Random planar maps : An overview

Journées MAS 2012,

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The Brownian paradigm

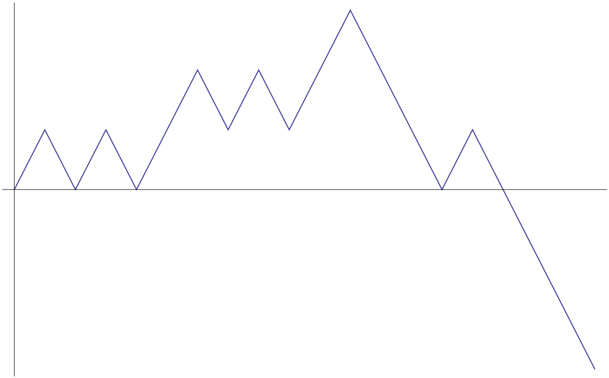

Figure: A random path of length 20

The Brownian paradigm

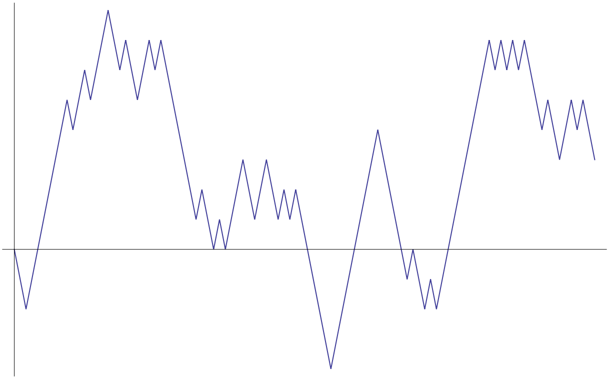

Figure: A random path of length 100

The Brownian paradigm

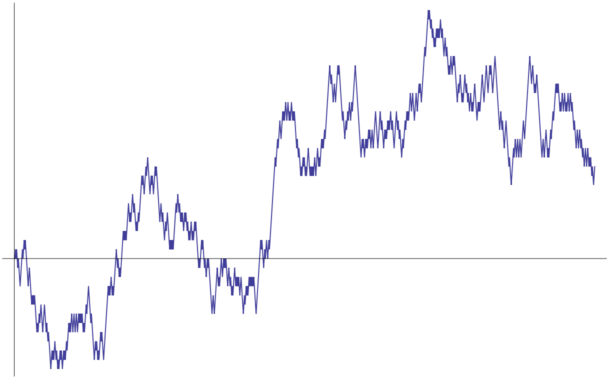

Figure: A random path of length 1000

The Brownian paradigm

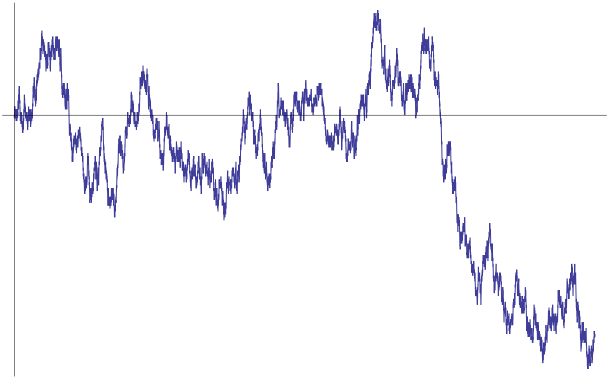

Figure: A random path of length ? ? ?

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Goal : Play the same game with other discrete structures, namely random planar graphs.

Plan

1. Planar maps
2. Scaling and local limits
3. A beautiful bijection
4. Large-scale properties

1. Planar maps

Planar maps

Definition

A planar map is a finite connected planar graph embedded in the two-dimensional sphere seen up to deformations that preserve the orientation.

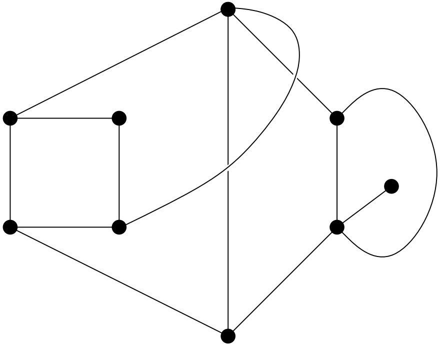

Figure: Not a map

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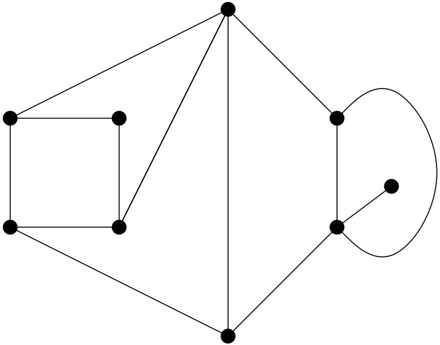

Figure: "A map"

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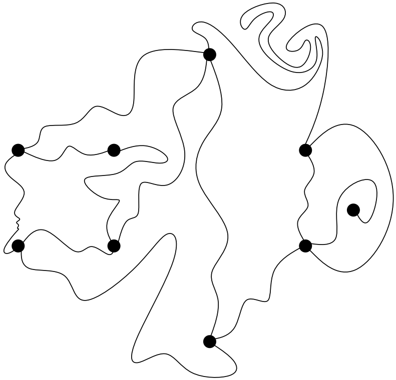

Figure: The same one

A few definitions

Classes of maps :

- All maps with n edges (finite set),
- Triangulations with n faces,
- Quadrangulations with n faces,
- ... (Universality)

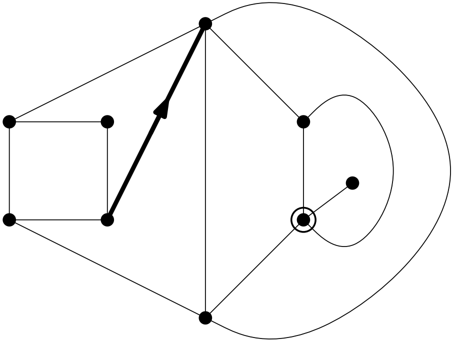

Figure: A rooted and pointed quadrangulation ($\rightarrow$) with 7 faces

Planar maps versus planar graphs

- Maps are more rigid because the embedding is "fixed"
- $\longrightarrow$ rigidity, surgery,...
- $\longrightarrow$ easier to enumerate [Tutte, 't Hooft, Schaeffer].

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$$\#\{\to, \bullet \text{ quadrangulations with } n \text{ faces}\} = 3^n \frac{2}{n+1}\binom{2n}{n}.$$

Enumeration of (decorated) maps is still a very active subject.

Goal : Understand the "geometry" of large random planar maps. (In the scaling limit we believe that uniform planar graphs $\approx$ uniform planar maps)

Motivations

Statistical Mechanics Models

Gromov-Hausdorff

Enumeration

SRW

Analytic combinatorics

Conformal Invariants

2D Quantum Gravity

Surgery

Stable Processes

Tricks

Combinatorics

Brownian Snake

K

GEODESICS

Universality

Baby universes

Gaussian Free Field

Beautiful Bijections

Fractal Geometry

P

SLE Processes

RANDOM TREES

Z

Measured Equivalence Relations

Ergodic Theory

Probability

Circle Packings

Higher genus topology

Unsolved open problems

2. Limits

Large scale structure

Let $\mathcal{Q}_n$ be the set of all (rooted) quadrangulations with n faces and denote by Q_n a uniform random element of $\mathcal{Q}_n$.

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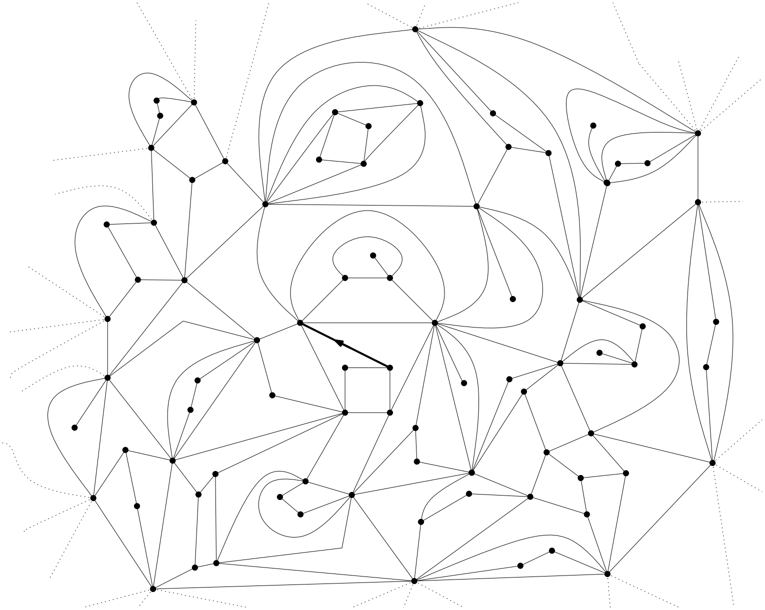
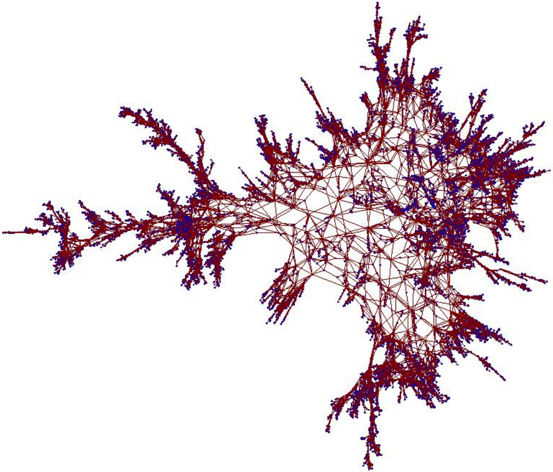
Two ways to proceed :

1. Scaling limit (Bird's-eye view). Rescale the whole map (i.e. multiply the length of all edges by some factor) so that the diameter of the quadrangulation remains bounded (in probability).
2. Local limit (Worm's-eye view). Do not rescale and understand the random infinite network obtained as $n \to \infty$.

A large quadrangulation



After rescaling



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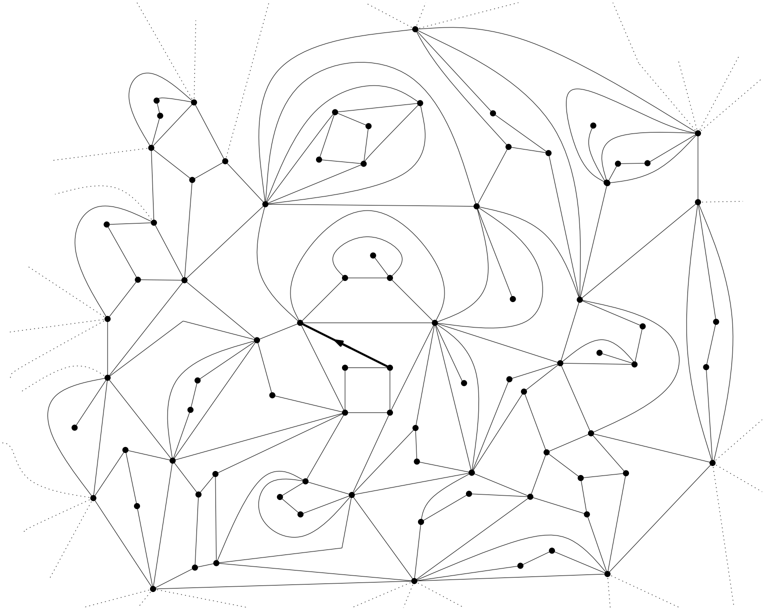
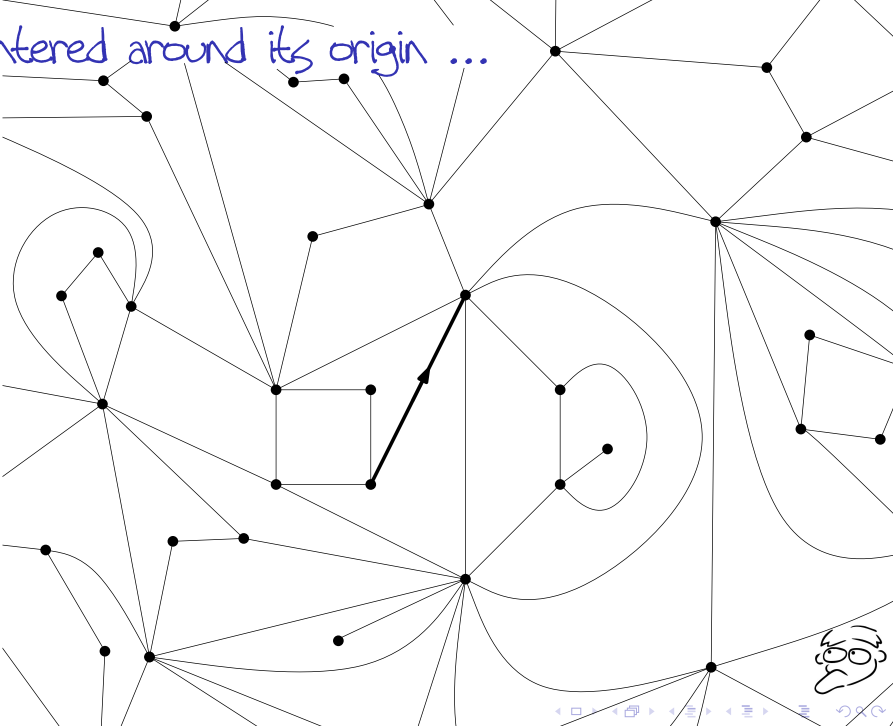
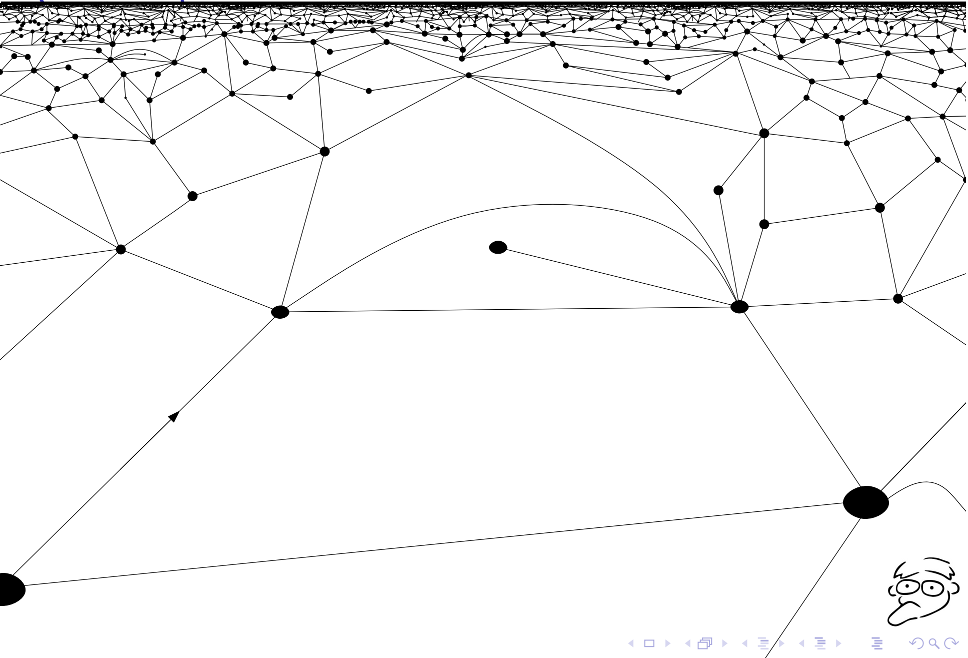


Centered around its origin ...



Another point of view



Scaling limit

Theorem (Le Gall ∐ Miermont (11))

We have the following convergence

$$\left(Q_n, n^{-1/4}\mathrm{d}_{\mathrm{gr}}\right) \xrightarrow[n\to\infty]{(d)} Cst.(\mathrm{m}_\infty, D^*),$$

in distribution for the Gromov-Hausdor topology. The object (m_∞, D^*) *is a random compact metric space called*

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a.s. homeomorphic to $\mathbb{S}_2$ [Le Gall & Paulin 06] and [Miermont 08]
Notice the "strange" $1/4$ to be explained later on.

Local limit

Theorem (Krikun (05), after Angel & Schramm (03))

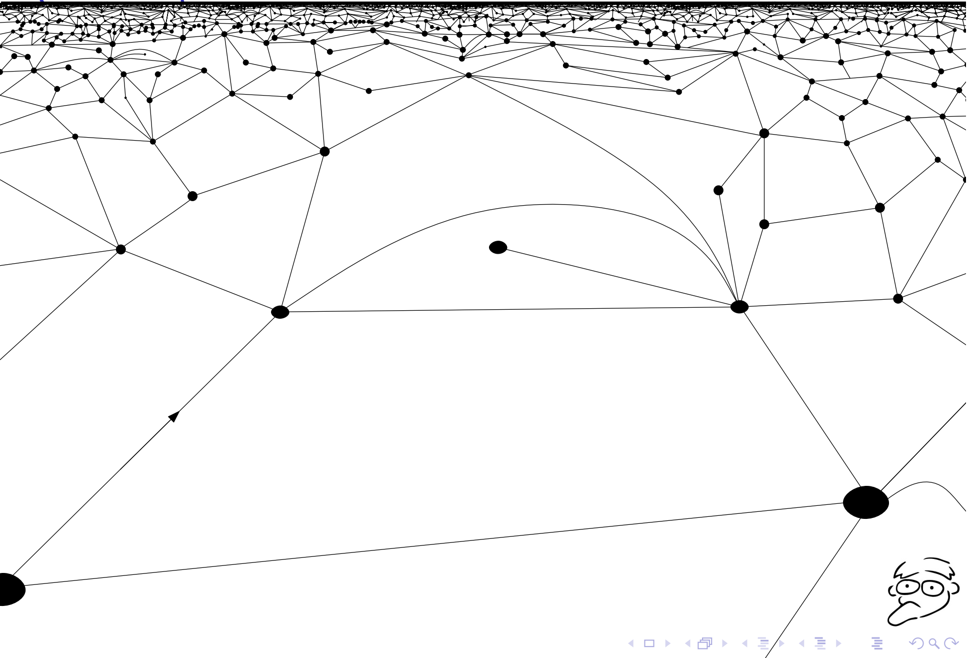
For every $r \geqslant 0$, we have the following convergence in distribution

$$\mathrm{Ball}(Q_n, r) \xrightarrow[n\to\infty]{(d)} \mathrm{Ball}(Q_\infty, r).$$

The object Q_∞ is a random (rooted) infinite quadrangulation called

" the Uniform Infinite Planar Quadrangulation (UIPQ)".

Another point of view



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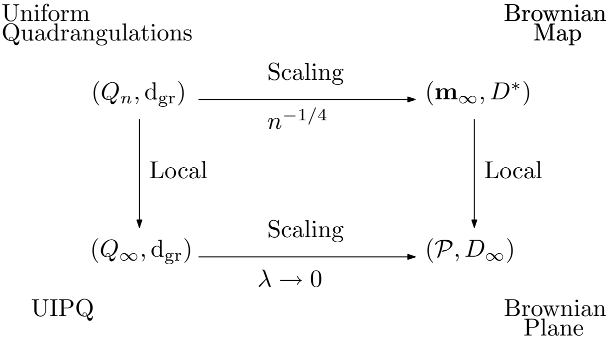
This result is much easier than the convergence towards the Brownian map and follows from enumerative formulæ.

We have come full circle

Theorem (C. & Le Gall (12))

The following diagram commutes

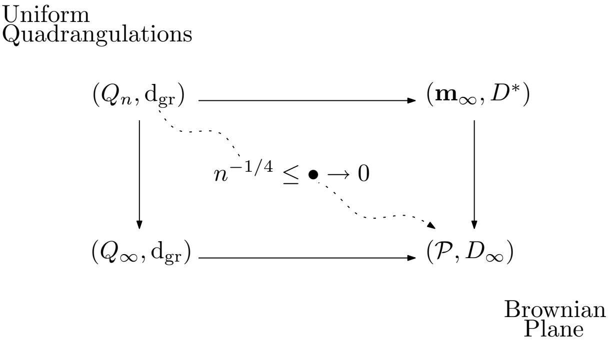


The Brownian Plane $(\mathcal{P}, D_\infty)$ is a random (locally compact) metric space that is homeomorphic to the plan $\mathbb{R}^2$ and of Hausdor dimension 4. Furthermore its distribution is invariant under dilation.

We have come full circle

Theorem (C. & Le Gall (12))

We also have



3. A beautiful bijection

Theorem (Cori-Vauquelin (81), Schaeffer (98))

There exists a bijection (with wonderful properties) between the set of all rooted and pointed quadrangulations with n faces and labeled planar trees with n edges plus a coin flip.

Recall :

- rooted = distinguished oriented edge,
  pointed = distinguished vertex
- planar trees = genealogical trees,
  labeling : 1-Lipschitz map ℓ : Tree $\rightarrow \mathbb{Z}$ with $\ell(\mathrm{root}) = 0$.

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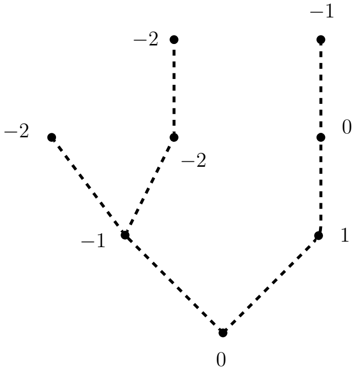
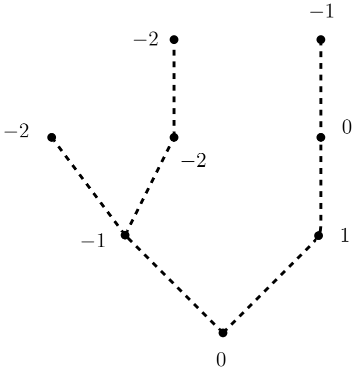
This proves

$$\#\{\to, \bullet \text{ quadrangulations with } n \text{ faces}\} = \underbrace{3^n}_{\textit{labels}} \cdot \underbrace{2}_{\textit{coin}} \cdot \underbrace{\frac{1}{n+1}\binom{2n}{n}}_{\textit{trees}}.$$


−1
−2
−2
0
−2
−1
1
0

Recipe :

- Add a vertex ∂ outside the tree.



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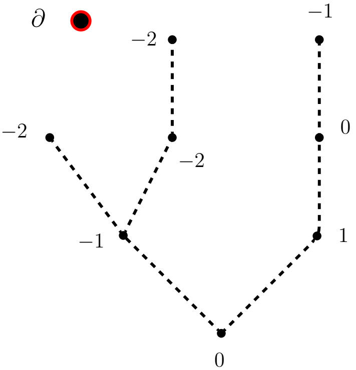

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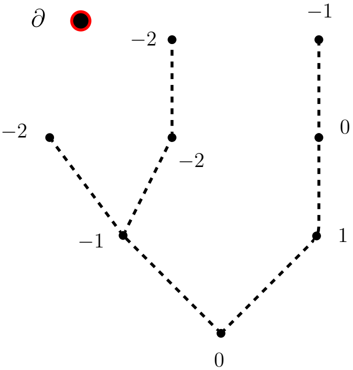
- Do the contour of the tree and link each corner of label i to the next corner of label $i-1$,
- If i is the minimal label, link it to ∂.



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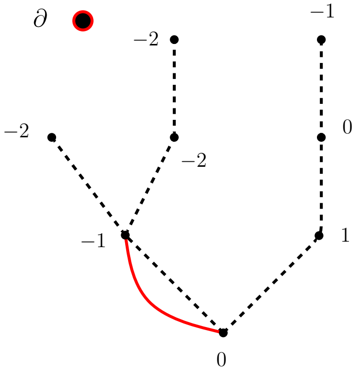

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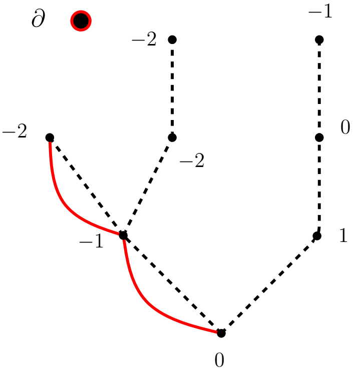

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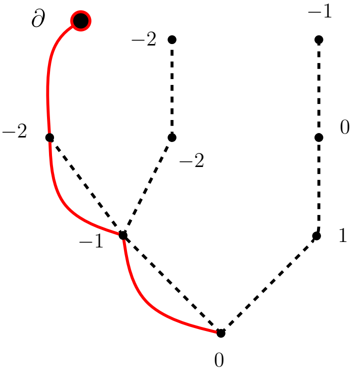

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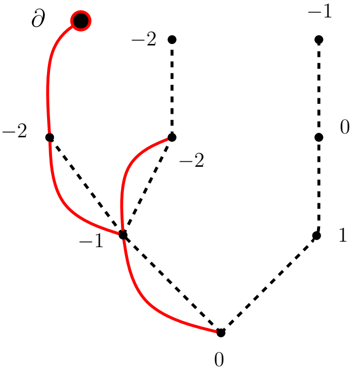

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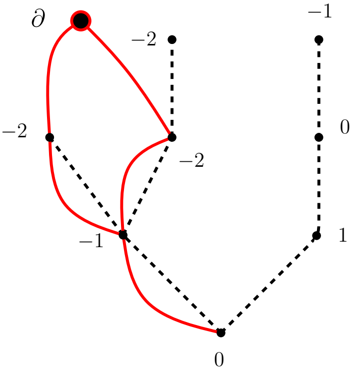

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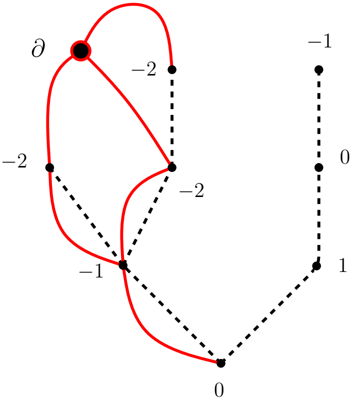

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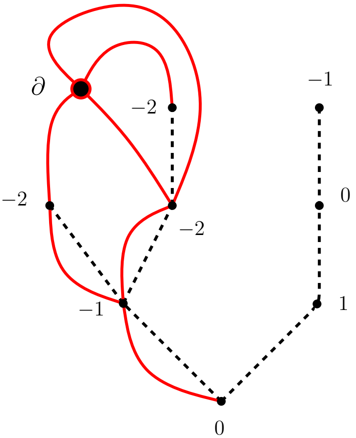

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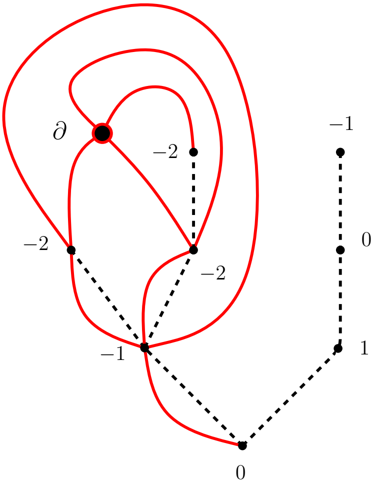

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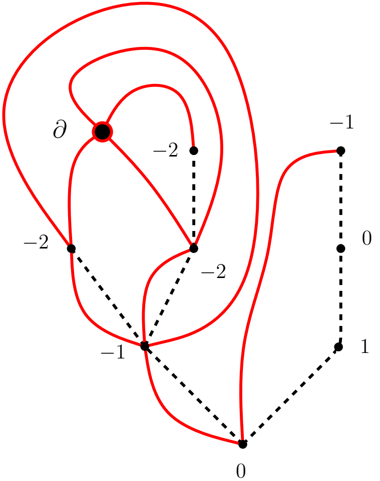

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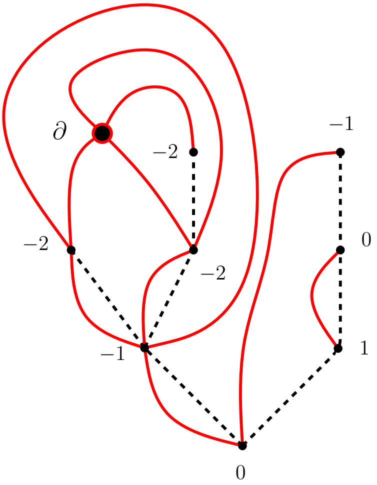

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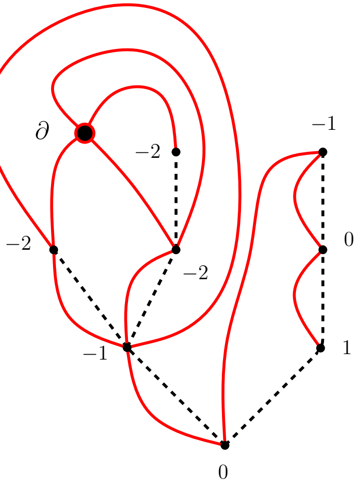

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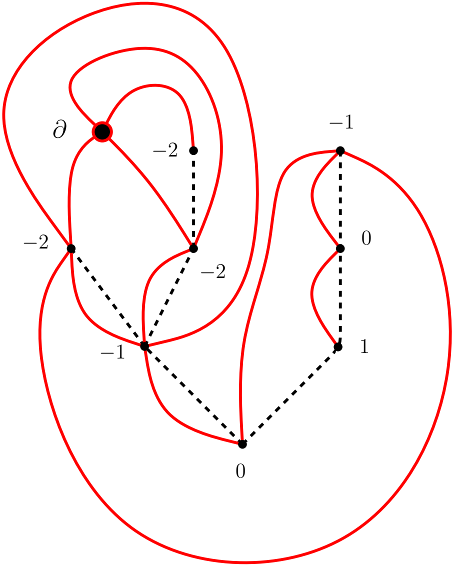

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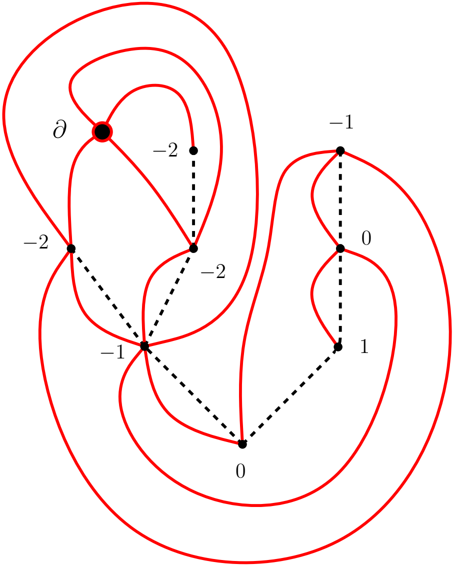

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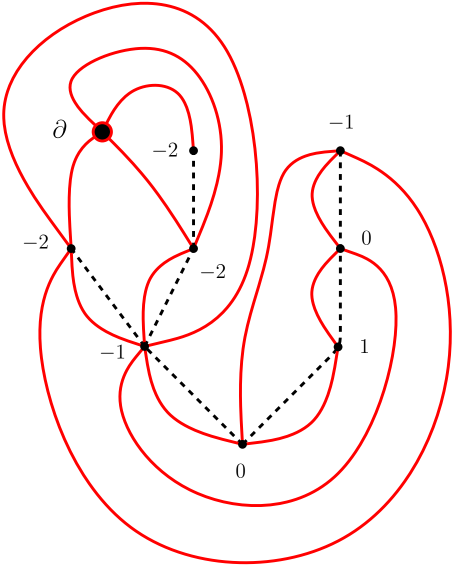

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- Add a vertex ∂ outside the tree.
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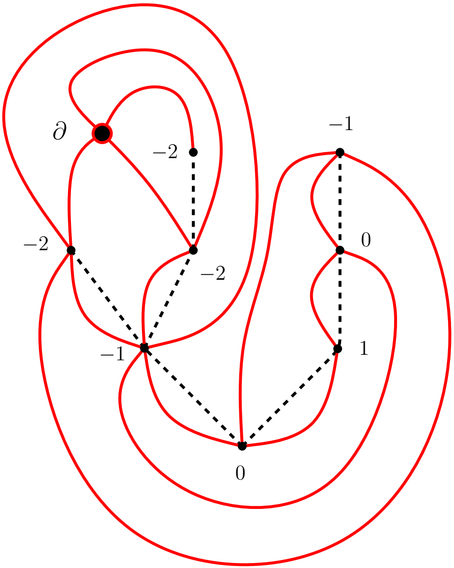
- Root the first edge drawn according to the coin flip.



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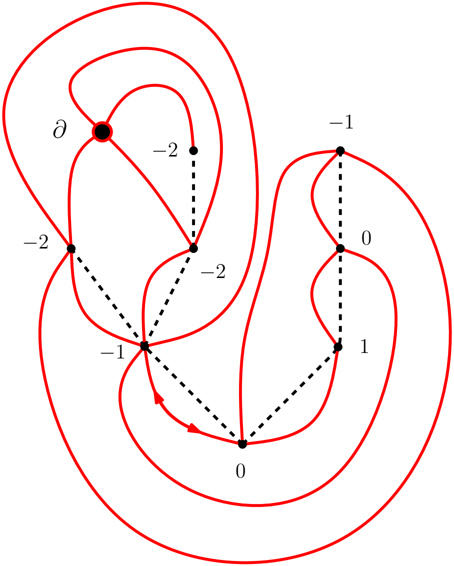
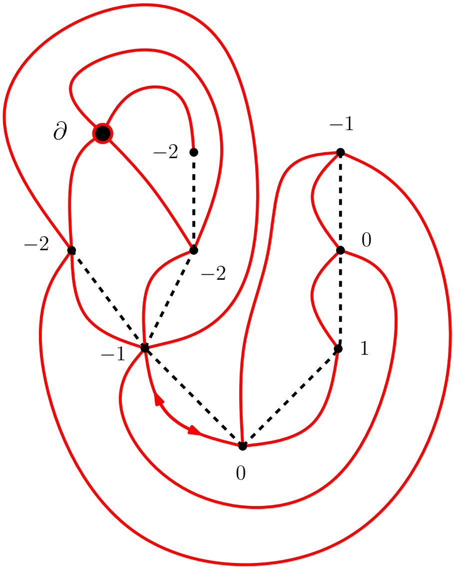


Main property :

For every vertex $u \in \mathrm{Tree}$ we have

$$d_{gr}^{quad}(u, \partial) =$$

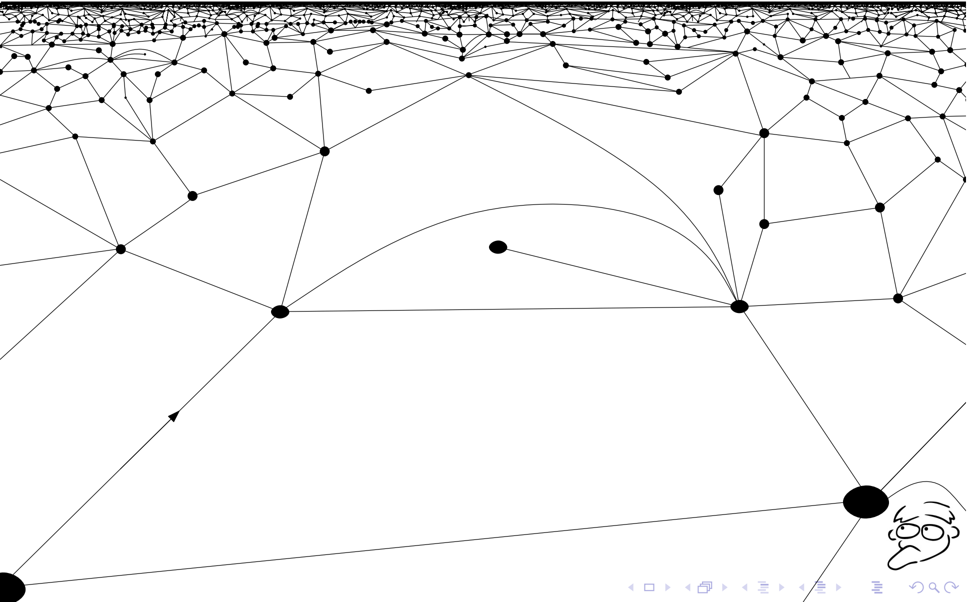
$$\ell(u) - \min \ell + 1.$$



Consequences

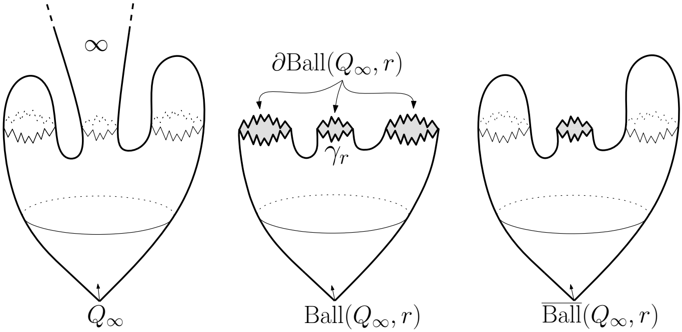
- Easy generation of large quadrangulations.
- Extension to the infinite setting to generate the UIPQ [Chassaing, Durhuus (06)] [C., Ménard, Miermont (12)].
- Explanation of the $n^{1/4}$ [Chassaing, Schaeffer (04)].
- Construction of the Brownian Map [Le Gall (07)].

4. More on the UTPQ



Geometric properties

Definition :



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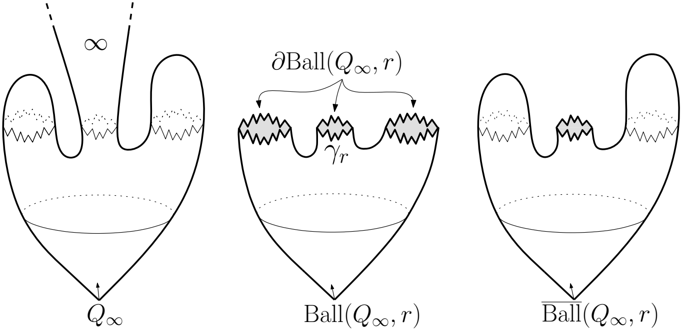
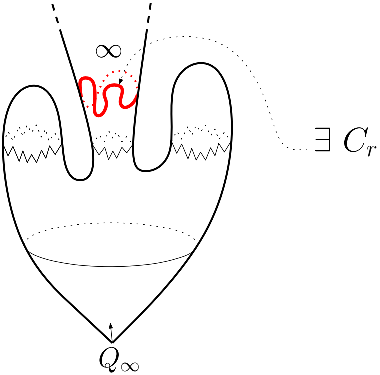


$$\begin{aligned}
|\mathrm{Ball}(Q_\infty, r)| &\approx r^{4}, \\
|\overline{\mathrm{Ball}}(Q_\infty, r)| &\approx r^{4}, \\
|\partial\mathrm{Ball}(Q_\infty, r)| &\approx r^{3}, \\
|\gamma_r| &\approx r^{2}.
\end{aligned}$$

[Angel] [Chassaing-Durhuus] [Krikun]

Isoperimetry



[Krikun] There exists a cycle at height $\approx r$ which separates the origin from ∞ whose length is $\approx r$. It is optimal [C., Le Gall]

Statistical models on the UIPQ

Theorem (Angel & C. (12+))

The critical parameter for bond percolation on the UIPQ is almost surely $p_c^{\mathrm{bond}} = \frac{2}{3}$.

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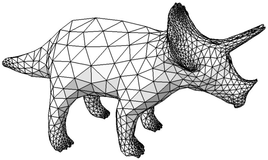
The critical parameter for bond percolation on the UIPQ is almost surely $p_c^{\mathrm{bond}} = \frac{2}{3}$. *Critical exponents (work in progress).*

Theorem (Benjamini & C. (12))

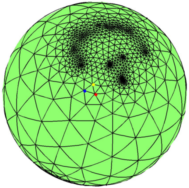
Conditionally on the UIPQ, let $(X_n)_{n \geqslant 0}$ *be a simple random walk started from the origin. Then we have*

$$\mathrm{d_{gr}}(X_0, X_n) \quad \preceq \quad n^{1/3}.$$

Perspectives

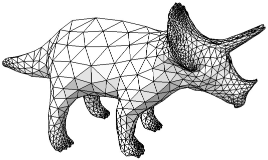


“uniformization” $\longrightarrow$

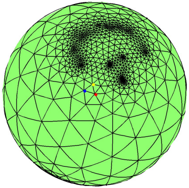


Pictures taken from Xianfeng David Gu.

Perspectives



“uniformization” $\longrightarrow$



Pictures taken from Xianfeng David Gu.

Replace the Triceratops by a random planar map and study the measure induced on $\mathbb{S}_2$ (KPZ).

Thank you for your attention !