Sciences Langues

Calculus

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Contents

Chapter 1 Real numbers 1
- I Supremum, infimum . . . 1
- II Integral part of a real number, density of $\mathbb{Q}$ in $\mathbb{R}$ . . . 3

Chapter 2 Study of functions 5
- A Intermediate Value Theorem . . . 5
- B Monotonic functions . . . 7
- C Graph of a function and of its inverse . . . 8
- D n-th root . . . 8
- E Inverse of a differentiable function . . . 9
- I Rolle's Theorem and the Mean Value Theorem . . . 9
- II Higher-order derivatives . . . 13
- III Complex valued functions . . . 15

Chapter 3 Classical functions and their inverses 16
- I Logarithm, powers and exponential . . . 16
  - A Logarithm . . . 16
  - B Exponential . . . 17
  - C Power functions . . . 18
  - D The general exponential function . . . 19
- II Indeterminate limits involving logarithms, exponentials and powers . . . 19
- III Inverses of trigonometric functions . . . 20
  - A The inverse sine function . . . 20
  - B The inverse cosine function . . . 21
  - C The inverse tangent function . . . 22
- IV The hyperbolic functions and their inverses . . . 23

Chapter 4 Study of recursive sequences $u_{n+1} = f(u_n)$ 25
- I Existence of all terms in the sequence . . . 25
  - A Stable intervals . . . 25
  - B Why do we need stable intervals? . . . 25
- II Potential limits . . . 26
  - A Fixed points . . . 26
  - B Limits . . . 26
- III Graphical representation . . . 26
- IV Monotony of the sequence . . . 27
  - A Sign of $f(x) - x$ . . . 27
  - B f is non-decreasing . . . 28
  - C f is non-increasing . . . 29
- V Summary . . . 30

Chapter 5 Study of equations $f(x) = 0$ 31
- I Dichotomy . . . 31
  - A Principle . . . 31
  - B Study of convergence . . . 31
- II Newton's method . . . 32
  - A Principle . . . 32
  - B Study of convergence . . . 33

Chapter 6 Polynomials 34
- I Arithmetics in $\mathbb{K}[X]$ . . . 34
  - A Greatest common divisors . . . 34
  - B Euclidean algorithm . . . 35
  - C Least common multiples . . . 37
  - D Coprime polynomials . . . 39
  - E Gcd and lcm of more than two polynomials . . . 40
- II Irreducible polynomials and factorisations . . . 42
  - A Irreducible polynomials in $\mathbb{C}[X]$ . . . 43
  - B Irreducible polynomials in $\mathbb{R}[X]$ . . . 45

III Lagrange interpolation . . . 46

Chapter 7 Rational fractions and partial fraction decomposition 47
I Rational fractions . . . 47
II Partial fraction decomposition . . . 49
A Partial fraction decomposition in $\mathbb{C}(X)$ . . . 49
B Partial fraction decomposition in $\mathbb{R}(X)$ . . . 50
C Simple poles . . . 51
D Partial fraction decomposition of $\frac{P'}{P}$ . . . 52
E Tricks . . . 52

Chapter 8 Integration 54
I Integration of step functions . . . 54
II Integrable functions . . . 56
III Properties of the integral . . . 58
IV Some generalisations . . . 61
A Complex valued functions . . . 61
B Piecewise continuous functions . . . 62
V Riemann sums . . . 62

Chapter 9 Primitives. Integration techniques. 65
I Primitive of a function . . . 65
II Classical primitives . . . 67
III Integration by parts . . . 67
IV Integration by substitution . . . 68
V Primitive of a rational function . . . 69
A Computation of $\int \frac{\mathrm{d}x}{(x-\alpha)^n}$ . . . 70
B Computation of $\int \frac{ax+b}{(x^2+px+q)^n}\,\mathrm{d}x$ . . . 70
VI Primitive of a rational function in sin, cos and tan . . . 71
VII Taylor's formula with integral remainder . . . 71
VIII Approximations . . . 72
A Trapezium rule . . . 72
B Simpson's rule . . . 73

Chapter 10 Improper integrals 75
I Fundamental examples . . . 76
II Convergence theorems . . . 77
A The comparison theorems . . . 78
B Consequences of the comparison theorems . . . 79
C Absolute convergence . . . 80
III Semi-convergence . . . 80
IV Substitution . . . 82

Chapter 11 Derivation and integration of functions defined on an interval of $\mathbb{R}$ with values in $\mathbb{R}^2$ 83
I Norms in $\mathbb{R}^2$ . . . 83
II Limits, continuity and differentiability of a vector function . . . 84
III Mean Value Inequality . . . 86
IV Taylor expansions of vector functions . . . 86
V Integration of vector functions . . . 87

Chapter 12 Functions of two variables with values in $\mathbb{R}$ or $\mathbb{R}^2$ 88
I Open subsets of $\mathbb{R}^2$ . . . 88
II Limits and continuity . . . 90
III Partial derivatives . . . 91
IV Computation of partial derivatives . . . 92
V Gradient . . . 94
VI Change of coordinates . . . 95
A Definition . . . 95
B Polar coordinates . . . 95
C Cylindrical coordinates . . . 96
VII Higher order partial derivatives . . . 96

Appendix A Trigonometric formulae 98
I Angles and properties of cos and sin . . . 98
II Circular trigonometric functions . . . 99
III Linearisation . . . 99
IV Hyperbolic trigonometric functions . . . 100

Appendix B Greek alphabet **101**

Appendix C Integer arithmetics **102**
I Greatest common divisor . . . 102
A Euclidean algorithm . . . 102
II Least common multiples . . . 103
III Coprime integers . . . 104
IV Solving equations $ax + by = c$ . . . 105
V Gcd and lcm of more than two integers . . . 106
VI Prime integers and factorisations . . . 107

Glossary **i**

Glossaire **v**

Chapter 1

Real numbers

Notation. ➢ $\mathbb{R}$ is the set[†] of real numbers.

➢ $\mathbb{N}$ is the set of integers that are non-negative[‡].
(The usual convention in English is that $\mathbb{N}$ is the set of positive[§] integers – we will not use it.)

➢ $\mathbb{Z}$ is the set of all integers; $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R}$.

➢ $\mathbb{Q}$ is the set of rational numbers; $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. It is the set of real numbers that can be written $\frac{p}{q}$ where $(p, q) \in \mathbb{Z}^2$ and $q \neq 0$.

➢ $\mathbb{D}$ is the set of decimal numbers; $\mathbb{Z} \subset \mathbb{D} \subset \mathbb{Q}$. It is the set of rational (or real) numbers that can be written $\frac{p}{10^n}$ where $(p, n) \in \mathbb{Z} \times \mathbb{N}$.

➢ $\mathbb{C}$ is the set of complex numbers; $\mathbb{R} \subset \mathbb{C}$.

➢ For $\mathbb{K}$ one of the sets above, $\mathbb{K}^*$ is the subset of non-zero elements.

I. Supremum, infimum

Definition 1. *A set A of real numbers is* ***bounded above***[a] *if there is a number x such that $x \geqslant a$ for all $a \in A$. Such a number x is called an* ***upper bound***[b] *for A.*

[a] majoré
[b] majorant

Example. The sets $A = [5, 7]$ and $B =]-\infty, 10[$ are bounded above 10, 13 and 3002 for instance are upper bounds for both sets, while 7 and 9 are upper bounds for A but not for B.
The sets $]-3; +\infty[$ and $\mathbb{N}$ are not bounded above.

Definition 2. *A number x is a* ***supremum***[a] *or* ***least upper bound*** *of A if x is an upper bound for A and for any upper bound y for A, we have $x \leqslant y$. We denote it by $\sup A$ or $\sup_{a \in A}\{a\}$ or $\sup\{a; a \in A\}$*

[a] borne supérieure

Example. The sets $A = [5, 7]$ and $B =]-\infty, 10[$ have supremums, and $\sup A = 7$ and $\sup B = 10$.

Lemma 3. A supremum, if it exists, is unique.

Proof. Let A be a set and let x and y be two supremums of A. They are in particular upper bounds for A so that

➢ since x is a supremum and y is an upper bound, we have $x \leqslant y$, and

➢ since y is a supremum and x is an upper bound, we have $x \geqslant y$

therefore $x = y$. ✓

[†] ensemble
[‡] positif (ou nul)
[§] strictement positif

Remark. Characterisation of the supremum. Let α be an upper bound for A. Then $\alpha = \sup A$ if and only if for any $t < \alpha$ there exists $a \in A$ such that $t < a \leqslant \alpha$ (or equivalently, if and only if for any $\varepsilon > 0$ there exists $a \in A$ such that $\alpha - \varepsilon < a \leqslant \alpha$).

With quantifiers, this becomes:

$$\begin{aligned} \alpha = \sup A &\iff \forall t < \alpha, \exists a \in A \text{ such that } t < a \leqslant \alpha \\ &\iff \forall \varepsilon > 0, \exists a \in A \text{ such that } \alpha - \varepsilon < a \leqslant \alpha. \end{aligned}$$

Indeed, if $\alpha = \sup A$, any $t < \alpha$ is not an upper bound for A so that there is an element $a \in A$ with $t < a$ (and of course, $a \leqslant \alpha$ since α is an upper bound for A).

Conversely, if $\alpha > \sup A$ take $t = \sup A$. By assumption there exists $a \in A$ such that $\sup A = t < a \leqslant \alpha$. This is a contradiction since $\sup A$ is an upper bound for A.

Remark. Sequential characterisation of the supremum. Let α be an upper bound for A. Then $\alpha = \sup A$ if and only if there exists a sequence[†] (u_n) of elements of A that converges to α.

Indeed, if $\alpha = \sup A$ then for any $n \in \mathbb{N}^*$ there exists $u_n \in A$ such that $\alpha - \frac{1}{n} < u_n \leqslant \alpha$ using the previous characterisation. Then (u_n) converges to α by the Comparison Theorem.

Conversely, assume that there exists a sequence (u_n) of elements of A that converges to α. Note that since α is an upper bound for A and $u_n \in A$, we have $u_n \leqslant \alpha$ for any n.

Fix $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geqslant N$, we have $|u_n - \alpha| < \varepsilon$. Since $|u_n - \alpha| = \alpha - u_n$, this implies that $\alpha - \varepsilon < u_n \leqslant \alpha$. Therefore $\alpha = \sup A$ using the previous characterisation.

Example. The set $\mathbb{R}$ itself is not bounded above, neither is $\mathbb{N}$ (subset[‡] of $\mathbb{R}$), therefore it cannot have a supremum.

The interval $[0,1]$ is bounded above (for instance, 2 and 35 are upper bounds of $[0,1]$); it also has a supremum, which is 1 (1 is an upper bound of $[0,1]$ and any other upper bound y satisfies $1 \leqslant y$).

Note that a supremum is not always an element of the set A. For instance, $\sup[0,1[= 1$. (Indeed, 1 is clearly an upper bound of $[0,1[$ and if there existed an upper bound y with $y < 1$, then $y < \frac{1+y}{2}$ and $\frac{1+y}{2} \in [0,1[$, which contradicts the fact that y is an upper bound of $[0,1[$).

Definition 4. *A number x is a **maximum**[a] of A if x is an upper bound of A and $x \in A$. It is denoted by $x = \max A$ or $x = \max_{a \in A}\{a\}$.*

[a] maximum *ou* plus grand élément

Remark. A maximum of A, if it exists, is necessarily the supremum of A.

Remark. Every non-empty finite set has a maximum.

The proof is by induction on the number of elements in the set.

Property 5. Any non-empty subset A of $\mathbb{R}$ that is bounded above has a supremum.

Remark. Property 5 is not necessarily true in ordered sets other than $\mathbb{R}$.

[Not done in class.] Let $A = \{x \in \mathbb{Q};\ x \geqslant 0 \text{ and } x^2 < 2\}$ (subset of $\mathbb{Q}$). Clearly, A is non-empty (it contains 0 and 1) and is bounded above (by 2) in $\mathbb{Q}$.

However, A does not have a supremum in $\mathbb{Q}$. This is essentially due to the fact that $\sqrt{2} \notin \mathbb{Q}$ (in $\mathbb{R}$, $\sqrt{2}$ would be the supremum of A).

Proof. Assume for a contradiction that A has a supremum α in $\mathbb{Q}$. We can write $\alpha = \frac{p}{q}$ with $q \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$ (since $\alpha \geqslant 1$).

- If $\alpha \in A$ then $\alpha^2 < 2$ so $p^2 - 2q^2 < 0$ and therefore $p^2 - 2q^2 \leqslant -1$. Set $\beta = \frac{4p^2+1}{4pq} \in \mathbb{Q}$. Then $\beta > \alpha$ and $\beta^2 < 2$ so $\beta \in A$: this contradicts the fact that $\alpha = \sup A$.
- Therefore $\alpha \notin A$. In particular, $\alpha^2 \geqslant 2$. We know that 2 is not a square in $\mathbb{Q}$ so $\alpha^2 > 2$ and therefore $p^2 - 2q^2 \geqslant 1$. Set $\gamma = \frac{2p^2-1}{2pq} \in \mathbb{Q}$. Then $0 < \gamma < \alpha$ and $\gamma^2 > 2$ so γ is an upper bound for A that is smaller than α; this contradicts the fact that $\alpha = \sup A$.

Therefore A does not have a supremum α in $\mathbb{Q}$. ✓

[†] suite
[‡] sous-ensemble

Definition 6. *A set A of real numbers is* ***bounded below***[a] *if there is a number y such that $y \leqslant a$ for all $a \in A$.*
Such a number y is called a ***lower bound***[b] *for A.*
A number x is an ***infimum***[c] *or* ***greatest lower bound*** *of A if x is a lower bound of A and for any lower bound y of A, we have $x \geqslant y$. If it exists, it is unique and we denote it by $\inf A$ or $\inf_{a \in A}\{a\}$.*
A number x is a ***minimum***[d] *of A if x is a lower bound of A and $x \in A$. It is denoted by $x = \min A$ or $x = \min_{a\in A}\{a\}$. When it exists, it is equal to the infimum of A.*
The set A is ***bounded***[e] *if it is bounded above and below.*

[a] minoré
[b] minorant
[c] borne inférieure
[d] minimum *ou* plus petit élément
[e] borné

Remark. Characterisations of the infimum. Let β be a lower bound for A. Then

$$\begin{aligned}\beta = \inf A &\iff \forall t > \beta,\ \exists a \in A \text{ st } \beta \leqslant a < t \\ &\iff \forall \varepsilon > 0,\ \exists a \in A \text{ st } \beta \leqslant a < \beta + \varepsilon.\end{aligned}$$

Moreover, $\beta = \inf A$ if and only if there exists a sequence (u_n) of elements of A that converges to β.

Property 7. Any non-empty subset A of $\mathbb{R}$ that is bounded below has an infimum.

Proof. Take $B = \{-a; a \in A\} \subset \mathbb{R}$. Since A has a lower bound y, B has an upper bound $x = -y$, hence B has a supremum α. Then $\beta = -\alpha$ is clearly a lower bound for A. Let us check that β is an infimum for A.

If $t > \beta$, then $-t < -\alpha$ so that there exists $b \in B$ such that $-t < b \leqslant \alpha$. Therefore $\beta = -\alpha < -b < t$ with $-b \in A$. Therefore A has an infimum, equal to $\beta = -\alpha$. ✓

Remark. Recall from the first semester that a sequence was called *bounded* (*resp. bounded above*, *resp. bounded below*) if the set $\{u_n; n \in \mathbb{N}\}$ is bounded (*resp.* bounded above, *resp.* bounded below).

Similarly, a map[†] $f : I \to \mathbb{R}$ was called *bounded* (*resp. bounded above*, *resp. bounded below*) if the set $f(I) = \{f(x); x \in I\}$ is bounded (*resp.* bounded above, *resp.* bounded below). We will usually write $\sup_I f = \sup_{x\in I} f(x) = \sup_{x \in I}\{f(x)\}$ for $\sup\{f(x); x \in I\}$ when it exists, or $\sup f$ when I is clear. Similarly, we will write $\inf_I f$, $\max_I f$ and $\min_I f$ when they exist.

Exercise 1. Prove that if $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are two maps such that $f \leqslant g$ (that is, for all $x \in I$ we have $f(x) \leqslant g(x)$), then $\sup f \leqslant \sup g$ and $\inf f \geqslant \inf g$.

Remark. Let A be a non-empty subset of $\mathbb{Z}$ that is bounded above in $\mathbb{R}$. Then A has a maximum.

Idea/summary of proof. Essentially, this is a consequence of the fact that any convergent sequence in $\mathbb{Z}$ is stationary[‡].

Proof. Since A is non-empty and bounded above in $\mathbb{R}$, it has a supremum $\alpha \in \mathbb{R}$. We know that there exists a sequence (u_n) of elements of A (hence of $\mathbb{Z}$) that converges to α. Therefore this sequence is stationary: there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $u_n = u_N$. In particular, $\alpha = u_N \in A$.

To prove that a convergent sequence of integers is stationary, use the definition of the limit with $\varepsilon = \frac{1}{2}$, so that there exists $N \in \mathbb{N}$ such that for any $n \geqslant N$ we have $|u_n - \alpha| < \frac{1}{2}$; then for any $n \geqslant N$ we have $0 \leqslant |u_n - u_N| \leqslant |u_n - \alpha| + |\alpha - u_N| < \frac{1}{2} + \frac{1}{2} = 1$. Since u_n and u_N are integers, we must have $u_n = u_N$ for all $n \geqslant N$. ✓

Remark. Any non-empty subset of $\mathbb{Z}$ that is bounded below in $\mathbb{R}$ has a minimum. In particular, any non-empty subset of $\mathbb{N}$ has a minimum.

II. Integral part of a real number, density of $\mathbb{Q}$ in $\mathbb{R}$

Proposition 8. For any real number x, there exists a unique integer n such that $n \leqslant x < n+1$. We denote $n = E(x)$ or $n = [x]$ or $n = \lfloor x \rfloor$ and call it the **integral part**[a] of x.

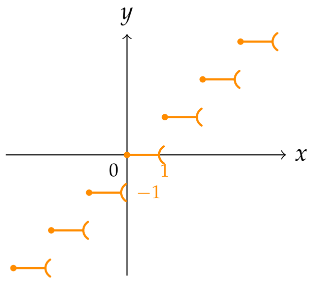
[a] partie entière

Proof.
- ➢ We first prove uniqueness. Let n and n' be two integers such that $n \leqslant x < n+1$ and $n' \leqslant x < n'+1$. Then $n \leqslant x < n'+1$ so $n < n'+1$ and $n \leqslant n'$. Similarly, $n' \leqslant n$ so $n = n'$.
- ➢ We now prove existence. The set $A := \{k \in \mathbb{Z}; k \leqslant x\}$ is non-empty and bounded above in $\mathbb{R}$, hence by a remark on page 3, it has a maximum $n \in A$. Then we have $n \leqslant x$ and $n+1 \notin A$ so $n+1 > x$. ✓

[†] application
[‡] stationnaire

Remark. $\lfloor x \rfloor$ is also the smallest integer satisfying $x < \lfloor x \rfloor + 1$ (the set $A' = \{k \in \mathbb{Z}; x < k+1\}$ has a minimum n that satisfies $n \leqslant x < n+1$; the uniqueness of such an integer proves that $n = \lfloor x \rfloor$).

Graph of E**.**



Definition 9. *Let x be a real number and take $n \in \mathbb{N}$. A decimal number d is a* ***decimal approximation by default***[a] *(resp.* ***decimal approximation by excess***[b]*)* ***accurate within*** 10^{-n}[c] *if $d \leqslant x \leqslant d + 10^{-n}$ (resp. if $d - 10^{-n} \leqslant x \leqslant d$).*

[a] valeur approchée par défaut
[b] valeur approchée par excès
[c] à 10^{-n} près

Remark. Let x be a real number. Take $n \in \mathbb{N}$ and set $q_n = [10^n x] \in \mathbb{Z}$. Then q_n is the unique integer such that $\frac{q_n}{10^n} \leqslant x < \frac{q_n+1}{10^n} = \frac{q_n}{10^n} + 10^{-n}$. Therefore $\frac{q_n}{10^n}$ is a decimal approximation of x by default accurate within 10^{-n}. In fact, $\frac{q_n}{10^n}$ is the number **x truncated to n decimal places**[†].

Proposition 10. Any real number is the limit of a sequence of decimal numbers.
In particular, every real number is the limit of a sequence of rational numbers. We say that $\mathbb{Q}$ is **dense**[a] in $\mathbb{R}$.

[a] dense

Proof. Use the notation in the previous remark. We have $(*)$ $q_{n+1} \leqslant x\,10^{n+1} < q_{n+1} + 1$ and $10q_n \leqslant x\,10^{n+1} < 10(q_n+1)$ therefore $10q_n \leqslant q_{n+1} \leqslant x\,10^{n+1} < q_{n+1} + 1 \leqslant 10(q_n + 1)$ (the first inequality follows from the definition of $[10^{n+1}x]$ and $(*)$ and the second follows from the remark above and $(*)$) so

$$\frac{q_n}{10^n} \leqslant \frac{q_{n+1}}{10^{n+1}} \leqslant x < \frac{q_{n+1}+1}{10^{n+1}} \leqslant \frac{q_n+1}{10^n}.$$

Consequently, the sequences $\left(\frac{q_n}{10^n}\right)_{n\in\mathbb{N}}$ and $\left(\frac{q_n+1}{10^n}\right)$ are adjacent and their common limit is x. ✓

Corollary 11. Let x and y be two real numbers with $x < y$. Then there exists $q \in \mathbb{Q}$ such that $x < q < y$.

Proof. By the previous proposition, there exists a sequence $(q_n)_n$ of rational numbers that converges to $\dfrac{x+y}{2}$. Therefore there exists an integer N such that for all $n \geqslant N$ we have $\left|q_n - \dfrac{x+y}{2}\right| < \dfrac{y-x}{2}$. In particular, $-\frac{y-x}{2} + \frac{x+y}{2} < q_N < \frac{y-x}{2} + \frac{x+y}{2}$ so that $x < q_N < y$. ✓

Proposition 12. The set $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$: for any $x \in \mathbb{R}$ there exists a sequence (t_n) of non-rational real numbers that converges to x.

Proof. We know that $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.
Set $y = \dfrac{x}{\sqrt{2}}$. We know by Proposition 10 that there exists a sequence (q_n) of rational numbers that converges to y. Put $t_n = q_n\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$. Then (t_n) converges to $y\sqrt{2} = x$. ✓

Corollary 13. Let x and y be two real numbers with $x < y$. Then there exists $t \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < t < y$.

Proof. Same proof as that of Corollary 11, or use Corollary 11 and the same trick as in the proof of Proposition 12. ✓

[†] tronqué à n décimales

Chapter 2

Study of functions

We shall now consider functions, building on what you have done in the first semester. We shall start with properties of continuous functions, then differentiable functions.

A. Intermediate Value Theorem

Theorem 1. Let f be a continuous function on $[a,b]$. If $f(a)f(b) < 0$ (*ie.* $f(a)$ and $f(b)$ have opposite signs), then there is some $x \in]a,b[$ such that $f(x) = 0$.

Idea/summary of proof. Assume that $f(a) < 0 < f(b)$. We prove that there is an $\alpha \in]a,b[$ such that f is negative on $]a,\alpha[$ and such that f is not negative on any $]a,c[$ with $c > \alpha$. We then prove that $f(\alpha) = 0$.

Proof. Assume that $f(a) < 0 < f(b)$ (the other case is obtained from this one by considering $-f$). Define the set $A = \{x;\, a \leqslant x \leqslant b \text{ and } f \text{ is negative on the interval } [a,x]\}$. Clearly, $A \neq \varnothing$ since $a \in A$. Moreover, b is an upper bound for A (since $A \subset [a,b]$), therefore A has a supremum. Set $\alpha = \sup A$. We shall prove that $a < \alpha < b$ and that $f(\alpha) = 0$.

➢ We first prove that $\alpha > a$. For this, we show that there is some $\delta > 0$ such that $[a, a+\delta] \subset A$.

Recall from the first semester Proposition 241: if $\lim_{x\to a} f(x) < c$, then f is bounded above by c in a neighbourhood[†] of a. We apply this here with $c = \frac{1}{2}f(a)$. Indeed, since f is continuous at a, we have $\lim_{x\to a} f(x) = f(a)$ and $f(a) < c$ because $f(a) < 0$. This means that there exists $\delta > 0$ such that for all $x \in [a, a+\delta]$ we have $f(x) \leqslant c$, and in particular $f(x) < 0$. Finally we have $[a; a+\delta] \subset A$.

In particular, $\alpha \geqslant a + \delta > a$.

➢ Let us now check that $\alpha < b$. Assume for a contradiction that $\alpha = b$. Since $f(b) > 0$, we can show as in the previous step that there is a $\delta' > 0$ such that f is positive on $[b-\delta'; b]$. We have assumed that $b = \alpha = \sup A$ and we have $b - \delta' < b$, therefore there exists $y \in A$ with $b - \delta' < y \leqslant b$. Therefore $f(y) > 0$ (because $y \in [b-\delta'; b]$) and $f(y) < 0$ (because $y \in A$), a contradiction. Therefore $\alpha < b$.

➢ We will now show that $f(\alpha) = 0$. Assume for a contradiction that $f(\alpha) \neq 0$. Then there are two cases:

- First case: $f(\alpha) < 0$. Then there exists $\delta > 0$ such that f is negative on the neighbourhood $[\alpha-\delta, \alpha+\delta]$ of α. Now there is some $x_0 \in A$ satisfying $\alpha - \delta < x_0 \leqslant \alpha$ (by definition of the supremum α) so f is negative on $[a, x_0]$. But if $x_1 \in]\alpha, \alpha+\delta]$ then f is negative on $[x_0, x_1]$ so f is negative on $[a, x_1]$ and therefore $x_1 \in A$. This contradicts the fact that α is an upper bound for A.
- Second case: $f(\alpha) > 0$. Then there exists $\delta > 0$ such that f is positive on $[\alpha-\delta, \alpha+\delta]$. Once again we know that there is an $x_0 \in A$ satisfying $\alpha - \delta < x_0 < \alpha$. But this means that f is negative on $[a, x_0]$, which is impossible since $x_0 \in [\alpha-\delta, \alpha+\delta]$. We have a contradiction.

Therefore $f(\alpha) = 0$. ✓

Corollary 2 (Intermediate Value Theorem[a]). Let f be a continuous function on $[a,b]$. If c is a real number that is strictly between $f(a)$ and $f(b)$, then there is some $x \in]a,b[$ such that $c = f(x)$.

[a] théorème des valeurs intermédiaires (TVI)

Proof. Define $g : [a,b] \to \mathbb{R}$ by $g(x) = f(x) - c$. Then g is continuous. If $f(a) < c < f(b)$ then $g(a) < 0$ and $g(b) > 0$. If $f(a) > c > f(b)$ then $g(a) > 0$ and $g(b) < 0$. In both cases, $g(a)g(b) < 0$ so there exists $x \in]a,b[$ such that $g(x) = 0$, that is, $f(x) = c$. ✓

Corollary 3. A polynomial of odd degree has at least one real root.

[†] voisinage

Proof. *cf.* TD. ✓

Definition 4. *An* ***interval***[a] *I is a subset of* $\mathbb{R}$ *satisfying the following property:*

$$\forall a \in I,\ \forall b \in I, a < b,\ [a,b] \subset I.$$

[a]intervalle

Remark. The intervals that we know, that is, of the form $[a,b]$, $]a,b[$, $[a,b[$ and $]a,b]$ (with a and b real numbers or possibly $\pm\infty$ in some cases) are precisely all the intervals in $\mathbb{R}$. (Exercise.)

Corollary 5. Let I be an interval and $f : I \to \mathbb{R}$ a function. If f is continuous then $f(I)$ is an interval.

Proof. Let x and y be elements in $f(I)$ such that $x < y$; we must prove that $[x,y] \subset f(I)$. By definition of $f(I)$, there exist u and v in I such that $x = f(u)$ and $y = f(v)$. Let z be an element of $]x,y[$. Then by the Intermediate Value Theorem, there is an element w between u and v such that $z = f(w)$. But I is an interval so $w \in I$. Therefore $z \in f(I)$. Moreover, if $z = x$ or $z = y$ then clearly $z \in f(I)$. We have proved that $[x,y] \subset f(I)$. Therefore $f(I)$ is an interval. ✓

Examples. **(1)** The image of $\sin : \mathbb{R} \to \mathbb{R}$ is contained in $[-1;1]$, it is an interval since sin is continuous, and it contains $1 = \sin\frac{\pi}{2}$ and $-1 = \sin\frac{3\pi}{2}$, therefore $\sin(\mathbb{R}) = [-1;1]$.

(2) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + x + 1$. We have $f(x) = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ so that the image of f is contained in $[\frac{3}{4};+\infty[$. Moreover, it is an interval since f is continuous, it contains $f(-\frac{1}{2}) = \frac{3}{4}$, and it contains arbitrarily large real numbers since $\lim_{x\to+\infty} f(x) = +\infty$. Therefore $f(\mathbb{R}) = [\frac{3}{4};+\infty[$.

(3) Consider the function $f : I \to \mathbb{R}$ where $\left[0, \frac{\pi}{2}\right[$ defined by $f(x) = \tan(x)$. We have $f(x) \geqslant 0$ for all $x \in I$ so that the image of f is contained in $[0;+\infty[$. Moreover, it is an interval since f is continuous, it contains $f(0) = 0$, and it contains arbitrarily large real numbers since $\lim_{x\to\frac{\pi}{2}} f(x) = +\infty$. Therefore $f(I) = [0;+\infty[$.

Note that $f(I)$ is not bounded although I is bounded.

Remark. The image of an interval need not be an interval of the same type. See the previous examples.

Proposition 6. Let $f : [a,b] \to \mathbb{R}$ be a *continuous* function defined on a *closed bounded* interval. Then f is bounded.

Idea/summary of proof. We assume for a contradiction that f is not bounded above, so that the sets $A_n := \{x \in [a,b]; f(x) \geqslant n\}$ are non-empty. We prove that each of these sets has a maximum, s_n, that the sequence (s_n) converges to $\ell \in [a,b]$, and we finally show that $(f(s_n))$ converges and diverges, a contradiction. Therefore f is bounded above.
To prove that f is bounded below, either adapt this proof or consider $-f$.

Proof. Assume for a contradiction that f is not bounded above. Then for any $n \in \mathbb{N}$ there exists $x_n \in [a,b]$ such that $f(x_n) \geqslant n$. Therefore the set $A_n := \{x \in [a,b]; f(x) \geqslant n\}$ is non-empty.

➢ We prove that A_n has a maximum, s_n.

Since A_n is a subset of $[a,b]$, it is bounded above (by b) and therefore it has a supremum $s_n = \sup A_n$. Moreover, $a \leqslant s_n \leqslant b$. There exists a sequence (t_k) of elements in A_n that converges to s_n. For all k we have $f(t_k) \geqslant n$. Moreover, f is continuous at s_n, so that $(f(t_k))$ converges to $f(s_n)$, and therefore $f(s_n) \geqslant n$. We have shown that $s_n \in A_n$, hence $s_n = \max A_n$.

➢ Since $f(s_n) \geqslant n$ for all n, we have $\lim_{n\to+\infty} f(s_n) = +\infty$.

➢ All the s_n are in $[a,b]$, therefore the sequence (s_n) is bounded below. Moreover, $A_{n+1} \subset A_n$ for all n : indeed, if $f(x) \geqslant n+1$ then $f(x) \geqslant n$. Therefore $s_{n+1} \leqslant s_n$ and the sequence $(s_n)_n$ is non-increasing. Consequently, the sequence (s_n) converges to some ℓ; moreover, since $a \leqslant s_n \leqslant b$ for all n, we have $\ell \in [a,b]$. Hence f is continuous at ℓ and therefore $(f(s_n))$ converges to $f(\ell)$.

We have obtained a contradiction.

Applying this to $-f$ (also continuous on $[a,b]$), we see that $-f$ is bounded above so that f is bounded below and therefore bounded. ✓

Theorem 7. Let $f : [a,b] \to \mathbb{R}$ be a *continuous* function on a closed bounded interval. Then f is bounded and there exist c and d in $[a,b]$ such that $\inf_{x\in[a,b]} f(x) = f(c)$ and $\sup_{x\in[a,b]} f(x) = f(d)$. In other words, $f([a,b])$ is a closed bounded interval.

Proof. (Not done in class.) We know that since f is continuous, $f([a,b])$ is an interval. Moreover, $f([a,b])$ is bounded by the previous proposition. Let m be its infimum and M its supremum. We must show that they are both in $f([a,b])$.

Assume for a contradiction that M is not in $f([a,b])$. We then have $f(t) < M$ for all $t \in [a,b]$, so that $M - f(t) \neq 0$. Let us consider $g : [a,b] \to \mathbb{R}$ defined by $g(t) = \frac{1}{M-f(t)}$. Since $M - f$ is continuous and does not vanish, g is continuous. Therefore by the previous proposition it is bounded and in particular there exists $K \in \mathbb{R}$ such that $g(t) \leqslant K$ for all $t \in [a,b]$.

On the other hand, since M is the supremum of f, there exists a sequence $(y_n)_n$ contained in $f([a,b])$ and such that $\lim_{n\to+\infty} y_n = M$. For every n there exists $x_n \in [a,b]$ such that $y_n = f(x_n)$. We then have $g(x_n) = \frac{1}{M-f(x_n)} = \frac{1}{M-y_n}$ and since $M - y_n > 0$ and has limit 0, we have $\lim_{n\to+\infty} g(x_n) = +\infty$, a contradiction.

Therefore $M \in f([a,b])$. The proof that $m \in f([a,b])$ is similar, so $f([a,b]) = [m,M]$. ✓

Corollary 8. Let $f : [a,b] \to \mathbb{R}$ be a *continuous* function on a *closed bounded* interval. If $f(x) > 0$ for all $x \in [a,b]$, then there exists $m > 0$ such that $f(x) \geqslant m$ for all $x \in [a,b]$.

Proof. m is the minimum of f, and $m > 0$ by assumption. ✓

Remark. The results above require the assumption that the interval is closed and bounded. For instance, tan is continuous but not bounded on $\left]-\frac{\pi}{2};\frac{\pi}{2}\right[$.

B. Monotonic functions

Proposition 9. Let I be an interval and let $f : I \to \mathbb{R}$ be a *continuous* and *injective* function. Then f is strongly monotonic.

Proof. (Not required.) Assume for a contradiction that f is not strongly monotonic. Then there exist elements a, b, c, d in I such that

$$a < b \text{ and } f(a) \leqslant f(b); \qquad\qquad c < d \text{ and } f(c) \geqslant f(d).$$

Now consider the function $g : [0,1] \to \mathbb{R}$ defined by $g(t) = f((1-t)b + td) - f((1-t)a + tc)$. This function is continuous. Moreover, $g(0) = f(b) - f(a) \geqslant 0$ and $g(1) = f(d) - f(c) \leqslant 0$. Therefore, either $g(0) = 0$ or $g(1) = 0$, or by the Intermediate Value Theorem, there exists $t_0 \in]0,1[$ such that $g(t_0) = 0$. In all cases, there exists $t_0 \in [0,1]$ such that $g(t_0) = 0$, that is, $f((1-t_0)b + t_0 d) = f((1-t_0)a + t_0 c)$. Since f is injective, we must have $(1-t_0)b + t_0 d = (1-t_0)a + t_0 c$ hence $(1-t_0)(b-a) = t_0(c-d)$. However, $(1-t_0)(b-a) \geqslant 0$ and $t_0(c-d) \leqslant 0$, and they are not simultaneously equal to 0, a contradiction.

Therefore f is strongly monotonic. ✓

Lemma 10. Let I be an interval and let $f : I \to \mathbb{R}$ be a strongly monotonic function. Then f is injective.

Proof. Let us prove the result when f is e.g. increasing. Let x and y be distinct elements in I. Then either $x < y$ and then $f(x) < f(y)$, or $x > y$ and then $f(x) > f(y)$. In both cases we have $f(x) \neq f(y)$. Therefore f is injective. ✓

Theorem 11. Let I be an interval and let $f : I \to \mathbb{R}$ be a continuous and strongly monotonic function. Then:

(i) $f(I)$ is an interval and f defines a bijection from I to $f(I)$.

(ii) The inverse of f is a continuous function and it is strongly monotonic. Moreover, if f is increasing then so is f^{-1} and if f is decreasing then so is f^{-1}.

Proof. (Proof of continuity not done in class.)

(i) This follows from the results above: f is continuous, so $f(I)$ is an interval; f is increasing so f is injective hence defines a bijection from I to $f(I)$.

(ii) Let $g = f^{-1} : f(I) \to I$ denote the inverse function. It is characterised by $y = f(x) \iff g(y) = x$.

We first show that g is increasing. Let $y < y'$ be elements in $f(I)$. There exist x and x' in I such that $y = f(x)$ and $y' = f(x')$. Note that $x = g(y)$ and $x' = g(y')$. Since f is increasing, if $x \geqslant x'$ we would have $y \geqslant y'$, a contradiction. Therefore $x < x'$, that is, $g(y) < g(y')$.

We must now prove that g is continuous at every $y_0 \in f(I)$. We shall do it when $I = [a,b[$, the other cases are similar.

➢ First case: $y_0 = f(a)$. Fix $\varepsilon > 0$. We may assume (replacing ε by a smaller $\varepsilon' > 0$ if necessary) that $a < a + \varepsilon < b$. Applying f to $a < a + \varepsilon$ yields $y_0 = f(a) < f(a+\varepsilon)$. Therefore it is possible to choose $\eta > 0$ such that $y_0 + \eta < f(a+\varepsilon)$. For any y with $y_0 < y < y_0 + \eta$ we have $f(a) < y < f(a+\varepsilon)$. Apply g to this (g is increasing): $a < g(y) < a + \varepsilon < a + \varepsilon$. Finally, $0 < y - y_0 < \eta \Rightarrow 0 < g(y) - g(y_0) < \varepsilon$, so that g is continuous at $y_0 = f(a)$.

➢ Second case: $y_0 > f(a)$. Set $x_0 = g(y_0)$. Again, we may assume that $a < x_0 - \varepsilon < x_0 + \varepsilon < b$. Applying f to $x_0 - \varepsilon < x_0 < x_0 + \varepsilon$ give $f(x_0 - \varepsilon) < y_0 < f(x_0 + \varepsilon)$. We can then choose $\eta > 0$ such that $f(x_0 - \varepsilon) < y_0 - \eta$ and $y_0 + \eta < f(x_0 + \varepsilon)$. For any y with $y_0 - \eta < y < y_0 + \eta$ we have $f(x_0 - \varepsilon) < y < f(x_0 + \varepsilon)$. Appplying g gives $x_0 - \varepsilon < g(y) < x_0 + \varepsilon$ and hence $x_0 - \varepsilon < g(y) < x_0 + \varepsilon$. Therefore g is continuous at y_0. ✓

Proposition 12. Let I be an interval and let $f : I \to \mathbb{R}$ be a continuous and strongly monotonic function.
Then $f(I)$ is an interval of the same type (closed on both sides, one side or none).
Moreover, if a and b are the endpoints of I with a and b either real numbers or $\pm\infty$, then the endpoints of $f(I)$ are $\lim_{x\to a} f(x)$ and $\lim_{x\to b} f(x)$.

Proof. (Not done in class.) We shall prove the result for f increasing (if f is decreasing, apply this to $-f$).

If $I = [a, b]$ then we have seen above that f defines a bijection from $[a, b]$ to $[f(a), f(b)]$. Since f is continuous, $f(a) = \lim_{x\to a} f(x)$ and $f(b) = \lim_{x\to b} f(x)$ so we have the result in this case.

If $I = [a, b[$ where a is a real number and b is either a real number or $+\infty$, then again we have $f(a) = \lim_{x\to a} f(x)$.

➢ First case: f is bounded above. Since f is increasing, the limit $\ell = \lim_{x\to b} f(x)$ exists and we know that $\ell = \sup f(I)$. For all $x \in [a, b[$ we have $f(x) \in f(I)$ so $f(a) \leqslant f(x) \leqslant \ell$. We will show that $f(I) = [f(a), \ell[$.

- We first prove that $f(I) \subset [f(a), \ell[$, that is, for all $x \in I$ we have $f(x) < \ell$.

  Let us assume for a contradiction that there exists $u \in [a, b[$ such that $f(u) = \ell$. For every $x \in [a, b[$ we then have $f(x) \leqslant f(u)$ and hence $x \leqslant u$ because f is increasing. Since $u < b$, there exist elements $x \in [a, b[$ such that $u < x < b$. But then $\ell = f(u) < f(x) \leqslant \ell$, a contradiction. Therefore $f(x) < \ell$ for all $x \in [a, b[$, so that $f(I) \subset [f(a), \ell[$.

- We now prove the other inclusion: take $y \in [f(a), \ell[$. We must prove that $y \in f(I)$.

  We have $y < \ell$ and $\ell = \sup f(I)$ so that there exists $y' \in f(I)$ such that $y < y' < \ell$. Since $y' \in f(I)$, there exists $x' \in I$ such that $f(x') = y'$. Consider the interval $[a, x'] \subset I$. Since f is continuous and increasing, f defines a bijection from $[a, x']$ to $[f(a), f(x')] = [f(a), y']$. But $y \in [f(a), y']$ therefore there exists $x \in [a, x']$ such that $y = f(x)$. Since $x \in I$, we have $y \in f(I)$. Therefore we have proved that $[f(a), \ell[\subset f(I)$ and we finally have the equality.

➢ Second case: f is not bounded above. Since f is increasing, we know that $\lim_{x\to b} f(x) = +\infty$. Moreover, for every $x \in I$ we have $f(x) \geqslant f(a)$. Therefore $f(I) \subset [f(a), +\infty[$.

We now prove the other inclusion. Take $y \geqslant f(a)$. Since $\lim_{x\to b} f(x) = +\infty$, there exists $x' \in I$ such that $f(x') > y$. As before, f defines a bijection from $[a, x']$ to $[f(a), f(x')]$ so that there exists $x \in [a, x']$ such that $y = f(x)$. But $x \in I$ so $y \in f(I)$. Therefore $[f(a), +\infty[= f(I)$.

The case $I =]a, b]$ is similar, and the final case can be deduced from these ($]a, b[=]a, c] \cup [c, b[$ for any $c \in]a, b[$ and $f(]a, b[) = f(]a, c]) \cup f([c, b[) =]\lim_{x\to a} f(x), f(c)] \cup [f(c), \lim_{x\to b} f(x)[=]\lim_{x\to a} f(x), \lim_{x\to b} f(x)[$). ✓

Example. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{1+x^2}$ is continuous therefore $f(\mathbb{R})$ is an interval. The function is even so $f(\mathbb{R}) = f([0; +\infty[)$. On the interval $[0; +\infty[$, the function f is decreasing, therefore $f(\mathbb{R}) = f([0; +\infty[) =]\lim_{x\to+\infty} f(x); f(0)] =]0; 1]$.

C. Graph of a function and of its inverse

Let I and J be intervals and $f : I \to J$ a function. Recall that the **graph**[†] of f is the subset G of $\mathbb{R}^2$ defined by $G = \{(x, f(x)) \in \mathbb{R}^2; x \in I\}$. If f is bijective, then the graph of f^{-1} is the set $G' = \{(y, f^{-1}(y)); y \in J\}$. Then clearly we have

$$(x, y) \in G \iff (y, x) \in G'.$$

Consequently, the graph of f^{-1} is obtained from the graph of f by applying the symmetry with respect to the line with equation $y = x$.

D. n-th root

Let $n \geqslant 2$ be an integer. The function $x \mapsto x^n$ is continuous and increasing on $[0, +\infty[$. Its value at 0 is 0 and we have $\lim_{x\to+\infty} x^n = +\infty$. Therefore $x \mapsto x^n$ defines a bijection from $[0, +\infty[$ to $[0, +\infty[$.

If n is odd, the function $x \mapsto x^n$ is continuous and increasing on $\mathbb{R}$ and $\lim_{x\to-\infty} x^n = -\infty$. In this case, $x \mapsto x^n$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

In both cases, the inverse function is continuous and increasing.

Definition 13. *The bijection inverse to $x \mapsto x^n$ above is called the **n-th root**[a] function and is denoted by $x \mapsto \sqrt[n]{x}$. If $n = 2$ it is simply the **square root**[b] denoted by $x \mapsto \sqrt{x}$.*

[a] racine n-ième
[b] racine carrée

[†] graphe

Remark. The n-th root function is defined on $[0,+\infty[$ if n is even and on $\mathbb{R}$ if n is odd. It is continuous and increasing.

- For $x \geqslant 0$, $y \geqslant 0$, we have $(y = x^n \iff x = \sqrt[n]{y})$.
- If $x \in [0,1]$ then $x^n \leqslant x$ so $x \leqslant \sqrt[n]{x}$.
- If $x \geqslant 1$ then $x \leqslant x^n$ so $\sqrt[n]{x} \leqslant x$.
- If n is odd then the function $x \mapsto x^n$ is odd and so is $x \mapsto \sqrt[n]{x}$.

E. Inverse of a differentiable function

Proposition 14 (Derivative of the inverse of a function)**.** Let I be an open interval and let $f\colon I \to \mathbb{R}$ be a function that is differentiable[a] and strongly monotonous on I. Set $J = f(I)$ and let $f^{-1}\colon J \to I$ the inverse of the bijection $I \to J$ defined by f. If $f'(t) \neq 0$ for all $t \in I$, then f^{-1} is differentiable on J and we have

$$(f^{-1})'(y) = \frac{1}{f' \circ f^{-1}(y)} \qquad \text{for all } y \in J.$$

[a] dérivable

Proof. Set $y_0 = f(x_0)$. Define $g\colon I \to \mathbb{R}$ by $g(x) = \frac{f(x)-f(x_0)}{x-x_0}$ if $x \neq x_0$ and $g(x_0) = f'(x_0)$, so that $f(x) = f(x_0) + (x - x_0)g(x)$ for all $x \in I$ and $\lim_{x\to x_0} g(x) = f'(x_0) \neq 0$. Then for $y \in J$ we have

$$y = f(f^{-1}(y)) = y_0 + \Big(f^{-1}(y) - f^{-1}(y_0)\Big)g(f^{-1}(y)).$$

Since f^{-1} is continuous, we have $\lim_{y\to y_0} g(f^{-1}(y)) = f'(x_0) \neq 0$, so there is an interval J' contained in J and containing y_0 such that for all $y \in J'$ except $y = y_0$ we have $g(f^{-1}(y)) \neq 0$. For all $y \in J'$, $y \neq y_0$, we have

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{g(f^{-1}(y))} \underset{y\to y_0}{\longrightarrow} \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

Therefore f^{-1} is differentiable at y_0 and $\left(f^{-1}\right)'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$. ✓

Example. If $f :]0,+\infty[\to]0;+\infty[$ is defined by $f(x) = x^n$ for some integer $n \geqslant 2$, then f is a bijection with inverse $g = \sqrt[n]{\ }$. Moreover, f is differentiable and $f'(x) = nx^{n-1} \neq 0$ for all $x \in]0;+\infty[$. Therefore $g = \sqrt[n]{\ }$ is differentiable and

$$g'(x) = \frac{1}{n\left(\sqrt[n]{x}\right)^{n-1}} = \frac{1}{n}x^{1/n-1}.$$

I. Rolle's Theorem and the Mean Value Theorem

Theorem 15 (Rolle's Theorem[a])**.** Let a, b be real numbers with $a < b$, and let $f\colon [a,b] \to \mathbb{R}$ be a function. Assume that f is continuous on $[a,b]$ and differentiable on $]a,b[$ and that $f(a) = f(b)$. Then there exists a real number $c \in]a,b[$ such that $f'(c) = 0$.

[a] théorème de Rolle

Proof. Since f is continuous on the closed bounded interval $[a,b]$, f has a maximum and a minimum (absolute) on $[a,b]$. If one of them occurs at a point $c \in]a,b[$ then by Theorem 279 of the first semester, we have $f'(c) = 0$. Otherwise, both the maximum and the minimum occur at the endpoints. Since $f(a) = f(b)$, the function is constant on $[a,b]$ and we can choose any $c \in]a,b[$. ✓

Example. Let a_0, a_1, a_2, a_3 be real numbers such that $\frac{1}{4}a_3 + \frac{1}{3}a_2 + \frac{1}{2}a_1 + a_0 = 0$. We want to prove that the polynomial function $P : x \mapsto a_3x^3 + a_2x^2 + a_1x + a_0$ has at least one real root in $]0,1[$.

Consider the polynomial function $Q : x \mapsto \frac{1}{4}a_3x^4 + \frac{1}{3}a_2x^3 + \frac{1}{2}a_1x^2 + a_0x$. Then $Q(0) = 0$ and $Q(1) = 0$ by assumption. Since Q is differentiable on $\mathbb{R}$, it is continuous on $[0,1]$ and differentiable on $]0,1[$ so that Rolle's Theorem applies: there exists $c \in]0,1[$ such that $Q'(c) = 0$. We have $P = Q'$, therefore P has a real root in $]0,1[$.

Theorem 16 (Mean Value Theorem[a])**.** Let a, b be real numbers with $a < b$, and let $f\colon [a,b] \to \mathbb{R}$ be a function. Assume that f is continuous on $[a,b]$ and differentiable on $]a,b[$. Then there exists a real number $c \in]a,b[$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

[a] théorème des accroissements finis

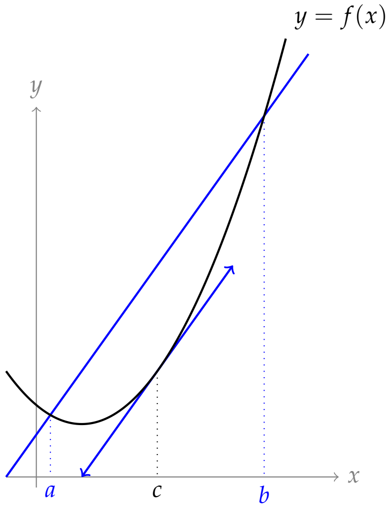
Proof. Define $h\colon [a,b] \to \mathbb{R}$ by $h(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)$. Clearly, h is continuous on $[a,b]$ and differentiable on $]a,b[$, and $h(a) = f(a)$ and $h(b) = f(a)$ so that $h(a) = h(b)$. We may therefore apply Rolle's theorem to h: there exists $c \in]a,b[$ such that $h'(c) = 0$. But $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$ so $f'(c) = \frac{f(b)-f(a)}{b-a}$ as required. ✓

Remark.

- ➢ **Kinematic interpretation.** If the variable x is time and if $f(x)$ is the position of a particle in a straight-line motion[†], you have seen in the first semester that $\dfrac{f(b)-f(a)}{b-a}$ is the average velocity of the particle between the instants a and b of time, and that $f'(c)$ is the velocity of the particle at the instant c of time.

  The Mean Value Theorem states that there is an instant c of time at which the velocity[‡] of the particle is equal to its average[§] velocity between the instants a and b of time.

- ➢ **Geometric interpretation.** Let C_f be the graph of f. The Mean Value Theorem states that there is a point c between a and b at which the tangent line is parallel to the line through $(a, f(a))$ and $(b, f(b))$.



  If the chord[¶] between $(a, f(a))$ and $(b, f(b))$ is horizontal, then Rolle's theorem says that there is a point $(c, f(c))$ in between at which the tangent is horizontal.

Corollary 17. If f is defined and differentiable on an *interval* I and $f'(x) = 0$ for all x in I, then f is constant on I.

Proof. Let a, b be two elements of I with $a \neq b$; we may assume for instance that $a < b$. Since I is an interval, $[a,b] \subset I$. Then there is some $c \in]a,b[$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ by the Mean Value Theorem. Moreover by assumption $f'(c) = 0$, therefore $f(a) = f(b)$. This is true for any two elements of I, hence f is constant on I. ✓

Corollary 18. Let I be an *interval* and let $f\colon I \to \mathbb{R}$ be a differentiable function. If $f'(x) > 0$ (*resp.* $f'(x) \geqslant 0$, *resp.* $f'(x) < 0$, *resp.* $f'(x) \leqslant 0$) for all $x \in I$ then f is increasing (*resp.* non-decreasing, *resp.* decreasing, *resp.* non-increasing) on I.

Proof. We will prove the case where $f'(x) > 0$ for all $x \in I$. Let a, b be two elements of I with $a < b$. Since I is an interval, $[a,b] \subset I$. Then there is some $c \in]a,b[$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ by the Mean Value Theorem. Moreover by assumption $f'(c) > 0$, therefore $f(a) < f(b)$. This is true for any two elements of I, hence f is increasing on I. ✓

Definition 19. *Let $f : I \to \mathbb{R}$ be a function and let K be a non-negative real number. The function f is called* ***Lipschitz continuous with Lipschitz constant K***[a] *if*

$$\forall x \in I,\ \forall y \in I,\ |f(x) - f(y)| \leqslant K|x-y|.$$

The function f is called ***contracting***[b] *if there exists a constant $K < 1$ such that f is Lipschitz continuous with Lipschitz constant K.*

[a] K-lipschitzienne
[b] contractante

[†] trajectoire rectiligne
[‡] vitesse
[§] moyenne
[¶] corde

Proposition 20 (Mean Value Inequality[a]). Let I be an open interval and $f\colon I \to \mathbb{R}$ a differentiable function. Assume that there exists a real number $K > 0$ such that $|f'(t)| \leqslant K$ for all $t \in I$. Then f is Lipschitz continuous with Lipschitz constant K, that is,

$$\forall x \in I,\ \forall y \in I,\ |f(x) - f(y)| \leqslant K|x - y|.$$

[a]inégalité des accroissements finis

Proof. If $x = y$ the result is clear. Otherwise we may assume without loss of generality that $x < y$. Since f is differentiable on I, f is continuous on I hence on $[x, y]$ and differentiable on $]x, y[$. Therefore there exists $c \in]x, y[$ such that $f'(c) = \frac{f(x)-f(y)}{x-y}$. We get:

$$|f(x) - f(y)| = \left|f'(c)\right||x - y| \leqslant K|x - y|$$

as required. ✓

Theorem 21. Let $f\colon I \to \mathbb{R}$ be function defined on an interval I and let x_0 be an element of I. Suppose that f is continuous on I and differentiable on $I \setminus \{x_0\}$. Assume also that $\ell = \lim_{x\to x_0} f'(x)$ exists.
Then $\dfrac{f(x) - f(x_0)}{x - x_0}$ has limit ℓ when x goes to x_0.
In particular, if $\ell \in \mathbb{R}$, then f is differentiable at x_0 and $f'(x_0) = \lim_{x\to x_0} f'(x)$.

Proof.

- First case: $\ell = +\infty$. Fix $A \in \mathbb{R}$. There exists $\delta > 0$ such that $|x - x_0| \leqslant \delta \Rightarrow f'(x) \geqslant A$. We may choose δ small enough that f is differentiable on $]x_0 - \delta, x_0[\cup]x_0, x_0 + \delta[$. Moreover, f is continuous on $[x_0 - \delta, x_0 + \delta]$. For any $x \in]x_0, x_0 + \delta[$, by the Mean Value Theorem applied on the interval $[x_0, x]$, there exists $c_x \in]x_0, x[$ such that $f'(c_x) = \frac{f(x)-f(x_0)}{x-x_0}$ and therefore $\frac{f(x)-f(x_0)}{x-x_0} = f'(c_x) \geqslant A$. Finally we have proved:

  $$\forall x,\ 0 < x - x_0 \leqslant \delta \ \Rightarrow\ \frac{f(x) - f(x_0)}{x - x_0} \geqslant A.$$

  Similarly,

  $$\forall x,\ 0 < x_0 - x \leqslant \delta \ \Rightarrow\ \frac{f(x) - f(x_0)}{x - x_0} \geqslant A$$

  so that

  $$\forall x,\ 0 < |x - x_0| \leqslant \delta \ \Rightarrow\ \frac{f(x) - f(x_0)}{x - x_0} \geqslant A,$$

  that is, $\lim_{x\to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = +\infty$.

- The proof when $\ell = -\infty$ is similar – or replace f with $-f$.

- Last case: $\ell \in \mathbb{R}$. Fix $\varepsilon > 0$. There exists $\delta > 0$ such that $|x - x_0| \leqslant \delta \Rightarrow |f'(x) - \ell| \leqslant \varepsilon$. We may choose δ small enough so that $]x_0 - \delta; x_0 + \delta[\subset I$ and therefore f is differentiable on $]x_0 - \delta, x_0[\cup]x_0, x_0 + \delta[$. Moreover, f is continuous on $[x_0 - \delta, x_0 + \delta]$. For any $x \in]x_0, x_0 + \delta[$, by the Mean Value Theorem applied on the interval $[x_0, x_0 + \delta]$, there exists $c_x \in]x_0, x[$ such that $f'(c_x) = \frac{f(x)-f(x_0)}{x-x_0}$ and therefore $\left|\frac{f(x)-f(x_0)}{x-x_0} - \ell\right| = |f'(c_x) - \ell| \leqslant \varepsilon$. Finally we have proved:

  $$\forall x,\ 0 < x - x_0 \leqslant \delta \ \Rightarrow\ \left|\frac{f(x) - f(x_0)}{x - x_0} - \ell\right| \leqslant \varepsilon.$$

  Similarly,

  $$\forall x,\ 0 < x_0 - x \leqslant \delta \ \Rightarrow\ \left|\frac{f(x) - f(x_0)}{x - x_0} - \ell\right| \leqslant \varepsilon$$

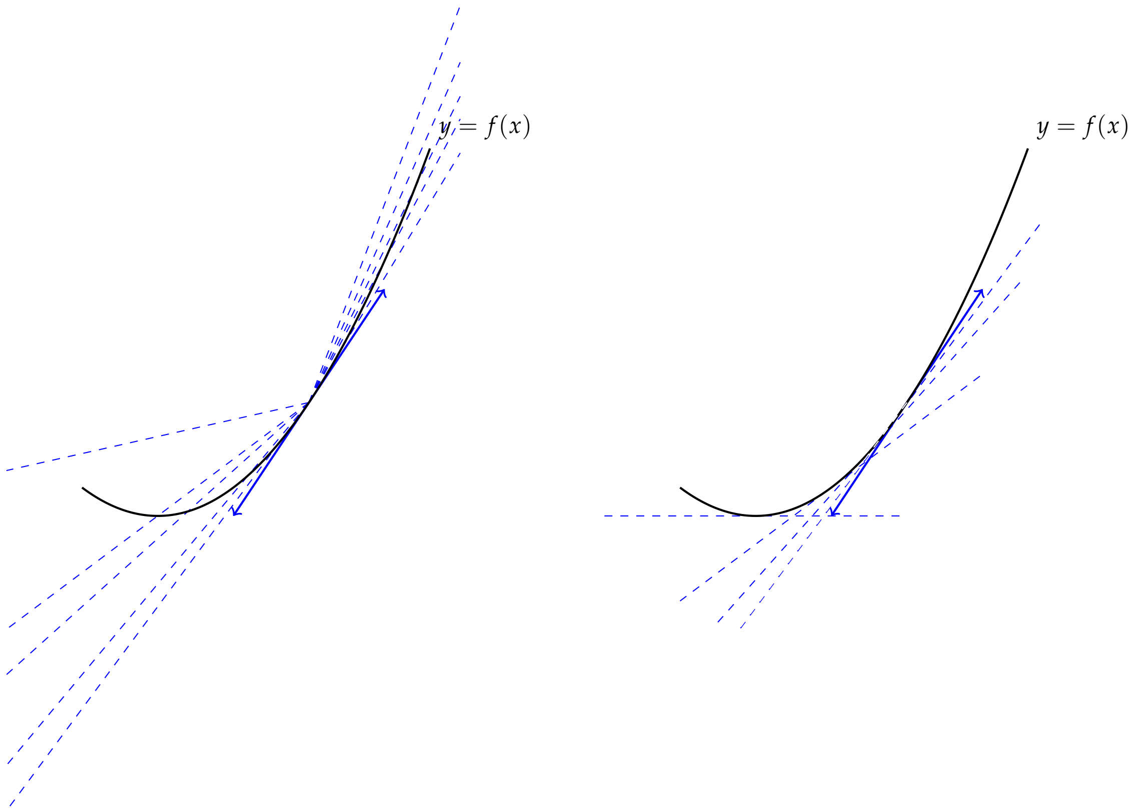
  so that

  $$\forall x,\ 0 < |x - x_0| \leqslant \delta \ \Rightarrow\ \left|\frac{f(x) - f(x_0)}{x - x_0} - \ell\right| \leqslant \varepsilon,$$

  that is, f is differentiable at x_0 and $f'(x_0) = \ell$. ✓

Remark. Geometrically, the derivative $f'(x_0)$ is the slope[†] of the tangent line below. The first picture on the left shows the limit of the chords through the point $(x_0, f(x_0))$. The second picture on the right shows the limit of the derivatives (slopes of tangent lines) from below at $(x_0, f(x_0))$.

[†]coefficient directeur (pente)



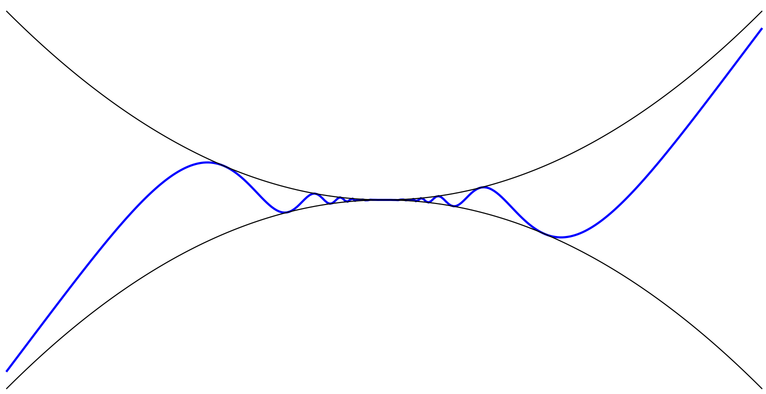
Remark. We can have a differentiable function f on $]a, b[$, $x_0 \in]a, b[$ and $\lim_{x \to x_0} f'(x) \neq f'(x_0)$ (that is, f' need not be continuous at x_0).

Let $f \colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

- The function f is differentiable on $]-\infty, 0[$ and on $]0, +\infty[$, and for every $x \neq 0$ we have $f'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2\left(-\frac{1}{x^2}\right)\cos\left(\frac{1}{x}\right) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$.
- For every $x \neq 0$ we have $\left|\sin\left(\frac{1}{x}\right)\right| \leqslant 1$ therefore $|f(x)| \leqslant x^2$. Since $\lim_{x \to 0} x^2 = 0$ we have $\lim_{x \to 0} f(x) = 0 = f(0)$. The function f is therefore continuous at 0.
- Similarly, if $x \neq 0$ we have $\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = x \sin\left(\frac{1}{x}\right)$ and $\left|x\sin\left(\frac{1}{x}\right)\right| \leqslant |x|$ so that $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = 0$. Therefore f is differentiable at 0 and $f'(0) = 0$.
- However we have $f'\left(\frac{1}{2n\pi}\right) = -1$ and $f'\left(\frac{1}{(2n+1)\pi}\right) = 1$ so by Proposition 234 in the first semester, f' does not have a limit at 0.

Below is the graph of this function, enclosed between the graphs of $x \mapsto x^2$ and $x \mapsto -x^2$. Although the slopes of the tangent lines will oscillate (faster and faster) between -1 and 1 as x goes to 0, the slopes of the chords between $(0, 0)$ and $(x, f(x))$ oscillate around 0 but within a range that reduces as x goes to 0.



Let I be an interval and $f\colon I \to \mathbb{R}$ a differentiable function. Therefore there is a function $f'\colon I \to \mathbb{R}$. If this function is in turn differentiable, we set $f'' = (f')'$; it is called the **second derivative**[†]. More generally, we define the **higher-order derivatives**[‡] recursively: $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(p+1)} = (f^{(p)})'$ for any $p \in \mathbb{N}$.

Definition 22. *Let $f\colon I \to \mathbb{R}$ be a function defined on an interval I.*
We say that f is ***of class $\mathcal{C}^k$***[a] *if all the derivatives $f', f'', \dots, f^{(k)}$ exist and are continuous.*
The function f is ***of class $\mathcal{C}^\infty$ or smooth***[b] *if it has derivatives of all orders (in this case, all the derivatives are continuous).*

[a] de classe $\mathcal{C}^k$
[b] de classe $\mathcal{C}^\infty$

Remark. Let f be a function which has derivatives up to order k such that $f^{(k)}$ is continuous. Then f is of class $\mathcal{C}^k$.
Indeed, for $1 \leqslant i < k$, the function $f^{(i)}$ is differentiable (since $f^{(i+1)}$ exists by assumption) and therefore continuous. Moreover, $f^{(k)}$ is continuous by assumption. Therefore all the derivatives up to order k are continuous and f is of class $\mathcal{C}^k$.

Remark. The functions of class $\mathcal{C}^0$ are precisely the continuous functions.

Proposition 23. Linear combinations, products, quotients and compositions (when defined) of functions of class $\mathcal{C}^k$ are of class $\mathcal{C}^k$.
If $f : I \to \mathbb{R}$ is of class $\mathcal{C}^k$ and strongly monotonic, and if f' does not vanish on I, then f^{-1} is of class $\mathcal{C}^k$ on $f(I)$.

Idea/summary of proof. All the proofs are by induction on $k \geqslant 1$, applying the induction hypothesis to the first derivative.

Proof. Let f and g be functions of class $\mathcal{C}^k$ on an interval I.

➢ If $\lambda$and μ are real numbers, let h be the function $h := \lambda f + \mu g$. Then h is of class $\mathcal{C}^k$: this is proved by induction on k (exercise).

➢ We know that if f and g are differentiable on I, then fg is differentiable on I and that $(fg)' = f'g + fg'$. We prove by induction that is f and g are of class $\mathcal{C}^k$ then so is fg.

- For $k = 1$, we have just said that fg is differentiable. Moreover, the function $(fg)' = f'g + fg'$ is continuous, therefore fg is of class $\mathcal{C}^1$.
- Inductively, assume that if f and g are of class $\mathcal{C}^k$ with $k \geqslant 1$ then so is fg.

  Let f and g be functions of class $\mathcal{C}^{k+1}$. Then fg is of class $\mathcal{C}^1$ with $(fg)' = f'g + fg'$. Moreover, f' and g are both of class $\mathcal{C}^k$ therefore by induction hypothesis $f'g$ is of class $\mathcal{C}^k$, and fg' is of class $\mathcal{C}^k$ similarly. Since a sum of functions of class $\mathcal{C}^k$ is also of class $\mathcal{C}^k$, it follows that $(fg)'$ is of class $\mathcal{C}^k$ and therefore fg is of class $\mathcal{C}^{k+1}$ as required.

➢ If g does not vanish on I, we know that $\frac{f}{g}$ is differentiable on I and $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

- For $k = 1$, the function $\left(\frac{f}{g}\right)'$ above is continuous, therefore $\frac{f}{g}$ is of class $\mathcal{C}^1$.
- Inductively, assume that if f and g are of class $\mathcal{C}^k$ with $k \geqslant 1$ then so is $\frac{f}{g}$.

  Let f and g be functions of class $\mathcal{C}^{k+1}$. Then $\frac{f}{g}$ is of class $\mathcal{C}^1$ with $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$. The latter is a quotient of functions of class $\mathcal{C}^k$, so that by the induction hypothesis, it is of class $\mathcal{C}^k$. Therefore $\frac{f}{g}$ is of class $\mathcal{C}^{k+1}$.

➢
- For $k = 1$, we know that if f and g are of class $\mathcal{C}^1$ then $g \circ f$ is differentiable and $(g \circ f)' = (g' \circ f)f'$ is continuous, hence $g \circ f$ is of class $\mathcal{C}^1$.
- Inductively, assume that if f and g are of class $\mathcal{C}^k$ then $(g \circ f)$ is of class $\mathcal{C}^k$.

  Let f and g be functions of class $\mathcal{C}^{k+1}$. Then $g \circ f$ is of class $\mathcal{C}^1$ and $(g \circ f)' = (g' \circ f)f'$. Moreover, $g' \circ f$, which is a composition of functions of class $\mathcal{C}^k$, is of class $\mathcal{C}^k$ by the induction hypothesis, and f' is of class $\mathcal{C}^k$, therefore the product $(g' \circ f)f'$ is also of class $\mathcal{C}^k$. Finally, $(g \circ f)'$ is of class $\mathcal{C}^k$ and $g \circ f$ is of class $\mathcal{C}^{k+1}$.

➢
- For $k = 1$, we know that since f is differentiable and strongly monotonic, then it defines a bijection from I to $J := f(I)$, and since moreover f' does not vanish on I the function $g := f^{-1} : J \to I$ is differentiable with differential $g' = \frac{1}{f' \circ g}$. This function is continuous, therefore g is of class $\mathcal{C}^1$.

[†] dérivée seconde
[‡] dérivées d'ordres supérieurs

- Inductively, assume that if f is of class $\mathcal{C}^k$, strongly monotonic and f' does not vanish on I, then g is of class $\mathcal{C}^k$.

  Let f be of class $\mathcal{C}^{k+1}$ and strongly monotonic and assume f' does not vanish on I. Then $g := f^{-1} : J \to I$ is differentiable with differential $g' = \frac{1}{f' \circ g}$. This function is of class $\mathcal{C}^k$ (g is of class $\mathcal{C}^k$ by induction hypothesis, and we have already seen that the composition and the quotient of functions of class $\mathcal{C}^k$ is of class $\mathcal{C}^k$), therefore g is of class $\mathcal{C}^{k+1}$. ✓

Example. A polynomial is of class $\mathcal{C}^\infty$.

Proposition 24 (Leibniz's Formula[a]). If f and g are two functions that are n times differentiable at x_0, then fg is also n times differentiable at x_0 and

$$(fg)^{(n)}(x_0) = \sum_{p=0}^{n} \binom{n}{p} f^{(p)}(x_0) g^{(n-p)}(x_0).$$

[a]formule de Leibniz

Proof. The proof is by induction on $n \geqslant 1$. For $n = 1$ the result is already known.

Now assume that the product of two functions which are n times differentiable is n times differentiable for some fixed $n \geqslant 1$. Let f and g be two functions which are $n+1$ times differentiable. They are in particular differentiable (once) and $(fg)' = f'g + fg'$ is a sum of products of functions which are (at least) n times differentiable, hence $(fg)'$ is n times differentiable and fg is $n+1$ times differentiable. Moreover,

$$\begin{aligned}
(fg)^{(n+1)}(x_0) &= \sum_{p=0}^{n} \binom{n}{p} (f^{(p)} g^{(n-p)})'(x_0) \\
&= \sum_{p=0}^{n} \binom{n}{p} f^{(p+1)}(x_0) g^{(n-p)}(x_0) + \sum_{p=0}^{n} \binom{n}{p} f^{(p)}(x_0) g^{(n-p+1)}(x_0) \\
&= \sum_{p=1}^{n+1} \binom{n}{p-1} f^{(p)}(x_0) g^{(n-p+1)}(x_0) + \sum_{p=0}^{n} \binom{n}{p} f^{(p)}(x_0) g^{(n-p+1)}(x_0) \\
&= \binom{n}{n} f^{(n+1)}(x_0) g^{(0)}(x_0) + \sum_{p=1}^{n} \left[\binom{n}{p-1} + \binom{n}{p} \right] f^{(p)}(x_0) g^{(n-p+1)}(x_0) \\
&\quad + \binom{n}{0} f^{(0)}(x_0) g^{(n+1)}(x_0) \\
&= f^{(n+1)}(x_0) g(x_0) + \sum_{p=1}^{n} \binom{n+1}{p} f^{(p)}(x_0) g^{(n+1-p)}(x_0) + f(x_0) g^{(n+1)}(x_0) \\
&= \sum_{p=0}^{n+1} \binom{n+1}{p} f^{(p)}(x_0) g^{(n+1-p)}(x_0).
\end{aligned}$$

The formula is true at rank $n+1$, therefore by induction it is true for all $n \geqslant 1$. ✓

Theorem 25. [a] Let $f \colon I \to \mathbb{R}$ be a function defined on an interval I and let x_0 be an element of I. Suppose that f is continuous on I and of class $\mathcal{C}^k$ on $I \setminus \{x_0\}$. Assume also that for all i with $1 \leqslant i \leqslant k$, the limit $\ell_i = \lim_{x \to x_0} f^{(i)}(x)$ exists and is finite.
Then f is of class $\mathcal{C}^k$ on I and $f^{(i)}(x_0) = \ell_i$ for all i with $1 \leqslant i \leqslant k$.

[a]Théorème de classe $\mathcal{C}^k$ par prolongement.

Proof. We prove it by induction on k.

➢ If $k = 1$, we know by Theorem 21 that f is differentiable at x_0 and that $f'(x_0) = \ell_1 = \lim_{x \to x_0} f'(x)$ hence f' is continuous. Therefore f is of class $\mathcal{C}^1$.

➢ Fix i with $1 \leqslant i < k$ and assume that f is of class $\mathcal{C}^i$ with $f^{(j)}(x_0) = \ell_j$ for all j with $1 \leqslant j \leqslant i$.

The function $f^{(i)}$ satisfies the assumptions of Theorem 21, therefore it is differentiable at x_0 and $f^{(i+1)}(x_0) = \left(f^{(i)}\right)'(x_0) = \ell_{i+1} = \lim_{x \to x_0} f^{(i+1)}(x)$ so that $f^{(i+1)}$ is continuous on I. Finally, f is of class $\mathcal{C}^{i+1}$. ✓

Now for functions of class $\mathcal{C}^k$, we have a generalisation of the Mean Value Inequality.

Theorem 26 (Taylor's inequality)**.** Let $f\colon I \to \mathbb{R}$ be a function of class $\mathcal{C}^{n+1}$ on an interval I. Suppose that a and b are elements in I. If $\left|f^{(n+1)}(t)\right| \leqslant M$ for all t between a and b, then

$$\left|f(b) - \left(f(a) + \frac{b-a}{1!}f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^n}{n!}f^{(n)}(a)\right)\right| \leqslant M\frac{|b-a|^{n+1}}{(n+1)!}.$$

This inequality is called **Taylor's inequality**[a] at a of order n.

[a] inégalité de Taylor-Lagrange

Proof. See Chapter 9. ✓

III. Complex valued functions

Definition 27. *Let $f : I \to \mathbb{C}$ be a complex valued function. We can write $f = f_1 + if_2$ where f_1 and f_2 are functions from I to $\mathbb{R}$. We say that f is* ***continuous*** *(resp.* ***differentiable****, resp.* ***of class $\mathcal{C}^k$****) if f_1 and f_2 are both continuous (resp. differentiable, resp. of class $\mathcal{C}^k$).*
If moreover f is of class $\mathcal{C}^k$ with $k \geqslant 1$, we define the k-th derivative of f to be $f_1^{(k)} + if_2^{(k)}$.

A number of results we have seen still hold for complex valued functions, but not all.

- Linear combinations, products, quotients of functions of class $\mathcal{C}^k$, $k \geqslant 0$, (*resp.* differentiable) are of class $\mathcal{C}^k$ (*resp.* differentiable).
- If f is a real valued function of class $\mathcal{C}^k$ (*resp.* differentiable) and g a complex valued function of class $\mathcal{C}^k$ (*resp.* differentiable), then $g \circ f$ is of class $\mathcal{C}^k$ (*resp.* differentiable).

  Warning! We do *not* differentiate a function whose variable is complex (or an integer...).
- A differentiable function f is constant on an interval I if and only if $f' = 0$.

  Warning! The derivative no longer has a "sign", and it does not make sense to say that a complex valued function is increasing...
- Rolle's Theorem and the Mean Value Theorem do *not* hold for complex valued functions.

  For instance, the function $f : \mathbb{R} \to \mathbb{C}$ defined by $f(x) = e^{ix}$ satisfies $f(0) = f(2\pi)$, but its derivative, which is $f'(x) = ie^{ix}$, never vanishes.

  However, the Mean Value Inequality (using modulus instead of absolute value where appropriate) does still hold.

Theorem 28 (Mean Value Inequality[a])**.** Let I be an open interval and $f\colon I \to \mathbb{C}$ be a complex valued function of class $\mathcal{C}^1$. Assume that there exists a real number $K > 0$ such that $|f'(t)| \leqslant K$ for all $t \in I$. Then f is Lipschitz continuous with Lipschitz constant K, that is,

$$\forall x \in I,\ \forall y \in I,\ |f(x) - f(y)| \leqslant K|x-y|.$$

[a] inégalité des accroissements finis

Proof. This result will be proved in Chapter 8 (and also in Chapter 11). ✓

Chapter 3

Classical functions and their inverses

I. Logarithm, powers and exponential

A. Logarithm

Later on we shall be able to use integration theory to prove that there is a unique differentiable function $\ln\colon]0\,;+\infty[\, \to \mathbb{R}$ such that $\ln'(x) = \frac{1}{x}$ and $\ln 1 = 0$. This function is called the **logarithm**[†]. Its main properties are as follows.

Properties 1. (i) $\ln(ab) = \ln a + \ln b$ for any positive real numbers a and b. Moreover, $\ln(a^n) = n \ln a$ for any positive real number a and any integer $n \in \mathbb{Z}$.

(ii) The logarithm is an increasing continuous bijection from $]0\,;+\infty[$ to $\mathbb{R}$; we have

$$\lim_{x\to 0} \ln x = -\infty \quad \text{and} \quad \lim_{x\to+\infty} \ln x = +\infty.$$

(iii) $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1.$

(iv) The function $x \mapsto \ln(x+1)$ has a Taylor expansion[a] of order $n \in \mathbb{N}$ at 0 given by

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + o(x^n) = \sum_{k=0}^{n} (-1)^{k-1}\frac{x^k}{k} + o(x^n).$$

[a]développement limité (DL)

Proof. (i) Let a be a positive real number. Define $f\colon]0\,;+\infty[\, \to \mathbb{R}$ by $f(x) = \ln(ax)$. Then $f'(x) = a\frac{1}{ax} = \frac{1}{x} = \ln'(x)$, so that the derivative of $f - \ln$ vanishes on the interval $]0\,;+\infty[$ and therefore the function $f - \ln$ is constant, there exists $k \in \mathbb{R}$ such that $f(x) = k + \ln x$ for all $x > 0$. We then have $f(1) = \ln a$ by definition and $f(1) = k + \ln 1 = k$, so that $k = \ln a$. Therefore for any $x > 0$ we have $\ln(ax) = \ln a + \ln x$. Note that in particular, $\ln a + \ln\left(\frac{1}{a}\right) = \ln\left(a\frac{1}{a}\right) = \ln 1 = 0$ so that $\ln\left(\frac{1}{a}\right) = -\ln a$.

The second formula is proved by induction on n for $n \in \mathbb{N}^*$. For $n = 0$, the formula is simply $\ln 1 = 0$. Finally, for $n < 0$, since $-n > 0$ we have $\ln(a^n) = -\ln(a^{-n}) = -(-n)\ln a = n \ln a$.

(ii) Since the logarithm is differentiable, it is continuous. Moreover, the derivative is positive so that $\ln$ is increasing. In particular, $\ln 2 > \ln 1 = 0$ so that the sequence $(n \ln 2)_n$ has limit $+\infty$. Since $n \ln 2 = \ln(2^n)$, the function $\ln$ is not bounded above and we then know that $\lim_{x\to+\infty} \ln x = +\infty$. We also have $\lim_{x\to 0} \ln x = \lim_{t\to+\infty} \ln\left(\frac{1}{t}\right) = \lim_{t\to+\infty} -\ln t = -\infty$. Therefore $\ln$ is a bijection from $]0\,;+\infty[$ to $\mathbb{R}$.

(iii) By definition, $\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\ln(1+x)-\ln(1)}{x-0}$ is the derivative of the function $\ln$ at 1, therefore it is equal to 1.

(iv) You have seen in the first semester that we can take the primitive of a Taylor expansion (Proposition 312). Let $f :]-1,+\infty[\to \mathbb{R}$ be the function defined by $f(x) = \ln(x+1)$. Then f is differentiable and $f'(x) = \frac{1}{1+x}$.

Moreover, you have seen in the first semester that $\frac{1}{1+x} = \sum_{k=0}^{n-1}(-1)^k x^k + o(x^{n-1})$. Since $f(0) = 0$, the result follows. ✓

[†]logarithme

B. Exponential

We have just seen that the logarithm is a bijection from $]0\,;+\infty[$ to $\mathbb{R}$, therefore it has an inverse, the **exponential**† function $\exp\colon \mathbb{R} \to]0\,;+\infty[$. Therefore

$$\begin{cases} \exp(\ln x) = x & \text{for all } x > 0 \\ \ln(\exp x) = x & \text{for all } x \in \mathbb{R}. \end{cases}$$

The properties of the exponential can be deduced from those of the logarithm.

Properties 2. (i) $\exp(a+b) = (\exp a)(\exp b)$ for any real numbers a and b, and $\exp(na) = (\exp a)^n$ for any real number a and any integer $n \in \mathbb{Z}$.

(ii) The exponential defines an increasing and continuous bijection from $\mathbb{R}$ to $]0\,;+\infty[$; we have $\lim\limits_{x\to-\infty} \exp x = 0$ and $\lim\limits_{x\to+\infty} \exp x = +\infty$.

(iii) The exponential function is differentiable and $\exp' = \exp$.

(iv) The function exp has a Taylor expansion of order $n \in \mathbb{N}$ at 0 given by

$$\exp x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n).$$

Proof. (i) Set $x = \exp a$ and $y = \exp b$. Then we know that $\ln(xy) = \ln x + \ln y$ hence $\ln((\exp a)(\exp b)) = \ln(\exp a) + \ln(\exp b) = a + b$ and, applying exp, we get $(\exp a)(\exp b) = \exp(a+b)$.

(ii) By construction, exp is the inverse function of the continuous and increasing function $\ln\colon]0;+\infty] \to \mathbb{R}$, which is bijective by the properties of the logarithm. Therefore it is a continuous and increasing function from $\mathbb{R}$ to $]0;+\infty]$.

(iii) Moreover, since the derivative of ln never vanishes, by Proposition 14 in Chapter 2, the function exp is differentiable and we have

$$\exp'(x) = \frac{1}{\ln'(\exp x)} = \frac{1}{\frac{1}{\exp x}} = \exp x.$$

(iv) *[cf. First semester.]* It is easy to use Taylor-Young's formula here, since $\exp^{(n)} x = \exp x$ for all $n \geqslant 0$ (by induction), therefore $\exp^{(n)}(0) = 1$ for all $n \in \mathbb{N}$ and finally

$$\exp x = \sum_{k=0}^{n} x^k \frac{\exp^{(k)}(0)}{k!} + o(x^n) = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n). \qquad \checkmark$$

Notation. The real number $\exp 1$ is denoted by e; therefore we have $\ln e = 1$. Note that since exp is strictly increasing, $e = \exp 1 > \exp 0 = 1$.

Remark. For any integer $n \in \mathbb{Z}$, we have $\exp(n \ln a) = (\exp(\ln a))^n = a^n$.

Suppose that $n \geqslant 2$ is an integer and set $y = \exp\left(\frac{1}{n}\ln a\right)$. Then $y^n = a$; since $y > 0$ we conclude that $y = \sqrt[n]{a}$ by definition of the n-th root. Therefore

$$\sqrt[n]{a} = \exp\left(\frac{1}{n}\ln a\right) \quad \text{for all } a > 0.$$

Definition 3. *Let a be a positive real number and let b be a real number. We define the real number a^b, called **a to the b or a to the power of b**[a] by*

$$a^b = \exp(b \ln a).$$

[a] *a puissance b*

Remark. If b is a positive integer, then $\exp(b\ln a) = \overbrace{a\cdots a}^{b \text{ terms}}$ which is the usual power a to the b. If $n \geqslant 2$ is an integer, we also have $a^{1/n} = \sqrt[n]{a}$. The definition above allows us to elevate any positive real number to the power of any real number.

Proposition 4.
- ➢ For any real number b we have $1^b = 1$.
- ➢ For any real number b and any positive real number a, we have $\ln(a^b) = b\ln a$.
- ➢ For any positive real number x and any real numbers b and c we have $x^{(b+c)} = x^b x^c$ and $(x^b)^c = x^{bc}$.
- ➢ For any positive real numbers x and y and for any real number c we have $(xy)^c = x^c y^c$.

† exponentielle

Proof. Let x be a positive real number. Then

- $1^b = \exp(b \ln 1) = \exp 0 = 1$.
- $\ln(a^b) = \ln(\exp(b \ln a)) = b \ln a$.
- $x^{(b+c)} = \exp((b+c)\ln x) = \exp(b \ln x + c \ln x) = (\exp(b \ln x))(\exp(c \ln x)) = x^b x^c$.
- We have $\ln((x^b)^c) = c \ln(x^b) = cb \ln x$ so $(x^b)^c = \exp(\ln((x^b)^c)) = \exp(bc \ln x) = x^{bc}$.
- We have $(xy)^c = \exp(c \ln(xy)) = \exp(c \ln x + c \ln y) = \exp(c \ln x)\exp(c \ln y) = x^c y^c$. ✓

Proposition 5. Let a be any real number. Then

$$\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^x = \exp a.$$

In particular, the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)_n$ converges to e.

Proof.
- *Using Taylor expansions.*

Put $t = \dfrac{1}{x}$: when x approches $+\infty$, t nears 0. We then have

$$\left(1+\frac{a}{x}\right)^x = \exp\left(x\ln\left(1+\frac{a}{x}\right)\right) = \exp\left(\frac{1}{t}\ln(1+at)\right).$$

We shall now write a Taylor expansion of order 1 at $t=0$ of $t \mapsto \exp\left(\frac{1}{t}\ln(1+at)\right)$.

We have $\ln(1+at) = at - \dfrac{(at)^2}{2} + o(t^2)$ (we need order 2 here since we divide by t afterwards).

Next, $\dfrac{1}{t}\ln(1+at) = a - \dfrac{a^2t}{2} + o(t)$. Therefore $g(t) := \dfrac{1}{t}\ln(1+at) - a = -\dfrac{a^2t}{2} + o(t)$.

We will now compose with exp. We may do this because $\lim_{t\to 0} g(t) = 0$. Since $\exp u = 1 + u + o(u)$ we have $\exp(g(t)) = 1 - \frac{a^2}{2}t + o(t)$.

Finally, $\exp\left(\frac{1}{t}\ln(1+at)\right) = \exp(a+g(t)) = (\exp a)(\exp g(t)) = (\exp a)\left(1-\frac{a^2}{2}t+o(t)\right)$, whose limit is $\exp a$ when $t\to 0$.

- *More elementary proof.*

Put $u(x) = 1+\frac{a}{x}$. If $x > |a|$ then $\left|\frac{a}{x}\right| < 1$ so that $\frac{a}{x} > -1$ and $u(x) > 0$. Therefore we can consider $u(x)^x$ for x large enough (and in particular the limit as x approaches $+\infty$). If $a = 0$ then $u(x)^x = 1 = \exp 0$ for all x. Suppose that $a \neq 0$. Then $\ln(u(x)^x) = x\ln(u(x))$ and $\ln(u(x))$ has limit 0 when x approaches $+\infty$ so we have an indeterminate form.

$$\lim_{x\to+\infty} x\ln u(x) = \lim_{t\to 0^+}\frac{1}{t}\ln\left(u\left(\frac{1}{t}\right)\right) = \lim_{t\to 0^+}\frac{\ln(1+at)}{t} = a\lim_{t\to 0^+}\frac{\ln(1+at)}{at}.$$

Since $\lim_{x\to 0}\dfrac{\ln(1+x)}{x} = 1$, we get $\lim_{t\to 0^+}\dfrac{\ln(1+at)}{at} = 1$ so that $\lim_{x\to+\infty} x\ln u(x) = a$. It follows that $\lim_{x\to+\infty} u(x)^x = \exp a$ since $u(x)^x = \exp(x\ln u(x))$.

The second statement follows from the first one (with $a=1$). ✓

C. Power functions

Definition 6. *Let b be a real number. The function $u\colon]0;+\infty[\to \mathbb{R}$ defined by $u(x) = x^b = \exp(b\ln x)$ is called* ***power function***[a].

[a] fonction puissance

Remark. For $x > 0$,

$$\begin{aligned} u'(x) &= \exp(b\ln x)b\frac{1}{x} = b\exp(b\ln x)\exp(\ln\frac{1}{x}) \\ &= b\exp(b\ln x)\exp(-\ln x) = b\exp((b-1)\ln x) = bx^{b-1}. \end{aligned}$$

Properties 7. ➢ For any $x > 0$ we have $x^{b-1} > 0$ so $u'(x) > 0$ if $b > 0$ and $u'(x) < 0$ if $b < 0$. Consequently, the function u is increasing if $b > 0$ and decreasing if $b < 0$.

➢ Assume that $b > 0$. We have $\lim_{x\to+\infty} b\ln x = +\infty$ therefore $\lim_{x\to+\infty} x^b = \lim_{t\to+\infty}\exp t = +\infty$. Since $\lim_{x\to 0}\ln x = -\infty$ we have $\lim_{x\to 0} b\ln x = -\infty$ and therefore $\lim_{x\to 0} x^b = \lim_{t\to-\infty}\exp t = 0$. The function u has a continuous extension[a] at 0 that is defined by $u(0) = 0$.

➢ If $b > 1$, then $\frac{u(x)}{x} = x^{b-1}$ nears 0 as x approaches 0. Therefore the function u extended to $[0\,;+\infty[$ has a right-hand derivative at 0 equal to 0. In particular, there is a horizontal tangent line at 0.

➢ If $0 < b < 1$, then $\frac{u(x)}{x}$ goes to $+\infty$ as x approaches 0, therefore there is a vertical tangent line at 0.

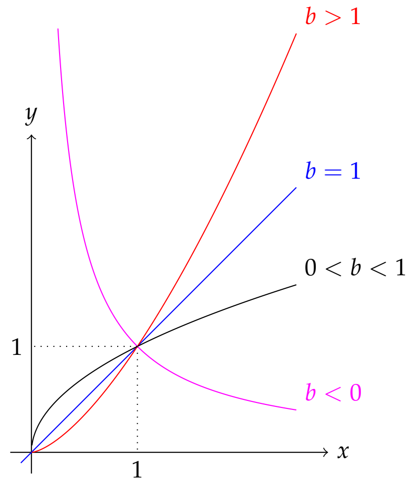
➢ If $b < 0$ we have $\lim_{x\to 0} b\ln x = +\infty$ so $\lim_{x\to 0} x^b = +\infty$. Moreover, $\lim_{x\to+\infty} b\ln x = -\infty$ so $\lim_{x\to+\infty} x^b = \lim_{t\to-\infty}\exp t = 0$.

➢ If $b = 0$, the function u is constant equal to 1 on $]0\,;+\infty[$.

[a] admet un prolongement par continuité

We summarise some of these properties.

Proposition 8. If $b > 0$, the function $x \mapsto x^b$ is a continuous and increasing bijection from $[0\,;+\infty[$ to $[0\,;+\infty[$. If $b < 0$, the function $x \mapsto x^b$ is a continuous and decreasing bijection from $]0\,;+\infty[$ to $]0\,;+\infty[$.



D. The general exponential function

Definition 9. *Let a be a positive real number. The function $v\colon \mathbb{R} \to \mathbb{R}$ defined by $v(x) = a^x$ is called the **general exponential function**[a] **with base** a.*

[a] exponentielle de base a

Remark. ➢ $v(x) = a^x = \exp(x\ln a)$ so $v'(x) = \ln a \exp(x\ln a) = (\ln a)a^x$ for all $x \in \mathbb{R}$.

➢ For $a = e$ we have $e^x = \exp x$: the general exponential with base e is the usual exponential function.

Proposition 10. If $a \neq 1$, the function $x \mapsto a^x$ is a continuous bijection from $\mathbb{R}$ to $]0\,;+\infty[$. If $a > 1$, this bijection is increasing, if $a < 1$ this bijection is decreasing.

Proof. Exercise ✓

II. Indeterminate limits involving logarithms, exponentials and powers[†]

Lemma 11. $\lim\limits_{x\to+\infty} \dfrac{\ln x}{x} = 0$ and $\lim\limits_{x\to+\infty} \dfrac{e^x}{x} = +\infty$.
In other words, $\ln x = o(x)$ at $+\infty$ and $x = o(e^x)$ at $+\infty$.

[†] 'croissances comparées'

Proof. The study of the function $x \mapsto \ln(x) - x + 1$ shows that for any positive $x \in \mathbb{R}$ we have $\ln x \leqslant x - 1$ so that $\ln x < x$. In particular, for any $x > 0$ we have $\dfrac{\ln \sqrt{x}}{\sqrt{x}} \leqslant 1$. If $x > 1$ we get

$$0 \leqslant \frac{\ln x}{x} = \frac{2 \ln \sqrt{x}}{x} = 2\frac{\ln \sqrt{x}}{\sqrt{x}} \frac{1}{\sqrt{x}} \leqslant \frac{2}{\sqrt{x}}.$$

Since $\lim_{x \to +\infty} \dfrac{1}{\sqrt{x}} = 0$, we have $\lim_{x \to +\infty} \dfrac{\ln x}{x} = 0$.

We have in particular $\displaystyle\lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{\ln(e^x)}{e^x} = \lim_{t \to +\infty} \frac{\ln t}{t} = 0$, therefore $\displaystyle\lim_{x \to +\infty} \frac{e^x}{x} = +\infty$. ✓

Proposition 12.
- ➢ If $b > 0$ is a real number, then $\displaystyle\lim_{x \to +\infty} \frac{\ln x}{x^b} = 0$ and $\displaystyle\lim_{x \to 0^+} (x^b \ln x) = 0$.
- ➢ If $a > 1$ and $b > 0$ are real numbers, then $\displaystyle\lim_{x \to +\infty} \frac{a^x}{x^b} = +\infty$ and $\displaystyle\lim_{x \to -\infty} (x^n a^x) = 0$ for any integer $n \in \mathbb{Z}$.

In other words, we have $\ln x = o(x^b)$ at $+\infty$, $\ln x = o\left(\dfrac{1}{x^b}\right)$ at 0, $x^b = o(a^x)$ at $+\infty$ and $a^x = o\left(\dfrac{1}{x^n}\right)$ at $-\infty$.

Proof. For any $x > 0$ we have the equality $\ln x = \dfrac{1}{b} \ln(x^b)$ therefore $\dfrac{\ln x}{x^b} = \dfrac{1}{b} \dfrac{\ln(x^b)}{x^b}$. Since $b > 0$ we know that x^b goes to $+\infty$ as x goes to $+\infty$. By the previous lemma, we get $\lim_{x \to +\infty} \dfrac{\ln(x^b)}{x^b} = 0$ and therefore $\displaystyle\lim_{x \to +\infty} \frac{\ln x}{x^b} = 0$.

For any $x > 0$, we have $x^b \ln x = -\dfrac{\ln(\frac{1}{x})}{(\frac{1}{x})^b}$. We have $\lim_{x \to 0^+} \dfrac{1}{x} = +\infty$ and $\lim_{t \to +\infty} \dfrac{\ln t}{t^b} = 0$ so $\lim_{x \to 0} x^b \ln x = 0$.

For any $x \neq 0$, we have $\dfrac{a^x}{x} = \dfrac{e^{x \ln a}}{x} = (\ln a) \dfrac{e^{x \ln a}}{x \ln a}$. Since $a > 1$, $\ln a$ is positive and $x \ln a$ goes to $+\infty$ as x goes to $+\infty$, therefore $\displaystyle\lim_{x \to +\infty} \frac{e^{x \ln a}}{x \ln a} = \lim_{t \to +\infty} \frac{e^t}{t} = +\infty$. Since $b > 0$, we know that $x \mapsto x^b$ is a bijection from $]1, +\infty[$ to $]1, +\infty[$; therefore there exists a real number $\alpha > 1$ such that $a = \alpha^b$. For any $x > 0$ we then have $a^x = (\alpha^b)^x = \alpha^{bx} = (\alpha^x)^b$ and $\dfrac{a^x}{x^b} = \dfrac{(\alpha^x)^b}{x^b} = \left(\dfrac{\alpha^x}{x}\right)^b$. Since $\displaystyle\lim_{x \to +\infty} \frac{\alpha^x}{x} = +\infty$ and $b > 0$ we get $\displaystyle\lim_{x \to +\infty} \frac{a^x}{x^b} = \lim_{x \to +\infty} \left(\frac{\alpha^x}{x}\right)^b = +\infty$.

For any real number x, we have $a^{-x} = \dfrac{1}{a^x}$. If $n \in \mathbb{N}$, we have $\displaystyle\lim_{x \to -\infty} |x^n a^x| = \lim_{x \to -\infty} \frac{|-x|^n}{a^{-x}} = \lim_{x \to +\infty} \frac{x^n}{a^x} = 0$ by the previous result. If $n \in \mathbb{Z}$, $n < 0$, x^n and a^x near 0 as x goes to $-\infty$ and so does their product. ✓

III. Inverses of trigonometric functions

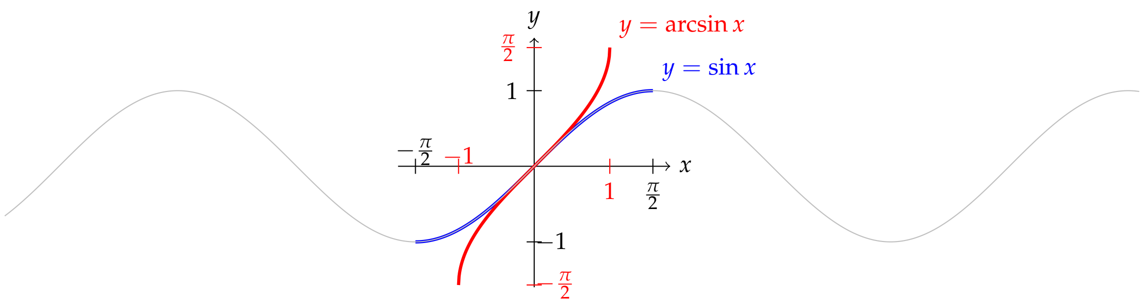
A. The inverse sine function

The sine function is continuous and differentiable, and if $x \in]-\frac{\pi}{2}\,;\frac{\pi}{2}[$ we have $\sin'(x) = \cos x > 0$. Therefore the sine function is increasing on $[-\frac{\pi}{2}\,;\frac{\pi}{2}]$. Moreover, $\sin(-\frac{\pi}{2}) = -1$ and $\sin(\frac{\pi}{2}) = 1$. Therefore the sine function defines a bijection from $[-\frac{\pi}{2}\,;\frac{\pi}{2}]$ to $[-1, 1]$.

Definition 13. *The inverse of the sine function on $[-\frac{\pi}{2}\,;\frac{\pi}{2}]$ is called* ***arcsine***[a]*:*

$$\arcsin\colon [-1\,;1] \to \left[-\frac{\pi}{2}\,;\frac{\pi}{2}\right].$$

[a] arcsinus



Remark. By definition we have:

$$\begin{cases} \sin(\arcsin x) = x & \text{for all } x \in [-1,1] \\ \arcsin(\sin x) = x & \text{for all } x \in [-\frac{\pi}{2}\,;\frac{\pi}{2}]. \end{cases}$$

Moreover, $\cos^2(\arcsin x) = 1 - \sin^2(\arcsin x) = 1 - x^2$ and $\cos(\arcsin x) \geqslant 0$ (since $\arcsin x \in [-\frac{\pi}{2}\,;\frac{\pi}{2}]$) so

$$\cos(\arcsin x) = \sqrt{1-x^2} \quad \text{for all } x \in [-1,1].$$

Properties 14.
- ➢ arcsin is continuous and increasing (inverse of sin which is continuous and increasing).
- ➢ arcsin is differentiable on $]-1\,;1[$ since sin is differentiable with non-vanishing derivative on $]-\frac{\pi}{2}\,;\frac{\pi}{2}[$. Moreover,

$$\arcsin' x = \frac{1}{\sqrt{1-x^2}} \quad \text{for all } x \in]-1\,;1[.$$

- ➢ The graph of arcsin is symmetric to the graph of sin with respect to the line with equation $y = x$.
- ➢ arcsin is an odd function (as is sin).
- ➢ The Taylor expansion of arcsin at 0 can be obtained by taking a primitive of that of $x \mapsto \dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$.

  We have

$$\arcsin(x) = x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3}{2^2\cdot 2!}\frac{x^5}{5} + \cdots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}\frac{x^{2n+1}}{2n+1} + o(x^{2n+1})$$

  (you should not try to remember this formula, but you should know how to find it).

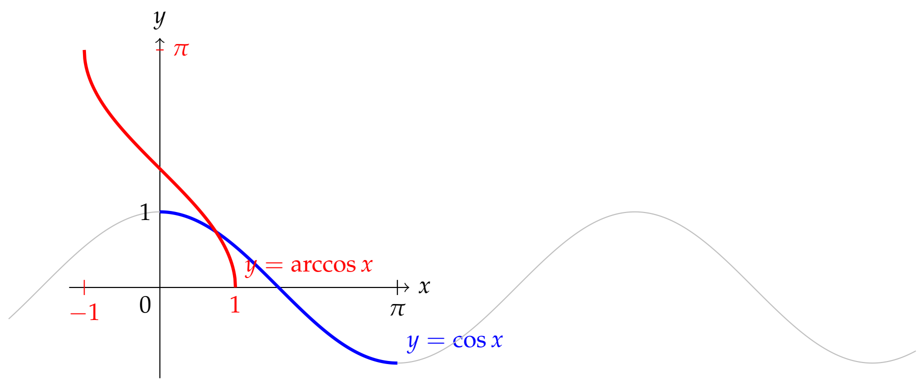
B. The inverse cosine function

The cosine function is continuous and differentiable, and if $x \in]0;\pi[$ we have $\cos'(x) = -\sin x < 0$. Therefore the cosine function is decreasing on $[0;\pi]$. Moreover, $\cos(0) = 1$ and $\cos(\pi) = -1$. Therefore the cosine function defines a bijection from $[0;\pi]$ to $[-1,1]$.

Definition 15. *The inverse of the cosine function on* $[0;\pi]$ *is called* ***arccosine***[a]*:*

$$\arccos\colon [-1,1] \to [0;\pi].$$

[a] arccosinus



Remark. By definition we have:

$$\begin{cases} \cos(\arccos x) = x & \text{for all } x \in [-1,1] \\ \arccos(\cos x) = x & \text{for all } x \in [0;\pi]. \end{cases}$$

Moreover, $\cos(\frac{\pi}{2} - \arcsin x) = \sin(\arcsin x) = x$ for $x \in [-1,1]$ and $0 \leqslant \frac{\pi}{2} - \arcsin x \leqslant \pi$ so $\arccos x = \arccos(\cos(\frac{\pi}{2} - \arcsin x)) = \frac{\pi}{2} - \arcsin x$ so that

$$\arccos x + \arcsin x = \frac{\pi}{2} \quad \text{for } x \in [-1,1].$$

Properties 16.

- ➢ arccos is continuous and decreasing (inverse of cos which is continuous and decreasing).
- ➢ arccos is differentiable on $]-1\,;1[$ since cos is differentiable with non-vanishing derivative on $]0,\pi[$. Moreover,

$$\arccos' x = -\arcsin' x = -\frac{1}{\sqrt{1-x^2}} \quad \text{for all } x \in\,]-1\,;1[.$$

- ➢ The graph of arccos is symmetric to the graph of cos with respect to the line with equation $y = x$.
- ➢ We have $\arccos = \frac{\pi}{2} - \arcsin$. In particular, the Taylor expansion of arccos at 0 is given by

$$\arccos(x) = \frac{\pi}{2} - x - \frac{x^3}{2\cdot 3} - \frac{1\cdot 3}{2^2\cdot 2!}\,\frac{x^5}{5} - \cdots - \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}\,\frac{x^{2n+1}}{2n+1} + o(x^{2n+1}).$$

C. The inverse tangent function

Recall that the tangent function is defined by $\tan x = \dfrac{\sin x}{\cos x}$ for all x different from $(2k+1)\frac{\pi}{2}$, $k\in\mathbb{Z}$. Recall also that

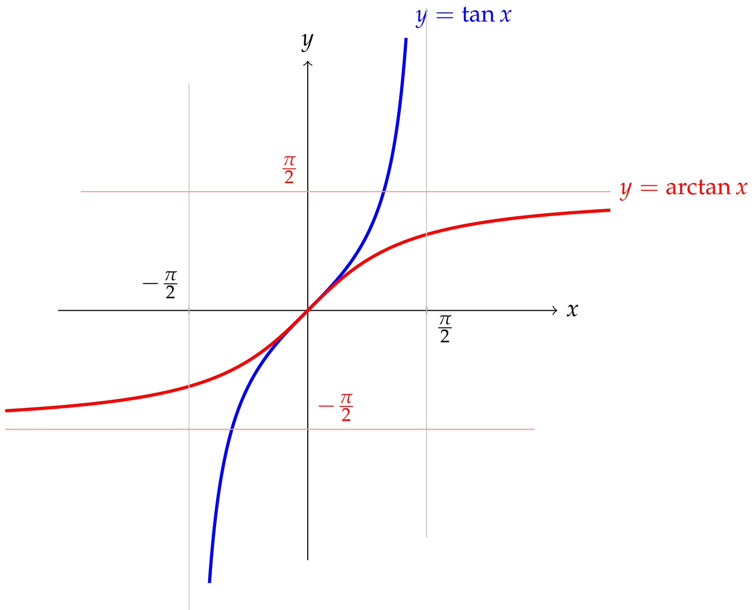
$$\tan' x = \frac{1}{\cos^2 x} = 1 + \tan^2 x.$$

The tangent function is continuous and increasing on $]-\frac{\pi}{2}\,;\frac{\pi}{2}[$. Moreover $\lim_{x\to\frac{\pi}{2}} \tan x = +\infty$ and $\lim_{x\to-\frac{\pi}{2}} \tan x = -\infty$ so that it defines a bijection from $]-\frac{\pi}{2}\,;\frac{\pi}{2}[$ to $\mathbb{R}$.

Definition 17. *The inverse of the tangent function on $]-\frac{\pi}{2}\,;\frac{\pi}{2}[$ is the **arctangent**[a] function:*

$$\arctan\colon \mathbb{R} \to \left]-\frac{\pi}{2}\,;\frac{\pi}{2}\right[$$

[a] arctangente



Remark. By definition we have:

$$\begin{cases} \tan(\arctan x) = x & \text{for all } x \in \mathbb{R} \\ \arctan(\tan x) = x & \text{for all } x \in\,]-\frac{\pi}{2}\,;\frac{\pi}{2}[. \end{cases}$$

Properties 18.
- ➢ arctan is continuous and increasing (inverse of tan which is continuous and increasing).
- ➢ arctan is differentiable on $\mathbb{R}$ since tan is differentiable with non-vanishing derivative on the interval $]-\frac{\pi}{2};\frac{\pi}{2}[$. Moreover,
$$\arctan' x = \frac{1}{1+x^2} \quad \text{for all } x \in \mathbb{R}.$$
- ➢ The graph of arccos is symmetric to the graph of cos with respect to the line with equation $y = x$.
- ➢ arctan is an odd function (as is tan).
- ➢ The Taylor expansion of arctan at 0 can be obtained by taking a primitive of that of $x \mapsto \frac{1}{1+x^2} = (1+x^2)^{-1}$. We have
$$\arctan(x) = x - \frac{x^3}{3} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + o(x^{2n+1}).$$

IV. The hyperbolic functions and their inverses

Definition 19. *The* ***hyperbolic cosine***[a] *function, denoted by* ch *or* cosh*, and the* ***hyperbolic sine***[b] *function, denoted by* sh *or* sinh*, are defined for all $x \in \mathbb{R}$ by:*
$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \textit{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

[a] cosinus hyperbolique
[b] sinus hyperbolique

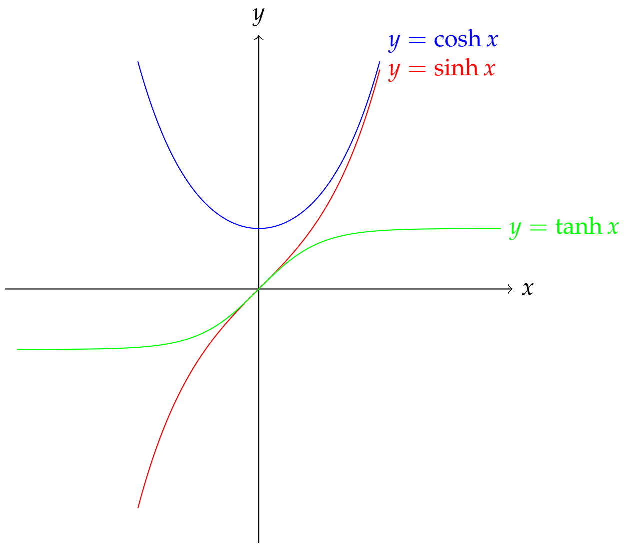
The following properties are easy to prove.

Properties 20.
- ➢ The functions cosh and sinh are continuous functions.
- ➢ cosh is even and sinh is odd.
- ➢ $\cosh x + \sinh x = e^x$ and $\cosh x - \sinh x = e^{-x}$ for all $x \in \mathbb{R}$. Therefore $\cosh^2 x - \sinh^2 x = 1$ for all $x \in \mathbb{R}$.
- ➢ The functions cosh and sinh are differentiable functions. Moreover, $\cosh' = \sinh$ and $\sinh' = \cosh$.
- ➢ $\cosh x - \sinh x = e^{-x} > 0$ so $\cosh x > \sinh x$ and $\lim_{x\to+\infty}(\cosh x - \sinh x) = 0$.
- ➢ The Taylor expansions at 0 are given by
$$\cosh(x) = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + o(x^{2n}) \quad \text{and} \quad \sinh(x) = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+1}).$$

For any $x > 0$, we have $x > -x$ so $e^x > e^{-x}$ and therefore $\sinh x > 0$. Since sinh is odd, $\sinh x < 0$ for $x < 0$. Therefore cosh is increasing on $[0;+\infty[$ and decreasing on $]-\infty;0]$. Moreover, $\cosh 0 = 1$ so $\cosh x > 1$ for any $x \neq 0$. In particular, cosh is positive so sinh is increasing on $\mathbb{R}$. Note also that $\sinh 0 = 0$.

We also have $\lim_{x\to+\infty} \sinh x = +\infty = \lim_{x\to+\infty} \cosh x$, $\lim_{x\to-\infty} \sinh x = -\infty$ and $\lim_{x\to-\infty} \cosh x = +\infty$. Therefore

Lemma 21. $\sinh\colon \mathbb{R} \to \mathbb{R}$ is an increasing bijection and cosh defines an increasing bijection from $[0;+\infty[$ to $[1;+\infty[$.



Definition 22. *The* ***hyperbolic tangent***[a] *function, denoted by* th *or* tanh*, is defined by* $\tanh x = \dfrac{\sinh x}{\cosh x}$ *for all* $x \in \mathbb{R}$.

[a] tangente hyperbolique

Properties 23.
- tanh is continuous, differentiable and odd. For all $x \in \mathbb{R}$ we have $\tanh' x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}$.
- tanh is increasing on $\mathbb{R}$.
- $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^x(1-e^{-2x})}{e^x(1+e^{-2x})} = \dfrac{1-e^{-2x}}{1+e^{-2x}}$. Therefore $\lim_{x\to+\infty} \tanh x = 1$ and since tanh is odd we have $\lim_{x\to-\infty} \tanh x = -1$.
- It follows from the above that tanh defines an increasing bijection from $\mathbb{R}$ to $]-1\,;1[$.
- The beginning of the Taylor expansion of tanh at 0 is given by

$$\tanh(x) = x - \frac{x^3}{3} + 2\frac{x^5}{15} + o(x^6).$$

We summarise and complete some of the properties above:

Definition-Proposition 24. *(i) The function* $\sinh\colon \mathbb{R} \to \mathbb{R}$ *is an increasing bijection. Its inverse* $\operatorname{Argsh}\colon \mathbb{R} \to \mathbb{R}$ *is continuous, differentiable, odd, and* $\operatorname{Argsh}' x = \frac{1}{\sqrt{1+x^2}}$ *for* $x \in \mathbb{R}$.

(ii) The function cosh *defines an increasing bijection from* $[0\,;+\infty[$ *to* $[1\,;+\infty[$ *whose inverse* $\operatorname{Argch}\colon [1\,;+\infty[\to [0\,;+\infty[$ *is continuous, differentiable on* $]1\,;+\infty[$*, and* $\operatorname{Argch}' x = \frac{1}{\sqrt{x^2-1}}$ *for* $x > 1$.

(iii) The function tanh *defines an increasing bijection from* $\mathbb{R}$ *to* $]-1\,;1[$ *whose inverse* $\operatorname{Argth}\colon]-1\,;1[\to \mathbb{R}$ *is continuous, differentiable, odd, and* $\operatorname{Argth}' x = \frac{1}{1-x^2}$ *for* $x \in]-1\,;1[$.

Proof. Exercise. ✓

Chapter 4

Study of recursive sequences $u_{n+1} = f(u_n)$

We shall not study these types of sequences extensively, as it is a difficult subject. But there are some useful tools that we will see in this chapter.

Example. A trivial example: Given $u_0 \in \mathbb{R}$, we want to know whether there exists a sequence $(u_n)_{n\geqslant 0}$ such that $u_{n+1} = au_n$ for every $n \in \mathbb{N}$ (where $a \in \mathbb{R}$ is fixed) and if so, whether it converges.

Clearly, the only sequence that satisfies $u_{n+1} = au_n$ is the sequence defined by $u_n = a^n u_0$ for all $n \geqslant 0$ (a simple induction will prove this). Moreover, it always converges (to 0) if $u_0 = 0$, and when $u_0 \neq 0$ it converges if and only if $a \in]-1, 1]$.

Example. A little less trivial. Fix u_0, a, b in $\mathbb{R}$ and consider sequences $(u_n)_{n\geqslant 0}$ such that $u_{n+1} = au_n + b$ for every $n \in \mathbb{N}$.

Assume that $(u_n)_n$ converges to ℓ. Then $\ell = \lim_{n\to+\infty} u_n = \lim_{n\to+\infty} u_{n+1} = \lim_{n\to+\infty}(au_n + b) = a\ell + b$ so that $\ell = a\ell + b$, *ie.* $(1-a)\ell = b$.

If $a = 1$, then $u_{n+1} = u_n + b$ so that $u_n = u_0 + nb$. Therefore $\lim_{n\to+\infty} u_n = \begin{cases} u_0 & \text{if } b = 0 \\ +\infty & \text{if } b > 0 \\ -\infty & \text{if } b < 0. \end{cases}$

If $a \neq 1$ and if $(u_n)_n$ converges then the limit must be $\ell = \frac{b}{1-a}$. Note also that

$$\begin{cases} u_{n+1} = au_n + b \\ \ell = a\ell + b \end{cases}$$

and subtracting these equalities gives $(u_{n+1} - \ell) = a(u_n - \ell)$. If we define $v_n = u_n - \ell$, then we are in the situation of the previous example with $v_{n+1} = av_n$ (and $a \neq 1$).

Example. Even less trivial: $u_0, a, b, c, d \in \mathbb{R}$ and $u_{n+1} = \frac{au_n+b}{cu_n+d}$ (and $c \neq 0$ since we have already considered the case $c = 0$). This expression is not very complicated but we already have a serious difficulty: if $u_n = -\frac{c}{d}$ then u_{n+1} is not defined. How can we be sure that we will never have $u_n = -\frac{d}{c}$? In practice, this kind of difficulty does arise – we shall therefore exclude this case.

We shall now see some conditions to ensure that a sequence given by a recursion $u_{n+1} = f(u_n)$ is well defined and then see how to study it.

In all this chapter, f will be a function defined on an *interval* I.

I. Existence of all terms in the sequence

A. Stable intervals

Definition 1. *An interval $J \subset I$ is said to be* ***stable***[a] *under f if $f(J) \subset J$.*

[a] stable

Example. The study of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x - x^2$ shows that $f([0;1]) = [0;\frac{1}{4}] \subset [0;1]$ so that $[0;1]$ is stable under f.

B. Why do we need stable intervals?

As we have seen in the introduction, it can happen that u_{n+1} is not defined. Therefore we introduce stable intervals in order to avoid this problem.

Lemma 2. Assume that $J \subset I$ is an interval stable under f and that $u_0 \in J$. Then we can define recursively a sequence (u_n) such that $u_{n+1} = f(u_n)$ for all $n \in \mathbb{N}$. Moreover, $u_n \in J$ for all $n \in \mathbb{N}$.

Proof. We prove by induction on n that u_n is defined and that $u_n \in J$ for all $n \in \mathbb{N}$. Recall that f is defined on I and hence on J.

- ✦ By assumption, $u_0 \in J$.
- ✦ Now assume that u_n is defined and that $u_n \in J$ for some $n \in \mathbb{N}$. Then $f(u_n)$ is defined because f is defined on J. Therefore $u_{n+1} = f(u_n)$ is defined, and moreover $u_{n+1} \in f(J) \subset J$ because J is stable under f.
- ✦ Therefore, for all $n \in \mathbb{N}$, u_n is defined and $u_n \in J$. ✓

II. Potential limits

We assume that we have a well-defined sequence (u_n) with $u_{n+1} = f(u_n)$ for all $n \in \mathbb{N}$.

A. Fixed points

Definition 3. *A real number $a \in I$ is called a **fixed point**[a] of f if $f(a) = a$.*

[a] point fixe

Remark. We have seen in the exercises that if f is a *continuous* function from $[0, 1]$ to $[0, 1]$, then f has a fixed point (apply the Intermediate Value Theorem to the function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = f(x) - x$).

More generally, if an interval of the form $[a, b]$ is stable under f and if f is continuous on $[a, b]$, then f has a fixed point in $[a, b]$.

B. Limits

Recall from the first semester (Proposition 234) that if (v_n) is a sequence that converges to a and if h is a function which has limit b at a, with $b \in \mathbb{R}$ or $b = \pm\infty$, then the sequence $(h(v_n))$ has limit b. In particular, if h is continuous at a, then $(h(v_n))$ converges to $h(a)$.

Theorem 4. Assume that the sequence (u_n) converges to ℓ and that f is continuous at ℓ. Then ℓ is a fixed point of f.

Proof. We have assumed that (u_n) converges and that $\lim_{n\to+\infty} u_n = \ell$. Then we also have $\lim_{n\to+\infty} u_{n+1} = \ell$. Taking limits when n goes to $+\infty$ in the equality $u_{n+1} = f(u_n)$ gives $\ell = \lim_{n\to+\infty} f(u_n)$. Moreover, since f is continous at ℓ and (u_n) converges to ℓ, we have $\lim_{n\to+\infty} f(u_n) = f(\ell)$ by the result recalled above. Therefore, $f(\ell) = \ell$ as required. ✓

Example. We fix $u_0 \geqslant 0$ and set $u_{n+1} = \sqrt{u_n + 2}$ for $n \in \mathbb{N}$.

The first step is to find f. We know that $x \mapsto \sqrt{x+2}$ is defined for all $x \geqslant -2$ and that $\sqrt{x+2} \geqslant 0$. In order to make sure that the sequence is well-defined, we must have a stable interval containing u_0, we choose

$$\begin{array}{rcl} f : [0;+\infty[& \longrightarrow & [0;+\infty[\\ x & \mapsto & \sqrt{x+2}. \end{array}$$

(we could have chosen the interval $]-2\,;+\infty[$). We have $u_{n+1} = f(u_n)$, therefore the sequence (u_n) is well-defined.

The function f is clearly continuous. Consequently, **if** (u_n) converges to ℓ we must have $\ell = \sqrt{\ell+2}$, that is, $\ell \geqslant 0$ and $\ell^2 - \ell - 2 = 0$, *ie.* $\ell = 2$.

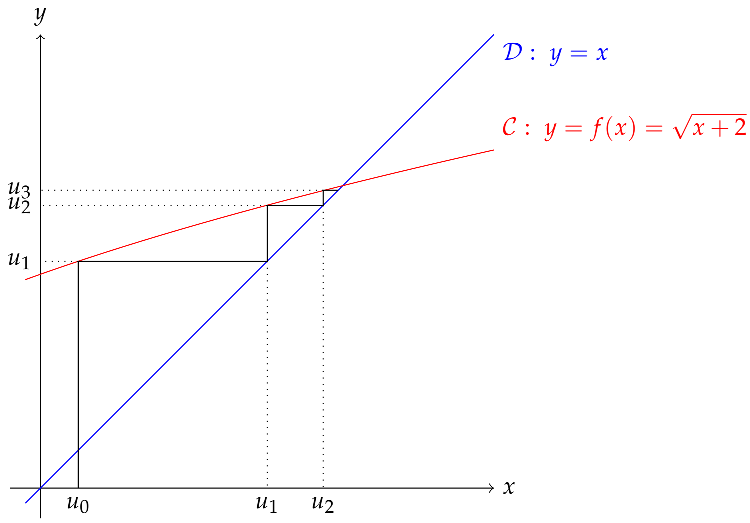
But this is not enough to prove that (u_n) converges.

We shall continue this example later.

III. Graphical representation

We use the graph $\mathcal{C}$ of f to represent the terms u_n on the x-axis. The line $\mathcal{D}$ with equation $y = x$ is used to transfer points from the y-axis to the x-axis. Moreover, if f is continuous and if (u_n) converges, then the limit is a fixed point of f, that is, either one of the coordinates of a point at the intersection of $\mathcal{C}$ and $\mathcal{D}$.

Example. For the sequence (u_n) defined by $u_0 \geqslant 0$ and $u_{n+1} = \sqrt{u_n + 2}$ for $n \in \mathbb{N}$, the graphical representation is as follows when $u_0 < 2$:



Example. We fix $u_0 \in \mathbb{R}$ and set $u_{n+1} = \cos u_n$ for $n \in \mathbb{N}$.

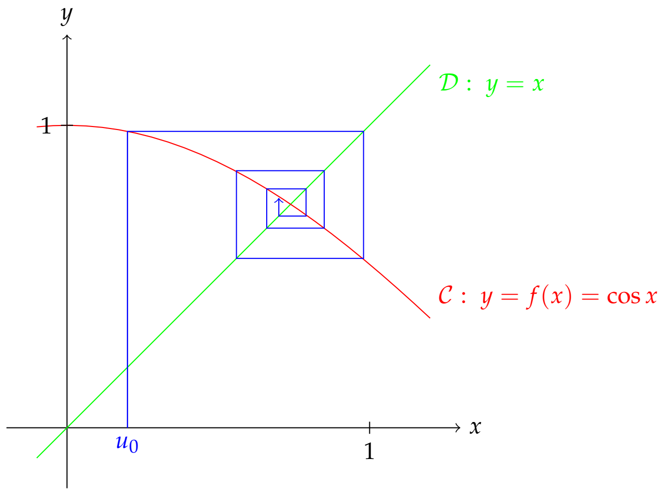
We first determine f. The function $\cos : \mathbb{R} \to \mathbb{R}$ works well, but we will simplify the discussion by noting that $\cos(\mathbb{R}) = [-1,1]$, so that for any $u_0 \in \mathbb{R}$ we have $u_1 \in [-1,1] \subset]-\frac{\pi}{2}; \frac{\pi}{2}[$ and $u_2 = f(u_1) \in [0,1]$; moreover, $[0,1]$ is stable under cos, therefore the sequence $(u_n)_{n\geqslant 2}$ is well-defined and contained in $[0,1]$. Therefore we may assume without loss of generality that $u_0 \in [0,1]$ (we need only forget the first two terms and shift the indices).

Therefore we define

$$\begin{array}{rcl} f : [0,1] & \to & [0,1] \\ x & \mapsto & \cos x \end{array}$$

which is continuous.

If u_0 is "small enough", the graphical representation is as follows:



IV. Monotony of the sequence

We still have to see whether the sequence converges or not and prove it. We shall often use the theorems on the convergence of monotonic sequences seen in the first semester. Therefore we need some methods to prove that a sequence is monotonic (or not).

We shall assume that f is continuous on the interval I, that I is stable under f, that $u_0 \in I$ and that $u_{n+1} = f(u_n)$.

A. Sign of $f(x) - x$

Lemma 5. Let $f : I \to I$ be a continuous function, take $u_0 \in I$ and set $u_{n+1} = f(u_n)$ for all $n \in \mathbb{N}$.
Assume that $f(x) - x$ has constant sign on I.
Then the sequence (u_n) is monotonic.
More precisely, (u_n) is non-decreasing if $f(x) - x \geqslant 0$ for all $x \in I$ and (u_n) is non-increasing if $f(x) - x \leqslant 0$ for all $x \in I$.

Proof. For any $n \in \mathbb{N}$, we have $u_{n+1} - u_n = f(u_n) - u_n$. Therefore

- if $f(x) - x \geqslant 0$ for all $x \in I$, then $u_{n+1} - u_n \geqslant 0$ for all $n \in \mathbb{N}$ (recall that $u_n \in I$ for all $n \in \mathbb{N}$), therefore the sequence is non-decreasing;

✦ if $f(x) - x \leqslant 0$ for all $x \in I$, then $u_{n+1} - u_n \leqslant 0$ for all $n \in \mathbb{N}$, therefore the sequence is non-increasing. ✓

B. f is non-decreasing

Lemma 6. Let $f : I \to I$ be a continuous function, take $u_0 \in I$ and set $u_{n+1} = f(u_n)$ for all $n \in \mathbb{N}$.
Assume that the function f is non-decreasing.
Then the sequence (u_n) is monotonic.
More precisely, (u_n) is non-decreasing if $u_1 = f(u_0) \geqslant u_0$ and (u_n) is non-increasing if $u_1 \leqslant u_0$.

Proof. ➢ First assume that $u_1 \geqslant u_0$. We shall prove by induction that for all $n \in \mathbb{N}$ we have $u_{n+1} \geqslant u_n$.

- ✦ The result is true for $n = 0$ by assumption.
- ✦ Assume that $u_{n+1} \geqslant u_n$ for some $n \geqslant 0$. Since the function f is non-decreasing, applying f gives $f(u_{n+1}) \geqslant f(u_n)$, that is, $u_{n+2} \geqslant u_{n+1}$ using the definition of (u_n). Therefore the result is true for $n+1$.
- ✦ By induction, we have $u_{n+1} \geqslant u_n$ for all $n \in \mathbb{N}$. Therefore the sequence (u_n) is non-decreasing.

➢ Now assume that $u_1 \leqslant u_0$. We shall prove by induction that for all $n \in \mathbb{N}$ we have $u_{n+1} \leqslant u_n$.

- ✦ The result is true for $n = 0$ by assumption.
- ✦ Assume that $u_{n+1} \leqslant u_n$ for some $n \geqslant 0$. Since the function f is non-decreasing, applying f gives $f(u_{n+1}) \leqslant f(u_n)$, that is, $u_{n+2} \leqslant u_{n+1}$ using the definition of (u_n). Therefore the result is true for $n+1$.
- ✦ By induction, we have $u_{n+1} \leqslant u_n$ for all $n \in \mathbb{N}$. Therefore the sequence (u_n) is non-increasing. ✓

Example. Let us come back to the example

$$\begin{cases} u_0 \geqslant 0 \\ u_{n+1} = \sqrt{u_n + 2} \quad \text{for all } n \in \mathbb{N}. \end{cases}$$

We have seen that if this sequence converges, then its limit must be 2, the only non-negative fixed point of the function $f\colon [0\,;+\infty[\to [0\,;+\infty[$ defined by $f(x) = \sqrt{x+2}$.

The function f is increasing (as the composition of the increasing functions $x \mapsto x+2$ and $x \mapsto \sqrt{x}$). Therefore the sequence (u_n) is monotonic; to prove convergence, if (u_n) is non-decreasing we must prove that it is bounded above, and if it is non-increasing we must prove that it is bounded below. Therefore we want to know whether $(u_n)_n$ is non-decreasing or non-increasing, and for this we need the sign of $u_1 - u_0 = f(u_0) - u_0$. Therefore we shall study $g = f - \mathrm{id}$.

The function g is differentiable and we have $g'(x) = \frac{1}{2\sqrt{x+2}} - 1 \leqslant \frac{1}{2\sqrt{2}} - 1 < 0$ for all $x \geqslant 0$ so we get the following table

x	0	2	$+\infty$
$f(x)$	$\sqrt{2}$ $\nearrow$	2 $\nearrow$	$+\infty$
$f(x) - x$	$\sqrt{2}$ $\searrow$	0 $\searrow$	$-\infty$

We can now conclude:

➢ If $u_0 = 2$ then $u_n = 2$ for all $n \in \mathbb{N}$ (induction: if $u_n = 2$ then $u_{n+1} = f(u_n) = f(2) = 2$).

➢ If $0 \leqslant u_n < 2$: since $f([0,2[) \subset [0,2[$, we have $u_n \in [0,2[$ for all $n \in \mathbb{N}$. Since $f(x) - x > 0$ for all $x \in [0,2[$, we have $u_1 - u_0 = f(u_0) - u_0 > 0$ and $u_{n+1} - u_n = f(u_n) - u_n > 0$ therefore $(u_n)_n$ is increasing. As $(u_n)_n$ is bounded above (by 2), it converges. The only possible limit is 2 as we have seen before, so $(u_n)_n$ converges to 2.

➢ If $u_0 > 2$: since $f(]2,+\infty[) \subset]2,+\infty[$, we have $u_n > 2$ for all $n \in \mathbb{N}$. Since $f(x) - x < 0$ for all $x > 2$, we can show as before that $(u_n)_n$ is decreasing. As $(u_n)_n$ is bounded below (by 2), it converges. The only possible limit is 2 as we have seen before, so $(u_n)_n$ converges to 2.

Note that it appears from the graphical representation to converge quite fast. Let us look into this. Using the Mean Value Theorem, we have

$$u_{n+1} - 2 = f(u_n) - f(2) = (u_n - 2) f'(\theta_n)$$

for some θ_n between 2 and u_n. Moreover, $f'(x) = \frac{1}{2\sqrt{x+2}} \leqslant \frac{1}{2\sqrt{2}}$ and $f'(x) > 0$ for all $x \geqslant 0$, so $|f'(\theta_n)| \leqslant \frac{1}{2\sqrt{2}}$. We then get $|u_{n+1} - 2| \leqslant \frac{1}{2\sqrt{2}} |u_n - 2|$ for all $n \in \mathbb{N}$ hence

$$|u_n - 2| \leqslant \left(\frac{1}{2\sqrt{2}} \right)^n |u_0 - 2|.$$

Remark. We have used here the Mean Value Theorem to prove that f is contracting, and used this to see that (u_n) converges to 2 at least as fast as the geometric sequence $\left(\left(\frac{1}{2\sqrt{2}} \right)^n \right)$ converges to 0.

C. f is non-increasing

Lemma 7. Let $f : I \to I$ be a continuous function, take $u_0 \in I$ and set $u_{n+1} = f(u_n)$ for all $n \in \mathbb{N}$.
Assume that the function f is non-increasing.
Then the sequence (u_n) is not monotonic (unless it is constant). However, the sequences $(u_{2n})_n$ and $(u_{2n+1})_n$ are monotonic (one is non-decreasing and the other is non-increasing).

Proof. We have $(f \circ f)(u_n) = f(f(u_n)) = f(u_{n+1}) = u_{n+2}$ therefore

- $u_{2(n+1)} = u_{2n+2} = f \circ f(u_{2n})$ and
- $u_{2(n+1)+1} = u_{2n+3} = f \circ f(u_{2n+1})$.

Moreover, the function $f \circ f$ is non-decreasing. Therefore, using Lemma 6 with the function $f \circ f$, we see that $(u_{2n})_n$ and $(u_{2n+1})_n$ are both monotonic.

Moreover, assume that $(u_{2n})_n$ is non-decreasing. Then $u_2 \geqslant u_0$. Applying the non-increasing function f gives $u_3 = f(u_2) \leqslant f(u_0) = u_1$ and therefore $(u_{2n+1})_n$ is non-increasing by Lemma 6. The other case is similar. ✓

Example. Let us return to the example with $u_0 \in [0, 1]$ and $u_{n+1} = \cos u_n$ for $n \in \mathbb{N}$. We had defined

$$\begin{array}{rcl} f : [0,1] & \to & [0,1] \\ x & \mapsto & \cos x \end{array}$$

which is continuous and differentiable. We now study f and $g = f - \mathrm{id}$. We have $g'(x) = -\sin x - 1 < 0$ so g is decreasing. Since $g(0) = 1 > 0$ and $g(1) = \cos 1 - 1 < 0$, by the Intermediate Value Theorem there exists $\alpha \in]0, 1[$ such that $g(\alpha) = 0$ (*ie.* $f(\alpha) = \alpha$).

x	0	α	1
$f(x)$	1 $\searrow$	α	$\searrow$ $\cos 1$
$f(x) - x$	1 $\searrow$	0	$\searrow$ $\cos 1 - 1$

If $u_0 = \alpha$ then $u_n = \alpha$ for all n.

We know that in our situation the sequences $(u_{2n})_n$ and $(u_{2n+1})_n$ are monotonic and since they are bounded (both are contained in $[0, 1]$), they converge. Set $\ell_1 = \lim_{n\to+\infty} u_{2n}$ and $\ell_2 = \lim_{n\to+\infty} u_{2n+1}$.

If $\ell_1 = \ell_2$ then $(u_n)_n$ converges to $\ell_1 = \ell_2$ so that, since f is continuous, $\ell_1 = \ell_2 = \alpha$.

Can we have $\ell_1 \neq \ell_2$?

If $u_0 > \alpha$ then $u_1 = f(u_0) < \alpha$ so up to a shift in indices we can assume that $u_0 < \alpha$. The graphical representation we had seen represented this situation.

It seems that there is a spiral converging to α, so that $\ell_1 = \ell_2$.

Since f is continuous, so is $f \circ f$, therefore it seems natural to look for fixed points of $f \circ f$, that is, to study $h : x \mapsto \cos(\cos(x)) - x$. The function h is differentiable and $h'(x) = \sin x \sin(\cos x) - 1 < 0$ for $x \in [0, 1]$. Hence h is decreasing from $h(0) = \cos 1 > 0$ to $h(1) = \cos(\cos 1) - 1 < 0$ and there exists a unique $\beta \in]0, 1[$ such that $\cos(\cos \beta) = \beta$. But we know that $\cos \alpha = \alpha$ so $\cos(\cos \alpha) = \cos \alpha = \alpha$ and therefore $\alpha = \beta$ (more generally, if $f(\alpha) = \alpha$ then $f \circ f(\alpha) = \alpha$). Therefore the only possible value for ℓ_1 and for ℓ_2 is $\ell_1 = \ell_2 = \alpha$ and finally $(u_n)_n$ converges to α.

Remark. Note that any fixed point of f is a fixed point of $f \circ f$, but $f \circ f$ could have more fixed points than f.

Application. We cannot give an exact value for α, but this sequence gives a way to approximate α.

Remark. In general, the computation of $f \circ f$ can be complicated. In order to prove that $\ell_1 = \ell_2$ we can, in some cases, use another trick. In our example, we have $u_{2n+1} = \cos u_{2n}$ and $u_{2n+2} = \cos u_{2n+1}$ so taking limits gives $\ell_2 = \cos \ell_1$ and $\ell_1 = \cos \ell_2$ so that

$$\ell_2 - \ell_1 = \cos \ell_1 - \cos \ell_2 = (\ell_1 - \ell_2)(-\sin \theta)$$

for some θ between ℓ_1 and ℓ_2 by the Mean Value Theorem. We get $(\ell_2 - \ell_1)(1 - \sin \theta) = 0$ and since $\sin \theta \neq 1$ we finally have $\ell_2 = \ell_1$.

We fix a function $f : I \to \mathbb{R}$ where I is an interval of $\mathbb{R}$ and $u_0 \in I$. If I is stable under f then the recursive formula $u_{n+1} = f(u_n)$ for all $n \in \mathbb{N}$ defines a sequence $(u_n)_{n\in\mathbb{N}}$ whose terms are all in I.

- We study the continuity of f on I and solve the equation $f(x) - x = 0$. This may require the study of $f - \mathrm{id}$ on I if we do not know how to solve this equation algebraically.
- The study of $f - \mathrm{id}$ gives other useful results: if it has constant sign on I then (u_n) is monotonic.
- If $(u_n)_n$ converges to ℓ with $\ell \in I$ and if f is continuous on I (or at ℓ), then $\ell = f(\ell)$.
- If f is non-decreasing (on I) then $(u_n)_n$ is monotonic.
- If $(u_n)_n$ is monotonic and I is bounded, then $(u_n)_n$ converges.
- If f is non-increasing (on I) then the sequences $(u_{2n})_n$ and $(u_{2n+1})_n$ are monotonic (one is non-decreasing and the other is non-increasing).

Note that even if I is stable under f, it could happen that none of these properties are true on I. It will then be necessary to work on intervals $J \subset I$ which are stable under f and on which f has some nice properties.

Chapter 5

Study of equations $f(x) = 0$

I. Dichotomy

A. Principle

Consider a function f that is continuous on an interval $[a, b]$. Assume that f has a unique root α in $]a, b[$ and that $f(a)f(b) < 0$. Let $c = \frac{a+b}{2}$ be the middle of the interval.

(1) If $f(c) = 0$, then c is the root of f and the problem is solved.

(2) If $f(c) \neq 0$, one of the following holds.
   (a) If $f(a)f(c) < 0$, then $\alpha \in]a, c[$.
   (b) If $f(c)f(b) < 0$, then $\alpha \in]c, b[$.

We repeat the procedure with the interval $[a, c]$ in the first case and the interval $[c, b]$ in the second case. In this way we construct inductively three sequences $(a_n)_n$, $(b_n)_n$ and $(c_n)_n$ such that $a_0 = a$, $b_0 = b$, and for all $n \geqslant 0$:

(i) $c_n = \frac{a_n+b_n}{2}$

(ii) If $f(c_n)f(b_n) < 0$ then $a_{n+1} = c_n$ and $b_{n+1} = b_n$

(iii) If $f(c_n)f(a_n) < 0$ then $a_{n+1} = a_n$ and $b_{n+1} = c_n$.

Definition 1. *The algorithm above is called the* ***dichotomy algorithm***[a].

[a] algorithme de dichotomie

B. Study of convergence

Theorem 2. Let f be a continuous function on $[a, b]$ satisfying $f(a)f(b) < 0$ and assume that the equation $f(x) = 0$ has a unique solution $\alpha \in]a, b[$. If the dichotomy algorithm can be applied up to the n-th stage, then

$$|\alpha - c_n| \leqslant \frac{b-a}{2^{n+1}}.$$

Consequently, the sequence $(c_n)_n$ converges to α.

Proof. Note that at each stage, the interval is halved, that is, $|a_{n+1} - b_{n+1}| = \dfrac{|a_n - b_n|}{2}$. Therefore, by induction, we have $|a_n - b_n| = \dfrac{|a_0 - b_0|}{2^n}$. It follows that $|\alpha - c_n| \leqslant \dfrac{|a_n - b_n|}{2} = \dfrac{|a_0 - b_0|}{2 \cdot 2^n} = \dfrac{|a-b|}{2^{n+1}}$. ✓

Remark. In order that c_n be an approximation of α with a precision of $\varepsilon > 0$, it is enough that n satisfies:

$$\frac{b-a}{2^{n+1}} \leqslant \varepsilon.$$

Then we have

$$|\alpha - c_n| \leqslant \frac{b-a}{2^{n+1}} \leqslant \varepsilon$$

so that we can compute beforehand the maximal number $n_0 \in \mathbb{N}$ of stages needed to have a precision of ε.

$$\frac{b-a}{2^{n+1}} \leqslant \varepsilon \iff \frac{b-a}{\varepsilon} \leqslant 2^{n+1} \iff n \geqslant \frac{\ln \frac{b-a}{\varepsilon}}{\ln 2} - 1.$$

Example. We can use the dichotomy algorithm on $f : [1,2] \to \mathbb{R}$ defined by $f(x) = x^2 - 2$ to approach $\alpha = \sqrt{2}$. We have a sequence $(c_n)_n$ that satisfies $|\alpha - c_n| \leqslant \frac{1}{2^{n+1}}$. If we wanted $\sqrt{2}$ with a precision of 10^{-2}, it would be enough to compute 6 terms, since

$$\frac{1}{2^{n+1}} \leqslant 10^{-2} \iff 2^{n+1} \geqslant 100 \iff (n+1)\ln 2 \geqslant \ln(100) \iff n \geqslant \frac{\ln 100}{\ln 2} - 1 \cong 5,6.$$

In detail, we have

n	a_n	b_n	c_n	$f(a_n)$	$f(b_n)$	$f(c_n)$
0	1	2	$\frac{3}{2}$	$-$	$+$	$+$
1	1	$\frac{3}{2}$	$\frac{5}{4}$	$-$	$+$	$-$
2	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{11}{8}$	$-$	$+$	$-$
3	$\frac{11}{8}$	$\frac{3}{2}$	$\frac{23}{16}$	$-$	$+$	$+$
4	$\frac{11}{8}$	$\frac{23}{16}$	$\frac{45}{32}$	$-$	$+$	$-$
5	$\frac{45}{32}$	$\frac{23}{16}$	$\frac{91}{64}$	$-$	$+$	$+$
6	$\frac{45}{32}$	$\frac{91}{64}$	$\frac{181}{64}$			

therefore $c_6 = \frac{181}{64} \cong 1.41$ is an approximation of $\sqrt{2}$ with a precision of 10^{-2}.

II. Newton's method

A. Principle

Proposition 3 (Newton's method[a]). Let $f : [a,b] \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ such that $f(a) < 0$, $f(b) > 0$, $f'(x) > 0$ and $f''(x) > 0$ for all $x \in [a,b]$.
Then there exists a unique $\alpha \in]a,b[$ such that $f(\alpha) = 0$ and α is the limit of the sequence $(u_n)_n$ defined by $u_0 \in [\alpha, b]$ and $u_{n+1} = g(u_n)$ where $g(x) = x - \frac{f(x)}{f'(x)}$ for all $x \in [a,b]$.

[a] méthode de Newton

Proof. Since f is of class $\mathcal{C}^2$ and f' does not vanish on $[a,b]$, the function g is of class $\mathcal{C}^1$ on $[a,b]$ and $g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$. Since f is continuous and increasing on $[a,b]$, there exists a unique $\alpha \in]a,b[$ such that $f(\alpha) = 0$ by the Intermediate Value Theorem. Moreover, $f(x) < 0$ for $x \in [a,\alpha[$ and $f(x) > 0$ for $x \in]\alpha, b]$. Therefore $g'(x) > 0$ for $x \in]\alpha, b]$ so that g is increasing on this interval, from $g(\alpha) = \alpha$ to $g(b) = b - \frac{f(b)}{f'(b)} \in]\alpha, b[$. In particular, $g([\alpha, b]) \subset [\alpha, b]$.

Set $I = [\alpha, b]$. Then I is stable under g. It follows from Chapter 4 that we can define a sequence $(u_n)_n$ by $u_0 \in I$ and $u_{n+1} = g(u_n)$. Moreover, we have seen in Chapter 4 that $u_n \in I$ for all n, so that $(u_n)_n$ is bounded, and that (u_n) is monotonic because g is increasing. Therefore $(u_n)_n$ converges to $\ell \in [\alpha, b]$. Since g is continuous at ℓ, we have $\ell = g(\ell)$, that is, $f(\ell) = 0$ hence $\ell = \alpha$. The sequence $(u_n)_n$ is therefore non-increasing and its limit is α. ✓

Corollary 4. Let $f : [a,b] \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ such that $f(a)f(b) < 0$, $f'(x) \neq 0$ and $f''(x) \neq 0$ for all $x \in [a,b]$.
Then there exists a unique $\alpha \in]a,b[$ such that $f(\alpha) = 0$ and α is the limit of the sequence $(u_n)_n$ defined by $u_0 \in [a,b]$ such that $(u_0 - \alpha)f(b)f'' > 0$ and $u_{n+1} = g(u_n)$ where $g(x) = x - \frac{f(x)}{f'(x)}$ for all $x \in [a,b]$.

Proof. Note that f' and f'' are continuous and do not vanish on $[a,b]$, therefore they have constant sign by the Intermediate Value Theorem. Moreover, the sign of f' is the same as the sign of $f(b)$ (if $f' > 0$ then f is increasing so that the hypotheses imply $f(a) < 0$ and $f(b) > 0$, and similarly when $f' < 0$). There are four cases:

- First case: $f' > 0$ and $f'' > 0$. This is the case in the proposition.
- Second case: $f' < 0$ and $f'' < 0$. This case is obtained from the previous one by replacing f with $-f$, which does not change g. Moreover, in this case u_0 is also chosen in $]\alpha, b]$, and we have $(u_0 - \alpha)f(b)f'' > 0$.
- Third case: $f' > 0$ and $f'' < 0$. Here we follow the proof of the proposition, adapting it where necessary. In this case, $g' > 0$ on the interval $[a, \alpha[$ and $g([a,\alpha]) \subset [a, \alpha]$, and we change I to $[a, \alpha]$.
- Fourth case: $f' < 0$ and $f'' > 0$. This case is obtained from the previous one by replacing f with $-f$, which does not change g. Moreover, in this case u_0 is also chosen in $[a, \alpha[$, and we have $(u_0 - \alpha)f(b)f'' > 0$. ✓

Remark. The final condition says only that u_0 must be chosen in a sub-interval of $[a,b]$ on which f and f'' have the same sign (*ie.* $f(u_0)f''(u_0) > 0$).

Example. We can use Newton's method to compute $\sqrt{2}$.

Consider $x \mapsto x^2 - 2$. It is of class $\mathcal{C}^2$, with derivative $x \mapsto 2x$ and second derivative $x \mapsto 2 > 0$. We restrict our study to $[1, 2]$ so as to satisfy the hypotheses in the proposition.

Therefore we define $f\colon [1, 2] \to \mathbb{R}$ by $f(x) = x^2 - 2$. We have $g(x) = x - \frac{x^2-2}{2x} = \frac{1}{2}\left(x + \frac{2}{x}\right)$. We know that f has a unique root $\alpha = \sqrt{2}$ and that it is the limit of a sequence $(u_n)_n$ defined by $u_0 \in [\sqrt{2}, 2]$ and $u_{n+1} = g(u_n) = \frac{1}{2}(u_n + \frac{2}{u_n})$.

(In this example, we can choose *eg.* $u_0 = 2$ (or anything larger than $\sqrt{2}$).)

B. Study of convergence

The functions $|f'|$ and $|f''|$ are continuous and positive on $[a, b]$, hence

- there exists $M = \sup_{x \in [a,b]} |f''(x)| = \max_{x \in [a,b]} |f''(x)| > 0$;
- there exists $m = \inf_{x \in [a,b]} |f'(x)| = \min_{x \in [a,b]} |f'(x)| > 0$.

By Taylor's inequality (Corollary 9.26), we have

$$\left|f(\alpha) - f(u_n) - (\alpha - u_n)f'(u_n)\right| \leqslant \frac{1}{2!}|\alpha - u_n|^2 M$$

and therefore, dividing by $|f'(u_n)|$ and using the fact that $f(\alpha) = 0$ we get

$$|\alpha - u_{n+1}| = \left|\alpha - u_n + \frac{f(u_n)}{f'(u_n)}\right| \leqslant \frac{M}{2}\frac{|\alpha - u_n|^2}{|f'(u_n)|} \leqslant \frac{M}{2m}|\alpha - u_n|^2.$$

It then follows by induction that

$$|u_n - \alpha| \leqslant \left(\frac{1}{2}\frac{M}{m}\right)^{2^n - 1}(u_0 - \alpha)^{2^n}$$

for all $n \in \mathbb{N}$.

The convergence is said to be **quadratic** (the power of $u_n - \alpha$ is 2). The convergence is fast if $|u_0 - \alpha|$ is chosen small enough to compensate for $\frac{1}{2}\frac{M}{m}$ if it is large.

Example. If $f : [1, 2] \to \mathbb{R}$ is defined by $f(x) = x^2 - 2$ and if we take $u_0 = 2$, then the number of terms we would need to compute to approach α with a precision of 10^{-2} is 3 since we have $M = 2$, $m = 2$, $0 < u_0 - \alpha < b - a = 1$ so that $0 \leqslant u_n - \alpha \leqslant \frac{1}{2^{2^n - 1}}(2 - \alpha)^{2^n} < \frac{1}{2^{2^n-1}}$ and $\frac{1}{2^{2^n-1}} \leqslant 10^{-2} \iff n \geqslant \frac{\ln\left(\frac{2\ln 10 + \ln 2}{\ln 2}\right)}{\ln 2} \cong 2{,}9 \iff n \geqslant 3$.

We would then have $u_0 = 2$, $u_1 = \frac{3}{2}$, $u_2 = \frac{17}{12}$ and $u_3 = \frac{577}{408} \cong 1{,}41$.

Since $f'' = 2 > 0$, the remark above says that we must choose u_0 such that $f(u_0) > 0$. Note that this implies that $u_0 > \alpha$ since f is increasing.

For example, we could choose $u_0 = \frac{3}{2}$. In this case, we would only need to compute 2 terms. Indeed, this time $0 < u_0 - \alpha < u_0 - a = \frac{1}{2}$ so that $0 \leqslant u_n - \alpha \leqslant \frac{1}{2^{2^n-1}}(2 - \alpha)^{2^n} < \frac{1}{2^{2^n-1}}\frac{1}{2^{2^n}} = \frac{1}{2^{2^{n+1}-1}}$ and $\frac{1}{2^{2^{n+1}-1}} \leqslant 10^{-2} \iff n + 1 \geqslant \frac{\ln\left(\frac{2\ln 10 + \ln 2}{\ln 2}\right)}{\ln 2} \cong 2{,}9 \iff n \geqslant 2$.

We would then have $u_0 = \frac{3}{2}$, $u_1 = \frac{17}{12}$, $u_2 = \frac{577}{408} \cong 1{,}41$.

Chapter 6

Polynomials

In this chapter, $\mathbb{K}$ is equal to $\mathbb{C}$ or $\mathbb{R}$ or $\mathbb{Q}$.

We shall study arithmetic properties of $\mathbb{K}[X]$, similar to those of $\mathbb{Z}$, and then consider rational functions. These will be useful for integration (as well as other things you shall see next year, *eg.* linear algebra).

Note that most of the results and definitions that follow are the same as those for integers, replacing polynomials by integers and degrees of polynomials by absolute values of integers. See Appendix C for more details.

I. Arithmetics in $\mathbb{K}[X]$

A. Greatest common divisors

Notation. Let A be a polynomial[†] in $\mathbb{K}[X]$. We denote by $\mathcal{D}(A)$ the set of divisors of A in $\mathbb{K}[X]$. Recall from the first semester that

- $\mathcal{D}(0) = \mathbb{K}[X]$;
- if $A \neq 0$ then any polynomial B in $\mathcal{D}(A)$ has degree at most $\deg A$.

Recall that a polynomial is called **monic**[‡] if its leading coefficient[§] is 1.

Recall that two non-zero polynomials A and B are said to be **associate**[¶] if there exists $\lambda \in \mathbb{K}^*$ such that $B = \lambda A$. Given two non-zero polynomials A and B, the following properties are equivalent.

(i) A and B are associate.

(ii) A divides B and B divides A.

(iii) A divides B and $\deg A \geqslant \deg B$.

Proof. First assume that *(i)* is satisfied. Then there exists $\lambda \in \mathbb{K}^*$ such that $B = \lambda A$. In particular, A divides B. Moreover, $A = \frac{1}{\lambda} B$ so that B divides A. Therefore *(ii)* is satisfied.

Next assume that *(ii)* is satisfied. Then obviously A divides B. In particular, $\deg A \leqslant \deg B$. Moreover, since B divides A we also have $\deg B \leqslant \deg A$. Therefore $\deg A = \deg B$. Therefore *(iii)* is satisfied.

Finally, assume that *(iii)* is satisfied. Since A divides B there exists a non-zero polynomial C such that $B = AC$. Moreover, $\deg B = \deg(AC) = \deg A + \deg C$ and by assumption $\deg A \geqslant \deg B$ therefore $\deg C \leqslant 0$; since $C \neq 0$ we have $\deg C = 0$ that is, C is a non-zero constant. Therefore A and $B = AC$ are associate so that *(i)* is satisfied. ✓

If A is a non-zero polynomial, there is a unique monic polynomial which is associate to A. We shall denote it by A°.

In the sequel, we shall consider two polynomials in $\mathbb{K}[X]$ with at least one of them non-zero.

Remark. Let A and B be two polynomials in $\mathbb{K}[X]$, at least one of which is non-zero. Then $\mathcal{D}(A) \cap \mathcal{D}(B)$ contains only non-zero polynomials, and they all have degree at most $\min(\deg A, \deg B)$ if A and B are non-zero, or $\deg A$ if $B = 0$. In particular, the set $\{\deg C; C \in \mathcal{D}(A) \cap \mathcal{D}(B)\}$ is a non-empty subset of $\mathbb{N}$ which is bounded above in $\mathbb{R}$, therefore it has a maximum, $d \geqslant 0$. This means that there exists a polynomial $D \in \mathcal{D}(A) \cap \mathcal{D}(B)$ whose degree is equal to d.

Definition 1. *Let A and B be two polynomials in $\mathbb{K}[X]$ with at least one of them non-zero.*
*Any polynomial $D \in \mathcal{D}(A) \cap \mathcal{D}(B)$ with maximum degree d is called a **greatest common divisor**[a] (or **gcd**[b]) of A and B.*

[a] plus grand commun diviseur
[b] pgcd

Two polynomials have many greatest common divisors.

[†] polynôme
[‡] unitaire
[§] coefficient dominant
[¶] associé

Lemma 2. Let A and B be two polynomials in $\mathbb{K}[X]$, at least one of which is non-zero. Let D be a gcd of A and B. Let P be a polynomial in $\mathbb{K}[X]$.
If P and D are associates, then P is a gcd of A and B.
In particular, P and D have at least one monic gcd, D°.

Proof. If P is an associate of D, then we have $P = \lambda D$ with $\lambda \in \mathbb{K}^*$. By assumption, $A = DA_1$, $B = DB_1$ and $\deg D = d$, therefore $A = P(\lambda^{-1}A_1)$, $B = P(\lambda^{-1}B_1)$ so that $P \in \mathcal{D}(A) \cap \mathcal{D}(B)$, and $\deg P = \deg D = d$ is maximal therefore P is a gcd of A and B. ✓

B. Euclidean algorithm

Recall that for any polynomial A and any polynomial $B \neq 0$, there exists a unique pair of polynomials (Q, R) such that $\begin{cases} A = QB + R & \text{and} \\ \deg R < \deg B. \end{cases}$

Proposition 3. Let A, B and Q be three polynomials. Then $\mathcal{D}(A) \cap \mathcal{D}(B) = \mathcal{D}(B) \cap \mathcal{D}(A - QB)$.
In particular, if R is the remainder[a] of the Euclidean divison of A by B, then $\mathcal{D}(A) \cap \mathcal{D}(B) = \mathcal{D}(B) \cap \mathcal{D}(R)$.

[a] reste

Proof. If $P \in \mathcal{D}(A) \cap \mathcal{D}(B)$ then P divides A and P divides B, therefore P divides $A - QB$. Therefore $P \in \mathcal{D}(B) \cap \mathcal{D}(A - QB)$. We have proved that $\mathcal{D}(A) \cap \mathcal{D}(B) \subset \mathcal{D}(B) \cap \mathcal{D}(A - QB)$.

If $P \in \mathcal{D}(B) \cap \mathcal{D}(A - QB)$ then P divides A and B divides $A - QB$, therefore P divides $QB + (A - QB) = A$. Therefore $P \in \mathcal{D}(B) \cap \mathcal{D}(A)$. We have proved that $\mathcal{D}(B) \cap \mathcal{D}(A - QB) \subset \mathcal{D}(A) \cap \mathcal{D}(B)$.

Finally we have the required equality. ✓

Proposition 4 (Euclidean algorithm[a]). Let A and B be two non-zero polynomials.
Define the following sequence of polynomials, defined inductively by: $R_0 = A$ and $R_1 = B$. For $k \geqslant 1$, assume that R_{k-1} and R_k are known; if $R_k = 0$, set $R_{k+1} = 0$; if $R_k \neq 0$, let R_{k+1} be the remainder of the Euclidean division of R_{k-1} by R_k, so that $R_{k-1} = Q_k R_k + R_{k+1}$ and $\deg R_{k+1} < \deg R_k$.
Then there exists $n \in \mathbb{N}$ such that $R_n \neq 0$ and $R_{n+1} = 0$. Moreover, R_n is a gcd for A and B.

[a] algorithme d'Euclide

Proof. (Not done in class.)

Assume for a contradiction the for all $n \in \mathbb{N}$ we have $R_n \neq 0$. We then have a decreasing sequence $(\deg R_n)_n$ of elements in $\mathbb{N}$, which is impossible (every non-increasing sequence of elements in $\mathbb{N}$ converges – because it is bounded below –, therefore it is stationary – since it consists of integers –). Therefore there is an integer n such that $R_{n+1} = 0$, and choosing n minimal we have $R_n \neq 0$ (by assumption, there are some non-zero R_n, namely R_0 and R_1).

We now prove that R_n is a gcd for $R_0 = A$ and $R_1 = B$.

➢ We have $R_{n-1} = QR_n$ so that R_n divides R_{n-1}. Therefore R_n divides R_k for all $k \geqslant n-1$. Now take $N \leqslant n-1$ and assume that R_n divides R_k for all $k \geqslant N$. We have $R_{N-1} = Q_N R_N + R_{N+1}$ and R_n divides R_N and R_{N+1}, therefore R_n divides R_{N_1}. Therefore, inductively, R_n divides all the R_k for $k \geqslant 0$.

  In particular, R_n divides A and B.

➢ We must prove that $\deg R_n$ is maximal among the degrees of elements in $\mathcal{D}(A) \cap \mathcal{D}(B)$. Let P be an element in $\mathcal{D}(A) \cap \mathcal{D}(B)$. Then P divides $A = R_0$ and $B = R_1$ therefore it divides R_2 by the previous proposition. Now take $N \geqslant 1$ and assume that P divides R_k for all $k \leqslant N$. Then R_{N+1} is the remainder of the Euclidean division of R_{N-1} by R_N and P divides R_{N-1} and R_N, so that P divides R_{N+1}. By induction, P divides all the R_k and therefore P divides R_n.

  In particular, $\deg P \leqslant \deg R_n$. Therefore $\deg R_n$ is an upper bound for $\{\deg P; P \in \mathcal{D}(A) \cap \mathcal{D}(B)\}$ and since $R_n \in \mathcal{D}(A) \cap \mathcal{D}(A)$, $\deg R_n$ is a maximum. ✓

Example. We want to find a gcd of $A = X^5 - 2X^4 + X^3$ and $B = X^3 - X$.
The Euclidean division of A by B gives $A = BQ_1 + R_2$ with $Q_1 = X^2 - 2X + 2$ and $R_2 = -2X^2 + 2X$.
The Euclidean division of B by R_2 gives $B = R_2 Q_2 + R_3$ with $Q_2 = \frac{-1}{2}X - \frac{1}{2}$ and $R_3 = 0$.
Therefore $R_2 = -2X^2 + 2X$ is a gcd for A and B (and $X^2 - X$ is a monic gcd for A and B).

Corollary 5. Let A and B be two polynomials in $\mathbb{K}[X]$ with at least one of them non-zero. Let D be a gcd of A and B. Then $\mathcal{D}(D) = \mathcal{D}(A) \cap \mathcal{D}(B)$.

Proof. *(Not done in class.)*
We already know that $\mathcal{D}(D) \subset \mathcal{D}(A) \cap \mathcal{D}(B)$.
Now let R_n be defined as in the previous proposition (Euclidean algorithm). We have seen in the proof that R_n is a gcd of A and B, therefore $\mathcal{D}(R_n) \subset \mathcal{D}(A) \cap \mathcal{D}(B)$, and also that any $P \in \mathcal{D}(A) \cap \mathcal{D}(B)$ divides R_n hence $\mathcal{D}(A) \cap \mathcal{D}(B) \subset \mathcal{D}(R_n)$. Therefore $\mathcal{D}(A) \cap \mathcal{D}(B) = \mathcal{D}(R_n)$.
In particular, $\mathcal{D}(D) \subset \mathcal{D}(R_n)$ so that D divides R_n. Moreover, they both have the same degree, so that D and R_n are associates. Hence $\mathcal{D}(D) = \mathcal{D}(R_n) = \mathcal{D}(A) \cap \mathcal{D}(B)$. ✓

Remark. The previous result shows that a gcd D of A and B can be characterised by the two properties:

(i) D divides A and B, and

(ii) if P is any polynomial that divides A and B, then P divides D.

Proof. We must prove that D is a gcd of A and B if, and only if, it satisfies *(i)* and *(ii)*.

- ➢ First assume that D is a gcd of A and B. Then *(i)* is satisfied by definition. Moreover, we have $\mathcal{D}(D) = \mathcal{D}(A) \cap \mathcal{D}(B)$, therefore, if P is a common divisor of A and B, then $P \in \mathcal{D}(A) \cap \mathcal{D}(B)$ hence $P \in \mathcal{D}(D)$ and finally P divides D and *(ii)* is satisfied.

- ➢ Conversely, assume that *(i)* and *(ii)* are satisfied. Then D divides A and B by assumption, and we must prove that $\deg D$ is maximal among the degrees of divisors of A and B. But if P divides A and B, then P divides D by assumption *(ii)*, so that $\deg P \leqslant \deg D$, as required. ✓

Definition-Proposition 6. *Let A and B be two polynomials in $\mathbb{K}[X]$, at least one of which is non-zero. Let D be a gcd of A and B. Let P be a polynomial in $\mathbb{K}[X]$.*
Then P is a gcd of A and B if, and only if, P and D are associates.
In particular, A and B have a unique monic gcd, D°, denoted by $A \wedge B$.
By convention, we set $0 \wedge 0 = 0$.

Proof. This follows from the proof of the previous result. ✓

Properties 7. **(a)** For any polynomials A and B we have $A \wedge B = B \wedge A$.

(b) For any non-zero polynomial A, we have $A \wedge 0 = A^\circ$.

(c) For any polynomial A, we have $A \wedge 1 = 1$.

(d) For any non-zero polynomial A, $A \wedge B = A^\circ$ if, and only if, A divides B.

(e) The polynomial $A \wedge B$ and its divisors are divisors common to A and B.

Proof. **(a)** This is clear, since $\mathcal{D}(A) \cap \mathcal{D}(B) = \mathcal{D}(B) \cap \mathcal{D}(A)$.

(b) We know that $\mathcal{D}(0) = \mathbb{K}[X]$ so that $\mathcal{D}(A) \cap \mathcal{D}(0) = \mathcal{D}(A)$, therefore $A \wedge 0$ is the monic polynomial that divides A and whose degree is $\deg A$, therefore it is A°.

(c) We have $\mathcal{D}(1) = \mathbb{K}$ hence $\mathcal{D}(A) \cap \mathcal{D}(1) = \mathbb{K}$ and therefore $A \wedge 1$ is the monic polynomial in $\mathbb{K}$, that is, 1.

(d) Assume that A divides B. Then $\mathcal{D}(A) \subset \mathcal{D}(B)$ so that $\mathcal{D}(A) \cap \mathcal{D}(B) = \mathcal{D}(A)$ and therefore $A \wedge B = A^\circ$.
Conversely, if $A \wedge B = A^\circ$ then A° divides B, and therefore A (which is an associate of A°) divides B.

(e) Clear. ✓

Proposition 8. Let A and B be two polynomials in $\mathbb{K}[X]$. For any non-zero polynomial P, we have $(PA) \wedge (PB) = P^\circ(A \wedge B)$.

Proof. Set $D = A \wedge B$ and let Δ be a gcd of PA and PB.
Since D divides A and B, PD divides PA and PB, therefore PD divides Δ. We must prove that $\deg(PD) = \deg \Delta$.
Since PD divides Δ, in particular P divides Δ, so that we can write $\Delta = PC$ for some polynomial C. Moreover, $PC = \Delta$ divides PA and PB, therefore C divides A and B and finally C divides D. In particular, $\deg C \leqslant \deg D$, so that $\deg(PD) \geqslant \deg PC = \deg \Delta$, as required.
We have proved that PD is a gcd of PA and PB, so that $(PA) \wedge (PB) = (PD)^\circ = P^\circ D$ since D is already monic. ✓

Example. Going back to the example following Proposition 4, we wanted to find a greatest common divisor for $A = X^5 - 2X^4 + X^3$ and $B = X^3 - X$. Note that we have $A = X^3(X^2 - 2X + 1) = X^3(X-1)^2$ and $B = X(X^2-1) = X(X-1)(X+1)$. Therefore, with $P = X(X-1)$ (monic), we have $A \wedge B = X(X-1) \cdot \left(X^2(X-1) \wedge (X+1)\right)$.

The Euclidean division of $X^2(X-1) = X^3 - X^2$ by $X+1$ gives $X^3 - X^2 = (X+1)(X^2 - 2X + 2) - 2$. Since the remainder is a constant, it must be a greatest common divisor for $X^2(X-1)$ and $X+1$, therefore $X^2(X-1) \wedge (X+1) = 1$ (we shall see later other ways of proving this).

Finally, $A \wedge B = X(X-1)$.

Proposition 9. Let A and B be two polynomials in $\mathbb{K}[X]$. Then there exist polynomials U and V such that $A \wedge B = UA + VB$.
This is called a **Bézout relation**[a] between A and B and the polynomials U and V are called the **Bézout coefficients**[b] for A and B.

[a] relation de Bézout
[b] coefficients de Bézout

Proof. (Not done in class.) We use the notation in the statement and proof of the Euclidean algorithm. We know that R_n is a gcd for A and B. We prove by (descending) induction that for all k with $1 \leqslant k \leqslant n-1$, we have $R_n = U_k R_k + V_k R_{k-1}$ for some polynomials U_k and V_k.

For $k = n-1$, we have $R_n = -Q_{n-1}R_{n-1} + R_{n-2}$ as required.

Now assume that it is true for some k. We must prove it for $k-1$.

We know that $R_k = -Q_{k-1}R_{k-1} + R_{k-2}$, therefore

$$R_n = U_k R_k + V_k R_{k-1} = (V_k - U_k Q_{k-1})R_{k-1} + U_k R_{k-2}$$

as required (put $U_{k-1} = V_k - U_k Q_{k-1}$ and $V_{k-1} = U_k$.)

In particular, this is true for $k=1$, that is, $R_n = U_1 R_1 + V_1 R_0 = U_1 B + V_1 A$.

Finally, we divide by the leading coefficient of R_n, so that $A \wedge B = V_1{}^\circ A + U_1{}^\circ B$. ✓

Remark. It follows from the proof that, to find the Bézout coefficients for A and B, we can use the Euclidean algorithm then work backwards, as we do for integers.

Example. We want tof find $A \wedge B$ and the Bézout coefficients for $A = X^4 - 2X^3 + X^2 + 3X - 1$ and $B = X^3 - 2X^2 + 3$.

We have

$$\begin{aligned} A &= XB + (X^2 - 1) \\ B &= (X-2)(X^2-1) + (X+1) \\ X^2 - 1 &= (X-1)(X+1) + 0 \end{aligned}$$

therefore $A \wedge B = X + 1$ (the last non-zero remainder, normalised). Moreover, working up the equalities above, we get:

$$\begin{aligned} X + 1 &= B - (X-2)(X^2-1) \\ &= B - (X-2)(A - XB) \\ &= -(X-2)A + (X^2 - 2X + 1)B. \end{aligned}$$

C. Least common multiples

In the sequel, we shall consider non-zero polynomials A and B.

Notation. The set $\mathcal{M}(A) = A\mathbb{K}[X]$ is the set of multiples of A.

Remark. Let A and B be two non-zero polynomials in $\mathbb{K}[X]$.

The set $\mathcal{M}(A) \cap \mathcal{M}(B)$ is the set of common multiples of A and B. It contains 0 as well as some non-zero polynomials (such as AB). The non-zero polynomials have degree at least $\max(\deg A, \deg B)$.

In particular, the set $\{\deg C; C \in \mathcal{M}(A) \cap \mathcal{M}(B), C \neq 0\}$ is a non-empty subset of $\mathbb{N}$ which is bounded below in $\mathbb{R}$, therefore it has a minimum, $m \geqslant 0$. This means that there exists a polynomial $M \in \mathcal{M}(A) \cap \mathcal{M}(B)$ whose degree is equal to m.

Definition 10. *Let A and B be two non-zero polynomials in $\mathbb{K}[X]$.*
Any polynomial $M \in \mathcal{M}(A) \cap \mathcal{M}(B)$ with minimum degree m is called a ***least common multiple***[a] *(or* ***lcm***[b]*) of A and B.*

[a] plus petit commun multiple
[b] ppcm

Two polynomials have many least common multiples.

Definition-Proposition 11. *Let A and B be two non-zero polynomials in $\mathbb{K}[X]$. Let M be an lcm of A and B. Let P be a polynomial in $\mathbb{K}[X]$.*
Then P is an lcm of A and B if, and only if, P and M are associates.
In particular, A and B have a unique monic lcm, M°, denoted by $A \vee B$.

Proof. If P and M are associates, then P is clearly a common multiple of A and B with the same degree as M so that P is an lcm of A and B.

Conversely, assume that P is an lcm of A and B. There exist polynomials Q and R such that $P = QM + R$ with $\deg R < \deg M$. Moreover, P and M are multiples of A and B, therefore R is a multiple of A and B. If $R \neq 0$, then by minimality of degree of M we must have $\deg R \geqslant \deg M$, a contradiction. Therefore $R = 0$ so that $P = QM$, that is, M divides P. Moreover, $\deg M = \deg P$ therefore M and P are associates. ✓

Notation. By convention, we set $A \vee 0 = 0$ for $A \neq 0$.

Properties 12.

- ➢ For any polynomials A and B, one of which is non-zero, we have $A \vee B = B \vee A$.
- ➢ For any non-zero polynomial A, we have $A \vee 1 = A^\circ$.
- ➢ For any non-zero polynomial A, $A \vee B = A^\circ$ if, and only if, B divides A.

Proof. Exercise. ✓

Proposition 13. Let A and B be two non-zero polynomials in $\mathbb{K}[X]$, and let C be any polynomial.
Then C is an lcm for A and B if, and only if, $\mathcal{M}(C) = \mathcal{M}(A) \cap \mathcal{M}(B)$.

Proof. Let M be an lcm for A and B.

- ➢ Since M is a common multiple of A and B, it is clear that all its multiples are common multiples of A and B, therefore $\mathcal{M}(M) \subset \mathcal{M}(A) \cap \mathcal{M}(B)$.

  Conversely, take $P \in \mathcal{M}(A) \cap \mathcal{M}(B)$. Then there exist polynomials Q and R such that $P = QM + R$ with $\deg R < \deg M$. The same argument as in the proof ot the previous proposition shows that M divides P, that is, $P \in \mathcal{M}(M)$. We have proved that $\mathcal{M}(A) \cap \mathcal{M}(B) \subset \mathcal{M}(M)$.

- ➢ Now assume that C is a polynomial such that $\mathcal{M}(C) = \mathcal{M}(A) \cap \mathcal{M}(B)$. We have just shown that also have $\mathcal{M}(M) = \mathcal{M}(A) \cap \mathcal{M}(B)$. Therefore C and M are multiples of each other. Consequently, they are associates. Since M is an lcm for A and B, so is C.

✓

Remark. Let A and B be two non-zero polynomials in $\mathbb{K}[X]$. The polynomial $A \vee B$ is the unique monic polynomial such that $\mathcal{M}(A \vee B) = \mathcal{M}(A) \cap \mathcal{M}(B)$.

Remark. The previous result shows that an lcm M of A and B can be characterised by the two properties:

(i) A and B divide M, and

(ii) if P is any polynomial such that A and B divide P, then M divides P.

[Not done in class.]

Proof. We must prove that M is an lcm of A and B if, and only if, it satisfies *(i)* and *(ii)*.

- ➢ First assume that M is an lcm of A and B. Then *(i)* is satisfied by definition. Moreover, we have $\mathcal{M}(M) = \mathcal{M}(A) \cap \mathcal{M}(B)$, therefore, if P is a common multiple of A and B, then $P \in \mathcal{M}(A) \cap \mathcal{M}(B)$ hence $P \in \mathcal{M}(M)$ and finally M divides P and *(ii)* is satisfied.

- ➢ Conversely, assume that *(i)* and *(ii)* are satisfied. Then M is a multiple of A and B by assumption, and we must prove that $\deg M$ is minimal among the degrees of non-zero multiples of A and B. But if P is a multiple of A and B, then P is a multiple of M by assumption *(ii)*, so that $\deg P \geqslant \deg M$, as required. ✓

Proposition 14. Let A and B be two non-zero polynomials in $\mathbb{K}[X]$. For any non-zero polynomial P, we have $(PA) \vee (PB) = P^\circ(A \vee B)$.

Proof. *[Not done in class.]* Set $M = A \vee B$ and let N be an lcm of PA and PB.

Since M is a multiple of A and B, PM is a multiple of PA and PB. We must prove that its degree is minimal.

Let C be a common multiple of PA and PB. Then P divides C, wo that $C = PC_1$ for some polynomial C_1. Moreover, A divides C_1 and B divides C_1, therefore $C_1 \in \mathcal{M}(A) \cap \mathcal{M}(B) = \mathcal{M}(M)$. In particular, $\deg M \leqslant \deg C_1$ so that $\deg(PM) \leqslant \deg(PC_1) = \deg C$. This proves that $\deg(PM)$ is minimal among the degrees of the non-zero multiples of PA and PB, therefore PM is an lcm for PA and PB.

Finally, $(PA) \vee (PB) = (PM)^\circ = P^\circ M$ since M is already monic. ✓

Proposition 15. Let A and B be two non-zero polynomials in $\mathbb{K}[X]$. Then

$$(A \wedge B) \cdot (A \vee B) = (AB)^\circ.$$

Proof. *[Not done in class.]* Set $M = A \vee B$ and $D = A \wedge B$. Since AB is a common multiple of A and B, it is a multiple of M. We can write $AB = MC$. To prove the result, we need only prove that C and D are associates, that is, that C is a gcd of A and B.

First, let us check that C is a common divisor of A and B. We can write $M = AA_1$, therefore $AB = MC = AA_1C$ and $B = A_1C$ so that C divides B; similarly, C divides A.

Now let P be a divisor of A and B. Then P also divides AB. We can write $A = PA_2$, $B = PB_2$ and $AB = PQ$.

We then have $PQ = AB = PA_2B = APB_2$, hence $Q = A_2B = AB_2$, so that A and B divide Q. Therefore M divides Q, and we can write $Q = MQ_1$.

We now have $MC = AB = PQ = PMQ_1$, therefore $C = PQ_1$, that is, P divides C.

We have proved that any common divisor of A and B divides C, therefore C is a gcd of A and B. ✓

D. Coprime polynomials

Definition 16. *Let A and B be two polynomials. We say that A and B are* ***coprime***[a] *if $A \wedge B = 1$.*
In other words, the only common divisors of A and B are the non-zero constant polynomials.

[a] premiers entre eux

Theorem 17 (Bézout Theorem[a])**.** Let A and B be two polynomials. Then A and B are coprime if, and only if, there exist two polynomials U and V such that $AU + BV = 1$.

[a] théorème de Bézout

Proof. If A and B are coprime, that is, 1 is a gcd of A and B, we have already seen that there exist U and V such that $1 = AU + BV$.

Now assume that there exist U and V such that $1 = AU + BV$. Let P be a common divisor of A and B. Then P divides $UA + BV$ and therefore P divides 1, so that P must be a constant. Therefore $A \wedge B = 1$. ✓

Proposition 18 (Gauss' Lemma[a])**.** Let A, B and C be three polynomials.
If A divides BC and if A and B are coprime, then A divides C.

[a] lemme de Gauss

Proof. Since A and B are coprime, there exist U and V such that $1 = AU + BV$. Multiplying by C gives $C = AUC + BVC = A \cdot UC + BC \cdot V$. Since A divides A and BC, it divides $A \cdot UC + BC \cdot V$, therefore A divides C. ✓

Proposition 19. Let A, B and C be three polynomials. The following are equivalent:

(i) A and B are coprime and A and C are coprime;

(ii) A and BC are coprime.

More generally, let $A_1, \ldots, A_p$ and $B_1, \ldots, B_n$ be $p + n$ polynomials. The following are equivalent:

(i) A_j and B_k are coprime for all j, k with $1 \leqslant j \leqslant p$ and $1 \leqslant k \leqslant n$;

(ii) $A_1 \cdots A_p$ and $B_1 \cdots B_n$ are coprime.

Proof. We first prove the case with three polynomials.

(i) Assume that *(i)* holds. Then there exist polynomials U, V, W, T such that $AU + BV = 1$ and $AW + CT = 1$. Multiplying these two identities gives $A(AUW + UCT + VWB) + (BC)(VT) = 1$, so that A and BC are coprime.

(ii) Conversely, assume that *(ii)* holds. Let P be a common divisor of A and B. Then P divides A and BC so that by assumption P must be a constant. Therefore A and B are coprime. Similarly, A and C are coprime.

In the general case *[not done in class]*, the fact that *(ii)⇒(i)* is similar to the one above.

Next, assume that $p = 1$ and $n \geqslant 2$ (or the opposite). The proof that *(i)⇒(ii)* can be done in a similar way to the first case, or by induction on $n \geqslant 2$, the first case being the case $n = 2$.

Finally, when $p \geqslant 2$ and $n \geqslant 2$, the case $p = 1$ shows that each A_j is coprime to $B_1 \cdots B_n$, therefore, applying the case $n = 1$, $B_1 \cdots B_n$ and $A_1 \cdots A_p$ are coprime. ✓

Definition 20. *Let $A_1, \ldots, A_n$ be a family of polynomials. We say that they are* ***pairwise coprime***[a] *if any two of them are coprime, that is,*

$$\forall i, j,\ 1 \leqslant i < j \leqslant n,\ A_i \text{ and } A_j \text{ are coprime.}$$

[a] premiers entre eux deux à deux

Proposition 21. Let A, B and C be three polynomials. Assume that A and B are coprime.
The polynomial C is a multiple of A and B if, and only if, it is a multiple of AB.
More generally, if $A_1, \ldots, A_n$ is a family of pairwise coprime polynomials, then C is a multiple of each of the A_k if, and only if, it is a multiple of their product $A_1 A_2 \cdots A_n$.

Proof. If AB divides C then A divides C and B divides A (with no assumption on $A \wedge B$).

Conversely, assume that A and B are coprime and that A and B both divide C. Then $A \vee B$ divides C. But we know that $AB = (A \wedge B)(A \vee B) = A \vee B$. Therefore AB divides C.

For the general case *[not done in class]*, one implication is always true, we prove the other one by induction on $n \geqslant 2$. We have already done the case $n = 2$.

Assume the result true at the stage $n \geqslant 2$. Let $A_1, \ldots, A_{n+1}$ be a family of pairwise coprime polynomials such that C is a multiple of each of the A_k. We must prove that C is a multiple of their product $A_1 A_2 \cdots A_{n+1}$.

Set $B = A_2 \ldots A_{n+1}$. The polynomials $A_2, \ldots, A_{n+1}$ are n pairwise coprime polynomials such that each A_k with $k \geqslant 2$ divides C. Therefore by induction hypothesis, B divides C. Moreover, A_1 and B are coprime by Proposition 19, both divide C, therefore $A_1 B$ divides C (case $n = 2$), that is, $A_1 A_2 \cdots A_{n+1}$ divides C. ✓

Remark. Let A and B be two non-zero polynomials. There exist polynomials A_1 and B_1 such that $A = (A \wedge B) A_1$ and $B = (A \wedge B) B_1$.

Then A_1 and B_1 are coprime.

Indeed, setting $D = A \wedge B$, if P divides A_1 and B_1, then DP divides $DA_1 = A$ and $DB_1 = B$, therefore DP divides D and finally P divides 1, that is, P is a constant. Therefore $A_1 \wedge B_1 = 1$.

Proposition 22. Let a and b be two distinct elements of $\mathbb{K}$. Then $(X - a) \wedge (X - b) = 1$.
More generally, let $a_1, \ldots, a_n, b_1, \ldots, b_p$ be $n + p$ pairwise distinct elements of $\mathbb{K}$. Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_p$ $n + p$ be non-negative integers. Then the polynomials $A = (X - a_1)^{\alpha_1} (X - a_2)^{\alpha_2} \cdots (X - a_n)^{\alpha_n}$ and $B = (X - b_1)^{\beta_1} (X - b_2)^{\beta_2} \cdots (X - b_p)^{\beta_p}$ are coprime.

Proof. If $D = (X - a) \wedge (X - b)$, then $X - a = DA$ and $X - b = DB$, so that $\deg D \leqslant 1$. Moreover, if $\deg D = 1$, then A and B must be constants. The polynomials involved are all monic, therefore $X - a = D = X - b$ and $a = b$, a contradiction. Therefore D is a constant, that is, $D = 1$.

To prove the general case *[not done in class]*, apply Proposition 19 (each of the A_j is a polynomial of the form $X - a_t$ and each of the B_k is a polynomial of the form $X - b_s$, with $a_t \neq b_s$). ✓

Example. Consider $A = X^2(X - 1)$ and $B = X(X + 1)^3$. Then $A \wedge B = X \cdot (X(X - 1)) \wedge ((X + 1)^3)$. The proposition above tells us that $X(X - 1)$ and $(X + 1)^3$ are coprime. Therefore $A \wedge B = X$.

E. Gcd and lcm of more than two polynomials

In order to avoid problems, in the sequel we assume that all polynomials are non-zero.

Proposition 23. For any three polynomials A, B and C, we have

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C$$
$$A \vee (B \vee C) = (A \vee B) \vee C.$$

In other words, gcds and lcms are associative.

Proof. [Not done in class.] The intersection is associative (check this!). Therefore $\mathcal{D}(A \wedge (B \wedge C)) = \mathcal{D}(A) \cap \mathcal{D}(B \wedge C) = \mathcal{D}(A) \cap (\mathcal{D}(B) \cap \mathcal{D}(C)) = (\mathcal{D}(A) \cap \mathcal{D}(B)) \cap \mathcal{D}(C) = \mathcal{D}(A \wedge B) \cap \mathcal{D}(C) = \mathcal{D}((A \wedge B) \wedge C)$. The result for gcds follows.

Similarly, replacing $\mathcal{D}$ by $\mathcal{M}$ and $\wedge$ by $\vee$, we get the result for lcms. ✓

Consequence 24. In particular, for any family of n polynomials $A_1, \ldots, A_n$, we may consider $A_1 \wedge A_2 \wedge \cdots \wedge A_n$ and $A_1 \vee A_2 \vee \cdots \vee A_n$ (without brackets).

Definition 25. *Let $A_1, \dots, A_n$ be a family of n polynomials, with $n \geqslant 2$.*

- *The polynomial $A_1 \wedge A_2 \wedge \cdots \wedge A_n$ is called the* ***greatest common divisor (gcd)*** *of the polynomials $A_1, \dots, A_n$.*
- *The polynomial $A_1 \vee A_2 \vee \cdots \vee A_n$ is called the* ***least common multiple (lcm)*** *of the polynomials $A_1, \dots, A_n$.*

Proposition 26. The gcd and lcm of a family of polynomials are characterised as follows.

- $D = A_1 \wedge A_2 \wedge \cdots \wedge A_n$ is the unique monic polynomial such that $\mathcal{D}(D) = \mathcal{D}(A_1) \cap \cdots \cap \mathcal{D}(A_n)$.
- $M = A_1 \vee A_2 \vee \cdots \vee A_n$ is the unique monic polynomial such that $\mathcal{M}(M) = \mathcal{M}(A_1) \cap \cdots \cap \mathcal{M}(A_n)$.

Proof. [Not done in class.] We prove these characterisations by induction on $n \geqslant 2$. The initial cases, for $n = 2$, are known.

We shall do the induction step for gcds, the proof for lcms is similar.

Let n be an integer with $n \geqslant 2$ and assume that the result is true for n polynomials. Let $A_1, \dots, A_{n+1}$ be $n+1$ polynomials.

The polynomial $A_1 \wedge A_2 \wedge \cdots \wedge A_n$ is the unique monic polynomial such that $\mathcal{D}(D) = \mathcal{D}(A_1) \cap \cdots \cap \mathcal{D}(A_n)$ by induction hypothesis.

The polynomial $A_1 \wedge A_2 \wedge \cdots \wedge A_{n+1} = (A_1 \wedge A_2 \wedge \cdots \wedge A_n) \wedge A_{n+1}$ is the unique monic polynomial such that $\mathcal{D}(A_1 \wedge A_2 \wedge \cdots \wedge A_{n+1}) = \mathcal{D}(A_1 \wedge A_2 \wedge \cdots \wedge A_n) \cap \mathcal{D}(A_{n+1})$ (case $n = 2$).

Therefore, combining both facts, $A_1 \wedge A_2 \wedge \cdots \wedge A_{n+1}$ is the unique monic polynomial such that $\mathcal{D}(A_1 \wedge A_2 \wedge \cdots \wedge A_{n+1}) = \cap_{j=1}^{n+1} \mathcal{D}(A_j)$. ✓

We can extend some of the results for the gcd and lcm of two polynomials.

Proposition 27. Let $A_1, \dots, A_n$ be a family of n polynomials, with $n \geqslant 2$, and let P be a non-zero polynomial. Then

$$(PA_1) \wedge (PA_2) \wedge \cdots \wedge (PA_n) = P^\circ (A_1 \wedge A_2 \wedge \cdots \wedge A_n)$$
$$(PA_1) \vee (PA_2) \vee \cdots \vee (PA_n) = P^\circ (A_1 \vee A_2 \vee \cdots \vee A_n)$$

Proof. [Not done in class.] We prove these results by induction on $n \geqslant 2$. The initial cases, for $n = 2$, are known.

We shall do the induction step for lcms, the proof for gcds is similar.

Let n be an integer with $n \geqslant 2$ and assume that the result is true for n polynomials. Let $A_1, \dots, A_{n+1}$ be $n+1$ polynomials. Then

$$\begin{aligned}
(PA_1) \vee (PA_2) \vee \cdots \vee (PA_{n+1}) &= (PA_1) \vee ((PA_2) \vee \cdots \vee (PA_{n+1})) \\
&= (PA_1) \vee (P^\circ((A_2) \vee \cdots \vee (A_{n+1}))) \quad \text{(induction hypothesis)} \\
&= (P^\circ A_1) \vee (P^\circ((A_2) \vee \cdots \vee (A_{n+1}))) \quad \text{(normalise first polynomial)} \\
&= P^\circ(A_1 \vee ((A_2) \vee \cdots \vee (A_{n+1}))) \quad \text{(case } n = 2\text{)} \\
&= P^\circ(A_1 \vee A_2 \vee \cdots \vee A_{n+1})
\end{aligned}$$
✓

Proposition 28. Let $A_1, \dots, A_n$ be a family of n polynomials, with $n \geqslant 2$. Then there exist polynomials $U_1, \dots, U_n$ such that

$$A_1 \wedge A_2 \wedge \cdots \wedge A_n = A_1 U_1 + A_2 U_2 + \cdots + A_n U_n.$$

Proof. [Not done in class.] We prove this result by induction on $n \geqslant 2$. The initial case, for $n = 2$, is known.

Let n be an integer with $n \geqslant 2$ and assume that the result is true for n polynomials. Let $A_1, \dots, A_{n+1}$ be $n+1$ polynomials.

By induction hypothesis, there exist n polynomials $V_1, \dots, V_n$ such that $A_1 \wedge A_2 \wedge \cdots \wedge A_n = A_1 V_1 + A_2 V_2 + \cdots + A_n V_n$. Using the case $n = 2$, there exist two polynomials U and U_{n+1} such that $(A_1 \wedge A_2 \wedge \cdots \wedge A_n) \wedge A_{n+1} = (A_1 \wedge A_2 \wedge \cdots \wedge A_n) U + A_{n+1} U_{n+1}$.

Taking $U_j = V_j U$ for $1 \leqslant j \leqslant n$ gives the result. ✓

Definition 29. *Let $A_1, \dots, A_n$ be a family of n polynomials, with $n \geqslant 2$. We say that they are* ***relatively prime***[a] *if $A_1 \wedge A_2 \wedge \cdots \wedge A_n = 1$.*

[a] premiers entre eux dans leur ensemble

Proposition 30. Let $A_1, \dots, A_n$ be a family of n polynomials, with $n \geqslant 2$. The following are equivalent:

(i) the polynomials $A_1, \dots, A_n$ are relatively prime;

(ii) there exist n polynomials $U_1, \dots, U_n$ such that $A_1 U_1 + A_2 U_2 + \cdots + A_n U_n = 1$.

Proof. *[Not done in class.]* The implication *(i)⇒(ii)* is true by the previous result.

For the converse, assume that P divides A_i for all i. Then P divides $A_1U_1 + A_2U_2 + \cdots + A_nU_n = 1$ so that P is invertible. Therefore $A_1 \wedge A_2 \wedge \cdots \wedge A_n = 1$ as required. ✓

Example. Consider $A = X^2 + X$, $B = X^2 - 1$ and $C = X^2 - X$. We have

$$A = B + (X+1) \quad \text{and} \quad B = (X+1)(X-1) + 0$$

so that $A \wedge B = X + 1$ and $X + 1 = A - B$.

We have

$$C = (X-2)(X+1) + 2$$

so that $1 = (X+1) \wedge C$ and $1 = \frac{1}{2}C - \frac{1}{2}(X-2)(X+1)$.

Finally, $1 = \frac{1}{2}C - \frac{1}{2}(X-2)A + \frac{1}{2}(X-2)B$.

Remark. The equality $(A \wedge B) \cdot (A \vee B) = (AB)^\circ$ does not generalise to more than two polynomials.

Take for instance $A = B = C = X$. Then $A \wedge B \wedge C = X$ and $A \vee B \vee C = X$ so that $(A \wedge B \wedge C) \cdot (A \vee B \vee C) = X^2$, but $ABC = X^3$.

However, if the polynomials $A_1, \ldots, A_n$ are *pairwise* coprime, then $A_1 \vee A_2 \wedge \cdots \wedge A_n = A_1A_2 \cdots A_n$. This follows from Propositions 21 and 26.

II. Irreducible polynomials and factorisations

Definition 31. *Let P be a polynomial in $\mathbb{K}[X]$. We say that P is* ***irreducible***[a] *if it is not constant and if its only divisors are*

- *the non-zero constant polynomials, and*
- *the polynomials which are associates of P, that is, the λP for $\lambda \in \mathbb{K}^*$.*

A polynomial is ***reducible***[b] *if it is not irreducible.*

[a] irréductible
[b] réductible

Remark. A polynomial P is reducible if, and only if, there exist Q and R such that $\deg Q < \deg P$, $\deg R < \deg P$ and $P = QR$.

Properties 32.

- Any polynomial of degree 1 is irreducible.
- If P is a polynomial of degree 2 or 3, then it is irreducible if, and only if, it has no root[a] in $\mathbb{K}$.
- If P is an irreducible polynomial and if P does not divide a polynomial A, then P and A are coprime.
- Let P be an irreducible polynomial. Let $A_1, \ldots, A_n$ be a family of polynomials. Then P divides the product $A_1 \cdots A_n$ if, and only if, P divides one of the A_i.
- If a polynomial is irreducible, then so are its associates. Therefore if two irreducible polynomials are not associates, then they are coprime. In particular, two distinct monic irreducible polynomials are coprime.

[a] racine

Proof.

- Let P be a polynomial of degree 1. It is not a constant. Moreover, if Q divides P, then $0 \leqslant \deg Q \leqslant 1$.

  If $\deg Q = 0$, then Q is a constant. If $\deg Q = 1$, then P and Q are associates ($P = QR$ for some R with $1 = \deg P = \deg Q + \deg R = 1 + \deg R$ so that R is a constant).

  Therefore the only divisors of P are constants and associates of P.

- Let P be a polynomial of degree 2 or 3.

  First assume that P is irreducible. Assume for a contradiction that P has a root $a \in \mathbb{K}$. Then $X - a$ is a non-constant polynomial that divides P and which is not an associate of P (different degrees). Therefore P is reducible, a contradiction. Therefore P does not have a root in $\mathbb{K}$.

  Now assume that P does not have a root in $\mathbb{K}$. Assume for a contradiction that P is reducible in $\mathbb{K}[X]$. Then there exist Q and R such that $P = QR$, $\deg Q < \deg P$ and $\deg R < \deg P$. Note that neither Q nor R are constants (otherwise the other polynomial would have the same degree as P). Therefore we have

$$\begin{cases} 1 \leqslant \deg Q \deg P \in \{2, 3\} \\ 1 \leqslant \deg R \deg P \in \{2, 3\} \\ \deg Q + \deg R = \deg P \in \{2, 3\} \end{cases}$$

and it follows that one of Q and R has degree 1, for instance $\deg Q = 1$. Set $Q = aX + b$ with $a \neq 0$. Then $-\frac{b}{a}$ is a root of P, a contradiction. Therefore P is irreducible.

➢ We assume that P is an irreducible polynomial that does not divide A. We must prove that $P \wedge A = 1$. Let Q be a polynomial that divides both P and A. We must prove that Q is a constant.

Assume for a contradiction that Q is not a constant. Then, since Q divides P and P is irreducible, Q is an associate of P. But Q divides A, therefore P divides A, a contradiction.

Therefore Q is a constant, as required.

➢ We prove it by induction on $n \geqslant 2$.

- First assume that $n = 2$. The polynomial P is irreducible and divides $A_1 A_2$. If P divides A_1, there is nothing to prove. If P does not divide A_1, then the previous property shows that P and A_1 are coprime. Therefore by Gauss' Lemma, P divides A_2.1
- *[Not done in class.]* Now assume that the result is true for n polynomials. Let $A_1, \ldots, A_{n+1}$ be $n+1$ polynomials such that P divides $A_1 \cdots A_n A_{n+1}$. If P divides one of the A_i for $1 \leqslant i \leqslant n$, there is nothing to prove. Assume therefore that P does not divides any of the A_i for $1 \leqslant i \leqslant n$. Then by induction hypothesis (its contrapositive), P does not divide $A_1 \cdots A_n$. Therefore P and $A_1 \cdots A_n$ are coprime, and by Gauss' Lemma, P divides A_{n+1}.

➢ The first assertion is clear. Now if P and Q are two irreducible polynomials that are not associates, in particular P does not divide Q, therefore (since P is irreducible) P and Q are coprime.

Finally, two distinct monic polynomials cannot be associates. ✓

Example. The polynomial $X^2 + 1$ is irreducible in $\mathbb{R}[X]$ since it has no root in $\mathbb{R}$ and has degree 2. (It is also irreducible in $\mathbb{Q}[X]$ for the same reason).

However, it is reducible in $\mathbb{C}[X]$ since $X^2 + 1 = (X + i)(X - i)$ with $\deg(X + i) = 1 < 2$ and $\deg(X - i) = 1 < 2$ (or because it has a root in $\mathbb{C}$).

Therefore the irreducibility of a polynomial depends on $\mathbb{K}$.

Example. We shall see later that there are no irreducible polynomials of degree greater than 2 in $\mathbb{R}[X]$ or in $\mathbb{C}[X]$. However, there are in $\mathbb{Q}[X]$.

For instance, the polynomial $X^3 + 2$ does not have a rational root, otherwise there would be coprime integers p and q (with $q > 0$) such that $\dfrac{p^3}{q^3} = -2$, that is, $p^3 = -2q^3$. Therefore p^3 would be even, hence p also, write $p = 2r$, then $8r^3 = -2q^3$ hence $4r^3 = -q^3$, therefore q would be even also, which contradicts the fact that p and q are coprime. Consequently, since $\deg(X^3 + 2) = 3$, it must be irreducible in $\mathbb{Q}[X]$.

Example. The polynomial $(X^2 + 1)^2$ is reducible in $\mathbb{R}[X]$ but has no root in $\mathbb{R}$: the property above does not hold for polynomials of degree 4 or more.

Remark. Any non-constant polynomial has an irreducible divisor.

This can be proved by induction on the degree.

➢ Any polynomial of degree 1 is irreducible, hence has an irreducible divisor.

➢ Assume that any non-constant polynomial Q of degree $\deg Q \leqslant d$ for some $d \geqslant 1$ has an irreducible divisor. Let P be a non-constant polynomial of degree $d + 1$.

If P is irreducible, there is nothing to prove.

If P is reducible, then P has a non-constant divisor Q with $\deg Q < \deg P$, that is, $\deg Q \leqslant d$. By induction hypothesis, Q has an irreducible divisor, hence so has P.

In particular, to prove that two polynomials A and B are coprime, it is enough to prove that they have no common irreducible divisor.

Indeed, if P divides A and B and if P is not constant, then P has an irreducible divisor which is a common divisor of A and B.

A. Irreducible polynomials in $\mathbb{C}[X]$

Theorem 33 (Fundamental Theorem of Algebra[a]). Any non-constant polynomial in $\mathbb{C}[X]$ has at least one root in $\mathbb{C}$.

[a] théorème de d'Alembert-Gauss

Proof. Admitted. ✓

Corollary 34. The irreducible polynomials in $\mathbb{C}[X]$ are the polynomials of degree 1.

Proof. We already know that any polynomial of degree 1 is irreducible.

Conversely, let P be an irreducible polynomial in $\mathbb{C}[X]$. We must have $\deg P \geqslant 1$ (P is not constant). If $\deg P > 1$ then P has a root in $\mathbb{C}$, therefore there exists Q with $\deg Q < \deg P$ such that Q divides P. The polynomial Q is not constant and not an associate of P. This contradicts the irreducibility of P. Therefore $\deg P = 1$. ✓

The irreducible polynomials in $\mathbb{C}[X]$ are the 'building blocks' for polynomials in $\mathbb{C}[X]$ (just like prime numbers are for integers).

Corollary 35. Let A be a non-constant polynomial in $\mathbb{C}[X]$. Then A can be written uniquely (up to reordering of the factors)

$$A = \lambda \prod_{k=1}^{p} (X - a_k)^{n_k} = \lambda (X - a_1)^{n_1} (X - a_2)^{n_2} \cdots (X - a_p)^{n_p}$$

where λ is the leading coefficient[a] of A, the scalars $a_1, \ldots, a_p$ are the distincts roots of A in $\mathbb{C}$ and the positive integers n_k are their respective multiplicities.

[a] coefficient dominant

Proof. The existence is proved by induction on $\deg A \geqslant 1$.

If $\deg A = 1$, the result is clear.

Assume that $\deg A = d > 1$ and assume that the result is true for polynomials of degree $\leqslant d - 1$. By the Fundamental Theorem of Algebra (d'Alembert-Gauss), the polynomial A has a root $\alpha \in \mathbb{C}$. Then $A = (X - \alpha)B$ with $\deg B = d - 1$. By induction hypothesis, we can write $B = \lambda \prod_{k=1}^{p} (X - a_k)^{n_k} = (X - a_1)^{n_1} (X - a_2)^{n_2} \cdots (X - a_p)^{n_p}$ where λ is the leading coefficient[†] of B, the scalars $a_1, \ldots, a_p$ are the distincts roots of B in $\mathbb{C}$ and the positive integers m_k are their respective multiplicities. The result for A then follows.

[The remainder of the proof was not done in class.]

We now prove uniqueness. Assume that $A = \lambda \prod_{k=1}^{p} (X - a_k)^{n_k} = \mu \prod_{\ell=1}^{q} (X - b_\ell)^{m_\ell}$. It is clear that λ and μ are the leading coefficient of A hence $\lambda = \mu$.

For $\ell \in \{1, \ldots, q\}$, the polynomial $X - b_\ell$ is irreducible and divides the product $\prod_{k=1}^{p} (X - a_k)^{n_k}$ hence $X - b_\ell$ divides one of the factors $X - a_k$. It follows that $b_\ell = a_k$. Therefore $\{b_1, \ldots, b_q\} \subset \{a_1, \ldots, a_p\}$. The other inclusion is proved in the same way, hence we have equality and $p = q$. We now have $A = \lambda \prod_{k=1}^{p} (X - a_k)^{n_k} = \lambda \prod_{k=1}^{p} (X - a_k)^{m_k}$. Assume for a contradiction that $n_k < m_k$ for some k. To simplify notation, say $k = 1$. Then $\prod_{k=2}^{p} (X - a_k)^{n_k} = (X - a_1)^{m_1 - n_1} \prod_{k=2}^{p} (X - a_k)^{m_k}$ so that $X - a_1$ divides $\prod_{k=2}^{p} (X - a_k)^{n_k}$ and therefore $X - a_1$ (irreducible) divides one of the a_k for $k \geqslant 2$ and $a_1 = a_k$ for some $k \geqslant 2$, a contradiction. This proves that $n_k = m_k$ for all k so that the decomposition is unique.

It is clear that the a_k are the roots of A. Moreover, the multiplicity of a_k as a root of A is necessarily at least n_k, and it cannot be greater (same argument as above), hence it is the multiplicity of a_k as a root of A. ✓

Definition 36. *A non-constant polynomial A in $\mathbb{K}[X]$ is said to be* ***split***[a] *in $\mathbb{K}[X]$ if it is a product of polynomials of degree 1. In other words, the number of roots of A in $\mathbb{K}$ is exactly $\deg A$.*

[a] scindé

Remark. The existence part of the result above can be expressed as follows: any non-constant polynomial in $\mathbb{C}[X]$ is split.

Example. Roots of unity. Let $n \geqslant 1$ be an integer. The polynomial $X^n - 1$ has n roots in $\mathbb{C}$. The elements $\omega_k = e^{2ik\pi/n}$ for $0 \leqslant k \leqslant n - 1$ are roots of this polynomial and are pairwise distinct. Indeed, if $\omega_k = \omega_\ell$, then $1 = \omega_k \omega_\ell^{-1} = \exp\left(\frac{2ik\pi}{n} - \frac{2i\ell\pi}{n}\right) = \exp\left(\frac{2i(k-\ell)\pi}{n}\right)$, so that $\frac{k-\ell}{n}$ must be an integer; however, $-n - 1 \leqslant k - \ell \leqslant n - 1$ so that we must have $k - \ell = 0$ as required. Therefore they are all the roots of $X^n - 1$. In particular, $X^n - 1 = \prod_{k=0}^{n-1} (X - \omega_k)$.

Proposition 37. Let A and B be two polynomials in $\mathbb{K}[X]$ which are split. Set $A = \lambda \prod_{k=1}^{p} (X - a_k)^{n_k}$ and $B = \mu \prod_{k=1}^{p} (X - a_k)^{m_k}$ with n_k and m_k non-negative integers (a_k need not be a root of A or B if $n_k = 0$ or $m_k = 0$).
Then A divides B if, and only if, $n_k \leqslant m_k$ for all k.

Proof. If $n_k \leqslant m_k$ for all k, then $B = A\mu\lambda^{-1} \prod_{k=1}^{p} (X - a_k)^{m_k - n_k}$ hence A divides B.

Conversely, assume that A divides B. Since the polynomial $(X - a_k)^{n_k}$ divides A, it divides B. If $n_k = 0$ then necessarily $m_k \geqslant 0 = n_k$. Otherwise, this means that a_k is a root of B with multiplicity at least n_k. But we know that the multiplicity of a_k as a root of B is m_k and therefore $m_k \geqslant n_k$. ✓

Proposition 38. Let A and B be two polynomials in $\mathbb{K}[X]$ which are split. Set $A = \lambda \prod_{k=1}^{p} (X - a_k)^{n_k}$ and $B = \mu \prod_{k=1}^{p} (X - a_k)^{m_k}$ with n_k and m_k non-negative integers. For each k, set $u_k = \min(n_k, m_k)$ and $v_k = \max(n_k, m_k)$.
Then $A \wedge B = \prod_{k=1}^{p} (X - a_k)^{u_k}$ and $A \vee B = \prod_{k=1}^{p} (X - a_k)^{v_k}$.

[†] coefficient dominant

Proof. *[Not done in class.]* Set $D = \prod_{k=1}^{p}(X-a_k)^{u_k}$ and $M = \prod_{k=1}^{p}(X-a_k)^{v_k}$. The previous result shows that D divides A and B and that M is a multiple of A and B.

We have $A = DA_1$ and $B = DB_1$ with $A_1 = \lambda\prod_{k=1}^{p}(X-a_k)^{n_k-u_k}$ and $B_1 = \mu\prod_{k=1}^{p}(X-a_k)^{m_k-u_k}$. Moreover, if P is an irreducible polynomial that divides A_1 and B_1, it divides one of their factors, say $X-a_k$. But $X-a_k$ is irreducible, therefore P and $X-a_k$ are associates, so that $X-a_k$ divides both A_1 and B_1. This implies that $n_k-u_k>0$ and $m_k-u_k>0$ which is impossible. Therefore A_1 and B_1 have no non-constant divisor, hence they are coprime. Consequently, D is a gcd of A and B and it is monic therefore $D = A\wedge B$.

Finally, $MD = \prod_{k=1}^{p}(X-a_k)^{u_k+v_k} = \prod_{k=1}^{p}(X-a_k)^{n_k+m_k} = (AB)^\circ$ (since for each k we have $n_k+m_k = u_k+v_k$), therefore $M = \dfrac{(AB)^\circ}{D} = \dfrac{(AB)^\circ}{A\wedge B} = A\vee B$. ✓

Example. Let us return to the example following Proposition 4, where we wanted to find a greatest common divisor for $A = X^5-2X^4+X^3 = X^3(X-1)^2$ and $B = X^3-X = X(X-1)(X+1)$. We may now immediately say, using the result above, that $A\wedge B = X(X-1)$. We also have $A\vee B = X^3(X-1)^2(X+1)$.

B. Irreducible polynomials in $\mathbb{R}[X]$

Lemma 39. Let A be a polynomial in $\mathbb{R}[X]$. Let $z\in\mathbb{C}$ be a root of A. Then the conjugate $\bar{z}$ is also a root of A. Moreover, the multiplicity of $\bar{z}$ is equal to the multiplicity of z.
In other words, the non-real complex roots of A are pairwise conjugate.

Proof. Write $A = \sum_{k=0}^{d} a_kX^k$ with $a_k\in\mathbb{R}$. Then we have $A(\bar{z}) = \sum_{k=0}^{d} a_k\bar{z}^k = \sum_{k=0}^{d}\overline{a_k}\bar{z}^k = \overline{\sum_{k=0}^{d} a_kz^k} = \overline{A(z)} = \bar{0} = 0$ so that $\bar{z}$ is a root of A. Hence z is a root of A of and only if $\bar{z}$ is a root of A.

Similarly, for any $k\geqslant 1$, z is a root of the derivative $A^{(k)}$ if, and only if, $\bar{z}$ is a root of $A^{(k)}(\bar{z})$. Therefore the multiplicities of z and $\bar{z}$ as roots of A are the same (Corollary 159 in the first semester). ✓

Proposition 40. The irreducible polynomials in $\mathbb{R}[X]$ are exactly

➢ the polynomials of degree 1, and

➢ the polynomials of degree 2 with no roots in $\mathbb{R}$.

Proof. We already know that polynomials of degree 1 and polynomials of degree 2 with no roots in $\mathbb{R}$ are irreducible.
Now let A be a polynomial in $\mathbb{R}[X]$.

➢ If $\deg A>1$ is odd, then by the Intermediate Value Theorem, A has a root in $\mathbb{R}$. Therefore A is not irreducible.

➢ If $\deg A>2$ is even, we may assume that A has no real root (otherwise it is clearly not irreducible in $\mathbb{R}[X]$). As a polynomial in $\mathbb{C}[X]$, using the previous lemma, we have $A = \lambda\prod_{k=1}^{p}(X-a_k)^{n_k}(X-\overline{a_k})^{n_k}$ where the $a_k,\overline{a_k}$ in $\mathbb{C}$ for $1\leqslant k\leqslant p$ are the pairwise distinct roots of A, $\lambda\in\mathbb{R}$ is the leading coefficient of A and n_k is the multiplicity of a_k. Therefore $A = \lambda\prod_{k=1}^{p}(X^2-2\Re a_kX+|a_k|^2)^{n_k}$ in $\mathbb{R}[X]$. Since $\deg A>2$, we must have either $p>2$ or $p=1$ and $n_1>2$. Therefore A is reducible in $\mathbb{R}[X]$.

The result then follows. ✓

Theorem 41. Let A be a non-constant polynomial in $\mathbb{R}[X]$. Then A can be written uniquely (up to reordering of the factors)

$$\begin{aligned}A &= \lambda\prod_{k=1}^{p}(X-a_k)^{n_k}\prod_{\ell=1}^{s}(X^2+b_\ell X+c_\ell)^{m_\ell}\\ &= (X-a_1)^{n_1}(X-a_2)^{n_2}\cdots(X-a_p)^{n_p}(X^2+b_1X+c_1)^{m_1}(X^2+b_2X+c_2)^{m_2}\cdots(X^2+b_sX+c_s)^{m_\ell}\end{aligned}$$

where λ is the leading coefficient[a] of A, the scalars $a_1,\ldots,a_p$ are the distincts roots of A in $\mathbb{R}$ and the positive integers n_k are their respective multiplicities, the polynomials $X^2+b_\ell X+c_\ell$ are pairwise distinct with negative discriminant and the m_ℓ are positive integers.
Each polynomial $X^2+b_\ell X+c_\ell$ can be written $X^2+b_\ell X+c_\ell = (X-\beta_\ell)(X-\overline{\beta_\ell})$ where β_ℓ is a non-real complex root of A; the integer m_ℓ it then the multiplicity of β_ℓ as a root of A.

[a] coefficient dominant

Proof. *[Not done in class.]* Let $a_1,\ldots,a_p$ be the distinct roots of A, with multiplicities $n_1,\ldots,n_p$ respectively. Then each polynomial $(X-a_k)^{n_k}$ divides A. Moreover, the polynomials $(X-a_1)^{n_1},\ldots,(X-a_p)^{n_p}$ are pairwise coprime (indeed, if P is an irreducible polynomial that divides $(X-a_j)^{n_j}$ and $(X-a_k)^{n_k}$ ($k\neq j$), then P divides $X-a_j$ and $X-a_k$, therefore, since all three polynomials are irreducible, we have $X-a_j = P = X-a_k$, a contradiction). Therefore A is a multiple of $\prod_{k=1}^{p}(X-a_k)^{n_k}$, that is, we have $A = \lambda\prod_{k=1}^{p}(X-a_k)^{n_k}Q$ where λ is the leading coefficient of A and Q is a polynomial in $\mathbb{R}[X]$ with no real roots.

The same argument as in the proof of the previous proposition shows that we can write $Q = \prod_{\ell=1}^{s}(X-\beta_\ell)^{m_\ell}(X-\overline{\beta_\ell})^{m_\ell} = \prod_{\ell=1}^{s}(X^2+b_\ell X+c_\ell)^{m_\ell}$ with the notation of the statement. This proves existence.

Uniqueness follows from the uniqueness of the decomposition of A in $\mathbb{C}[X]$. ✓

III. Lagrange interpolation

In this section we fix an integer $n \geqslant 1$ and n *distinct* elements $x_1, \ldots, x_n$ in $\mathbb{K}$.

Given any n elements $y_1, \ldots, y_n$, the aim is to find and characterise polynomials A such that for all i we have $A(x_i) = y_i$.

Definition 42. *For any $k \in \{1, \ldots, n\}$, define the polynomial*

$$L_k(X) = \prod_{\substack{j=1\\ j\neq k}} \frac{X - x_j}{x_k - x_j}.$$

It is the k^{th} ***Lagrange interpolation polynomial***[a]

[a] polynôme d'interpolation de Lagrange

Lemma 43. We have $\deg L_k = n-1$, $L_k(x_k) = 1$ and $L_k(x_j) = 0$ if $j \neq k$.

Proof. Clear. ✓

Proposition 44. Let $x_1, \ldots, x_n$ be a family of pairwise distinct elements in $\mathbb{K}$ and let $y_1, \ldots, y_n$ be a family of elements in $\mathbb{K}$.
There exists a unique polynomial A with $\deg A \leqslant n-1$ and, for all $k \in \{1, \ldots, n\}$, $A(x_k) = y_k$.
This polynomial is given by $A = \sum_{k=1}^{n} y_k L_k$.

Proof. Put $A = \sum_{k=1}^{n} y_k L_k$. Then $\deg A \leqslant n-1$ and $A(x_j) = \sum_{k=1}^{n} y_k L_k(x_j) = y_j L_j(x_j) = y_j$ for all j as required. Therefore the polynomial exists.

We now prove uniqueness. Assume that A and B are two polynomials of degree at most $n-1$ such that $A(x_i) = y_i = B(x_i)$ for all $i \in \{1, \ldots, n\}$. Then x_i is a root of $A-B$ for all i, so that $A-B$ has at least n distinct roots. But we know that the number of roots of $A-B$ is at most $\deg(A-B)$ if $A-B \neq 0$, and $\deg(A-B) \leqslant n-1$ by assumption, therefore $A = B$. The polynomial A is therefore unique. ✓

Proposition 45. Let $x_1, \ldots, x_n$ be a family of pairwise distinct elements in $\mathbb{K}$ and let $y_1, \ldots, y_n$ be a family of elements in $\mathbb{K}$.
Let P be a polynomial such that for all $k \in \{1, \ldots, n\}$ we have $P(x_k) = y_k$.
Then $P(X) = A(X) + Q(X)\prod_{k=1}^{n}(X - x_k)$ where A is the polynomial in Proposition 44 and Q is any polynomial.

Proof. The Euclidean division of P by $(X-x_1)\cdots(X-x_n)$ gives $P = Q\prod_{k=1}^{n}(X-x_k) + A$ with $\deg A \leqslant n-1$. Moreover, $A(x_k) = P(x_k) = y_k$ for all k, therefore by the previous proposition, $A = \sum_{k=1}^{n} y_k L_k$.

We could also say that x_j is a root of $P - A$, with $A = \sum_{k=1}^{n} y_k L_k$, for all j, therefore since the polynomials $X - x_1, \ldots, X - x_n$ are pairwise coprime (the x_j are pairwise distinct), the polynomial $(X-x_1)\cdots(X-x_n)$ divides $P - A$, as required. ✓

Chapter 7

Rational fractions and partial fraction decomposition

I. Rational fractions

We consider the set $\mathbb{K}(X)$ of fractions of the form $F = \dfrac{A}{B}$ where A and B are polynomials in $\mathbb{K}[X]$ with $B \neq 0$. Such a fraction is called a **rational fraction**†, the polynomial A is the **numerator**‡ of $\dfrac{A}{B}$ and the polynomial B is the **denominator**§ of $\dfrac{A}{B}$.

Definition 1. *Two such fractions $\dfrac{A_1}{B_1}$ and $\dfrac{A_2}{B_2}$ are said to be equal if, and only if, $A_1B_2 = A_2B_1$. In particular, for any non-zero polynomial C, we have $\dfrac{A}{B} = \dfrac{AC}{BC}$.*

Remark. Note that any polynomial may be viewed as a rational fraction: if A is a polynomial in $\mathbb{K}[X]$ then $\dfrac{A}{1}$ is a rational fraction in $\mathbb{K}(X)$.

Definition-Proposition 2. *We define two operations on $\mathbb{K}(X)$,*

- ***addition***[a]*, defined by* $\dfrac{A}{B} + \dfrac{C}{D} = \dfrac{AD+BC}{BD}$
- ***multiplication***[b]*, defined by* $\dfrac{A}{B} \cdot \dfrac{C}{D} = \dfrac{AC}{BD}$.

[a] addition
[b] multiplication

Proof. We must check that these operations are well defined, that is, different expressions of the rational fractions $\dfrac{A}{B}$ and $\dfrac{C}{D}$ gives the same final result.

Assume that $\dfrac{A}{B} = \dfrac{A_1}{B_1}$ and $\dfrac{C}{D} = \dfrac{C_1}{D_1}$. This means that $AB_1 = A_1B$ and $CD_1 = C_1D$.

Then

$$\begin{aligned}(A_1D_1 + B_1C_1)BD - (AD+BC)B_1D_1 &= A_1BD_1D + B_1BC_1D - AB_1D_1D - B_1BCD_1 \\ &= (A_1B - AB_1)D_1D + B_1B(C_1D - CD_1) = 0\end{aligned}$$

so that $\dfrac{AD+BC}{BD} = \dfrac{A_1D_1 + B_1C_1}{B_1D_1}$, and

$$\begin{aligned}A_1C_1BD - ACB_1D_1 &= A_1C_1BD - AC_1B_1D + AC_1B_1D - ACB_1D_1 \\ &= (A_1B - AB_1)C_1D + AB_1(C_1D - CD_1) = 0\end{aligned}$$

so that $\dfrac{A_1C_1}{B_1D_1} = \dfrac{AC}{BD}$. ✓

† fraction rationnelle
‡ numérateur
§ dénominateur

Definition-Proposition 3. *The set $\mathbb{K}(X)$ is a* commutative field[a], *that is, the operations above satisfy, for all F, G and H in $\mathbb{K}(X)$:*

(i) $F + G = G + F$ *and* $FG = GF$ *(the operations are* ***commutative***[b]*);*

(ii) $(F + G) + H = F + (G + H)$ *and* $(FG)H = F(GH)$ *(the operations are* ***associative***[c]*);*

(iii) $0 + F = F$ *and* $1F = F$ *(addition and multiplication have an* ***indentity element***[d]*);*

(iv) $F + (-1)F = 0$ *and, if* $F \neq 0$, $F\dfrac{1}{F} = 1$ *(elements in $\mathbb{K}(X)$ are* ***invertible***[e] *for addition and non-zero elements in $\mathbb{K}(X)$ are* ***invertible***[f] *for multiplication);*

(v) $F(G + H) = FG + FH$ *(multiplication is* ***distributive***[g] *with respect to addition).*

[a] corps commutatif
[b] commutative
[c] associative
[d] élément neutre
[e] inversible
[f] inversible
[g] distributive

Proof. Exercise. ✓

Definition-Proposition 4. *For any $F \in \mathbb{K}(X)$, there exist coprime polynomials A and B such that $F = \dfrac{A}{B}$. This expression for F is called the* ***irreducible form***[a] *of F.*

[a] forme irréductible

Proof. Set $F = \dfrac{C}{D}$. Then $C = (C \wedge D)A$ and $D = (C \wedge D)B$ with A and B coprime. Moreover, $F = \dfrac{(C \wedge D)A}{(C \wedge D)B} = \dfrac{A}{B}$. ✓

Definition 5. *Let $F = \dfrac{A}{B}$ be a rational fraction in irreducible form. We associate to F a function defined on a subset of $\mathbb{K}$ with values in $\mathbb{K}$, again denoted by F, which sends x to $F(x) := \dfrac{A(x)}{B(x)}$ (where A and B are viewed as* polynomial functions[a]*). This function is called* ***rational function***[b] *associated to F. It is defined on $\mathbb{K} \setminus \{\text{roots of } B\}$.*

[a] fonctions polynomiales
[b] fonction rationnelle

Definition-Proposition 6. *Let $F = \dfrac{A}{B}$ be a rational fraction. The* ***degree***[a] *of F is the integer $\deg F = \deg A - \deg B \in \mathbb{Z}$.*
This definition does not depend on the choice of A and B such that $\dfrac{A}{B} = F$.
We put $\deg 0 = -\infty$, as for polynomials.

[a] degré

Proof. We must check that the degree does not depend on the choice of expression for F.
If $F = \dfrac{A}{B} = \dfrac{C}{D}$, then $AD = BC$ therefore $\deg A + \deg D = \deg B + \deg C$ and finally $\deg A - \deg B = \deg C - \deg D$ as required. ✓

Properties 7.

- If F is a polynomial, then $\deg F$ is the degree of F viewed as a polynomial.
- For any rational fractions F and G we have

$$\begin{cases} \deg(F+G) \leqslant \max(\deg F, \deg G) \\ \deg(FG) = \deg F + \deg G. \end{cases}$$

Proof. The first part is clear since $\deg 1 = 0$.

Set $F = \dfrac{A}{B}$ and $G = \dfrac{C}{D}$.

We have $F + G = \dfrac{AD + BC}{BD}$ so that $\deg(F + G) = \deg(AD + BC) - \deg(BD) \leqslant \max(\deg(AD); \deg(BC)) - \deg B - \deg D$.

Assume that $\deg F \geqslant \deg G$. Then $\deg A - \deg B \geqslant \deg C - \deg D$ so that $\deg(AD) \geqslant \deg(BC)$. We then have $\deg(F + G) \leqslant \deg(AD) - \deg B - \deg D = \deg A - \deg B = \deg F$. Similarly, if $\deg F \leqslant \deg G$, we get $\deg(F + G) \leqslant \deg G$. Therefore $\deg(F + G) \leqslant \max(\deg F, \deg G)$.

Finally, $\deg(FG) = \deg \frac{AC}{BD} = \deg(AC) - \deg(BD) = \deg A + \deg C - \deg B - \deg D = \deg F + \deg G$. ✓

Proposition 8. Let F be a rational fraction in $\mathbb{K}(X)$. Then F can be written uniquely as $F = E_F + G$ where E_F is a polynomial in $\mathbb{K}[X]$ and G is a rational fraction in $\mathbb{K}(X)$ with $\deg G < 0$.

Proof. Set $F = \dfrac{A}{B}$ with A, B in $\mathbb{K}[X]$, $B \neq 0$. We can do the Euclidean division of A by B, si that $A = QB + R$ with $\deg R < \deg B$. Therefore we get $F = Q + \dfrac{R}{B}$. Taking $E_F := Q$ and $G := \dfrac{R}{B}$ gives the existence.

Now assume we can write $F = P + H$ where P is a polynomial and H is a rational fraction with $\deg H < 0$. Then we have $G - H = P - E_F$. The previous result gives $\deg(G - H) \leqslant \max(\deg G, \deg H) < 0$. However, $P - E_F$ is a polynomial, hence has non-negative degree or is zero. Therefore we get $P - E_F = 0$ and $G - H = 0$. Finally, the expression in the statement is unique. ✓

Definition 9. *The polynomial E_F in the Proposition above is called the* ***integral part***[a] *of the rational fraction F.*

[a]partie entière

Remark. Put $F = \dfrac{A}{B}$. We have $E_F = 0$ if, and only if, $\deg F < 0$. Otherwise, $\deg E_F = \deg A - \deg B$.

If $\deg A = \deg B$ then E_F is the quotient of the leading coefficients of A and B.

Proof. Exercise. ✓

Example. The rational fraction $F = \dfrac{X^3 + 3X + 2}{X^2 - 1}$ has degree 1. Therefore $\deg E_F = 1$. The Euclidean division of $X^3 + 3X + 2$ by $X^2 - 1$ gives $X^3 + 3X + 2 = X(X^2 - 1) + 4X + 2$ so that $F = \dfrac{X(X^2 - 1) + 4X + 2}{X^2 - 1} = X + \dfrac{4X + 2}{X^2 - 1}$. On a $E_F = X$ qui est bien un polynôme de degré 1 et $G = \dfrac{4X + 2}{X^2 - 1}$ qui est bien une fraction rationnelle de degré $-1 < 0$.

Definition 10. *Let $F = \dfrac{A}{B}$ be a rational fraction in irreducible form. In particular, the polynomials A and B have no common root.*
An element $\alpha \in \mathbb{K}$ is a ***root***[a] *of F, with* ***multiplicity***[b] *$m \in \mathbb{N}^*$, if α is a root of the polynomial A with multiplicity m.*
An element $\alpha \in \mathbb{K}$ is a ***pole***[c] *of F, with* ***multiplicity*** *$m \in \mathbb{N}^*$, if α is a root of the polynomial B with multiplicity m. The pole is called* ***simple***[d] *(resp.* ***double***[e]*) if $m = 1$ (resp. $m = 2$).*

[a]racine
[b]multiplicité
[c]pôle
[d]simple
[e]double

II. Partial fraction decomposition

A. Partial fraction decomposition in $\mathbb{C}(X)$

Theorem 11. Let $F = \dfrac{A}{B}$ be a rational fraction in $\mathbb{C}(X)$, in irreducible form. Let $a_1, \ldots, a_p$ be the distinct poles of F, with respective multiplicities $n_1, \ldots, n_p$.
Then F can be written uniquely in the form

$$F = E_F + \sum_{k=1}^{p} \left(\sum_{\ell=1}^{n_k} \frac{\lambda_{k,\ell}}{(X - a_k)^\ell} \right)$$

where E_F is the integral part of F and the $\lambda_{k,\ell}$ are complex numbers.

Proof. Admitted. ✓

Definition 12. *The expression of F given in the theorem is called* ***partial fraction decomposition***[a] *of F in $\mathbb{C}(X)$.*

[a] décomposition en éléments simples

Example. We may consider $F = \dfrac{X^{13}+1}{X^3(X-i)^2(X+j)^5}$. The roots of the denominator are 0, i and $-j$, which are not roots of the numerator $X^{13}+1$. Therefore F is in irreducible form and 0, i and $-j$ are the poles of F.

We know that $\deg E_F = \deg F = 13 - 10 = 3$. Moreover, the pole 0 has multiplicity 3, the pole i has multiplicity 2 and the pole j has multiplicity 5. Therefore the partial fraction decomposition of F is of the following form:

$$\begin{aligned} F &= aX^3 + bX^2 + cX + d \\ &+ \frac{\lambda_{1,3}}{X^3} + \frac{\lambda_{1,2}}{X^2} + \frac{\lambda_{1,1}}{X} \\ &+ \frac{\lambda_{2,2}}{(X-i)^2} + \frac{\lambda_{2,1}}{X-i} \\ &+ \frac{\lambda_{3,5}}{(X+j)^5} + \frac{\lambda_{3,4}}{(X+j)^4} + \frac{\lambda_{3,3}}{(X+j)^3} + \frac{\lambda_{3,2}}{(X+j)^2} + \frac{\lambda_{3,1}}{X+j}. \end{aligned}$$

To find the coefficients, we can then reduce to the same denominator and identify, but that is usually long and technical. We shall see later a few methods to find the coefficients, although they will not necessarily yield all the coefficients.

B. Partial fraction decomposition in $\mathbb{R}(X)$

Proposition 13. Let $F = \dfrac{A}{B}$ be a rational fraction in $\mathbb{R}(X)$, in irreducible form. Assume that B is split in $\mathbb{R}[X]$. Let $a_1, \dots, a_p$ be the distinct poles of F, with respective multiplicities $n_1, \dots, n_p$.
Then F can be written uniquely in the form

$$F = E_F + \sum_{k=1}^{p} \left(\sum_{\ell=1}^{n_k} \frac{\lambda_{k,\ell}}{(X-a_k)^\ell} \right)$$

where E_F is the integral part of F and the $\lambda_{k,\ell}$ are real numbers.

More generally,

Theorem 14. Let $F = \dfrac{A}{B}$ be a rational fraction in $\mathbb{R}(X)$, in irreducible form.
Let $B = \lambda \prod_{k=1}^{p} (X-a_k)^{n_k} \prod_{t=1}^{s} (X^2 + b_t X + c_t)^{m_t}$ be the factorisation of B as a product of irreducible polynomials in $\mathbb{R}[X]$.
Then F can be written uniquely in the form

$$F = E_F + \sum_{k=1}^{p} \left(\sum_{\ell=1}^{n_k} \frac{\lambda_{k,\ell}}{(X-a_k)^\ell} \right) + \sum_{t=1}^{s} \left(\sum_{j=1}^{m_t} \frac{\beta_{t,j} X + \gamma_{t,j}}{(X^2 + b_t X + c_t)^j} \right)$$

where E_F is the integral part of F and the $\lambda_{k,\ell}$, $\beta_{t,j}$ and $\gamma_{t,j}$ are real numbers.

Proof. Admitted. ✓

Definition 15. *The expression of F given in the theorem is called* ***partial fraction decomposition***[a] *of F in $\mathbb{R}(X)$.*

[a] décomposition en éléments simples

Remark. The terms in the partial fraction decomposition of $F = \dfrac{A}{B}$ that are not polynomials are all of the form $\dfrac{P}{Q^n}$ where Q is an irreductible factor in B, P is a polynomial with $\deg P \leqslant \deg Q - 1$ and n is a positive integer which is between 1 and the power at which Q appears in the factorisation of B (in other words, the multiplicity of the complex roots of Q in B).

Example. Consider $F = \dfrac{X^{12}+1}{X^2(X-1)(X^2+1)^5(X^2+X+1)^2}$; it is in irreducible form (the irreducible divisors of the denominator are X, $X-1$, X^2+1 and X^2+X+1, which do not divide the numerator).

We know that $\deg F = 12 - 17 = -5 < 0$ hence $E_F = 0$. Moreover, the pole 0 has multiplicity 2, the pole 1 has multiplicity 1, the irreducible factor X^2+1 appears to the power 5 and the irreducible factor X^2+X+1 appears to the power 2 (in other words, the poles $\pm i$ in $\mathbb{C}$ have multiplicity 5 and the poles $\frac{-1\pm i\sqrt{3}}{2}$ have multiplicity 2). Therefore the partial fraction decomposition of F is of the following form:

$$
\begin{aligned}
F = {}& aX^2+bX+c \\
&+ \frac{\lambda_{1,2}}{X^2} + \frac{\lambda_{1,1}}{X} \\
&+ \frac{\lambda_{2,1}}{X-1} \\
&+ \frac{\beta_{1,5}X+\gamma_{1,5}}{(X^2+1)^5} + \frac{\beta_{1,4}X+\gamma_{1,4}}{(X^2+1)^4} + \frac{\beta_{1,3}X+\gamma_{1,3}}{(X^2+1)^3} + \frac{\beta_{1,2}X+\gamma_{1,2}}{(X^2+1)^2} + \frac{\beta_{1,1}X+\gamma_{1,1}}{X^2+1} \\
&+ \frac{\beta_{2,2}X+\gamma_{2,2}}{(X^2+X+1)^2} + \frac{\beta_{2,1}X+\gamma_{2,1}}{X^2+X+1}.
\end{aligned}
\tag{7.1}
$$

C. Simple poles

Proposition 16. Let $F = \dfrac{A}{B}$ be a rational function in $\mathbb{K}(X)$ in irreducible form, and let a be a simple pole of F. Then the coefficient of $\dfrac{1}{X-a}$ in the partial fraction decomposition of F is $\dfrac{A(a)}{B'(a)}$.

Proof. Since a is a simple pole of F, we have $F = \dfrac{\lambda}{X-a} + G$ where G is a rational fraction such that a is not a pole of G. Moreover, we have $B = (X-a)C$ where C is a polynomial such that a is not a root of C.

Multiplying $F = \dfrac{A}{(X-a)C} = \dfrac{\lambda}{X-a} + G$ by $X-a$ gives $\dfrac{A}{C} = \lambda + (X-a)G$ and evaluating at a gives $\lambda = \dfrac{A(a)}{C(a)}$. Moreover, $B' = C + (X-a)C'$ so that $B'(a) = C(a)$. Finally, $\lambda = \dfrac{A(a)}{B'(a)}$. ✓

Example. Let us return to the previous example $F = \dfrac{X^{12}+1}{X^2(X-1)(X^2+1)^5(X^2+X+1)^2}$. The only simple pole of F is 1. The method in the proof will enable us to find the corresponding coefficient $\lambda_{2,1}$. Set $G = F - \dfrac{\lambda_{2,1}}{X-1}$. Multiplying equation (7.1) by $X-1$ gives

$$
\begin{aligned}
\frac{X^{12}+1}{X^2(X^2+1)^5(X^2+X+1)^2} &= (X-1)F \\
&= \lambda_{2,1} \\
&\quad + (X-1)\Bigg[aX^2+bX+c \\
&\qquad + \frac{\lambda_{1,2}}{X^2} + \frac{\lambda_{1,1}}{X} \\
&\qquad + \frac{\beta_{1,5}X+\gamma_{1,5}}{(X^2+1)^5} + \frac{\beta_{1,4}X+\gamma_{1,4}}{(X^2+1)^4} + \frac{\beta_{1,3}X+\gamma_{1,3}}{(X^2+1)^3} + \frac{\beta_{1,2}X+\gamma_{1,2}}{(X^2+1)^2} + \frac{\beta_{1,1}X+\gamma_{1,1}}{X^2+1} \\
&\qquad + \frac{\beta_{2,2}X+\gamma_{2,2}}{(X^2+X+1)^2} + \frac{\beta_{2,1}X+\gamma_{2,1}}{X^2+X+1}\Bigg]
\end{aligned}
$$

then we evaluate at 1 to get $\dfrac{2}{2^5 3^2} = \lambda_{2,1}$ and finally $\lambda_{2,1} = \dfrac{1}{144}$.

Example. Consider $F = \frac{1}{X^3-1}$ in $\mathbb{C}(X)$. It is in irreducible form. Its poles are the third roots of unity, 1, $j = e^{2i\pi/3}$ and j^2. They are all simple. Moreover, $\deg F < 0$. Therefore the partial fraction decomposition of F has the form

$$F = \frac{a}{X-1} + \frac{b}{X-j} + \frac{c}{X-j^2}.$$

To find the coefficients, since F is in irreducible form, we may apply the proposition. We have $B = X^3-1$ and $B' = 3X^2$ so that $B'(1) = 3$, $B'(j) = 3j^2$ and $B'(j^2) = 3j$. Moreover, $A = 1$. Therefore $a = \frac{1}{3}$, $b = \frac{1}{3j^2} = \frac{1}{3}j$ and $c = \frac{1}{3j} = \frac{1}{3}j^2$ (we use the fact that $j^3 = 1$ so that $j^2 = j^{-1}$).

Finally, $F = \dfrac{1}{3}\left(\dfrac{1}{X-1} + \dfrac{j}{X-j} + \dfrac{j^2}{X-j^2}\right)$.

Note that F is in $\mathbb{R}(X)$ so that we may consider its partial fraction decomposition in $\mathbb{R}(X)$, which is $F = \dfrac{1}{3}\left(\dfrac{1}{X-1} + \dfrac{-X-2}{X^2+X+1}\right)$ (adding the last two terms in the complex decomposition, using the relation $1+j+j^2=0$).

To find the coefficients a, b and c, we could also have used the method outline in the proof of the proposition. Note that $F = \dfrac{1}{(X-1)(X-j)(X-j^2)} = \dfrac{a}{X-1} + \dfrac{b}{X-j} + \dfrac{c}{X-j^2}$.

- Multiplying by $X-1$ gives $\dfrac{1}{(X-j)(X-j^2)} = a + (X-1)\left(\dfrac{b}{X-j} + \dfrac{c}{X-j^2}\right)$, then evaluating at 1 gives $\dfrac{1}{(1-j)(1-j^2)} = a$; we have $(1-j)(1-j^2) = 1-j-j^2+j^3 = 1+1+1 = 3$ so that $a = \frac{1}{3}$.
- Multiplying by $X-j$ gives $\dfrac{1}{(X-1)(X-j^2)} = b + (X-j)\left(\dfrac{a}{X-1} + \dfrac{c}{X-j^2}\right)$, then evaluating at j gives $\dfrac{1}{(j-1)(j-j^2)} = b$; we have $(j-1)(j-j^2) = j^2-j-j^3+j^2 = 2j^2-j-1 = 3j^2 = 3j^{-1}$ so that $b = \frac{1}{3}j$.
- Multiplying by $X-j^2$ gives $\dfrac{1}{(X-1)(X-j)} = c + (X-j^2)\left(\dfrac{a}{X-1} + \dfrac{b}{X-j}\right)$, then evaluating at j^2 gives $\dfrac{1}{(j^2-1)(j^2-j)} = c$; we have $(j^2-1)(j^2-j) = j^4-j^3-j^2+j = j-1-j^2+j = 3j = 3j^{-2}$ so that $a = \frac{1}{3}j^2$.

Example. Consider $F = \frac{X+1}{X(X-1)(X-2)(X-3)(X-4)}$ in $\mathbb{R}(X)$. It is in irreducible form, its roots are 0, 1, 2, 3 and 4, all simple, and $\deg F < 0$. Therefore

$$F = \frac{a}{X} + \frac{b}{X-1} + \frac{c}{X-2} + \frac{d}{X-3} + \frac{e}{X-4} \tag{7.2}$$

for some real numbers a, b, c, d, e.

Now multiply (7.2) by X, which gives $\dfrac{X+1}{(X-1)(X-2)(X-3)(X-4)} = a + \dfrac{bX}{X-1} + \dfrac{cX}{X-2} + \dfrac{dX}{X-3} + \dfrac{eX}{X-4}$, then evaluate at 0, which gives $\frac{1}{24} = a$.

Similarly, multiply (7.2) by $X-1$ then evaluate at 1, which gives $b = -\frac{1}{3}$.
Similarly, multiply (7.2) by $X-2$ then evaluate at 2, which gives $c = \frac{3}{4}$.
Similarly, multiply (7.2) by $X-3$ then evaluate at 3, which gives $d = -\frac{2}{3}$.
Similarly, multiply (7.2) by $X-4$ then evaluate at 4, which gives $e = \frac{5}{24}$.

Finally, $F = \dfrac{1}{24}\dfrac{1}{X} - \dfrac{1}{3}\dfrac{1}{X-1} + \dfrac{3}{4}\dfrac{1}{X-2} - \dfrac{2}{3}\dfrac{1}{X-3} + \dfrac{5}{24}\dfrac{1}{X-4}$.

D. Partial fraction decomposition of $\dfrac{P'}{P}$

Proposition 17. Let P be a split polynomial in $\mathbb{K}[X]$. Let $a_1, \dots, a_p$ be its distinct roots, with multiplicities $n_1, \dots, n_p$ respectively.

Then the partial fraction decomposition of $\dfrac{P'}{P}$ in $\mathbb{K}(X)$ is

$$\frac{P'}{P} = \sum_{k=1}^{p} \frac{n_k}{X-a_k}.$$

Proof. We prove the result by induction on the number p of distinct roots of P.

If $p=1$, then $P = (X-a)^n$ so that $P' = n(X-a)^{n-1}$ and $\dfrac{P'}{P} = \dfrac{n}{X-a}$ as required.

If $p > 1$, then we can write $P = (X-a_1)^{n_1}Q_1$ where Q_1 is a polynomial such that a_1 is not a root of Q_1. Then $P' = n_1(X-a_1)^{n_1-1}Q_1 + (X-a_1)^{n_1}Q_1'$ so that $\dfrac{P'}{P} = \dfrac{n_1}{X-a_1} + \dfrac{Q_1'}{Q_1}$. We may apply the induction hypothesis to $\dfrac{Q_1'}{Q_1}$ (the polynomial Q_1 has $p-1$ distinct roots, $a_2, \dots, a_p$) to get the result. ✓

Example. Take $P = X(X-1)^2(X+2)^5(X-13)^3$ in $\mathbb{R}[X]$. Then $\dfrac{P'}{P} = \dfrac{1}{X} + \dfrac{2}{X-1} + \dfrac{5}{X+2} + \dfrac{3}{X-13}$.

If $P = (X-i)^2X^7(X+j)^3$ then $\dfrac{P'}{P} = \dfrac{2}{X-i} + \dfrac{7}{X} + \dfrac{3}{X+j}$.

E. Tricks

1. Partial fraction decomposition in $\mathbb{C}(X)$ of a rational fraction in $\mathbb{R}(X)$

If $F \in \mathbb{R}(X)$, it can be viewed in $\mathbb{C}(X)$ and therefore we can write its partial fraction decomposition in $\mathbb{C}(X)$. The coefficients in elements of the same degree corresponding to conjugate poles must be conjugate. This halves the number of complex unknowns. (See example below.)

2. Use of $\lim_{x\to a}(x-a)^n F(x)$

Let $F = \dfrac{A}{B}$ be in irreducible form, with $\deg F < 0$ (we can always reduce to this case). Let a be a pole of F of order n. Then we have $F = \dfrac{A}{(X-a)^n C}$ where a is not a root of C. We then have

$$F = \sum_{k=1}^{n} \frac{\lambda_k}{(X-a)^k} + G$$

where G is a rational fraction such that a is not a pole of G (we have isolated the part of the partial fraction decomposition of F corresponding to the pole a).

Then, in order to find λ_n, we can multiply the equation above by $(X-a)^n$ then evaluate at a.

Example. Consider $F = \frac{X+1}{X^2(X-1)^2}$ in $\mathbb{R}(X)$. It is in irreducible form, it has two double poles, 0 and 1 in $\mathbb{R}$, therefore we know that $F = \frac{a}{X^2} + \frac{b}{X} + \frac{c}{(X-1)^2} + \frac{d}{X-1}$. The method above enables us to find a and c.

Multiply by X^2 then evaluate at 0. This gives $a = 1$.

Multiply by $(X-1)^2$ then evaluate at 1. This gives $c = 2$.

We get $F = \frac{1}{X^2} + \frac{b}{X} + \frac{2}{(X-1)^2} + \frac{d}{X-1}$.

We shall continue this example later.

3. Use of parity

If F is even or odd, this gives extra equations on the coefficients of the partial fraction decomposition.

Example. Consider $F = \frac{1}{(X^2+1)^2}$ in $\mathbb{R}(X)$. It is even.

In $\mathbb{C}(X)$, the poles of F are i and $-i$ of multiplicity 2. Therefore, in $\mathbb{C}(X)$ we have

$$F = \frac{a}{(X-i)^2} + \frac{b}{X-i} + \frac{c}{(X+i)^2} + \frac{d}{X+i} \tag{7.3}$$

with a, b, c, d in $\mathbb{C}$.

Conjugating (7.3) gives $F = \dfrac{\bar{a}}{(X+i)^2} + \dfrac{\bar{b}}{X+i} + \dfrac{\bar{c}}{(X-i)^2} + \dfrac{\bar{d}}{X-i}$ so that $a = \bar{c}$ and $b = \bar{d}$.

Moreover, since F is even, we have $F(X) = F(-X) = -\dfrac{a}{(X+i)^2} - \dfrac{b}{X+i} - \dfrac{c}{(X-i)^2} - \dfrac{d}{X-i}$ so that $a = c$ and $-b = d$. Therefore a is a real number and b is an imaginary number.

Multiplying (7.3) by $(X-i)^2$ then evaluating at i gives $\frac{1}{-4} = a$. Evaluating at 0 gives $1 = -2a + i(b - \bar{b})$ so that $b = -\frac{i}{4}$.

Finally, we get $F = -\dfrac{1}{4}\dfrac{1}{(X-i)^2} - \dfrac{i}{4}\dfrac{1}{X-i} + \dfrac{i}{4}\dfrac{1}{X+i} - \dfrac{1}{4}\dfrac{1}{(X+i)^2}$.

Note that F is its own partial fraction decomposition in $\mathbb{R}(X)$.

4. Evaluating at elements in $\mathbb{K}$

When there are few coefficients remaining to be found, it can be helfpul to evaluate at an element $\alpha \in \mathbb{K}$ which is not a pole of F. Note that evaluating the expression of $F \in \mathbb{R}(X)$ at an element $\alpha \in \mathbb{C}$ gives two equations.

Example. Consider $F = \frac{1}{X^2(X+1)}$ in $\mathbb{R}(X)$. The poles are -1 (simple) and 0 (double). Therefore $F = \frac{a}{X+1} + \frac{b}{X^2} + \frac{c}{X}$.

Multiplying by $X+1$ then evaluating at -1 gives $a = 1$. Evaluating at i gives $-\frac{1}{2}(1-i) = \frac{a(1-i)}{2} - b - ci$ which gives two equations, $\frac{a}{2} - b = -\frac{1}{2}$ and $-\frac{a}{2} - c = \frac{1}{2}$. Therefore $a = 1$, $b = 1$ and $c = -1$.

Finally, $F = \frac{1}{X+1} + \frac{1}{X^2} - \frac{1}{X}$.

5. Use of $\lim_{x\to+\infty} xF(x)$

We assume here that $\deg F < 0$ (the integral part is 0). Then multiplying the expression of F by x then taking the limit when x goes to $+\infty$ gives an equation between some of the coefficients.

Example. Consider $F = \frac{X+1}{X^2(X-1)^2}$ in $\mathbb{R}(X)$. Recall that we had found $F = \frac{1}{X^2} + \frac{b}{X} + \frac{2}{(X-1)^2} + \frac{d}{X-1}$.

Now multiply by X; this gives $\dfrac{X+1}{X(X-1)^2} = \dfrac{1}{X} + b + \dfrac{2X}{(X-1)^2} + \dfrac{dX}{X-1}$. Taking the limit when X goes to $+\infty$ gives $0 = b + d$. Therefore $b = -d$.

To complete the decomposition, we can evaluate for instance at -1. This gives $0 = 1 - b + \frac{1}{2} - \frac{d}{2} = \frac{3}{2} - \frac{b}{2}$ so that $b = 3$.

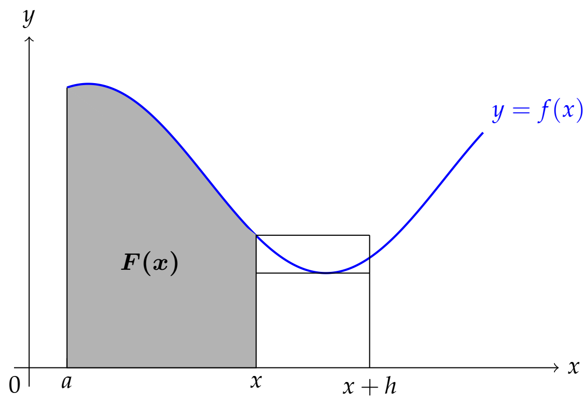
Finally, $F = \dfrac{1}{X^2} + \dfrac{3}{X} + \dfrac{2}{(X-1)^2} - \dfrac{3}{X-1}$.

Chapter 8

Integration

Given a function $f : I \to \mathbb{R}$, we would like to find a differentiable function F such that $F' = f$. This will not always be possible of course, since a derivative function is not just any function. For instance if $f : [0,2] \to \mathbb{R}$ is defined by $f(x) = \begin{cases} 0 & \text{if } 0 \leqslant x < 1 \\ 1 & \text{if } x = 1 \\ 0 & \text{if } 1 < x \leqslant 2 \end{cases}$ then if F exists we must have $F(x) = a$ for $x \in [0,1[$ and $F(x) = b$ for $x \in]1,2]$ for some constants a and b. Since F is differentiable, it is in particular continuous, so that $\lim_{x\to 1^-} F(x) = \lim_{x\to 1^+} F(x) = F(1)$ and therefore $a = b = F(1)$, so that F is constant on $[a,b]$ and $F' = 0 \neq f$.

Now assume that f is continuous. Provided we can give a precise meaning to the area between the x axis, the graph of f and the vertical lines at a and at x, the shaded area $F(x)$ is a good candidate.



Fix real numbers a and b with $a < b$. We shall consider functions defined on $[a,b]$.

I. Integration of step functions

Definition 1. *A* ***partition***[a] *of $[a,b]$ is a finite collection S of points in $[a,b]$, one of which is a and the other is b. We shall write $S = \{s_0, \ldots, s_n\}$ where $a = s_0 < s_1 < \ldots < s_{n-1} < s_n = b$. The* ***mesh***[b] *of the partition S is $\max\limits_{0\leqslant i<n} (s_{i+1} - s_i)$.*

[a] subdivision
[b] pas

Definition 2. *A function $\varphi : [a,b] \to \mathbb{R}$ is called a* ***step function***[a] *if there exists a partition $S = \{s_0, \ldots, s_n\}$ of $[a,b]$ such that for each i with $0 \leqslant i \leqslant n-1$, the function φ is constant on $]s_i, s_{i+1}[$.*
Such a partition is said to be ***adapted***[b] *to φ.*

[a] fonction en escalier *ou* étagée
[b] adaptée

Lemma 3. Let $\varphi : [a,b] \to \mathbb{R}$ be a step function. If $S = \{s_0, \ldots, s_n\}$ is a partition adapted to φ and if $\varphi(x) = m_i$ on the interval $]s_i, s_{i+1}[$ for all i with $0 \leqslant i \leqslant n-1$, then the number $\sum_{i=0}^{n-1} m_i(s_{i+1} - s_i)$ does not depend on the choice of S (adapted to φ).

Proof. ➢ Special case: $S \subset T$. To simplify notation, we shall assume that $T = S \cup \{u\}$ with $s_i < u < s_{i+1}$. Since φ is constant on $]s_i, s_{i+1}[$ we have $\varphi(x) = m_i$ on $]s_i, u[$ and on $]u, s_{i+1}[$. Then

$$\begin{aligned}\sum_{j=0}^{n-1} m_j(s_{j+1} - s_j) &= \sum_{j=0}^{i-1} m_j(s_{j+1} - s_j) + m_i(u - s_i) + m_i(s_{i+1} - u) + \sum_{j=i+1}^{n-1} m_j(s_{j+1} - s_j) \\ &= \sum_{k=0}^{n} p_k(t_{k+1} - t_k)\end{aligned}$$

where $T = \{t_0, \ldots, t_{n+1}\}$ and p_k is the value of φ on $]t_k, t_{k+1}[$, as required.

When $S \subset T$ in general, since both sets are finite, we can write $T = S \cup \{u_1, \ldots, u_r\}$ and do an induction on r (the case $r = 1$ is done).

➢ General case: S and T are not necessarily contained in each other. Set $U = S \cup T$. Then U is a partition of $[a, b]$ adapted to φ, $S \subset U$ and $T \subset U$. If we set $U = \{u_0, \ldots, u_r\}$, m_i the value of φ on $]s_i, s_{i+1}[$, p_k the value of φ on $]t_k, t_{k+1}[$, and q_j the value of φ on $]u_j, u_{j+1}[$, then the special case shows that

$$\sum_{i=0}^{n-1}(s_{i+1} - s_i)m_i = \sum_{j=0}^{r-1}(u_{j+1} - u_j)q_j = \sum_{k=0}^{m-1}(t_{k+1} - t_k)p_k. \qquad ✓$$

Definition 4. *Let φ be a step function. Then, with the notations of the previous definition and lemma, we define the* ***integral***[a] *of φ on $[a, b]$ by:*

$$\int_a^b \varphi = \int_a^b \varphi(x)\,\mathrm{d}x := \sum_{i=0}^{n-1} m_i(s_{i+1} - s_i).$$

[a] intégrale

Remark. This definition is independant of the choice of partition of $[a, b]$ adapted to φ by the previous lemma.

Remark. This is the (signed) area under the graph of φ as we know it.

Notation. Given any map $h : X \to Y$ between two sets X and Y and any subset A of X, the **restriction** of h to A is the map $h_{|A} : A \to Y$ defined by $h_{|A}(a) = h(a)$ for all $a \in A$.

If φ is a step function on $[a, b]$ and if $c \in]a, b[$, then $\varphi_{|[a,c]}$ is a step function on $[a, c]$ and we define $\displaystyle\int_a^c \varphi = \int_a^c \varphi_{|[a,c]}$.

Proposition 5. Let φ and ψ be two step functions on $[a, b]$.

(1) If $\varphi \leqslant \psi$ we have $\displaystyle\int_a^b \varphi(x)\,\mathrm{d}x \leqslant \int_a^b \psi(x)\,\mathrm{d}x$,

(2) The function $\varphi + \psi$ is a step function and $\displaystyle\int_a^b (\varphi(x) + \psi(x))\,\mathrm{d}x = \int_a^b \varphi(x)\,\mathrm{d}x + \int_a^b \psi(x)\,\mathrm{d}x$

(3) For any real number λ, the function $\lambda\varphi$ is a step function and $\displaystyle\int_a^b (\lambda\varphi(x))\,\mathrm{d}x = \lambda \int_a^b \varphi(x)\,\mathrm{d}x$,

(4) Two step functions that are equal except at a finite number of points have the same integral.

(5) For any $c \in]a, b[$, we have $\displaystyle\int_a^b \varphi(x)\,\mathrm{d}x = \int_a^c \varphi(x)\,\mathrm{d}x + \int_c^b \varphi(x)\,\mathrm{d}x$ (Chasles relation).

Proof. Let S (*resp.* T) be a partition of $[a, b]$ adapted to φ (*resp.* ψ). Replacing S and T by $S \cup T$ if necessary (which does not change the integrals), we may assume that $S = T = \{s_0, \ldots, s_n\}$. Let m_i (*resp.* p_i) be the value of φ (*resp.* ψ) on $]s_i, s_{i+1}[$ for all i.

(1) Since $\varphi \leqslant \psi$ we have $m_i \leqslant p_i$ for all i. Then $\displaystyle\int_a^b \varphi(x)\,\mathrm{d}x = \sum_{i=0}^{n-1}(s_{i+1} - s_i)m_i \leqslant \sum_{i=0}^{n-1}(s_{i+1} - s_i)p_i = \int_a^b \psi(x)\,\mathrm{d}x$.

(2) The function $\varphi + \psi$ is constant, equal to $m_i + p_i$, on $]s_i, s_{i+1}[$, therefore it is a step function. Moreover,

$$\begin{aligned}\int_a^b (\varphi(x) + \psi(x))\,\mathrm{d}x &= \sum_{i=0}^{n-1}(m_i + p_i)(s_{i+1} - s_i) \\ &= \sum_{i=0}^{n-1} m_i(s_{i+1} - s_i) + \sum_{i=0}^{n-1} p_i(s_{i+1} - s_i) \\ &= \int_a^b \varphi(x)\,\mathrm{d}x + \int_a^b \psi(x)\,\mathrm{d}x.\end{aligned}$$

(3) *[Not done in class]* The function $\lambda\varphi$ is constant, equal to λm_i, on $]s_i, s_{i+1}[$, therefore it is a step function. Moreover,

$$\int_a^b (\lambda\varphi(x))\,\mathrm{d}x = \sum_{i=0}^{n-1} (\lambda m_i)(s_{i+1} - s_i) = \lambda \sum_{i=0}^{n-1} m_i(s_{i+1} - s_i) = \lambda \int_a^b \varphi(x)\,\mathrm{d}x.$$

(4) If φ and ψ are equal except at a finite number of points, then $\varphi - \psi$ is a step function which is zero except at a finite number of points. Let $T = \{t_0, t_1, \dots, t_p\}$ be the set of points at which $\varphi - \psi$ is non-zero to which we have added a and b if necessary. Then T is a partition of $[a, b]$ that is adapted to $\varphi - \psi$. Therefore

$$\int_a^b (\varphi - \psi)(x)\,\mathrm{d}x = \sum_{j=0}^{p-1} 0(t_{j+1} - t_j) = 0.$$

Using **(2)** and **(3)** we get $\int_a^b \varphi(x)\,\mathrm{d}x = \int_a^b \psi(x)\,\mathrm{d}x$

(5) *[Not done in class]* By Lemma 3 we may assume that $c \in S$ (adding it to the partition will not change the integrals). Say $c = s_r$. Then $\{s_0, \dots, s_r\}$ is a partition of $[a, c]$ adapted to $\varphi_{|[a,c]}$ and $\{s_r, \dots, s_n\}$ is a partition of $[c, b]$ adapted to $\varphi_{|[c,b]}$. We then have

$$\begin{aligned}\int_a^c (\varphi(x))\,\mathrm{d}x + \int_c^b (\varphi(x))\,\mathrm{d}x &= \sum_{i=0}^{r-1} m_i(s_{i+1} - s_i) + \sum_{i=r}^{n-1} m_i(s_{i+1} - s_i) \\ &= \sum_{i=0}^{n-1} m_i(s_{i+1} - s_i) = \int_a^b (\varphi(x))\,\mathrm{d}x.\end{aligned}$$

✓

II. Integrable functions

Definition-Proposition 6. *A function f on $[a, b]$ is* ***integrable***[a] *if it is bounded and*

$$\sup\left\{\int_a^b \varphi(x)\,\mathrm{d}x; \varphi \text{ step function and } \varphi \leqslant f\right\} = \inf\left\{\int_a^b \psi(x)\,\mathrm{d}x; \psi \text{ step function and } \psi \geqslant f\right\}.$$

[a] intégrable

Proof. We must verify that this definition makes sense. Set $A_f := \left\{\int_a^b \varphi; \varphi \text{ step function and } \varphi \leqslant f\right\}$ and $B_f = \left\{\int_a^b \psi; \psi \text{ step function and } \psi \geqslant f\right\}$. We must prove that A_f has a supremum and that B_f has an infimum.

First note that both sets are non-empty: since f is bounded, the constant functions $\varphi_0 = \inf f$ and $\psi_0 = \sup f$ satisfy $\varphi_0 \leqslant f \leqslant \psi_0$ so that $\int_a^b \varphi_0$ is in A_f and $\int_a^b \psi_0$ is in B_f.

Moreover, A_f is bounded above by $\int_a^b \psi_0$ therefore it has a supremum, and B_f is bounded below by $\int_a^b \varphi_0$ therefore it has an infimum. ✓

Remark. Note that we always have $\sup A_f \leqslant \inf B_f$. Indeed, for any step functions φ and ψ with $\varphi \leqslant g \leqslant \varphi$, we have $\int_a^b \varphi \leqslant \int_a^b \psi$ so that $\int_a^b \psi$ is an upper bound for A_f and therefore $\sup A_f \leqslant \int_a^b \psi$. Therefore $\sup A_f$ is a lower bound for B_f and we have $\sup A_f \leqslant \inf B_f$.

Proposition 7. A step function is integrable.

Proof. If φ is a step function, then it is bounded because it only takes a finite number of values, and clearly $\sup A_\varphi \geqslant \int_a^b \varphi \geqslant \inf B_\varphi \geqslant \sup A_\varphi$ so that $\sup A_\varphi = \inf B_\varphi = \int_a^b \varphi$. ✓

Proposition 8. Let $f : [a, b] \to \mathbb{R}$ be a function. Then f is integrable if and only if for any $\varepsilon > 0$ there exist step functions φ and ψ such that $\varphi \leqslant f \leqslant \psi$ and $\int_a^b (\psi - \varphi) \leqslant \varepsilon$.

Proof. ➢ First assume that f is integrable so that $\sup A_f = \inf B_f$. Fix $\varepsilon > 0$.

By definition of the supremum and the infimum, there exist step functions φ and ψ such that $\varphi \leqslant f \leqslant \psi$ and $\sup A_f - \frac{\varepsilon}{2} \leqslant \int_a^b \varphi \leqslant \sup A_f$ and $\inf B_f \leqslant \int_a^b \psi \leqslant \inf B_f + \frac{\varepsilon}{2}$. Therefore we have $\int_a^b (\psi - \varphi) = \int_a^b \psi - \int_a^b \varphi \leqslant (\inf B_f + \frac{\varepsilon}{2}) - (\sup A_f - \frac{\varepsilon}{2}) = \varepsilon$.

➢ Now assume that for any $\varepsilon > 0$ there exist step functions φ and ψ such that $\varphi \leqslant f \leqslant \psi$ and $\int_a^b (\psi - \varphi) \leqslant \varepsilon$. In particular, f is bounded (since any step function is bounded). Fix $\varepsilon > 0$ and let φ and ψ be as above. Clearly $\sup A_f \geqslant \int_a^b \varphi$ and $\inf B_f \leqslant \int_a^b \psi$ so that $0 \leqslant \inf B_f - \sup A_f \leqslant \int_a^b \psi - \int_a^b \varphi = \int_a^b (\psi - \varphi)$ and therefore $0 \leqslant \inf B_f - \sup A_f \leqslant \varepsilon$. This is true for any $\varepsilon > 0$, therefore $\sup A_f = \inf B_f$ and f is integrable. ✓

Definition 9. *If f is an integrable function on $[a, b]$, the* ***integral***[a] *of f on $[a, b]$ is*

$$\int_a^b f = \int_a^b f(t)\,\mathrm{d}t := \sup\left\{ \int_a^b \varphi; \varphi \text{ step function}, \varphi \leqslant f \right\} = \inf\left\{ \int_a^b \psi; \psi \text{ step function}, \psi \geqslant f \right\}.$$

[a] intégrale

Remark. Note that the integral just defined for an integrable function generalises the integral of a step function.

Example. There exist functions that are not integrable. For instance, let $f : [0, 1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Let φ be a step function such that $\varphi \leqslant f$. Let $S = \{s_0, s_1, \ldots, s_n\}$ be a partition adapted to φ. For each i with $0 \leqslant i \leqslant n-1$, there exists $r_i \in \,]s_i, s_{i+1}[$ such that $r_i \notin \mathbb{Q}$ (because $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$). In particular, $\varphi(r_i) \leqslant f(r_i) = 0$. But φ is constant on $]s_i, s_{i+1}[$, therefore $\varphi_{|\,]s_i, s_{i+1}[} \leqslant 0$. Finally $\varphi \leqslant 0$.

Similarly, using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, any step function ψ with $\psi \geqslant f$ must satisfy $\psi \geqslant 1$.

In particular, for any step functions φ and ψ such that $\varphi \leqslant f \leqslant \psi$, we have $\int_0^1 (\psi - \varphi) \geqslant 1$. By Proposition 8, f is not integrable.

We could also note that $\sup A_f \leqslant 0 < 1 \leqslant \inf B_f$ and use the definition to prove that f is not integrable.

Proposition 10. Any monotonic function on $[a, b]$ is integrable.

Proof. [Not done in class] We prove it for a non-decreasing function (the case of a non-increasing function is obtained in a similar way or by changing f to $-f$). Note that f is bounded below by $f(a)$ and bounded above by $f(b)$ hence f is bounded. A constant function is integrable (step function), so we may assume that f is not constant and therefore $f(a) < f(b)$.

If S is any partition of $[a, b]$, define step functions $\varphi_{f,S}$ and $\psi_{f,S}$ by

$$\begin{aligned}
\varphi_{f,S}(t) &= \inf_{[s_i, s_{i+1}[} f \quad \text{if } t \in [s_i, s_{i+1}[\text{ for } 0 \leqslant i < n \\
\psi_{f,S}(t) &= \sup_{[s_i, s_{i+1}[} f \quad \text{if } t \in [s_i, s_{i+1}[\text{ for } 0 \leqslant i < n \\
\varphi_{f,S}(b) &= f(b) = \psi_{f,S}(b).
\end{aligned}$$

For $x < y$ in $[a, b]$ and any $t \in [x, y[$, we have $f(x) = \inf_{[x,y[} f \leqslant f(t) \leqslant \sup_{[x,y[} f \leqslant f(y)$ hence $\varphi_{f,S} \leqslant f \leqslant \psi_{f,S}$. Moreover, if $t \in [s_i, s_{i+1}[$ we have $\varphi_{f,S}(t) = f(s_i)$ and $\psi_{f,S}(t) \leqslant f(s_{i+1})$.

For any $\varepsilon > 0$, choose an integer $n \geqslant \dfrac{(b-a)(f(b) - f(a))}{\varepsilon}$. Consider the partition of $[a, b]$ defined by $s_i = a + i\frac{b-a}{n}$ for $i = 0, \ldots, n$ (any partition S with $\eta = \operatorname{mesh}(S) \leqslant \frac{\varepsilon}{f(b) - f(a)}$ works) and the corresponding step functions $\varphi_{f,S}$ and $\psi_{f,S}$. Then we have

$$0 \leqslant \int_a^b \psi_{f,S} - \int_a^b \varphi_{f,S} \leqslant \sum_{i=0}^{n-1} (s_{i+1} - s_i)(f(s_{i+1}) - f(s_i)) \leqslant \eta \sum_{i=0}^{n-1} (f(s_{i+1}) - f(s_i)) = \eta(f(b) - f(a)) \leqslant \varepsilon.$$

Therefore, for any $\varepsilon > 0$, we have $0 \leqslant \int_a^b \psi_{f,S} - \int_a^b \varphi_{f,S} \leqslant \varepsilon$. Hence f is integrable by Proposition 8. ✓

Theorem 11. If f is continuous on $[a, b]$ then f is integrable on $[a, b]$.

Proof. We will accept this result without proof. ✓

Theorem 12. Let f and g be integrable functions on $[a,b]$ and let λ be a real number. Then:

(1) $f+g$ and λf are integrable on $[a,b]$ and we have

$$\int_a^b (f(x)+g(x)\,\mathrm{d}x) = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x \quad \text{and} \quad \int_a^b (\lambda f(x))\,\mathrm{d}x = \lambda \int_a^b f(x)\,\mathrm{d}x.$$

(2) if for all $x \in [a,b]$ we have $f(x) \geqslant 0$, then $\int_a^b f(x)\,\mathrm{d}x \geqslant 0$. In particular, if $f \leqslant g$ on $[a,b]$ we have $\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x$.

(3) If h is a function on $[a,b]$ that is equal to f except at a finite number of points, then h is integrable and $\int_a^b h = \int_a^b f$.

(4) (Chasles relation) For any $c \in]a,b[$, the function f is integrable on $[a,b]$ if and only if it is integrable on $[a,c]$ and on $[c,b]$, and when f is integrable on $[a,b]$ we have $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$.

Proof. (1) ➢ *[Not done in class.]* We first prove that $f+g$ is integrable and that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. Fix $\varepsilon > 0$. Since f and g are integrable, there exist step functions φ_1, ψ_1, φ_2 and ψ_2 such that

$$\varphi_1 \leqslant f \leqslant \psi_1,\ \varphi_2 \leqslant g \leqslant \psi_2,\ \int_a^b (\psi_1 - \varphi_1) \leqslant \frac{\varepsilon}{2} \text{ and } \int_a^b (\psi_2 - \varphi_2) \leqslant \frac{\varepsilon}{2}.$$

Set $\varphi_3 = \varphi_1 + \varphi_2$ and $\psi_3 = \psi_1 + \psi_2$. Then φ_3 and ψ_3 are step functions, $\varphi_3 \leqslant f+g \leqslant \psi_3$ and $\int_a^b (\psi_3 - \varphi_3) = \int_a^b (\psi_1 - \varphi_1) + \int_a^b (\psi_2 - \varphi_2) \leqslant \varepsilon$. Therefore $f+g$ is integrable.

Since $\varphi_3 \leqslant f+g \leqslant \psi_3$ we have

$$\int_a^b \varphi_1 + \int_a^b \varphi_2 = \int_a^b \varphi_3 \leqslant \int_a^b (f+g) \leqslant \int_a^b \psi_3 = \int_a^b \psi_1 + \int_a^b \psi_2.$$

Moreover, we also have

$$\int_a^b \varphi_1 + \int_a^b \varphi_2 \leqslant \int_a^b f + \int_a^b g \leqslant \int_a^b \psi_1 + \int_a^b \psi_2.$$

Subtracting these inequalities gives

$$\int_a^b \varphi_1 + \int_a^b \varphi_2 - \int_a^b \psi_1 - \int_a^b \psi_2 \leqslant \int_a^b (f+g) - \int_a^b f - \int_a^b g \leqslant \int_a^b \psi_1 + \int_a^b \psi_2 - \int_a^b \varphi_1 - \int_a^b \varphi_2.$$

The term on the right is equal to $\int_a^b (\psi_1 - \varphi_1) + \int_a^b (\psi_2 - \varphi_2) \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and similarly the term on the left is $\geqslant -\varepsilon$. Therefore

$$-\varepsilon \leqslant \int_a^b (f+g) - \int_a^b f - \int_a^b g \leqslant \varepsilon.$$

Since this is true for any $\varepsilon > 0$ we get $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

➢ We now prove that λf is integrable and that $\int_a^b \lambda f = \lambda \int_a^b f$.

- If $\lambda = 0$ the result is clear so we assume that $\lambda \neq 0$.
- We prove that $-f$ is integrable and that $\int_a^b -f = -\int_a^b f$. Fix $\varepsilon > 0$. Since f is integrable, there exist step functions φ and ψ such that

$$\varphi \leqslant f \leqslant \psi \text{ and } \int_a^b (\psi - \varphi) \leqslant \varepsilon.$$

Then $-\varphi$ and $-\psi$ are step functions and

$$-\psi \leqslant -f \leqslant -\varphi \text{ and } \int_a^b ((-\varphi) - (-\psi)) = \int_a^b \psi - \varphi \leqslant \varepsilon$$

therefore $-f$ is integrable. Moreover,

$$-\int_a^b (\psi - \varphi) = \int_a^b (-\psi) + \int_a^b \varphi \leqslant \int_a^b (-f) + \int_a^b f \leqslant \int_a^b (-\varphi) + \int_a^b \psi = \int_a^b \psi - \varphi$$

so that $-\varepsilon \leqslant \int_a^b (-f) + \int_a^b f \leqslant \varepsilon$ for all $\varepsilon > 0$ and finally $\int_a^b (-f) = -\int_a^b f$.

Consequently, it is enough to prove the required result for $\lambda > 0$.

- Now assume that $\lambda > 0$. Fix $\varepsilon > 0$. Since f is integrable, there exist step functions φ and ψ such that

$$\varphi \leqslant f \leqslant \psi \text{ and } \int_a^b (\psi - \varphi) \leqslant \frac{\varepsilon}{\lambda}.$$

Then $\lambda\varphi$ and $\lambda\psi$ are step functions and

$$\lambda\varphi \leqslant \lambda f \leqslant \lambda\psi \text{ and } \int_a^b ((\lambda\psi) - (\lambda\varphi) = \lambda \int_a^b (\psi - \varphi) \leqslant \varepsilon$$

therefore λf is integrable. Moreover,

$$-\lambda \int_a^b (\psi - \varphi) = \int_a^b (\lambda\varphi) - \lambda \int_a^b \psi \leqslant \int_a^b (\lambda f) - \lambda \int_a^b f \leqslant \int_a^b (\lambda\psi) - \lambda \int_a^b \varphi = \lambda \int_a^b (\psi - \varphi)$$

so that $-\varepsilon \leqslant \int_a^b (\lambda f) - \lambda \int_a^b f \leqslant \varepsilon$ for all $\varepsilon > 0$ and finally $\int_a^b (\lambda f) = \lambda \int_a^b f$.

(2) Assume that f is integrable and $f \geqslant 0$. The zero function is a step function that is less than f, therefore $0 = \int_a^b 0 \leqslant \int_a^b f$.

If f and g are integrable with $f \geqslant g$ then $f - g$ is integrable (by the previous two points) and $f - g \geqslant 0$ therefore $\int_a^b (f - g) \geqslant 0$ and (again by the previous two points) $\int_a^b f - \int_a^b g \geqslant 0$.

(3) Assume that f and h are equal except at a finite number of points. Define $\varphi = h - f$; it is equal to zero except at a finite number of points hence it is a step function. Then $h = f + \varphi$ is integrable as the sum of two integrable functions. Moreover, φ is a step function that is 0 except at a finite number of points, therefore $\int_a^b h = \int_a^b (f + \varphi) = \int_a^b f + \int_a^b \varphi = \int_a^b f$.

(4) *[Not done in class.]*

➢ First assume that f is integrable on $[a, b]$. We must prove that f is integrable on $[a, c]$ and on $[c, b]$, that is, that $f_{|[a,c]}$ and $f_{|[c,b]}$ are integrable.

Fix $\varepsilon > 0$. There exist step functions φ and ψ such that $\varphi \leqslant f \leqslant \psi$ and $\int_a^b (\psi - \varphi) \leqslant \varepsilon$.

We then have $\varphi_{|[a,c]} \leqslant f_{|[a,c]} \leqslant \psi_{|[a,c]}$ and $\varphi_{|[c,b]} \leqslant f_{|[c,b]} \leqslant \psi_{|[c,b]}$. We also have

$$\int_a^c (\psi_{|[a,c]} - \varphi_{|[a,c]}) + \int_c^b (\psi_{|[c,b]} - \varphi_{|[c,b]}) = \int_a^c (\psi - \varphi) + \int_c^b (\psi - \varphi) = \int_a^b (\psi - \varphi) \leqslant \varepsilon$$

using the Chasles relation for step functions. Since both terms on the left are non-negative, we have $\int_a^c (\psi_{|[a,c]} - \varphi_{|[a,c]}) \leqslant \varepsilon$ and $\int_c^b (\psi_{|[c,b]} - \varphi_{|[c,b]}) \leqslant \varepsilon$. Therefore $f_{|[a,c]}$ and $f_{|[c,b]}$ are integrable.

Moreover, $\int_a^b \varphi \leqslant \int_a^b f \leqslant \int_a^b \psi$ and

$$\begin{aligned}\int_a^b \varphi = \int_a^c \varphi + \int_c^b \varphi = \int_a^c \varphi_{|[a,c]} + \int_c^b \varphi_{|[c,b]} &\leqslant \int_a^c f_{|[a,c]} + \int_c^b f_{|[c,b]} = \int_a^c f + \int_c^b f \\ &\leqslant \int_a^c \psi_{|[a,c]} + \int_c^b \psi_{|[c,b]} = \int_a^c \psi + \int_c^b \psi = \int_a^b \psi\end{aligned}$$

so that

$$\int_a^b (\varphi - \psi) \leqslant \int_a^b f - \left(\int_a^c f + \int_c^b f\right) \leqslant \int_a^b (\psi - \varphi)$$

hence

$$-\varepsilon \leqslant \int_a^b f - \left(\int_a^c f + \int_c^b f\right) \leqslant \varepsilon$$

and since this is true for all $\varepsilon > 0$ we get $\int_a^b f = \int_a^c f + \int_c^b f$.

➢ Now assume that f is integrable on $[a, c]$ and on $[c, b]$. We must prove that f is integrable on $[a, b]$. Fix $\varepsilon > 0$. Then there are step functions φ_1 and ψ_1 on $[a, c]$ and φ_2 and ψ_2 on $[c, b]$ such that

$$\varphi_1 \leqslant f_{|[a,c]} \leqslant \psi_1 \text{ and } \int_a^b (\psi_1 - \varphi_1) \leqslant \frac{\varepsilon}{2}$$

$$\varphi_2 \leqslant f_{|[c,b]} \leqslant \psi_2 \text{ and } \int_a^b (\psi_2 - \varphi_2) \leqslant \frac{\varepsilon}{2}.$$

Define step functions on $[a,b]$ by $\varphi = \varphi_1 + \varphi_2$ and $\psi = \psi_1 + \psi_2$ (that is, $\varphi(x) = \varphi_1(x)$ for $x \in [a,c[$, $\varphi(x) = \varphi_2(x)$ for $x \in]c,b]$ and $\varphi(c) = \varphi_1(c) + \varphi_2(c)$). Let χ be the step function which is equal to 0 everywhere except at c where $\chi(c) = f(c)$. Then

$$\varphi \leqslant f + \chi \leqslant \psi \text{ and } \int_a^b (\psi - \varphi) + \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $f + \chi$ is integrable on $[a,b]$ and so is f. ✓

Notation. We have defined $\int_a^b f$ when $a < b$. We set

$$\int_a^a f = 0 \quad \text{and} \quad \int_b^a f = -\int_a^b f \text{ when } a < b$$

so that the Chasles relation is always true (not only if $a < c < b$).

Theorem 13. Let $f : [a,b] \to \mathbb{R}$ be a *continuous* function that does not change sign. If $\int_a^b f = 0$ then $f = 0$.

Proof. We prove the result when $f(x) \geqslant 0$ for all $x \in [a,b]$ (the case $f \leqslant 0$ can be deduced from it by considering $-f$).

Assume for a contradiction that $f \neq 0$. Then there exists $\alpha \in [a,b]$ such that $f(\alpha) > 0$. Since f is continuous, there is an interval $[c,d] \subset [a,b]$ containing α and such that $f(x) \geqslant \frac{f(\alpha)}{2}$ for all $x \in [c,d]$. Note that $\int_a^c f \geqslant 0$ and $\int_d^b f \geqslant 0$ since f is non-negative. We then have

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \int_a^c f(x)\,\mathrm{d}x + \int_c^d f(x)\,\mathrm{d}x + \int_d^b f(x)\,\mathrm{d}x \\ &\geqslant \int_c^d f(x)\,\mathrm{d}x \geqslant \int_c^d \frac{f(\alpha)}{2}\,\mathrm{d}x \\ &= (d-c)\frac{f(\alpha)}{2} > 0,\end{aligned}$$

a contradiction. Therefore $f = 0$. ✓

Proposition 14 (Mean Value Theorem for integrals[a]). Let $f : [a,b] \to \mathbb{R}$ be an integrable function. If m and M are real numbers such that $m \leqslant f(x) \leqslant M$ for all $x \in [a,b]$, then

$$m \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant M.$$

[a] formule de la moyenne

Proof. We have $m(b-a) = \int_a^b m\,\mathrm{d}t \leqslant \int_a^b f(t)\,\mathrm{d}t \leqslant \int_a^b M\,\mathrm{d}t = M(b-a)$. Since $b - a > 0$ we get the result. ✓

Proposition 15. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a,b]$ such that $\int_a^b f(x)\,\mathrm{d}x = (b-a)f(c)$.

Proof. Since f is continuous on a closed bounded interval, it has a maximum M and a minimum m and $f([a,b]) = [m,M]$. The previous proposition shows that $\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$ is in $[m,M]$, therefore there exists $c \in [a,b]$ such that $\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = f(c)$. ✓

More generally, we have

Proposition 16. Let f be a continuous function on $[a,b]$ and g a non-negative continuous function on $[a,b]$. Then there exists $\theta \in [a,b]$ such that $\int_a^b (fg) = f(\theta)\int_a^b g$.

Proof. Since f is continuous on a closed bounded interval, it is bounded: $f([a,b]) = [m,M]$. Since g is non-negative, we have $mg \leqslant fg \leqslant Mg$ so that $m\int_a^b g \leqslant \int_a^b (fg) \leqslant M\int_a^b g$. If $g = 0$ the result is clear. If g is non-zero, then $\int_a^b g > 0$ (g is continuous and non-negative) so that $m \leqslant \dfrac{\int_a^b (fg)}{\int_a^b g} \leqslant M$. Therefore there exists $\theta \in [a,b]$ such that $\dfrac{\int_a^b (fg)}{\int_a^b g} = f(\theta)$. ✓

Proposition 17. Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Then $|f|$ is integrable and

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \int_a^b |f(t)|\,\mathrm{d}t.$$

Proof. For any function h on $[a, b]$, define h^+ and h^- on $[a, b]$ by

$$h^+(x) = \begin{cases} h(x) & \text{if } h(x) \geqslant 0 \\ 0 & \text{if } h(x) < 0 \end{cases} \quad \text{and} \quad h^-(x) = \begin{cases} 0 & \text{if } h(x) > 0 \\ -h(x) & \text{if } h(x) \leqslant 0. \end{cases}$$

Then we have

$$h = h^+ - h^- \quad \text{and} \quad |h| = h^+ + h^-.$$

Moreover, if h is a step function, then so are h^+ and h^-.

Fix $\varepsilon > 0$. Since f is integrable, there exist step functions φ and ψ such that $\varphi \leqslant f \leqslant \psi$ and $\int_a^b (\psi - \varphi) \leqslant \varepsilon$. Then we have

$$\begin{aligned} \varphi^+ &\leqslant f^+ \leqslant \psi^+ \\ \varphi^- &\geqslant f^- \geqslant \psi^-. \end{aligned}$$

Therefore $\varphi^+ + \psi^-$ and $\varphi^- + \psi^+$ are step functions such that

$$\varphi^+ + \psi^- \leqslant f^+ + f^- = |f| \leqslant \varphi^- + \psi^+$$

$$\text{and } \int_a^b ((\varphi^- + \psi^+) - (\varphi^+ + \psi^-)) = \int_a^b ((\psi^+ - \psi^-) - (\varphi^+ - \varphi^-)) = \int_a^b (\psi - \varphi) \leqslant \varepsilon$$

therefore $|f|$ integrable.

Moreover, $-|f| \leqslant f \leqslant |f|$ so $-\int_a^b |f| \leqslant \int_a^b f \leqslant \int_a^b |f|$. Since $|f| \geqslant 0$ and therefore $\int_a^b |f| \geqslant 0$, we have $\left|\int_a^b f\right| \leqslant \int_a^b |f|$. ✓

Corollary 18. For any x, y in $[a, b]$, we have

$$\left|\int_x^y f\right| \leqslant \left|\int_x^y |f|\right|.$$

Proof. If $x < y$ the inequality is true by the previous result. If $x = y$ we have 0 on both sides so the inequality is true. If $x > y$, then $\left|\int_y^x f\right| \leqslant \int_y^x |f|$

$$\left|\int_x^y f\right| = \left|-\int_y^x f\right| = \left|\int_y^x f\right| \leqslant \int_y^x |f| = \left|\int_x^y |f|\right|.$$ ✓

IV. Some generalisations

A. Complex valued functions

Definition 19. *Let $f : [a, b] \to \mathbb{C}$ be a complex valued function. We can write $f = f_1 + i f_2$ where f_1 and f_2 are functions from $[a, b]$ to $\mathbb{R}$. We say that f is* ***integrable***[a] *if f_1 and f_2 are both integrable, and we define the* ***integral***[b] *of f to be*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f_1(x)\,\mathrm{d}x + i\int_a^b f_2(x)\,\mathrm{d}x.$$

[a] intégrable
[b] intégrale

Remark. Note that $\int_a^b f_1(x)\,\mathrm{d}x = \Re\left(\int_a^b f(x)\,\mathrm{d}x\right)$ and $\int_a^b f_2(x)\,\mathrm{d}x = \Im\left(\int_a^b f(x)\,\mathrm{d}x\right)$. This means that

$$\int_a^b \Re(f) = \Re\left(\int_a^b f\right) \text{ and } \int_a^b \Im(f) = \Im\left(\int_a^b f\right)$$

Proposition 20. Linearity and the Chasles relation remain true for integrable complex valued functions. Moreover, if $f : [a, b] \to \mathbb{C}$ is an integrable function, then so is $|f|$ and we have

$$\left|\int_a^b f\right| \leqslant \int_a^b |f|.$$

Proof. The first part is easy to check.

We shall accept without proof that if $f = f_1 + if_2$ is integrable then $|f| = \sqrt{f_1^2 + f_2^2}$ is integrable. Note that if f is continuous, then so is $|f|$ and therefore $|f|$ is integrable in this case (the main one we shall consider).

When f is integrable, $z := \int_a^b f$ is a complex number: $\int_a^b f = |z|e^{i\theta}$. We then have

$$\begin{aligned} \left|\int_a^b f\right| &= e^{-i\theta}z = \Re\left(e^{-i\theta}z\right) \\ &= \Re\left(e^{-i\theta}\int_a^b f\right) = \Re\left(\int_a^b e^{-i\theta}f\right) = \int_a^b \Re\left(e^{-i\theta}f\right) \\ &\leqslant \int_a^b \left|e^{-i\theta}f\right| = \int_a^b |f|. \end{aligned}$$

✓

B. Piecewise continuous functions

Definition 21. *A function $f : [a,b] \to \mathbb{R}$ is* ***piecewise continuous***[a] *if it is continuous on $[a,b]$ except at a finite number of points where it has finite limits from above and from below.*

[a] continue par morceaux

Example. Continuous functions and step functions are piecewise continuous.

Remark. A function f is piecewise continuous if, and only if, there exists a partition $S = \{s_0, \dots, s_n\}$ of $[a,b]$ such that f is continuous on each $]s_i, s_{i+1}[$ and has limits from above and from below at each s_i (from above only at s_0 and from below only at s_n). Moreover, if f is piecewise continuous, then for each i there is a continuous extension g_i of $f_{|]s_i, s_(i+1)[}$ to $[s_i, s_{i+1}]$.

Proposition 22. A piecewise continuous function is integrable and (with the notations in the remark above)

$$\int_a^b f = \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} f.$$

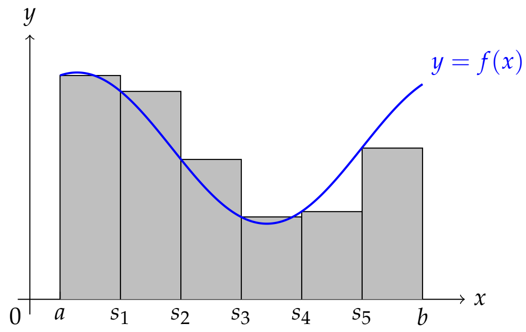
Proof. Define functions f_i for $0 \leqslant i < n$ by $f_i = g_i$ on $[s_i, s_{i+1}]$ and 0 elsewhere. Set $\tilde{f} = f_0 + f_1 + \cdots + f_{n-1}$. Then $\tilde{f}$ is equal to f except perhaps at each s_i. Moreover, the f_i are integrable (f_i is equal to the sum of a continuous function an a step function on $[a,b]$). Therefore $\tilde{f}$ integrable and so is f. Moreover, $\int_a^b f = \int_a^b \tilde{f} = \sum_{i=0}^{n-1} \int_a^b f_i = \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} f.$ ✓

V. Riemann sums

Definition 23. *If f is a function on $[a,b]$, a* ***Riemann sum***[a] *of f is a sum of the form $R_n(f) = \frac{b-a}{n} \sum_{k=0}^{n-1} f(s_k)$ where n is an integer and, for each k, $s_k = a + k\frac{b-a}{n}$.*

[a] somme de Riemann

Remark. The figure below shows the geometric interpretation of a Riemann sum; it is the total area of n rectangles that lie partly below the graph of f and partly above it. Because of the arbitrary way in which the heights of the rectangles have been picked, we cannot say whether a particular Riemann sum is less than or greater than the integral $\int_a^b f$. But the theorem below shows that this does not matter; provided the mesh of the partition n is large enough (that is, the bases of the rectangles are small enough), the Riemann sum is close to the integral.



Theorem 24. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the sequence $(R_n(f))_n$ converges to $\displaystyle\int_a^b f$.

In particular, if $f : [0, 1] \to \mathbb{R}$ is a continuous function, then the sequence $(R_n(f))_n$ defined by $R_n(f) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)$ converges to $\displaystyle\int_0^1 f$.

Proof. *[Admitted.]*

Set $s_k = a + k\frac{b-a}{n}$. Then for a fixed integer n, $S_n = \{s_0, s_1, \ldots, s_n\}$ is a partition of $[a, b]$. Set $m_k = \inf_{[s_k, s_{k+1}]} f$ and $M_k = \sup_{[s_k, s_{k+1}]} f$. For any partition $T = \{t_0, \ldots, t_p\}$ of $[a, b]$, define $\varphi_{f,T}$ and $\psi_{f,T}$ as in Proposition 10. Clearly, for any integer n we have $\displaystyle\int_a^b \varphi_{f,S_n} \leqslant R_n(f) \leqslant \int_a^b \psi_{f,S_n}$.

Fix $\varepsilon > 0$. We know that there exist step functions φ and ψ such that $\varphi \leqslant f \leqslant \psi$ and $\displaystyle\int_a^b (\psi - \varphi) \leqslant \frac{\varepsilon}{2}$. If T is any partition adapted to φ and to ψ, we have $\varphi \leqslant \varphi_{f,T} \leqslant f \leqslant \psi_{f,T} \leqslant \psi$ and therefore $\displaystyle\int_a^b (\psi_{f,T} - \varphi_{f,T}) \leqslant \frac{\varepsilon}{2}$.

Set $M = \sup_{[a,b]} f - \inf_{[a,b]} f \geqslant 0$ and $\eta = \frac{\varepsilon}{2p(M+1)} > 0$. Fix $N \in \mathbb{N}$ such that $\frac{b-a}{N} < \eta$. Then for any $n \geqslant N$ we also have $\frac{b-a}{n} < \eta$. There are at most p intervals $[s_j, s_{j+1}[$ that contain a t_i. For these intervals, we have

$$(s_{j+1} - s_j)\left(\sup_{[s_j, s_{j+1}[} f - \inf_{[s_j, s_{j+1}[} f\right) \leqslant \eta M.$$

All the other intervals $[s_j, s_{j+1}[$ are contained in an $[t_i, t_{i+1}[$ so that $(s_{j+1} - s_j)\left(\sup_{[s_j, s_{j+1}[} f - \inf_{[s_j, s_{j+1}[} f\right) \leqslant (t_{i+1} - t_i)\left(\sup_{[t_i, t_{i+1}[} f - \inf_{[t_i, t_{i+1}[} f\right)$ and summing over j gives

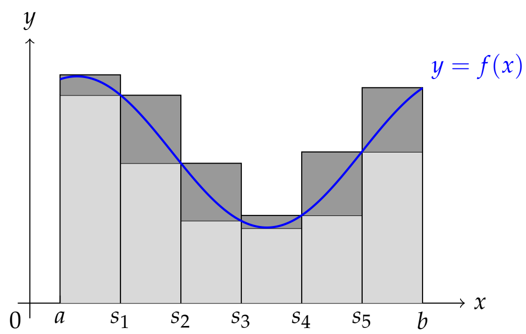
$$\int_a^b (\psi_{f,S_n} - \psi_{f,S_n}) \leqslant \int_a^b (\psi_{f,T} - \psi_{f,T}) + p\eta M < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We have $\displaystyle\int_a^b \varphi_{f,S_n} \leqslant \int_a^b f \leqslant \int_a^b \psi_{f,S_n}$ and $\displaystyle\int_a^b \varphi_{f,S_n} \leqslant R_n(f) \leqslant \int_a^b \psi_{f,S_n}$ so that

$$-\varepsilon < \int_a^b (\psi_{f,S_n} - \psi_{f,S_n}) \leqslant R_n(f) - \int_a^b f \leqslant \int_a^b (\psi_{f,S_n} - \psi_{f,S_n}) < \varepsilon$$

as required. ✓

Remark. The figure below shows a geometric interpretation of $\displaystyle\int_a^b \varphi_{f,S_6}$ (the lightly shaded area) and $\displaystyle\int_a^b \psi_{f,S_6}$ (the whole shaded area (light and dark)).



Application. We want to find the limit (if there is one!) of the sequence u defined by $u_n = \sum_{k=0}^{n-1} \dfrac{1}{2n+3k}$.

We have $u_n = \dfrac{1}{n} \sum_{k=0}^{n-1} \dfrac{1}{2+3\frac{k}{n}} = \dfrac{1}{n} \sum_{k=0}^{n-1} f\left(\dfrac{k}{n}\right)$ where $f : [0,1] \to \mathbb{R}$ is defined by $f(x) = \dfrac{1}{2+3x}$ and is continuous. The proposition above shows that $(u_n)_n$ converges to $\displaystyle\int_0^1 f = \int_0^1 \frac{1}{2+3x}\,\mathrm{d}x = \frac{1}{3}\ln\frac{5}{2}$.

Remark. Assume that $f\colon [a\,;b] \to \mathbb{R}$ is continous. For $n \in \mathbb{N}^*$, define $S_n(f) = \frac{b-a}{n}\sum_{k=1}^{n} f(s_k)$ with $s_k = a + k\frac{b-a}{n}$ for each k. Then the sequence $(S_n(f))_n$ converges to $\displaystyle\int_a^b f$.

Indeed, we have $S_n(f) - R_n(f) = \frac{b-a}{n}(f(1) - f(0))$ which has limit 0 when n goes to $+\infty$.

Chapter 9

Primitives. Integration techniques.

I. Primitive of a function

Definition 1. *Let I be an interval and $f : I \to \mathbb{R}$ be a function.*

- *If I is an open interval, an* ***antiderivative or primitive***[a] *of f is a differentiable function $F : I \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in I$.*
- *If $I = [a,b[$ or $]a,b]$ or $[a,b]$, an* ***antiderivative or primitive***[b] *of f is a continuous function $F : I \to \mathbb{R}$ that is differentiable on $]a,b[$ and such that $F'(x) = f(x)$ for all $x \in]a,b[$.*

[a] primitive
[b] primitive

Proposition 2. Let I be an interval. If F and G are two primitives of a function $f : I \to \mathbb{R}$, the function $F - G$ is constant on I.

Proof. If I is an open interval, then $F - G$ is differentiable on I and we have $(F-G)' = F' - G' = f - f = 0$, therefore $F - G$ is constant on I.

If I is not an open interval, then $F - G$ is differentiable on $]a,b[$ and $(F-G)' = F' - G' = f - f = 0$ on $]a,b[$, therefore $F - G$ is constant on $]a,b[$. Since $F - G$ is continuous on I, we finally have $F - G$ constant on I. ✓

Remark. Primitives do not always exist. If we consider the function f in the introduction of the previous chapter, there does not exist a differentiable function F such that $F' = f$ (the function f is integrable). However, for continuous functions they do, as the next theorem shows.

Theorem 3. Let I be an interval and let $f : I \to \mathbb{R}$ be a continuous function. For any $a \in I$, the function $F : I \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f = \int_a^x f(t)\,\mathrm{d}t \text{ for any } x \in I$$

is a primitive for f.

Proof. If a and x are in I, the closed bounded interval J with endpoints a and x is contained in I; since f is continuous on I, it is continuous and hence integrable on J and therefore the integral $F(x) = \int_a^x f$ is well defined for all $x \in I$. Fix $x_0 \in I$. We have, for any $x \in I$,

$$F(x) - F(x_0) = \int_a^x f - \int_a^{x_0} f = \int_{x_0}^x f$$

$$\text{and } (x - x_0) f(x_0) = \int_{x_0}^x f(x_0)\,\mathrm{d}t$$

(integral of the constant function equal to $f(x_0)$) so that

$$F(x) - F(x_0) - (x - x_0) f(x_0) = \int_{x_0}^x f(t)\,\mathrm{d}t - \int_{x_0}^x f(x_0)\,\mathrm{d}t = \int_{x_0}^x (f(t) - f(x_0))\,\mathrm{d}t.$$

Taking absolute values gives

$$|F(x) - F(x_0) - (x - x_0) f(x_0)| = \left| \int_{x_0}^x (f(t) - f(x_0))\,\mathrm{d}t \right| \leqslant \left| \int_{x_0}^x |f(t) - f(x_0)|\,\mathrm{d}t \right|.$$

Fix $\varepsilon > 0$. Since f is continuous at x_0, there exists $\eta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $x \in I$ and $|x - x_0| < \eta$. Let $x \in I$ be such that $|x - x_0| < \eta$. If t is between x_0 and x then $t \in I$ so that $|t - x_0| < \eta$ and therefore $|f(t) - f(x_0)| < \varepsilon$. If $x \geqslant x_0$ then we have

$$0 \leqslant \int_{x_0}^{x} |f(t) - f(x_0)| \,\mathrm{d}t \leqslant \int_{x_0}^{x} \varepsilon \,\mathrm{d}t = \varepsilon(x - x_0)$$

and if $x \leqslant x_0$ we have

$$0 \leqslant \int_{x}^{x_0} |f(t) - f(x_0)| \,\mathrm{d}t \leqslant \int_{x}^{x_0} \varepsilon \,\mathrm{d}t = \varepsilon(x_0 - x)$$

so that

$$0 \geqslant \int_{x_0}^{x} |f(t) - f(x_0)| \,\mathrm{d}t = -\int_{x}^{x_0} |f(t) - f(x_0)| \,\mathrm{d}t \geqslant -\varepsilon(x_0 - x).$$

In all cases we have

$$\left| \int_{x_0}^{x} |f(t) - f(x_0)| \,\mathrm{d}t \right| \leqslant \varepsilon |x - x_0|$$

for any $x \in I$ such that $|x - x_0| < \eta$.

Therefore, for any ε there exists $\eta > 0$ such that for any $x \in I$ with $|x - x_0| < \eta$ we have

$$|F(x) - F(x_0) - (x - x_0) f(x_0)| \leqslant \varepsilon |x - x_0|. \tag{9.1}$$

In particular, it follows that $F(x)$ has limit $F(x_0)$ when x goes to x_0, so that F is continuous at x_0 and hence on I.

Now assume that x_0 is not an endpoint of I. Dividing (9.1) by $|x - x_0|$ gives

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \leqslant \varepsilon$$

whenever $x \neq x_0$ and $|x - x_0| < \eta$. Therefore we have $\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$ so that F is differentiable at x_0 and $F'(x_0) = f(x_0)$. ✓

Corollary 4. Let I be an interval and $f : I \to \mathbb{R}$ be a continuous function.

- There exist primitives for f.
- If $a \in I$, the function $x \mapsto \int_a^x f$ is the unique primitive for f that vanishes at a.
- **(Fundamental Theorem of Calculus[a])** If F is a primitive for f, then for any a and x in I we have

$$\int_a^x f = \int_a^x f(t) \,\mathrm{d}t = F(x) - F(a)$$

[a] Théorème fondamental de l'analyse

Proof. Fix $a \in I$. Then the theorem above shows that $G : x \mapsto \int_a^x f$ is a primitive for f. Moreover, $G(a) = \int_a^a f = 0$.

If F is another primitive for f, then we know that $F - G = c$ is constant. Moreover $c = F(a) - G(a) = F(a)$. Therefore $F(x) = G(x) + F(a) = \int_a^x f + F(a)$. In particular, if F vanishes at a, then $F = G$ so that G is the unique primitive for f that vanishes at a. ✓

Notation. If f is continuous, a primitive for f will be denoted by $\displaystyle\int f$ or $\displaystyle\int f(t) \,\mathrm{d}t$. In this notation, t and $\mathrm{d}t$ are symbols. This notation is only defined up to a constant.

Warning: this notation is dangerous, you must be aware of its meaning, but it is very convenient.

We shall also write $F(t) = \displaystyle\int f = \int f(t) \,\mathrm{d}t$ for a primitive for f. It is an abuse of notation, since $F(t)$ is a real number, not a function.

We can now prove the Mean Value Inequality for complex valued functions. Let us recall the statement.

Theorem (Mean Value Inequality[a] – Theorem 2.28)**.** Let I be an open interval and $f\colon I \to \mathbb{C}$ be a complex valued function of class $\mathcal{C}^1$. Assume that there exists a real number $K > 0$ such that $|f'(t)| \leqslant K$ for all $t \in I$. Then f is Lipschitz continuous with Lipschitz constant K, that is,

$$\forall x \in I,\ \forall y \in I,\ |f(x) - f(y)| \leqslant K|x - y|.$$

[a] inégalité des accroissements finis

Proof. Take $x \in I$ and $y \in I$ with $x \geqslant y$. The function f' is continuous and f is a primitive of f', therefore $f(x) - f(y) = \displaystyle\int_y^x f'$ and it follows that

$$|f(x) - f(y)| \leqslant \int_y^x |f'| \leqslant \int_y^x K = K(x - y) = K|x - y|.$$

Now if $y \geqslant x$, exchanging x and y in the line above yields $|f(y) - f(x)| \leqslant K|y - x|$ and therefore $|f(x) - f(y)| \leqslant K|x - y|$. ✓

Remark. If $n \in \mathbb{N}$, then a primitive of $x \mapsto x^n$ is $x \mapsto \dfrac{x^{n+1}}{n+1}$. Therefore any primitive of a polynomial function is a polynomial function.

However this is not true in general of rational functions, for instance the primitives of $x \to \frac{1}{x}$ are not rational functions. This leads to the following definition.

Definition 5. *The **logarithm**[a] **function**, denoted by* $\ln$*, is the primitive of* $x \to \frac{1}{x}$ *defined on* $]0;+\infty[$ *that vanishes at* 1*:*

$$\ln x = \int_1^x \frac{1}{t}\,\mathrm{d}t \quad \textit{for all } x \in]0;+\infty[.$$

[a] logarithme

Examples.

- The function $x \to -\cos x$ is a primitive of the sine function. The function $x \to 1-\cos x$ is the primitive of sin that vanishes at 0 and $x \to 2-\cos x$ is a positive primitive of sin. We can then write $\int \sin t\,\mathrm{d}t = -\cos t$ or $\int \sin t\,\mathrm{d}t = 2-\cos t$.
- The function $x \to 1+\cos x$ is periodic, but none of its primitives are periodic.

II. Classical primitives

The following primitives must be known.

$\displaystyle\int (t+a)^b\,\mathrm{d}t = \begin{cases} \dfrac{(t+a)^{b+1}}{b+1} & \text{if } b \neq -1 \\ \ln\lvert t+a\rvert & \text{if } b = -1 \end{cases}$ b can be any real number if $t+a>0$ on the domain	
$\displaystyle\int \cos(at)\,\mathrm{d}t = \frac{\sin(at)}{a}$	$\displaystyle\int \sin(at) = -\frac{\cos(at)}{a}$ if $a \neq 0$
$\displaystyle\int \cosh(at)\,\mathrm{d}t = \frac{\sinh(at)}{a}$	$\displaystyle\int \sinh(at) = \frac{\cosh(at)}{a}$ if $a \neq 0$
$\displaystyle\int e^{at}\,\mathrm{d}t = \frac{e^{at}}{a}$ if $a \neq 0$	
$\displaystyle\int a^t\,\mathrm{d}t = \frac{a^t}{\ln a}$ if $a > 0$ and $a \neq 1$	
$\displaystyle\int \frac{\mathrm{d}t}{\cos^2 t} = \tan t$	$\displaystyle\int \frac{\mathrm{d}t}{\cosh^2 t} = \tanh t$
$\displaystyle\int \frac{\mathrm{d}t}{1+t^2} = \arctan t$	

Remark. Note that these formulas are only true on some intervals that should be specified.

Remark. Sometimes we want to find a primitive of a function of the form $u(x)^b u'(x)$ where $b \in \mathbb{R}$. This is given by $\frac{1}{b+1}(u(x))^{b+1}$ if $b \neq -1$ and $\ln\lvert u(x)\rvert$ if $b = -1$.

III. Integration by parts

When u and v are differentiable functions, then uv is differentiable and $(uv)' = u'v + uv'$. If u and v are of class $\mathcal{C}^1$, then u', v', $u'v$, uv' and $(uv)'$ are continuous and therefore have primitives. We then have $uv = \int u'v + \int uv'$. The principle of integration by parts is to use this formula to find $\int u'v$.

Theorem 6 (Integration by parts[a]). If u and v are of class $\mathcal{C}^1$, then

$$\int u'v = uv - \int uv'$$

$$\int_a^b u'(t)v(t)\,\mathrm{d}t = [u(x)v(x)]_a^b - \int_a^b u(t)v'(t)\,\mathrm{d}t.$$

[a] intégration par parties (IPP)

Example. We want $I = \int_0^{\pi} t \sin t \, dt$. Set $u'(t) = \sin t$ and $v(t) = t$. Then $u(t) = -\cos t$ and $v'(t) = 1$. Therefore

$$I = [-t\cos t]_0^{\pi} - \int_0^{\pi} -\cos t \, dt = [-t\cos t]_0^{\pi} - [-\sin t]_0^{\pi} = \pi - 0 + 0 - 0 = \pi.$$

Example. This method enables us to find primitives for ln. Set $u' = 1$ and $v = \ln$. Then $u(t) = t$ and $v'(t) = \frac{1}{t}$. Therefore

$$\int \ln t \, dt = t\ln t - \int t\frac{1}{t} \, dt = t\ln t - \int 1 \, dt = t\ln t - t + C.$$

Example. We want a primitive of arctan, that is $\int \arctan$. We do an integration by parts, setting $u' = 1$ and $v = \arctan$ so that $u(t) = t$ and $v'(t) = \frac{1}{1+t^2}$. This gives

$$\begin{aligned}
\int \arctan t \, dt &= t\arctan t - \int \frac{t \, dt}{1+t^2} \\
&= t\arctan t - \frac{1}{2}\int \frac{2t \, dt}{1+t^2} \\
&= t\arctan t - \frac{1}{2}\ln\left|1+t^2\right| + C \\
&= t\arctan t - \frac{1}{2}\ln(1+t^2) + C
\end{aligned}$$

since $\dfrac{2t}{1+t^2}$ is of the form $\dfrac{w'(t)}{w(t)}$.

IV. Integration by substitution

We now introduce a new technique to find integrals or primitives, that arises from the chain rule, called integration by substitution†.

Theorem 7. Let $f : [a,b] \to \mathbb{R}$ be a continuous function and let $\varphi : [\alpha,\beta] \to \mathbb{R}$ be a function of class $\mathcal{C}^1$ such that $\varphi([\alpha,\beta]) \subset [a,b]$. Then

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(t) \, dt = \int_{\alpha}^{\beta} f(\varphi(x))\varphi'(x) \, dx.$$

Proof. Note that $(f\circ\varphi)\varphi'$ is continuous and therefore integrable on $[\alpha,\beta]$. Set $F(x) = \int_{\varphi(\alpha)}^{x} f(t) \, dt$ so that $F' = f$. Now consider $u := F\circ\varphi$. Then by the chain rule we have

$$u'(x) = (F\circ\varphi)'(x) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x) \quad \text{for } x \in [\alpha,\beta].$$

Therefore u is a primitive of $(f\circ\varphi)\varphi'$ so that

$$\int_{\alpha}^{\beta} f(\varphi(x))\varphi'(x) \, dx = u(\beta) - u(\alpha) = F(\varphi(\beta)) - F(\varphi(\alpha)) = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t) \, dt. \qquad ✓$$

Remark. This formula can be used in two ways.

➢ From right to left. We want to compute $\int_a^b g(x) \, dx$ and we notice that $g(x)$ can be written in the form $g(x) = f(\varphi(x))\varphi'(x)$. We then set $t = \varphi(x)$ and write $dt = \varphi'(x) \, dx$.
This essentially means that we recognise that g is the derivative of the composition $F\circ\varphi$ where F is a primitive of f, but the change of variable can still be useful to simplify or clarify our computations.

Example. Say we want to compute $\int_1^2 \frac{\ln x}{x} \, dx$. We see that $\dfrac{\ln x}{x}$ is of the form $f(\varphi(x))\varphi'(x)$ with $f = \mathrm{id}$ and $\varphi(x) = \ln x$. Therefore

$$\int_1^2 \frac{\ln x}{x} \, dx = \int_{\ln 1}^{\ln 2} t \, dt = \left[\frac{t^2}{2}\right]_0^{\ln 2} = \frac{1}{2}(\ln 2)^2.$$

➢ From left to right. In this case, we should check that φ is bijective from $[\alpha,\beta]$ to $\varphi([\alpha,\beta])$ so as to recover the boundaries α and β from $\varphi(\alpha)$ and $\varphi(\beta)$. This is used in practise to make sure that $\int_a^b f(t) \, dt$ has a more tractable form. In this case, it is more difficult to find φ, but practising helps...

† changement de variable

Example. We want to compute $\int_0^1 \sqrt{1-t^2}\,\mathrm{d}t$. We set $\varphi(x) = \cos x$. The function φ is of class $\mathcal{C}^1$ and is bijective from $[0, \frac{\pi}{2}]$ to $[0, 1]$. We have $\varphi'(x) = -\sin x$. Using the theorem, and the fact that $\sin \geqslant 0$ on $[0, \frac{\pi}{2}]$, we get

$$\int_0^1 \sqrt{1-t^2}\,\mathrm{d}t = \int_{\pi/2}^0 \sqrt{1-\cos^2 x}(-\sin x)\,\mathrm{d}x = \int_0^{\pi/2} \sqrt{\sin^2 x}\sin x\,\mathrm{d}x = \int_0^{\pi/2} \sin^2 x\,\mathrm{d}x$$
$$= \int_0^{\pi/2} \frac{1}{2}(1-\cos(2x))\,\mathrm{d}x = \left[\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\right]_0^{\pi/2} = \frac{\pi}{4}.$$

Example. We want to compute $\int_{1/\sqrt{2}}^{\sqrt{3}/2} \frac{1}{t^2\sqrt{1-t^2}}\,\mathrm{d}t$. Define $\varphi : [\frac{\pi}{4}; \frac{\pi}{3}] \to \mathbb{R}$ by $\varphi(x) = \sin x$ so that $\varphi'(x) = \cos x > 0$. We then have

$$\int_{1/\sqrt{2}}^{\sqrt{3}/2} \frac{1}{t^2\sqrt{1-t^2}}\,\mathrm{d}t = \int_{\pi/4}^{\pi/3} \frac{\cos x}{\sin^2 x \cos x}\,\mathrm{d}x = \int_{\pi/4}^{\pi/3} \frac{1}{\sin^2 x}\,\mathrm{d}x = \left[-\frac{1}{\tan x}\right]_{\pi/4}^{\pi/3} = 1 - \frac{1}{\sqrt{3}}.$$

We could also have made the change of variable $\varphi(x) = \cos x$.

Remark. In practise, in the example above, we set $u = \sin x$, so that $\mathrm{d}u = \cos x\,\mathrm{d}x$ and we adjust the bounds of the integral.

More generally, to use the substitution formula $\int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,\mathrm{d}t = \int_\alpha^\beta f(\varphi(x))\varphi'(x)\,\mathrm{d}x$, we set

$$\boxed{t = \varphi(x), \quad \mathrm{d}t = \varphi'(x)\,\mathrm{d}x \quad \text{and change the bounds of the integral.}}$$

Examples. **(1)** To compute $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$, we substitute $x = \tan t$ so that $\mathrm{d}x = (1+\tan^2 t)\,\mathrm{d}t$ hence $\mathrm{d}t = \frac{1}{1+x^2}\mathrm{d}x$ and therefore

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \int_0^{\pi/4} \mathrm{d}t = \frac{\pi}{4}.$$

Note that we could have computed this integral directly using a primitive of $\frac{1}{1+x^2}$:

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = [\arctan x]_0^1 = \frac{\pi}{4}.$$

(2) Substitution can also be used (if we are careful!) to compute primitives. In particular, it requires φ to be bijective.

Let us compute $\int \frac{1}{\sqrt{3+4t-4t^2}}\,\mathrm{d}t$.

Note that this only makes sense if $3+4t-4t^2 > 0$. We have $3+4t-4t^2 = -4(t^2-t-\frac{3}{4}) = -4((t-\frac{1}{2})^2-1)$. Let us put $x = t - \frac{1}{2}$. Then $3+4t-4t^2 = -4(x^2-1) = 4(1-x^2)$ and $\mathrm{d}x = \mathrm{d}t$. We work only when $-\frac{1}{2} \leqslant t \leqslant \frac{3}{2}$ or equivalently $-1 \leqslant x \leqslant 1$. We then have

$$\int \frac{1}{\sqrt{3+4t-4t^2}}\,\mathrm{d}t = \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{1}{2}\arcsin x + C = \frac{1}{2}\arcsin(t - \frac{1}{2}) + C.$$

(We must not forget to go back to the original variable in order to really have a primitive of the original function.)

V. Primitive of a rational function

Let $F = \frac{P}{Q}$ be a rational fraction in $\mathbb{R}(X)$. Then we know that F decomposes into a partial fraction decomposition, that is, F is the sum

- of a polynomial,
- of fractions $\frac{a}{(x-\alpha)^n}$ where a and α are real numbers and $n \in \mathbb{N}^*$,
- of fractions $\frac{ax+b}{(x^2+px+q)^n}$ where a, b, p and q are real numbers such that $p^2 - 4q < 0$ and $n \in \mathbb{N}^*$.

Since we already know the primitives of polynomial functions, we need to know how to find primitives of $x \to \frac{a}{(x-\alpha)^n}$ and $x \to \frac{ax+b}{(x^2+px+q)^n}$.

A. Computation of $\displaystyle\int \frac{\mathrm{d}x}{(x-\alpha)^n}$

The function $x \to \dfrac{1}{(x-\alpha)^n}$ is of the form $\dfrac{u'(x)}{u(x)^n}$ so that

$$\int \frac{1}{(x-\alpha)^n}\,\mathrm{d}x = \begin{cases} -\dfrac{1}{(n-1)(x-\alpha)^{n-1}} & \text{if } n \neq 1 \\ \ln|x-\alpha| & \text{if } n = 1 \end{cases}$$

(up to a constant).

B. Computation of $\displaystyle\int \frac{ax+b}{(x^2+px+q)^n}\,\mathrm{d}x$

Set $u(x) = x^2+px+q$.

➢ First note that $u'(x) = 2x+p$ so that $ax+b = \frac{a}{2}u'(x) - \frac{ap}{2} + b = cu'(x)+d$. Therefore,

$$\frac{ax+b}{(x^2+px+q)^n} = c\frac{u'(x)}{u(x)^n} + d\frac{1}{u(x)^n}.$$

Moreover, we know the primitives of $\dfrac{u'}{u^n}$.

➢ We now consider $\dfrac{1}{u}$ (case $n=1$). Write $x^2+px+q = \left(x+\frac{p}{2}\right)^2 + \frac{4q-p^2}{4}$ and put and $\alpha^2 = \frac{4q-p^2}{4}$ (this is possible since $4q-p^2>0$). Then $u(x) = \alpha^2\left(\left(\frac{1}{\alpha}(x+\frac{p}{2})\right)^2+1\right)$.

We shall now use substitution. Set $t = \frac{1}{\alpha}(x+\frac{p}{2})$ (that is, $\varphi(x) = \frac{1}{\alpha}(x+\frac{p}{2})$). Then $\mathrm{d}t = \frac{1}{\alpha}\,\mathrm{d}x$ (since $\varphi'(x) = \frac{1}{\alpha}$), therefore

$$\int \frac{\mathrm{d}x}{u(x)} = \int \frac{\alpha\,\mathrm{d}t}{\alpha^2(t^2+1)} = \frac{1}{\alpha}\int \frac{\mathrm{d}t}{t^2+1} = \frac{1}{\alpha}\arctan(t) = \frac{1}{\alpha}\arctan\left(\frac{1}{\alpha}\left(x+\frac{p}{2}\right)\right).$$

➢ It is possible to do find the primitives of $\dfrac{1}{u^n}$ for $n \geqslant 2$ inductively, but we shall not do it in this course.

Example. We want to find $\displaystyle\int \frac{\mathrm{d}t}{t^2-1}$.

First step: partial fraction decomposition of $\dfrac{1}{t^2-1}$. We get

$$\frac{1}{t^2-1} = \frac{1}{2}\left(\frac{1}{t-1} - \frac{1}{t+1}.\right)$$

Therefore

$$\int \frac{\mathrm{d}t}{t^2-1} = \frac{1}{2}(\ln|t-1| - \ln|t+1|) + C = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C.$$

Example. We want to find $\displaystyle\int \frac{t-2}{t^2-t+1}\,\mathrm{d}t$. This rational fraction is in irreducible form, its degree is negative and the denominator is irreducible.

We must now apply the method described above.

The derivative of $t \mapsto t^2-t+1$ sends t to $2t-1$ and we have $t-2 = \frac{1}{2}(2t-1) - \frac{3}{2}$ so that

$$\frac{t-2}{t^2-t+1} = \frac{1}{2}\frac{2t-1}{t^2-t+1} - \frac{3}{2}\frac{1}{t^2-t+1}$$

and therefore

$$\int \frac{t-2}{t^2-t+1}\,\mathrm{d}t = \frac{1}{2}\ln\left|t^2-t+1\right| - \frac{3}{2}\int \frac{\mathrm{d}t}{t^2-t+1}.$$

Next, to compute $\displaystyle\int \frac{\mathrm{d}t}{t^2-t+1}$ we write $t^2-t+1 = \left(t-\dfrac{1}{2}\right)^2 + \dfrac{3}{4} = \dfrac{3}{4}\left(\left(\dfrac{2t-1}{\sqrt{3}}\right)^2+1\right)$ and put $u = \dfrac{2t-1}{\sqrt{3}}$. We then have $\mathrm{d}u = \frac{2}{\sqrt{3}}\,\mathrm{d}t$ so that

$$\int \frac{\mathrm{d}t}{t^2-t+1} = \int \frac{\frac{\sqrt{3}}{2}\,\mathrm{d}u}{\frac{3}{4}(u^2+1)} = \frac{2}{\sqrt{3}}\int \frac{\mathrm{d}u}{u^2+1} = \frac{2}{\sqrt{3}}\arctan u = \frac{2}{\sqrt{3}}\arctan\left(\frac{2t-1}{\sqrt{3}}\right).$$

Finally, we have $\displaystyle\int \frac{t-2}{t^2-t+1}\,\mathrm{d}t = \frac{1}{2}\ln\left|t^2-t+1\right| - \sqrt{3}\arctan\left(\frac{2t-1}{\sqrt{3}}\right) + C$.

Example. We want to find $\displaystyle\int \frac{dt}{t^3+1}$.

First step: partial fraction decomposition of $\dfrac{1}{t^3+1}$. We get

$$\frac{1}{t^3+1} = \frac{1}{3}\frac{1}{t+1} - \frac{1}{3}\frac{t-2}{t^2-t+1}$$

so that $\displaystyle\int \frac{dt}{t^3+1} = \frac{1}{3}\ln|t+1| - \frac{1}{3}\int \frac{t-2}{t^2-t+1}\,dt$.

Using the previous example, we have $\displaystyle\int \frac{dt}{t^3+1} = \frac{1}{3}\ln|t+1| - \frac{1}{6}\ln\left|t^2-t+1\right| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2t-1}{\sqrt{3}}\right) + C$.

VI. Primitive of a rational function in sin, cos and tan

This is usually done by substitution, then integration of an ordinary rational function as in the previous section.

If $f(x)$ is a rational function in $\sin x$, $\cos x$ and $\tan x$, the substitution $t = \tan\frac{x}{2}$ always changes $\int f$ into the primitive of a rational functiun in t, but different (and more efficient) substitutions can sometimes be found.

It is important to know the trigonometry formulas.

Example. To compute $F = \displaystyle\int \frac{1}{\sin x + \cos x}\,dx$, we set $t = \tan\frac{x}{2}$. Then $dx = \frac{2}{1+t^2}\,dt$ so that $F = 2\displaystyle\int \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}\frac{1}{1+t^2}\,dt = 2\int \frac{1}{-t^2+2t+1}\,dt$. The partial fraction decomposition of $\frac{1}{-t^2+2t+1}$ is $\frac{1}{-t^2+2t+1} = \frac{1}{2\sqrt{2}}\left(\frac{1}{t-\alpha} - \frac{1}{t-\beta}\right)$ where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$. Therefore $\displaystyle\int \frac{1}{-t^2+2t+1}\,dt = \frac{1}{2\sqrt{2}}\ln\left|\frac{t-\alpha}{t-\beta}\right|$ so that $F(x) = \frac{1}{\sqrt{2}}\ln\left|\dfrac{\tan\left(\frac{x}{2}\right)-1-\sqrt{2}}{\tan\left(\frac{x}{2}\right)-1+\sqrt{2}}\right|$. This formula is valid on the interval $]0,\pi[$ or in fact in any interval that does not contain $k\pi$, $k \in \mathbb{Z}$.

VII. Taylor's formula with integral remainder

Theorem 8 (Taylor's formula with integral remainder)**.** Let $f : I \to \mathbb{R}$ be a function of class $\mathcal{C}^{n+1}$ on an open interval I for some $n \in \mathbb{N}$. For any elements a and b of I, we have

$$f(b) = f(a) + \frac{b-a}{1!}f'(a) + \cdots + \frac{(b-a)^n}{n!}f^{(n)}(a) + \int_a^b \frac{(b-t)^n}{n!}f^{(n+1)}(t)\,dt.$$

This is the formula of order n at a. The term $\int_a^b \frac{(b-t)^n}{n!}f^{(n+1)}(t)\,dt$ is called the **remainder**[a].

[a] reste

Proof. We shall prove it by induction on n.

For $n = 0$, this is the Fundamental Theorem of Calculus.

Now assume that the result is true at order $n-1$ for functions of class $\mathcal{C}^n$ on I for some $n \geqslant 1$, and let f be a function of class $\mathcal{C}^{n+1}$. Then, since f is also of class $\mathcal{C}^n$ we have

$$f(b) = f(a) + \frac{b-a}{1!}f'(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \int_a^b \frac{(b-t)^{n-1}}{(n-1)!}f^{(n)}(t)\,dt.$$

For any $t \in I$, set

$$u(t) = -\frac{(b-t)^n}{n!},\ v(t) = f^{(n)}(t) \ \text{ and } \ R = \int_a^b \frac{(b-s)^{n-1}}{(n-1)!}f^{(n)}(s)\,ds.$$

We have $u'(t) = \dfrac{n(b-t)^{n-1}}{n!} = \dfrac{(b-t)^{n-1}}{(n-1)!}$ so $R = \displaystyle\int_a^b u'(t)v(t)\,dt$. The function u is of class $\mathcal{C}^\infty$ (it is a polynomial function) and the function v is of class $\mathcal{C}^1$ since $v' = f^{(n+1)}$ is continuous by assumption. We can therefore do an integration by parts

$$\begin{aligned}
R &= [u(t)v(t)]_a^b - \int_a^b u'(t)v(t)\,dt \\
&= u(b)v(b) - u(a)v(a) - \int_a^b -\frac{(b-t)^n}{n!}f^{(n+1)}(t)\,dt \\
&= -u(a)v(a) + \int_a^b \frac{(b-t)^n}{n!}f^{(n+1)}(t)\,dt \\
&= \frac{(b-a)^n}{n}f^{(n)}(a) + \int_a^b \frac{(b-t)^n}{n}f^{(n+1)}(t)\,dt
\end{aligned}$$

Finally, we have

$$\begin{aligned}
f(b) &= f(a) + \frac{b-a}{1!} f'(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R \\
&= f(a) + \frac{b-a}{1!} f'(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^n}{n} f^{(n)}(a) + \int_a^b \frac{(b-t)^n}{n} f^{(n+1)}(t)\,\mathrm{d}t
\end{aligned}$$

which is the formula of order n. ✓

As a corollary, we get Taylor's inequality.

Theorem (Taylor's inequality – Theorem 2.26)**.** Let $f\colon I \to \mathbb{R}$ be a function of class $\mathcal{C}^{n+1}$ on an interval I. Suppose that a and b are elements in I. If $\left|f^{(n+1)}(t)\right| \leqslant M$ for all t between a and b, then

$$\left| f(b) - \left(f(a) + \frac{b-a}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^n}{n!} f^{(n)}(a) \right) \right| \leqslant M \frac{|b-a|^{n+1}}{(n+1)!}.$$

This equality is called **Taylor's inequality**[a] at a of order n.

[a] inégalité de Taylor-Lagrange

Proof. By the Taylor formula with integral remainder, we have

$$\begin{aligned}
\left| f(b) - \left(f(a) + \frac{b-a}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^n}{n!} f^{(n)}(a) \right) \right| &= \left| \int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t \right| \\
&\leqslant \left| \int_a^b \left| \frac{(b-t)^n}{n!} f^{(n+1)}(t) \right| \mathrm{d}t \right| \\
&\leqslant \left| \int_a^b \frac{(b-t)^n}{n!} M\,\mathrm{d}t \right| \\
&= \frac{M}{(n+1)!} |b-a|^{n+1}.
\end{aligned}$$

✓

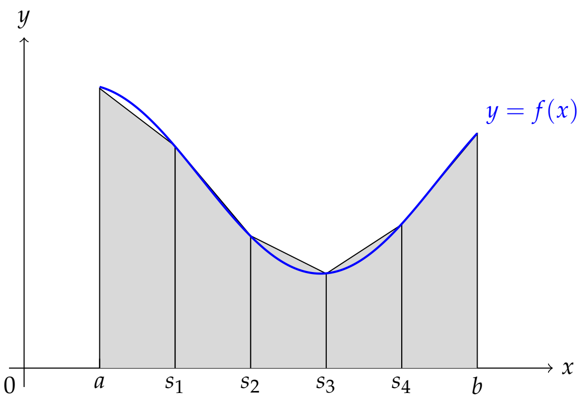
Remark. You have already seen a Taylor formula: Taylor-Young's formula (first semester, Théorème 156). That formula was of a *local* nature: it gives an approximation of f by a polynomial (the larger the n, the better the approximation), but only in an immediate neighbourhood of a point in I.

The results we have just seen (Taylor's formula with integral remainder and Taylor's inequality) are of a *global* nature. They require assumptions on $[a,b]$, and the results are valid on $[a,b]$.

VIII. Approximations

A. Trapezium rule

The idea is to approximate the graph of f by a broken line.



Consider the partition S of $[a,b]$ given by $s_k = a + k\frac{b-a}{n}$ (divide $[a,b]$ into n intervals of equal length $\frac{b-a}{n}$). Let f be a function on $[a,b]$. Define

$$T_n = \frac{b-a}{n} \sum_{k=0}^{n-1} \frac{1}{2}\left(f(s_k) + f(s_{k+1})\right) = \frac{b-a}{n} \left(\sum_{k=0}^{n} f(s_k) - \frac{1}{2}\left(f(a) + f(b)\right) \right)$$

(this is the shaded area above).

Theorem 9. If f is of class $\mathcal{C}^2$ on $[a,b]$, then

$$\left|T_n - \int_a^b f\right| \leqslant \frac{(b-a)^3}{12n^2}\sup_{[a,b]}|f''|.$$

Proof. *[Not done in class.]*

➢ We first prove that if $a \leqslant \alpha < \beta \leqslant b$, there exists $\gamma \in]\alpha,\beta[$ such that

$$\int_\alpha^\beta f = \underbrace{\frac{\beta-\alpha}{2}(f(\alpha)+f(\beta))}_{\text{area of the trapezium}} - \underbrace{\frac{1}{12}(\beta-\alpha)^3 f''(\gamma)}_{\text{error}}.$$

Define $F(x) = \displaystyle\int_\alpha^x f(t)\,\mathrm{d}t - \frac{(x-\alpha)}{2}(f(\alpha)+f(x)) + (x-\alpha)^3 K$ where K is chosen so that $F(\beta) = 0$. We then have $F(\alpha) = F(\beta) = 0$ and F is twice differentiable. By Rolle's theorem there exists $\delta \in]\alpha,\beta[$ such that $F'(\delta) = 0$. We have $F'(x) = f(x) - \frac{1}{2}(f(\alpha)+f(x)) - \frac{1}{2}(x-\alpha)f'(x) + 3(x-\alpha)^2 K$ so that $F'(\alpha) = F'(\delta) = 0$. Therefore by Rolle's theorem again, there exists $\gamma \in]\alpha,\delta[$ such that $F''(\gamma) = 0$. We have $F''(x) = -\frac{1}{2}(x-\alpha)f''(x) + 6(x-\alpha)K$ hence $K = \frac{f''(\gamma)}{12}$. Finally, $F(\beta) = 0$ becomes $\int_\alpha^\beta f = \frac{\beta-\alpha}{2}(f(\alpha)+f(\beta)) - \frac{1}{12}(\beta-\alpha)^3 f''(\gamma)$.

➢ We then have $\gamma_k \in]s_k, s_{k+1}[$ for all k such that

$$\begin{aligned}\left|T_n - \int_a^b f\right| &\leqslant \sum_{k=0}^{n-1}\left|\frac{b-a}{2n}(f(s_k)+f(s_{k+1})) - \int_{s_k}^{s_{k+1}} f\right| \\ &= \frac{(b-a)^3}{12n^3}\sum_{k=0}^{n-1}|f''(\gamma_k)| \quad \text{by the first part of the proof} \\ &\leqslant \frac{(b-a)^3}{12n^3}\sum_{k=0}^{n-1}\sup_{[a,b]}|f''| = \frac{(b-a)^3}{12n^2}\sup_{[a,b]}|f''|.\end{aligned}$$

✓

Example. We know that $\displaystyle\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \arctan 1 - \arctan 0 = \frac{\pi}{4} \approx 0.785$.

We want to apply the result above to get approximations of $\displaystyle\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$. We have $f'(x) = -\frac{2x}{(x^2+1)^2}$, $f''(x) = 2\frac{3x^2-1}{(x^2+1)^3}$ and $f'''(x) = \frac{24x(1-x)(1+x)}{(x^2+1)^4} \geqslant 0$ so that f'' is non-decreasing.

If we apply the result above with $n = 1$ we have $f''(0) = -2$, $f''(1) = \frac{1}{2}$ and f'' is non-decreasing so that $|f''| \leqslant 2$; we then get $\displaystyle\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x T_1 - R_1 == \frac{1}{2}(f(0)+f(1)) - R_1 = \frac{1}{2}\frac{3}{4} - R_1 = 0.75 - R_1$ with $|R_1| \leqslant \frac{1}{6}$ so that the error in $\displaystyle\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = 0.75$ is less than $\frac{1}{6}$.

If we apply the result above with $n = 2$ we still have $|f''| \leqslant 2$; we then get $\displaystyle\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = T_2 - R_2 = \frac{1}{4}\left(f(0) + f(\frac{1}{2})\right) + \frac{1}{4}\left(f(\frac{1}{2}) + f(1)\right) - R_2 = \frac{1}{4}\left(\frac{3}{2} + \frac{8}{5}\right) - R_2 = 0.775 - R_2$ with $|R_2| \leqslant \frac{2}{12\cdot 4} \approx 0.04$ so that the error in $\displaystyle\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = 0.775$ is less than 0.04.

B. Simpson's rule

The idea here is to approach the graph of f by a (broken) quadric.

Note that if g is a polynomial of degree 2 then $\displaystyle\int_a^b g(t)\,\mathrm{d}t = \frac{b-a}{6}\left(g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right)$. Moreover, given three distinct real numbers α, β and γ in $[a,b]$ (such as a, b and $\frac{a+b}{2}$), there is a unique polynomial g of degree 2 such that $g(\alpha) = f(\alpha)$, $g(\beta) = f(\beta)$ and $g(\gamma) = f(\gamma)$ (use Lagrange interpolation).

Theorem 10. Let f be a function of class $\mathcal{C}^4$ on $[a,b]$. Then

$$\left|\int_a^b f - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \leqslant \frac{(b-a)^5}{2880}\sup_{[a,b]}\left|f^{(4)}\right|.$$

Proof. (Admitted – not difficult but technical.)

To simplify notation, set $c = \dfrac{a+b}{2}$ and $h = \dfrac{b-a}{2}$. Note that $b - c = h = c - a$.

We use integration by parts in each of the following equalities:

$$\begin{aligned}
A := \int_c^b &\left(\frac{(b-x)^4}{24} - h\frac{(b-x)^3}{18}\right) f^{(4)}(x)\,\mathrm{d}x \\
&= -\left(\frac{h^4}{24} - \frac{h^4}{18}\right) f^{(3)}(c) + \int_c^b \left(\frac{(b-x)^3}{6} - h\frac{(b-x)^2}{6}\right) f^{(3)}(x)\,\mathrm{d}x \\
&= \frac{h^4}{72} f^{(3)}(c) - \left(\frac{h^3}{6} - \frac{h^3}{6}\right) f^{(2)}(c) + \int_c^b \left(\frac{(b-x)^2}{2} - h\frac{b-x}{3}\right) f''(x)\,\mathrm{d}x \\
&= \frac{h^4}{72} f^{(3)}(c) - \left(\frac{h^2}{2} - \frac{h^2}{3}\right) f'(c) + \int_c^b \left(b - x - \frac{h}{3}\right) f'(x)\,\mathrm{d}x \\
&= \frac{h^4}{72} f^{(3)}(c) - \frac{h^2}{6} f'(c) - \frac{h}{3} f(b) - \frac{2h}{3} f(c) + \int_c^b f \\
&= \frac{h^4}{72} f^{(3)}(c) - \frac{h^2}{6} f'(c) - \frac{h}{3}(f(b) + 2f(c)) + \int_c^b f.
\end{aligned}$$

Similarly,

$$B := \int_a^c \left(\frac{(a-x)^4}{24} - h\frac{(a-x)^3}{18}\right) f^{(4)}(x)\,\mathrm{d}x = -\frac{1}{72}h^4 f^{(3)}(c) + \frac{1}{6}h^2 f(c) - \frac{1}{3}h(f(a) + 2f(c)) + \int_a^c f.$$

Adding these equalities gives

$$\int_a^b f - \frac{h}{3}(f(a) + 4f(c) + f(b)) = A + B.$$

We must now find an upper bound for $A + B$.

We use Proposition 8.16: there exists $\theta \in [c, b]$ such that

$$\begin{aligned}
|A| \leqslant \int_c^b \left| \left(\frac{(b-x)^4}{24} - h\frac{(b-x)^3}{18}\right) f^{(4)}(x) \right| \mathrm{d}x &= \int_c^b \left(-\frac{(b-x)^4}{24} + h\frac{(b-x)^3}{18}\right) \left|f^{(4)}(x)\right| \mathrm{d}x \\
&= \left|f^{(4)}(\theta)\right| \int_c^b \left(-\frac{(b-x)^4}{24} + h\frac{(b-x)^3}{18}\right) \mathrm{d}x \\
&= \left|f^{(4)}(\theta)\right| \left[\frac{(b-x)^5}{120} - h\frac{(b-x)^4}{72}\right]_c^b \\
&= \left|f^{(4)}(\theta)\right| \frac{h^5}{180} \leqslant \sup_{[a,b]} \left|f^{(4)}\right| \frac{h^5}{180}.
\end{aligned}$$

Similarly, $|B| \leqslant \dfrac{h^5}{180} \sup_{[a,b]} \left|f^{(4)}\right|$. The result then follows. ✓

Remark. In practise, we divide $[a, b]$ into smaller intervals and apply Simpson's rule to each, thus obtaining a better approximation.

Example. We use Simpson's rule with three intervals to compute $\displaystyle\int_1^4 \sqrt{1 + x^3}\,\mathrm{d}x$. Set $f(x) = \sqrt{1 + x^3}$. We have $f^{(4)}(x) = \dfrac{9x^2(56x^3 + x^6 - 80)}{16(1 + x^3)^{7/2}}$.

$$\begin{aligned}
\int_1^4 \sqrt{1 + x^3}\,\mathrm{d}x &= \frac{1}{6}\left(f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right) + R \\
&= \frac{1}{6}\left(\sqrt{2} + 4\frac{\sqrt{70}}{4} + 2 \cdot 3 + 4\frac{\sqrt{266}}{4} + 2\sqrt{28} + 4\frac{3\sqrt{78}}{4} + \sqrt{65}\right) + R \approx 12.871 + R
\end{aligned}$$

with $R \leqslant \dfrac{1}{2880}\dfrac{9}{16}\left(\dfrac{2^2(56 \cdot 2^3 + 2^6 - 80)}{2^{7/2}} + \dfrac{3^2(56 \cdot 3^3 + 3^6 - 80)}{3^7} + \dfrac{4^2(56 \cdot 4^3 + 4^6 - 80)}{28^{7/2}}\right) \leqslant 0.032.$

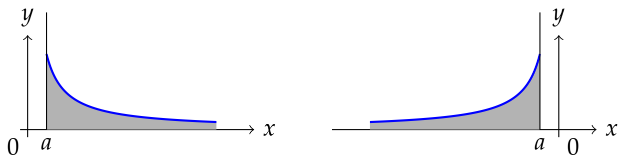
Chapter 10

Improper integrals

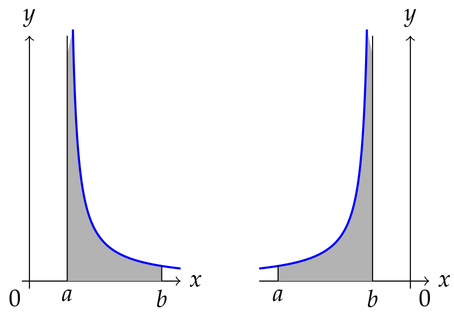
We have defined the integral $\int_a^b f$ of an integrable function on an interval $[a,b]$. Recall that it represents the area under the graph of f between a and b. We shall now study conditions under which we shall be able to define $\int_a^b f$ when f defined on $[a,b[$ or $]a,b]$.

More precisely, we want to generalise (when it is possible) in two ways:

➢ define the area under the graph between a real number a and $+\infty$ or between $-\infty$ and a (the interval is no longer bounded).



➢ define the area under the graph between two real numbers when the function is unbounded near a or b.



Definition 1. *Let $f : I \to \mathbb{R}$ be a function where I is an interval that is not necessarily closed or bounded.*

➢ *We say that f is **piecewise continuous**[a] on I if f is piecewise continuous on any closed bounded interval $[a,b] \subset I$.*

➢ *More generally, we say that f is **locally integrable**[b] on I if it is integrable on any closed bounded interval $[a,b] \subset I$.*

[a] continue par morceaux
[b] localement intégrable

Definition 2. *Let $f : [a,b[\to \mathbb{R}$ be a locally integrable function with $b \in \mathbb{R}$ or $b = +\infty$.*
*If the function $x \to \int_a^x f$ has a finite limit when x goes to b, we say that the **improper integral**[a] $\int_a^b f$ is **convergent**[b]. This limit is also denoted by $\int_a^b f$.*
*If the limit does not exist or is infinite, we say that the improper integral $\int_a^b f$ is **divergent**[c].*
Similarly, let $g :]a,b] \to \mathbb{R}$ be a locally integrable function with $a \in \mathbb{R}$ or $a = -\infty$. If the function $x \to \int_x^b g$ has a finite limit when x goes to a, this limit is denoted by $\int_a^b g$ and we use the same terminology as for f.

[a] intégrale impropre
[b] convergente
[c] divergente

Warning! The notation $\int_a^b f$ stands for the improper integral and, when it converges, for its value. Use of the notation $\int_a^b f$ does not imply that the limit exists. Writing $\int_a^b f = \cdots$ has no meaning until the existence of the limit has been proved.

Example. Is the improper integral $\int_0^1 \frac{\ln t}{t}\,\mathrm{d}t$ convergent?

The function $g\colon]0,1] \to \mathbb{R}$ defined by $g(t) = \frac{\ln t}{t}$ is continuous and of the form uu' so that it is easy to find a primitive: $G(t) = \frac{1}{2}(\ln t)^2$.

We then have, for $x \in]0,1]$, $\int_x^1 \frac{\ln t}{t}\,\mathrm{d}t = -\frac{1}{2}(\ln x)^2$ whose limit when x approaches 0 is $+\infty$.

Therefore the improper integral $\int_0^1 \frac{\ln t}{t}\,\mathrm{d}t$ is divergent.

Remark. Let f be a function defined on $[a,b[$ and take $c \in]a,b[$. Then $\int_a^x f = \int_a^c f + \int_c^x f$ and $\int_a^c f$ is an ordinary integral. Clearly, $\int_a^x f$ is convergent if and only if $\int_c^x f$ is convergent. Therefore the fact that an improper integral is convergent or divergent only depends on the bound where the function is not defined.

Definition 3. *Let $f\colon]a,b[\to \mathbb{R}$ be a locally integrable function. Fix any $c \in]a,b[$. We say that the improper integral $\int_a^b f$ is convergent if the improper integrals $\int_a^c f$ and $\int_c^b f$ are both convergent. It does not depend on the choice of c. We then define*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Examples.

➢ $\int_0^{+\infty} \frac{1}{1+t^2}\,\mathrm{d}t$ converges. Indeed, $\int_0^x \frac{1}{1+t^2}\,\mathrm{d}t = \arctan(x) \xrightarrow[x\to+\infty]{} \frac{\pi}{2}$.

Similarly, $\int_{-\infty}^0 \frac{1}{1+t^2}\,\mathrm{d}t$ converges and $\int_{-\infty}^0 \frac{1}{1+t^2}\,\mathrm{d}t = \frac{\pi}{2}$.

Therefore $\int_{-\infty}^{+\infty} \frac{1}{1+t^2}\,\mathrm{d}t$ converges and $\int_{-\infty}^{+\infty} \frac{1}{1+t^2}\,\mathrm{d}t = \pi$.

➢ $\int_0^{+\infty} \sin t\,\mathrm{d}t$ diverges. Indeed, $\int_0^x \sin t\,\mathrm{d}t = 1 - \cos x$ has no limit when x goes to $+\infty$.

Note that for any $n \in \mathbb{N}$, we have $\int_0^{2n\pi} \sin t\,\mathrm{d}t = 0$.

Remark. 'Falsely improper integrals'.

If f is continuous on $[a,b[$ and has a finite limit ℓ when x nears b, then f can be extended to a continuous function $\tilde{f}$ on $[a,b]$. Therefore the function $\tilde{f}$ is integrable on $[a,b]$. Moreover, by the Fundamental Theorem of Calculus we have $\int_a^x f = F(x) - F(a) \xrightarrow[x\to b]{} F(b) - F(a)$ for a primitive F of $\tilde{f}$.

Therefore the improper integral $\int_a^b f$ is convergent.

Example. Consider $f\colon]0\,;\pi] \to \mathbb{R}$ defined by $f(t) = \frac{\sin t}{t}$. This function is not defined at 0 but $\lim_{t\to 0} \frac{\sin t}{t} = 1$, therefore $\int_0^\pi f(t)\,\mathrm{d}t$ is falsely improper and hence converges.

I. Fundamental examples

Proposition 4. (1) Let a be a positive real number. Then

(a) the improper integral $\int_a^{+\infty} \frac{1}{t^\alpha}\,\mathrm{d}t$ converges if $\alpha > 1$ and diverges if $\alpha \leqslant 1$,

(b) the improper integral $\int_0^a \frac{1}{t^\alpha}\,\mathrm{d}t$ converges if $\alpha < 1$ and diverges if $\alpha \geqslant 1$.

(2) Let $a < b$ be real numbers. Then the improper integral $\int_a^b \frac{1}{(t-a)^\alpha}\,\mathrm{d}t$ converges if $\alpha < 1$ and diverges if $\alpha \geqslant 1$.

Proof. If $\alpha \neq 1$, $\displaystyle\int \frac{1}{t^\alpha}\,\mathrm{d}t = \frac{1}{1-\alpha}t^{1-\alpha}$ so that

$$I_x = \int_a^x \frac{1}{t^\alpha}\,\mathrm{d}t = \frac{1}{1-\alpha}\left(x^{1-\alpha} - a^{1-\alpha}\right) \text{ and } J_x = \int_x^a \frac{1}{t^\alpha}\,\mathrm{d}t = \frac{1}{1-\alpha}\left(a^{1-\alpha} - x^{1-\alpha}\right)$$

and $\lim_{x\to+\infty} I_x = \begin{cases} +\infty & \text{if } 1-\alpha > 0 \\ \frac{a^{1-\alpha}}{\alpha-1} & \text{if } 1-\alpha < 0 \end{cases}$ and $\lim_{x\to 0} J_x = \begin{cases} +\infty & \text{if } 1-\alpha < 0 \\ \frac{a^{1-\alpha}}{\alpha-1} & \text{if } 1-\alpha > 0 \end{cases}$.

Now if $\alpha = 1$, then $\displaystyle\int_a^x \frac{1}{t}\,\mathrm{d}t = \ln x - \ln a$ which goes to $+\infty$ as x goes to $+\infty$, and $\displaystyle\int_x^a \frac{1}{t}\,\mathrm{d}t = \ln a - \ln x$ which goes to $+\infty$ as x goes to 0, therefore the improper integrals are both divergent.

The proof of the last statement is similar. ✓

Remark. These examples will be very important once we have developed some tools to study the convergence of improper integrals.

Proposition 5. Let $f, g\colon [a,b[\to \mathbb{R}$ be locally integrable functions. We assume that the improper integrals $\displaystyle\int_a^b f$ and $\displaystyle\int_a^b g$ are convergent. Then for any $\lambda \in \mathbb{R}$, the improper integrals $\displaystyle\int_a^b (f+g)$ and $\displaystyle\int_a^b (\lambda f)$ are convergent. Moreover, $\displaystyle\int_a^b (f+g) = \int_a^b f + \int_a^b g$ and $\displaystyle\int_a^b (\lambda f) = \lambda \int_a^b f$.

Proof. We have $\displaystyle\int_a^x (f+g) = \int_a^x f + \int_a^x g$ and $\displaystyle\int_a^x (\lambda f) = \lambda\int_a^x f$. We then take limits when $x \to b$. ✓

Remark. There is a similar statement for locally integrable functions on $]a,b]$.

We shall need more tools to study convergence of improper integrals, as this example shows.

Example. What is the nature of $\displaystyle\int_0^{+\infty} t^{\alpha-1}e^{-t}\,\mathrm{d}t$? This integral is an improper integral at $+\infty$ but also at 0 if $\alpha < 1$. Therefore we study separately the improper integrals $\displaystyle\int_0^1 t^{\alpha-1}e^{-t}\,\mathrm{d}t$ and $\displaystyle\int_1^{+\infty} t^{\alpha-1}e^{-t}\,\mathrm{d}t$.

➢ Study of $\displaystyle\int_0^1 t^{\alpha-1}e^{-t}\,\mathrm{d}t$.

If $\alpha \geqslant 1$, then $t \to t^{\alpha-1}e^{-t}$ is continuous on $[0,1]$ so that it is a falsely improper integral.

If $0 < \alpha < 1$, then $\displaystyle\int_x^1 t^{\alpha-1}e^{-t}\,\mathrm{d}t \leqslant \int_x^1 t^{\alpha-1}\,\mathrm{d}t \leqslant \int_0^1 t^{\alpha-1}\,\mathrm{d}t$ the last integral being an convergent integral. Moreover, the function $y \to \displaystyle\int_{1/y}^1 t^{\alpha-1}e^{-t}\,\mathrm{d}t$ is increasing (the integrand is positive), and since it is bounded above it has a finite limit when $y \to +\infty$, that is, when $x \to 0$. Therefore $\displaystyle\int_0^1 t^{\alpha-1}e^{-t}\,\mathrm{d}t$ converges.

If $\alpha \leqslant 0$ then $\displaystyle\int_x^1 t^{\alpha-1}e^{-t}\,\mathrm{d}t \geqslant \int_x^1 t^{-1}e^{-1}\,\mathrm{d}t = \frac{1}{e}\ln\frac{1}{x} \xrightarrow[x\to 0]{} +\infty$, therefore $\displaystyle\int_0^1 t^{\alpha-1}e^{-t}\,\mathrm{d}t$ diverges.

➢ Study of $\displaystyle\int_1^{+\infty} t^{\alpha-1}e^{-t}\,\mathrm{d}t$.

For any α we have $\lim_{t\to+\infty} t^2 t^{\alpha-1}e^{-t} = 0$. Therefore the function $t \to t^2t^{\alpha-1}e^{-t}$ is bounded on $[1,+\infty[$, so that there exists $M \in \mathbb{R}$ such that for all $t \geqslant 1$ we have $t^{\alpha-1}e^{-t} \leqslant \dfrac{M}{t^2}$.

Therefore, for any $x \geqslant 1$ we have $\displaystyle\int_1^x t^{\alpha-1}e^{-t}\,\mathrm{d}t \leqslant \int_1^x \frac{M}{t^2}\,\mathrm{d}t = M\left(1 - \frac{1}{x}\right) \leqslant M$. The function $x \to \displaystyle\int_1^x t^{\alpha-1}e^{-t}\,\mathrm{d}t$ is therefore increasing and bounded above, so that it has a finite limit when x goes to $+\infty$. Finally the improper integral $\displaystyle\int_1^{+\infty} t^{\alpha-1}e^{-t}\,\mathrm{d}t$ converges.

Therefore the improper integral $\displaystyle\int_0^{+\infty} t^{\alpha-1}e^{-t}\,\mathrm{d}t$ converges if and only if $\alpha > 0$.

II. Convergence theorems

We shall state all results for locally integrable functions defined on an interval $[a,b[$, but the corresponding results for locally integrable functions defined on an interval $]a,b]$ are true.

A. The comparison theorems

Proposition 6. Let f be a locally integrable *non-negative* function defined on $[a,b[$. The improper integral $\int_a^b f$ converges if and only if $x \to \int_a^x f$ is bounded above on $[a,b[$.

Proof. Consider the function $F\colon x \to \int_a^x f$. Then for $x < y$ we have $F(y) = F(x) + \int_x^y f$. Since f is non-negative, $\int_x^y f \geqslant 0$ and therefore F is non-decreasing. Therefore F has a limit when x nears b if and only if it is bounded above. ✓

Proposition 6 bis. Let f be a locally integrable *non-negative* function defined on $]a,b]$. The improper integral $\int_a^b f$ converges if and only if $x \to \int_x^b f$ is bounded above on $]a,b]$.

Proof. Consider the function $F\colon \left]\frac{1}{b-a}, +\infty\right[\to \mathbb{R}$ defined by $y \to \int_{a+1/y}^b f$. Then for $y < y'$, that is, $\frac{1}{y'} < \frac{1}{y}$, we have $F(y') = F(y) + \int_{a+1/y'}^{a+1/y} f$. Since f is non-negative, $\int_{a+1/y'}^{a+1/y} f \geqslant 0$ and therefore F is non-decreasing. Therefore F has a limit when y nears $+\infty$ if and only if it is bounded above. ✓

Theorem 7. Let f and g be locally integrable functions defined on $[a,b[$ (or $]a,b]$) such that $0 \leqslant f \leqslant g$.

- If the improper integral $\int_a^b g$ is convergent, then so is $\int_a^b f$ and we have $\int_a^b f \leqslant \int_a^b g$.
- If the improper integral $\int_a^b f$ is divergent, then so is $\int_a^b g$.

Proof. For $x \in [a,b[$, define $F(x) = \int_a^x f$ and $G(x) = \int_a^x g$. Since $f \leqslant g$ we have $F(x) \leqslant G(x)$ for any $x \in [a,b[$. If the improper integral $\int_a^b g$ is convergent, since g is non-negative this means by the previous proposition that G is bounded above, so that F is bounded above, and since f is non-negative the previous proposition shows that $\int_a^b f$ is convergent. Moreover if this is the case, since $F(x) \leqslant G(x)$ for all x, we have $\int_a^b f = \lim_{x\to b} F(x) \leqslant \lim_{x \to b} G(x) = \int_a^b g$.

The second statement is the contrapositive of the first. The proof in the case of $]a,b]$ is similar, using $\int_{a+1/y}^b f$ and $\int_{a+1/y}^b g$. ✓

We know that the convergence of an integral only depends on the local behaviour of f as x approaches b. Therefore we have a more general version of the previous theorem.

Theorem 8. Let f and g be locally integrable functions defined on $[a,b[$ (*resp.* $]a,b]$). Assume that there exists $c \in]a,b[$ such that for all $x \in [c,b[$ (*resp.* for all $x \in]a,c]$) we have $0 \leqslant f(x) \leqslant g(x)$.

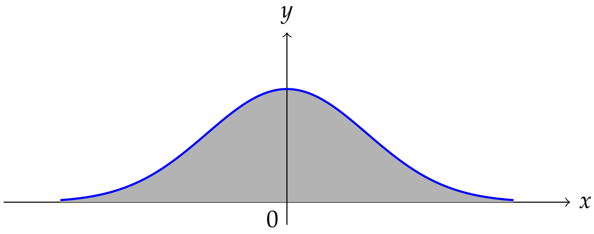
- If the improper integral $\int_a^b g$ is convergent, then so is $\int_a^b f$.
- If the improper integral $\int_a^b f$ is divergent, then so is $\int_a^b g$.

Remark. If f is non-positive, we can apply the previous results to $-f$.

Example. Let us prove that $\int_{-\infty}^{+\infty} e^{-t^2}\,\mathrm{d}t$ converges. We study separately $\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t$ and $\int_{-\infty}^0 e^{-t^2}\,\mathrm{d}t$.

- For any $t \geqslant 1$, we have $0 \leqslant e^{-t^2} \leqslant e^{-t}$ since $t^2 \geqslant t$, and $\int_1^x e^{-t}\,\mathrm{d}t = \frac{1}{e} - \frac{1}{e^x} \xrightarrow[x\to+\infty]{} \frac{1}{e}$. Therefore $\int_1^{+\infty} e^{-t^2}\,\mathrm{d}t$ converges, and $\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t$ also ($t \to e^{-t^2}$ is continuous on $[0,1]$).
- For any $t \leqslant -1$, we have $0 \leqslant e^{-t^2} \leqslant e^t$ since $-t^2 \leqslant t$, and $\int_x^1 e^t\,\mathrm{d}t = e - e^x \xrightarrow[x\to-\infty]{} e$. Therefore $\int_{-\infty}^{-1} e^{-t^2}\,\mathrm{d}t$ converges, and $\int_{+\infty}^0 e^{-t^2}\,\mathrm{d}t$ also ($t \to e^{-t^2}$ is continuous on $[-1,0]$).

Therefore $\int_{-\infty}^{+\infty} e^{-t^2}\,\mathrm{d}t$ converges.



The total area under the graph is finite.

Example. Let us prove that $\displaystyle\int_0^{+\infty}\frac{1-\cos t}{t^2}\,\mathrm{d}t$ converges.

We have, for any $t \geqslant 1$, $0 \leqslant \frac{1-\cos t}{t^2} \leqslant \frac{2}{t^2}$ and $\displaystyle\int_1^{+\infty}\frac{1}{t^2}\,\mathrm{d}t$ converges therefore $\displaystyle\int_1^{+\infty}\frac{1-\cos t}{t^2}\,\mathrm{d}t$ converges. Moreover, $\lim_{t\to 0}\frac{1-\cos t}{t^2} = \frac{1}{2}$, so that $\displaystyle\int_0^{1}\frac{1-\cos t}{t^2}\,\mathrm{d}t$ is a falsely improper integral. Therefore, $\displaystyle\int_0^{+\infty}\frac{1-\cos t}{t^2}\,\mathrm{d}t$ converges.

B. Consequences of the comparison theorems

Proposition 9. Let f and g be locally integrable functions on $[a,b[$ (*resp.* on $]a,b]$) that are positive. If $f{\sim}g$ at b (*resp.* at a), then the improper integrals $\int_a^b f$ and $\int_a^b g$ have the same behaviour (both convergent or both divergent).

Proof. By assumption, $\lim_{x\to b}\frac{f(x)}{g(x)} = 1$ so there exists $A \in [a,b[$ such that $\frac{1}{2} \leqslant \frac{f(x)}{g(x)} \leqslant \frac{3}{2}$ for $x \geqslant A$. Since $g(x) > 0$ for all x, we get $\frac{1}{2}g(x) \leqslant f(x) \leqslant \frac{3}{2}g(x)$ if $A \leqslant x < b$. Therefore, since both functions are non-negative:

- the first inequality shows that if if $\int_a^b g$ is divergent, so is $\int_a^b f$ and if $\int_a^b f$ is convergent so is $\int_a^b g$;
- the second inequality shows that if if $\int_a^b g$ is convergent, so is $\int_a^b f$ and if $\int_a^b f$ is divergent so is $\int_a^b g$. ✓

Example. $\displaystyle\int_0^{+\infty}\frac{x-1}{1+x^3}\,\mathrm{d}x$ converges. Indeed, it converges if and only if $\displaystyle\int_2^{+\infty}\frac{x-1}{1+x^3}\,\mathrm{d}x$ converges; we have $\displaystyle\frac{x-1}{1+x^3} \sim \frac{1}{x^2}$ at $+\infty$ with both functions positive on $[2,+\infty[$, and $\displaystyle\int_2^{+\infty}\frac{1}{x^2}\,\mathrm{d}x$ converges, therefore $\displaystyle\int_2^{+\infty}\frac{x-1}{1+x^3}\,\mathrm{d}x$ converges.

Proposition 10. Let f and g be locally integrable functions on $[a,b[$ (*resp.* on $]a,b]$) such that f is non-negative and g is positive. If $f = o(g)$ at b (*resp.* at a), then if the improper integral $\int_a^b g$ converges, so does $\int_a^b f$ and if the improper integral $\int_a^b f$ diverges, so does $\int_a^b g$.

Proof. By assumption, $\lim_{x\to b}\frac{f(x)}{g(x)} = 0$ so there exists $A \in [a,b[$ such that $0 \leqslant \frac{f(x)}{g(x)} \leqslant 1$ for $x \geqslant A$. Since $g(x) > 0$ for all x, we get $0 \leqslant f(x) \leqslant g(x)$ if $A \leqslant x < b$. Therefore, since both functions are non-negative, Theorem 8 gives the result. ✓

Examples.

- $\displaystyle\int_0^{+\infty} t^{10}e^{-\sqrt{t}}\,\mathrm{d}t$ converges, since $t^{10}e^{-\sqrt{t}} = o\left(\frac{1}{t^2}\right)$ at $+\infty$ and $\displaystyle\int_1^{+\infty}\frac{1}{t^2}\,\mathrm{d}t$ converges.
- $\displaystyle\int_0^{1}|\ln t|\,\mathrm{d}t$ converges, since $|\ln t| = o\left(\frac{1}{t^{1/2}}\right)$ at 0 and $\displaystyle\int_0^{1}\frac{1}{t^{1/2}}\,\mathrm{d}t$ converges.

More generally, we can obtain the following criterion for 'Bertrand integrals'.

Proposition 11. The improper integral $\displaystyle\int_2^{+\infty}\frac{1}{t^\alpha(\ln t)^\beta}\,\mathrm{d}t$ converges if and only if $\alpha > 1$ or $\alpha = 1$ and $\beta > 1$.

Proof. Note that the function $t \to \dfrac{1}{t^\alpha(\ln t)^\beta}$ is positive for $t \geqslant 2$.

- First case: $\alpha > 1$. Fix α' with $1 < \alpha' < \alpha$. Then $\dfrac{t^{\alpha'}}{t^\alpha(\ln t)^\beta} = \frac{1}{t^{\alpha-\alpha'}(\ln t)^\beta} \xrightarrow[t\to+\infty]{} 0$ so that $\frac{1}{t^\alpha(\ln t)^\beta} = o\left(\frac{1}{t^{\alpha'}}\right)$. Moreover, we know that $\displaystyle\int_2^{+\infty}\frac{1}{t^{\alpha'}}\,\mathrm{d}t$ is convergent, therefore $\displaystyle\int_2^{+\infty}\frac{1}{t^\alpha(\ln t)^\beta}\,\mathrm{d}t$ converges.

➢ Second case: $\alpha = 1$. Then, if $\beta \neq 1$, we have $\displaystyle\int_2^x \frac{1}{t(\ln t)^\beta}\,\mathrm{d}t = \frac{1}{1-\beta}\left(\frac{1}{(\ln x)^{\beta-1}} - \frac{1}{(\ln 2)^{\beta-1}}\right)$ and this has a finite limit when x goes to $+\infty$ if and only if $\beta > 1$; if $\beta = 1$, we have $\displaystyle\int_2^x \frac{1}{t\ln t}\,\mathrm{d}t = \ln(\ln x) - \ln(\ln 2) \xrightarrow[x\to+\infty]{} +\infty$. Therefore the improper integral $\displaystyle\int_2^{+\infty} \frac{1}{t(\ln t)^\beta}\,\mathrm{d}t$ converges if and only if $\beta > 1$.

➢ Third case: $\alpha < 1$. Fix α' with $1 > \alpha' > \alpha$. Then $\frac{1}{t^{\alpha'}} = o\left(\frac{1}{t^\alpha(\ln t)^\beta}\right)$ at $+\infty$ and the improper integral $\displaystyle\int_2^{+\infty} \frac{1}{t^{\alpha'}}\,\mathrm{d}t$ is divergent, therefore $\displaystyle\int_2^{+\infty} \frac{1}{t^\alpha(\ln t)^\beta}\,\mathrm{d}t$ diverges. ✓

C. Absolute convergence

We now consider the case where f does not have constant sign.

Theorem 12. Let f be a locally integrable function defined on $[a,b[$ (or $]a,b]$). If the improper integral $\displaystyle\int_a^b |f|$ is convergent, then so is $\displaystyle\int_a^b f$ and we have $\displaystyle\left|\int_a^b f\right| \leqslant \int_a^b |f|$.

Proof. For any real number r we have $0 \leqslant |r| - r \leqslant 2|r|$ so that for any $t \in [a,b[$ we have $0 \leqslant |f(t)| - f(t) \leqslant 2|f(t)|$. Set $g = |f| - f$. We have $0 \leqslant g \leqslant 2|f|$.

By assumption, $\displaystyle\int_a^b |f|$ is convergent, therefore so is $\displaystyle\int_a^b 2|f|$, and since g is non-negative and $g \leqslant 2|f|$, the improper integral $\displaystyle\int_a^b g$ is convergent. We have $f = |f| - g$, so $\displaystyle\int_a^b f$ is convergent. Moreover,

$$\left|\int_a^b f\right| = \lim_{x\to+\infty}\left|\int_a^x f\right| \leqslant \lim_{x\to+\infty}\int_a^x |f| = \int_a^b |f|.$$

✓

Definition 13. *If the improper integral $\displaystyle\int_a^b |f|$ is convergent, we say that the improper integral $\displaystyle\int_a^b f$ is* ***absolutely convergent***[a]*.*
The theorem above then states that an absolutely convergent improper integral is convergent.

[a] absolument convergente

Remark. When trying to prove absolute convergence, we can use the results of the previous section since $|f|$ is non-negative.

Example. Consider the improper integral $\displaystyle\int_0^{+\infty} \frac{\sin t}{1+t^2}\,\mathrm{d}t$. The integrand does not have constant sign. However, $\displaystyle\left|\frac{\sin t}{1+t^2}\right| \leqslant \frac{1}{1+t^2}$. Moreover, for any $x \geqslant 0$ we have $\displaystyle\int_0^x \frac{1}{1+t^2}\,\mathrm{d}t = \arctan x \xrightarrow[x\to+\infty]{} \frac{\pi}{2}$ so that the improper integral $\displaystyle\int_0^{+\infty} \frac{1}{1+t^2}\,\mathrm{d}t$ is convergent and $\displaystyle\int_0^{+\infty} \frac{\sin t}{1+t^2}\,\mathrm{d}t$ is therefore absolutely convergent hence convergent.

We could have used the fact that $\frac{1}{1+t^2} \sim \frac{1}{t^2}$ at $+\infty$ to prove convergence of $\displaystyle\int_0^{+\infty} \frac{1}{1+t^2}\,\mathrm{d}t$ (both functions are positive on $[1,+\infty[$).

III. Semi-convergence

There exist improper integrals that are convergent but not absolutely convergent.

Definition 14. *An improper integral that is convergent but not absolutely convergent is called* ***semi-convergent***[a]*.*

[a] semi-convergente

Example. Consider $\displaystyle\int_0^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t$.
This improper integral is convergent:

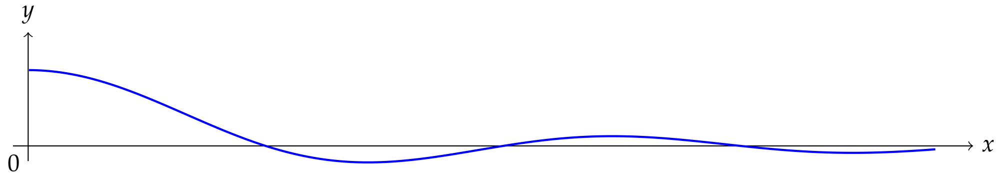
➢ At 0, since $\lim_{t\to 0} \dfrac{\sin t}{t} = 1$, it is a falsely improper integral.

➢ At $+\infty$, integration by parts gives $\displaystyle\int_1^x \frac{\sin t}{t}\,\mathrm{d}t = \left[-\frac{\cos t}{t}\right]_1^x - \int_1^x \frac{\cos t}{t^2}\,\mathrm{d}t$. The first term has a finite limit when x goes to $+\infty$, the second term is an absolutely convergent improper integral.

However, the integral is not absolutely convergent. We have $\left|\dfrac{\sin t}{t}\right| \geqslant \dfrac{\sin^2 t}{t}$ so it is enough to prove that $\displaystyle\int_1^{+\infty} \frac{\sin^2 t}{t}\,\mathrm{d}t$ diverges. We have, using integration by parts,

$$\begin{aligned}\int_1^x \frac{\sin^2 t}{t}\,\mathrm{d}t &= \int_1^x \frac{1-\cos 2t}{2t}\,\mathrm{d}t \\ &= \frac{1}{2}\left[\frac{1}{t}\left(t - \frac{1}{2}\sin(2t)\right)\right]_1^x + \frac{1}{2}\int_1^x \frac{1}{t^2}\left(t - \frac{1}{2}\sin 2t\right)\mathrm{d}t \\ &= -\frac{\sin 2x}{4x} + \frac{\sin 2}{4} + \frac{1}{2}\ln x - \frac{1}{4}\int_1^x \frac{\sin 2t}{t^2}\,\mathrm{d}t.\end{aligned}$$

The first two terms have finite limits when $x \to +\infty$ and so does the last one (absolutely convergent improper integral), and $\lim_{x\to+\infty} \frac{1}{2}\ln x = +\infty$ therefore $\lim_{x\to+\infty} \displaystyle\int_1^x \frac{\sin^2 t}{t}\,\mathrm{d}t = +\infty$.



The areas compensate.

This example shows that semi-convergent improper integrals exist.

Example. Note that as in the previous example, we can prove that $\displaystyle\int_2^{+\infty} \frac{\sin t}{\sqrt{t}}\,\mathrm{d}t$ converges. Moreover, $\dfrac{\sin t}{\sqrt{t}} \sim \dfrac{\sin t}{\sqrt{t}+\sin t}$ at $+\infty$. However, we shall now prove that $\displaystyle\int_2^{+\infty} \frac{\sin t}{\sqrt{t}+\sin t}\,\mathrm{d}t$ diverges.

We have $\dfrac{\sin t}{\sqrt{t}+\sin t} = \dfrac{\sin t}{\sqrt{t}}\dfrac{1}{1+\frac{\sin t}{\sqrt{t}}}$ and $\lim_{t\to+\infty} \frac{\sin t}{\sqrt{t}} = 0$. Set $u = \frac{\sin t}{\sqrt{t}}$, then $\dfrac{\sin t}{\sqrt{t}+\sin t} = u\frac{1}{1+u} = u(1-u+o(u)) = u - u^2 + o(u^2) = \dfrac{\sin t}{\sqrt{t}} + g(t)$. Note that as t approaches $+\infty$, we have $g(t) < 0$ so that we can use equivalent functions to determine convergence. We have $g(t) \sim -\dfrac{\sin^2 t}{t}$ at $+\infty$, and we have seen that $\displaystyle\int_2^{+\infty} -\frac{\sin^2 t}{t}\,\mathrm{d}t$ diverges so that $\displaystyle\int_2^{+\infty} \frac{\sin t}{\sqrt{t}+\sin t}\,\mathrm{d}t$ diverges.

This example shows that equivalent functions cannot be used to determine convergence when the functions do not have constant sign.

Remark. We have seen that integration by parts and Taylor expansions at infinity are useful tools when determining the nature of an improper integral (especially if the function does not have constant sign).

Example. We wish to determine the nature of $\displaystyle\int_2^{+\infty} \ln\left(1+\frac{\sin t}{t}\right)\mathrm{d}t$. We have

$$\ln\left(1+\frac{\sin t}{t}\right) = \frac{\sin t}{t} - \frac{\sin^2 t}{2t^2} + o\left(\frac{1}{t^2}\right).$$

We have seen that $\displaystyle\int_2^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t$ converges, and the function $-\dfrac{\sin^2 t}{2t^2} + o\left(\dfrac{1}{t^2}\right)$ is negative and equivalent to $-\dfrac{\sin^2 t}{2t^2}$ at $+\infty$; moreover, $\displaystyle\int_2^{+\infty} \frac{\sin^2 t}{2t^2}\,\mathrm{d}t$ is (absolutely) convergent.

Therefore $\displaystyle\int_2^{+\infty} \ln\left(1+\frac{\sin t}{t}\right)\mathrm{d}t$ is convergent.

We shall now see one more tool to determine the nature of an improper integral.

Theorem 15. Let f be a continuous function on $[a,b[$ (*resp.* $]a,b]$) Let φ be an increasing function of class $\mathcal{C}^1$ on $[\alpha,\beta[$ such that $a=\varphi(\alpha)$ and $b=\lim_{x\to\beta}\varphi(x)$ (*resp.* such that $a=\lim_{x\to\alpha}\varphi(x)$ and $b=\varphi(\beta)$). Then the improper integrals $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ and $\displaystyle\int_\alpha^\beta f(\varphi(t))\varphi'(t)\,\mathrm{d}t$ have the same nature and if they converge we have

$$\int_a^b f(x)\,\mathrm{d}x=\int_\alpha^\beta f(\varphi(t))\varphi'(t)\,\mathrm{d}t.$$

Remark. There is a similar statement with φ decreasing, in which the roles of α and β are reversed.

Proof. If $\alpha\leqslant v<\beta$ then $\displaystyle\int_{\varphi(\alpha)}^{\varphi(v)} f(x)\,\mathrm{d}x=\int_\alpha^v f(\varphi(t))\varphi'(t)\,\mathrm{d}t$. Define $F(u)=\displaystyle\int_{\varphi(\alpha)}^u f(x)\,\mathrm{d}x$ for $u\in[a,b[$ and $G(v)=\displaystyle\int_\alpha^v f(\varphi(t))\varphi'(t)\,\mathrm{d}t$ for $v\in[\alpha,\beta[$. We then have $F\circ\varphi(v)=G(v)$.

If $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ converges then the limit of $F(u)$ as u nears b exists, therefore so does the limit of $G(v)=F(\varphi(v))$ as v nears β and we have $\lim_{v\to\beta}G(v)=\lim_{v\to\beta}F(\varphi(v))=\lim_{u\to b}F(u)$ as required.

The converse is proved similarly, using the relation $F=G\circ\varphi^{-1}$ (φ is continuous and strongly monotonic on an interval therefore it defines a bijection on its image). ✓

Example. Fix $\alpha>2$. We want the nature of $\displaystyle\int_1^{+\infty} t\sin(t^\alpha)\,\mathrm{d}t$.

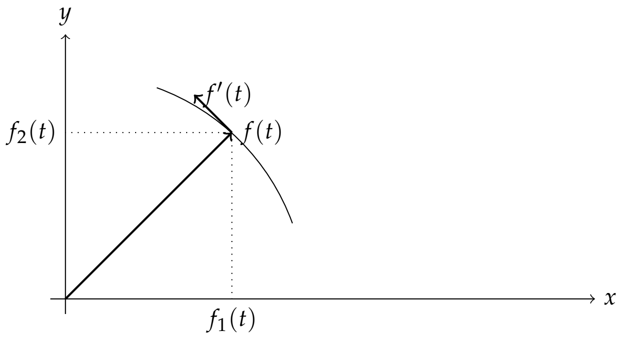
Set $u=t^\alpha$, that is, $\varphi(t)=t^\alpha$. The function φ is of class $\mathcal{C}^1$ and defines an increasing bijection from $[1\,;+\infty[$ to $[1\,;+\infty[$. We have $t=u^{1/\alpha}$, therefore $\mathrm{d}t=\frac{1}{\alpha}u^{1/\alpha-1}\,\mathrm{d}u=\frac{1}{\alpha u^{1-\frac{1}{\alpha}}}\,\mathrm{d}u$. Moreover, $t\sin(t^\alpha)=u^{1/\alpha}\sin u$. It then follows from the theorem that the improper integrals $\displaystyle\int_1^{+\infty} t\sin(t^\alpha)\,\mathrm{d}t$ and $\displaystyle\int_1^{+\infty}\frac{\sin u}{\alpha u^{1-\frac{2}{\alpha}}}\,\mathrm{d}u$ have the same nature. Moreover, the improper integral $\displaystyle\int_1^{+\infty}\frac{\sin u}{\alpha u^{1-\frac{2}{\alpha}}}\,\mathrm{d}u$ is convergent (similar to a previous example, use integration by parts – or take $\alpha=4$ to get exactly the previous example). Therefore $\displaystyle\int_1^{+\infty} t\sin(t^\alpha)\,\mathrm{d}t$ is convergent.

This is the end of the lectures given in the University year 2014-2015. The sequel is contained in the syllabus of the second semester of the second year and was intended as an introduction.

Chapter *11*

Derivation and integration of functions defined on an interval of $\mathbb{R}$ with values in $\mathbb{R}^2$

In this chapter, all maps are defined on an interval I in $\mathbb{R}$ and take values in $\mathbb{R}^2$; such a map is called a **vector function**†. We shall extend what we have done for functions from a subset of $\mathbb{R}$ to $\mathbb{R}$ to this situation.



A vector function not only gives a curve, it also gives the way it is drawn. For instance, the curve described by the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(t) = (\cos t, \sin t)$ is a circle around which we go an infinite number of times, and the curve described by the function $g\colon [0; 2\pi] \to \mathbb{R}$ defined by $g(t) = (\cos t, \sin t)$ is the same circle around which we go once.

We first need the notion of norm.

I. Norms in $\mathbb{R}^2$

Definition 1. *A* ***norm***[a] *on $\mathbb{R}^2$ is a map $N : \mathbb{R}^2 \to \mathbb{R}$ such that:*

- $\forall v \in \mathbb{R}^2$, $N(v) \geqslant 0$,
- $\forall v \in \mathbb{R}^2$, $N(v) = 0 \Rightarrow v = 0$,
- $\forall \lambda \in \mathbb{R}$ *and* $v \in \mathbb{R}^2$, $N(\lambda v) = |\lambda| N(v)$,
- $\forall (u, v) \in \mathbb{R}^2 \times \mathbb{R}^2$, $N(u + v) \leqslant N(u) + N(v)$ *(**triangle inequality**[b]).*

[a] norme
[b] inégalité du triangle

Remark. The triangle inequality can also be written (as is the case for the absolute value): $|N(u) - N(v)| \leqslant N(u - v)$.

Example. The **Euclidean norm**‡ on $\mathbb{R}^2$ is defined by $N(x, y) = \|(x, y)\|_2 = \sqrt{x^2 + y^2}$. The fact that it is indeed a norm is left as an exercise.

Example. The **supremum norm**§ on $\mathbb{R}^2$ is defined by $N(x, y) = \|(x, y)\|_\infty = \sup\{|x|, |y|\} = \max\{|x|, |y|\}$.

The first three properties of a norm are easy to check (exercise). For the last one, let (x, y) and (x', y') be two vectors in $\mathbb{R}^2$. Then $|x + x'| \leqslant |x| + |x'| \leqslant \|(x, y)\|_\infty + \|(x', y')\|_\infty$ and similarly $|y + y'| \leqslant \|(x, y)\|_\infty + \|(x', y')\|_\infty$ so that $\|(x, y) + (x', y')\|_\infty = \|(x + x', y + y')\|_\infty \leqslant \|(x, y)\|_\infty + \|(x', y')\|_\infty$ as required.

Remark. We can define many norms on $\mathbb{R}^2$. However we shall only consider the supremum norm and the Euclidean norm in the sequel. The notation $\|\cdot\|$ will stand for one of those two norms.

† fonction à valeurs vectorielles
‡ norme euclidienne
§ norme sup

In this chapter, we aim to extend the definitions of limit, continuity, differentiability, etc. to functions that takes values in $\mathbb{R}^2$. We shall replace the absolute value wherever necessary by a norm. It is therefore very useful to know that we can use whichever norm we prefer.

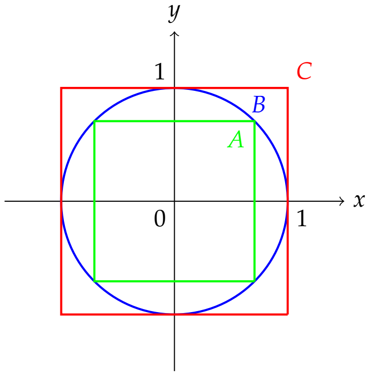
Proposition 2. The norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent, that is, there exist real numbers $\alpha > 0$ and $\beta > 0$ such that for every $v \in \mathbb{R}^2$ we have

$$\|v\|_2 \leqslant \alpha \|v\|_\infty \quad \text{and} \quad \|v\|_\infty \leqslant \beta \|v\|_2.$$

Proof. We can take $\alpha = \sqrt{2}$ and $\beta = 1$. ✓

Remark. The figure below represents this equivalence:

$$A := \left\{(x,y) \in \mathbb{R}^2; \|(x,y)\|_\infty \leqslant \frac{1}{\sqrt{2}}\right\} \subset B := \left\{(x,y) \in \mathbb{R}^2; \|(x,y)\|_2 \leqslant 1\right\}$$
$$\subset C := \left\{(x,y) \in \mathbb{R}^2; \|(x,y)\|_\infty \leqslant 1\right\}.$$



Remark. We can identify $\mathbb{C}$ with $\mathbb{R}^2$. The modulus on $\mathbb{C}$ corresponds via this identification to the Euclidean norm $\|\cdot\|_2$. Everything that follows can therefore be applied to complex-valued functions. In fact, we have already seen some of the results in this case.

To simplify notation, we only consider functions with values in $\mathbb{R}^2$, but everything can be immediately generalised to functions with values in $\mathbb{R}^n$, as you will see next year.

II. Limits, continuity and differentiability of a vector function

Definition-Proposition 3. *Let $f : I \to \mathbb{R}^2$ be a vector function and let ℓ be an element of $\mathbb{R}^2$. Let t_0 be an element of I or an endpoint of I. Let $\|\cdot\|$ be one of the norms $\|\cdot\|_\infty$ or $\|\cdot\|_2$.*

- *We say that f has* ***limit***[a] *ℓ when t nears t_0 if the function $t \to \|f(t) - \ell\|$ has limit 0 when t nears t_0. In other words,*

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ 0 < |t - t_0| < \eta \Rightarrow \|f(t) - \ell\| < \varepsilon.$$

  We write $\lim_{t\to t_0} f(t) = \ell$.

- *We say that f is* ***continuous***[b] *at $t_0 \in I$ if $f(t)$ has limit $f(t_0)$ when t nears t_0.*

- *We say that f is* ***differentiable***[c] *at $t_0 \in I$ if $\dfrac{f(t) - f(t_0)}{t - t_0}$ has a finite limit when t nears t_0; this limit is denoted by $f'(t_0)$ and is called the* ***derived vector***[d] *of f at t_0.*

[a] limite
[b] continue
[c] dérivable
[d] vecteur dérivé

Proof. We must check that the definition makes sense, that is, that it does not depend on the norm we have chosen.

- Assume that f has limit ℓ when t nears t_0 for the norm $\|\cdot\|_\infty$. This means that $\lim_{t\to t_0} \|f(t) - \ell\|_\infty = 0$. We have $0 \leqslant \|f(t) - \ell\|_2 \leqslant \sqrt{2}\|f(t) - \ell\|_\infty$ so that $\lim_{t\to t_0} \|f(t) - \ell\|_2 = 0$. Therefore f has limit ℓ when t nears t_0 for the norm $\|\cdot\|_2$.
- Assume that f has limit ℓ when t nears t_0 for the norm $\|\cdot\|_2$. This means that $\lim_{t\to t_0} \|f(t) - \ell\|_2 = 0$. We have $0 \leqslant \|f(t) - \ell\|_\infty \leqslant \|f(t) - \ell\|_2$ so that $\lim_{t\to t_0} \|f(t) - \ell\|_\infty = 0$. Therefore f has limit ℓ when t nears t_0 for the norm $\|\cdot\|_\infty$. ✓

Example. Consider the function $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (t^2 - e^t, \cos t)$. Then $\lim_{t\to 0} f(t) = (-1, 1)$.
Indeed, we have $\|f(t) - (-1,1)\|_\infty = \max\big(|t^2 - e^t + 1|, |\cos t - 1|\big)$, $|t^2 - e^t + 1| \xrightarrow[t\to 0]{} 0$ and $|\cos t - 1| \xrightarrow[t\to 0]{} 0$ so that $\|f(t) - (-1,1)\|_\infty \xrightarrow[t\to 0]{} 0$. Note that $f(0) = (-1,1)$ so that f is in fact continuous at 0.

Remark. If we view $f : I \to \mathbb{R}^2$ as a complex-valued function, the definitions above differ from those we gave earlier. However, they are equivalent, as we shall now see.

Definition 4. *Let $f : I \to \mathbb{R}^2$ be a function. Define the linear projections $\pi_i : \mathbb{R}^2 \to \mathbb{R}$ for $i = 1, 2$ by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. The* ***coordinate functions***[a] *of f are the functions $f_i = \pi_i \circ f : I \to \mathbb{R}$ for $i = 1, 2$.*

[a] fonctions coordonnées

Proposition 5. Let $f : I \to \mathbb{R}^2$ be a vector function, defined by $f(t) = (f_1(t), f_2(t))$ for $t \in I$, where $f_i = \pi_i \circ f$. Let $\ell = (\ell_1, \ell_2)$ be in $\mathbb{R}^2$ and let t_0 be an element of I or an endpoint of I.
Then f has limit ℓ as t nears t_0 if and only if for $i = 1, 2$ the function $f_i = \pi_i \circ f$ has limit ℓ_i as t nears t_0.

Proof. By definition of $\|\cdot\|_\infty$ we have $0 \leqslant |f_i(t) - \ell_i| \leqslant \|f(t) - \ell\|_\infty$. Therefore it is clear that if f has limit ℓ as t nears t_0 then f_i has limit ℓ_i as t nears t_0 for $i = 1, 2$.
Conversely, since $\|f(t) - \ell\|_\infty = \max(|f_1(t) - \ell_1|, |f_2(t) - \ell_2|)$, if both $|f_1(t) - \ell_1|$ and $|f_2(t) - \ell_2|$ near 0 as t goes to t_0, then $\|f(t) - \ell\|_\infty$ nears 0 as t goes to t_0. ✓

Corollary 6. Let $f : I \to \mathbb{R}^2$ be a vector function, defined by $f(t) = (f_1(t), f_2(t))$ for $t \in I$, where $f_i = \pi_i \circ f$. Then f is continuous (*resp.* differentiable) at $t_0 \in I$ if and only if f_1 and f_2 are continuous (*resp.* differentiable) at t_0. Moreover, if f is differentiable at t_0, we have $f'(t_0) = (f_1'(t_0), f_2'(t_0))$.

Proof. Exercise. ✓

Definition 7. *Let $f : I \to \mathbb{R}^2$ be a vector function, defined by $f(t) = (f_1(t), f_2(t))$ for $t \in I$. If $t_0 \in I$ we say that f is* ***of class*** $\mathcal{C}^k$ *at t_0 if the functions f_1 and f_2 are of class $\mathcal{C}^k$ at t_0. The set of all functions of class $\mathcal{C}^k$ is denoted by $\mathcal{C}^k(I, \mathbb{R}^2)$.*

Definition 8. *Let $u = (x, y)$ and $u' = (x', y')$ be two vectors in $\mathbb{R}^2$. We denote by $\langle u, u' \rangle$ the* ***scalar product***[a] *of u and v, that is, $\langle u, v \rangle = xx' + yy'$.*

[a] produit scalaire

Remark. Note that $\langle u, u \rangle = \|u\|_2$.

Corollary 9. Let f and g be functions from I to $\mathbb{R}^2$ and α a function from I to $\mathbb{R}$. Let t_0 be a point in I.

- If $\lim_{t\to t_0} f(t) = \ell$ and $\lim_{t\to t_0} g(t) = \ell'$ then $\lim_{t\to t_0}(f(t) + g(t)) = \ell + \ell'$.
- If $\lim_{t\to t_0} f(t) = \ell$ and $\lim_{t\to t_0} \alpha(t) = a$, then $\lim_{t\to t_0}(\alpha(t) f(t)) = a\ell$.
- If $\lim_{t\to t_0} f(t) = \ell$ and $\lim_{t\to t_0} g(t) = \ell'$ then $\lim_{t\to t_0}\langle f(t), g(t) \rangle = \langle \ell, \ell' \rangle$.
- If f, g and α are continuous (*resp.* differentiable, *resp.* of class $\mathcal{C}^k$), then so are $f + g$, αf and $\langle f, g \rangle$. Moreover, when they are differentiable, we have

$$(f+g)' = f' + g', \quad (\alpha f)' = \alpha' f + \alpha f' \quad \text{and} \quad \langle f, g \rangle' = \langle f', g \rangle + \langle f, g' \rangle.$$

Proof. Exercise. ✓

Theorem 10. Let $a < b$ be two real numbers and let $f : [a,b] \to \mathbb{R}^2$ be a continuous function that is differentiable on $]a,b[$. Assume that there is a real number M such that, for all $t \in]a,b[$, we have $\|f'(t)\| \leqslant M$. Then

$$\|f(b) - f(a)\| \leqslant M(b-a).$$

Proof. Fix $\varepsilon > 0$ and consider the function $\varphi_\varepsilon : [a,b] \to \mathbb{R}$ defined by $\varphi_\varepsilon(t) = \|f(t) - f(a)\| - (M+\varepsilon)t$. This function is continuous, therefore it has a minimum at $c \in [a,b]$.

We shall now prove that $c = b$. Assume for a contradiction that $c \in [a,b[$; this means that there exist real numbers $t \in [a,b]$ with $t > c$.

There exists $\eta > 0$ such that for $t \in [a,b]$ with $0 < |t-c| < \eta$ we have $\left\| \frac{f(t)-f(c)}{t-c} - f'(c) \right\| < \varepsilon$. Now let $t \in [a,b]$ be such that $c < t < c + \eta$. Using the triangle inequality $\|u\| - \|v\| \leqslant \|u - v\|$ and multiplying by $t - c > 0$ we get

$$\|f(t) - f(c)\| < \left\|f'(c)\right\|(t-c) + \varepsilon(t-c) \leqslant M(t-c) + \varepsilon(t-c) = (M+\varepsilon)(t-c).$$

We then have

$$\begin{aligned}\varphi_\varepsilon(t) - \varphi_\varepsilon(c) &= \|f(t) - f(a)\| - \|f(c) - f(a)\| - (M+\varepsilon)t + (M+\varepsilon)c \\ &\leqslant \|f(t) - f(c)\| - g(t) - (M+\varepsilon)(t-c) < 0,\end{aligned}$$

which is a contradiction since $\varphi_\varepsilon(c)$ is a minimum value of φ_ε on $[a,b]$.

Therefore $\varphi_\varepsilon(b)$ is a minimum value of φ_ε on $[a,b]$. In particular, $\varphi_\varepsilon(b) \leqslant \varphi_\varepsilon(a)$, that is, $\|f(b) - f(a)\| \leqslant (M + \varepsilon + (b-a)$. This is true for any $\varepsilon > 0$, so that the result follows. ✓

Remark. There is no equivalent of Rolle's Theorem for functions with values in $\mathbb{R}^2$. For instance, if $f : [0, 2\pi] \to \mathbb{R}^2$ is defined by $f(t) = (\cos t, \sin t)$ then we have $f(0) = f(2\pi)$ but $f'(t) = (-\sin t, \cos t)$ never vanishes.

Corollary 11. Let $f : I \to \mathbb{R}^2$ be a differentiable function whose derived vector function is 0 on I. Then f is a constant function.

Proof. We can take $M = 0$ in the previous theorem. ✓

IV. Taylor expansions of vector functions

Assume that the coordinate functions of a function $f : I \to \mathbb{R}^2$ have a Taylor expansion of order p at t_0 :

$$f_k(t) = a_{k,0} + a_{k,1}(t-t_0) + \ldots + a_{k,p}(t-t_0)^p + o((t-t_0)^p).$$

For each j with $0 \leqslant j \leqslant p$ let A_j denote the vector $(a_{1,j}; a_{2,j})$. Then

$$f(t) = A_0 + A_1(t-t_0) + \cdots + A_p(t-t_0)^p + o((t-t_0)^p) \tag{11.1}$$

where $o((t-t_0)^p)$ is a function h such that $\lim_{t\to t_0} \frac{h(t)}{(t-t_0)^p} = 0$. The expression (11.1) is called a **Taylor expansion**[†] of f of order p at t_0.

Example. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(t) = (2t^3 - t\sin t, t^3 + \cos t)$. Then we have the Taylor expansions of order 4 at 0 :

$$\begin{aligned}2t^3 - t\sin t &= -t^2 + 2t^3 + \frac{1}{6}t^4 + o(t^4) \\ t^3 + \cos t &= 1 - \frac{1}{2}t^2 + t^3 + \frac{1}{24}t^4 + o(t^4) \\ f(t) &= (0,1) + t^2\left(-1, -\frac{1}{2}\right) + t^3(2,1) + t^4\left(\frac{1}{6}, \frac{1}{24}\right) + o(t^4).\end{aligned}$$

In the same way as for functions with values in $\mathbb{R}$, if f is sufficiently regular, Taylor expansions for f exist.

Theorem 12. (Taylor-Young formula[a]) Let $f : I \to \mathbb{R}^2$ be a function of class $\mathcal{C}^p$ and let t_0 be an element of I. Then

$$\forall x \in I,\ f(x) = f(t_0) + (t-t_0)f'(t_0) + \cdots + (t-t_0)^p \frac{f^{(p)}(t_0)}{p!} + o((t-t_0)^p).$$

[a] formule de Taylor-Young

[†] développement limité

Proof. The result is true for each of the coordinate functions hence it is true for f. ✓

Remark. A continuous function from I to $\mathbb{R}^2$ are also called a **parametric curve**[†]. Taylor expansions are used for the local study of these curves.

V. Integration of vector functions

Definition 13. *Let $f : [a, b] \to \mathbb{R}^2$ be a piecewise continuous function (that is, each coordinate function f_k is piecewise continuous). The* ***integral***[a] *of f on $[a, b]$ is*

$$\int_a^b f(t)\,\mathrm{d}t = \left(\int_a^b f_1(t)\,\mathrm{d}t, \int_a^b f_2(t)\,\mathrm{d}t\right).$$

[a]intégrale

Remark. This definition makes sense since each of the f_k is integrable.

Properties 14. Let $f, g : [a, b] \to \mathbb{R}^2$ be piecewise continuous functions. Then

- for any constant λ we have $\int_a^b (f(t) + \lambda g(t))\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t + \lambda \int_a^b g(t)\,\mathrm{d}t$;
- for any $c \in [a, b]$ we have $\int_a^b f(t)\,\mathrm{d}t = \int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t$;

Theorem 15. Let $f : [a, b] \to \mathbb{R}^2$ be a piecewise continuous function. Then

$$\left\| \int_a^b f(t)\,\mathrm{d}t \right\| \leqslant \int_a^b \|f(t)\|\,\mathrm{d}t.$$

Proof. Admitted. ✓

We clearly have the following result (since $x \to \|f(x)\|$ is a function from $[a, b]$ to $\mathbb{R}$).

Proposition 16. Let $f : [a, b] \to \mathbb{R}^2$ be a piecewise continuous function. If M is a real number such that $\|f(x)\| \leqslant M$ for all $x \in [a, b]$, then

$$\frac{1}{b-a}\int_a^b \|f(x)\|\,\mathrm{d}x \leqslant M.$$

The next results are consequences of the corresponding results for functions $I \to \mathbb{R}$ applied to the coordinate functions.

Theorem 17. Let $f : I \to \mathbb{R}^2$ be a continuous function and let $\varphi : [\alpha, \beta] \to I$ be a function of class $\mathcal{C}^1$. Then

$$\int_\alpha^\beta f(\varphi(u))\varphi'(u)\,\mathrm{d}u = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,\mathrm{d}t.$$

Theorem 18. (Taylor formula with integral remainder[a]) Let $f : I \to \mathbb{R}^2$ be a function of class $\mathcal{C}^{p+1}$ and a an element of I. Then for all $t \in I$ we have

$$f(t) = f(a) + (t-a)f'(a) + \cdots + (t-a)^p \frac{f^{(p)}(a)}{p!} + \int_a^t \frac{(t-u)^p}{p!} f^{(p+1)}(u)\,\mathrm{d}u.$$

This is called the **Taylor formula of order p at a with integral remainder**.

[a]formule de Taylor avec reste intégral

[†]courbe paramétrée

Chapter 12

Functions of two variables with values in $\mathbb{R}$ or $\mathbb{R}^2$

We shall now consider functions of two variables. We need to replace the intervals of $\mathbb{R}$ by a new object that we shall define now.

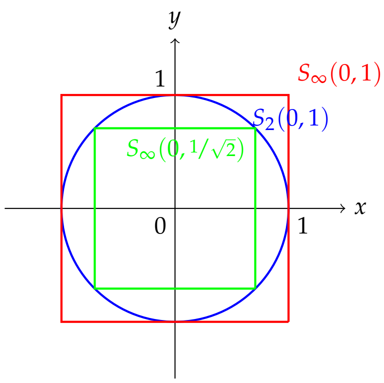
I. Open subsets of $\mathbb{R}^2$

Definition 1. *Fix a norm $\|\cdot\|$ on $\mathbb{R}^2$.*
The ***open disk***[a] *centered at $v_0 \in \mathbb{R}^2$ of radius*[b] *$r \in \mathbb{R}^+$ is the set $B(v_0, r) := \{v \in \mathbb{R}^2; \|v - v_0\| < r\}$.*
The ***closed disk***[c] *centered at $v_0 \in \mathbb{R}^2$ of radius $r \in \mathbb{R}^+$ is the set $\bar{B}(v_0, r) := \{v \in \mathbb{R}^2; \|v - v_0\| \leqslant r\}$.*
The ***sphere***[d] *centered at $v_0 \in \mathbb{R}^2$ of radius $r \in \mathbb{R}^+$ is the set $S(v_0, r) := \{v \in \mathbb{R}^2; \|v - v_0\| = r\} = \bar{B}(v_o, r) \setminus B(v_0, r)$.*
A ***unit disk***[e] *is a disk of radius 1.*

[a] disque (*ou* boule) ouvert
[b] rayon
[c] disque (*ou* boule) fermé
[d] sphère
[e] disque (*ou* boule) unité

Remark. We shall write B_2 or B_∞ to specify which of the norms we are using.

We represent below some spheres for our two norms.



We now define the parts of $\mathbb{R}^2$ that will replace open intervals in $\mathbb{R}$.

Definition 2. *A subset of $\mathbb{R}^2$ is called an* ***open subset***[a] *if it is empty or a union of open disks.*

[a] ouvert (*ou* partie ouverte)

Proposition 3. Let Ω be a non-empty subset of $\mathbb{R}^2$. The following statements are equivalent:

(i) Ω is open

(ii) for every $v \in \Omega$ there exists an open disk B such that $v \in B \subset \Omega$

(iii) for every $v \in \Omega$ there exists $r > 0$ such that $B(v, r) \subset \Omega$.

Proof. ➢ (i)$\Rightarrow$(ii) By assumption we have $\Omega = \cup_{i \in I} B_i$ for some set I and some open disks B_i. Therefore if $v \in \Omega$, there exists $i \in I$ such that $v \in B_i \subset \Omega$ as required.

➢ (ii)⇒(iii) Take $v \in \Omega$. By assumption, there exist $u \in \mathbb{R}^2$ and $\rho \in \mathbb{R}_+^*$ such that $v \in B(u,\rho) \subset \Omega$. If $v = u$ the result is proved. Otherwise, set $r = \min(\|v-u\|, \rho - \|v-u\|) > 0$. We need only prove that $B(v,r) \subset B(v,\rho)$ to conclude. Therefore let w be a point in $B(v,r)$. Then $\|w-u\| \leqslant \|w-v\| + \|v-u\| < \rho - \|v-u\| + \|v-u\| = \rho$ so that $w \in B(u,\rho)$.

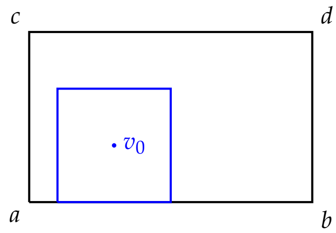
➢ (iii)⇒(i) Our assumption is that for every $v \in \Omega$, there exists $r_v > 0$ such that $B(v,r_v) \subset \Omega$. We then have $\cup_{v\in\Omega} B(v,r_v) \subset \Omega$. But the other inclusion is clearly true, therefore $\Omega = \cup_{v\in\Omega} B(v,r_v)$ so that Ω is open. ✓

Remark. Although the definition of an open subset of $\mathbb{R}^2$ seems to depend on the norm we choose, this is in fact not the case.

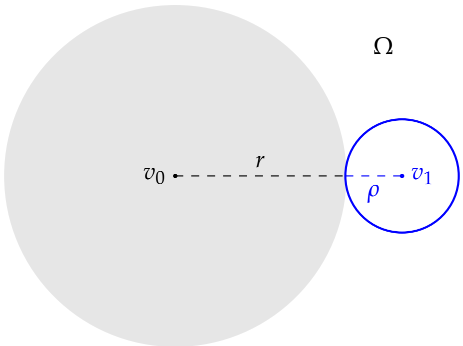
Indeed, suppose that Ω is an open set for $\|\cdot\|_\infty$. Let v be a point in Ω. Then we know that there is an open disk $B_\infty(v,r) \subset \Omega$. We also know that $B_2(v, \frac{r}{\sqrt{2}}) \subset B_\infty(v,r) \subset \Omega$ and this proves that Ω is open for $\|\cdot\|_2$. The converse is similar.

Examples. ➢ The open disks are obviously open sets by definition.

➢ A rectangle $\Omega =]a,b[\times]c,d[$ is an open subset of $\mathbb{R}^2$. Indeed, if $v_0 = (x_0,y_0) \in \Omega$, set $r = \min(x_0 - a, b - x_0, y_0 - c, d - y_0)$; then $B_\infty(v_0,r) \subset \Omega$.



➢ Fix $v_0 \in \mathbb{R}^2$ and $r \in \mathbb{R}_+$. Then the set $\Omega = \mathbb{R}^2 \setminus \bar{B}_2(v_0,r) = \{v \in \mathbb{R}^2; \|v - v_0\|_2 > r\}$ is open. Indeed, let v_1 be a point in Ω and set $\rho = \|v_1 - v_0\|_2 - r > 0$. We prove that $B_2(v_1,\rho) \subset \Omega$: if $v \in B_2(v_1,\rho)$ we have $\|v - v_0\|_2 \geqslant \|v_0 - v_1\|_2 - \|v_1 - v\|_2 > \|v_0 - v_1\|_2 - \rho = r$.



In this section, Ω is a non-empty open subset of $\mathbb{R}^2$.

Definition 4. *Let $f : \Omega \to \mathbb{R}$ be a function and let v_0 be a point in Ω. Fix one of the norms $\|\cdot\|$. We say that:*

- *f has **limit** $\ell \in \mathbb{R}$ at v_0 if $f(v)$ has limit ℓ when $\|v - v_0\|$ nears 0, that is,*

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \|v - v_0\| < \eta \Rightarrow |f(v) - \ell| < \varepsilon.$$

- *f is **continuous** at v_0 if f has limit $f(v_0)$ at v_0.*

- *f is **continuous** on Ω if f is continous at every point in Ω. The set of all functions from Ω to $\mathbb{R}$ that are continuous is denoted by $\mathcal{C}^0(\Omega, \mathbb{R})$.*

Let $g : \Omega \to \mathbb{R}^2$ be a function and let v_0 be a point in Ω. Fix a norm $\|\cdot\|$ on $\mathbb{R}^2$ at the origin and a norm $\|\cdot\|'$ at the target (they may be different). We say that:

- *g has limit $w \in \mathbb{R}^2$ at v_0 if $\|g(v) - w\|'$ has limit 0 when $\|v - v_0\|$ nears 0, that is,*

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \|v - v_0\| < \eta \Rightarrow \|g(v) - w\|' < \varepsilon.$$

- *g is continuous at v_0 if g has limit $g(v_0)$ at v_0.*

- *g is continuous on Ω if g is continous at every point in Ω. The set of all functions from Ω to $\mathbb{R}^2$ that are continuous is denoted by $\mathcal{C}^0(\Omega, \mathbb{R}^2)$.*

Proposition 5. Let f be a function defined on Ω with values in $\mathbb{R}$ (*resp.* $\mathbb{R}^2$), let v_0 be a point in Ω and let ℓ be an element of $\mathbb{R}$ (*resp.* $\mathbb{R}^2$). Then the fact that f has limit ℓ as v nears v_0 does not depend on the choice of norms.

Proof. We do the proof when f takes values in $\mathbb{R}^2$, the other case is left as an exercise. Assume that we have two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ at the origin and two norms $\|\cdot\|'_a$ and $\|\cdot\|'_b$ at the target, and that $f(v)$ has limit ℓ when v nears v_0 for the norms $\|\cdot\|_a$ and $\|\cdot\|'_a$. We know that there exist real numbers $\alpha > 0$ and $\beta > 0$ such that $\|\cdot\|_a \leqslant \alpha\|\cdot\|_b$ and $\|\cdot\|'_b \leqslant \beta\|\cdot\|'_a$.

Fix $\varepsilon > 0$. Then there exists $\eta > 0$ such that $\|v - v_0\|_a < \eta \Rightarrow \|f(v) - \ell\|'_a < \frac{\varepsilon}{\beta}$. Therefore

$$\|v - v_0\|_b < \frac{\eta}{\alpha} \Rightarrow \|v - v_0\|_a < \eta \Rightarrow \|f(v) - \ell\|'_a < \frac{\varepsilon}{\beta} \Rightarrow \|f(v) - \ell\|'_b < \varepsilon$$

as required. ✓

Remark. The properties of limits and of continuous functions (uniqueness of the limit, linear combinations, products, compositions) are still true and the proofs are similar.

Definition 6. *Let $f : \Omega \to \mathbb{R}^2$ be a function. Define the linear projections $\pi_i : \mathbb{R}^2 \to \mathbb{R}$ for $i = 1, 2$ by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. The **coordinate functions** of f are the functions $f_i = \pi_i \circ f : \Omega \to \mathbb{R}$ for $i = 1, 2$.*

Proposition 7. Let $f : \Omega \to \mathbb{R}^2$ be a function, let $\ell = (\ell_1, \ell_2)$ be in $\mathbb{R}^2$ and let v_o be a point in Ω. Then f has limit ℓ as v nears v_0 if and only if for $i = 1, 2$ its coordinate function f_i has limit ℓ_i as v nears v_0. In particular, f is continuous at v_0 if and only if its coordinate functions f_1 and f_2 are continous at v_0.

Proof. We work with $\|\cdot\|_\infty$. For $i = 1, 2$, for any $v \in \Omega$, we have $|f_i(v) - \ell_i| \leqslant \|f(v) - \ell\|_\infty$.

- First assume that f has limit ℓ at v_0. Then $\|f(v) - \ell\|_\infty$ goes to 0 as v nears v_0 so for $i = 1, 2$, $|f_i(v) - \ell_i|$ goes to 0 as v nears v_0 as required.

- Now assume that f_1 has limit ℓ_1 and f_2 has limit ℓ_2 at v_0. Fix $\varepsilon > 0$. Then there exist $\eta_1 > 0$ and $\eta_2 > 0$ such that $\|v - v_0\|_\infty < \eta_i \Rightarrow |f_i(v) - \ell_i| < \varepsilon$. Set $\eta = \min(\eta_1, \eta_2)$ so that $\|v - v_0\|_\infty < \eta \Rightarrow |f_i(v) - \ell_i| < \varepsilon$. Then we have immediately that $\|v - v_0\|_\infty < \eta \Rightarrow \|f(v) - \ell\|_\infty < \varepsilon$.

  Therefore f has limit ℓ at v_0. ✓

Remark. Assume that f has a limit ℓ (in $\mathbb{R}$ or $\mathbb{R}^2$) as v nears $v_0 = (x_0, y_0)$. Then in particular f has limit ℓ as (x_0, y) nears (x_0, y_0); since $|y - y_0| = \|(x_0, y) - (x_0, y_0)\|$, this means that $f(x_0, y)$ has limit ℓ as y nears y_0. This means that if f has a limit as v nears v_0, then f has the same limit as v nears v_0 *along the vertical line with equation $x = x_0$*.

There are similar statements as v approaches v_0 along any line through v_0. They are mostly useful to prove that f does *not* have a limit as v nears v_0, using their contrapositives.

Examples. ➢ The norm is a continuous function. Indeed, it follows from the triangle inequality $|\|v\| - \|v_0\|| \leqslant \|v - v_0\|$.

➢ $\lim_{(x,y)\to(3,1)} \left(\frac{3xy^2}{7+y} + \frac{1}{2}xy\right) = \frac{3(3)(1)}{7+1} + \frac{1}{2}(3)(1) = \frac{21}{8}$.

➢ We want to prove that $\lim_{(x,y)\to(0,0)} \dfrac{3xy^2}{x^2+y^2} = 0$. We have

$$\left|\frac{3xy^2}{x^2+y^2} - 0\right| = \left|\frac{3xy^2}{x^2+y^2}\right| = 3|x|\left|\frac{y^2}{x^2+y^2}\right| \leqslant 3|x| \leqslant 3\|(x,y)\|$$

where the final norm is either of our two norms. Therefore, when $\|(x,y)\| = \|(x,y) - (0,0)\|$ goes to 0, so does $\left|\dfrac{3xy^2}{x^2+y^2} - 0\right|$.

➢ Let us show that $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ does not have a limit as (x,y) nears $(0,0)$.

We have seen that if such a limit ℓ exists, then there are limits as (x,y) nears $(0,0)$ along the x-axis and along the y-axis, and they are both equal to ℓ.

Along the x-axis, we have $f(x,0) = 1$ so that $\ell = 1$.

Along the y-axis, we have $f(0,y) = -1$ so that $\ell = -1$.

We have a contradiction, therefore the limit does not exist.

➢ Let us show that $f(x,y) = \left(\dfrac{x^2-y^2}{x^2+y^2}\right)^2$ does not have a limit as (x,y) nears $(0,0)$.

Assume that this limit exists and is equal to ℓ.

Along the x-axis and the y-axis, the limit is equal to 1.

Now consider the line with equation $y = x$ that goes through $(0,0)$. Along this line, we have $f(x,x) = 0$ so that $\ell = 0$.

We have a contradiction, therefore the limit does not exist.

Proposition 8. Let $\varphi : \mathbb{R}^p \to \mathbb{R}^q$ with $p, q \in \{1,2\}$ be a linear function. Then φ is continuous.

Proof. If $q = 2$, we need only prove that the coordinate functions, each of which is linear (since π_i is linear), is continuous. Therefore we may assume that $q = 1$.

If $p = 1$, then $\varphi(x) = ax$ for some $a \in \mathbb{R}$ so that φ is clearly continuous.

Now assume that $p = 2$. Let $\{e_1, e_2\}$ be the canonical basis of $\mathbb{R}^2$. Set $M = \max(|\varphi(e_1)|, |\varphi(e_2)|)$. Then for any vector $u = (x,y) = xe_1 + ye_2$ we have

$$|\varphi(u)| = |x\varphi(e_1) + y\varphi(e_2)| \leqslant |x||\varphi(e_1)| + |y||\varphi(e_2)| \leqslant 2\|u\|M.$$

Let v_0 be a point in $\mathbb{R}^2$. Then $|\varphi(v) - \varphi(v_0)| = |\varphi(v - v_0)| \leqslant 2\|v - v_0\|M$. Therefore when v nears v_0, that is, when $\|v - v_0\|$ nears 0, then $|\varphi(v) - \varphi(v_0)|$ nears 0 so that $\varphi(v)$ has limit $\varphi(v_0)$. Therefore φ is continuous at v_0. ✓

III. Partial derivatives

In this section, Ω is a non-empty open subset of $\mathbb{R}^2$.

Let $v_0 = (x_0, y_0) \in \Omega$ be a point, we know that there exists $r > 0$ such that $B(v_0, r) \subset \Omega$. For any $x \in]x_0 - r; x_0 + r[$ we have $\|(x, y_0) - (x_0, y_0)\| = \|(x - x_0, 0)\| = |x - x_0|$ (for both our norms) so that $(x, y_0) \in B(v_0, r) \subset \Omega$. We may therefore consider the function $p_1\colon\ ]x_0 - r; x_0 + r[\to \Omega$ defined by $p_1(x) = (x, y_0)$. Similarly, we can consider the function $p_2\colon\ ]y_0 - r; y_0 + r[\to \Omega$ defined by $p_2(y) = (x_0, y)$.

Now if $f : \Omega \to \mathbb{R}$ is a function, then $f \circ p_1\colon\ ]x_0 - r; x_0 + r[\to \mathbb{R}$ is the map obtained by fixing $y = y_0$ in the expression of f and $f \circ p_2\colon\ ]y_0 - r; y_0 + r[\to \mathbb{R}$ is the map obtained by fixing $x = x_0$ in the expression of f.

Definition 9. *Let $f : \Omega \to \mathbb{R}$ be a function and v_0 a point in Ω.*
If $f \circ p_1\colon\ ]x_0 - r; x_0 + r[\to \mathbb{R}$ is differentiable at x_0, the number $(f \circ p_1)'(x_0)$ is called the ***partial derivative***[a] ***of f with respect to x*** *at x_0 and is denoted by $\dfrac{\partial f}{\partial x}(v_0)$.*
If $f \circ p_2\colon\ ]y_0 - r; y_0 + r[\to \mathbb{R}$ is differentiable at y_0, the number $(f \circ p_2)'(y_0)$ is called the ***partial derivative of f with respect to y*** *at y_0 and is denoted by $\dfrac{\partial f}{\partial y}(v_0)$.*
If for every $v \in \Omega$ the function f has a partial derivative with respect to x (resp. y) at v, then the function $\Omega \to \mathbb{R}$ defined by $v \to \dfrac{\partial f}{\partial x}(v)$ (resp. $\dfrac{\partial f}{\partial y}(v)$) is the partial derivative function with respect to x (resp. y) and is denoted by $\dfrac{\partial f}{\partial x}$ (resp. $\dfrac{\partial f}{\partial y}$).

[a] dérivée partielle

Remark. If we write (x_1, x_2) instead of (x, y) for a point in Ω, then $\frac{\partial f}{\partial x}$ becomes $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial y}$ becomes $\frac{\partial f}{\partial x_2}$.

Remark. Note that if $v = (x, y)$, we may write $\frac{\partial f}{\partial x}(x, y)$. The two letters x do not have the same meaning. The x in $\frac{\partial f}{\partial x}$ means that we differentiate with respect to the first variable, whereas the other letter x is a real number, the first component of the point (x, y).

Example.
- ➢ Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = y^2 \sin(x)$. Then $\frac{\partial f}{\partial x}(x, y) = y^2 \cos(x)$ and $\frac{\partial f}{\partial y}(x, y) = 2y \sin(x)$
- ➢ Now consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = y^2 \sin(xy)$. Then $\frac{\partial f}{\partial x}(x, y) = y^3 \cos(xy)$ and $\frac{\partial f}{\partial y}(x, y) = 2y \sin(xy) + y^2 x \cos(xy)$.

Definition 10. *A function $f : \Omega \to \mathbb{R}$ is of class $\mathcal{C}^1$ if its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are defined and continuous on Ω.*
A function $f : \Omega \to \mathbb{R}^2$ is of class $\mathcal{C}^1$ if its two coordinate functions are of class $\mathcal{C}^1$.

Theorem 11. Let $f : \Omega \to \mathbb{R}$ a function of class $\mathcal{C}^1$ and let v_0 be a point in Ω. Then there exists a real number $r > 0$ and a function $\varepsilon : B(0, r) \to \mathbb{R}$, continuous at 0 and such that $\varepsilon(0) = 0$ and satisfying the following property: for any $h = (h_1, h_2) \in B(0, r)$ we have

$$f(v_0 + h) = f(v_0) + h_1 \frac{\partial f}{\partial x}(v_0) + h_2 \frac{\partial f}{\partial y}(v_0) + \|h\| \varepsilon(h).$$

Proof. Admitted. ✓

Remark. If $f : I \to \mathbb{R}$ is a function defined on an interval $I \subset \mathbb{R}$, then f is differentiable at $a \in I$ if and only if $f(a + h) = f(a) + hf'(a) + h\varepsilon(h)$ with ε continuous and $\varepsilon(0) = 0$. The theorem above is a generalisation of this.

Corollary 12. Let $f : \Omega \to \mathbb{R}$ or $f : \Omega \to \mathbb{R}^2$ be a function of class $\mathcal{C}^1$. Then f is continuous.

Proof. We need only consider the case $f : \Omega \to \mathbb{R}$ (the other follows from this one for the coordinate functions).
Let v_0 be a point in Ω. By the theorem there exist $r > 0$ and $\varepsilon : B(0, r) \to \mathbb{R}$ continuous at 0 with $\varepsilon(0) = 0$ such that for all $h \in B(0, r)$ we have

$$f(v_0 + h) = f(v_0) + h_1 \frac{\partial f}{\partial x}(v_0) + h_2 \frac{\partial f}{\partial y}(v_0) + \|h\| \varepsilon(h).$$

The map $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\varphi(h) = h_1 \frac{\partial f}{\partial x}(v_0) + h_2 \frac{\partial f}{\partial y}(v_0)$ is linear hence continuous. The norm is continuous therefore $h \to \varphi(h) + \|h\| \varepsilon(h)$ is continuous at 0. The constant functions are clearly continuous, therefore $h \to f(v_0 + h)$ is continous at 0, and finally f is continuous at v_0. ✓

IV. Computation of partial derivatives

Proposition 13. If f and g are functions from Ω to $\mathbb{R}$ and if $v_0 \in \Omega$ is a point at which both functions have partial derivatives, then $f + g$ and fg have partial derivatives at v_0 and

$$\frac{\partial(f+g)}{\partial x}(v_0) = \frac{\partial f}{\partial x}(v_0) + \frac{\partial g}{\partial x}(v_0)$$
$$\frac{\partial(fg)}{\partial x}(v_0) = \left(\frac{\partial f}{\partial x}(v_0)\right) g(v_0) + f(v_0) \left(\frac{\partial g}{\partial x}(v_0)\right)$$

and similarly for the partial derivatives with respect to y.

Proof. Clear. ✓

Proposition 14. The partial derivatives of compositions are described as follows.

(1) Let $f : \Omega \to \mathbb{R}$ be a function of class $\mathcal{C}^1$ and let $g : I \to \mathbb{R}$ be a function of class $\mathcal{C}^1$ defined on an open interval in $\mathbb{R}$ such that $f(\Omega) \subset I$. Then $g \circ f$ is of class $\mathcal{C}^1$ and

$$\frac{\partial(g \circ f)}{\partial x_i}(v) = g'(f(v))\frac{\partial f}{\partial x_i}(v).$$

(2) Let $f : I \to \mathbb{R}^2$ be a function of class $\mathcal{C}^1$ on an interval I of $\mathbb{R}$ and let $g : \Omega \to \mathbb{R}$ be a function of class $\mathcal{C}^1$ defined on an open subset $\Omega \subset \mathbb{R}^2$ such that $f(I) \subset \Omega$. Let f_1 and f_2 be the coordinate functions of f. Then $g \circ f$ is of class $\mathcal{C}^1$ and

$$(g \circ f)'(a) = \frac{\partial g}{\partial x_1}(f(a))f_1'(a) + \frac{\partial g}{\partial x_2}(f(a))f_2'(a).$$

(3) Let $f : U \to \mathbb{R}^2$ be a function of class $\mathcal{C}^1$ on an open subset U of $\mathbb{R}^2$ and let $g : \Omega \to \mathbb{R}$ be a function of class $\mathcal{C}^1$ defined on an open subset $\Omega \subset \mathbb{R}^2$ such that $f(U) \subset \Omega$. Let f_1 and f_2 be the coordinate functions of f. Then $g \circ f$ is of class $\mathcal{C}^1$ and

$$\frac{\partial(g \circ f)}{\partial x_k}(v) = \frac{\partial g}{\partial x_1}(f(v))\frac{\partial f_1}{\partial x_k}(v) + \frac{\partial g}{\partial x_2}(f(v))\frac{\partial f_2}{\partial x_k}(v).$$

Proof. (1) We use the notation at the beginning of Section III. Then for $i = 1, 2$, $g \circ f \circ p_i$ is differentiable at the appropriate component of v_0 and we have

$$\begin{aligned}
\frac{\partial(g \circ f)}{\partial x_1}(v_0) &= (g \circ f \circ p_1)'(x_0) = g'(f(p_1(x_0)))(f \circ p_1)'(x_0) = g'(f(v_0))\frac{\partial f}{\partial x_1}(v_0) \\
\frac{\partial(g \circ f)}{\partial x_2}(v_0) &= (g \circ f \circ p_2)'(y_0) = g'(f(p_2(y_0)))(f \circ p_2)'(y_0) = g'(f(v_0))\frac{\partial f}{\partial x_2}(v_0).
\end{aligned}$$

(2) Fix $a \in I$ and $b = (b_1, b_2) = f(a) \in \Omega$. We know by Theorem 11 that

$$g(v) = g(b) + (x_1 - b_1)\frac{\partial g}{\partial x_1}(b) + (x_2 - b_2)\frac{\partial g}{\partial x_2}(b) + \|v - b\|_\infty \varepsilon(v)$$

for some function ε continuous at b with $\varepsilon(b) = 0$. For $v = f(t)$ we get

$$g(f(t)) = g(b) + (f_1(t) - f_1(a))\frac{\partial g}{\partial x_1}(f(a)) + (f_2(a) - f_2(a))\frac{\partial g}{\partial x_2}(f(a)) + \|f(t) - f(a)\|_\infty \varepsilon(f(t)).$$

Now f_1 and f_2 are also of class $\mathcal{C}^1$ so that for $i = 1, 2$ we have

$$f_i(t) = f_i(a) + (t - a)f_i'(a) + (t - a)\varepsilon_i(t)$$

for some functions ε_i continuous at a and such that $\varepsilon_i(a) = 0$. Therefore

$$g(f(t)) = g(f(a)) + (t - a)\left(f_1'(a)\frac{\partial g}{\partial x_1}(f(a)) + f_2'(a)\frac{\partial g}{\partial x_2}(f(a))\right) + R(t)$$

with

$$R(t) = \frac{\partial g}{\partial x_1}(f(a))(t - a)\varepsilon_1(t) + \frac{\partial g}{\partial x_2}(f(a))(t - a)\varepsilon_2(t) + \|f(t) - f(a)\|_\infty \varepsilon(f(t)).$$

To conclude, we need only prove that $R(t) = o(t - a)$. We have

$$\frac{R(t)}{t - a} = \frac{\partial g}{\partial x_1}(f(a))\varepsilon_1(t) + \frac{\partial g}{\partial x_2}(f(a))\varepsilon_2(t) + \left\|\frac{f(t) - f(a)}{t - a}\right\|_\infty \frac{|t - a|}{t - a}\varepsilon(f(t))$$

in which the first two terms clearly go to 0 when t nears a. Moreover,

- $\left\|\frac{f(t)-f(a)}{t-a}\right\|_\infty$ goes to $\|(f_1'(a), f_2'(a))\|_\infty$ as t nears a, therefore it is bounded near a,
- $\frac{|t-a|}{t-a}$ is bounded and
- $\varepsilon(f(t))$ goes to 0 as t nears a

so that finally the third term and hence $R(t)$ goes to 0 as t nears a as required.

(3) We use again the notation at the beginning of Section III. Using the previous case, each of the functions $g \circ f \circ p_i$ is of class $\mathcal{C}^1$ and

$$\begin{aligned}
\frac{\partial(g \circ f)}{\partial x_i}(v_0) = (g \circ f \circ p_i)'(x_i) &= \frac{\partial g}{\partial x_1}(f(v_0))(f \circ p_i)'(x_i) + \frac{\partial g}{\partial x_2}(f(v_0))(f \circ p_i)'(x_i) \\
&= \frac{\partial g}{\partial x_1}(f(v_0))\frac{\partial f}{\partial x_i}(v_0) + \frac{\partial g}{\partial x_2}(f(v_0))\frac{\partial f}{\partial x_i}(v_0).
\end{aligned}$$

✓

Definition 15. *Let n and p be in $\{1;2\}$, let Ω be an open subset (interval if $n = 1$) of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^p$ a function of class $\mathcal{C}^1$. Denote by f_i the coordinate functions of f.*
The ***jacobean matrix***[a] *of f at v_0 is the matrix $J_f(v_0)$ with p rows and n columns in which the coefficient at row i and column j is $\dfrac{\partial f_i}{\partial x_j}(v_0)$ (or $f_i'(v_0)$ if $n = 1$).*

[a] matrice jacobienne

The previous proposition can then be rewritten in the form of a "chain rule".

Corollary 16. With the same assumptions as in Proposition 14, we have

$$J_{g\circ f}(v_0) = J_g(f(v_0))J_f(v_0).$$

V. Gradient

Definition 17. *If $f : \Omega \to \mathbb{R}$ is of class $\mathcal{C}^1$, the* ***gradient***[a] *of f at $v_0 \in \Omega$ is the vector $\nabla_{v_0} f = \left(\dfrac{\partial f}{\partial x}(v_0), \dfrac{\partial f}{\partial y}(v_0)\right)$.*

[a] gradient

Let $f : \Omega \to \mathbb{R}$ be a function of class $\mathcal{C}^1$. The set $\{(x, y) \in \Omega\,;\, f(x, y) = 0\}$ is called the **curve**† with equation $f(x, y) = 0$.
Let $v_0 = (x_0, y_0)$ be a point in Ω such that $\nabla_{v_0} f \neq 0$. Then we define the **tangent line** to the curve with equation $f(x, y) = 0$ at v_0 to be the line with equation

$$(x - x_0)\frac{\partial f}{\partial x}(v_0) + (y - y_0)\frac{\partial f}{\partial y}(v_0) = 0.$$

Note that by Proposition 11, the curve goes nearer to the tangent line as v approaches v_0.
The equation of the tangent line may be expressed as the scalar product $\langle v - v_0, \nabla_{v_0} f\rangle = 0$.

Remark. If $f(x, y) = g(x) - y$ (that is, the curve is given by an equation $y = g(x)$) with g differentiable at x_0, then (noting that v_0 is on the curve so that $0 = f(v_0) = g(x_0) - y_0$) we have

$$(x - x_0)\frac{\partial f}{\partial x}(v_0) + (y - y_0)\frac{\partial f}{\partial y}(v_0) = (x - x_0)g'(x_0) + (y - g(x_0))(-1)$$

so that we recover the usual equation of a tangent line.

Remark. The vector $\nabla_{v_0} f$ is orthogonal to the tangent line at v_0.

Definition 18. *Let $f : \Omega \to \mathbb{R}$ be a function. We say that it has*

- *a* ***local maximum***[a] *at v_0 if there exists $r > 0$ such that $\forall v \in \Omega$, $\|v - v_0\| < r \Rightarrow f(v) \leqslant f(v_0)$;*
- *a* ***local minimum***[b] *at v_0 if there exists $r > 0$ such that $\forall v \in \Omega$, $\|v - v_0\| < r \Rightarrow f(v) \geqslant f(v_0)$;*
- *a* ***local extremum***[c] *at v_0 if it has either a local maximum or a local minimum at v_0.*

In each case, we say that v_0 is an ***extremal point****.*

[a] maximum local
[b] minimum local
[c] extremum local

Proposition 19. Let f be a function of class $\mathcal{C}^1$ on Ω. If f has a local extremum at $v_0 \in \Omega$ then $\nabla_{v_0} f = (0, 0)$ (we say that v_0 is a **critical point** for f).

Proof. Let us do the proof for a local maximum. Set $v_0 = (x_0, y_0)$. There exists $r > 0$ such that $\forall v \in \Omega$, $\|v - v_0\| < r \Rightarrow f(v) \leqslant f(v_0)$. Since Ω is an open set, we may assume that $\bar{B}(v_0, r) \subset \Omega$.
Then the functions $t \to f(x_0 + t, y_0)$ and $t \to f(x_0, y_0 + t)$ are defined on $[-r, r]$ and have a maximum at $t = 0$, therefore their derivatives at 0 vanish, that is, the partial derivatives of f at v_0 vanish. ✓

† courbe

Example. As in the case of functions of one variable, critical points are not necessarily extremal points.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^2 - y^2$. Then $\frac{\partial f}{\partial x}(x,y) = 2x$ and $\frac{\partial f}{\partial y}(x,y) = -2y$ so that $v_0 = (0,0)$ is a critical point for f. However, v_0 is not an extremal point for f since $f(x,0) > 0$ as soon as $x \neq 0$ and $f(0,y) < 0$ as soon as $y \neq 0$ (and points of the form $(x,0)$ or $(0,y)$ are in every open disk around v_0)

VI. Change of coordinates

A. Definition

Definition 20. *Let Ω be an open subset of $\mathbb{R}^2$. A* ***change of coordinates***[a] *is a function $\varphi : \Omega \to \mathbb{R}^2$ satisfying:*

- *f is of class $\mathcal{C}^1$;*
- *f is injective;*
- *the jacobean matrix at any point of Ω is invertible.*

[a] changement de coordonnées

Example. The map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \to (x+y, x+2y)$ is a change of coordinates (it is clearly of class $\mathcal{C}^1$, the jacobean matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ whose determinant is $1 \neq 0$ and injectivity is easily checked).

It simplifies the resolution of the equation $2\dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial y} = 0$ on $\mathbb{R}^2$. Set $g = f \circ \varphi^{-1}$ so that $g(u,v) = f(2u - v, v - u)$. Since $f = g \circ \varphi$, we have $\dfrac{\partial f}{\partial x} = \dfrac{\partial g}{\partial u}\dfrac{\partial \varphi_1}{\partial x} + \dfrac{\partial g}{\partial v}\dfrac{\partial \varphi_2}{\partial x} = \dfrac{\partial g}{\partial u} + \dfrac{\partial g}{\partial v}$ and $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial u} + 2\dfrac{\partial g}{\partial v}$ so that $2\dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial u}$. The equation then becomes $\dfrac{\partial g}{\partial u} = 0$ so that $g(u,v) = \psi(v)$ for some function $\psi : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^1$, that is, $f(x,y) = \psi(x + 2y)$.

B. Polar coordinates

Let A be the set of points in the Euclidean plane that are not of the form $(x,0)$ with $x \leqslant 0$ and set $B :=]0; +\infty[\times] - \pi; \pi[$.

The map $\varphi : B \to \mathbb{R}^2$ defined by $\varphi(r,\theta) = (r\cos\theta, r\sin\theta)$ is a change of coordinates:

- φ is clearly of class $\mathcal{C}^1$ and

$$\frac{\partial \varphi_1}{\partial r} = \cos\theta \qquad \frac{\partial \varphi_1}{\partial \theta} = -r\sin\theta$$
$$\frac{\partial \varphi_2}{\partial r} = \sin\theta \qquad \frac{\partial \varphi_2}{\partial \theta} = r\cos\theta$$

- φ is injective: assume that $\varphi(r,\theta) = \varphi(r',\theta')$, that is,

$$r\cos\theta = r'\cos\theta' \tag{12.1}$$
$$r\sin\theta = r'\sin\theta' \tag{12.2}$$

  Then $(12.1)^2 + (12.2)^2$ gives $r^2 = r'^2$ so that $r = r'$. We then get $\theta = \theta'$.

- $J_{(r,\theta)}(\varphi) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$ has determinant $r \neq 0$ hence is invertible.

We can see that $\varphi(B) = A$ so that φ induces a bijection $\varphi : B \to A$ whose inverse is given by $\varphi^{-1}(x,y) = \left(\sqrt{x^2+y^2}; 2\arctan\frac{y}{x+\sqrt{x^2+y^2}}\right)$.

To see this, note that $\frac{y}{x} = \tan\theta$, but $\theta \in] - \pi, \pi[$ so we cannot just apply arctan. We shall therefore consider $\frac{\theta}{2}$ to which we can apply arctan.

Note that $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\cos^2\frac{\theta}{2}\tan\frac{\theta}{2}$ and that $2\cos^2\frac{\theta}{2} = 1 + \cos\theta$ so that $\tan\frac{\theta}{2} = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{\frac{y}{r}}{1+\frac{x}{r}} = \dfrac{y}{r+x} = \dfrac{y}{\sqrt{x^2+y^2}+x}$.

The map φ^{-1} is of class $\mathcal{C}^1$ on B and we have

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$
$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}.$$

Now if f is a function of class $\mathcal{C}^1$ on A, the function $F : B \to \mathbb{R}$ defined by $F(r,\theta) = f(r\cos\theta, r\sin\theta)$ is of class $\mathcal{C}^1$ and we have

$$\frac{\partial F}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}$$
$$\frac{\partial F}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}.$$

Conversely, if F is a function of class $\mathcal{C}^1$ on B, then the function $f : A \to \mathbb{R}$ defined by $f(x,y) = F(\sqrt{x^2+y^2}; 2\arctan\frac{y}{x+\sqrt{x^2+y^2}})$ is of class $\mathcal{C}^1$ and we can compute its partial derivatives.

C. Cylindrical coordinates

We can extend polar coordinates to **cylindrical coordinates**[†] on parts of $\mathbb{R}^3$ via the following bijections:

$$A \times \mathbb{R} \to B \times \mathbb{R}$$
$$(x,y,z) \to \left(\sqrt{x^2+y^2}, 2\arctan\frac{y}{x+\sqrt{x^2+y^2}}, z\right)$$

$$B \times \mathbb{R} \to A \times \mathbb{R}$$
$$(r,\theta,z) \to (r\cos\theta, r\sin\theta, z).$$

If we project on the xy-plane, we get the polar coordinates.

The map $\psi : B \times \mathbb{R} \to \mathbb{R}^2$ defined by $\psi(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$ is indeed a change of coordinates:

➢ It is clearly of class $\mathcal{C}^1$, and $J_\psi(r,\theta,z) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has determinant $r \neq 0$ hence is invertible; moreover, it is easy to check that ψ is injective, using the fact that φ defines a bijection from B to A.

VII. Higher order partial derivatives

Definition 21. *Let $f : \Omega \to \mathbb{R}$ be a function of class $\mathcal{C}^1$. If the function $\frac{\partial f}{\partial x} : \Omega \to \mathbb{R}$ has partial derivatives, then*

➢ *its first partial derivative is denoted by $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ or $\frac{\partial^2 f}{\partial x^2}$*

➢ *its second partial derivative is denoted by $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ or $\frac{\partial^2 f}{\partial y \partial x}$.*

Similarly, when they exist, the partial derivatives of $\frac{\partial f}{\partial y}$ are denoted by $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$.

Definition 22. *A function $f : \Omega \to \mathbb{R}$ is said to be **of class** $\mathcal{C}^2$ on Ω if its four second order partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial y^2}$ exist and are continuous on Ω.*

Theorem 23. If f is a function of class $\mathcal{C}^2$ on Ω then

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Proof. Admitted. ✓

Example. Let us consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \frac{(x^2-y^2)xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

This function is clearly continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$. Moreover, $|f(x,y)| \leqslant \frac{\|(x,y)\|_2^4}{\|(x,y)\|_2^2} = \|(x,y)\|_2^2$ so that f is also continuous at $(0,0)$. We have

[†] coordonnées cylindriques

- $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{(3yx^2 - y^3)(x^2+y^2) - xy(x^2-y^2)2x}{(x^2+y^2)^2}$ if $(x,y) \neq (0,0)$,
- $\dfrac{\partial f}{\partial x}(0,y) = -y$ if $y \neq 0$,
- $\dfrac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = 0$,
- $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{(x^3 - 3xy^2)(x^2+y^2) - xy(x^2-y^2)2y}{(x^2+y^2)^2}$ if $(x,y) \neq (0,0)$,
- $\dfrac{\partial f}{\partial y}(x,0) = x$ if $x \neq 0$,
- $\dfrac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \dfrac{f(0,h) - f(0,0)}{h} = 0$.

Now let us compute the second order partial derivatives at $(0,0)$.

- $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ is the derivative of $x \to \dfrac{\partial f}{\partial y}(x,0)$ at 0, so $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) = 1$.
- $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$ is the derivative of $y \to \dfrac{\partial f}{\partial x}(0,y)$ at 0, so $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = -1$.

In particular, we can say that f is not of class $\mathcal{C}^2$.

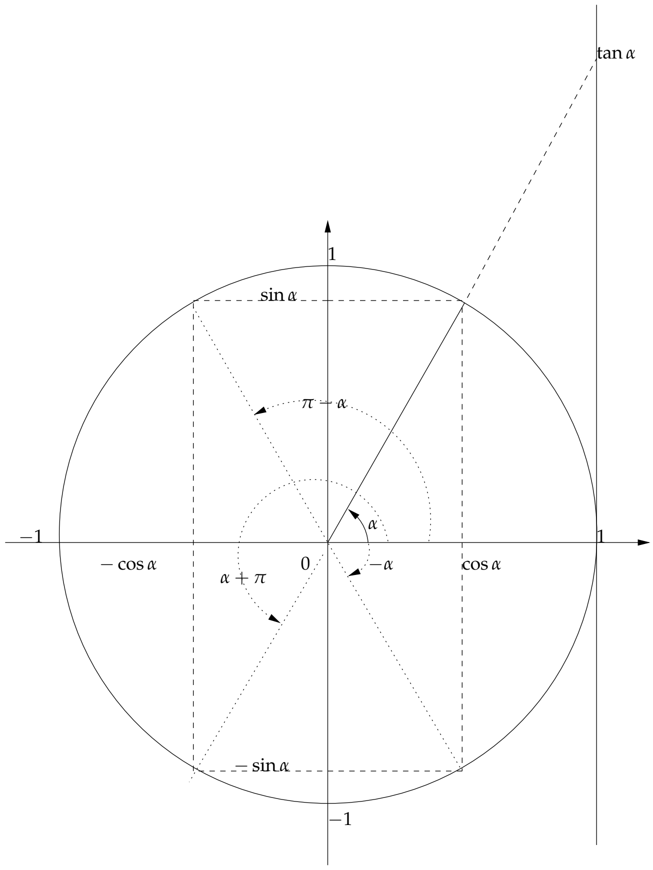
We can repeat the process of taking partial derivatives to obtain the following definition.

Definition 24. *Let $f : \Omega \to \mathbb{R}$ be a function defined on Ω. We say that f is* ***of class*** $\mathcal{C}^k$ *if all its partial derivatives up to order k (inclusive) exist and are continuous. We denote by $\mathcal{C}^k(\Omega)$ the set of all these functions. We say that f is* ***of class $\mathcal{C}^\infty$ or smooth***[a] *if it is of class $\mathcal{C}^k$ for all $k \in \mathbb{N}$.*

[a] de classe $\mathcal{C}^\infty$

Appendix *A*

Trigonometric formulae



The sine, cosine and tangent of an angle can be read off the trigonometric circle. One full turn of the circle represents an angle of 2π.

I. Angles and properties of cos and sin

It is useful to know the cosine, sine and tangent of a few remarkable angles (the tangent may be found easily from the other two, since $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$).

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Next, we can find the cosine, sine and tangent of other angles, such as $\frac{2\pi}{3}$ or $-\frac{\pi}{6}$, using formulae from Section II, as well as the following relations (that can be seen on the trigonometric circle).

$$\begin{aligned}
\cos(-\alpha) &= \cos\alpha & \sin(-\alpha) &= -\sin\alpha \\
\cos(\pi-\alpha) &= -\cos\alpha & \sin(\pi-\alpha) &= \sin\alpha \\
\cos(\alpha+\pi) &= -\cos\alpha & \sin(\alpha+\pi) &= -\sin\alpha
\end{aligned}$$

So for instance

α	$\frac{5\pi}{6}$	$\frac{3\pi}{4}$	$\frac{2\pi}{3}$	π	$\frac{7\pi}{6}$
$\cos\alpha$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	-1	$-\frac{\sqrt{3}}{2}$
$\sin\alpha$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	0	$\frac{1}{2}$

II. Circular trigonometric functions

The following formulae *must* be known (although formulae (A.2) and (A.3) may be recovered from $e^{i(a+b)} = e^{ia}e^{ib}$).

$$\cos^2 a + \sin^2 a = 1 \tag{A.1}$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b \tag{A.2}$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b. \tag{A.3}$$

We deduce immediately that

$$\begin{array}{ll} \cos(-\alpha) = \cos\alpha & \sin(-\alpha) = -\sin\alpha \\ \cos(\pi-\alpha) = -\cos\alpha & \sin(\pi-\alpha) = \sin\alpha \\ \cos(\alpha+\pi) = -\cos\alpha & \sin(\alpha+\pi) = -\sin\alpha \end{array}$$

Moreover, from these formulae, many others may be found (even if you do not remember them, you must know that they exist in order to recover them – for instance, you must know that there are formulae relating $\cos^2 a$ and $\cos(2a)$...). They are especially useful to compute integrals and primitives.

Many of the formulae below may also be found using the Euler formulae, complex exponentials and (A.1).

- $\cos(a-b) = \cos a \cos b + \sin a \sin b$ (Replace b by $-b$ in (A.2)).
- $\sin(a-b) = \sin a \cos b - \cos a \sin b$ (Replace b by $-b$ in (A.3)).
- $\cos(2a) = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$ (Using formulae (A.2) and (A.1)).
- $\cos^2 a = \dfrac{1+\cos(2a)}{2}$ (From the previous line).
- $\sin^2 a = \dfrac{1-\cos(2a)}{2}$ (Similar).
- $\sin(2a) = 2\sin a \cos a$ (Formula (A.3) with $a = b$).
- $1 + \tan^2 a = \dfrac{1}{\cos^2 a}$ (Using formula (A.1)).
- $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$ (Using formulae (A.2) and (A.3)).
- $\tan(2a) = \dfrac{2\tan a}{1-\tan^2 a}$ (From the previous line).
- $\tan a = \dfrac{2\tan(\frac{a}{2})}{1-\tan^2(\frac{a}{2})}$ (From the previous line).
- $\cos a = \dfrac{1-\tan^2(\frac{a}{2})}{1+\tan^2(\frac{a}{2})}$ (indeed, $\cos a = \cos^2\frac{a}{2} - \sin^2\frac{a}{2} = \cos^2\frac{a}{2}(1-\tan^2\frac{a}{2}) = \dfrac{1-\tan^2\frac{a}{2}}{1+\tan^2\frac{a}{2}}$).
- $\sin a = \dfrac{2\tan(\frac{a}{2})}{1+\tan^2(\frac{a}{2})}$ (indeed, $\sin a = 2\sin\frac{a}{2}\cos\frac{a}{2} = 2\tan\frac{a}{2}\cos^2\frac{a}{2} = 2\dfrac{\tan\frac{a}{2}}{1+\tan^2\frac{a}{2}}$).

III. Linearisation

When we want to compute primitives and integrals, it is useful to know how to linearise the (even) powers of cos and sin. We can use for instance the Euler formulae to linearise. For example, the primitives of $\cos^2 x = \dfrac{1+\cos(2x)}{2}$ are $\dfrac{x}{2} + \dfrac{\sin 2x}{4} + C$.

The hyperbolic functions have been defined in Chapter 3 by the formulae $\operatorname{ch} x = \dfrac{e^x + e^{-x}}{2}$ and $\operatorname{sh} x = \dfrac{e^x - e^{-x}}{2}$. Note that they are similar to the Euler formulae, and we can use them in the same way to recover the relations below.

The following formulae must be known.

$$\operatorname{ch}^2 a - \operatorname{sh}^2 a = 1 \tag{A.4}$$
$$\operatorname{ch}(a+b) = \operatorname{ch} a \operatorname{ch} b + \operatorname{sh} a \operatorname{sh} b \tag{A.5}$$
$$\operatorname{sh}(a+b) = \operatorname{sh} a \operatorname{ch} b + \operatorname{ch} a \operatorname{sh} b. \tag{A.6}$$

From there, the following formulae can be recovered (or use the definitions of ch and sh, as well as exp).

- $\operatorname{ch}(a-b) = \operatorname{ch} a \operatorname{ch} b - \operatorname{sh} a \operatorname{sh} b$ (Replace b by $-b$ in (A.5)).
- $\operatorname{sh}(a-b) = \operatorname{sh} a \operatorname{ch} b - \operatorname{ch} a \operatorname{sh} b$ (Replace b by $-b$ in (A.6)).
- $\operatorname{ch}(2a) = \operatorname{ch}^2 a + \operatorname{sh}^2 a = 2\operatorname{ch}^2 a - 1 = 1 + 2\operatorname{sh}^2 a$ (Using formulae (A.5), (A.6) and (A.4)).
- $\operatorname{ch}^2 a = \dfrac{1 + \operatorname{ch}(2a)}{2}$ (From the previous line).
- $\operatorname{sh}^2 a = \dfrac{\operatorname{ch}(2a) - 1}{2}$ (Similar).
- $\operatorname{sh}(2a) = 2 \operatorname{sh} a \operatorname{ch} a$ (Formula (A.6) with $a = b$).

Appendix *B*

Greek alphabet

Nom	**Lower case**	**Upper case**
alpha	α	A
beta	β	B
gamma	γ	Γ
delta	δ	Δ
epsilon	ε	E
zeta	ζ	Z
eta	η	H
theta	θ	Θ
iota	ι	I
kappa	κ	K
lambda	λ	Λ
mu	μ	M
nu	ν	N
xi	ξ	Ξ
omicron	o	O
pi	π, ϖ	Π
rho	ρ, ϱ	R
sigma	σ, ς	Σ
tau	τ	T
upsilon	υ	U
phi	φ, ϕ	Φ
chi	χ	X
psi	ψ	Ψ
omega	ω	Ω

The lighter coloured letters are not used in mathematics since they are the same as the latin ones.

Appendix C

Integer arithmetics

This appendix is a reminder on the arithmetics of integers. These are not on the syllabus, but are considered to be known. (Most of) the proofs are left as exercises.

I. Greatest common divisor

Notation. Let a be an integer in $\mathbb{Z}$. We denote by $\mathcal{D}(a)$ the set of divisors of a in $\mathbb{Z}$. Recall that

➢ $\mathcal{D}(0) = \mathbb{Z}$;

➢ if $a \neq 0$ then for any integer b in $\mathcal{D}(a)$ we have $|b| \leqslant |a|$.

In the sequel, we shall consider two integers in $\mathbb{Z}$ with at least one of them non-zero.

Remark. Let a and b be two integers in $\mathbb{Z}$, at least one of which is non-zero. Then $\mathcal{D}(a) \cap \mathcal{D}(b)$ contains only non-zero integers, and they all have absolute value at most $\min(|a|, |b|)$ if a and b are both non-zero, or at most $|a|$ if $b = 0$. In particular, the set $\{|c|; c \in \mathcal{D}(a) \cap \mathcal{D}(b)\}$ is a non-empty subset of $\mathbb{N}$ which contains 1 and is bounded above in $\mathbb{R}$, therefore it has a maximum, $d \geqslant 1$.

Definition 1. *Let a and b be two integers in $\mathbb{Z}$ with at least of them non-zero. Then $d = \max\{|c|; c \in \mathcal{D}(a) \cap \mathcal{D}(b)\}$ is called the **greatest common divisor**[a] (or **gcd**[b]) of a and b. It is denoted by $a \wedge b$.*
By convention, we set $0 \wedge 0 = 0$.

[a] plus grand commun diviseur
[b] pgcd

A. Euclidean algorithm

Theorem 2 (Euclidean division[a]). For any integer a and any integer $b \neq 0$, there exists a unique pair of integers (q, r) such that $\begin{cases} a = qb + r & \text{and} \\ 0 \leqslant r < |b|. \end{cases}$
The integer r is called the **remainder**[b] and the integer q is called the **quotient**[c] of the division.

[a] division euclidienne
[b] reste
[c] quotient

Proposition 3. Let a, b and q be three integers. then $\mathcal{D}(a) \cap \mathcal{D}(b) = \mathcal{D}(b) \cap \mathcal{D}(a - qb)$.
In particular, if r is the remainder of the Euclidean divison of a by b, then $\mathcal{D}(a) \cap \mathcal{D}(b) = \mathcal{D}(b) \cap \mathcal{D}(r)$.

Proposition 4 (Euclidean algorithm[a]). Let a and b be two non-zero integers.
Define the following sequence of integers, defined inductively by: $r_0 = a$ and $r_1 = b$. For $k \geqslant 1$, assume that r_{k-1} and r_k are known; if $r_k = 0$, set $r_{k+1} = 0$; if $r_k \neq 0$, let r_{k+1} be the remainder of the Euclidean division of r_{k-1} by r_k, so that $r_{k-1} = q_k r_k + r_{k+1}$ and $0 \leqslant r_{k+1} < |r_k|$.
Then there exists $n \in \mathbb{N}$ such that $r_n \neq 0$ and $r_{n+1} = 0$. Moreover, $r_n = a \wedge b$.

[a] algorithme d'Euclide

Example. We want to find a gcd of $a = 530$ and $b = 280$. We do successive Euclidean divisions until the remainder vanishes:

$$\begin{aligned} 530 &= 1 \cdot 280 + 250 \\ 280 &= 1 \cdot 250 + 30 \\ 250 &= 8 \cdot 30 + 10 \\ 30 &= 3 \cdot 10 + 0 \end{aligned}$$

therefore $a \wedge b = 10$ (the last non-zero remainder).

Corollary 5. Let a and b be two integers in $\mathbb{Z}$ with at least one of them non-zero. Set $d = a \wedge b$. Then $\mathcal{D}(d) = \mathcal{D}(a) \cap \mathcal{D}(b)$.

Remark. The previous result shows that the gcd d of a and b can be characterised by the three properties:

(i) $d > 0$ and

(ii) d divides a and b, and

(iii) if c is any integer that divides a and b, then c divides d.

Properties 6. **(a)** For any integers a and b, one of which is non-zero, we have $a \wedge b = b \wedge a$.

(b) For any non-zero integer a, we have $a \wedge 0 = |a|$.

(c) For any integer a, we have $a \wedge 1 = 1$.

(d) $a \wedge b = |a|$ if, and only if, a divides b.

(e) The integer $a \wedge b$ and its divisors are all the divisors common to a and b.

Proposition 7. Let a and b be two integers in $\mathbb{Z}$ with at least one of them non-zero. For any non-zero integer c, we have $(ca) \wedge (cb) = |c|(a \wedge b)$.

Proposition 8. Let a and b be two integers in $\mathbb{Z}$, at least one of which is non-zero. Then there exist integers u and v such that $a \wedge b = ua + vb$.
This is called a **Bézout relation**[a] between a and b and the integers u and v are called the **Bézout coefficients**[b] for a and b.

[a] relation de Bézout
[b] coefficients de Bézout

Remark. It follows from the proof that, to find the Bézout coefficients for a and b, we can use the Euclidean algorithm then work backwards, as we do for integers.

Example. In the example above, we have seen that $530 \wedge 280 = 10$.
Moreover, working up the Euclidean algorithm, we have

$$\begin{aligned} 10 &= 250 - 8 \cdot 30 \\ &= 250 - 8 \cdot (280 - 250) = 9 \cdot 250 - 8 \cdot 280 \\ &= 9 \cdot (530 - 280) - 8 \cdot 280 \\ 10 &= -17 \cdot 280 + 9 \cdot 530. \end{aligned}$$

II. Least common multiples

In the sequel, we shall consider non-zero integers a and b.

Notation. The set $\mathcal{M}(a) = a\mathbb{Z}$ is the set of multiples of a.

Remark. Let a and b be two non-zero integers in $\mathbb{Z}$.
The set $\mathcal{M}(a) \cap \mathcal{M}(b)$ is the set of common multiples of a and b. It contains 0 as well as some non-zero integers (such as ab). The non-zero integers have degree at least $\max(|a|, |b|)$.
In particular, the set $\{|c|; c \in \mathcal{M}(a) \cap \mathcal{M}(b), c \neq 0\}$ is a non-empty subset of $\mathbb{N}^*$ which is bounded below in $\mathbb{R}$, therefore it has a positive minimum, $m > 0$.

Definition 9. *Let a and b be two non-zero integers in $\mathbb{Z}$.*
The integer $m = \min\{|c|; c \in \mathcal{M}(a) \cap \mathcal{M}(b), c \neq 0\} > 0$ is called the ***least common multiple***[a] *(or* ***lcm***[b]*) of a and b. It is denoted by $m = a \vee b$.*
By convention, if a or b is zero, then $a \vee b = 0$.

[a] plus petit commun multiple
[b] ppcm

Notation. By convention, we set $A \vee 0 = 0$ for $A \neq 0$.

Properties 10.

- For any integers a and b, one of which is non-zero, we have $a \vee b = b \vee a$.
- For any non-zero integer a, we have $a \vee 1 = |a|$.
- $a \vee b = |a|$ if, and only if, b divides a.

Proposition 11. Let a and b be two non-zero integers in $\mathbb{Z}$, and let c be any integer.
Then c is the lcm for a and b if, and only if, $\mathcal{M}(c) = \mathcal{M}(a) \cap \mathcal{M}(b)$.

Remark. Let a and b be two non-zero integers in $\mathbb{Z}$. The integer $a \vee b$ is the unique positive integer such that $\mathcal{M}(a \vee b) = \mathcal{M}(a) \cap \mathcal{M}(b)$.

Proposition 12. Let a and b be two non-zero integers in $\mathbb{Z}$. For any non-zero integer c, we have $(ca) \vee (cb) = |c|(a \vee b)$.

Proposition 13. Let a and b be two non-zero integers in $\mathbb{Z}$. Then

$$(a \wedge b) \cdot (a \vee b) = |ab|.$$

III. Coprime integers

Definition 14. *Let a and b be two integers. We say that a and b are* ***coprime***[a] *if $a \wedge b = 1$.*
In other words, the only common divisors of a and b are ± 1.

[a] premiers entre eux

Theorem 15 (Bézout Theorem[a])**.** Let a and b be two integers. Then a and b are coprime if, and only if, there exist two integers u and v such that $au + bv = 1$.

[a] théorème de Bézout

Proposition 16 (Gauss' Lemma[a])**.** Let a, b and c be three integers.
If a divides bc and if a and b are coprime, then a divides c.

[a] lemme de Gauss

Proposition 17 (Euclid's Lemma[a])**.** Let a and b be two integers and let p be a prime integer.
If p divides ab then p divides a or p divides b.

[a] lemme d'Euclide

Proposition 18. Let a, b and c be three integers. The following are equivalent:

(i) a and b are coprime and a and c are coprime;

(ii) a and bc are coprime.

More generally, let $a_1, \ldots, a_p$ and $b_1, \ldots, b_n$ be $p+n$ integers. The following are equivalent:

(i) a_j and b_k are coprime for all j, k with $1 \leqslant j \leqslant p$ and $1 \leqslant k \leqslant n$;

(ii) $a_1 \cdots a_p$ and $b_1 \cdots b_n$ are coprime.

Definition 19. *Let $a_1, \ldots, a_n$ be a family of integers. We say that they are* ***pairwise coprime***[a] *if any two of them are coprime, that is,*

$$\forall i, j,\ 1 \leqslant i < j \leqslant n,\ a_i \text{ and } a_j \text{ are coprime.}$$

[a] premiers entre eux deux à deux

Proposition 20. Let a, b and c be three integers. Assume that a and b are coprime.
The integer c is a multiple of a and b if, and only if, it is a multiple of ab.
More generally, if $a_1, \ldots, a_n$ is a family of pairwise coprime integers, then c is a multiple of each of the a_k if, and only if, it is a multiple of their product $a_1 a_2 \cdots a_n$.

Remark. Let a and b be two non-zero integers. There exist integers a_1 and b_1 such that $a = (a \wedge b)a_1$ and $b = (a \wedge b)b_1$. Then a_1 and b_1 are coprime.

IV. Solving equations $ax + by = c$

Given three integers a, b and c, we would like to find, if possible, all the pairs of integers $(x, y) \in \mathbb{Z}^2$ such that $ax + by = c$.

Lemma 21. The equation (E) $ax + by = c$ has a solution if, and only if, $a \wedge b$ divides c.

Proof. Assume that (E) has a solution (x, y). Then $a \wedge b$ divides $ax + by = c$.
Conversely, assume that $d := a \wedge b$ divides c, so that $c = dc'$. There exists $(u, v) \in \mathbb{Z}^2$ such that $d = au + bv$. Therefore $c = auc' + bvc'$. We have found a solution, (uc', vc'). ✓

Definition 22. *The* ***homogeneous equation***[a] *associated to (E) is (E_0) $ax + by = 0$.*

[a] équation homogène

Lemma 23. Given a solution (x_0, y_0) of (E), the set of all solutions of (E) is the set of $(x_0 + s, y_0 + t)$ with (s, t) solution of the homogeneous equation (E_0).

Proof. Let (s, t) be a solution of the homogeneous equation (E_0). Then $as + bt = 0$ so that $a(x_0 + s) + b(y_0 + t) = ax_0 + by_0 = c$, therefore $(x_0 + s, y_0 + t)$ is a solution of (E).
Conversely, let (x, y) be a solution of (E). Put $s = x - x_0$ and $t = y - y_0$. Then $as + bt = (ax + by) - (ax_0 + (by_0) = c - c = 0$ so that (s, t) is a solution of (E_0). ✓

Lemma 24. Let d be the gcd of a and b. The solutions of (E_0) are the pairs $\left(-\frac{b}{d}k, \frac{a}{d}k\right)$ with $k \in \mathbb{Z}$.

Proof. First note that for any $k \in \mathbb{Z}$, $\left(-\frac{a}{d}k, \frac{b}{d}k\right)$ is a solution of (E_0), since $a\left(-\frac{b}{d}k\right) + b\left(\frac{a}{d}k\right) = 0$.

Conversely, let (s, t) be a solution of (E_0). Then $as + bt = 0$. Divide by d, we get $\frac{a}{d}s = -\frac{b}{d}t$. Then $\frac{a}{d}$ divides $\frac{b}{d}t$, But $\frac{a}{d}$ and $\frac{b}{d}t$ are coprime, therefore by Gauss' Lemma, $\frac{a}{d}$ divides t. We can write $t = \frac{a}{d}k$ for some $k \in \mathbb{Z}$. It then follows that $s = -\frac{b}{d}k$. ✓

Combining everything, we get the following result.

Proposition 25. The equation (E) $ax + by = c$ has a solution if, and only if, $d := a \wedge b$ divides c.
Assuming d divides c, the solutions of (E) are the sum of one solution of (E) and of all the solutions of the homogeneous equation (E_0) $ax + by = 0$ associated to (E), and the solutions of (E_0) are the $\left(-\frac{b}{d}k, \frac{a}{d}k\right)$ with $k \in \mathbb{Z}$. Moreover, a solution of (E) may be obtained from the Bézout coefficients of a and b.

Example. The equation $4x + 6y = 7$ has no solutions, since $4 \wedge 6 = 2$ does not divide 7.

Example. Consider the equation $12x + 8y = 28$. Then $12 \wedge 8 = 4$ divides 28, therefore it has solutions.
We first look for one solution. We have $28 = 4 \cdot 7$ and $4 = 12 - 8$ therefore $28 = 12 \cdot 7 - 8 \cdot 7$. Hence one solution is given by $(x_0, y_0) = (7, -7)$.
Now consider the homogeneous equation $12x + 8y = 0$. Dividing by $4 = 12, 8\wedge$ gives the equivalent equation $3x + 2y = 0$. We then have $3x = -2y$ so that 3 divides $2y$. But 3 and 2 are coprime, therefore by Gauss' Lemma 3 must divide y, so that $y = 3k$ for some $k \in \mathbb{Z}$. Therefore $3x = -6k$ and $x = -2k$. The solutions of the equation $12x + 8y = 0$ are the $(-2k, 3k)$ for $k \in \mathbb{Z}$.
Finally, the solutions of the equation $12x + 8y = 28$ are $(7 - 2k, -7 + 3k)$ for $k \in \mathbb{Z}$.

V. Gcd and lcm of more than two integers

Proposition 26. For any three integers a, b and c, we have

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c$$
$$a \vee (b \vee c) = (a \vee b) \vee c.$$

In other words, gcds and lcms are associative.

Consequence 27. In particular, for any family of n integers $a_1, \ldots, a_n$, we may consider $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ and $a_1 \vee a_2 \vee \cdots \vee a_n$ (without brackets).

Definition 28. *Let $a_1, \ldots, a_n$ be a family of n integers, with $n \geqslant 2$.*

- *The integer $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ is called the **greatest common divisor (gcd)** of the integers $a_1, \ldots, a_n$.*
- *The integer $a_1 \vee a_2 \vee \cdots \vee a_n$ is called the **least common multiple (lcm)** of the integers $a_1, \ldots, a_n$.*

Proposition 29.
- $d = a_1 \wedge a_2 \wedge \cdots \wedge a_n$ is the unique positive integer such that $\mathcal{D}(d) = \mathcal{D}(a_1) \cap \cdots \cap \mathcal{D}(a_n)$.
- $m = a_1 \vee a_2 \vee \cdots \vee a_n$ is the unique positive integer such that $\mathcal{M}(m) = \mathcal{M}(a_1) \cap \cdots \cap \mathcal{M}(a_n)$.

We can extend some of the results for the gcd and lcm of two integers.

Proposition 30. Let $a_1, \ldots, a_n$ be a family of n integers, with $n \geqslant 2$, and let c be a non-zero integer. Then

$$(ca_1) \wedge (ca_2) \wedge \cdots \wedge (ca_n) = |c|(a_1 \wedge a_2 \wedge \cdots \wedge a_n)$$
$$(ca_1) \vee (ca_2) \vee \cdots \vee (ca_n) = |c|(a_1 \vee a_2 \vee \cdots \vee a_n)$$

Proposition 31. Let $a_1, \ldots, a_n$ be a family of n integers, with $n \geqslant 2$. Then there exist integers $u_1, \ldots, u_n$ such that

$$a_1 \wedge a_2 \wedge \cdots \wedge a_n = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n.$$

Definition 32. *Let $a_1, \ldots, a_n$ be a family of n integers, with $n \geqslant 2$. we say that they are **relatively prime**[a] if $a_1 \wedge a_2 \wedge \cdots \wedge a_n = 1$.*

[a] premiers entre eux dans leur ensemble

Proposition 33. let $a_1, \ldots, a_n$ be a family of n integers, with $n \geqslant 2$. The following are equivalent:

(i) the integers $a_1, \ldots, a_n$ are relatively prime;

(ii) there exist n integers $u_1, \ldots, u_n$ such that $a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 1$.

Remark. The equality $(a \wedge b) \cdot (a \vee b) = |ab|$ does not generalise to more than two integers.
However, if the integers $a_1, \dots, a_n$ are *pairwise* coprime, then $a_1 \wedge a_2 \wedge \cdots \wedge a_n = a_1 a_2 \cdots a_n$.

VI. Prime integers and factorisations

Definition 34. *Let p be an integer in $\mathbb{Z}$. We say that p is* ***prime***[a] *if it is not constant and if its only divisors are ± 1 and $\pm p$.*

[a] premier

Remark. An integer p is not prime if, and only if, there exist q and r such that $|q| < |p|$, $|r| < |p|$ and $p = qr$.

Properties 35.
- ± 2 are the only even prime numbers.
- If p is an prime integer and if p does not divide an integer a, then p and a are coprime.
- Let p be an prime integer. Let $a_1, \dots, a_n$ be a family of integers. Then p divides the product $a_1 \cdots a_n$ if, and only if, p divides one of the a_i.

Proposition 36. There is are infinitely many distinct prime numbers.

Remark. Any integer a with $|a| > 1$ has an prime divisor.
In particular, to prove that two integers a and b are coprime, it is enough to prove that they have no common prime divisor.

Corollary 37. Let a be a non-zero integer. Then a can be written uniquely (up to reordering of the factors)

$$a = \lambda \prod_{k=1}^{r} p_k^{n_k} = \lambda p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

where $\lambda = \pm 1$ is the sign of a, the integers $p_1, \dots, p_r$ are positive prime integers and the n_k are positive integers.

Proposition 38. Let a and b be two integers in $\mathbb{Z}$ which are not in $\{-1; 0; 1\}$. Set $a = \lambda \prod_{k=1}^{r} p_k^{n_k}$ and $b = \mu \prod_{k=1}^{r} p_k^{m_k}$ with n_k and m_k non-negative integers (p_k need not occur in the decomposition of a or b if $n_k = 0$ or $m_k = 0$).
Then a divides b if, and only if, $n_k \leqslant m_k$ for all k.

Proposition 39. Let a and b be two integers in $\mathbb{Z}$ which are not in $\{-1; 0; 1\}$. Set $a = \lambda \prod_{k=1}^{r} p_k^{n_k}$ and $b = \mu \prod_{k=1}^{r} p_k^{m_k}$ with n_k and m_k non-negative integers (not both zero). For each k, set $u_k = \min(n_k, m_k)$ and $v_k = \max(n_k, m_k)$.
Then $a \wedge b = \prod_{k=1}^{r} p_k^{u_k}$ and $a \vee b = \prod_{k=1}^{r} p_k^{v_k}$.

Glossary

A

accurate within 10^{-n} – à 10^{-n} près 4
antiderivative or primitive – primitive 65
associative – associative 48
average – moyenne 10

B

bounded – borné 3
bounded above – majoré 1
bounded below – minoré 3

C

change of coordinates – changement de coordonnées 95
chord – corde 10
closed disk – disque (*ou* boule) fermé 88
commutative – commutative 48
commutative field – corps commutatif 48
contracting – contractante 10
coordinate function – fonction coordonnée 85
curve – courbe 94
cylindrical coordinates – coordonnées cylindriques 96

D

decimal approximation by default – valeur approchée par défaut 4
decimal approximation by excess – valeur approchée par excès 4
dense – dense 4
dichotomy algorithm – algorithme de dichotomie 31
distributive – distributive 48

E

exponential – exponentielle 17

F

fixed point – point fixe 26
function – fonction
  ***n*-th root** – racine *n*-ième 8
  arccosine – arccosinus 21
  arcsine – arcsinus 20
  arctangent – arctangente 22
  continuous – continue 15
  cosine – cosinus 21
  derivative – dérivée
    higher-order derivative – dérivée d'ordre supérieur 13
    second derivative – dérivée seconde 13
  differentiable – dérivable 9, 15
  graph – graphe 8
  has a continuous extension – admet un prolongement par continuité 19
  hyperbolic cosine – cosinus hyperbolique 23
  hyperbolic sine – sinus hyperbolique 23
  hyperbolic tangent – tangente hyperbolique 24
  integrable – intégrable 56, 61
  locally integrable – localement intégrable 75

odd – impaire 21
of class $\mathcal{C}^\infty$ or smooth – de classe $\mathcal{C}^\infty$ 13, 97
of class $\mathcal{C}^k$ – de classe $\mathcal{C}^k$ 13, 15
piecewise continuous – continue par morceaux 62, 75
power function – fonction puissance 18
sine – sinus 20
square root – racine carrée 8
step function – fonction en escalier *ou* étagée 54
tangent – tangente 22
Fundamental Theorem of Algebra – théorème de d'Alembert-Gauss 43
Fundamental Theorem of Calculus – Théorème fondamental de l'analyse 66

G

general exponential function – exponentielle de base a 19
gradient – gradient 94
greatest lower bound – borne inférieure 3

I

improper integral – intégrale impropre 75
absolutely convergent – absolument convergente 80
convergent – convergente 75
divergent – divergente 75
semi-convergent – semi-convergente 80
indentity element – élément neutre 48
induction – récurrence 14
infimum – borne inférieure 3
integer – entier
Bézout coefficients – coefficients de Bézout 103
Bézout relation – relation de Bézout 103
Euclid's Lemma – lemme d'Euclide 104
Euclidean algorithm – algorithme d'Euclide 102
Euclidean division – division euclidienne 102
Gauss' Lemma – lemme de Gauss 104
gcd – pgcd 102
greatest common divisor – plus grand commun diviseur 102
homogeneous equation – équation homogène 105
pairwise coprime – premiers entre eux deux à deux 105
prime – premier 107
quotient – quotient 102
remainder – reste 102
integral – intégrale 55, 57, 61
integration by parts – intégration par parties (IPP) 67
integration by substitution – changement de variable 68
integral part – partie entière 3
Intermediate Value Theorem – théorème des valeurs intermédiaires (TVI) 5
interval – intervalle 6
stable – stable 25
invertible – inversible 48

J

jacobean matrix – matrice jacobienne 94

L

least upper bound – borne supérieure 1
Leibniz's Formula – formule de Leibniz 14
Lipschitz continuous with Lipschitz constant K – K-lipschitzienne 10
local extremum – extremum local 94
local maximum – maximum local 94
local minimum – minimum local 94
logarithm – logarithme 16, 67
lower bound – minorant 3

M

map – application 3
maximum – maximum *ou* plus grand élément 2

Mean Value Inequality – inégalité des accroissements finis 11, 15, 66
Mean Value Theorem – théorème des accroissements finis 9
Mean Value Theorem for integrals – formule de la moyenne 60
minimum – minimum *ou* plus petit élément 3

N

neighbourhood – voisinage 5
Newton's method – méthode de Newton 32
non-empty – non vide 2
non-negative – positif (ou nul) 1
norm – norme 83
  euclidean norm – norme euclidienne 83
  supremum norm – norme sup 83

O

open disk – disque (*ou* boule) ouvert 88
open subset – ouvert (*ou* partie ouverte) 88

P

parametric curve – courbe paramétrée 87
partial derivative – dérivée partielle 91
partition – subdivision 54
  adapted – adaptée 54
  mesh – pas 54
polynomial – polynôme 34
  associate – associé 34
  Bézout coefficients – coefficients de Bézout 37
  Bézout relation – relation de Bézout 37
  Bézout Theorem – théorème de Bézout 39, 104
  coprime – premiers entre eux 39, 104
  Euclidean algorithm – algorithme d'Euclide 35
  Gauss' Lemma – lemme de Gauss 39
  gcd – pgcd 34
  greatest common divisor – plus grand commun diviseur 34
  irreducible – irréductible 42
  Lagrange interpolation polynomial – polynôme d'interpolation de Lagrange 46
  lcm – ppcm 37, 104
  leading coefficient – coefficient dominant 34, 44, 45
  least common multiple – plus petit commun multiple 37, 104
  monic – unitaire 34
  pairwise coprime – premiers entre eux deux à deux 40
  reducible – réductible 42
  relatively prime – premiers entre eux dans leur ensemble 41, 106
  remainder – reste 35
  root – racine 5, 42
  split – scindé 44
polynomial function – fonction polynomiale 48
positive – strictement positif 1
a **to the** b **or** a **to the power of** b – a puissance b 17

R

radius – rayon 88
rational fraction – fraction rationnelle 47
  addition – addition 47
  degree – degré 48
  denominator – dénominateur 47
  integral part – partie entière 49
  irreducible form – forme irréductible 48
  multiplication – multiplication 47
  multiplicity – multiplicité 49
  numerator – numérateur 47
  partial fraction decomposition – décomposition en éléments simples 50
  pole – pôle 49
    double – double 49
    simple – simple 49
  root – racine 49

rational function – fonction rationnelle 48
recursively – par récurrence 13
Riemann sum – somme de Riemann 62
Rolle's Theorem – théorème de Rolle 9

S

scalar product – produit scalaire 85
sequence – suite 2
  stationary – stationnaire 3
set – ensemble 1
slope – coefficient directeur (pente) 11
sphere – sphère 88
straight-line motion – trajectoire rectiligne 10
subset – sous-ensemble 2
supremum – borne supérieure 1
symmetry with respect to – symétrie par rapport à 8

T

Taylor expansion – développement limité (DL) 16
Taylor's formula with integral remainder – formule de Taylor avec reste intégral 71
  remainder – reste 71
Taylor's inequality – inégalité de Taylor-Lagrange 15, 72
triangle inequality – inégalité du triangle 83
x **truncated to** n **decimal places** – tronqué à n décimales 4

U

unit disk – disque (*ou* boule) unité 88
upper bound – majorant 1

V

vector function – fonction à valeurs vectorielles 83
  continuous – continue 84
  derived vector – vecteur dérivé 84
  differentiable – dérivable 84
  integral – intégrale 87
  limit – limite 84
  Taylor expansion – développement limité 86
  Taylor formula with integral remainder – formule de Taylor avec reste intégral 87
  Taylor-Young formula – formule de Taylor-Young 86
velocity – vitesse 10

Glossaire

A

à 10^{-n} **près** – accurate within 10^{-n} 4
algorithme de dichotomie – dichotomy algorithm 31
application – map 3
associative – associative 48

B

borné – bounded 3
borne inférieure – infimum 3
borne supérieure – supremum 1

C

changement de coordonnées – change of coordinates 95
coefficient directeur (pente) – slope 11
commutative – commutative 48
contractante – contracting 10
coordonnées cylindriques – cylindrical coordinates 96
corde – chord 10
corps commutatif – commutative field 48
courbe – curve 94
courbe paramétrée – parametric curve 87

D

dense – dense 4
dérivée partielle – partial derivative 91
développement limité (DL) – Taylor expansion 16
disque (*ou* boule) fermé – closed disk 88
disque (*ou* boule) ouvert – open disk 88
disque (*ou* boule) unité – unit disk 88
distributive – distributive 48

E

élément neutre – indentity element 48
ensemble – set 1
entier – integer
  algorithme d'Euclide – Euclidean algorithm 102
  coefficients de Bézout – Bézout coefficients 103
  division euclidienne – Euclidean division 102
  équation homogène – homogeneous equation 105
  lemme d'Euclide – Euclid's Lemma 104
  lemme de Gauss – Gauss' Lemma 104
  pgcd – gcd 102
  plus grand commun diviseur – greatest common divisor 102
  premier – prime 107
  premiers entre eux deux à deux – pairwise coprime 105
  quotient – quotient 102
  relation de Bézout – Bézout relation 103
  reste – remainder 102
exponentielle – exponential 17
exponentielle de base a – general exponential function 19
extremum local – local extremum 94

F

fonction – function
admet un prolongement par continuité – has a continuous extension 19
arccosinus – arccosine 21
arcsinus – arcsine 20
arctangente – arctangent 22
continue par morceaux – piecewise continuous 62, 75
cosinus hyperbolique – hyperbolic cosine 23
de classe $\mathcal{C}^\infty$ – of class $\mathcal{C}^\infty$ or smooth 13, 97
de classe $\mathcal{C}^k$ – of class $\mathcal{C}^k$ 13
dérivable – differentiable 9
dérivée – derivative
dérivée d'ordre supérieur – higher-order derivative 13
dérivée seconde – second derivative 13
fonction en escalier *ou* étagée – step function 54
fonction puissance – power function 18
graphe – graph 8
intégrable – integrable 56, 61
localement intégrable – locally integrable 75
racine carrée – square root 8
racine n-ième – n-th root 8
sinus hyperbolique – hyperbolic sine 23
tangente hyperbolique – hyperbolic tangent 24
fonction à valeurs vectorielles – vector function 83
continue – continuous 84
dérivable – differentiable 84
développement limité – Taylor expansion 86
formule de Taylor avec reste intégral – Taylor formula with integral remainder 87
formule de Taylor-Young – Taylor-Young formula 86
intégrale – integral 87
limite – limit 84
vecteur dérivé – derived vector 84
fonction coordonnée – coordinate function 85
fonction polynomiale – polynomial function 48
fonction rationnelle – rational function 48
formule de la moyenne – Mean Value Theorem for integrals 60
formule de Leibniz – Leibniz's Formula 14
formule de Taylor avec reste intégral – Taylor's formula with integral remainder
reste – remainder 71
fraction rationnelle – rational fraction 47
addition – addition 47
décomposition en éléments simples – partial fraction decomposition 50
degré – degree 48
dénominateur – denominator 47
forme irréductible – irreducible form 48
multiplication – multiplication 47
multiplicité – multiplicity 49
numérateur – numerator 47
partie entière – integral part 49
pôle – pole 49
double – double 49
simple – simple 49
racine – root 49

G

gradient – gradient 94

I

inégalité de Taylor-Lagrange – Taylor's inequality 15, 72
inégalité des accroissements finis – Mean Value Inequality 11, 15, 66
inégalité du triangle – triangle inequality 83
intégrale – integral 55, 57, 61
changement de variable – integration by substitution 68
intégration par parties (IPP) – integration by parts 67
intégrale impropre – improper integral 75
absolument convergente – absolutely convergent 80
convergente – convergent 75
divergente – divergent 75
semi-convergente – semi-convergent 80

intervalle – interval 6
stable – stable 25
inversible – invertible 48

L

K-**lipschitzienne** – Lipschitz continuous with Lipschitz constant *K* 10
logarithme – logarithm 16, 67

M

majorant – upper bound 1
majoré – bounded above 1
matrice jacobienne – jacobean matrix 94
maximum local – local maximum 94
maximum *ou* plus grand élément – maximum 2
méthode de Newton – Newton's method 32
minimum local – local minimum 94
minimum *ou* plus petit élément – minimum 3
minorant – lower bound 3
minoré – bounded below 3
moyenne – average 10

N

norme – norm 83
norme euclidienne – euclidean norm 83
norme sup – supremum norm 83

O

ouvert (*ou* partie ouverte) – open subset 88

P

partie entière – integral part 3
point fixe – fixed point 26
polynôme – polynomial 34
algorithme d'Euclide – Euclidean algorithm 35
associé – associate 34
coefficients de Bézout – Bézout coefficients 37
coefficient dominant – leading coefficient 34, 44, 45
irréductible – irreducible 42
lemme de Gauss – Gauss' Lemma 39
pgcd – gcd 34
plus grand commun diviseur – greatest common divisor 34
plus petit commun multiple – least common multiple 37, 104
polynôme d'interpolation de Lagrange – Lagrange interpolation polynomial 46
ppcm – lcm 37, 104
premiers entre eux – coprime 39, 104
premiers entre eux dans leur ensemble – relatively prime 41, 106
premiers entre eux deux à deux – pairwise coprime 40
racine – root 42
réductible – reducible 42
relation de Bézout – Bézout relation 37
reste – remainder 35
scindé – split 44
théorème de Bézout – Bézout Theorem 39, 104
unitaire – monic 34
positif (ou nul) – non-negative 1
primitive – antiderivative or primitive 65
produit scalaire – scalar product 85
a **puissance** *b* – *a* to the *b* 17

R

rayon – radius 88

S

somme de Riemann – Riemann sum 62

sous-ensemble – subset 2
sphère – sphere 88
strictement positif – positive 1
subdivision – partition 54
  adaptée – adapted 54
  pas – mesh 54
suite – sequence 2
  stationnaire – stationary 3

T

théorème de d'Alembert-Gauss – Fundamental Theorem of Algebra 43
théorème de Rolle – Rolle's Theorem 9
théorème des accroissements finis – Mean Value Theorem 9
théorème des valeurs intermédiaires (TVI) – Intermediate Value Theorem 5
Théorème fondamental de l'analyse – Fundamental Theorem of Calculus 66
trajectoire rectiligne – straight-line motion 10
tronqué à n décimales – x truncated to n decimal places 4

V

valeur approchée par défaut – decimal approximation by default 4
valeur approchée par excès – decimal approximation by excess 4
vitesse – velocity 10
voisinage – neighbourhood 5