j_{u,i} + u - 1 \equiv 0$.

It then follows that the quiver of $\mathcal{D}(\Lambda_{n,d})$ has $\frac{n^2}{d}$ isolated vertices which correspond to the simple projective modules, and $\frac{n(d-1)}{2}$ copies of the quiver
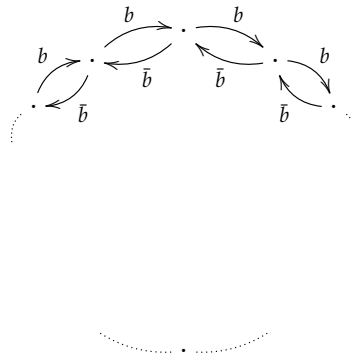
with $\frac{2n}{d}$ vertices and $\frac{4n}{d}$ arrows. The relations on this quiver are bb, $\bar{b}\bar{b}$ and $b\bar{b} - \bar{b}b$ (there are $\frac{6n}{d}$ relations on each of these quivers). The vertices in this quiver correspond to the simple modules $L(u,j)$, $L(u,\sigma_u(j))$, $L(u,\sigma_u^2(j))$, $\dots$, $L(u,\sigma_u^{\frac{2n}{d}-1}(j))$.

7.2.3 Classification of indecomposable $\mathcal{D}(\Lambda_{n,d})$-modules

The blocks $\mathbb{B}_{u,i}$ are special biserial and tame. Their indecomposable representations are of two types: string modules (of odd or even length) and band modules (of even length).

The indecomposable modules of odd length.

They are precisely the syzygies of the non projective simple modules, that is, the $\Omega^m(L(u,i))$ for $m \in \mathbb{Z}$ and $(u,i) \in \mathbb{Z}_n^2$ where d does not divide $2i + u - 1$.

The indecomposable string modules of even length.

For each $0 \leqslant p \leqslant \frac{2n}{d} - 1$ and for each $\ell \geqslant 1$, there are two indecomposable modules of length 2ℓ which we call $M_{2\ell}^{\pm}(u, \sigma_u^p(i))$:

- The module $M_{2\ell}^+(u,i)$ has top composition factors $L(u,i)$, $L(u,\sigma_u^2(i))$, ..., $L(u,\sigma_u^{2(\ell-1)}(i))$ and socle composition factors $L(u,\sigma_u(i))$, $L(u,\sigma_u^3(i))$, ..., $L(u,\sigma_u^{2(\ell-1)+1}(i))$:

$$
\begin{array}{ccccccc}
L(u,i) & & & L(u,\sigma_u^2(i)) & & L(u,\sigma_u^{2(\ell-1)}(i)) & \\
 & \searrow & \nearrow & & \searrow & \cdots \nearrow & \\
 & L(u,\sigma_u(i)) & & & \cdots & & L(u,\sigma_u^{2(\ell-1)+1}(i))
\end{array}
$$

  The lines joining the simple modules are given by multiplication by the appropriate b-arrow or $\bar{b}$-arrow (in the case $n = d$, when there is an ambiguity, the first line is multiplication by $\gamma^{\dim L(u,i)}$, the next one is multiplication by a non-zero scalar multiple of $X^{d-\dim L(u,i)}$, and so on, up to non-zero scalars).

- The module $M_{2\ell}^-(u,i)$ has top composition factors $L(u,i)$, $L(u,\sigma_u^{-2}(i))$, ..., $L(u,\sigma_u^{-2(\ell-1)}(i))$ and socle composition factors $L(u,\sigma_u^{-1}(i))$, $L(u,\sigma_u^{-3}(i))$, ..., $L(u,\sigma_u^{-2(\ell-1)+1}(i))$:

$$
\begin{array}{ccccccc}
 & L(u,\sigma_u^{-2(\ell-1)}(i)) & & L(u,\sigma_u^{-2}(i)) & & & L(u,i) \\
\nearrow & & \searrow & & \nearrow & & \nearrow \\
L(u,\sigma_u^{-2(\ell-1)+1}(i)) & & & \cdots & & L(u,\sigma_u^{-1}(i)) &
\end{array}
$$

  As for the other string modules, the lines represent multiplication by an appropriate b or $\bar{b}$ arrow, and in the case $n = d$ the first one from the left is multiplication by a power of X and so on.

In both cases, indices are taken modulo $\frac{2n}{d}$.

  These modules are periodic of period $\frac{2n}{d}$. Moreover, the Auslander-Reiten sequences of the string modules $M_{2\ell}^{\pm}(u,i)$ are given by

$$\mathcal{A}(M_{2\ell}^+(u,i)): \qquad 0 \to M_{2\ell}^+(u,i-d) \to M_{2\ell+2}^+(u,i-d) \oplus M_{2\ell-2}^+(u,i) \to M_{2\ell}^+(u,i) \to 0,$$

$$\mathcal{A}(M_{2\ell}^-(u,i)): \qquad 0 \to M_{2\ell}^-(u,i+d) \to M_{2\ell+2}^-(u,i+d) \oplus M_{2\ell-2}^-(u,i) \to M_{2\ell}^-(u,i) \to 0$$

(where $M_0^{\pm}(u,i) = 0$).

The indecomposable band modules (of even length).

For each $\lambda \neq 0$ in k, and for each $\ell \geqslant 1$, there is an indecomposable module of length $\frac{2n}{d}\ell$, which we denote by $C_\lambda^\ell(u,i)$. It is defined as follows. Denote by ϵ_p, for $p \in \mathbb{Z}_{2n/d}$, the vertices in the quiver of $\mathbb{B}_{u,i}$, with ϵ_0 corresponding to $L(u,i)$. The arrow b_p goes from ϵ_p to ϵ_{p+1} and the arrow $\bar{b}_p$ goes from ϵ_{p+1} to ϵ_p. Let V be an ℓ-dimensional vector space. Then $C_\lambda^\ell(u,i)$ has underlying space $C = \bigoplus_{p=0}^{\frac{2n}{d}-1} C_p$ with $C_p = V$ for all p. The action of the idempotents ϵ_p is such that $\epsilon_p C = C_p$. The action of the arrows $\bar{b}_{2p}$ and b_{2p+1} is zero. The action of the arrows $\bar{b}_{2p+1}$ is the identity of V. The action of the arrows b_{2p} with $p \neq 0$ is also the identity. Finally, the action of b_0 is given by the indecomposable Jordan matrix $J_\ell(\lambda)$.

  It is periodic of period 2 and its Auslander-Reiten sequence is given by

$$\mathcal{A}(C_\lambda^\ell(u,i)): \qquad 0 \to C_\lambda^\ell(u,i-d) \to C_\lambda^{\ell+1}(u,i) \oplus C_\lambda^{\ell-1}(u,i) \to C_\lambda^\ell(u,i) \to 0$$

(where $C_\lambda^0(u,i) = 0$).

  Note that $\mathrm{soc}(C) = \mathrm{rad}(C) = \bigoplus_p \epsilon_{2p+1} C$ and that $C/\mathrm{rad}(C) = \bigoplus_p \epsilon_{2p} C$.

7.3 The stable Green ring of the algebra $\mathcal{D}(\Lambda_{n,d})$

We now consider the tensor products of indecomposable $\mathcal{D}(\Lambda_{n,d})$-modules up to projectives. Chen (the first author of [36]) asked us about some of the proofs and statements on tensor products involving modules of even length in our paper [EGST06], and we realised that we had made some mistakes. We therefore returned to the question of determining these tensor products in [EGST19], with different, more homological methods, based on exploiting stable module homomorphisms. We also realised that many of the proofs could be simplified. This enabled us to describe completely the products in the stable Green ring of the algebra $\mathcal{D}(\Lambda_{n,d})$.

The tensor product of two simple modules was correctly determined in [EGST06].

In [EGST19], we then used

➤ properties of splitting trace modules (that is, modules M such that the trivial module $L(0,0)$ is a direct summand in $\mathrm{End}_k(M)$), in particular in relation with Auslander-Reiten sequences,

➤ the $\mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}$ between even modules of the same kind (M^+, M^- or C),

➤ an adjunction formula involving stable homomorphisms (which required finding the k-duals of the non projective indecomposable $\mathcal{D}(\Lambda_{n,d})$-modules in order to be applied) and

➤ some induction

in order to determine the tensor products between any two modules up to projectives. Moreover, in the case of band modules, we were able to define reliably and consistently the parameter μ of a module $C^\ell_\mu(u,i)$ having fixed those of the $C^\ell_\lambda(0,0)$.

Our results on the tensor products are summarised in Table 7.1 on page 59. The stable Green ring is commutative so that we may assume for instance that $\ell \geqslant t$.

We were able to simplify many of our proofs by working with $\mathbb{B}_{0,0}$-modules, then tensoring with the appropriate simple module to obtain the general result. This approach can be formalised as follows.

Theorem 7.3.1. *[EGST19, Theorem 5.10] Let $L(u,i)$ be a non-projective simple module. Then $- \otimes L(u,i)$ induces a stable equivalence between $\mathbb{B}_{0,0}$ and $\mathbb{B}_{u,i}$.*

Note that since $- \otimes L(u,i)$ is an exact functor from $\mathbb{B}_{0,0}$- mod to $\mathbb{B}_{u,i}$- mod, where the algebras are selfinjective, this stable equivalence is in fact a stable equivalence of Morita type by [131, Theorem 3.2].

Using these tensor products, we were also able to classify endotrivial and algebraic $\mathcal{D}(\Lambda_{n,d})$-modules.

If H is a finite-dimensional ribbon Hopf algebra (see for instance [31, Section 4.2.C]), then a finite-dimensional H-module M is **endotrivial** if there is an isomorphism $M \otimes_k M^* \cong k \oplus P$ where k is the trivial H-module and P is projective. Tensoring with an endotrivial module induces an equivalence of the stable module category, and such equivalences form a subgroup of the auto-equivalences of the stable module category. When $H = kG$ is the group algebra of a finite group G, endotrivial modules have been studied extensively. They have also been studied for finite group schemes in [27, 28]. However, we have not seen any results on endotrivial modules for other Hopf algebras.

Proposition 7.3.2. *[EGST19, Proposition 7.3] The indecomposable endotrivial $\mathcal{D}(\Lambda_{n,d})$-modules are precisely the syzygies of the simple modules of dimension 1 or $d-1$.*

The concept of an algebraic module is quite natural; it was introduced as a kG-module satisfying a polynomial equation with coefficients in $\mathbb{Z}$ in the Green ring of kG, for a finite group G. We shall use the following equivalent definition: a kG-module M is called **algebraic** if the

Table 7.1: Description of the product in the stable Green ring of $\mathcal{D}(\Lambda_{n,d})$ [EGST19]

$\otimes$	$\Omega^m(L(v,j))$	$M^+_{2\ell}(v,j)$	$M^-_{2\ell}(v,j)$	$C^\ell_\mu(v,j)$
$\Omega^n(L(u,i))$	$\bigoplus\limits_{\theta\in\mathfrak{I}} \Omega^{m+n}(L(w,i+j+\theta))$	$\bigoplus\limits_{\theta\in\mathfrak{I}} M^+_{2\ell}(w,\sigma_w^{-n}(i+j+\theta))$	$\bigoplus\limits_{\theta\in\mathfrak{I}} M^-_{2\ell}(w,\sigma_w^{n}(i+j+\theta))$	n even: $\bigoplus\limits_{\theta\in\mathfrak{I}} C^\ell_\mu(w,i+j+\theta)$ n odd: $\bigoplus\limits_{\theta\in\mathfrak{I}} C^\ell_\mu(w,\sigma_w(i+j+\theta))$
$M^+_{2t}(u,i)$	$\bigoplus\limits_{\theta\in\mathfrak{I}} M^+_{2t}(w,\sigma_w^{-m}(i+j+\theta))$	$\bigoplus\limits_{\theta\in\mathfrak{I}}\left(M^+_{2t}(w,i+j+\theta)\oplus M^+_{2t}(w,\sigma_w^{2\ell-1}(i+j+\theta)) \right)$	0	0
$M^-_{2t}(u,i)$	$\bigoplus\limits_{\theta\in\mathfrak{I}} M^-_{2t}(w,\sigma_w^{m}(i+j+\theta))$	0	$\bigoplus\limits_{\theta\in\mathfrak{I}}\left(M^-_{2t}(w,i+j+\theta)\oplus M^-_{2t}(w,\sigma_w^{-(2\ell-1)}(i+j+\theta)) \right)$	0
$C^t_\lambda(u,i)$	m even: $\bigoplus\limits_{\theta\in\mathfrak{I}} C^t_\lambda(w,i+j+\theta)$ m odd: $\bigoplus\limits_{\theta\in\mathfrak{I}} C^t_\lambda(w,\sigma_w(i+j+\theta))$	0	0	$\lambda=\mu$: $\bigoplus\limits_{\theta\in\mathfrak{I}}\left(C^t_\lambda(w,i+j+\theta)\oplus C^t_\lambda(w,\sigma_w(i+j+\theta)) \right)$ $\lambda\neq\mu$: 0

where, for (u,i) and (v,j) in $\mathbb{Z}_n^2$, we have put

$$\mathfrak{I} = \begin{cases} \{\theta;\, 0\leqslant\theta\leqslant \min(\dim L(u,i),\dim L(v,j))-1\} \text{ if } \dim L(u,i)+\dim L(v,j)\leqslant d \\ \{\theta;\, \dim L(u,i)+\dim L(v,j)-d\leqslant\theta\leqslant\min(\dim L(u,i),\dim L(v,j))-1\} \text{ otherwise.} \end{cases}$$

and $w=u+v$. We assume that $\ell\geqslant t$.

number of non-isomorphic indecomposable summands of the set of modules $M^{\otimes_k t}$, when $t \geqslant 1$ varies, is finite. Such modules occur in particular in the study of the Auslander-Reiten quiver of kG. For a study of algebraic kG-modules, see for instance [40] and the references there. This definition can be generalised to any Hopf algebra instead of kG. In the case of $\mathcal{D}(\Lambda_{n,d})$, we got the following classification.

Proposition 7.3.3. *[EGST19, Remark 7.6 and Proposition 7.7] The indecomposable algebraic $\mathcal{D}(\Lambda_{n,d})$-modules are the projective modules, the simple modules and the modules of even length.*

7.4 Application: the Gerstenhaber-Schack cohomology of $\Lambda_{n,d}$

The bialgebra cohomology $\mathrm{H}_b^*(H, H)$ for a Hopf algebra H was defined by Gerstenhaber and Schack in [66] in order to understand deformations of bialgebras and Hopf algebras (in a similar way that Hochschild cohomology is a tool in the study of deformations of associative algebras). This cohomology has also been useful to establish results on classification or structure of Hopf algebras (see [50] and [137]).

The bialgebra cohomology is endowed with a graded algebra structure which is graded commutative (see [141, 142]), but it is difficult to compute in general. Some results are known (mainly for commutative or cocommutative algebras): Gerstenhaber and Schack [66] showed that if G is a discrete group and k is a field, then $\mathrm{H}_b^*(kG, kG) \cong \mathrm{H}^*(G, k)$ (since kG is coseparable); Parshall and Wang [118] established some results for the function algebras of some affine algebraic groups, and various results were obtained for the enveloping algebra of a Lie algebra (see [66, 67, 133]) that enabled the authors to deduce some information on deformations of the enveloping algebra of a Lie algebra; Bichon [17] described it for $\mathcal{O}(\mathrm{SL}_q(2))$ for q generic.

In [T07], I computed $\mathrm{H}_b^*(\Lambda_{n,d}, \Lambda_{n,d})$ where $\Lambda_{n,d}$ is the dual of a generalised Taft algebra, which is a finite-dimensional Hopf algebra that is neither commutative nor cocommutative.

For this, I used Theorem 7.2.6 as well as the following result.

Theorem 7.4.1. *[141, Remark, end of Section 4] and [142, Remark 3.10]. Let H be a finite-dimensional Hopf algebra and let $\mathcal{D}(H)$ be its Drinfeld double. Denote by k the trivial simple left module over $\mathcal{D}(H)$. Then $\mathrm{H}_b^*(H, H)$ is isomorphic to $\mathrm{Ext}_{\mathcal{D}(H)}^*(k, k)$ as a graded algebra (the product on the algebra $\mathrm{Ext}_{\mathcal{D}(H)}^*(k, k)$ is the Yoneda product).*

Here, the trivial module k is the simple module $L(0,0)$. Moreover, $\mathrm{Ext}_{\mathcal{D}(H)}^*(L(0,0), L(0,0)) \cong \mathrm{Ext}_{\mathbb{B}_{0,0}}^*(L(0,0), L(0,0))$ is isomorphic to $\mathrm{Ext}_B^*(k, k)$ where B is the basic algebra that is Morita equivalent to $\mathbb{B}_{0,0}$ (and whose quiver and relations is given in Theorem 7.2.6; B is the algebra $A_{2\frac{n}{d},1}$ of Section 2.1.2) and S is a simple B-module. It is not difficult to find a minimal projective left B-module resolution of S and the vector spaces $\mathrm{Ext}_B^r(S, S)$. I then gave bases of cochain maps of the $\mathrm{Ext}_B^r(S, S)$, whose composition give the cup-product on $\mathrm{Ext}_B^*(S, S)$.

The final result is as follows.

Theorem 7.4.2. *[T07, Proposition 3.3 and Corollary 3.4] The dimensions of the $\mathrm{H}_b^r(\Lambda_{n,d}, \Lambda_{n,d})$ for $r \in \mathbb{N}$ are given by:*

$$\dim_k \mathrm{H}_b^r(\Lambda_{n,d}, \Lambda_{n,d}) = \begin{cases} 0 & \text{if } r \text{ is odd} \\ 2\lfloor \frac{rd}{2n} \rfloor + 1 & \text{if } r \text{ is even} \end{cases}$$

where $\lfloor \ \rfloor$ denotes the lower integer part.

Moreover, there is a graded algebra isomorphism:

$$\mathrm{H}_b^*(\Lambda_{n,d}, \Lambda_{n,d}) \cong \frac{k[x, y, z]}{(z^{\frac{2n}{d}} - xy)}$$

with x and y of degree $\frac{2n}{d}$ and z of degree 2.

Chapter 8

More recent results: Koszul calculus of preprojective algebras

Summary

In joint work with Berger [BT], we adapted the notion of Koszul calculus of [13] to quiver algebras, in order to study the Koszul calculus of preprojective algebras. It satisfies a number of remarkable properties: a duality between Koszul homology and cohomology, Calabi-Yau properties and a Poincaré duality in particular. We also computed it explicitly for the non Koszul preprojective algebras.

8.1 Introduction

Koszul calculus was introduced by Berger, Lambre and Solotar [13] for quadratic connected k-algebras, as a new tool in the study of these algebras. If the algebra is not Koszul, its Koszul calculus provides different information to Hochschild calculus. They used this calculus to prove a Koszul duality between Koszul cohomologies of any quadratic connected algebra and its Koszul dual.

Berger and I [BT] were interested in studying this Koszul calculus for preprojective algebras. This required that the definition and some properties of Koszul calculus be adapted to quadratic quiver algebras. We then observed a remarkable duality property in the case of preprojective algebras. Moreover, the Koszul calculus for preprojective algebras also satisfies a Poincaré duality, which led us to introduce a definition of Koszul complex Calabi-Yau algebra (which is different to the usual Calabi-Yau property if the algebra is not Koszul). We finally computed explicitly the whole of the Koszul calculus for all the non Koszul preprojective algebras.

8.2 Koszul calculus for quadratic quiver algebras

Let Q be a finite quiver. Denote by Q_j the set of paths of length j, by E the semisimple algebra $E = kQ_0$ and by V the E-bimodule $V = kQ_1$. Then kQ_j identifies with $V_E^{\otimes j}$ and the path algebra kQ identifies with the tensor algebra $T_E(V)$.

Let R be a sub-E-bimodule of $V \underset{E}{\otimes} V \cong kQ_2$. Then $A := T_E(V)/(R) \cong kQ/(R)$ is a quadratic algebra; the grading induced by the length of paths will be called the **weight**.

As in [13], the Koszul complex $K(A)$ of A is the subcomplex of the bar resolution defined by the sub-A-bimodules $A \underset{E}{\otimes} W_p \underset{E}{\otimes} A$ of $A \underset{E}{\otimes} A_E^{\otimes p} \underset{E}{\otimes} A$ where $W_0 = k$, $W_1 = V$ and, for $p \geqslant 2$,
$W_p = \bigcap_{i+2+j=p} V_E^{\otimes i} \underset{E}{\otimes} R \underset{E}{\otimes} V_E^{\otimes j}$. The differential becomes

$$d(a \otimes_k x_1 \dots x_p \otimes_k a') = ax_1 \otimes_k x_2 \dots x_p \otimes_k a' + (-1)^p a \otimes_k x_1 \dots x_{p-1} \otimes_k x_p a'$$

for a, a' in A and $x_1 \dots x_p$ in W_p. It is a complex of finitely generated projective modules.

Definition 8.2.1. For any A-bimodule M, Koszul homology and cohomology are defined by

$$\text{HK}_\bullet(A, M) = H_\bullet(M \otimes_{A^e} K(A)) \text{ and } \text{HK}^\bullet(A, M) = H^\bullet(\text{Hom}_{A^e}(K(A), M)).$$

We set $\text{HK}_\bullet(A) = \text{HK}_\bullet(A, A)$ and $\text{HK}^\bullet(A) = \text{HK}^\bullet(A, A)$.

The embedding of $K(A)$ into the bar complex induces maps $\text{HK}_p(A, M) \to \text{HH}_p(A, M)$ and $\text{HH}^p(A, M) \to \text{HK}^p(A, M)$, that are always isomorphisms for $p = 0$ and $p = 1$, and if A is Koszul they are isomorphisms for all p. Moreover, for $p = 2$, the first one is surjective and the second one is injective ([BT, Proposition 2.11]).

The restriction of the cup and cap products on Hochschild cohomology and homology induce Koszul cup and cap products on Koszul cohomology, described on (co)chains as follows. Let P, Q and M be A-bimodules. For $f \in \text{Hom}_{k^e}(W_p, P)$, $g \in \text{Hom}_{k^e}(W_q, Q)$ and $z = m \otimes_{k^e} x_1 \ldots x_q \in M \otimes_{k^e} W_q$, we define $f \underset{K}{\smile} g \in \text{Hom}_{k^e}(W_{p+q}, P \otimes_A Q)$, $f \underset{K}{\frown} z \in (P \otimes_A M) \otimes_{k^e} W_{q-p}$ and $z \underset{K}{\frown} f \in (M \otimes_A P) \otimes_{k^e} W_{q-p}$ by

$$(f \underset{K}{\smile} g)(x_1 \ldots x_{p+q}) = (-1)^{pq} f(x_1 \ldots x_p) \otimes_A g(x_{p+1} \ldots x_{p+q}),$$

$$f \underset{K}{\frown} z = (-1)^{(q-p)p}(f(x_{q-p+1} \ldots x_q) \otimes_A m) \otimes_{k^e} x_1 \ldots x_{q-p},$$

$$z \underset{K}{\frown} f = (-1)^{pq}(m \otimes_A f(x_1 \ldots x_p)) \otimes_{k^e} x_{p+1} \ldots x_q.$$

The satisfy some associativity relations (see [BT, Subsection 2.3]).

The **general Koszul calculus** of A is defined by the following data: the spaces $\text{HK}^\bullet(A, P)$ and $\text{HK}_\bullet(A, M)$ for all A-bimodules P and M, endowed with the cup and cap products. The **Koszul calculus** of A consists of the graded associative algebra $(\text{HK}^\bullet(A), \underset{K}{\smile})$ and of the graded $\text{HK}^\bullet(A)$-bimodules $\text{HK}^\bullet(A, M)$ and $\text{HK}_\bullet(A, M)$ for all A-bimodules M.

The **restricted Koszul calculus** of A consists of $\text{HK}^\bullet(A)$ and $\text{HK}_\bullet(A)$ endowed with the cup and cap products. In this context, the cup and cap products induce graded Koszul brackets on Koszul (co)homology defined by

$$[f, g]_{\underset{K}{\smile}} = f \underset{K}{\smile} g - (-1)^{pq} g \underset{K}{\smile} f$$

$$[f, z]_{\underset{K}{\frown}} = f \underset{K}{\frown} z - (-1)^{pq} z \underset{K}{\frown} f.$$

At the level of (co)chains, these brackets are related to the Koszul differentials b_K and b^K.

Proposition 8.2.2. *Let $A = T_k(V)/(R)$ be a quadratic k-algebra over Q and let M be an A-bimodule. For any $\alpha \in \text{HK}^p(A, M)$ with $p = 0$ or $p = 1$, $\beta \in \text{HK}^q(A)$ and $\gamma \in \text{HK}_q(A)$, we have the identities*

$$[\alpha, \beta]_{\underset{K}{\smile}} = 0,$$

$$[\alpha, \gamma]_{\underset{K}{\frown}} = 0.$$

The second identity also holds if $p = q \notin \{0, 1\}$.

The Koszul calculus only depends on the algebra A and not on the presentation of A as $A \cong T_E(V)/(R)$ ([BT, Subsection 2.7]).

8.3 Koszul calculus of preprojective algebras

Let Δ be a finite connected graph. Let Q be a quiver whose underlying graph is Δ and let Q^* be the quiver obtained by reversing all the arrows. Let $\overline{Q}$ be the double quiver of Q, defined by $\overline{Q}_0 = Q_0$ and $\overline{Q}_1 = Q_1 \cup Q_1^*$. We shall view $(-)^*$ as an involution of $\overline{Q}_1$.

The preprojective algebra associated with the graph Δ is the algebra $k\overline{Q}/(R)$ where the E-bimodule R is generated by the

$$\sigma_i := \sum_{\substack{a \in Q_1 \\ \mathfrak{t}(a)=i}} aa^* - \sum_{\substack{a \in Q_1 \\ \mathfrak{s}(a)=i}} a^*a = \sum_{\substack{a \in \overline{Q}_1 \\ \mathfrak{t}(a)=i}} \varepsilon(a)aa^* \quad \text{for all } i \in Q_0,$$

where $\varepsilon(a) = 1$ if $a \in Q_1$, $\varepsilon(a) = -1$ if $a \in Q_1^*$. It does not depend on the orientation of Q (up to isomorphism), but only on Δ, and we denote it by $A(\Delta)$.

It is well known, if Δ is not of type A_1 or A_2, that $A(\Delta)$ is finite dimensional if, and only if, Δ is Dynkin of type ADE. Moreover, the finite dimensional preprojective algebras are precisely the non Koszul preprojective algebras (see [69, 109, 110]).

In computing the Koszul calculus of the preprojective algebras, our first step was to examine the Koszul complex.

Theorem 8.3.1. *[BT, Theorem 4.2] Let $A = A(\Delta)$ be a preprojective algebra over k with $\Delta \neq A_1$ and $\Delta \neq A_2$. Then the Koszul complex $K(A)$ of A has length 2. Consequently, $\mathrm{HK}^p(A, M) \cong \mathrm{HK}_p(A, M) = 0$ for all A-bimodules M and all $p \geqslant 3$.*

We then noted that there is a remarkable duality between Koszul homology and cohomology.

Theorem 8.3.2. *[BT, Theorem 4.4] Let $A = A(\Delta)$ be a preprojective algebra over k with $\Delta \neq A_1$ and $\Delta \neq A_2$. Consider the Koszul 2-cycle $\omega_0 = \sum_{i \in Q_0} e_i \otimes \sigma_i \in A \otimes_{E^e} R$. For each Koszul p-cochain f with coefficients in an A-bimodule M, we define the Koszul $(2-p)$-chain $\theta_M(f)$ with coefficients in M by*

$$\theta_M(f) = \omega_0 \underset{K}{\frown} f.$$

Then the equalities

$$\theta_{M \otimes_A N}(f \underset{K}{\smile} g) = \theta_M(f) \underset{K}{\frown} g = f \underset{K}{\frown} \theta_N(g)$$

hold for any Koszul cochains f and g with coefficients in A-bimodules M and N respectively.

Moreover $H(\theta_M) : \mathrm{HK}^\bullet(A, M) \to \mathrm{HK}_{2-\bullet}(A, M)$ is an isomorphism of graded $\mathrm{HK}^\bullet(A)$-bimodules.

Using these maps, we proved a property similar to the 2-Calabi-Yau property (involving Koszul instead of Hochschild cohomology).

Theorem 8.3.3. *[BT, Theorem 4.7] Let $A = A(\Delta)$ be a preprojective algebra over k with $\Delta \neq A_1$ and $\Delta \neq A_2$. Then the $\mathrm{HK}^\bullet(A)^e$-A^e-bimodules $\mathrm{HK}^\bullet(A, A^e)$ and $H_{2-\bullet}(K(A))$ are isomorphic. In particular, we have the following.*

(i) *The A-bimodule $\mathrm{HK}^2(A, A^e)$ is isomorphic to the A-bimodule A.*

(ii) $\mathrm{HK}^1(A, A^e) = 0$.

(iii) *The A-bimodule $\mathrm{HK}^0(A, A^e)$ is isomorphic to the A-bimodule $H_2(K(A))$.*

This led us to generalise the definition of Calabi-Yau algebras in the context of quadratic algebras.

Definition 8.3.4. Let $A = T_E(V)/(R)$ be a quadratic k-algebra. Let $n \geqslant 0$ be an integer. We say that A is **Koszul complex Calabi-Yau** (or Kc-Calabi Yau) of dimension n if

(i) the Koszul bimodule complex $K(A)$ of A has length n, and

(ii) $\mathrm{RHom}_{A^e}(K(A), A^e) \cong K(A)[-n]$ in the bounded derived category of A-bimodules.

Our results show that a preprojective algebra A with Δ not of type A_1 or A_2 is Kc-Calabi Yau of dimension 2. If moreover Δ is Dynkin of type ADE, then A is not Calabi-Yau in the usual sense of Ginzburg [68]. However, for Koszul algebras, the two notions coincide ([BT, Proposition 5.2]).

We then obtained a duality similar to the Van den Bergh duality for Calabi-Yau algebras.

Theorem 8.3.5. *[BT, Theorem 5.4] Let A be a Kc-Calabi Yau algebra of dimension n. Then for any A-bimodule M and any integer p with $0 \leqslant p \leqslant n$, the vector spaces $\mathrm{HK}^p(A, M)$ and $\mathrm{HK}_{n-p}(A, M)$ are isomorphic.*

We then computed explicitly the whole restricted Koszul calculus of preprojective non Koszul algebras in [BT, Section 6] in all characteristics, and compared the second Koszul (co)homology spaces to the second Hochschild (co)homology spaces (that have been computed in [47] in some cases).

As a consequence, we obtained the following characterisation of a preprojective algebra of type ADE.

Theorem 8.3.6. *[BT, Theorem 6.24 and Remark 6.25] Let A (resp. A') be the preprojective algebra of a Dynkin graph over k. Assume that A (resp. A') has type A_n with $n \geqslant 3$ (resp. $A_{n'}$ with $n' \geqslant 3$), or D_n with $n \geqslant 4$ (resp. $D_{n'}$ with $n' \geqslant 4$), or E_n with $n = 6, 7, 8$ (resp. $E_{n'}$ with $n' = 6, 7, 8$).*

If $\dim \mathrm{HK}^p(A) = \dim \mathrm{HK}^p(A')$ for $p = 0, 1$ and 2, then $n = n'$, and A and A' have the same type. This is no longer true if we only assume that $\dim \mathrm{HK}^p(A) = \dim \mathrm{HK}^p(A')$ for $p = 0$ and 1.

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