Concentration inequalities for Feynman-Kac particle models

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Some hyper-refs

- Feynman-Kac formulae, Genealogical & Interacting Particle Systems with appl., Springer (2004)
- Sequential Monte Carlo Samplers JRSS B. (2006). (joint work with A. Doucet & A. Jasra)
- A Backward Particle Interpretation of Feynman-Kac Formulae M2AN (2010). (joint work with A. Doucet & S.S. Singh)
- Concentration Inequalities for Mean Field Particle Models. Annals of Applied Probability (2011) (joint work with Emmanuel Rio).
- On the concentration of interacting processes. Foundations & Trends in Machine Learning [170p.] (2012). (joint work with Peng Hu & Liming Wu) [+ Refs]
- More references on the website http://www.math.u-bordeaux1.fr/∼delmoral/index.html [+ Links]

Basic notation

$\mathcal{P}(E)$ probability meas., $\mathcal{B}(E)$ bounded functions on E.

- $(\mu, f) \in \mathcal{P}(E) \times \mathcal{B}(E) \quad \longrightarrow \quad \mu(f) = \int \mu(dx)\ f(x)$
- $Q(x_1, dx_2)$ **integral operators** $x_1 \in E_1 \rightsquigarrow x_2 \in E_2$

$$\begin{aligned} Q(f)(x_1) &= \int Q(x_1, dx_2) f(x_2) \\ [\mu Q](dx_2) &= \int \mu(dx_1) Q(x_1, dx_2) \qquad (\Longrightarrow [\mu Q](f) = \mu[Q(f)]\ ) \end{aligned}$$

Basic notation

- **Boltzmann-Gibbs transformation** $G \geq 0$

$$\mu(dx) \quad \mapsto \quad \Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\ G(x)\ \mu(dx)$$

When $G \leq 1$ we have the (non unique) Markov transport equation

$$\Psi_G(\mu) = \mu S_{\mu,G}$$

with

$$S_{\mu,G}(x,dy) = G(x)\ \delta_x(dy) + (1 - G(x))\ \Psi_G(\mu)(dy)$$

Feynman-Kac models

Markov chain $X_n \in E_n$ **and a potential function** $G_n : E_n \to [0, \infty[$

$$d\mathbb{Q}_n := \frac{1}{\mathcal{Z}_n} \left\{ \prod_{0 \leq p < n} G_p(X_p) \right\} d\mathbb{P}_n \quad \textbf{with} \quad \mathbb{P}_n = \mathrm{Law}(X_0, \ldots, X_n)$$

- Transition/Excursions/Path spaces

$$X_n = (X'_n, X'_{n+1}) \qquad X_n = X'_{[T_n, T_{n+1}]} \qquad X_n = (X'_0, \ldots, X'_n)$$

- $\supset$ Continuous time models

$$X_n = X'_{[t_n, t_{n+1}]} \quad \& \quad G_n(X_n) = \exp \int_{t_n}^{t_{n+1}} V_t(X'_t) dt$$

Some examples

- **Confinements:** $X_n \in \mathbb{Z}^d \supset A \quad \& \quad G_n := 1_A$.

$$\mathbb{Q}_n = \text{Law}\left((X_0, \ldots, X_n) \mid X_p \in A, \ \forall 0 \leq p < n\right)$$

and

$$\mathcal{Z}_n = \text{Proba}\left(X_p \in A, \ \forall 0 \leq p < n\right)$$

- **SAW :** $X_n = (X'_p)_{0 \leq p \leq n} \quad \& \quad G_n(X_n) = 1_{X'_n \notin \{X'_0, \ldots, X'_{n-1}\}}$

$$\mathbb{Q}_n = \text{Law}\left((X'_0, \ldots, X'_n) \mid X'_p \neq X'_q, \ \forall 0 \leq p < q < n\right)$$

and

$$\mathcal{Z}_n = \text{Proba}\left(X'_p \neq X'_q, \ \forall 0 \leq p < q < n\right)$$

Some examples

- **Filtering :** $Y_n = H_n(X_n, V_n) \quad \& \quad G_n(x_n) := p_{Y_n|X_n}(y_n|x_n)$.

$$\mathbb{Q}_n = \mathrm{Law}\left((X_0, \ldots, X_n) \mid Y_p = y_p, \; \forall 0 \leq p < n\right)$$

and

$$\mathcal{Z}_n = p_{Y_0,\ldots,Y_n}(y_0, \ldots, y_n)$$

- **Multilevel splitting :** $A_n \downarrow$, with B non critical recurrent subset.

$$T_n := \inf \{t \geq T_{n-1} \;:\; X'_t \in (A_n \cup B)\}$$

$$X_n = (X'_t)_{t \in [T_n, T_{n+1}]} \quad \& \quad G_n(X_n) = 1_{A_{n+1}}(X'_{T_{n+1}})$$

$$\mathbb{Q}_n = \mathrm{Law}\left(X'_{[T_0,T_n]} \mid X' \text{ hits } A_{n-1} \text{ before } B\right)$$

and

$$\mathcal{Z}_n = \mathbb{P}(X' \text{ hits } A_{n-1} \text{ before } B)$$

Some examples

- **Absorption models :**

$$X_n^c \in E_n^c \xrightarrow{\text{absorption } \sim(1-G_n)} \widehat{X}_n^c \xrightarrow{\text{exploration } \sim M_{n+1}} X_{n+1}^c$$

$$\mathbb{Q}_n = \mathrm{Law}((X_0^c, \ldots, X_n^c) \mid T^{abs.} \geq n) \quad \& \quad \mathcal{Z}_n = \mathrm{Proba}\left(T^{abs.} \geq n\right)$$

- **Quasi-invariant measures :** $(G_n, M_n) = (G, M)$ **&** M μ**-reversible**

$$\frac{1}{n} \log \mathbb{P}\left(T^{abs.} \geq n\right) \simeq_{n\uparrow\infty} \lambda \;=\; \text{top spect. of} \;\; Q(x,dy) = G(x)M(x,dy)$$

[Frobenius theo] $Q(h) = \lambda h = \lambda \times$ eigenfunction (ground state)

$$\mathbb{P}(X_n^c \in dx \mid T^{abs.} > n) \simeq_{n\uparrow\infty} \frac{1}{\mu(h)}\; h(x)\; \mu(dx)$$

Some examples

- **Doob h-processes X^h :**

$$M^h(x,dy) = \frac{1}{\lambda} h^{-1}(x) Q(x,dy) h(y) = \frac{Q(x,dy)h(y)}{Q(h)(x)} = \frac{M(x,dy)h(y)}{M(h)(x)}$$

$$\Downarrow$$

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) \propto \mathrm{Proba}((X_0^h,\ldots,X_n^h) \in d(x_0,\ldots,x_n))\ h^{-1}(x_n)$$

Flow of n-marginals [X_n Markov with transitions M_n]

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f) := \mathbb{E}\left(f(X_n) \prod_{0 \leq p < n} G_p(X_p)\right)$$

$$\Updownarrow \left(\gamma_n(1) = \mathcal{Z}_n\right)$$

Nonlinear evolution equation :

$$\begin{array}{rcl} \eta_{n+1} & = & \Psi_{G_n}(\eta_n) M_{n+1} \\ \mathcal{Z}_{n+1} & = & \eta_n(G_n) \times \mathcal{Z}_n \end{array}$$

$$\downarrow$$

Nonlinear m.v.p. = Law of a Markov $\overline{X}_n$ (perfect sampler)

$$\begin{array}{rcl} \eta_{n+1} & = & \Phi_{n+1}(\eta_n) \\ & = & \eta_n \left(S_{n,\eta_n} M_{n+1}\right) = \eta_n \mathbf{K}_{\mathbf{n+1},\eta_\mathbf{n}} = \text{Law}(\overline{X}_{n+1}) \end{array}$$

Examples related to product models

$$\eta_n(dx) := \frac{1}{\mathcal{Z}_n} \left\{ \prod_{p=0}^{n} h_p(x) \right\} \lambda(dx) \quad \text{with} \quad h_p \geq 0$$

3 illustrations:

$$h_p(x) = e^{-(\beta_{p+1}-\beta_p)V(x)} \quad \beta_p \uparrow \quad \Longrightarrow \quad \eta_n(dx) = \frac{1}{\mathcal{Z}_n}\ e^{-\beta_n V(x)}\ \lambda(dx)$$

$$h_p(x) = 1_{A_{p+1}}(x) \quad A_p \downarrow \quad \Longrightarrow \quad \eta_n(dx) = \frac{1}{\mathcal{Z}_n}\ 1_{A_n}(x)\ \lambda(dx)$$

$$h_p(\theta) = p(y_p|\theta, y_0, \ldots, y_{p-1}) \quad \Longrightarrow \quad \eta_n(d\theta) = p(\theta|y_0, \ldots, y_n)$$

For any MCMC transitions M_n with target η_n, we have

$$\eta_{n+1} = \eta_{n+1} M_{n+1} = \Psi_{h_{n+1}}(\eta_n) M_{n+1} \subset \textbf{Feynman-Kac model}$$

Mean field particle model = Markov $\xi_n = (\xi_n^i)_{1\leq i\leq N} \in E_n^N$

$$\mathbb{P}\left(\xi_{n+1} \in dx \mid \xi_n\right) = \prod_{1\leq i\leq N} K_{n+1,\eta_n^N}(\xi_n^i, dx^i) \quad \text{with} \quad \eta_n^N = \frac{1}{N}\sum_{1\leq i\leq N} \delta_{\xi_n^i}$$

and the (unbiased) particle normalizing constants

$$\mathcal{Z}_{n+1}^N = \eta_n^N(G_n) \times \mathcal{Z}_n^N = \prod_{0\leq p\leq n} \eta_p^N(G_p)$$

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$$\mathcal{Z}_{n+1}^N = \eta_n^N(G_n) \times \mathcal{Z}_n^N = \prod_{0\leq p\leq n} \eta_p^N(G_p)$$

$$\Updownarrow \left[K_{n+1,\eta_n} = S_{n,\eta_n} M_{n+1}\right]$$

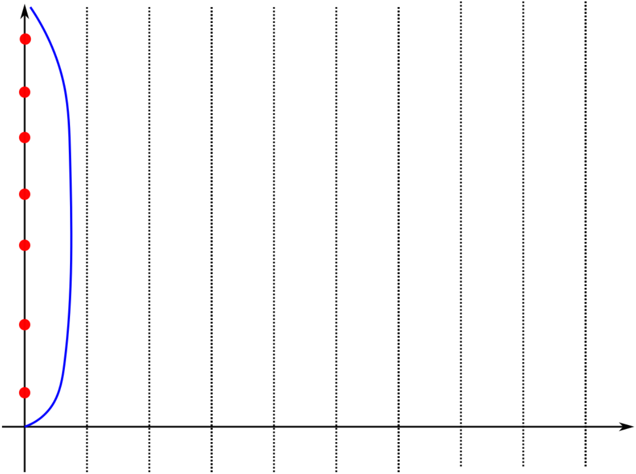
$\rightsquigarrow$ **Sequential particle simulation technique**

G_n-acceptance-rejection with recycling $\oplus$ M_{n+1}-propositions

$\rightsquigarrow$ **Genetic type branching particle model**

$$\xi_n = (\xi_n^i)_{1\leq i\leq N} \xrightarrow{G_n-selection} \widehat{\xi}_n = (\widehat{\xi}_n^i)_{1\leq i\leq N} \xrightarrow{M_n-mutation} \xi_{n+1} = (\xi_{n+1}^i)_{1\leq i\leq N}$$

Graphical illustration : $\eta_n \simeq \eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i}$



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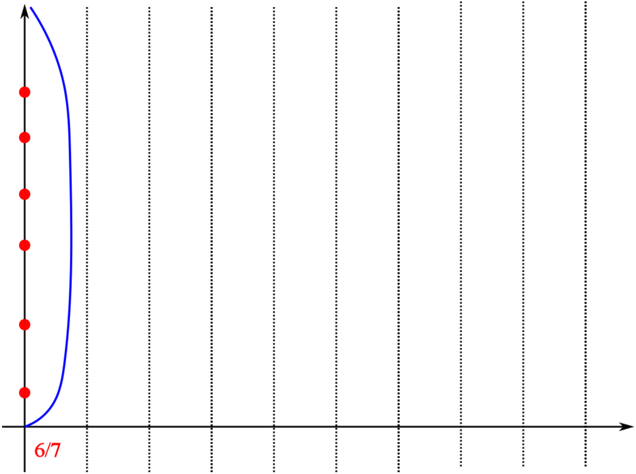

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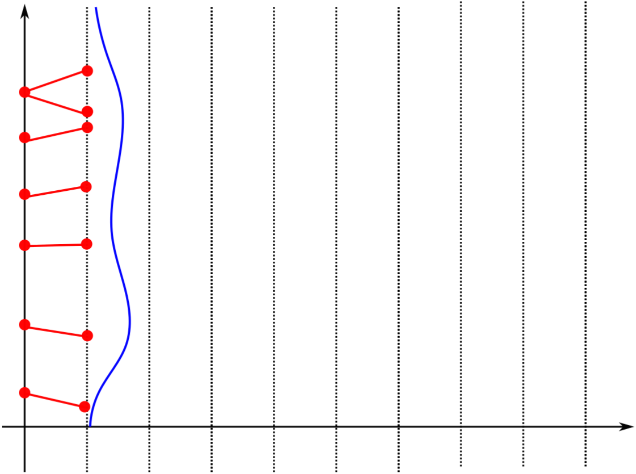

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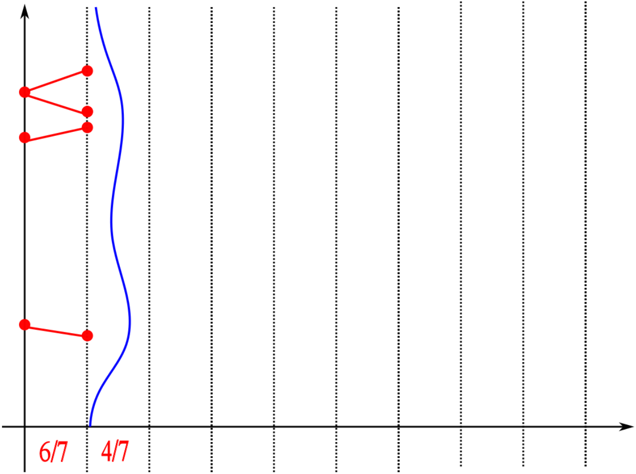

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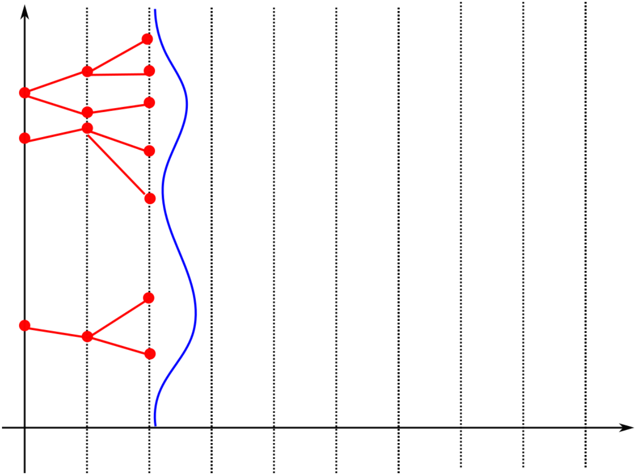

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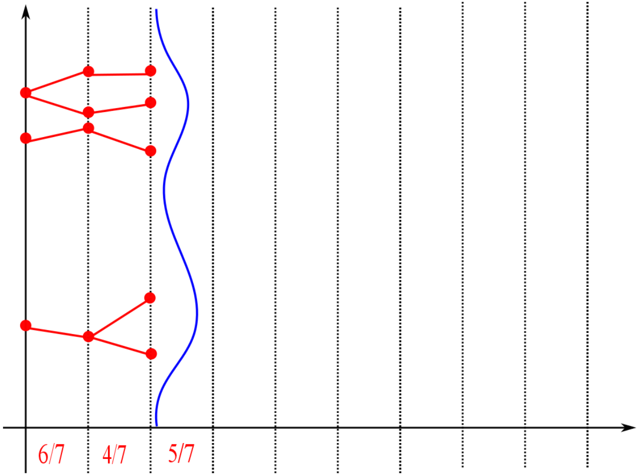

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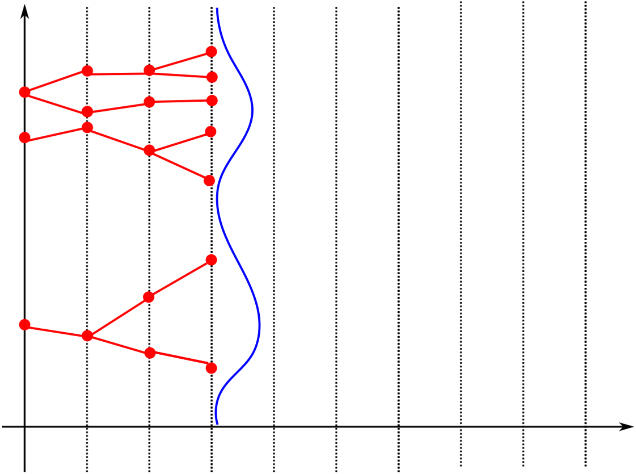

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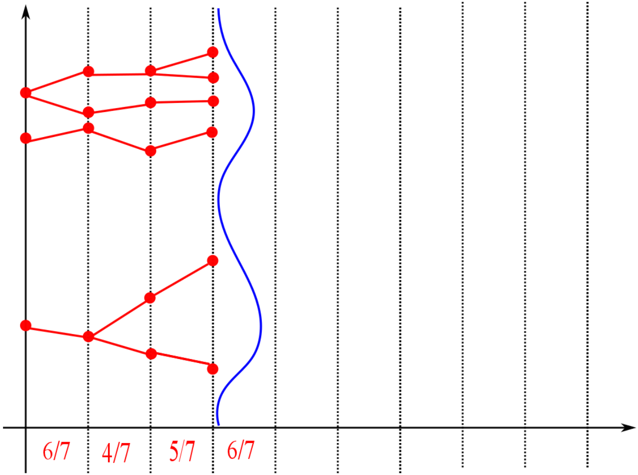

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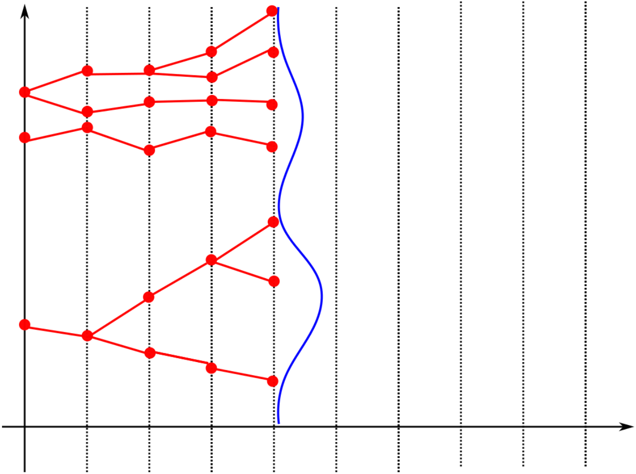

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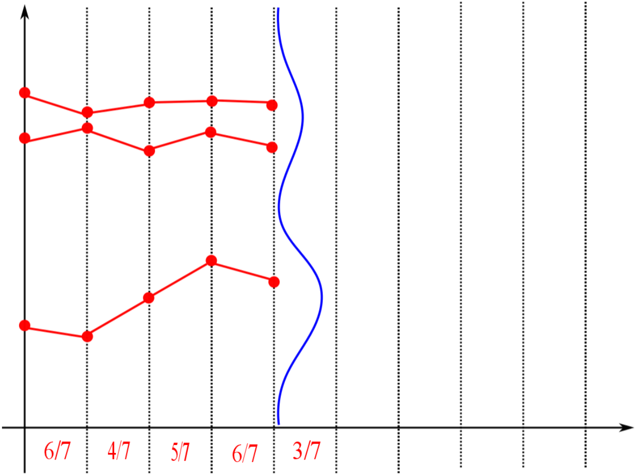

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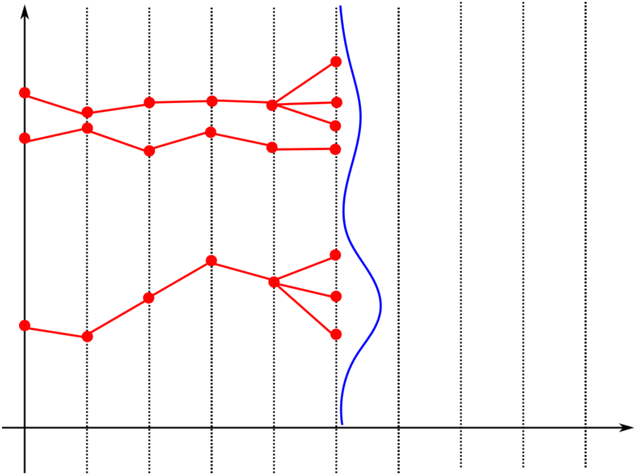

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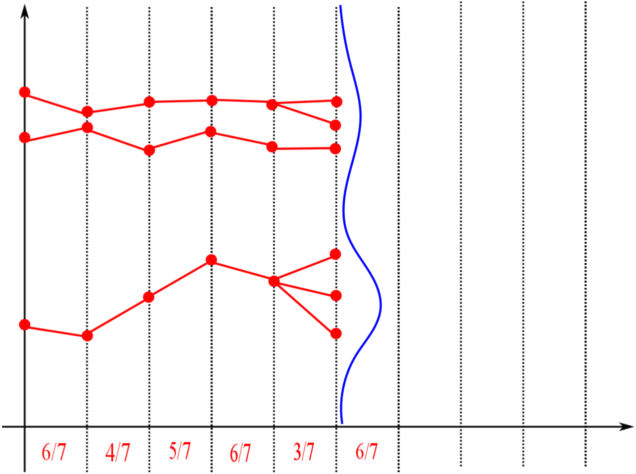

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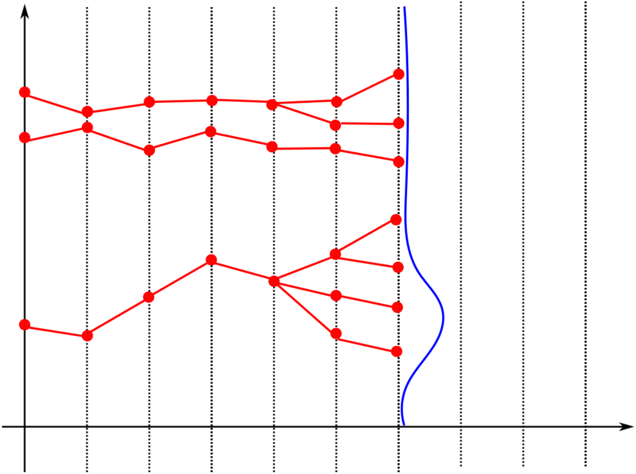

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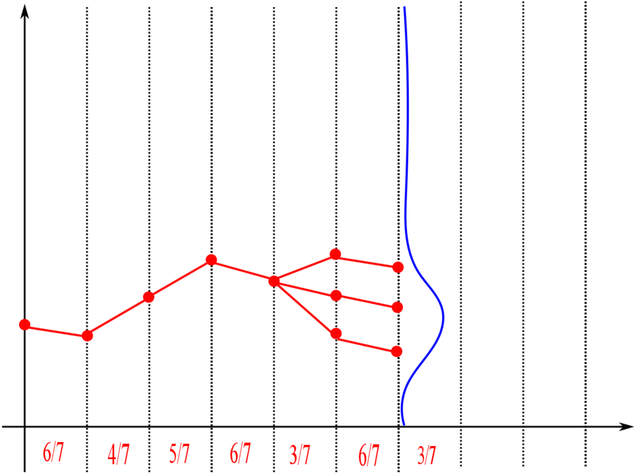

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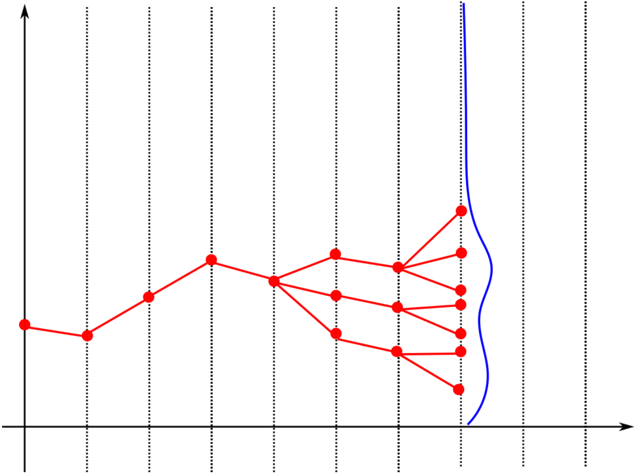

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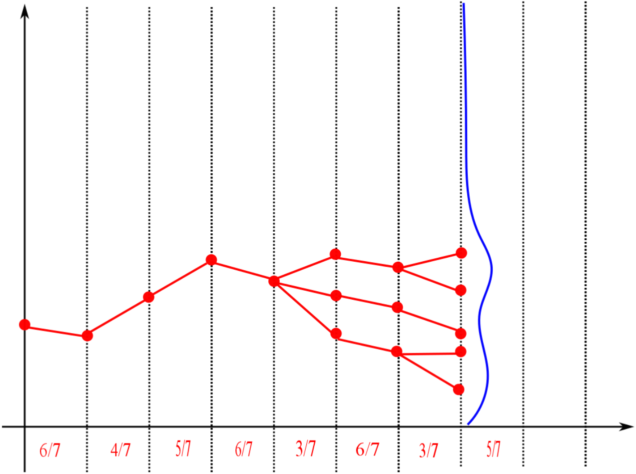

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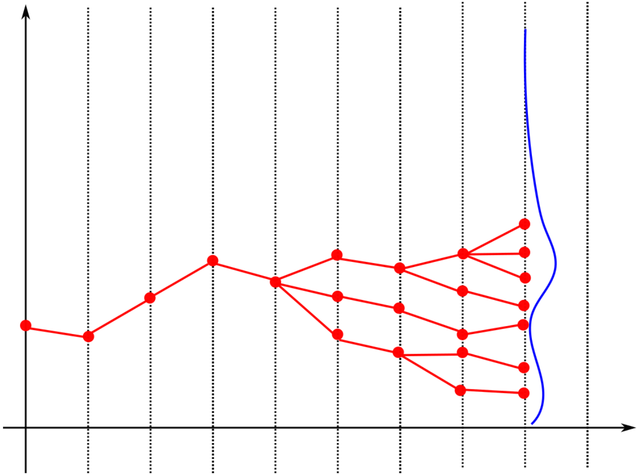

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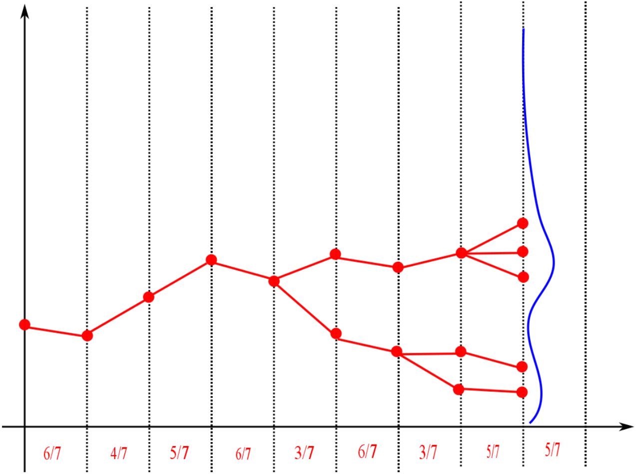

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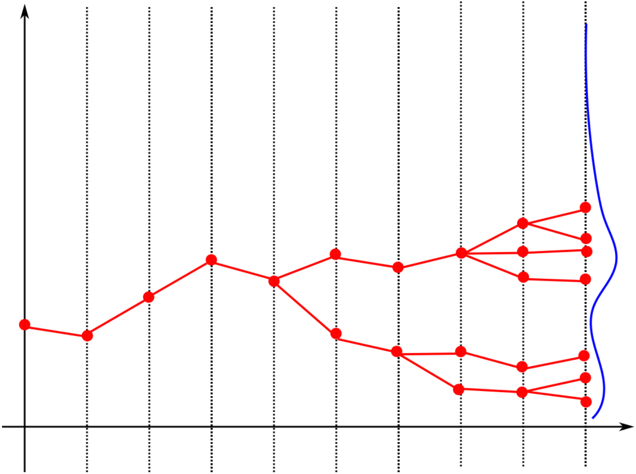

How to use the full ancestral tree model ?

$$G_{n-1}(x_{n-1})M_n(x_{n-1}, dx_n) \stackrel{hyp}{=} H_n(x_{n-1}, x_n)\ \nu_n(dx_n)$$

$$\Rightarrow \mathbb{Q}_n(d(x_0, \ldots, x_n)) = \eta_n(dx_n)\ \underbrace{\mathbb{M}_{n,\eta_{n-1}}(x_n, dx_{n-1})}_{\propto\ \eta_n(dx_n)\ H_{n+1}(x_n, x_{n+1})} \ldots \mathbb{M}_{1,\eta_0}(x_1, dx_0)$$

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Particle approximation = Random stochastic matrices

$$\mathbb{Q}_n^N(d(x_0,\ldots,x_n)) = \eta_n^N(dx_n)\ \mathbb{M}_{n,\eta_{n-1}^N}(x_n,dx_{n-1})\ldots\mathbb{M}_{1,\eta_0^N}(x_1,dx_0)$$

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Particle approximation $=$ Random stochastic matrices

$$\mathbb{Q}_n^N(d(x_0, \ldots, x_n)) = \eta_n^N(dx_n)\ \mathbb{M}_{n,\eta_{n-1}^N}(x_n, dx_{n-1}) \ldots \mathbb{M}_{1,\eta_0^N}(x_1, dx_0)$$

Ex.: Additive functionals $\quad \mathbf{f_n}(x_0, \ldots, x_n) = \frac{1}{n+1} \sum_{0\leq p\leq n} f_p(x_p)$

$$\mathbb{Q}_n^N(\mathbf{f_n}) := \frac{1}{n+1} \sum_{0\leq p\leq n} \eta_n^N \underbrace{\mathbb{M}_{n,\eta_{n-1}^N} \ldots \mathbb{M}_{p+1,\eta_p^N}(f_p)}_{\text{matrix operations}}$$

The example of the h-process and the absorption model

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) \propto \mathbb{P}((X_0^h,\ldots,X_n^h) \in d(x_0,\ldots,x_n))\ h^{-1}(x_n)$$

- *Invariant measure $\mu_h = \mu_h M^h$ & normalized additive functionals*

$$\mathbf{f}_n(x_0,\ldots,x_n) = \frac{1}{n+1} \sum_{0\leq p\leq n} f(x_p) \Longrightarrow \mathbb{Q}_n(\mathbf{f}_n) \simeq_n \mu_h(f)$$

- If $G = G^\theta$ depends on some $\theta \in \mathbb{R} \rightsquigarrow f := \frac{\partial}{\partial\theta} \log G^\theta$

$$\frac{\partial}{\partial\theta} \log \lambda^\theta \simeq_n \frac{1}{n+1} \frac{\partial}{\partial\theta} \log \mathcal{Z}_{n+1}^\theta = \mathbb{Q}_n(\mathbf{f}_n)$$

NB : Similar expression when M^θ depends on some $\theta \in \mathbb{R}$.

4 particle estimates

- Individuals ξ_n^i "almost" iid with law

$$\eta_n \simeq \eta_n^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i}$$

- Path space models $\rightsquigarrow$ Ancestral lines "almost" iid with law

$$\mathbb{Q}_n \simeq \eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\text{ancestral line}_n(i)}$$

- Backward particle model

$$\mathbb{Q}_n^N(d(x_0, \ldots, x_n)) = \eta_n^N(dx_n)\ \mathbb{M}_{n,\eta_{n-1}^N}(x_n, dx_{n-1}) \ldots \mathbb{M}_{1,\eta_0^N}(x_1, dx_0)$$

- Normalizing constants

$$\mathcal{Z}_{n+1} = \prod_{0 \leq p \leq n} \eta_p(G_p) \simeq_{N\uparrow\infty} \mathcal{Z}_{n+1}^N = \prod_{0 \leq p \leq n} \eta_p^N(G_p) \quad (\text{Unbiased})$$

How & Why it works

- (Computer Sci.) Stochastic adaptive grid approximation.
- (Stats) Universal acceptance-rejection-recycling sampling schemes.
- (Probab) Stochastic linearization/perturbation technique.

$$
\begin{aligned}
\eta_n &= \Phi_n\left(\eta_{n-1}\right) \\
\eta_n^N &= \Phi_n\left(\eta_{n-1}^N\right) + \frac{1}{\sqrt{N}}\ V_n^N
\end{aligned}
$$

Theorem: $(V_n^N)_n \simeq_{N\uparrow\infty} (V_n)_n$ **independent centered Gaussian fields.**

$\Phi_{p,n}(\eta_p) = \eta_n$ stable sg $\Longleftrightarrow$ No propagation of local errors

$\Longrightarrow$ Uniform control w.r.t. the time horizon

$\rightsquigarrow$ **New concentration inequalities for (general) interacting processes**

Key idea = First order expansions

Key telescoping decomposition

$$\eta_n^N - \eta_n = \sum_{p=0}^{n} \left[\Phi_{p,n}(\eta_p^N) - \Phi_{p,n}\left(\Phi_p(\eta_{p-1}^N)\right)\right]$$

$\oplus$ First order expansion

$$\sqrt{N}\ \left[\Phi_{p,n}(\eta_p^N) - \Phi_{p,n}\left(\Phi_p(\eta_{p-1}^N)\right)\right]$$

$$= \sqrt{N}\ \left[\Phi_{p,n}\left(\Phi_p(\eta_{p-1}^N) + \frac{1}{\sqrt{N}}\ V_p^N\right) - \Phi_{p,n}\left(\Phi_p(\eta_{p-1}^N)\right)\right]$$

$$\simeq V_p^N D_{p,n} + \frac{1}{\sqrt{N}}\ R_{p,n}^N$$

with $\underbrace{\text{a predictable } D_{p,n} - \text{first order operator}}_{\text{fluctuation term}} \oplus \underbrace{\text{2nd-order measure } R_{p,n}^N}_{\text{bias-term}}$

Stochastic perturbation model

Stochastic perturbation model

$$W_n^{\eta,N} := \sqrt{N}\left[\eta_n^N - \eta_n\right] = \sum_{0\leq p\leq n} V_p^N D_{p,n} + \frac{1}{\sqrt{N}}\, R_n^N$$

Under some mixing condition on the limiting FK semigroups $\Phi_{p,n}$

$$\mathrm{osc}\left(D_{p,n}(f)\right) \leq Cte\ e^{-(n-p)\alpha}$$

and

$$\mathbb{E}\left(|R_n^N(f)|^m\right) \leq Cte\ 2^{-m}(2m)!/m!$$

$$\Downarrow$$

Uniform concentration estimates w.r.t. the time parameter

Particle free energy

Multiplicative formulae

$$\mathcal{Z}_n^N = \prod_{0\leq p<n} \eta_p^N(G_p) = \gamma_n^N(1) \longrightarrow_{N\uparrow\infty} \gamma_n(1) = \prod_{0\leq p<n} \eta_p(G_p)$$

Taylor first order expansion

$$\forall x,y>0 \qquad \log y - \log x = \int_0^1 \frac{(y-x)}{x+t(y-x)}\,dt$$

$$\Downarrow$$

$$\log\left(\gamma_n^N(1)/\gamma_n(1)\right)$$

$$= \sum_{0\leq p<n}\left(\log \eta_p^N(G_p) - \log \eta_p(G_p)\right)$$

$$= \sum_{0\leq p<n}\left(\log\left(\eta_p(G_p) + \frac{1}{\sqrt{N}}\; W_p^{\eta,N}(G_p)\right) - \log \eta_p(G_p)\right)$$

$$= \frac{1}{\sqrt{N}} \sum_{0\leq p<n} \int_0^1 \frac{W_p^{\eta,N}(G_p)}{\eta_p(G_p) + \frac{t}{\sqrt{N}}\; W_p^{\eta,N}(G_p)}\,dt$$

$\rightsquigarrow$ first order expansion [exercice]

Current population models

Constants (c_1, c_2) related to (bias,variance), c universal constant
Test funct. $\|f_n\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$.

- The probability of the event

$$\left[\eta_n^N - \eta_n\right](f) \leq \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

  is greater than $1 - e^{-x}$.

- $x = (x_i)_{1 \leq i \leq d} \rightsquigarrow (-\infty, x] = \prod_{i=1}^{d} (-\infty, x_i]$ cells in $E_n = \mathbb{R}^d$.

$$F_n(x) = \eta_n\left(1_{(-\infty,x]}\right) \quad \text{and} \quad F_n^N(x) = \eta_n^N\left(1_{(-\infty,x]}\right)$$

  The probability of the following event

$$\sqrt{N}\ \left\|F_n^N - F_n\right\| \leq c\ \sqrt{d\ (x+1)}$$

  is greater than $1 - e^{-x}$.

Particle free energy models

Constants (c_1, c_2) related to (bias,variance), c universal constant
$\forall\ (x \geq 0, n \geq 0, N \geq 1)$

- Unbiased property

$$\mathbb{E}\left(\eta_n^N(f_n) \prod_{0\leq p<n} \eta_p^N(G_p)\right) = \mathbb{E}\left(f_n(X_n) \prod_{0\leq p<n} G_p(X_p)\right)$$

- For any $\epsilon \in \{+1, -1\}$, the probability of the event

$$\frac{\epsilon}{n} \log \frac{\mathcal{Z}_n^N}{\mathcal{Z}_n} \leq \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

is greater than $1 - e^{-x}$.

note $(0 \leq \epsilon \leq 1 \Rightarrow (1 - e^{-\epsilon}) \vee (e^{\epsilon} - 1) \leq 2\epsilon)$

$$e^{-\epsilon} \leq \frac{z^N}{z} \leq e^{\epsilon} \Rightarrow \left|\frac{z^N}{z} - 1\right| \leq 2\epsilon$$

Genealogical tree models $:= \eta_n^N$ (in path space)

Constants (c_1, c_2) related to (bias,variance), c universal constant
$\mathbf{f_n}$ test function $\|\mathbf{f_n}\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$.

- The probability of the event

$$\left[\eta_n^N - \mathbb{Q}_n\right](f) \leq c_1\ \frac{n+1}{N}\ \left(1 + x + \sqrt{x}\right) + c_2\ \sqrt{\frac{(n+1)}{N}}\ \sqrt{x}$$

  is greater than $1 - e^{-x}$.

- $\mathcal{F}_n =$ indicator fct. $\mathbf{f_n}$ of cells in $\mathbf{E_n} = \left(\mathbb{R}^{d_0} \times \ldots, \times \mathbb{R}^{d_n}\right)$
  The probability of the following event

$$\sup_{\mathbf{f_n} \in \mathcal{F}_n} \left|\eta_n^N(\mathbf{f_n}) - \mathbb{Q}_n(\mathbf{f_n})\right| \leq c\ (n+1)\ \sqrt{\frac{\sum_{0 \leq p \leq n} d_p}{N}\ (x+1)}$$

  is greater than $1 - e^{-x}$.

Backward particle models

Constants (c_1, c_2) related to (bias,variance), c universal constant.
$\mathbf{f_n}$ normalized additive functional with $\|f_p\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$

- The probability of the event

$$\left[\mathbb{Q}_n^N - \mathbb{Q}_n\right](\mathbf{f}_n) \leq c_1\ \frac{1}{N}\ (1 + (x + \sqrt{x})) + c_2\ \sqrt{\frac{x}{N(n+1)}}$$

- $\mathbf{f}_{a,n}$ normalized additive functional w.r.t. $f_p = 1_{(-\infty,a]}$, $a \in \mathbb{R}^d = E_n$
  .
  The probability of the following event

$$\sup_{a \in \mathbb{R}^d} \left|\mathbb{Q}_n^N(\mathbf{f_{a,n}}) - \mathbb{Q}_n(\mathbf{f_{a,n}})\right| \leq c\ \sqrt{\frac{d}{N}(x+1)}$$

  is greater than $1 - e^{-x}$.